Problem Set 5

Exercise 1: Present-Value Model of the Current Account

$$NO_t \equiv Y_t - I_t - G_t \quad \text{exogenous for households}$$

1(a)

$$B_{t+1} = (1+r)B_t + NO_t - C_t$$

$$\begin{aligned}
B_t &= \frac{C_t - NO_t}{1+r} + \frac{B_{t+1}}{1+r} \\
B_{t+1} &= \frac{C_{t+1} - NO_{t+1}}{1+r} + \frac{B_{t+2}}{1+r} \\
&\vdots \\
B_{t+T-1} &= \frac{C_{t+T-1} - NO_{t+T-1}}{1+r} + \frac{B_{t+T}}{1+r}
\end{aligned}$$

$$B_t = \sum_{s=0}^{T-1} \frac{C_{t+s} - NO_{t+s}}{(1+r)^{s+1}} + \frac{B_{t+T}}{(1+r)^T}$$

$$\text{transverality condition} \quad \lim_{T\to\infty} \frac{B_{t+T}}{(1+r)^T} = 0$$

$$B_t = \sum_{s=0}^{\infty} \frac{C_{t+s} - NO_{t+s}}{(1+r)^{s+1}}$$

$$\text{PVIBC} \qquad \underbrace{(1+r)B_t}_{\text{initial financial wealth}} + \underbrace{\sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}}_{\text{PV lifetime income}} = \underbrace{\sum_{s=0}^{\infty} \frac{C_{t+s}}{(1+r)^s}}_{\text{PV lifetime consumption}}$$

1(b)

$$\begin{aligned}
&\max_{C_{t+s}, B_{t+s+1}} \qquad U_t = \sum_{s=0}^{\infty} \beta^s u(C_{t+s}) \\
&\text{s.t.} \qquad B_{t+s+1} = (1+r)B_{t+s} + NO_{t+s} - C_{t+s}
\end{aligned}$$

$$\text{FOC:} \quad \frac{\partial U_t}{\partial B_{t+s+1}} = \beta^s u'(C_{t+s}) \cdot (-1) + \beta^{s+1} u'(C_{t+s+1})(1+r) = 0$$

$$\frac{\beta u'(C_{t+s+1})}{u'(C_{t+s})} = \frac{1}{1+r} \quad \textbf{Euler Equation}$$

Under $\beta = \frac{1}{1+r}$

$$u'(C_{t+s+1}) = u'(C_{t+s}) \implies C_{t+s+1} = C_{t+s} = \overline{C}$$

$$\sum_{s=0}^{\infty} \frac{C_{t+s}}{(1+r)^s} = (1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}$$

$$\sum_{s=0}^{\infty} \frac{\overline{C}}{(1+r)^s} = (1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}$$

$$\frac{1+r}{r}\overline{C} = (1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}$$

$$\overline{C} = \frac{r}{1+r}\left[(1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}\right]$$

$$\overline{C} = \frac{r}{1+r}\underbrace{\left[(1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}\right]}_{\text{net wealth}}$$

$$C_t = r \cdot \underbrace{\frac{w}{1+r}}_{\text{discounted net wealth}}$$

1(c)

(i)

$$B_{t+1} = (1+r)B_t + NO_t - C_t$$
$$\Delta B_{t+1} = r \cdot B_t + NO_t - C_t$$
$$CA_t = \Delta B_{t+1} = r \cdot B_t + NO_t - C_t$$

$$\begin{aligned} CA_t &= rB_t + NO_t - \frac{r}{1+r}\left[(1+r)B_t + \sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s}\right] \\ &= -\frac{r}{1+r}\left[\sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s} - \frac{1+r}{r}NO_t\right] \quad \underline{\sum_{s=0}^{\infty} \frac{1}{(1+r)^s} = \frac{1+r}{r}} \\ &= -\frac{r}{1+r}\left[\sum_{s=0}^{\infty} \frac{NO_{t+s}}{(1+r)^s} - \sum_{s=0}^{\infty} \frac{NO_t}{(1+r)^s}\right] \\ &= -\frac{r}{1+r}\left[\sum_{s=0}^{\infty} \frac{NO_{t+s} - NO_t}{(1+r)^s}\right] \end{aligned}$$

Exercise 2: Isoelastic Utility

$$X \sim \mathcal{N}(\mu_X, \sigma_X^2)$$

$\exp(X)$ is log-normal with mean $E[\exp(X)] = \exp\left(\mu_X + \frac{1}{2}\sigma_X^2\right)$

2(a)

$$\textbf{bond-Euler equation} \quad \underbrace{\beta E_t\left[\frac{u'(C_{t+1})}{u'(C_t)}\right]}_{\textbf{IMRS}} = \frac{1}{1+r} \quad \textbf{with} \quad \beta = \frac{1}{1+r}$$

$$u(C_t) = \frac{C_t^{1-\rho} - 1}{1-\rho} \implies \begin{cases} u'(C_t) = C_t^{-\rho} \\ u''(C_t) = -\rho C_t^{-\rho-1} \end{cases}$$

$$\beta E_t\left[\frac{u'(C_{t+1})}{u'(C_t)}\right] = \frac{1}{1+r}$$
$$E_t\left[\frac{u'(C_{t+1})}{u'(C_t)}\right] = 1$$
$$E_t\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\rho}\right] = 1$$
$$E_t\left[\exp\left(-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right)\right] = 1$$

$$\frac{C_{t+1}}{C_t} \text{ is log-normally distributed}$$
$$\ln\left(\frac{C_{t+1}}{C_t}\right) \text{ is normally distributed}$$
$$\exp\left(-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right) \text{ is log-normally distributed}$$

$$E_t\left[\exp\left(-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right)\right] = 1$$
$$\exp\left(E_t\left[-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right] + \frac{1}{2}Var_t\left[-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right]\right) = 1 \quad \text{take log of both sides}$$
$$E_t\left[-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right] + \frac{1}{2}Var_t\left[-\rho\ln\left(\frac{C_{t+1}}{C_t}\right)\right] = 0$$
$$E_t\left[\ln\left(\frac{C_{t+1}}{C_t}\right)\right] = \underbrace{\frac{1}{2}\rho Var_t\left[\ln\left(\frac{C_{t+1}}{C_t}\right)\right]}_{\text{constant}}$$

$$E_t[\ln(C_{t+1})] - E_t[\ln(C_t)] = a$$
$$E_t[\ln(C_{t+1})] = a + \ln C_t$$
$$\ln C_{t+1} = a + \ln C_t + \varepsilon_{t+1}$$
$$\textbf{Random Walk with dift } a$$

2(b)

$$\textbf{AR(1) with a drift}$$
$$\begin{cases} \ln C_{t+1} = a + \ln C_t + \varepsilon_{t+1} \\ \ln C_t = a + \ln C_{t-1} + \varepsilon_t \\ \qquad\vdots \\ \ln C_2 = a + \ln C_1 + \varepsilon_2 \end{cases}$$
$$\Downarrow$$
$$\ln C_{t+1} = a\cdot t + \ln C_1 + \sum_{i=2}^{t+1}\varepsilon_i$$

$$\ln C_{t+1} = \underbrace{a\cdot t}_{\text{deterministic trend}} + \underbrace{\ln C_1}_{\text{initial value}} + \underbrace{\sum_{i=2}^{t+1}\varepsilon_i}_{\text{stochastic trend}}$$
$$\textbf{Non-stationary}$$

$$\ln C_{t+1} = a + \ln C_t + \varepsilon_{t+1}$$
$$\text{First Differences:} \quad \Delta\ln C_{t+1} = a + \varepsilon_{t+1}$$
$$\textbf{Stationary}$$