Exercises

Social Choice Theory

Wenjie Tu

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Part 1: Preliminaries

Exercise 1.1

(a)

Let $X = \{x, y, z\}$. A binary relation R on the set X is as follows:

$$R = \{(x, y), (y, z), (z, y)\}$$

The asymmetric part is:

$$P = \{(x, y)\}$$

- The induced asymmetric part $P = \{(x, y)\}$ is transitive, hence R is quasi-transitive.
- $(x, y) \in R$ and $(y, z) \in R$ while $(x, z) \notin R$, a contradiction to transitivity.

Therefore $R = \{(x, y), (y, z), (z, y)\}$ is quasi-transitive but not transitive.

(b)

Let $X = \{x, y, z\}$. A binary relation R on the set X is as follows:

$$R = \{(x, y), (y, z)\}$$

Its asymmetric part P is:

$$P = \{(x, y), (y, z)\}$$

- Since $(x, z) \notin P$, P is not transitive and R is not quasi-transitive.
- Since $(z, x) \notin P$, P has no cycle and R is acyclical.

(c)

Need to show:

$$xIy \wedge yIz \implies xIz$$

Proof:

We can rewrite induced I into binary relation:

$$
\begin{aligned}
xIy &\implies [xRy \wedge yRx] \\
yIz &\implies [yRz \wedge zRy]
\end{aligned}
$$

$$
\begin{aligned}
xRy \wedge yRz &\implies xRz \quad \text{by transitivity} \\
yRx \wedge zRy &\implies zRx \quad \text{by transitivity}
\end{aligned}
$$

$$
xRz \wedge zRx \implies xIz
$$

Exercise 1.2

(a)

$\succeq$ is a preference

$$
\begin{gathered}
x \sim y \\
y \succ z \\
z \succ w
\end{gathered}
$$

$$
\begin{aligned}
G(\{w, x, y, z\}) &= \{x, y\} \\
M(\{w, x, y, z\}) &= \{x, y\}
\end{aligned}
$$

$$
R = \{(x, y), (y, x), (y, z), (z, w)\}
$$

(b)

$$
M(\{x, y, z\}, R) = \{\} = G(\{x, y, z\})
$$

Exercise 1.3

(a)

We know that x is chosen in the larger set (i.e. $C(\{x, y, z\}) = \{x\}$), and we need to check whether x is still chosen from all the possible subset containing x.

$$
\begin{aligned}
C(\{x, y\}) &= \{x\} \\
C(\{x, z\}) &= \{x\}
\end{aligned}
$$

Hence, property α is satisfied.

We cannot find any two alternatives that are generate by the choice function. Hence, we cannot find a counter example that violates property β (i.e. β is satisfied).

(b)

We know that z is chosen from the larger set (i.e. $C(\{x, y, z\}) = \{z\}$), and we need to check whether z is still chosen from all the possible subset containing z.

Base relation:

$$C(\{x, z\}) = \{x, z\}$$
$$C(\{y, z\}) = \{z\}$$

Preference:

$$x \sim z \succ y$$

Hence, property α is satisfied.

We see that $\{x, z\}$ is chosen from the smaller set $\{x, z\}$ (i.e. $C(\{x, z\}) = \{x, z\}$), and z is chosen from the larger set $\{x, y, z\}$ while x is not, hence a violation of property β.

(c)

We know that $\{y, z\}$ is chosen from the larger set $\{x, y, z\}$, and we need to check whether $\{y, z\}$ is still chosen from all the possible subset containing $\{y, z\}$.

We see:

$$C(\{y, z\}) = \{y, z\}$$

Hence, property α is satisfied.

We see that $\{y, z\}$ is chosen from the smaller set $\{y, z\}$, and $\{y, z\}$ is chosen from the larger set containing both y and z. Hence, property β is satisfied.

Exercise 1.4

Shorthands for choices:

- P: peatnuts
- A: apple juice
- M: mineral water
- B: beer

(a)

$$X = \{P, A, M, B\}$$

$$C(\{P, A, M\}) = \{P, A\}$$

$$C(\{P, A, M, B\}) = \{P, B\}$$

β is violated?

Better translation of the information:

$$X = \{PA, PM, PB\}$$

$$C(\{PA, PM\}) = \{PA\}$$

$$C(\{PA, PM, PB\}) = \{PB\}$$

β is satisfied.

Part 2: The Problem of Social Choice

Exercise 2.1

(a)

#	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
1	$z\ P\ x\ P\ y$

Pairwise comparison	Votes	Social preference
x vs.y	2 vs.1	xPy
x vs.z	1 vs.2	zPx
y vs.z	2 vs.1	yPz

This leads to an outcome that is not transitive and not acyclical.

There is no Condorcet winner (loser) as each alternative loses in the pairwise voting at once.

(b)

Fix any $R^* \in R$, let $\mathscr{A} = \{(R^*, R^*, \cdots, R^*)\}$. In other words, all individuals share the same preference. In this example, the pairwise majority voting is an SWF.

Exercise 2.2

When $m = 2$, PV, IR, PM, CO, and BC are identical. They all collapse to the majority voting method.

Exercise 2.3

#	preferences
2	$x\ P\ w\ P\ y\ P\ z$
2	$y\ P\ w\ P\ z\ P\ x$
1	$w\ P\ z\ P\ x\ P\ y$

PV:

In plurality voting, we only consider the top-ranked alternatives given by voters.

$$x \; I \; y \; P \; w \; P \; z$$

IR:

- Stage 1: z is eliminated
- Stage 2: w is eliminated
- Stage 3: x wins with 3 votes (majority)

PM:

pairwise comparison	votes	binary relation
$w : x$	$3 : 2$	wPx
$w : y$	$3 : 2$	wPy
$w : z$	$5 : 0$	wPz
$x : y$	$3 : 2$	xPy
$x : z$	$2 : 3$	zPx
$y : z$	$4 : 1$	yPz

The pairwise majority voting method delivers a Condorcet winner w but there exists a cycle in x, y, z.

CO:

	w	x	y	z	$\sum$
w		$+1$	$+1$	$+1$	3
x	-1		$+1$	-1	-1
y	-1	-1		$+1$	-1
x	-1	$+1$	-1		-1

Copeland method delivers the social preference $w \; P \; x \; I \; y \; I \; z$.

BC:

#	preferences	w	x	y	z
2	$x \; P \; w \; P \; y \; P \; z$	2	3	1	0
2	$y \; P \; w \; P \; z \; P \; x$	2	0	3	1
1	$w \; P \; z \; P \; x \; P \; y$	3	1	0	2
		11	7	8	4

Borda Count method delivers the social preference $wPyPxPz$

EM:

There exists at least one individual who ranks x, y, w as his/her top preference, hence the Pareto efficient set is $X^E = \{w, x, y\}$. All individuals strictly prefer w over z, hence z is strictly dominated by w. $X^I = \{z\}$.

Pareto Efficient Method delivers the social preference $wIxIyPz$.

PE:

pairwise comparison	Pareto extension rule
$w : x$	wIx
$w : y$	wIy
$w : z$	wPz
$x : y$	xIy
$x : z$	zIx
$y : z$	yIz

Pareto Extension Rule delivers an outcome that violates transitivity but is quasi-transitive. This is SDF but not SWF.

Exercise 2.4

(a)

#	preferences
2	$x\ P\ z\ P\ y$
2	$y\ P\ z\ P\ x$
1	$z\ P\ y\ P\ x$

Applying IR:

- Stage 1: z is eliminated.
- Stage 2: y wins with 3 votes (majority).

pairwise comparison	votes	binary relation
$x : y$	$2 : 3$	yPx
$x : z$	$2 : 3$	zPx
$y : z$	$2 : 3$	zPy

- z wins all pairwise comparison hence a Condorcet winner.
- IR is not a Condorcet method.

(b)

Proof by contradiction:

No, a Condorcet winner is Pareto efficient, otherwise it would lose at least one pairwise vote.

(c)

We can construct the following example resulting x to be a Condorcet winner:

#	preferences
2	$x\ P\ y$
1	$y\ P\ x$

There is no Pareto dominance between x and y so EM delivers social indifference between x and y (i.e. xIy).

Same argument for PE.

Exercise 2.5

(a)

For arbitrary values of m:

- Anti-plurality voting and rejection voting are equivalent.
- Nameless example I and nameless example II are equivalent.

For $m = 3$:

- Anti-plurality voting and rejection voting are equivalent.
- Borda count, nameless example I, and nameless example II are equivalent.

(b)

$$s = (m, m-1, \cdots, 1)$$

(c)

$$s = (m^2, (m-1)^2, \cdots, 1^2)$$

Exercise 2.6

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1	3	0	2
1	$y\,P\,z\,P\,v\,P\,x$	1	0	3	2
1	$v\,P\,z\,P\,y\,P\,x$	3	0	1	2
1	$x\,P\,y\,P\,v\,P\,z$	1	3	2	0
		6	6	6	6

s^1 delivers social preference $vIxIyIz$

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1/4	1	0	3/4
1	$y\,P\,z\,P\,v\,P\,x$	1/4	0	1	3/4
1	$v\,P\,z\,P\,y\,P\,x$	1	0	1/4	3/4
1	$x\,P\,y\,P\,v\,P\,z$	1/4	1	3/4	0
		7/4	8/4	8/4	9/4

s^2 delivers social preference $zPxIyPv$

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1/4	1	0	1/2
1	$y\,P\,z\,P\,v\,P\,x$	1/4	0	1	1/2
1	$v\,P\,z\,P\,y\,P\,x$	1	0	1/4	1/2
1	$x\,P\,y\,P\,v\,P\,z$	1/4	1	1/2	0
		7/4	8/4	7/4	6/4

s^3 delivers social preference $xPvIyPz$

Part 3: Arrow's Theorem

Exercise 3.1

(a) Plurality voting (PV)

[I] Independence of Irrelevant Alternatives Axiom is violated.

The counter example is given as follows:

$\mathbf{R}$	preferences		$\mathbf{R}'$	preferences
2	$x\,P\,y\,P\,z$		2	$z\,P\,x\,P\,y$
1	$y\,P\,x\,P\,z$	$\rightarrow$	1	$y\,P\,x\,P\,z$
1	$x\,P\,z\,P\,y$		1	$x\,P\,z\,P\,y$
f^{PV}	$x\,P\,y\,P\,z$		f^{PV}	$z\,P\,y\,I\,x$

Note: z is the irrelevant alternative. We change the preferences for z for the first two people while maintaining the relative positions for x and y (i.e. xPy). From $\mathbf{R} \rightarrow \mathbf{R}'$, we see social preference between x and y changes, hence a violation of [I].

[P] Weak Pareto Principle is violated.

The counter example is given as follows:

$\mathbf{R}$	preferences
2	$z\,P\,x\,P\,y$
f^{PV}	$z\,P\,x\,I\,y$

[U] Universality is satisfied.

- Plurality voting (PV) is well-defined method and always applicable as there are no constraints in definition/preferences.
- The resulting outcomes are preferences as plurality voting method is a ranking method based on numbers/votes. Numbers are transitive, hence outcomes are transitive (PV is a SWF).

[D] Non-dictatorship is satisfied.

Each individual has only one vote. In order to impose the own preference on the whole society, the dictator must have "majority" votes, which is not possible for plurality voting where each individual has one single vote.

Or proof by contradiction.

(b) Pairwise majority voting (PM)

[I] Independence of Irrelevant Alternatives Axiom is satisfied.

Pairwise majority voting method is a pairwise method by construction. To determine the ranking between x and y, we do pairwise comparison. If preferences between x and y do not change, the outcome does not change.

[D] Non-dictatorship is satisfied.

As long as the majority is against you, you cannot enforce your own preference on the society. This holds for everybody. Nobody can be a dictator.

[P] Weak Pareto Principle is satisfied.

If everybody has a strict preference for x over y, the outcome will be an unanimous vote for x over y as we only do pairwise comparison, hence a strict preference for x over y for the society.

[U] Universality is violated.

A counter example is constructed as follows:

#	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
1	$z\ P\ x\ P\ y$

The example given above leads to an acyclical preferences. This is not a SWF on the full domain. We can easily get violations of acyclicity and transitivity.

(c) Copeland method (CO)

[U] Universality is satisfied.

Copeland method by definition is a number-based voting method and hence it is always applicable as numbers are always transitive.

[D] Non-dictatorship is satisfied.

As long as the majority is against you, you cannot enforce your own preference on the society. This holds for everybody. Nobody can be a dictator.

[P] Weak Pareto Principle is satisfied.

Suppose x strictly Pareto dominates y, under Copeland method y loses every pairwise vote that x loses. In other words, whenever x loses, y loses as well.

$$CO(x, \mathbf{R}) \geq CO(y, \mathbf{R}) + 2$$

[I] Independence of Irrelevant Alternatives Axiom is violated.

A counter example is constructed as follows:

$\mathbf{R}$	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ x\ P\ z$
f^{CO}	$x\ I\ y\ P\ z$

$\rightarrow$

$\mathbf{R}'$	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
f^{CO}	$y\ P\ x\ P\ z$

Note: z is the irrelevant alternative. We change the preference for z for the second person while maintaining the relative positions for x and y (i.e. xPy). From $\mathbf{R} \rightarrow \mathbf{R}'$, we see social preference between x and y changes, hence a violation of [I].

(d) Borda count (BC)

[D] Non-dictatorship is satisfied.

Borda count method is formulated in a linear way for scoring vector and all votes use the same scoring vector, hence no dictatorship possible.

[U] Universality is satisfied.

Borda count is a number-based voting method so it always leads to transitive preferences. It is applicable to all preference profiles as there are no constraints in definition.

[P] Weak Pareto Principle is satisfied.

By definition, if $xP_iy\ \forall i$,

$$BC(x,\mathbf{R}) \geq BC(x,\mathbf{R}) + n$$

Hence, the society strictly prefers x over y.

[I] Independence of Irrelevant Alternatives Axiom is violated.

A counter example is constructed as follows:

$\mathbf{R}$	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ x\ P\ z$
f^{BC}	$x\ I\ y\ P\ z$

$\rightarrow$

$\mathbf{R}'$	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
f^{BC}	$y\ P\ x\ P\ z$

Note: z is the irrelevant alternative. We change the preference for z for the second person while maintaining the relative positions for x and y (i.e. xPy). From $\mathbf{R} \rightarrow \mathbf{R}'$, we see social preference between x and y changes, hence a violation of [I].

(e) Pareto Efficiency Method (EM)

[D] Non-dictatorship is satisfied.

Proof by contradiction: suppose there exists a dictator who strictly prefers x over y, the society is supposed to strictly prefer x over y. Now there is an individual who strictly prefers y over x, by definition of Pareto Efficiency Method, the society should be indifferent between x and y as there is no Pareto dominance between x and y.

[P] Weak Pareto Principle is violated.

#	preferences
2	$z\ P\ x\ P\ y$
f^{EM}	$z\ P\ x\ I\ y$

- $X^E = \{z\}$
- $X^I = \{x, y\}$

All individuals strictly prefer x over y but by applying Pareto Efficiency Method the society is indifferent between x and y, hence a violation of Weak Pareto Principle.

[I] Independence of Irrelevant Alternatives Axiom is violated.

A counter example is constructed as follows:

$\mathbf{R}$	preferences
2	$z\ P\ x\ P\ y$
f^{EM}	$z\ P\ x\ I\ y$

$\rightarrow$

$\mathbf{R}'$	preferences
2	$x\ P\ y\ P\ z$
f^{EM}	$x\ P\ y\ I\ z$

Note: z is the irrelevant alternative. We change the preferences for z while maintaining the relative positions for x and y (i.e. xPy) for all individuals. From $\mathbf{R} \rightarrow \mathbf{R}'$, we see social preference between x and y changes, hence a violation of [I].

[U] Universality is satisfied.

- It is well-defined and always applicable.
- It always gives us preferences with at most two indifference classes.

(f) The preference of society coincides with the preference of individual $h \in N$

[D] Non-dictatorship is violated.

By definition, the individual h can always impose the own preference on the whole society and hence a dictator.

[U] Universality is satisfied.

- There are obviously no constraints built into the method and it is always applicable no matter how people's preferences look like.
- The dictator's preference is a preference. By definition of this method, the social preference will be a preference.

[P] Weak Pareto Principle is satisfied.

If everybody strictly prefers x over y, then the dictator also strictly prefers x over y because the dictator is one of those people.

[I] Independence of Irrelevant Alternatives Axiom is satisfied.

Social preference is only determined by the dictator's preference. [I] holds for a single individual hence it also holds for the society with dictatorship.

(g) Construct an example where all axioms but [P] are satisfied

Fix any $R^* \in R$ and define $f(\mathbf{R}) = R^* \; \forall \, \mathbf{R} \in \mathscr{R}^U$

P is violated

I is satisfied

U is satisfied. The outcome will be a preference and the method is well-defined on the universal domain

D is satisfied. R^* is arbitrary. Only the rule maker decides the dictator. Voters themselves do not know which one of them will be the dictator.

Exercise 3.2

[$\bar{\text{M}}$] is violated.

A counter example is constructed as follows:

$\mathbf{R}$	preferences		$\mathbf{R}'$	preferences
2	$x\,P\,z\,P\,y$		2	$x\,P\,z\,P\,y$
2	$y\,P\,z\,P\,x$	$\rightarrow$	2	$z\,P\,y\,P\,x$
1	$z\,P\,x\,P\,y$		1	$z\,P\,x\,P\,y$
	x wins			z wins

[$\bar{\text{D}}$] is satisfied.

Proof by contradiction.

[$\bar{\text{U}}$] is satisfied.

- People are allowed to express whatever preferences they have. We have well-specified tie-breaking rules such that at the end there will always be a winner even if we go to the last stage.

- It is a well-specified method on the universal domain.

$[\bar{\mathrm{P}}]$ is satisfied.

Suppose $yP_ix \, \forall i$

1. As long as y is around, x gets zero votes
2. y can be eliminated only if it gets zero votes
3. Updated votes do not change after elimination of alternatives with zero votes
4. Either now or after some steps x will be eliminated
5. x cannot be the winner