Exam
Social Choice Theory
Spring 2017
Solution

Problem 1: Individual Choice Functions

(a) For α, we only need to check whether an alternative chosen from $\{x, y, z\}$ remains chosen in any binary set where it is still available. For β, we only need to check cases where more than one alternative is chosen from a binary set.

C^1: α is satisfied, because y is chosen whenever available.
β is satisfied, because $C^1(S)$ always contains exactly one element.

C^2: α is violated, because $x \in C^2(\{x, y, z\})$ but $x \notin C^2(\{x, z\})$.
β is satisfied, because $C^2(S)$ always contains exactly one element.

C^3: α is satisfied, because both x and z are chosen whenever available.
β is satisfied, because $C^3(\{x, z\}) = C^3(\{x, y, z\}) = \{x, z\}$.

C^4: α is satisfied, because z is chosen whenever available.
β is violated, because $C^4(\{y, z\}) = \{y, z\}$ but $C^4(\{x, y, z\}) = \{z\}$.

C^5: α is violated, because $x \in C^5(\{x, y, z\})$ but $x \notin C^5(\{x, z\})$.
β is violated, because $C^5(\{x, y\}) = \{x, y\}$ but $C^5(\{x, y, z\}) = \{x\}$.

(b) C^1: $R_{C^1} = \{(y, x), (z, x), (y, z), (x, x), (y, y), (z, z)\}$
C^2: $R_{C^2} = \{(x, y), (z, x), (y, z), (x, x), (y, y), (z, z)\}$
C^3: $R_{C^3} = \{(x, y), (x, z), (z, x), (z, y), (x, x), (y, y), (z, z)\}$
C^4: $R_{C^4} = \{(y, x), (z, x), (y, z), (z, y), (x, x), (y, y), (z, z)\}$
C^5: $R_{C^5} = \{(x, y), (y, x), (z, x), (z, y), (x, x), (y, y), (z, z)\}$

(c) C^1: R_{C^1} is transitive and rationalizes C^1, because α and β are satisfied.

C^2: R_{C^2} is not transitive, because $(x, y) \in R_{C^2}$ and $(y, z) \in R_{C^2}$ but $(x, z) \notin R_{C^2}$.
It does not rationalize C^2, because it would predict that $C^2(\{x, y, z\})$ is empty.

C^3: R_{C^3} is transitive and rationalizes C^1, because α and β are satisfied.

C^4: R_{C^4} is transitive. The only interesting case to verify is that $(y, x) \in R_{C^4}$ because $(y, z) \in R_{C^4}$ and $(z, x) \in R_{C^4}$. R_{C^4} does not rationalize C^4, because it would predict that $y \in C^4(\{x, y, z\})$.

C^5: R_{C^5} is transitive. The interesting cases to verify are that $(z, y) \in R_{C^5}$ because $(z, x) \in R_{C^5}$ and $(x, y) \in R_{C^5}$, and that $(z, x) \in R_{C^5}$ because $(z, y) \in R_{C^5}$ and $(y, x) \in R_{C^5}$. R_{C^5} does not rationalize C^5, because it would predict that $z \in C^5(\{x, y, z\})$.

Problem 2: Manipulability

(a) Arrow's axioms for SCFs:

[$\bar{\text{U}}$] Universality $\mathscr{A} = \mathscr{R}^3$ is satisfied by definition of the rule.

[$\bar{\text{M}}$] Monotonicity is satisfied.

In a given table, x maintaining position means moving left and/or up. Across tables, x maintaining position means moving left. The converse holds for y. It is then easy to see that a selected alternative remains selected whenever it maintains its position.

Alternatively, monotonicity also follows immediately once we observe that this SCF is just majority voting (with some tie-breaking rule).

[$\bar{\text{P}}$] The Weak Pareto Principle is satisfied, because x is selected when all voters have xP_iy, and y is selected when all voters have yP_ix.

[$\bar{\text{D}}$] Non-Dictatorship is satisfied, as the rule does not always select the most preferred alternative of one fixed voter.

(b) The SCF is surjective because each alternative is selected for some preference profile.

We have already verified [$\bar{\text{U}}$] and [$\bar{\text{D}}$].

Strategy-proofness [$\bar{\text{S}}$] is also satisfied. This follows from the fact that misrepresenting the own preference cannot impove the outcome for the respective agent with majority voting.

(c) The impossibility result states that there exists no surjective SCF that satisfies the axioms [$\bar{\text{U}}$], [$\bar{\text{D}}$], and [$\bar{\text{S}}$] when $m = 2$. This is no contradiction because we have $m = 2$ here.

Problem 3: Social Evaluation Functions

(a) The following table computes MAD for each alternative in the example:

	x	y	z
U_1	2	1	9
U_2	3	1	7
U_3	4	1	5
$\bar{U}$	3	1	7
MAD	2/3	0	4/3

Hence we obtain $y\ e_P^{\mathrm{MAD}}(\mathbf{U})\ x\ e_P^{\mathrm{MAD}}(\mathbf{U})\ z$.

(b) The SEF is consistent with CM-LC (and hence with RM-LC). Suppose $\mathbf{U}'$ is obtained from $\mathbf{U}$ by a common positive affine transformation $\varphi(u) = \alpha + \beta u$, where $\beta > 0$. Then, for any alternative $x \in X$ we have

$$\begin{aligned}\mathrm{MAD}(x, \mathbf{U}') &= \frac{1}{n}\sum_{i=1}^{n}\left|U_i'(x) - \bar{U}(x, \mathbf{U}')\right| \\ &= \frac{1}{n}\sum_{i=1}^{n}\left|\alpha + \beta U_i(x) - \alpha - \beta\bar{U}(x, \mathbf{U})\right| \\ &= \beta\mathrm{MAD}(x, \mathbf{U}).\end{aligned}$$

Hence the induced social preferences are identical for $\mathbf{U}$ and $\mathbf{U}'$.

The SEF is not consistent with any of the remaining information structures. To show this, counterexamples are now provided for OM-LC, CM-UC, and RM-NC.

OM-LC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by the common strictly increasing transformation $\varphi(u) = u^2$. We obtain $e^{\mathrm{MAD}}(\mathbf{U}) \neq e^{\mathrm{MAD}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	3	4
U_2	1	6
MAD	1	1

$\mathbf{U}'$	x	y
U_1'	9	16
U_2'	1	36
MAD	4	10

CM-UC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by positive affine transformations $\varphi_i(u) = \alpha_i + \beta u$ for $\beta = 1$, $\alpha_1 = 2$, and $\alpha_2 = 0$. We obtain $e^{\mathrm{MAD}}(\mathbf{U}) \neq e^{\mathrm{MAD}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	1	0
U_2	1	2
MAD	0	1

$\mathbf{U}'$	x	y
U_1'	3	2
U_2'	1	2
MAD	1	0

RM-NC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by positive linear transformations $\varphi_i(u) = \beta_i u$ for $\beta_1 = 2$ and $\beta_2 = 1$. We obtain $e^{\mathrm{MAD}}(\mathbf{U}) \neq e^{\mathrm{MAD}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	3	4
U_2	1	6
MAD	1	1

$\mathbf{U}'$	x	y
U_1'	6	8
U_2'	1	6
MAD	5/2	1