Problem Set 6

1. EU Theory

Prove

$$\overline{X} = \int x dF_L(x) \geq c(F_L, u) \quad \text{where } c(F_L, u) \text{ is certainty equivilence}$$

Jensens' inequality

If $u(\cdot)$ is concave:

$$u(\int x dF_L(x)) \geq \int u(x) dF_L(x)$$

Certainty equivalence

$$u(c(F_L, u)) = \int u(x) dF_L(x)$$

Take **Jensens' inequality** and **certainty equivalence** together

$$u(c(F_L, u)) = \int u(x) dF_L(x) \leq u(\int x dF_L(x))$$

$u(\cdot)$ is strictly increasing

$$u(c(F_L, u)) \leq u(\int x dF_L(x)) \iff c(F_L, u) \leq \int x dF_L(x)$$

2. State-Space Model

	probabilities	π_1	π_2
quantities	asset/state	state 1	state 2
x_1	asset 1	1	1
x_2	asset 2	0	$U[1,2]$

Utility functions in two states

$$\begin{cases} u_1(z) = \log(z) \\ u_2(z) = 2\log(z) \end{cases}$$

$$\begin{aligned} \max_{x_1, x_2} \quad & \pi_1 u_1(x_1 \times 1 + x_2 \times 0) + \pi_2 \int_1^2 u_2(x_1 \times 1 + x_2 \times r) dF(r) \\ \text{s.t.} \quad & \begin{cases} x_1 + x_2 \leq w \\ x_1 \geq 0 \\ x_2 \geq 0 \end{cases} \end{aligned}$$

$$u(\cdot) \text{ is strictly increasing in } x_1, x_2 \implies \text{Walras' law holds} \implies x_1 = w - x_2$$

$$
\begin{aligned}
\min_{x_1,x_2} \quad & -\pi_1\log(w-x_2)-(1-\pi_1)\int_1^2 2\log(w-x_2+rx_2)dr+\mu_1(x_2-w)+\mu_2(-x_2) \\
\text{s.t.} \quad & \begin{cases}\mu_1,\mu_2\geq 0 \\ \mu_1(x_2-w)=0, \mu_2(-x_2)=0 \quad \text{complementary slackness}\end{cases}
\end{aligned}
$$

Leibniz-rule

$$
\frac{d}{dx}\int_a^b u(x,t)dt=\int_a^b\frac{d}{dx}u(x,t)dt
$$

$$
L(x_2)=-\pi_1\log(w-x_2)-(1-\pi_1)\int_1^2 2\log(w-x_2+rx_2)dr+\mu_1(x_2-w)+\mu_2(-x_2)
$$

FOC:

$$
\frac{\partial L}{\partial x_2}=\frac{\pi_1}{w-x_2}-(1-\pi_1)\int_1^2\frac{2(r-1)}{w-x_2+rx_2}dr+\mu_1-\mu_2=0
$$

$$
\begin{cases}\mu_1,\mu_2\geq 0 \\ \mu_1(x_2-w)=0, \mu_2(-x_2)=0 \quad \text{complementary slackness}\end{cases}
$$

$$
\begin{gathered}
u(\cdot)\text{ is concave and constraint set is convex} \\
\Downarrow \\
\text{Convex optimization} \\
\Downarrow \\
\text{Solution is unique and KKT conditions are sufficient}
\end{gathered}
$$

Assume $x_2=0$

$$
\mu_1(x_2-w)=-\mu_1 w=0 \implies \mu_1=0
$$

FOC:

$$
\frac{\pi_1}{w}-(1-\pi_1)\int_1^2\frac{2(r-1)}{w}dr-\mu_2=0 \implies \frac{\pi_1}{w}=\frac{1-\pi_1}{w}+\mu_2
$$

$$
\begin{aligned}
\frac{1-\pi_1}{w}+\mu_2 &\geq \frac{1-\pi_1}{w} \\
\frac{\pi_1}{w} &\geq \frac{1-\pi_1}{w} \\
\pi_1 &\geq \frac{1}{2}
\end{aligned}
$$

3. Prospect Theory

$$
u(x)=\begin{cases}\sqrt{x}, & x\geq 0 \\ -\sqrt{-x}, & x<0\end{cases}
$$

3(a)

- DM is risk-averse in gain domain and risk-seeking in loss domain (satisfied)
- DM is loss-averse (not satisfied)
- Probability weighting (not satisfied)

It is consistent with Prospect Theory

3(b)

$$w = 0$$

$$L = \begin{cases} 10 & \text{with probability } \frac{1}{2} \\ -10 & \text{with probability } \frac{1}{2} \end{cases}$$

$$\begin{cases} u(w) = u(0) = \sqrt{0} = 0 \\ u(w+L) = \frac{1}{2}u(10) + \frac{1}{2}u(-10) = \frac{1}{2}\sqrt{10} + \frac{1}{2}(-\sqrt{10}) = 0 \end{cases}$$

$$u(w) = u(w+L)$$

The DM is indifferent about accepting the gamble if $w = 0$

3(c)

To show

$$\text{if } w \in (-10, 0) \implies u(w) < u(w+L) \quad \text{DM accepts L}$$

$$u(w) = -\sqrt{-w}$$

$$u(w+L) = \frac{1}{2}\sqrt{w+10} + \frac{1}{2}\left(-\sqrt{-(10-w)}\right) = \frac{1}{2}(\sqrt{w+10} - \sqrt{10-w})$$

Prove $-\sqrt{-w} < \frac{1}{2}(\sqrt{w+10} - \sqrt{10-w})$ always holds if $w \in (-10, 0)$

$$\begin{aligned} \underbrace{-\sqrt{-w}}_{\text{negative}} &< \underbrace{\frac{1}{2}(\sqrt{w+10} - \sqrt{10-w})}_{\text{negative}} \\ -w &> \frac{1}{4}(w + 10 + 10 - w - 2\sqrt{100-w^2}) \\ -4w &> 20 - 2\sqrt{100-w^2} \\ -2w &> 10 - \sqrt{100-w^2} \\ \sqrt{100-w^2} &> 10 + 2w \end{aligned}$$

We need to show $\sqrt{100-w^2} > 10 + 2w$ holds when $w \in (-10, 0)$

- $w \in (-10, -5)$

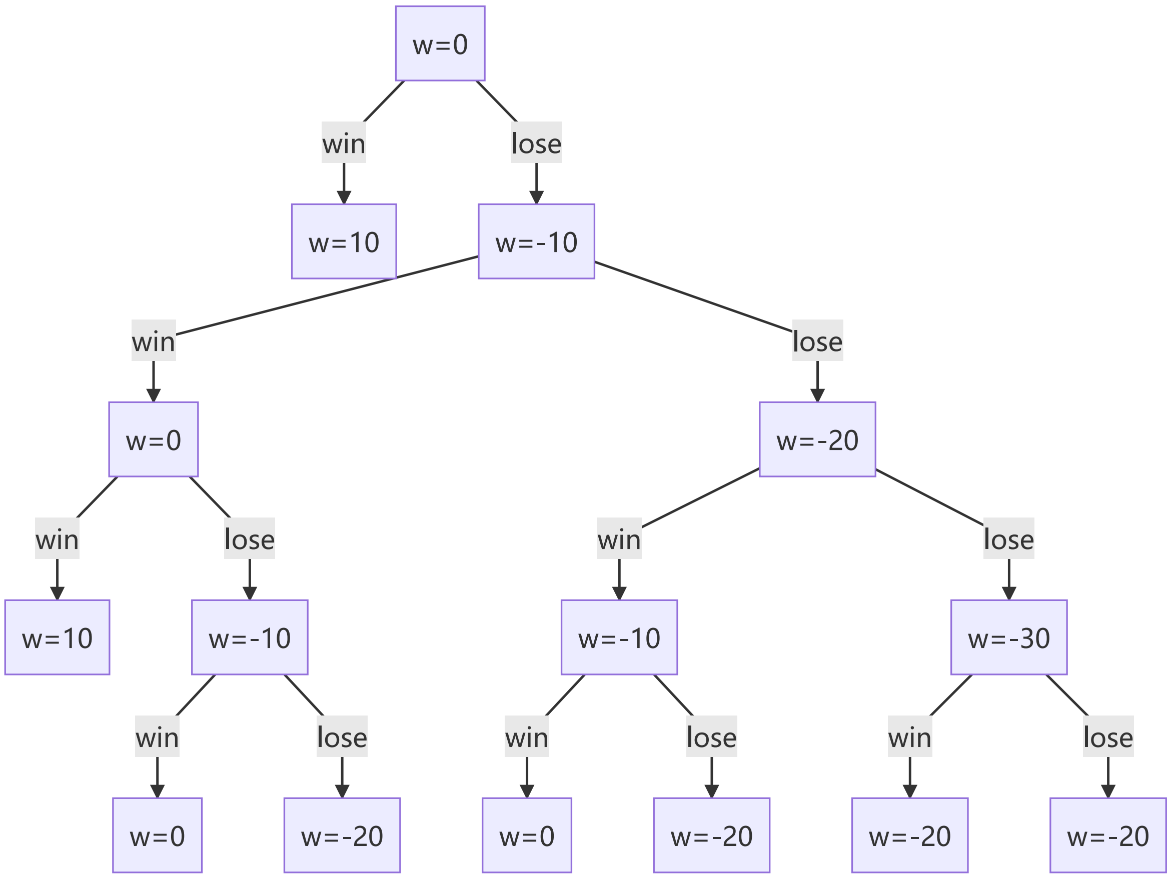
$$\underbrace{\sqrt{100-w^2}}_{\text{non-negative}} > \underbrace{10+2w}_{\text{negative}} \quad \text{always holds}$$

- $w \in [-5, 0)$

$$\begin{aligned} \sqrt{100-w^2} &> 10 + 2w \\ 100 - w^2 &> 100 + 40w + 4w^2 \\ 0 &> 40w + 5w^2 \qquad \text{always true when } w \in [-5, 0) \\ 0 &> 8w + w^2 \\ 0 &> w(8+w) \end{aligned}$$

3(d)

$$\begin{cases} \text{accept} & w \in (-\infty, -10) \\ \text{accept} & w \in (-10, 0) \\ \text{accept} & w = 0 \\ \text{reject} & w \in (0, 10) \\ \text{reject} & w \in (10, +\infty) \end{cases} \implies \begin{cases} \text{accept} & w \le 0 \\ \text{reject} & w > 0 \end{cases}$$



wealth	10	0	-20	-40
probability	$\frac{10}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{1}{16}$

$$L = \begin{cases} 10 & \frac{5}{8} \\ 0 & \frac{1}{8} \\ -20 & \frac{3}{16} \\ -40 & \frac{1}{16} \end{cases}$$

$$\begin{aligned} E(L) &= 10 \times \frac{5}{8} + 0 \times \frac{1}{8} - 20 \times \frac{3}{16} - 40 \times \frac{1}{16} \\ &= \frac{25}{4} - \frac{15}{4} - \frac{5}{2} \\ &= 0 \end{aligned}$$

3(e)

$$U(0) = 0$$

$$\begin{aligned} U(L) &= \frac{5}{8}\sqrt{10} + \frac{1}{8}\sqrt{0} + \frac{3}{16} \times (-\sqrt{20}) + \frac{1}{16} \times (-\sqrt{40}) \\ &\approx 0.74 > 0 \end{aligned}$$

DM (strictly) prefers to accept this gamble.

Rabin's Critique

Assume CARA (Constant Absolute Risk Aversion) $u(x) = -e^{-\sigma x}$

$$\begin{cases} u(w+L) = \frac{1}{2}(-e^{-\sigma(w+11)} - e^{-\sigma(w-10)}) \\ u(w) = -e^{-\sigma w} \end{cases}$$

$$\begin{aligned} u(w) &\geq u(w+L) \\ -e^{-\sigma w} &\geq \frac{1}{2}(-e^{-\sigma(w+11)} - e^{-\sigma(w-10)}) \\ 2e^{-\sigma w} &\leq e^{-\sigma(w+11)} + e^{-\sigma(w-10)} \\ 2 &\leq e^{-11\sigma} + e^{10\sigma} \\ \sigma &\geq 0.009 \end{aligned}$$

$$\begin{cases} u(w+L') = \frac{1}{2}(-e^{-\sigma(w+Y)} - e^{-\sigma(w-100)}) \\ u(w) = -e^{-\sigma w} \end{cases}$$

$$\begin{aligned} u(w) &\geq u(w+L') \\ -e^{-\sigma w} &\geq \frac{1}{2}(-e^{-\sigma(w+Y)} - e^{-\sigma(w-100)}) \\ 2e^{-\sigma w} &\leq e^{-\sigma(w+Y)} + e^{-\sigma(w-100)} \\ 2 &\leq e^{-\sigma Y} + e^{100\sigma} \quad \text{always holds} \end{aligned}$$

No matter how high Y is, DM always rejects L'