# Problem Set 10

## Exercise 1: Market Completeness and Risk Sharing

| | $t = 1$ | $t = 2$ |
|---|---|---|
| **Home** | $Y_1$ | $Y_2(s)$ |
| **Foreign** | $Y_1^\star$ | $Y_2^*(s)$ |

$s \in \{1, \cdots, \mathcal{S}\}$ state of nature that occurs with probability $\pi(s) > 0$ .

Asset markets are complete, i.e., for each state of the world, there is a state contingent Arrow-Debreu security.

- State contingent: payment depends on the state of nature
- Arrow-Debreu security: name for a security in a complete market
- Complete market: "I can always buy an umbrella when it rains"

$$\begin{cases} B_2(s) & \text{Arrow-Debreu security} \\ \widetilde{p}(s) & \text{price of Arrow-Debreu security} \\ p(s) & \text{state price} \\ r & \text{return on a risk-free bond} \end{cases}$$

### 1(a)

Household problem

$$\begin{aligned} \max_{C_1, C_2} \quad & \frac{C_1^{1-\rho} - 1}{1-\rho} + \beta \sum_{s=1}^{\mathcal{S}} \pi(s) \frac{(C_2(s))^{1-\rho} - 1}{1-\rho} \\ \text{s.t.} \quad & \text{period t=1 BC:} \quad C_1 + \sum_{s=1}^{\mathcal{S}} \frac{p(s)}{1+r} B_2(s) = Y_1 \\ & \text{period t=2 BC:} \quad C_2(s) = Y_2(s) + B_2(s) \end{aligned}$$

$$\begin{aligned} \max_{B_2} \quad & \frac{(Y_1 - \sum_{s=1}^{\mathcal{S}} \frac{p(s)}{1+r} B_2(s))^{1-\rho} - 1}{1-\rho} + \beta \sum_{s=1}^{\mathcal{S}} \pi(s) \frac{(Y_2(s) + B_2(s))^{1-\rho} - 1}{1-\rho} \\ \text{FOC } (B_2): \quad & C_1^{-\rho} \cdot \left( -\frac{p(s)}{1+r} \right) + \beta \pi(s) C_2^{-\rho} = 0 \implies \beta \pi(s) \left( \frac{C_1}{C_2(s)} \right)^\rho = \frac{p(s)}{1+r} \\ & \underbrace{\beta \pi(s) \left( \frac{C_1}{C_2(s)} \right)^\rho}_{\text{marginal rate of intertemporal substitution}} = \underbrace{\frac{p(s)}{1+r}}_{\text{price of Arrow-Debreu}} \end{aligned}$$

- Marginal rate of intertemporal substitution: How much $C_1$ I am willing to give up today for an extra unit of $C_2(s)$ tomorrow

**Intuition:**

My willingness to pay for one unit of $C_2(s)$ is equal to my willingness to give up one unit of $C_1$ to gain one unit of $C_2(s)$ tomorrow

### 1(b)

Assume that household cannot save (there is no saving technology)

World market clearing

$$\begin{aligned} C_1 + C_1^\star = Y_1 + Y_1^\star \equiv Y_1^W \\ C_2(s) + C_2^\star(s) = Y_2(s) + Y^\star(s) \equiv Y_2^W(s) \quad \forall s \in \{1, 2, \cdots, \mathcal{S}\} \end{aligned}$$

From (a), we derived

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^{\rho}=\frac{p(s)}{1+r} \implies C_2(s)=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}C_1$$

$$\begin{aligned} C_2(s) &= \left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}C_1 \\ C_2^{\star}(s) &= \left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}C_1^{\star} \end{aligned} \implies C_2(s)+C_2^{\star}(s)=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}(C_1+C_1^{\star})$$

$$\underbrace{C_2(s)+C_2^{\star}(s)}_{Y_2^W(s)}=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}\underbrace{(C_1+C_1^{\star})}_{Y_1^W}$$

$$\left(\frac{Y_2^W(s)}{Y_1^W}\right)^{\rho}=\beta\pi(s)\frac{1+r}{p(s)}$$

Similarly, for state $s'$

$$\left(\frac{Y_2^W(s')}{Y_1^W}\right)^{\rho}=\beta\pi(s')\frac{1+r}{p(s')}$$

$$\begin{cases} \left(\frac{Y_2^W(s)}{Y_1^W}\right)^{\rho}=\beta\pi(s)\frac{1+r}{p(s)} \\ \left(\frac{Y_2^W(s')}{Y_1^W}\right)^{\rho}=\beta\pi(s')\frac{1+r}{p(s')} \end{cases} \implies \left(\frac{Y_2^W(s)}{Y_2^W(s')}\right)^{\rho}=\frac{\pi(s)}{\pi(s')}\cdot\frac{p(s')}{p(s)} \implies \left(\frac{Y_2^W(s)}{Y_2^W(s')}\right)^{-\rho}\frac{\pi(s)}{\pi(s')}=\frac{p(s)}{p(s')}$$

**Intuition:**

If global output is the same in all states $s, s'$ (i.e., if agents perfectly insure themselves against any state of the world), $\frac{p(s')}{p(s)}=\frac{\pi(s')}{\pi(s)}$. Arrow-Debreu price fully reflects the state probabilities (i.e., prices are actuarially fair). If prices are actuarially fair, the agents insure themselves fully. However, if there is aggregate uncertainty (e.g., $Y_2^W(s)>Y_2^W(s')$), then $\frac{p(s')}{p(s)}>\frac{\pi(s')}{\pi(s)}$, i.e., consumption in state $s'$ is more expensive than the actuarially fair price.

## 1(c)

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^{\rho}=\frac{p(s)}{1+r} \implies \frac{C_2(s)}{C_1}=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}$$

By analogy

$$\frac{C_2^{\star}(s)}{C_1^{\star}}=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}$$

$$\frac{Y_2^W(s)}{Y_1^W}=\left(\beta\pi(s)\frac{1+r}{p(s)}\right)^{\frac{1}{\rho}}$$

$$\frac{C_2(s)}{C_1}=\frac{C_2^{\star}(s)}{C_1^{\star}}=\frac{Y_2^W(s)}{Y_1^W} \quad \forall s\in\{1,2,\cdots,\mathcal{S}\}$$

## 1(d)

**Interpretation:**

- Consumption growth across countries should be perfectly correlated.
- A country's consumption is correlated with the world output and not with the own country output.

**Empirical facts:** Backus, Kehoe, Kydland (1992)

Consumption correlation across countries is low.

Consumption Correlation Puzzle Cochrane (JPE 1991)

**Possible explanations:**

- Asset markets are not complete.
- No saving assumption is problematic: different saving preferences across countries may lower the correlation of consumption growth.

## 1(e)

$$u(C) = \log(C) \quad \text{and} \quad \mathcal{S} = 2$$

$$\begin{aligned} \max_{C_1, C_2} \quad & \log(C_1) + \beta \sum_{s=1}^{\mathcal{S}} \pi(s) \log(C_2(s)) \\ \text{s.t.} \quad & \text{period t=1 BC:} \quad C_1 + \sum_{s=1}^{\mathcal{S}} \frac{p(s)}{1+r} B_2(s) = Y_1 \\ & \text{period t=2 BC:} \quad C_2(s) = Y_2(s) + B_2(s) \end{aligned}$$

$$\begin{aligned} &\text{period t=1 BC:} \quad C_1 + \sum_{s=1}^{\mathcal{S}} \frac{p(s)}{1+r} B_2(s) = Y_1 \\ &\text{period t=2 BC:} \quad C_2(s) = Y_2(s) + B_2(s) \\ &\qquad\qquad\qquad\quad \Downarrow \\ &\text{period t=1 BC:} \quad C_1 + \frac{p(1)}{1+r} B_2(1) + \frac{p(2)}{1+r} B_2(2) = Y_1 \\ &\text{period t=2 BC:} \quad C_2(1) = Y_2(1) + B_2(1) \quad \text{or} \quad C_2(2) = Y_2(2) + B_2(2) \end{aligned}$$

$$C_1 + \frac{p(1)}{1+r}(C_2(1) - Y_2(1)) + \frac{p(2)}{1+r}(C_2(2) - Y_2(2)) = Y_1$$

$$\underbrace{C_1 + \frac{p(1)C_2(1) + p(2)C_2(2)}{1+r}}_{\text{expected PV of consumption}} = \underbrace{Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r}}_{\text{expected PV of endowment}}$$

**Euler equation:**

$$\beta\pi(s) \frac{C_1}{C_2(s)} = \frac{p(s)}{1+r} \implies \frac{p(s)C_2(s)}{1+r} = \beta\pi(s)C_1 \implies \begin{cases} \dfrac{p(1)C_2(1)}{1+r} = \beta\pi(1)C_1 \\ \dfrac{p(2)C_2(2)}{1+r} = \beta\pi(2)C_1 \end{cases}$$

$$\begin{aligned} C_1 + \frac{p(1)C_2(1) + p(2)C_2(2)}{1+r} &= Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r} \\ C_1 + \beta\pi(1)C_1 + \beta\pi(2)C_1 &= Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r} \\ (1+\beta)C_1 &= Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r} \\ C_1 &= \frac{1}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r}\right) \end{aligned}$$

Consumption is a constant fraction of the lifetime income.

$$\begin{aligned} CA_1 &= rB_1 + Y_1 - C_1 \\ &= Y_1 - C_1 \\ &= Y_1 - \frac{1}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r}\right) \\ &= \frac{\beta}{1+\beta} Y_1 - \frac{1}{1+\beta}\left(\frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r}\right) \end{aligned}$$

## 1(f)

Under autarky condition, there is no trade in A.D. security with other countries, i.e., $B_2(s) = 0$

$$\begin{cases} C_1 + \sum_{s=1}^{\mathcal{S}} \frac{p(s)}{1+r} B_2(s) = Y_1 \\ C_2(s) = Y_2(s) + B_2(s) \end{cases} \implies \begin{cases} C_1 = Y_1 \\ C_2(s) = Y_2(s) \quad \forall s \in \{1, 2\} \end{cases}$$

**Euler equation:**

$$\beta\pi(s) \frac{C_1}{C_2(s)} = \frac{p(s)}{1+r} \implies \beta\pi(s) \frac{Y_1}{Y_2(s)} = \frac{p(s)^A}{1+r^A} \implies \begin{cases} \beta\pi(1)Y_1 = \dfrac{p(1)^A}{1+r^A} Y_2(1) \\ \beta\pi(2)Y_1 = \dfrac{p(2)^A}{1+r^A} Y_2(2) \end{cases} \quad \forall s = \{1, 2\}$$

$$
\begin{aligned}
CA_1 &= \frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\left(\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
&= \frac{1}{1+\beta}\left(\beta Y_1 - \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
&= \frac{1}{1+\beta}\left(\beta\pi(1)Y_1 + \beta\pi(2)Y_1 - \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
&= \frac{1}{1+\beta}\left(\frac{p(1)^A}{1+r^A}Y_2(1) + \frac{p(2)^A}{1+r^A}Y_2(2) - \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
&= \frac{Y_2(1)}{1+\beta}\left(\frac{p(1)^A}{1+r^A} - \frac{p(1)}{1+r}\right) + \frac{Y_2(2)}{1+\beta}\left(\frac{p(2)^A}{1+r^A} - \frac{p(2)}{1+r}\right)
\end{aligned}
$$

**Interpretation:**

Under autarky condition, no trade happens.

However, the CA depends positively on the difference of autarky prices and world prices of A.D. security, implying that if $\frac{p(s)^A}{1+r^A} > \frac{p(s)}{1+r} \quad \forall s$ , then upon gaining market access, the country would like to buy the A.D. security, thus exporting consumption today (for consumption tomorrow). The CA would indeed be positive (CA surplus).

## 1(g)

We derived in (e)

$$
C_1 = \frac{1}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right)
$$

**Euler equation:**

$$
\beta\pi(s)\frac{C_1}{C_2(s)} = \frac{p(s)}{1+r} \implies C_1 = \frac{1}{\beta\pi(s)}\cdot\frac{p(s)}{1+r}C_2(s)
$$

$$
\begin{aligned}
C_1 &= \frac{1}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
\frac{1}{\beta\pi(s)}\cdot\frac{p(s)}{1+r}C_2(s) &= \frac{1}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
\frac{p(s)}{1+r}C_2(s) &= \frac{\beta\pi(s)}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{p(s)}{1+r}C_2(s) &= \frac{\beta\pi(s)}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
\frac{p(s)}{1+r}(Y_2(s)+B_2(s)) &= \frac{\beta\pi(s)}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) \\
\frac{p(s)}{1+r}B_2(s) &= \frac{\beta\pi(s)}{1+\beta}\left(Y_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) - \frac{p(s)}{1+r}Y_2(s) \\
\frac{p(s)}{1+r}B_2(s) &= \pi(s)\left(\frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\left(\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) - \frac{p(s)}{1+r}Y_2(s) \\
\frac{p(s)}{1+r}B_2(s) &= \pi(s)\left(CA_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) - \frac{p(s)}{1+r}Y_2(s)
\end{aligned}
$$

$s = 1$:

$$
\begin{aligned}
\frac{p(1)}{1+r}B_2(1) &= \pi(1)\left(CA_1 + \frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right) - \frac{p(1)}{1+r}Y_2(1) \\
&= \pi(1)CA_1 - \frac{p(1)}{1+r}Y_2(1)(1-\pi(1)) + \pi(1)\frac{p(2)Y_2(2)}{1+r} \\
&= \pi(1)CA_1 - \pi(2)\frac{p(1)Y_2(1)}{1+r} + \pi(1)\frac{p(2)Y_2(2)}{1+r} \\
&= \pi(1)CA_1 + \frac{\pi(2)p(2)Y_2(1)}{1+r}\left(\frac{\pi(1)}{\pi(2)}\cdot\frac{Y_2(2)}{Y_2(1)} - \frac{p(1)}{p(2)}\right) \\
&= \pi(1)CA_1 + \frac{\pi(2)p(2)Y_2(1)}{1+r}\left(\frac{p(1)^A}{p(2)^A} - \frac{p(1)}{p(2)}\right)
\end{aligned}
$$

$s = 2$:

$$\frac{p(2)}{1+r}B_2(2) = \pi(2)CA_1 - \frac{\pi(2)p(2)Y_2(1)}{1+r}\left(\frac{p(1)^A}{p(2)^A} - \frac{p(1)}{p(2)}\right)$$

**Interpretation:**

When $CA_1 = 0$ , the country spends on the A.D. security that has a relative price in autarky higher than in the open economy and sells the security that has a low relative price in autarky. This means the country insures itself by buying claims to receive consumption goods shoud a bad realize (high $p^A$) and by selling claims to export consumption should a good state realize.