Mock Exam
Social Choice Theory
Fall 2012
Solution

Problem 1: Preferences

(a) The Cartesian product $X \times X$ is the set of all ordered pairs of elements from X.
A binary relation R on X is a subset $R \subseteq X \times X$.
A preference on X is a binary relation R on X which satisfies the following properties:

Reflexivity: $(x, x) \in R$ for every $x \in X$.

Completeness: $(x, y) \in R$ or $(y, x) \in R$ (or both) for all $x, y \in X$, $x \neq y$.

Transitivity: If $(x, y) \in R$ and $(y, z) \in R$, then also $(x, z) \in R$, for all $x, y, z \in X$.

[Using the alternative notation xRy for $(x, y) \in R$ is of course also correct.]

(b) R^1: Reflexivity is satisfied, as there is a checkmark in every field on the main diagonal.
Completeness is satisfied. Whenever there is no checkmark in a field, then there is one in the field that corresponds to the opposite order of the two alternatives.
Transitivity is satisfied. For example, we have $(y, w) \in R^1$ and $(w, z) \in R^1$, and also $(y, z) \in R^1$. The same can also be verified for all other possibilities.
Hence R^1 is a preference.
It is also antisymmetric, because we never have both $(x, y) \in R^1$ and $(y, x) \in R^1$ for any $x \neq y$. Hence R^1 is a strict preference.

R^2: Reflexivity is satisfied.
Completeness is satisfied.
Transitivity is violated. We have $(z, x) \in R^2$ and $(x, w) \in R^2$, but not $(z, w) \in R^2$.
Hence R^2 is not a preference.

R^3: Reflexivity is satisfied.
Completeness is satisfied.
Transitivity is satisfied, which can be checked as described before.
Hence R^3 is a preference.
It is not antisymmetric, however, because we have both $(x, y) \in R^3$ and $(y, x) \in R^3$.
Hence R^3 is not a strict preference.

(c) R^1: We have $C(S, R^1) = \{y\}$.
Alternative y is a best element of S because $(y, w) \in R^1$ and $(y, z) \in R^1$.
Alternative w is not a best element of S because $(w, y) \notin R^1$.
Alternative z is not a best element of S because $(z, x) \notin R^1$, and also $(z, y) \notin R^1$.

R^2: We have $C(S, R^2) = \{w\}$. The arguments are analogous.

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Problem 2: Arrow's Theorem for SCFs

(a) Arrow's Impossibility Theorem for SCFs says that, when there are at least three alternatives, then there is no social choice function $c : \mathscr{A} \to X$ (which assigns a winning alternative $c(\mathbf{R})$ to each admissible preference profile $\mathbf{R} \in \mathscr{A} \subseteq \mathscr{R}^n$) that satisfies the following four axioms:

Universality: All preference profiles are admissible, $\mathscr{A} = \mathscr{R}^n$.

Monotonicity: If, for any $x \in X$, $c(\mathbf{R}) = x$ and x maintains its position from $\mathbf{R}$ to $\mathbf{R}' \in \mathscr{A}$, then $c(\mathbf{R}') = x$ must hold. Alternative x maintains its position from $\mathbf{R}$ to $\mathbf{R}'$ if it does not drop in anyone's preference, i.e. if xR_iy implies $xR_i'y$ and xP_iy implies $xP_i'y$, for all voters i and alternatives $y \in X$.

Weak Pareto Principle: If an alternative is strictly Pareto dominated, it should not be the winner. Formally, if xP_iy for some $x, y \in X$ and all voters i, then $y \neq c(\mathbf{R})$.

Non-Dictatorship: No voter should be able to always impose one of his most preferred alternatives on society. Formally, there is no voter h for which it always (i.e. for all preference profiles) holds that $c(\mathbf{R})R_hx$, for all $x \in X$.

(b) The votes are assigned as follows:

#	preferences	w	x	y	z
4	$w\ P\ x\ P\ y\ P\ z$	1	1	0	0
3	$x\ P\ y\ P\ z\ P\ w$	0	1	1	0
5	$y\ P\ z\ P\ w\ P\ x$	0	0	1	1
1	$z\ P\ w\ P\ x\ P\ y$	1	0	0	1
1	$x\ P\ w\ P\ z\ P\ y$	1	1	0	0
		6	8	8	6

Since both x and y get the maximal number of votes, alternative x is the winner.

(c) [$\bar{\text{U}}$] Universality $\mathscr{A} = \mathscr{P}^n$ is satisfied, because the method can always be applied and will (due to the tie-breaking rule) always deliver a unique winner.

[$\bar{\text{M}}$] Monotonicity is violated, as the following example illustrates:

#	profile $\mathbf{R}$	w	x	y	z
2	$w\ P\ x\ P\ y\ P\ z$	1	1	0	0
2	$x\ P\ w\ P\ y\ P\ z$	1	1	0	0
1	$y\ P\ z\ P\ x\ P\ w$	0	0	1	1
		4	4	1	1

#	profile $\mathbf{R}'$	w	x	y	z
2	$w\ P\ x\ P\ y\ P\ z$	1	1	0	0
2	$x\ P\ w\ P\ y\ P\ z$	1	1	0	0
1	$y\ P\ x\ P\ z\ P\ w$	0	1	1	0
		4	5	1	0

With preference profile $\mathbf{R}$, alternative w wins (using the tie-breaking rule).
Alternative w maintains its position from $\mathbf{R}$ to $\mathbf{R}'$.
But w does not win in $\mathbf{R}'$ (where x is the winner).

[$\bar{\text{P}}$] The Weak Pareto Principle is also violated:

#	preferences	w	x	y	z
2	$y\ P\ x\ P\ w\ P\ z$	0	1	1	0
		0	2	2	0

Here, alternative x wins (by the tie-breaking rule), even though it is strictly Pareto dominated by y.

[$\bar{\text{D}}$] The rule is clearly not dictatorial. Nobody can make sure that an own preferred alternative is always selected. If sufficiently many other voters have a diverging preference, then the alternative will not be selected.

(d) Fix an arbitrary alternative, say w, and consider the SCF that always selects w, i.e. $c(\mathbf{R}) = w$ for all $\mathbf{R} \in \mathscr{R}^n$.

- [$\bar{\text{U}}$] Universality is satisfied, by definition of the rule.
- [$\bar{\text{M}}$] Monotonicity is satisfied, because the winner is always the same alternative.
- [$\bar{\text{P}}$] The Weak Pareto Principle is violated, since w is selected even if xP_iw for some x and all voters i.
- [$\bar{\text{D}}$] Non-Dictatorship is satisfied, because the winner is always the same alternative.

Problem 3: Manipulability

$R_1 \backslash R_2$	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	y	y	z	z
xzy	x	x	y^*	z^*	z	z
yxz	y	y^*	y	y	z	z
yzx	y	z^*	y	y	z	z
zxy	z	z	z	z	z	z
zyx	z	z	z	z	z	z

(a) [Ū] Universality $\mathscr{A} = \mathscr{P}^2$ is satisfied, by definition of the rule.

[M̄] Monotonicity is violated. Consider, for instance, the following case:

If voter 1 has preference xzy and voter 2 has preference yxz, then y is selected.

If voter 1's preference changes to zxy and voter 2's preference remains unchanged, then y maintains its position. Still, alternative z is now selected, not y.

[P̄] The Weak Pareto Principle is satisfied, because this method always selects among the Pareto efficient alternatives.

[D̄] Non-Dictatorship is satisfied. It is easily verified that there are situations for each voter where the winner is not top-ranked for this voter.

(b) The method is clearly surjective, because each of the three alternatives wins for some preference profile.

We have already verified [Ū] and [D̄].

Axiom [S̄] is violated. The four profiles at which the rule can be manipulated by some voter are marked by an $*$ in the above table.

(c) The following four strict preferences are single peaked with respect to $x > y > z$: xyz, yxz, yzx, and zyx. The restricted SCF is given in the following table:

$R_1 \backslash R_2$	xyz	yxz	yzx	zyx
xyz	x	y	y	z
yxz	y	y	y	z
yzx	y	y	y	z
zyx	z	z	z	z

It is surjective, because each alternative wins for some (single-peaked) preference profile.

It is still not dictatorial, so [D̄] is satisfied. The argument is the same as for part (a) above.

It also satisfies [S̄], because the profiles where manipulation was possible have been exluded.

By definition, [Ū] is no longer satisfied.