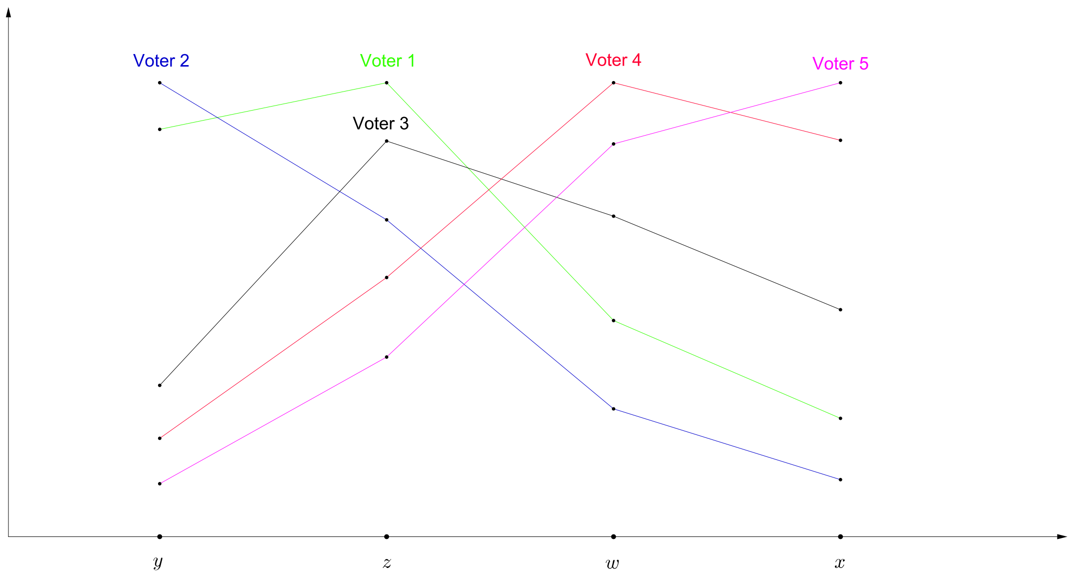
Exam
Social Choice Theory
Fall 2013
Solution

Problem 1: Arrow's Theorem for SWFs

(a) (i) The preferences are single-peaked with respect to $x > w > z > y$:



(ii) Voters 1 and 3 are median voters. They both have the peak z. There are $3 > n/2$ voters with peak z or y, and there are $4 > n/2$ voters with peak z, w or x.

(iii) The following table contains the number of votes for the row alternative in each of the pairwise comparisons:

	w	x	y	z
w		4	3	2
x			3	2
y				1
z				

We obtain the social preference $zPwPxPy$.
Alternative z wins every pairwise vote, hence is the Condorcet winner.
Alternative y loses every pairwise vote, hence is a Condorcet loser.

(b) U: This axiom is violated. A counterexample can be found in Exercise 2.1.

I: This axiom is satisfied, because the social ranking of any two alternatives depends only on their pairwise comparison.

P: This axiom is satisfied, because xP_iy for all $i \in N$ implies that x wins the pairwise vote against y unanimously.

D: This axiom is satisfied, because pairwise majority voting is clearly not a dictatorship.

(c) (i) Consider the following example:

	preferences
Voter 1	$w \; P_1 \; x \; P_1 \; y \; P_1 \; z$
Voter 2	$w \; P_2 \; y \; P_2 \; z \; P_2 \; x$
Voter 3	$w \; P_3 \; z \; P_3 \; x \; P_3 \; y$

Alternative w is top-ranked by all voters, hence is the Condorcet winner. Among the remaining alternatives we obtain a cycle (compare to Exercise 2.1).

(ii) If the outcome of pairwise majority voting is transitive, then the top-ranked alternatives *cannot be defeated* in any pairwise vote. However, a Condorcet winner must *win* every pairwise vote. There is a difference between the two statements if there are several top-ranked alternatives (among which society is indifferent), such that none of them is a Condorcet winner. Example:

#	preferences
2	$x \; I \; y \; P \; z$

Problem 2: May's Theorem

(a) The following table describes the procedure's progress, where the rows correspond to the iteration steps:

k	$\alpha^{(k)}$									$n-(k-1)$	n^+	n^-	stop?
1	0	+1	−1	+1	−1	+1	−1	−1	0	9	3	4	no
2		+1	−1	+1	−1	+1	−1	−1	0	8	3	4	no
3			−1	+1	−1	+1	−1	−1	0	7	2	4	no
4				+1	−1	+1	−1	−1	0	6	2	3	no
5					−1	+1	−1	−1	0	5	1	3	no
6						+1	−1	−1	0	4	1	2	no
7							−1	−1	0	3	0	2	yes

Hence the procedure yields the social preference yPx.

(b) See p. 102, 105 and 109 in the presentation slides (spring 2016).

(c) U: This axiom is satisfied. The procedure is always applicable and necessarily results in an outcome $f(\boldsymbol{\alpha}) \in \{-1, 0, +1\}$.

N: This axiom is satisfied. The procedure stops at the same step k for $\boldsymbol{\alpha}$ and $-\boldsymbol{\alpha}$, but yields an outcome of opposite sign.

PR: This axiom is violated. Counterexample:
$\boldsymbol{\alpha} = (+1, 0, -1, 0, 0)$ yields $f(\boldsymbol{\alpha}) = 0$.
$\boldsymbol{\alpha}' = (+1, +1, -1, 0, 0)$ still yields $f(\boldsymbol{\alpha}') = 0$.

A: This axiom is violated. Counterexample:
$\boldsymbol{\alpha} = (+1, +1, -1, -1)$ yields $f(\boldsymbol{\alpha}) = -1$.
$\boldsymbol{\alpha}' = (-1, -1, +1, +1)$ yields $f(\boldsymbol{\alpha}) = +1$, despite being a permutation of $\boldsymbol{\alpha}$.

Problem 3: Social Evaluation Functions

(a) See p. 171 and 172 in the presentation slides (spring 2016).

(b) (i) The following table computes RA for each alternative in the example:

	x	y	z
U_1	5	1	0
U_2	0	2	2
U_3	2	2	3
$\max\{U\}$	5	2	3
$\min\{U\}$	0	1	0
RA	5	1	3

Hence we obtain the social preference $y\, e^{RA}(\mathbf{U})\, z\, e^{RA}(\mathbf{U})\, x$.

(ii) The following example satisfies that x strictly Pareto-dominates y, while e^{RA} yields the opposite social ranking:

	x	y
U_1	2	0
U_2	1	0
$\max\{U\}$	2	0
$\min\{U\}$	1	0
RA	1	0

(c) Consider the three information structures separately:

CM-LC: Suppose $\mathbf{U}'$ is obtained from $\mathbf{U}$ by a common positive affine transformation $\varphi(u) = \alpha + \beta u$, where $\beta > 0$. Then, for any alternative $x \in X$ we have

$$\begin{aligned} RA(x, \mathbf{U}') &= \max\{U_1'(x), \dots, U_n'(x)\} - \min\{U_1'(x), \dots, U_n'(x)\} \\ &= \max\{\alpha + \beta U_1(x), \dots, \alpha + \beta U_n(x)\} - \min\{\alpha + \beta U_1(x), \dots, \alpha + \beta U_n(x)\} \\ &= \alpha + \beta \max\{U_1(x), \dots, U_n(x)\} - \alpha - \beta \min\{U_1(x), \dots, U_n(x)\} \\ &= \beta RA(x, \mathbf{U}). \end{aligned}$$

Hence the induced social preferences are identical for $\mathbf{U}$ and $\mathbf{U}'$.

CM-UC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by positive affine transformations $\varphi_i(u) = \alpha_i + \beta u$ for $\beta = 1$, $\alpha_1 = 0$, $\alpha_2 = 1$. We obtain $e^{RA}(\mathbf{U}) \neq e^{RA}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	2	0
U_2	1	0
RA	1	0

$\mathbf{U}'$	x	y
U_1	2	0
U_2	2	1
RA	0	1

OM-LC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by the common strictly increasing transformation $\varphi(u) = u^2$. We obtain $e^{RA}(\mathbf{U}) \neq e^{RA}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	2	4
U_2	1	5
RA	1	1

$\mathbf{U}'$	x	y
U_1	4	16
U_2	1	25
RA	3	9