# Problem Set 4

Program Evaluation and Causal Inference

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Names are listed in alphabetical order

# Instrumental Variables

### 1. Bias of the IV estimator

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

1(a)

$$Cov(y_i, z_i) = Cov(\beta_0 + \beta_1 x_i + u_i, z_i)$$

$$= \beta_1 Cov(x_i, z_i) + \underbrace{Cov(u_i, z_i)}_{0}$$

$$= \beta_1 Cov(x_i, z_i)$$

$$\beta_1 = \frac{Cov(y_i, z_i)}{Cov(x_i, z_i)}$$

where  $Cov(y_i, z_i)$  can be obtained from reduced-form equation and  $Cov(x_i, z_i)$  can be obtained from first-stage regression.

$$\begin{split} \hat{\beta}_{IV} &= \frac{\widehat{Cov(y_i, z_i)}}{\widehat{Cov(x_i, z_i)}} \\ &= \frac{\sum_{i=1}^n (z_i - \overline{z})(y_i - \overline{y})}{\sum_{i=1}^n (z_i - \overline{z})(x_i - \overline{x})} \\ &= \frac{\sum_{i=1}^n (z_i y_i - z_i \overline{y} - y_i \overline{z} + \overline{y} \overline{z})}{\sum_{i=1}^n (z_i x_i - z_i \overline{x} - x_i \overline{z} + \overline{x} \overline{z})} \\ &= \frac{\sum_{i=1}^n (z_i y_i - y_i \overline{z}) - \overline{y} \sum_{i=1}^n z_i + n \overline{y} \overline{z}}{\sum_{i=1}^n (z_i x_i - x_i \overline{z}) - \overline{x} \sum_{i=1}^n z_i + n \overline{x} \overline{z}} \\ &= \frac{\sum_{i=1}^n (z_i y_i - y_i \overline{z}) - n \overline{y} \overline{z} + n \overline{y} \overline{z}}{\sum_{i=1}^n (z_i x_i - x_i \overline{z}) - n \overline{x} \overline{z} + n \overline{x} \overline{z}} \\ &= \frac{\sum_{i=1}^n (z_i x_i - \overline{z}) y_i}{\sum_{i=1}^n (z_i - \overline{z}) x_i} \end{split}$$

1(b)

$$\hat{\beta}_{IV} = \frac{\sum_{i=1}^{n} (z_i - \overline{z}) y_i}{\sum_{i=1}^{n} (z_i - \overline{z}) x_i}$$

$$= \frac{\sum_{i=1}^{n} (z_i - \overline{z}) (\beta_0 + \beta_1 x_i + u_i)}{\sum_{i=1}^{n} (z_i - \overline{z}) x_i}$$

$$= \frac{\beta_0 \sum_{i=1}^{n} (z_i - \overline{z}) + \beta_1 \sum_{i=1}^{n} (z_i - \overline{z}) x_i + \sum_{i=1}^{n} (z_i - \overline{z}) u_i}{\sum_{i=1}^{n} (z_i - \overline{z}) u_i}$$

$$= \beta_1 + \underbrace{\sum_{i=1}^{n} (z_i - \overline{z}) u_i}_{\text{bias}}$$

$$\mathbb{E} \left[ \hat{\beta}_{IV} \mid x_i, z_i \right] = \beta_1 + \mathbb{E} \left[ \underbrace{\sum_{i=1}^{n} (z_i - \overline{z}) u_i}_{\sum_{i=1}^{n} (z_i - \overline{z}) x_i} \middle| x_i, z_i \right]$$

$$= \beta_1 + \underbrace{\sum_{i=1}^{n} (z_i - \overline{z})}_{\sum_{i=1}^{n} (z_i - \overline{z}) x_i} \underbrace{\mathbb{E} \left[ u_i \mid x_i, z_i \right]}_{\neq 0}$$

 $\mathbb{E}\left[u_i \mid x_i, z_i\right] \neq 0 \text{ since } u_i \not\perp x_i.$ 

1(c)

$$p \lim(\hat{\beta}_{IV} - \beta_1) = \frac{p \lim \sum_{i=1}^{n} (z_i - \overline{z}) u_i}{p \lim \sum_{i=1}^{n} (z_i - \overline{z}) u_i}$$

$$= \frac{p \lim \sum_{i=1}^{n} (z_i - \overline{z}) (u_i - \overline{u})}{p \lim \sum_{i=1}^{n} (z_i - \overline{z}) (x_i - \overline{x})}$$

$$= \frac{p \lim \frac{1}{n} \sum_{i=1}^{n} (z_i - \overline{z}) (u_i - \overline{u})}{p \lim \frac{1}{n} \sum_{i=1}^{n} (z_i - \overline{z}) (x_i - \overline{x})}$$

$$\approx \frac{Cov(z_i, u_i)}{Cov(z_i, x_i)} = 0$$

By Exogeneity Assumption,  $Cov(z_i, u_i) = 0$  and by Relevance Assumption,  $Cov(z_i, x_i) \neq 0$  Therefore,  $\hat{\beta}_{IV}$  is a consistent estimator of  $\beta_1$ .

1(d)

In a small sample,  $\hat{\beta}_{IV}$  is biased. But as the sample increases,  $\beta_{IV}$  will probability converge to the  $\beta_1$ . Therefore, in a large sample, IV estimator will be a consistent estimator of  $\beta_1$  regardless of whether there exists an endogeneity problem.

### 2. Derivation of the Wald estimator

2(a)

From question 1, we know that

$$\begin{split} \delta^W &= \hat{\beta}_1 = \frac{Cov(y_i, z_i)}{Cov(d_i, z_i)} \\ &= \frac{\mathbb{E}[y_i|z_i = 1] - \mathbb{E}[y_i|z_i = 0]}{\mathbb{E}[d_i|z_i = 1] - \mathbb{E}[d_i|z_i = 0]} \\ &= \frac{\mathbb{E}[\beta_1 + \beta_1 d_i + u_i|z_i = 1] - \mathbb{E}[\beta_1 + \beta_1 d_i + u_i|z_i = 0]}{\mathbb{E}[d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} \\ &= \frac{\beta_1 \mathbb{E}[d_i|z_i = 1] + \mathbb{E}[u_i|z_i = 1] - \beta_1 \mathbb{E}[d_i|z_i = 0] - \mathbb{E}[u_i|z_i = 0]}{\mathbb{E}[d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} \\ &= \frac{\beta_1 \left( \mathbb{E}[d_i|z_i = 1] - \mathbb{E}[d_i|z_i = 0] \right) + \mathbb{E}[u_i|z_i = 1] - \mathbb{E}[u_i|z_i = 0]}{\mathbb{E}[d_i|z_i = 1] - \mathbb{E}[u_i|z_i = 0]} \\ &= \beta_1 + \frac{\mathbb{E}[u_i|z_i = 1] - \mathbb{E}[u_i|z_i = 0]}{\mathbb{E}[d_i|z_i = 1] - \mathbb{E}[d_i|z_i = 0]} \end{split}$$

2(b)

In order to identify  $\beta_1$  using the instrument, we need

$$\frac{\mathbb{E}[u_i|z_i=1] - \mathbb{E}[u_i|z_i=0]}{\mathbb{E}[d_i|z_i=1] - \mathbb{E}[d_i|z_i=0]} = 0 \iff \begin{cases} \mathbb{E}[u_i|z_i=1] = \mathbb{E}[u_i|z_i=0] & \text{Exclusion Assumption} \\ \mathbb{E}[d_i|z_i=1] \neq \mathbb{E}[d_i|z_i=0] & \text{Relevance Assumption} \end{cases}$$

#### Assumptions

- SUTVA (Stable Unit Treatment Value Assumption): outcomes of the *i*th individual are independent of other individuals' outcome
- Exclusion restriction:  $\mathbb{E}[y_i|z=1,d] = \mathbb{E}[y_i|z=0,d] \quad \forall i=0,1$
- Relevance assumption:  $\mathbb{E}[d|z=1] \neq \mathbb{E}[d|z=0]$
- Monotonicity assumption:  $d_i[z_i = 1] \ge d_i[z_i = 0] \quad \forall i$

Only relevance assumption can be tested empirically. The vadility of other assumptions must be assessed on a case-by-case basis.

### 3. Self selection revisited

3(a)

$$= \mathbf{1}(\beta_1 + u_{1i} - u_{0i} > 0)$$

$$\Delta^{\text{ATE}} = \mathbb{E}(Y_{1i} - Y_{0i})$$

$$= \mathbb{E}[(\beta_0 + \beta_1 + u_{1i}) - (\beta_0 + u_{0i})]$$

$$= \mathbb{E}(\beta_1 + u_{1i} - u_{0i})$$

$$= \mathbb{E}(\beta_1) + \mathbb{E}(u_{1i}) + \mathbb{E}(u_{0i})$$

$$= \beta_1 > 0$$

 $D_i = \mathbf{1}(Y_{1i} - Y_{0i} > 0)$ 

$$\Delta^{\text{ATT}} = \mathbb{E}(Y_{1i} - Y_{0i} \mid D = 1)$$

$$= \mathbb{E}(\beta_1 + u_{1i} - u_{0i} \mid D = 1)$$

$$= \beta_1 + \mathbb{E}(u_{1i} - u_{0i} \mid D = 1)$$

$$= \Delta^{\text{ATE}} + \mathbb{E}(u_{1i} - u_{0i} \mid D = 1)$$

$$\mathbb{E}(u_{1i} - u_{0i} \mid D = 1) = \mathbb{E}(u_{1i} - u_{0i} \mid \beta_1 + u_{1i} - u_{0i} > 0)$$

$$= \mathbb{E}(u_{1i} - u_{0i} \mid u_{1i} - u_{0i} > -\beta_1) > 0$$

ATT is larger than ATE.

3(b)

$$\Delta^{\text{naive}} = \mathbb{E}(Y_{1i} \mid D=1) - \mathbb{E}(Y_{0i} \mid D=0)$$

$$= \mathbb{E}(Y_{1i} \mid D=1) - \mathbb{E}(Y_{0i} \mid D=0) - \mathbb{E}(Y_{0i} \mid D=1) + \mathbb{E}(Y_{0i} \mid D=1)$$

$$= \mathbb{E}(Y_{1i} \mid D=1) - \mathbb{E}(Y_{0i} \mid D=1) + \mathbb{E}(Y_{0i} \mid D=1) - \mathbb{E}(Y_{0i} \mid D=0)$$
selection bias

selection bias = 
$$\mathbb{E}(Y_{0i} \mid D = 1) - \mathbb{E}(Y_{0i} \mid D = 0)$$
  
=  $\mathbb{E}(\beta_0 + u_{0i} \mid \beta_1 + u_{1i} - u_{0i} > 0) - \mathbb{E}(\beta_0 + u_{0i} \mid \beta_1 + u_{1i} - u_{0i} \le 0)$   
=  $\mathbb{E}(u_{0i} \mid u_{0i} < u_{1i} + \beta_1) - \mathbb{E}(u_{0i} \mid u_{0i} \ge u_{1i} + \beta_1)$   
negative positive

If individuals can self-select themselves into the program, the naive estimator will be underestimated since the selection bias is negative  $(\mathbb{E}(Y_{0i}|D=1) < \mathbb{E}(Y_{0i}|D=0))$ 

3(c)

$$\begin{cases} D_{1i} = \mathbf{1}(Y_{1i} - Y_{0i} + Z_i > 0) & Z_i = 1 \\ D_{0i} = \mathbf{1}(Y_{1i} - Y_{0i} > 0) & Z_i = 0 \end{cases}$$

$$\begin{cases} D_{1i} = \mathbf{1}(u_{1i} - u_{0i} > -\beta_1 - Z_i) & Z_i = 1 \\ D_{0i} = \mathbf{1}(u_{1i} - u_{0i} > -\beta_1) & Z_i = 0 \end{cases}$$

In order to identify LATE, we need to determine the conditions for compliers. Compliers are those who are induced to switch treatment status as a result of the instrument. Therefore,  $D_{1i}>D_{0i}$  should hold for compliers.

LATE = 
$$\mathbb{E}(Y_{1i} - Y_{0i} \mid D_{1i} > D_{0i})$$

3(d)

$$D_{1i} > D_{0i} \implies \begin{cases} D_{1i} = 1 \\ D_{0i} = 0 \end{cases} \implies \begin{cases} u_{1i} - u_{0i} > -\beta_1 - Z_1 \\ u_{1i} - u_{0i} < -\beta_1 \end{cases} \implies -\beta_1 - Z_1 < u_{1i} + u_{0i} < -\beta_1 \end{cases}$$

LATE = 
$$\mathbb{E}(Y_{1i} - Y_{0i} \mid D_{1i} > D_{0i})$$
  
=  $\beta_1 + \mathbb{E}(u_{1i} - u_{0i} \mid D_{1i} > D_{0i})$   
=  $\beta_1 + \underbrace{\mathbb{E}(u_{1i} - u_{0i} \mid -\beta_1 - Z_1 < u_{1i} + u_{0i} < -\beta_1)}_{<-\beta_1}$   
 $< \beta_1 \equiv \text{ATE}$ 

# 4. Application: Angrist's (1990) study on military service

4(a)

The OLS estimate may be biased:

- There may be a self-selection bias from participants. People may self-select themselves into or not into the program.
- There may be a self-selection bias from experimenter. The military authority can select who can join the army and who cannot.

4(b)

#### Due to monotonicity:

In the observed Z=0 group, the individuals who received treatment (D=1) must be always-takers.

$$p_A = \mathbb{E}(D_i \mid Z_i = 0) = \frac{\sum_i \mathbf{1}(D_i[Z_i = 0] = 1)}{\sum_i \mathbf{1}(Z_i = 0)} = \frac{1400}{5928 + 1400} = 0.191$$

In the observed Z=1 group, the individuals who did not receive treatment (D=0) must be never-takers.

$$p_N = 1 - \mathbb{E}(D_i \mid Z_i = 1) = \frac{\sum_i \mathbf{1}(D_i[Z_i = 1] = 0)}{\sum_i \mathbf{1}(Z_i = 1)} = \frac{1875}{1875 + 863} = 0.685$$
$$p_C = \mathbb{E}(D_i \mid Z_i = 1) - \mathbb{E}(D_i \mid Z_i = 0)$$
$$= 1 - 0.685 - 0.191$$
$$= 0.124$$

#### Due to randomization:

The proportions of compliers, always-takers, and never-takers are the same between Z=0 and Z=1 group.

$$p_C = 1 - p_A - p_N = 0.124$$

Note:

- N denotes **never takers**
- $\bullet$  C denotes **compliers**
- A denotes always takers

**4(c)** 

- Average potential outcome for always-takers  $\mathbb{E}(Y_{1i} \mid D_i = 1, Z_i = 0) = 6.4076$
- Average potential outcome for never-takers  $\mathbb{E}(Y_{0i} \mid D_i = 0, Z_i = 1) = 6.4028$

Average potential outcome for compliers:

In Z = 0 group:

$$\underbrace{\mathbb{E}(Y_0 \mid D = 0, Z = 0)}_{6.4472} = \frac{p_C}{p_N + p_C} \times \mathbb{E}(Y_{0i} \mid D_{1i} > D_{0i}) + \frac{p_N}{p_N + p_C} \times \mathbb{E}(Y_{0i} \mid D_{1i} = 0)$$

$$\mathbb{E}(Y_{0i} \mid D_{1i} > D_{0i}) = 6.6867$$

In Z = 1 group:

$$\underbrace{\mathbb{E}(Y_1 \mid D = 1, Z = 1)}_{6.4289} = \frac{p_C}{p_A + p_C} \times \mathbb{E}(Y_{1i} \mid D_{1i} > D_{0i}) + \frac{p_A}{p_A + p_C} \times \mathbb{E}(Y_{1i} \mid D_{1i} = 0)$$

$$\mathbb{E}(Y_{1i} \mid D_{1i} > D_{0i}) = 6.4616$$

- Average potential outcome for untreated compliers  $\mathbb{E}(Y_0|Z=0,C)=6.6867$
- Average potential outcome for treated compliers  $\mathbb{E}(Y_1|Z=1,C)=6.4616$

4(d)

LATE = 
$$\mathbb{E}(Y_{1i} \mid D_{1i} > D_{0i}) - \mathbb{E}(Y_{0i} \mid D_{1i} > D_{0i})$$
  
=  $6.4616 - 6.6867$   
=  $-0.2251$ 

### 5. IV in action

5(a)

```
# load relevant libraries
library(haven) # read dta file
library(lattice) # density plot
library(stargazer) # print summary statistics
library(ggplot2) # plot
library(AER) # iv regression

d.mort <- read_dta('mortality.dta')

demo <- subset(d.mort, select = c('before67dead', 'dist65_ageATend4emp', 'Zd_during'))

subdata <- as.data.frame(
   subset(d.mort, select = c('before67dead', 'dist65_ageATend4emp', 'Zd_during'))
)
stargazer(subdata, header = F, title = 'Descriptive Statistics')</pre>
```

Table 3: Descriptive Statistics

Statistic	N	Mean	St. Dev.	Min	Pctl(25)	Pctl(75)	Max
before67dead	2,298	0.073	0.260	0	0	0	1
$dist65\_ageATend4emp$	2,298	6.499	2.248	-5	5	8	11
Zd_during	2,298	0.479	0.500	0	0	1	1

#### 5(b)

#### **5(c)**

As we can see, the coefficient on the treatment slightly increases from column 1 (with control variables) to column 2 (without control variables).

#### Significance

• With control variables, *p-value* is smaller than 10

Table 4: Comparison between control and non-control

	Dependent variable:  before67dead	
	Control	Non-control
	(1)	(2)
dist65_ageATend4emp	0.0046*	0.0049**
	(0.0026)	(0.0024)
Observations	2,298	2,298
Note:	*p<0.1; **r	p<0.05; ***p<0.

• Without control variables, *p-value* is smaller than 5

We can reject the null hypothesis in both cases but we are more confident to reject  $\beta_1 = 0$  with control variables.

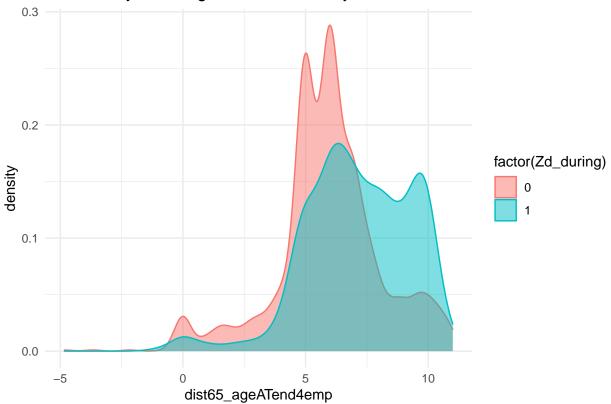
# **5(d)**

### Omitted-variable bias

Health status. If people are in a bad physical condition, they are more likely to spend less years in their early retirement or even die before retirement. Therefore, the estimator for  $\beta_1$  is biased upwards and we expect a positive bias.

## **5(e)**

# The density of endogeneous variable by instrument



**5(f)** 

5(g)

Table 5: First stage regression

_	$Dependent\ variable:$	
	$dist65\_ageATend4emp$	
Zd_during	$0.664^{*}$	
	(0.345)	
Zd_during:as.factor(halfyearOFbirth)-50	-0.448	
	(0.492)	
Zd_during:as.factor(halfyearOFbirth)-49	-0.070	
	(0.505)	
Zd_during:as.factor(halfyearOFbirth)-48	0.179	
	(0.506)	
Zd_during:as.factor(halfyearOFbirth)-47	-0.280	
	(0.492)	
Zd_during:as.factor(halfyearOFbirth)-46	0.424	
	(0.471)	
Zd_during:as.factor(halfyearOFbirth)-45	0.253	
	(0.509)	
Zd_during:as.factor(halfyearOFbirth)-44	0.525	
	(0.504)	
Zd_during:as.factor(halfyearOFbirth)-43	0.558	
	(0.495)	
Zd_during:as.factor(halfyearOFbirth)-42	0.150	
	(0.460)	
Zd_during:as.factor(halfyearOFbirth)-41	1.229***	
	(0.450)	
Zd_during:as.factor(halfyearOFbirth)-40	0.681	
	(0.451)	
Zd_during:as.factor(halfyearOFbirth)-39	0.608	
	(0.463)	
Zd_during:as.factor(halfyearOFbirth)-38	0.524	
	(0.479)	
Zd_during:as.factor(halfyearOFbirth)-37	0.345	
	(0.459)	
Observations	2,298	

Note:

\*p<0.1; \*\*p<0.05; \*\*\*p<0.01

Table 6: Comparsion between 2SLS and ivreg

	Dependent variable: before67dead		
	OLS	$instrumental\\variable$	
	(1)	(2)	
predict(iv.1st.stage)	-0.0143 (0.0109)		
$dist 65\_age AT end 4 emp$		-0.0143 (0.0110)	
Observations F Statistic	$ \begin{array}{c} 2,298 \\ 1.3992^* \text{ (df} = 32; 2265) \end{array} $	2,298	
Note:	*p<0.1; **p<0.05; ***p<0.01		

As we can see, **2SLS** and **ivreg** yield exactly the same estimate but with different standard errors.

5(h)

As expected, from column(1) to column(2), we see a decrease in the coefficient on  $dist65\_ageATend4emp$ , which verifies our statement in 5(d) - a positive bias in the OLS estimator.

Table 7: Comparison between OLS and 2SLS results

	Dependent variable: before67dead		
	OLS	$instrumental\\variable$	
	(1)	(2)	
dist65_ageATend4emp	$0.0046^* \ (0.0026)$	-0.0143 (0.0110)	
Observations F Statistic	$ \begin{array}{c} 2,298 \\ 1.4431^* \text{ (df} = 32; 2265) \end{array} $	2,298	
Note:		*p<0.1; **p<0.05; ***p<0.01	