Exam
Social Choice Theory
Spring 2016
Solution

Problem 1: Scoring Methods

(a) (i) The following table lists the voters' assignment of points to each alternative:

	preferences	x	y	z
Voter 1	$x\,P_1\,y\,P_1\,z$	a	b	c
Voter 2	$x\,P_2\,z\,P_2\,y$	a	c	b
Voter 3	$y\,P_3\,z\,P_3\,x$	c	a	b

The overall scores are $SC(x) = a + a + c$, $SC(y) = a + b + c$, and $SC(z) = b + b + c$. If $a > b$, we therefore obtain the social preference $xPyPz$. If $a = b$, we obtain $xIyIz$.

(ii) The following table lists the voters' assignment of points to each alternative:

	preferences	x	y	z
Voter 4	$x\,P_4\,y\,P_4\,z$	a	b	c
Voter 5	$y\,P_5\,z\,P_5\,x$	c	a	b
Voter 6	$z\,P_6\,x\,P_6\,y$	b	c	a

The overall scores are $SC(x) = a + b + c$, $SC(y) = a + b + c$, and $SC(z) = a + b + c$. We therefore always obtain $xIyIz$.

(iii) The following table lists the voters' assignment of points to each alternative:

	preferences	x	y	z
Voter 7	$x\,P_7\,y\,P_7\,z$	a	b	c
Voter 8	$z\,P_8\,y\,P_8\,x$	c	b	a

The overall scores are $SC(x) = a + c$, $SC(y) = b + b$, and $SC(z) = a + c$.
If $(a + c)/2 > b$, we therefore obtain $xIzPy$. If $(a + c)/2 < b$, we obtain $yPxIz$.
If $(a + c)/2 = b$, we obtain $xIyIz$.

(b) U: This axiom is satisfied. The method is always applicable and always yields a transitive preference, as the social ranking is based on numerical scores.

I: This axiom is satisfied if $a = b = c$. The social preference is then always $xIyIz$. The axiom is violated if at least one inequality is strict.

For the case when $a > b = c$, the following example illustrates a social preference change between y and z even though no voter's preference between y and z changed.

R	preferences	x	y	z
Voter 1	$x\,P_1\,y\,P_1\,z$	a	b	c
Voter 2	$x\,P_2\,y\,P_2\,z$	a	b	c

R′	preferences	x	y	z
Voter 1	$y\,P_1\,x\,P_1\,z$	b	a	c
Voter 2	$x\,P_2\,y\,P_2\,z$	a	b	c

The social preference is $xPyIz$ for $\mathbf{R}$ and $xIyPz$ for $\mathbf{R}'$.

For the case when $a = b > c$, the following example illustrates a social preference change between x and y even though no voter's preference between x and y changed.

R	preferences	x	y	z
Voter 1	$x\,P_1\,y\,P_1\,z$	a	b	c
Voter 2	$x\,P_2\,y\,P_2\,z$	a	b	c

$\mathbf{R}'$	preferences	x	y	z
Voter 1	$x\ P_1\ z\ P_1\ y$	a	c	b
Voter 2	$x\ P_2\ y\ P_2\ z$	a	b	c

The social preference is $xIyPz$ for $\mathbf{R}$ and $xPyIz$ for $\mathbf{R}'$.

For the case when $a > b > c$, the following example illustrates a social preference change between y and z even though no voter's preference between y and z changed.

$\mathbf{R}$	preferences	x	y	z
Voter 1	$x\ P_1\ y\ P_1\ z$	a	b	c
Voter 2	$x\ P_2\ z\ P_2\ y$	a	c	b

$\mathbf{R}'$	preferences	x	y	z
Voter 1	$y\ P_1\ x\ P_1\ z$	b	a	c
Voter 2	$x\ P_2\ z\ P_2\ y$	a	c	b

The social preference is $xPyIz$ for $\mathbf{R}$ and $xPyPz$ for $\mathbf{R}'$.

P: This axiom is satisfied if $a > b > c$, because the strictly Pareto-better alternative then gets strictly higher points from each voter.

The axiom is violated otherwise (i.e. with at least one parameter equality). It is then possible that two alternatives get the same points from each voter even though a strict Pareto-comparison between them is possible.

D: This axiom is satisfied, because scoring methods are clearly not a dictatorship.

(c) The social preference in this case is given by $xPyPz$.

This can be shown by adding the above derived scores for each alternative in the three groups. Alternatively, under the given parameters, both group (*ii*) and (*iii*) can be ignored, as each of them by itself leads to a social indifference between all three alternatives. Hence the social preference for the entire group coincides with the social preference for group (*i*).

Problem 2: Manipulability

$R_1 \backslash R_2$	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	y	y	z	z
xzy	x	x	z	z	z	z
yxz	y	z	y	y	z	z
yzx	y	z	y	y	z	z
zxy	z	z	z	z	z	z
zyx	z	z	z	z	z	z

(a) Arrow's axioms for SCFs:

[$\bar{\text{U}}$] Universality $\mathscr{A} = \mathscr{P}^2$ is satisfied by definition of the rule.

[$\bar{\text{M}}$] Monotonicity is violated.

Suppose voter 1's preference is yxz and voter 2's preference is xzy. Then the selected alternative is z.

Now suppose voter 1's preference changes to xzy and voter 2's preference remains xzy. Then the selected alternative becomes x.

Hence the winner has changed even though it maintained its position.

[$\bar{\text{P}}$] The Weak Pareto Principle is violated.

For instance, alternative z is selected if voter 1's preference is yxz and voter 2's preference is xzy, in which case z is strictly Pareto-dominated by x.

[$\bar{\text{D}}$] Non-Dictatorship is satisfied, as the rule does not always select the most preferred alternative of one fixed voter.

(b) The SCF is surjective because each alternative is selected for some preference profile.

We have already verified [$\bar{\text{U}}$] and [$\bar{\text{D}}$].

We can therefore conclude that [$\bar{\text{S}}$] must be violated.

Manipulation is possible when voter 1 has the preference yxz and voter 2 has the preference xzy. When voter 1 misreports xzy instead, she can obtain x instead of z. By symmetry, a manipulation is also possible when voter 1 has the preference xzy and voter 2 has the preference yxz. Manipulation is not possible at any other profile.

(c) The domain restriction implies a violation of [$\bar{\text{U}}$]. On the restricted domain, the SCF is still surjective and satisfies [$\bar{\text{D}}$]. Since the manipulable profile has now been excluded, it also satisfies [$\bar{\text{S}}$].

Problem 3: Social Evaluation Functions

(a) The following table computes WA for each alternative in the example:

	x	y	z
U_1	4	1	1
U_2	0	5	3
max	4	5	3
min	0	1	1
WA	4γ	$4\gamma+1$	$2\gamma+1$

The value for y is the largest. The ranking between x and z depends on the value of γ. We have $4\gamma \geq 2\gamma+1$ if and only if $\gamma \geq 1/2$. Hence, if $\gamma > 1/2$, we obtain the social preference $yPxPz$. If $\gamma < 1/2$, we obtain $yPzPx$. If $\gamma = 1/2$, we obtain $yPxIz$.

(b) The SEF is consistent with CM-LC (and hence with RM-LC). Suppose $\mathbf{U}'$ is obtained from $\mathbf{U}$ by a common positive affine transformation $\varphi(u) = \alpha + \beta u$, where $\beta > 0$. Then, for any alternative $x \in X$ we have

$$\begin{aligned}
\mathrm{WA}(x, \mathbf{U}') &= \gamma \max\{U_1'(x), \ldots, U_n'(x)\} + (1-\gamma)\min\{U_1'(x), \ldots, U_n'(x)\} \\
&= \gamma \max\{\alpha + \beta U_1(x), \ldots\} + (1-\gamma)\min\{\alpha + \beta U_1(x), \ldots\} \\
&= \alpha + \beta\,[\gamma \max\{U_1(x), \ldots\} + (1-\gamma)\min\{U_1(x), \ldots\}] \\
&= \alpha + \beta \mathrm{WA}(x, \mathbf{U}).
\end{aligned}$$

Hence the induced social preferences are identical for $\mathbf{U}$ and $\mathbf{U}'$.

The SEF is not consistent with any of the remaining information structures. To show this, counterexamples are now provided for OM-LC, CM-UC, and RM-NC. The examples all use the specific value $\gamma = 1/2$.

OM-LC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by the common strictly increasing transformation $\varphi(u) = u^2$. We obtain $e^{\mathrm{WA}}(\mathbf{U}) \neq e^{\mathrm{WA}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	4	3
U_2	0	1
WA	2	2

$\mathbf{U}'$	x	y
U_1	16	9
U_2	0	1
WA	8	5

CM-UC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by positive affine transformations $\varphi_i(u) = \alpha_i + \beta u$ for $\beta = 1$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = 3$. We obtain $e^{\mathrm{WA}}(\mathbf{U}) \neq e^{\mathrm{WA}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	5	5
U_2	2	1
U_3	0	0
WA	2.5	2.5

$\mathbf{U}'$	x	y
U_1	5	5
U_2	2	1
U_3	3	3
WA	3.5	3

RM-NC: In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by positive linear transformations $\varphi_i(u) = \beta_i u$ for $\beta_1 = 2$ and $\beta_2 = 1$. We obtain $e^{\mathrm{WA}}(\mathbf{U}) \neq e^{\mathrm{WA}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	4	3
U_2	0	1
WA	2	2

$\mathbf{U}'$	x	y
U_1	8	6
U_2	0	1
WA	4	3.5