Problem Set 2

1. (WARP and demand)

Let (p, w) and (p', w') be two price-wealth situations for which

$$(\underbrace{p \cdot x(p', w') \leq w}_{1} \quad and \quad \underbrace{x(p', w') \neq x(p, w)}_{2} \implies \underbrace{p' \cdot x(p, w) > w'}_{3})$$

1(a)

$p \cdot x \leq w : x$ is affordable at p, w

- $(p, w) \to x(p, w)$ state 1
- $(p', w') \to x(p', w')$ state 2

1. choice in state 2 is affordable in state 1
2. different choices in state 1 and state 2
3. choice in state 1 is not affordable in the state 2

1(b)

$$condition(1) \implies WARP$$
$$\neg WARP \implies \neg condition(1)$$

Assume $\exists x_1, x_2 \in B_1, B_2$ and $x_1 \in C(B_1), x_2 \in C(B_2)$ but $x_1 \notin C(B_2)$

$$B_1 = B(p, w) = \{x : p \cdot x \leq w\}$$
$$B_2 = B(p', w') = \{x : p' \cdot x \leq w'\}$$

$$x_1 \in B_2 \iff x_1 \in B(p', w') \iff p' \cdot x_1 \leq w'$$
$$x_2 \in B_1 \iff x_2 \in B(p, w) \iff p \cdot x_2 \leq w$$
$$x_1 \neq x_2 \quad x_1 = x(p, w) \quad x_2 = x(p', w')$$

$$p' \cdot x_1 = p' \cdot x(p, w) \leq w'$$
$$p \cdot x_2 = p \cdot x(p', w') \leq w$$
$$p' \cdot x(p, w) \neq p \cdot x(p', w')$$

Condition(1) is violated.

1(c)

2. (Non-continuous preferences)

2(a)

$$x \succeq y \quad \iff \quad [x_1 \succ y_1] \quad or \quad [x_1 = y_1 \wedge x_2 \succeq y_2]$$

Rational

- Completeness: assume $x \not\succeq y$

$$x \not\succeq y \quad \iff \quad [y_1 \succ x_1] \quad or \quad [y_1 = x_1 \wedge y_2 \succ x_2] \quad \iff \quad y \succeq x$$

- Transitivity: $\forall x, y, z$, assume $x \succeq y$ and $y \succeq z$

$$x \succeq y \iff [x_1 \succ y_1] \vee [x_1 = y_1 \wedge x_2 \succeq y_2]$$
$$y \succeq z \iff [y_1 \succ z_1] \vee [y_1 = z_1 \wedge y_2 \succeq z_2]$$

$$[x_1 \succ y_1] \wedge [y_1 \succ z_1] \implies x_1 \succ z_1$$
$$[x_1 \succ y_1] \wedge [y_1 = z_1 \wedge y_2 \succeq z_2] \implies x_1 \succ z_1$$
$$[x_1 = y_1 \wedge x_2 \succeq y_2] \wedge [y_1 \succ z_1] \implies x_1 \succ z_1$$
$$[y_1 = z_1 \wedge y_2 \succeq z_2] \wedge [x_1 = y_1 \wedge x_2 \succeq y_2] \implies x_1 = z_1 \wedge x_2 \succeq z_2$$
$$\therefore [x_1 \succ z_1] \wedge [x_1 = z_1 \wedge x_2 \succeq z_2] \iff x_1 \succeq z_1$$

Strong Monotonicity

$$\text{To show: } \begin{cases} x \geq y \quad (\forall i \quad x_i \geq y_i) \\ x \neq y \end{cases} \implies x \succ y$$

- Case 1:

$$\begin{cases} x_1 = y_1 \\ x_2 \geq y_2 \\ x_2 \neq y_2 \end{cases} \implies x \succ y$$

- Case 2:

$$\begin{cases} x_1 > y_1 \\ x_2 \geq y_2 \end{cases} \implies x \succ y$$

Convexity

$$\text{To show: } \begin{cases} y \succeq x \\ z \succeq x \end{cases} \implies \forall \lambda \in (0,1) : \lambda y + (1-\lambda) z \succ x$$

$$\underbrace{[y_1 > x_1]}_{1} \quad \text{or} \quad \underbrace{[y_1 = x_1 \wedge y_2 \geq x_2]}_{2}$$
$$\underbrace{[z_1 > x_1]}_{A} \quad \text{or} \quad \underbrace{[z_1 = x_1 \wedge z_2 \geq x_2]}_{B}$$

- Case 1 (1A or 1B or 2A):

$$\text{One of the cases} \begin{cases} [y_1 > x_1] \wedge [z_1 > x_1] \\ [y_1 > x_1] \wedge [z_1 = x_1] \\ [y_1 = x_1] \wedge [z_1 > x_1] \end{cases} \implies \lambda y_1 + (1-\lambda) z_1 > x_1 \implies \lambda y + (1-\lambda) z \succ x$$

- Case 2 (2B):

$$\begin{cases} y_1 = z_1 = x_1 \\ y_2 \geq x_2 \\ z_2 \geq x_2 \end{cases}$$

$$y \neq z \implies z_2 \neq y_2 \implies [z_2 > x_2] \vee [y_2 > x_2] \implies \lambda y_2 + (1-\lambda) z_2 > x_2 \implies \lambda y (1-\lambda) z \succ x$$

2(b)

No, we cannot find a utility function that represents these $\succeq$

Proof by contradiction

Assume $\exists u(\cdot)$ representing $\succeq$

$$\forall x_1 \rightarrow (x_1, 2) \succ (x_1, 1) \implies u(x_1, 2) > u(x_1, 1)$$

We know:

$$\begin{gathered}
\exists r(x_1) \in \mathbb{Q} \implies u(x_1, 2) > r(x_1) > u(x_1, 1) \\
\forall x_1, x_2 \in \mathbb{R}, \text{if } x_1 < x_2 \implies r(x_1) < u(x_1, 2) < u(x_2, 1) < r(x_2) \\
r(\cdot) \text{ is strictly increasing } \implies r \text{ is injective} \\
r(\cdot) : \mathbb{R} \rightarrow \mathbb{Q} \quad \text{contradiction}(|\mathbb{R}| > |\mathbb{Q}|)
\end{gathered}$$

2(c)

Continuity

$$\forall n, x_n \succeq y_n \implies \lim_{n\to\infty} x_n \succeq \lim_{n\to\infty} y_n$$

A counter example

$$\begin{cases} x_n = (\frac{1}{n}, 0) \\ y_n = (0, 1) \end{cases} \implies \forall n, x_n \succ y_n \implies \forall, x_n \succeq y_n$$

Note: if the first coordinate is different, then we only look at the first coordinate.

$$\begin{cases} \lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} (\frac{1}{n}, 0) = (0, 0) \\ \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} (0, 1) = (0, 1) \end{cases} \implies \lim_{n\to\infty} y_n \succeq \lim_{n\to\infty} x_n$$

3. (Preferences and Utility)

3(a)

$$\begin{gathered}
x \succeq y \Longleftrightarrow x \geq y \Longleftrightarrow x_l \geq y_l \quad \forall l \\
L = 1 \\
u(x) = x \\
u(x) \geq u(y) \\
x \succeq y \\
x \geq y \\
L = 2 \\
\succeq \quad \text{not rational} \\
\succeq \quad \text{not complete} \\
\text{Example:} \\
x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
x_1 > y_1 \wedge x_2 < y_2 \implies x \not\succeq y
\end{gathered}$$

Therefore, these preferences cannot be represented by a utility function.

3(b)

4. (Preferences and Utility)

$$u(x, y) = 3\ln(x) + 2\ln(y)$$

4(a)

- Convex $\Longleftrightarrow$ Quasi-concave
    - $\ln(\cdot)$ is strictly concave
    - sum of concave functions is concave
    - $u(x, y)$ is strictly concave
    - $u(x, y)$ is quasi-concave
    - Preferences are strictly convex
- Continuous
    - $\ln(\cdot)$ is continuous
    - $u(x, y)$ is continuous
- Monotonic

$$\frac{\partial u(x, y)}{\partial x} = \frac{3}{x} > 0$$
$$\frac{\partial u(x, y)}{\partial y} = \frac{2}{y} > 0$$

- LNS

  Monotonicity $\Longrightarrow$ LNS

4(b)

$$v = f(u) = \exp\left(\frac{u}{5}\right)$$
$$v(x, y) = \exp\frac{u(x, y)}{5} = \exp\left(\frac{3\ln(x) + 2\ln(y)}{5}\right) = x^{\frac{3}{5}} \cdot y^{\frac{2}{5}}$$
$$v(\lambda x, \lambda y) = (\lambda x)^{\frac{3}{5}} \cdot (\lambda y)^{\frac{2}{5}} = \lambda v(x, y)$$

MRS is homotheic of degree 1.

$$MRS(x, y) = \frac{\frac{\partial u(x,y)}{\partial x}}{\frac{\partial u(x,y)}{\partial y}} = \frac{3}{x} \cdot \frac{y}{2} = \frac{3y}{2x}$$
$$u(\lambda x, \lambda y) = 3\ln(\lambda x) + 2\ln(\lambda y)$$
$$MRS(\lambda x, \lambda y) = \frac{\frac{\partial u(\lambda x,\lambda y)}{\partial x}}{\frac{\partial u(\lambda x,\lambda y)}{\partial y}} = \frac{3}{\lambda x} \cdot \frac{\lambda y}{2} = \frac{3y}{2x}$$
$$MRS(x, y) = MRS(\lambda x, \lambda y)$$

MRS is homothetic of degree 0.

5. (Demand and Utility)

5(a)

Zero-homogeneity

$$x_l(\lambda p, \lambda w) = \frac{\lambda w}{\sum_{i=1}^{L} \lambda p_i} = \frac{w}{\sum_{i=1}^{L} p_i} = x_l(p, w)$$

Walras' Law

$$p \cdot x = \sum_{i=1}^{L} p_i \cdot x_l(p, w) = \sum_{l=1}^{L} p_l \cdot \frac{w}{\sum_{i=1}^{L} p_i} = w$$

5(b)

$$\begin{aligned} D_p x(p, w) &= \begin{pmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & \frac{\partial x_L(p,w)}{\partial p_L} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{w}{(\sum_{i=1}^{L} p_i)^2} & \cdots & -\frac{w}{(\sum_{i=1}^{L} p_i)^2} \\ \vdots & \ddots & \vdots \\ -\frac{w}{(\sum_{i=1}^{L} p_i)^2} & \cdots & -\frac{w}{(\sum_{i=1}^{L} p_i)^2} \end{pmatrix} \end{aligned}$$

5(c)

$$\frac{\partial x_l(p, w)}{\partial p_k} \geq 0 \implies \text{good } l \text{ is a gross substitute for good } k$$

5(d)

Leontief Utility Function