Problem Set 9

1. Profit Maximization

$q \in \mathbb{R}_+$ denotes the quantity produced (output)

$f(x)$ is a production function, where $x \in \mathbb{R}_+^{L-1}$ are inputs

The production possibility set

$$Y = \{(q, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{L-1} : q \leq f(x)\}$$

where $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is a strictly concave C^2-function

1(b)

<u>Notation</u>

- price of q is p_1
- input price $w = (p_2, \cdots, p_L)$

The profit-maximization problem (PMP)

$$\begin{aligned} \max_{x,q} \quad & p_1 \cdot q - w \cdot x \\ \text{s.t.} \quad & q \leq f(x) \\ & -x_l \leq 0 \quad \forall l = 2, \cdots, L \\ & -q \leq 0 \end{aligned}$$

We argue by contradiction that $q = f(x)$ at any optimum

Suppose (q', x') solves PMP and $q' < f(x')$

Define $q^* = f(x')$ such that (q^*, x') is also a feasible production plan

By optimality of (q', x'), we have

$$\begin{aligned} p_1 q' - w x' &\geq p_1 q^* - w x' \\ q' &\geq q^* \end{aligned}$$

However,

$$q' < f(x') \implies q' < q^* \quad \text{contradiction}$$

1(c)

Define FOC

$$q = f(x) \quad \text{at any optimum}$$

We can write the Lagrangian as

$$\mathcal{L} = p_1 \cdot f(x) - w \cdot x + \sum_{l=1}^{L} \mu_l x_l \quad \mu_l \text{ is the Lagrangian multipliers}$$

By Kuhn-Tucker, the FOCs are

$$p_1 \frac{\partial f(x)}{\partial x_l} - p_l + \mu_l = 0 \quad \forall l = 2, \cdots, L$$
$$p_1 \frac{\partial f(x)}{\partial x_l} = p_l - \mu_l$$

In vector notation

$$p_1 \nabla f(x) = w - \mu \quad \text{complementary slackness condition } \mu_l x_l = 0$$

Hence, if the solution is interior, we have $\mu_l = 0$ and FOC becomes $p_1 \nabla f(x) = w$

1(d)

$$\begin{cases} L = 2 \\ f(x) = \sqrt{x} \end{cases}$$

$$p_1 \nabla f(x) = w$$
$$p_1 \times \frac{1}{2} \times \frac{1}{\sqrt{x}} = w$$
$$\sqrt{x} = \frac{p_1}{2w}$$

We know that

$$f(x) = \sqrt{x} = q \implies \begin{cases} x = \dfrac{p_1^2}{4w^2} \\ q = \dfrac{p_1}{2w} \end{cases}$$

$$(x^*, q^*) = \left(\frac{p_1^2}{4w^2}, \frac{p_1}{2w} \right)$$

iso-profit line of Π^* (profit of optimal plan (x^*, q^*) is tangent to Y)

2. Exchange Economy

$$\begin{cases} u^1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha} & \alpha \in (0,1) \\ u^2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta} & \beta \in (0,1) \end{cases}$$

2(a)

Take income for consumer 1 $(w_1 > 0)$ as exogenously given

$$\max_{x_1, x_2} \quad \alpha \ln(x_1) + (1-\alpha) \ln(x_2)$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 \leq w_1$$

$$\mathcal{L} = \alpha \ln(x_1) + (1-\alpha) \ln(x_2) - \lambda(p_1 x_1 + p_2 x_2 - w_1)$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_1} = \dfrac{\alpha}{x_1} - \lambda p_1 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_2} = \dfrac{1-\alpha}{x_2} - \lambda p_2 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_1 + p_2 x_2 - w_1 = 0 \end{cases} \implies \begin{cases} \dfrac{p_1}{p_2} = \dfrac{\alpha}{1-\alpha} \cdot \dfrac{x_2}{x_1} \\ p_1 x_1 + p_2 x_2 = w_1 \end{cases} \implies \begin{cases} x_1 = \dfrac{\alpha}{p_1} w_1 \\ x_2 = \dfrac{1-\alpha}{p_2} w_1 \end{cases}$$

$x_1(p, w_1)$ is independent of p_2

2(b)

Endogenous income for consumer 1

$$w_1 = p_1\omega_{11} + p_2\omega_{21} \implies \begin{cases} x_1 = \dfrac{\alpha}{p_1}w_1 = \dfrac{\alpha}{p_1}(p_1\omega_{11} + p_2\omega_{21}) \\ x_2 = \dfrac{1-\alpha}{p_2}w_1 = \dfrac{1-\alpha}{p_2}(p_1\omega_{11} + p_2\omega_{21}) \end{cases}$$

$$x_1 = \frac{\alpha}{p_1}(p_1\omega_{11} + p_2\omega_{21})$$

x_1 depends on positively on p_2 . Demand behaves as if goods are gross substitutes.

Reason: with endogenous income, a change in prices changes real income, which triggers additional income effects

x_1 increases in p_2 is driven by:

- x_1 is a normal good
- Income increases if p_2 increases

2(c)

$$\text{endowments} = \begin{cases} (\omega_{11}, \omega_{21}) = (1, 0) \\ (\omega_{12}, \omega_{22}) = (0, 1) \end{cases}$$

$$\text{price} = \begin{cases} p_1 = 1 \\ p_2 = p \end{cases}$$

$$\text{demand for consumer 1} = \begin{cases} x_{11} = \dfrac{\alpha}{p_1}(p_1\omega_{11} + p_2\omega_{21}) = \alpha \\ x_{21} = \dfrac{1-\alpha}{p_2}(p_1\omega_{11} + p_2\omega_{21}) = \dfrac{1-\alpha}{p} \end{cases}$$

$$\text{demand for consumer 2} = \begin{cases} x_{12} = \dfrac{\beta}{p_1}(p_1\omega_{12} + p_2\omega_{22}) = \beta \cdot p \\ x_{22} = \dfrac{1-\beta}{p_2}(p_1\omega_{12} + p_2\omega_{22}) = 1 - \beta \end{cases}$$

Market clearing

$$\begin{aligned} x_{11} + x_{12} &= \omega_{11} + \omega_{12} \\ \alpha + \beta \cdot p &= 1 \\ p &= \frac{1-\alpha}{\beta} \end{aligned}$$

$$\textbf{Walrasian equilibrium} = \begin{cases} x_{11} = \alpha \\ x_{21} = \beta \\ x_{12} = 1 - \alpha \\ x_{22} = 1 - \beta \end{cases}$$

3. Exchange Economy

$$\begin{cases} u^1(x_{11}, x_{21}) = \min\{2x_{11}, x_{21}\} & \text{consumer 1} \\ u^2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\} & \text{consumer 2} \end{cases}$$

3(b)

$$\begin{cases} (\omega_{11}, \omega_{21}) = (1, 0) \\ (\omega_{12}, \omega_{22}) = (0, 1) \\ (p_1, p_2) = (0, 1) \end{cases}$$

Equilibrium

Consumer 1 has income 0 and consumer 2 has income 1

Consumer 2 will consume $x_{22} = 1$ and $x_{12} = 1$ to have $\min\{x_{11}, x_{22}\} = 1$

Consumer 1 chooses $x_{21} = 0$ and market clearing gives $x_{11} = 0$

$$\begin{cases} (x^*_{11}, x^*_{21}) = (0, 0) \\ (x^*_{12}, x^*_{22}) = (1, 1) \end{cases}$$