Exam
Social Choice Theory
Fall 2012
Solution

Problem 1: Choice Functions

(a) Let K be the set of all non-empty subsets of X.

A choice function on X is a mapping $C : K \to K$ with $C(S) \subseteq S$ for all $S \in K$.

The choice function summarizes how a decision-maker chooses in every situation. If exactly the alternatives in S are available, then the alternatives $C(S)$ are chosen. If $C(S)$ has more than one element, this means that the decision-maker chooses differently from time to time.

(b) C^1: Property α is satisfied. Alternative y is always chosen when available. Alternative x is chosen from $\{w, x, z\}$ and also from $\{x, z\}$ and $\{w, x\}$.

Property β is satisfied, because $C^1(S)$ always contains exactly one element.

C^2: Property α is violated. We have $x \in C^2(\{w, x, y\})$ but $x \notin C^2(\{w, x\})$.

Property β is violated. We have $C^2(\{w, y\}) = \{w, y\}$ but $C^2(\{w, x, y\}) = \{w, x\}$.

C^3: Property α is satisfied. Alternative w is always chosen when available. Alternative x is chosen from $\{x, y, z\}$ and also from $\{x, y\}$ and $\{x, z\}$. Alternative y is chosen from $\{x, y, z\}$ and also from $\{x, y\}$ and $\{y, z\}$.

Property β is satisfied. We have $C^3(\{x, y\}) = \{x, y\}$ and also $C^3(\{x, y, z\}) = \{x, y\}$, while neither x nor y are chosen from any other set S with $|S| \geq 3$.

(c) C^1: $R_{C^1} = \{(x, w), (y, w), (w, z), (y, x), (x, z), (y, z), (w, w), (x, x), (y, y), (z, z)\}$

This relation is transitive, as C^1 satisfies α and β.

C^2: $R_{C^2} = \{(w, x), (w, y), (y, w), (w, z), (x, y), (x, z), (y, z), (w, w), (x, x), (y, y), (z, z)\}$

This relation is not transitive: $(y, w) \in R_{C^2}$ and $(w, x) \in R_{C^2}$ but $(y, x) \notin R_{C^2}$.

C^3: $R_{C^3} = \{(w, x), (w, y), (w, z), (x, y), (y, x), (x, z), (y, z), (w, w), (x, x), (y, y), (z, z)\}$

This relation is transitive, as C^3 satisfies α and β.

Problem 2: Arrow's Theorem for SWFs

(a) Arrow's Impossibility Theorem for SWFs says that, when there are at least three alternatives, there is no social welfare function $f : \mathscr{A} \to \mathscr{R}$ (which assigns a preference $f(\mathbf{R})$ to each admissible preference profile $\mathbf{R} \in \mathscr{A} \subseteq \mathscr{R}^n$) that satisfies the following four axioms:

Universality [U]:
$\mathscr{A} = \mathscr{R}^n$.

Independence of Irrelevant Alternatives [I]:
For any pair of alternatives $x, y \in X$, if two profiles $\mathbf{R}, \mathbf{R}' \in \mathscr{A}$ satisfy

$$xR_iy \Leftrightarrow xR'_iy \text{ and } yR_ix \Leftrightarrow yR'_ix$$

for all voters i, then

$$xf(\mathbf{R})y \Leftrightarrow xf(\mathbf{R}')y \text{ and } yf(\mathbf{R})x \Leftrightarrow yf(\mathbf{R}')x$$

must hold.

Weak Pareto Principle [P]:
For any pair of alternatives $x, y \in X$, if $\mathbf{R} \in \mathscr{A}$ satisfies xP_iy for all voters i, then $xf_P(\mathbf{R})y$ must be true, where $f_P(\mathbf{R})$ is the asymmetric part of $f(\mathbf{R})$.

Non-Dictatorship [D]:
There is no voter i such that, for all $x, y \in X$ and $\mathbf{R} \in \mathscr{A}$, xP_iy implies $xf_P(\mathbf{R})y$.

(b) We obtain $X^E = \{x, z\}$ and $X^I = \{w, y\}$, as both w and y are Pareto dominated by z, while both x and z are ranked top by at least one voter. Hence we obtain the social preference $x\,I\,z\,P\,y\,P\,w$.

(c) [U] Universality is satisfied, because the method can always be applied and will always deliver a preference.

[I] Independence of Irrelevant Alternatives is violated:

#	profile $\mathbf{R}$	profile $\mathbf{R}'$
1	$x\ P\ y\ P\ z$	$x\ P\ z\ P\ y$
1	$y\ P\ x\ P\ z$	$z\ P\ y\ P\ x$
f	$x\ I\ y\ P\ z$	$x\ I\ z\ P\ y$

Even though the preference between x and y does not change for any voter from $\mathbf{R}$ to $\mathbf{R}'$, the social preference between x and y changes.

[P] The Weak Pareto Principle is satisfied. If xP_iy for all i, then $y \in X^I$. If $x \in X^E$, then $xf_P(\mathbf{R})y$ by definition of the method. If $x \in X^I$, then $xf_P(\mathbf{R})y$ by definition of the method because xP_1y by assumption.

[D] The rule is not dictatorial, since voter 1 dictates the outcome only among the Pareto-inefficient alternatives.

(d) Fix an arbitrary preference $R' \in \mathscr{R}$ and consider the SWF f that always prescribes R', i.e. $f(\mathbf{R}) = R'$ for all $\mathbf{R} \in \mathscr{R}^n$.

[U] Satisfied, by definition of the SWF.

[I] Satisfied, because the social preference never changes.

[P] Violated. For instance, assume $xR'y$ (without loss of generality). Then $xf(\mathbf{R})y$ even if yP_ix for all i.

[D] Satisfied, because no voter can always enforce the own strict preference.

Problem 3: Manipulability

$R_1 \backslash R_2$	xyz	xzy	yzx	yxz	zxy	zyx
xyz	x	x	z	x	z	z
xzy	x	x	z	x	z	z
yzx	x	x	z	x	z	z
yxz	x	x	z	x	z	z
zxy	x	x	z	x	z	z
zyx	x	x	z	x	z	z

(a) Arrow's axioms for SCFs:

[$\bar{\text{U}}$] Universality $\mathscr{A} = \mathscr{P}^2$ is satisfied by definition of the rule.

[$\bar{\text{M}}$] Monotonicity is satisfied.

Voter 1 can be ignored, as her preference never influences the outcome.

Alternative z is chosen when voter 2's preference is yzx. It maintains its position only when moving to zxy or zyx, in which case it is still chosen. Alternative x is chosen when voter 2's preference is yxz. It maintains its position only when moving to xyz or xzy, in which case it is still chosen. Voter 2's top ranked alternative is chosen in all other cases, so there is nothing else to check.

[$\bar{\text{P}}$] The Weak Pareto Principle is violated. For instance, alternative z is selected even if both voters' preference is yzx.

[$\bar{\text{D}}$] Non-Dictatorship is satisfied, as the rule does not always select the most preferred alternative of voter 2 (and clearly also not of voter 1).

(b) We have already verified [$\bar{\text{U}}$] and [$\bar{\text{D}}$].

Axiom [$\bar{\text{S}}$] is also satisfied. Voter 1 can clearly never manipulate this rule. Voter 2 cannot manipulate when her preference is xyz, xzy, zxy, or zyx, as her optimum is already selected in these cases. Manipulation at yzx or yxz is not possible either, because the preferred alternative y cannot be obtained.

These arguments do not contradict the Gibbard-Satterthwaite theorem, as the rule is not surjective, i.e., it never selects alternative y.

(c) The following four strict preferences are single peaked with respect to $x > y > z$: xyz, yzx, yxz, and zyx. Consider the following SCF:

$R_1 \backslash R_2$	xyz	yzx	yxz	zyx
xyz	x	y	y	y
yzx	y	y	y	y
yxz	y	y	y	y
zyx	y	y	y	z

It can easily be checked that this SCF is surjective and satisfies both [$\bar{\text{D}}$] and [$\bar{\text{S}}$].