Problem Set 3

Exercise 1: Logarithmic Case of Two-Country Endowment Economy

An open endowment economy

$$S_1 + S_1^\star = 0$$

Home country

$$\begin{aligned} \text{lifetime utility in home country:} \quad & U = \log(C_1) + \beta \log(C_2) \\ \text{intertemporal budget constraint:} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

Foreign country

$$\begin{aligned} \text{lifetime utility in foreign country:} \quad & U^\star = \log(C_1^\star) + \beta^\star \log(C_2^\star) \\ \text{intertemporal budget constraint:} \quad & C_1^\star + \frac{C_2^\star}{1+r} = Y_1^\star + \frac{Y_2^\star}{1+r} \end{aligned}$$

Solution to the intertemporal optimization problem

$$\begin{aligned} C_1 &= \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right) \\ C_2 &= \frac{\beta}{1+\beta}(1+r)\left(Y_1 + \frac{Y_2}{1+r}\right) \end{aligned}$$

1(a)

Savings schedule

$$\begin{aligned} S_1 &= Y_1 - C_1 \\ &= Y_1 - \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right) \\ &= \frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\cdot\frac{Y_2}{1+r} \\ S_1(r) &= \frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\cdot\frac{Y_2}{1+r} \\ \frac{\partial S_1(r)}{\partial r} &= \frac{Y_2}{(1+\beta)(1+r)^2} > 0 \end{aligned}$$

r : willingness to consume tomorrow。

Three channels through which a change in r can affect C_1 :

- *Substitution Effect (SE)*

  A rise in r makes present consumption more expensive in terms of future consumption. Ceteris paribus, this lead to a decline in $C_1 \rightarrow \frac{1}{(1+r)^\sigma}$

- *Income Effect (IE)*

  Given the present value of life time resources, higher r makes Y_1 more valuable in terms of consumption tomorrow $C_1 \rightarrow \frac{1}{(1+r)^{-1}}$

- *Wealth Effect (WE)*

  A rise in r decreases the present value of lifetime resources and therefore reduces C_1

With log utility function, substitution effect and income effect cancel out, and wealth effect dominates. Therefore, a rise in r decreases C_1 and increases S_1.

1(b)

World Equilibrium

$$\text{world equilibrium conditions} \begin{cases} \text{interest rates in both home country and foreign country are the same} \\ S_1 + S_1^\star = 0 \end{cases}$$

From 1(a), we have

$$\begin{aligned} S_1 &= \frac{\beta}{1+\beta} Y_1 - \frac{Y_2}{(1+\beta)(1+r)} \\ S_1^\star &= \frac{\beta^\star}{1+\beta^\star} Y_1^\star - \frac{Y_2^\star}{(1+\beta^\star)(1+r)} \end{aligned}$$

$$\begin{aligned} S_1 + S_1^\star = 0 \implies \frac{\beta}{1+\beta} Y_1 - \frac{Y_2}{(1+\beta)(1+r)} + \frac{\beta^\star}{1+\beta^\star} Y_1^\star - \frac{Y_2^\star}{(1+\beta^\star)(1+r)} &= 0 \\ \frac{\beta}{1+\beta} Y_1 + \frac{\beta^\star}{1+\beta^\star} Y_1^\star &= \frac{1}{1+r}\left(\frac{Y_2}{1+\beta} + \frac{Y_2^\star}{1+\beta^\star}\right) \\ 1 + r &= \frac{\frac{Y_2}{1+\beta} + \frac{Y_2^\star}{1+\beta^\star}}{\frac{\beta}{1+\beta} Y_1 + \frac{\beta^\star}{1+\beta^\star} Y_1^\star} \\ r &= \frac{\frac{Y_2}{1+\beta} + \frac{Y_2^\star}{1+\beta^\star}}{\frac{\beta}{1+\beta} Y_1 + \frac{\beta^\star}{1+\beta^\star} Y_1^\star} - 1 \end{aligned}$$

1(c)

Optimization problem in home country

$$\begin{aligned} \max_{C_1, C_2} \quad & U = \log(C_1) + \beta \log(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

FOC:

$$\frac{\beta u'(C_2)}{u'(C_1)} = \frac{\beta C_1}{C_2} = \frac{1}{1+r}$$

Under the *Autarky* condition, the agent consumes exactly the same amount of endowment in each period

$$\frac{\beta u'(Y_2)}{u'(Y_1)} = \frac{\beta Y_1}{Y_2} = \frac{1}{1+r^A} \implies Y_2 = \beta(1+r^A) Y_1$$

Similarly,

$$Y_2^\star = \beta^\star(1 + r^{A^\star}) Y_1^\star$$

Plug $\begin{cases} Y_2 = \beta(1+r^A) Y_1 \\ Y_2^\star = \beta^\star(1+r^{A^\star}) Y_1^\star \end{cases}$ into $1 + r = \dfrac{\frac{Y_2}{1+\beta} + \frac{Y_2^\star}{1+\beta^\star}}{\frac{\beta}{1+\beta} Y_1 + \frac{\beta^\star}{1+\beta^\star} Y_1^\star}$

$$
\begin{aligned}
1+r &= \frac{\frac{Y_2}{1+\beta}+\frac{Y_2^\star}{1+\beta^\star}}{\frac{\beta}{1+\beta}Y_1+\frac{\beta^\star}{1+\beta^\star}Y_1^\star} \\
&= \frac{\frac{\beta}{1+\beta}Y_1(1+r^A)+\frac{\beta^\star}{1+\beta^\star}Y_1^\star(1+r^{A^\star})}{\frac{\beta}{1+\beta}Y_1+\frac{\beta^\star}{1+\beta^\star}Y_1^\star} \\
&= \underbrace{\frac{\frac{\beta}{1+\beta}Y_1}{\frac{\beta}{1+\beta}Y_1+\frac{\beta^\star}{1+\beta^\star}Y_1^\star}}_{\alpha}(1+r^A)+\underbrace{\frac{\frac{\beta^\star}{1+\beta^\star}Y_1^\star}{\frac{\beta}{1+\beta}Y_1+\frac{\beta^\star}{1+\beta^\star}Y_1^\star}}_{(1-\alpha)}(1+r^{A^\star}) \\
&= \alpha(1+r^A)+(1-\alpha)(1+r^{A^\star})
\end{aligned}
$$

$$
1+r=\alpha(1+r^A)+(1-\alpha)(1+r^{A^\star}) \implies r=\alpha r^A+(1-\alpha)r^{A^\star}
$$

1(e)

(i)

$$
\frac{Y_2^\star}{Y_1^\star}=\beta^\star(1+r^{A^\star})
$$

Suppose that $Y_1^\star$ remains unchanged,

$$
\frac{Y_2^\star\uparrow}{Y_1^\star} \implies r^{A^\star}\uparrow \implies r\uparrow
$$

(ii)

$$
C_1=\frac{1}{1+\beta}\left(Y_1+\frac{Y_2}{1+r}\right) \implies \frac{\partial C_1}{\partial r}=-\frac{Y_2}{(1+\beta)(1+r)^2}
$$

$$
C_2=\frac{\beta}{1+\beta}(1+r)\left(Y_1+\frac{Y_2}{1+r}\right) \implies \frac{\partial C_2}{\partial r}=\frac{\beta}{1+\beta}\left(Y_1+\frac{Y_2}{1+r}\right)-\frac{\beta Y_2}{(1+\beta)(1+r)}=\frac{\beta}{1+\beta}Y_1
$$

$$
\begin{cases}
\frac{\partial C_1}{\partial r}=-\frac{Y_2}{(1+\beta)(1+r)^2}\\
\frac{\partial C_2}{\partial r}=\frac{\beta}{1+\beta}Y_1
\end{cases}
$$

$$
\begin{cases}
C_1=\frac{1}{1+\beta}\left(Y_1+\frac{Y_2}{1+r}\right)\\
C_2=\frac{\beta}{1+\beta}(1+r)\left(Y_1+\frac{Y_2}{1+r}\right)\\
Y_2=\beta(1+r^A)Y_1
\end{cases}
$$

$$
\begin{aligned}
\frac{\mathrm{d}U}{\mathrm{d}r} &= \frac{\partial U}{\partial C_1}\cdot\frac{\partial C_1}{\partial r}+\frac{\partial U}{\partial C_2}\cdot\frac{\partial C_2}{\partial r} \\
&= -\frac{1}{C_1}\cdot\frac{Y_2}{(1+\beta)(1+r)^2}+\frac{\beta}{C_2}\cdot\frac{\beta}{1+\beta}Y_1 \\
&= -\frac{1}{Y_1+\frac{Y_2}{1+r}}\cdot\frac{Y_2}{(1+r)^2}+\frac{1}{Y_1+\frac{Y_2}{1+r}}\cdot\frac{\beta Y_1}{1+r} \\
&= \frac{1}{Y_1+\frac{Y_2}{1+r}}\cdot\frac{\beta Y_1(r-r^A)}{(1+r)^2} \\
&= \frac{\beta}{1+r}\left(\frac{r-r^A}{1+r+\frac{Y_2}{Y_1}}\right) \\
&= \frac{\beta}{1+r}\left(\frac{r-r^A}{1+r+\beta(1+r^A)}\right)
\end{aligned}
$$

(iii)

From (i), we have known that higher foreign output growth will lead to an increase in world interest rate.

- If $r > r^*$, the home country is a creditor country and runs a current account surplus. From (ii), we know that an increase in world interest rate in will lead to a higher utility in home country. In this case, higher foreign output growth is beneficial for the home country.
- If $r < r^*$, the home country is a debtor country and runs a current account deficit. From (ii), we know that an increase in world interest rate will lead to a lower utility in home country. In this case, higher foreign output growth is not beneficial for the home country.