Problem Set 10

1

$$\begin{cases} u^1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha} & \text{consumer 1} \\ u^2(x_{12}, x_{22}) = x_{12}^{\beta} x_{22}^{1-\beta} & \text{consumer 2} \end{cases}$$

$$\begin{cases} (\omega_1, \omega_2) = (1, 1) \\ \alpha = \frac{1}{3} \\ \beta = \frac{2}{3} \end{cases}$$

Proposition 3.1

An allocation (x) is a Pareto Optimum if and only if (x) solves the optimization problem for some choice of $(\bar{u}_2, \cdots, \bar{u}_I)$

$$\begin{aligned} \max_{x \geq 0} \quad & u_1(x_1) \\ \text{s.t.} \quad & u_i(x_i) \geq \bar{u}_i \quad i > 1 \\ & \sum_i x_i \leq \bar{\omega} \end{aligned}$$

Apply Proposition 3.1 from the lecture

$$\begin{aligned} \max_{x_{11}, x_{21}, x_{12}, x_{22}} \quad & \alpha \ln x_{11} + (1-\alpha) \ln x_{21} \\ \text{s.t.} \quad & \begin{cases} \beta \ln x_{12} + (1-\beta) \ln x_{22} \geq \bar{u}_2 \\ x_{11} + x_{12} \leq \omega_1 \\ x_{21} + x_{22} \leq \omega_2 \end{cases} \end{aligned}$$

Both utility functions are strictly increasing

$$\mathcal{L} = \alpha \ln(1 - x_{12}) + (1-\alpha) \ln(1 - x_{22}) - \mu(\bar{u}_2 - \beta \ln x_{12} - (1-\beta) \ln x_{22})$$

FOC:

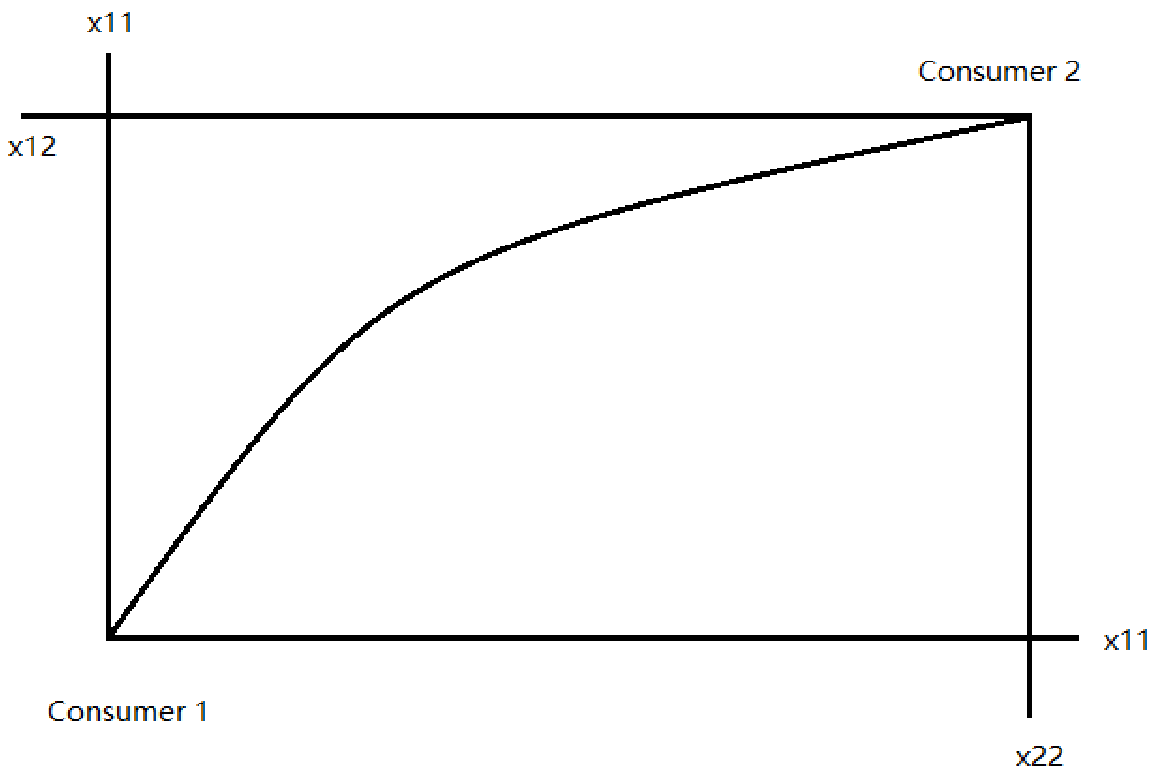
$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{12}} = -\dfrac{\alpha}{1 - x_{12}} + \mu \dfrac{\beta}{x_{12}} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_{22}} = -\dfrac{1-\alpha}{1 - x_{22}} + \mu \dfrac{1-\beta}{x_{22}} = 0 \end{cases} \implies \frac{\alpha}{\beta} \cdot \frac{x_{12}}{1 - x_{12}} = \frac{1-\alpha}{1-\beta} \cdot \frac{x_{22}}{1 - x_{22}}$$

$$\begin{aligned} \frac{\alpha}{\beta} \cdot \frac{x_{12}}{1 - x_{12}} &= \frac{1-\alpha}{1-\beta} \cdot \frac{x_{22}}{1 - x_{22}} \\ \frac{1}{2} \times \frac{x_{12}}{1 - x_{12}} &= 2 \times \frac{x_{22}}{1 - x_{22}} \\ x_{12}(1 - x_{22}) &= 4x_{22}(1 - x_{12}) \\ x_{12} - x_{12}x_{22} &= 4x_{22} - 4x_{12}x_{22} \\ x_{12} &= 4x_{22} - 3x_{12}x_{22} \\ x_{12} &= (4 - 3x_{12})x_{22} \\ x_{22} &= \frac{x_{12}}{4 - 3x_{12}} \end{aligned}$$

$$\begin{cases} x_{11} + x_{12} = 1 \\ x_{21} + x_{22} = 1 \end{cases} \implies \begin{cases} x_{12} = 1 - x_{11} \\ x_{22} = 1 - x_{21} \end{cases}$$

$$x_{22} = \frac{x_{12}}{4 - 3x_{12}}$$
$$1 - x_{21} = \frac{1 - x_{11}}{1 + 3x_{11}}$$
$$x_{21} = 1 - \frac{1 - x_{11}}{1 + 3x_{11}}$$
$$x_{21} = \frac{4x_{11}}{1 + 3x_{11}}$$

$$\text{Pareto set} = \begin{cases} x_{21} = \dfrac{4x_{11}}{1 + 3x_{11}} \\ x_{22} = \dfrac{x_{12}}{4 - 3x_{12}} \end{cases}$$



2

$$\omega = (\omega_1 \quad \omega_2 \quad \omega_3) = (0 \quad 1 \quad 0)$$

$$u = x_1 - \alpha x_3$$

$$\begin{cases} y_1 \le \sqrt{z_2} \\ y_3 = \beta z_2 \end{cases}$$

2(a)

- Good 1 is a normal good
- Good 2 can be interpreted as labor
- Good 3 can be interpreted as pollution/negative externalities

2(b)

The Pareto optimum can be characterized by solving

$$\max_{x_1,x_2,x_3} \quad x_1-\alpha x_3$$
$$\text{s.t.} \quad \begin{cases} x_1 \le \omega_1+y_1 \\ x_2+z_2 \le \omega_2 \\ x_3 \le \omega_3+y_3 \end{cases}$$

$$\begin{cases} x_1 \le \omega_1+y_1 \\ x_2+z_2 \le \omega_2 \\ x_3 \le \omega_3+y_3 \end{cases} \implies \begin{cases} x_1 = \omega_1+y_1 \\ x_2+z_2 = \omega_2 \\ x_3 = \omega_3+y_3 \end{cases} \implies \begin{cases} x_1 = \sqrt{z_2} \\ x_2+z_2 = 1 \\ x_3 = \beta z_2 \end{cases} \implies \begin{cases} x_1 = \sqrt{1-x_2} \\ x_3 = \beta(1-x_2) \end{cases}$$

$$\max_{x_1,x_2,x_3} \quad x_1-\alpha x_3$$
$$\text{s.t.} \quad \begin{cases} x_1 = \sqrt{1-x_2} \\ x_2 \in [0,1] \\ x_3 = \beta(1-x_2) \end{cases}$$

$$\max_{x_2\in[0,1]} \quad \sqrt{1-x_2}-\alpha\beta(1-x_2)$$

FOC:

$$-\frac{1}{2}\cdot\frac{1}{\sqrt{1-x_2}}+\alpha\beta=0$$
$$\sqrt{1-x_2}=\frac{1}{2\alpha\beta}$$
$$1-x_2=\frac{1}{4\alpha^2\beta^2}$$
$$x_2=1-\frac{1}{4\alpha^2\beta^2}$$

- x_2 is an interior solution

$$1-\frac{1}{4\alpha^2\beta^2}\ge 0$$
$$\frac{1}{4\alpha^2\beta^2}\le 1$$
$$\frac{1}{2\alpha\beta}\le 1$$
$$\alpha\beta\ge\frac{1}{2}$$
$$x_1=\sqrt{1-x_2}=\frac{1}{2\alpha\beta}$$
$$x_3=\beta(1-x_2)=\frac{1}{4\alpha^2\beta}$$

$$\text{Pareto set:} \quad \begin{cases} x_1^*=\dfrac{1}{2\alpha\beta} \\ x_2^*=1-\dfrac{1}{4\alpha^2\beta^2} \\ x_3^*=\dfrac{1}{4\alpha^2\beta} \end{cases}$$

- x_2 is a corner solution

$$\begin{aligned}1-\frac{1}{4\alpha^2\beta^2}&<0\\ \frac{1}{4\alpha^2\beta^2}&>1\\ \frac{1}{2\alpha\beta}&>1\\ \alpha\beta&<\frac{1}{2}\end{aligned}$$

$$\text{Pareto set:}\quad \begin{cases}x_1^*=1\\ x_2^*=0\\ x_3^*=\beta\end{cases}$$

2(c)

Given (p_1, p_2, p_3) , the firm's PMP is

$$\begin{aligned}\max_{y_1,y_2,y_3}\quad & p_1y_1+p_2y_2+p_3y_3\\ \text{s.t.}\quad & \begin{cases}y_1=\sqrt{z_2}\\ y_2=-z_2\\ y_3=\beta z_2\end{cases}\end{aligned}$$

$$\max_{z_2\geq 0}\quad p_1\sqrt{z_2}-p_2z_2+\beta p_3z_2$$

FOC:

$$\begin{aligned}p_1\frac{1}{2\sqrt{z_2}}-p_2+\beta p_3=0&\\ \sqrt{z_2}&=\frac{p_1}{2(p_2-\beta p_3)}\\ z_2&=\frac{p_1^2}{4(p_2-\beta p_3)^2}\end{aligned}$$

$$\begin{cases}y_1=\sqrt{z_2}\\ y_3=\beta z_2\end{cases}\implies\begin{cases}y_1=\dfrac{p_1}{2(p_2-\beta p_3)}\\ y_3=\dfrac{\beta p_1^2}{4(p_2-\beta p_3)^2}\end{cases}$$

Since good 3 cannot be priced, $p_3 = 0$

$$\begin{aligned}y_1&=\frac{p_1}{2(p_2-\beta p_3)}\implies y_1=\frac{p_1}{2p_2}\\ y_2&=-\frac{p_1^2}{4(p_2-\beta p_3)^2}\implies y_2=-\frac{p_1^2}{4p_2^2}\\ y_3&=\frac{\beta p_1^2}{4(p_2-\beta p_3)^2}\implies y_3=\frac{\beta p_1^2}{4p_2^2}\end{aligned}$$

For the consumer

$$x_2=\begin{cases}0 & p_2>0\\ (0,1) & p_2=0\\ 1 & p_2<0\end{cases}$$

Clearly $p_2 \leq 0$ cannot be supported as a price in a WE because the labor market would not clear.

Therefore $p_2 > 0$

$$\begin{cases} x_1 = \sqrt{1-x_2} \\ x_2 = 0 \\ x_3 = \beta(1-x_2) \end{cases} \implies \begin{cases} x_1 = 1 \\ x_2 = 0 \\ x_3 = \beta \end{cases}$$

$$x_2 = 1 - \frac{1}{4\alpha^2\beta^2} = 0 \implies 4\alpha^2\beta^2 = 1 \implies 2\alpha\beta = 1$$

$$\begin{cases} x_1 = \sqrt{z_2} \\ x_2 + z_2 = 1 \implies z_2 = 1 \\ x_3 = \beta z_2 \end{cases}$$

$$\begin{cases} y_1 = \sqrt{z_2} \\ y_3 = \beta z_2 \end{cases} \implies \begin{cases} y_1 = 1 \\ y_3 = \beta \end{cases}$$

$$y_1 = \frac{p_1}{2p_2} \implies p_1 = 2p_2 \quad \text{Price Equilibrium}$$

Since $p_2 > 0$, the consumer supplies all of their labor. The PE is PO if and only if $\alpha\beta \le \frac{1}{2}$

2(d)

$$\alpha = \beta = 1 \implies \alpha\beta = 1 > \frac{1}{2}$$

In (b), we know that x_2 is an interior solution

$$x_2 = 1 - \frac{1}{4\alpha^2\beta^2} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$\begin{cases} x_1 = \sqrt{1-x_2} \\ x_3 = \beta(1-x_2) \end{cases} \implies \begin{cases} x_1 = \frac{1}{2} \\ x_3 = \frac{1}{4} \end{cases}$$

$$x_2 + z_2 = 1 \implies z_2 = 1 - \frac{3}{4} = \frac{1}{4} \implies y_2 = -z_2 = -\frac{1}{4}$$

$$\begin{cases} y_1 = \sqrt{z_2} \\ y_3 = \beta z_2 \end{cases} \implies \begin{cases} y_1 = \frac{1}{2} \\ y_3 = \frac{1}{4} \end{cases}$$

Any PE that implements the PO $(x_2 > 0)$ must satisfy $p_2 = 0$

Consumer solves

$$\begin{aligned} \max_{x_1, x_3} \quad & x_1 - x_3 \\ \text{s.t.} \quad & p_1 x_1 + p_3 x_3 \le \omega \end{aligned}$$

$$\mathcal{L} = x_1 - x_3 - \lambda(p_1 x_1 + p_3 x_3 - \omega)$$

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = 1 - \lambda p_1 = 0 \implies \lambda = \frac{1}{p_1} \\ \frac{\partial \mathcal{L}}{\partial x_1} = -1 - \lambda p_3 = 0 \implies \lambda = -\frac{1}{p_3} \end{cases} \implies p_1 = -p_3$$

Normalize $p_1 = 1$

$$\begin{cases} (x_1, x_2, x_3) = (\frac{1}{2}, \frac{3}{4}, \frac{1}{4}) \\ (y_1, y_2, y_3) = (\frac{1}{2}, -\frac{1}{4}, \frac{1}{4}) \\ (p_1, p_2, p_3) = (1, 0, -1) \end{cases}$$

All markets clear and WE is a PO.

3

$$\begin{cases} x & \text{food} \\ e & \text{employment} \\ w & \text{wage rate} \\ \tau & \text{tax} \end{cases}$$

$$\begin{aligned} U &= \ln x + \ln(1-e) \\ x &= we + T + \Pi \\ T &= \tau e \end{aligned}$$

3(a)

Consumer Problem:

$$\begin{aligned} \max_{x,e} \quad & \ln x + \ln(1-e) \\ \text{s.t.} \quad & x = we + T + \Pi \end{aligned}$$

$$\max_{e} \quad \ln(e(w+\tau)+\Pi) + \ln(1-e)$$

FOC:

$$\frac{w+\tau}{e(w+\tau)+\Pi} - \frac{1}{1-e} = 0 \implies e = \frac{w+\tau-\Pi}{2(w+\tau)}$$

$$\begin{aligned} x &= we + T + \Pi \\ &= we + \tau e + \Pi \\ &= \frac{w+\tau-\Pi}{2} + \Pi \\ &= \frac{w+\tau+\Pi}{2} \end{aligned}$$

Firm Problem:

$$\max_{e} \quad \Pi = e - we - T \iff e(1-w-\tau)$$

$$e = \begin{cases} 0 & w > 1-\tau & \text{no production} \\ [0,+\infty) & w = 1-\tau & \text{only possible equilibrium} \\ +\infty & w < 1-\tau & \text{labor market does not clear} \end{cases}$$

Therefore:

$$w = 1-\tau \implies \Pi = 0$$

Market Clearing:

$$\begin{cases} e = \dfrac{w+\tau-\Pi}{2(w+\tau)} \\ x = \dfrac{w+\tau+\Pi}{2} \end{cases} \implies \begin{cases} e = \dfrac{1}{2} \\ x = \dfrac{1}{2} \end{cases}$$

- x, e are independent of τ
- w depends negatively on τ

3(b)

Welfare

$$U = \ln\left(\frac{1}{2}\right) + \ln\left(\frac{1}{2}\right) = 2\ln\left(\frac{1}{2}\right) \quad \text{is indepdent of } \tau$$

Explanation:

Any tax τ will redistribute firm profit to the consumer and it will be redistributed perfectly.

Wage responds perfectly to the tax increase. If tax τ goes up, wage goes down by the same amount, hence welfare is unchanged.

Key feature:

Production function is not strictly convex (it is linear). Hence, firm must make zero profit in linear case because otherwise the firm could just scale up labor input to infinity, then tax and labor expenses perfectly cancel each other out because the firm is entirely owned by the single consumer.

4

$$\begin{cases} u_1(x_{11}, x_{21}) = x_{11}^{\alpha} x_{21}^{1-\alpha} & \alpha \in (0,1) \\ u_2(x_{12}, x_{22}) = x_{12}^{\alpha} x_{22}^{1-\beta} & \beta \in (0,1) \end{cases}$$

$$(\omega_1, \omega_2) = (1,1)$$

$$W : \mathbb{R}^2 \to \mathbb{R}$$

$$W(u_1, u_2) \quad \text{Bergson-Samuelson social welfare function}$$

4(a)

$$W = \ln u_1 + \ln u_2$$

Social planner maximizes

$$\begin{aligned} \max_{x_{11}, x_{21}, x_{12}, x_{22}} \quad & \alpha \ln x_{11} + (1-\alpha)\ln x_{21} + \beta \ln x_{12} + (1-\beta)\ln x_{22} \\ \text{s.t.} \quad & x_{11} + x_{12} \le \omega_1, \quad x_{21} + x_{22} \le \omega_1 \end{aligned}$$

Constraints must hold with equality

$$\max_{x_{11}, x_{21}} \quad \alpha \ln x_{11} + (1-\alpha)\ln x_{21} + \beta \ln(1 - x_{11}) + (1-\beta)\ln(1 - x_{21})$$

FOC:

$$\begin{aligned} \frac{\alpha}{x_{11}} - \frac{\beta}{1 - x_{11}} &= 0 \implies x_{11} = \frac{\alpha}{\alpha + \beta} \\ \frac{1-\alpha}{x_{21}} - \frac{1-\beta}{1 - x_{21}} &= 0 \implies x_{21} = \frac{1-\alpha}{2 - \alpha - \beta} \end{aligned}$$

$$\begin{cases} x_{11} = \dfrac{\alpha}{\alpha+\beta} \\ x_{21} = \dfrac{1-\alpha}{2-\alpha-\beta} \end{cases} \implies \begin{cases} x_{12} = 1 - x_{11} = \dfrac{\beta}{\alpha+\beta} \\ x_{22} = 1 - x_{21} = \dfrac{1-\beta}{2-\alpha-\beta} \end{cases}$$

4(b)

$$\begin{cases} p_1 = 1 \\ p_2 = p \end{cases} \quad \begin{cases} \omega_{11} = 1 \\ \omega_{21} = 0.5 \end{cases} \quad \begin{cases} \omega_{12} = 0 \\ \omega_{22} = 0.5 \end{cases}$$

$$\begin{cases} w_1 = p_1\omega_{11} + p_2\omega_{21} \\ w_2 = p_1\omega_{12} + p_2\omega_{22} \end{cases} \implies \begin{cases} w_1 = 1 + 0.5p \\ w_2 = 0.5p \end{cases}$$

Consumer 1:

$$\begin{aligned} \max_{x_{11},x_{21}} \quad & \alpha \ln(x_{11}) + (1-\alpha)\ln(x_{21}) \\ \text{s.t.} \quad & p_1 x_{11} + p_2 x_{21} \le w_1 \end{aligned}$$

$$\mathcal{L} = \alpha \ln(x_{11}) + (1-\alpha)\ln(x_{21}) - \lambda(p_1 x_{11} + p_2 x_{21} - w_1)$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{11}} = \dfrac{\alpha}{x_{11}} - \lambda p_1 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_{21}} = \dfrac{1-\alpha}{x_{21}} - \lambda p_2 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \lambda} = p_1 x_{11} + p_2 x_{21} - w_1 = 0 \end{cases} \implies \begin{cases} \dfrac{p_1}{p_2} = \dfrac{\alpha}{1-\alpha} \cdot \dfrac{x_{21}}{x_{11}} \\ p_1 x_{11} + p_2 x_{21} = w_1 \end{cases} \implies \begin{cases} x_{11} = \dfrac{\alpha}{p_1} w_1 \\ x_{21} = \dfrac{1-\alpha}{p_2} w_1 \end{cases}$$

$$\text{demand for consumer 1} = \begin{cases} x_{11} = \dfrac{\alpha}{p_1} w_1 = \alpha(1 + \dfrac{1}{2}p) \\ x_{21} = \dfrac{1-\alpha}{p_2} w_1 = \dfrac{1-\alpha}{p}(1 + \dfrac{1}{2}p) \end{cases}$$

Similarly for consumer 2:

$$\text{demand for consumer 2} = \begin{cases} x_{12} = \dfrac{\beta}{p_1} w_2 = \dfrac{\beta p}{2} \\ x_{22} = \dfrac{1-\beta}{p_2} w_2 = \dfrac{1-\beta}{2} \end{cases}$$

Market clearing

$$\begin{aligned} x_{11} + x_{12} &= \omega_{11} + \omega_{12} \\ \alpha(1 + \frac{1}{2}p) + \frac{\beta p}{2} &= 1 \quad \text{note } (\alpha = \beta) \\ \alpha + \alpha p &= 1 \\ p &= \frac{1-\alpha}{\alpha} \end{aligned}$$

$$\begin{cases} x_{11} = \alpha(1 + \dfrac{1-\alpha}{2\alpha}) = \alpha + \dfrac{1-\alpha}{2} = \dfrac{1+\alpha}{2} \\ x_{21} = \dfrac{1+\alpha}{2} \\ x_{12} = \dfrac{1-\alpha}{2} \\ x_{22} = \dfrac{1-\alpha}{2} \end{cases}$$

4(c)

$$\begin{cases} w_1 = 1 + 0.5p + T_1 \\ w_2 = 0.5p + T_2 \end{cases}$$

$$\begin{cases} x_{11} = \alpha w_1 & \text{Walrasian Equilibrium} \\ x_{11} = \dfrac{\alpha}{\alpha+\beta} & \text{Optimal Planner Allocation} \end{cases} \implies \alpha w_1 = \frac{\alpha}{\alpha+\beta} \implies w_1 = \frac{1}{\alpha+\beta}$$

Similarly

$$\begin{cases} x_{12} = \beta w_2 & \text{Walrasian Equilibrium} \\ x_{12} = \dfrac{\beta}{\alpha+\beta} & \text{Optimal Planner Allocation} \end{cases} \implies \beta w_2 = \frac{\beta}{\alpha+\beta} \implies w_2 = \frac{1}{\alpha+\beta}$$

$$\begin{cases} w_1 = 1 + \dfrac{1}{2}p + T_1 \\ w_2 = \dfrac{1}{2}p + T_2 \end{cases} \implies \begin{cases} \dfrac{1}{\alpha+\beta} = 1 + \dfrac{1}{2}p + T_1 \\ \dfrac{1}{\alpha+\beta} = \dfrac{1}{2}p + T_2 \end{cases} \implies T_1 - T_2 = -1$$

$$\begin{cases} T_1 + T_2 = 0 \\ T_1 - T_2 = -1 \end{cases} \implies \begin{cases} T_1 = -\dfrac{1}{2} \\ T_2 = \dfrac{1}{2} \end{cases}$$

$$(T_1, T_2) = (-\frac{1}{2}, \frac{1}{2})$$