Problem Set 3

1. (**Walrasian Demand Function**) Let $X = \mathbb{R}^L_+$, and $x(p, w)$ be a continuously differentiable Walrasian demand function that satisfies Walras' Law.

   (a) Show that this implies $p \cdot D_w x(p, w) = 1$.

   (b) Show that at least one good must be a normal good, meaning that there is a good l for which $\frac{\partial x_l(p,w)}{\partial w} \geq 0$. (*Hint: Use* (*a*) *and argue by contradiction.*)

2. (**UMP and strictly convex preferences**) Let $X = \mathbb{R}^L_+$. Show that if $\succeq$ are represented by a strictly quasiconcave and continuous utility function, a unique solution $x(p, w)$ exists for any $p > 0$ and $w > 0$ (i.e., $x(p, w)$ is a demand **function**).

3. (**Kuhn-Tucker**) Let $X = \mathbb{R}^L_+$ and

$$u(x) = \left(\sum_{l=1}^{L} x_l^\sigma \right)^{1/\sigma},$$

   where $\sigma \in (0, 1)$ is a parameter.

   (a) Show that the preferences represented by this utility function are strictly convex and homothetic.

   (b) Derive the Walrasian demand $x(p, w)$ using the Kuhn-Tucker conditions for an interior solution. Explain the role of the preference parameter σ intuitively by looking at the limiting cases where $\sigma \to 0$ or $\sigma \to 1$.

(c) Consider the case $L = 2$ and derive the price effects $D_p x(p, w)$. Are the goods gross substitutes or gross complements?

(d) Sketch the wealth expansion path for $L = 2$ in a (x_1, x_2)-diagram. How do the consumption patterns of very rich and very poor people differ under these preferences?

4. **(Multiple Optima)** Suppose that $X = \mathbb{R}_+$ and let $\succeq$ be represented by a utility function

$$u(x) = \begin{cases} x, & x \in [0, 1) \\ 1, & x \geq 1 \end{cases}$$

(a) Is this $\succeq$ locally non-satiated? Continuous? Convex?

(b) Derive the Walrasian demand **correspondence** $x(p, w)$ for $p, w > 0$ by solving the utility maximization problem, and depict its image in the $(x, w/p)$-plane. Is $x(p, w)$ zero-homogeneous? Does $x(p, w)$ satisfy Walras' Law?

5. **(Homothetic Preferences)** Let $X = \mathbb{R}_+^L$ and $u(\cdot)$ be an increasing and strictly convex utility function representing homothetic preferences (i.e., $u(\lambda x) = \lambda u(x)$ for $\lambda > 0$). Let $p > 0$ and $w > 0$.

(a) Depict the indifference curves and the UMP graphically for the case where $L = 2$. Show graphically that the Walrasian demand $x(p, w)$ increases proportionally in income (i.e., $x(p, \lambda w) = \lambda x(p, w)$).

(b) Prove that $x(p, \lambda w) = \lambda x(p, w)$ for $\lambda > 0$ in general.

(*Hint: Try to argue by contradiction, exploiting that Walras Law applies and that $u(\lambda x) = \lambda u(x)$. Note that this is not a trivial exercise if you see it for the first time, so don't despair.*)