Exam
Social Choice Theory
Fall 2013

Instructions:

- The duration of the exam is 90 minutes.
  Please answer **all three** problems.
  You can achieve up to 90 points.
- Please solve and answer all problems on the answer sheet that is distributed separately.
  Write your name and your student ID number on the answer sheet.
- This question sheet has 4 pages (including cover page). Please check for completeness.
- You are not allowed to use any auxiliary equipment (except for dictionaries).
  Switch off all electronic devices.
- Please have your student identity card ready.

Points:

- Problem 1 (Arrow's Theorem for SWFs): 30 points
- Problem 2 (May's Theorem): 30 points
- Problem 3 (Social Evaluation Functions): 30 points

Good Luck!

Problem 1: Arrow's Theorem for SWFs (30 points)

(a) Let $X = \{w, x, y, z\}$ be the set of alternatives. Consider the following profile of preferences:

	preferences
Voter 1	$z\ P_1\ y\ P_1\ w\ P_1\ x$
Voter 2	$y\ P_2\ z\ P_2\ w\ P_2\ x$
Voter 3	$z\ P_3\ w\ P_3\ x\ P_3\ y$
Voter 4	$w\ P_4\ x\ P_4\ z\ P_4\ y$
Voter 5	$x\ P_5\ w\ P_5\ z\ P_5\ y$

(i) Find a linear order with respect to which these preferences are single-peaked. (5 points)

(ii) Identify the median voter(s). (3 points)

(iii) Apply pairwise majority voting. Identify Condorcet winner and loser. (4 points)

(b) Which of Arrow's axioms for SWFs are satisfied and which are violated by pairwise majority voting? When you think that an axiom is satisfied, give a brief argument. When you think an axiom is violated, give a counterexample. (8 points)

(c) (i) Construct an example of a preference profile for which a Condorcet winner exists but the outcome of pairwise majority voting is not transitive. (5 points)

(ii) Suppose that the outcome of pairwise majority voting is transitive, for some preference profile. Does this imply that a Condorcet winner exists? Argue briefly. (5 points)

注意细节区别，两两投票可以有无差异的，也就是x和y平局，可能x和y排下来都是第一名。但是孔多塞胜者要求严格赢下每一场，平局不算

Problem 2: May's Theorem (30 points)

Assume that there are n voters, $N = \{1, \dots, n\}$, and only two alternatives, $X = \{x, y\}$. Then voter i's preference can be written as $\alpha_i \in \{-1, 0, +1\}$, where $\alpha_i = +1$ denotes a strict preference for x over y, $\alpha_i = -1$ denotes a strict preference for y over x, and $\alpha_i = 0$ denotes indifference. Preference profiles are given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{-1, 0, +1\}^n$.

(a) Consider the following voting method. When at least two thirds of the voters strictly prefer x over y, then the outcome is a strict preference for x over y. When at least two thirds of the voters strictly prefer y over x, then the outcome is a strict preference for y over x. Otherwise, if no alternative is strictly preferred by at least two thirds of the voters, we eliminate voter 1. We then repeat the procedure, i.e, we check for a two thirds majority among the remaining $n-1$ voters $\{2, 3, \dots, n\}$. This procedure is iterated by eliminating voters in numerical order, until a two thirds majority is reached among the remaining voters, or until only voter n is left, in which case the outcome is equal to α_n.

Formally, for any $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, define $\boldsymbol{\alpha}^{(k)}$ to be the sub-profile starting with voter k, i.e., $\boldsymbol{\alpha}^{(k)} = (\alpha_k, \alpha_{k+1}, \dots, \alpha_n)$, for any $k \in \{1, \dots, n\}$. For any such $\boldsymbol{\alpha}^{(k)}$, let $n^+(\boldsymbol{\alpha}^{(k)})$ be the number of voters with a strict preference for x over y, and let $n^-(\boldsymbol{\alpha}^{(k)})$ be the number of voters with a strict preference for y over x. Then, at any iteration step k, when $k-1$ voters have already been eliminated, we check the following cases:

- If $k = n$ then $f(\boldsymbol{\alpha}) = \alpha_n$, and we stop.
- If $k < n$ then
  - If $n^+(\boldsymbol{\alpha}^{(k)}) \geq 2(n-(k-1))/3$ then $f(\boldsymbol{\alpha}) = +1$, and we stop.
  - If $n^-(\boldsymbol{\alpha}^{(k)}) \geq 2(n-(k-1))/3$ then $f(\boldsymbol{\alpha}) = -1$, and we stop.
  - Otherwise, k is increased by one and the next iteration step begins.

Apply this method to $\boldsymbol{\alpha} = (0, +1, -1, +1, -1, +1, -1, -1, 0)$. (8 points).

(b) State May's Theorem. Give a precise definition of the underlying axioms. (10 points)

(c) Which of May's axioms are satisfied and which are violated by the method described in part (a) of this problem? When you think that an axiom is satisfied, give a brief argument. When you think an axiom is violated, give a counterexample. (12 points)

一般这种和顺序有关系的淘汰制的投票方法，很可能不是匿名的，因为先后顺序淘汰可能影响结果的

一定要解释或者找反例。有点暗示说今年这种题维基百科查得到，考场上可以查查看

Problem 3: Social Evaluation Functions (30 points)

(a) Give a formal and precise definition of the concept of a social evaluation function (SEF). (10 points)

(b) Let X be a set of alternatives. For any profile of utility functions $\mathbf{U} = (U_1, \ldots, U_n) \in \mathscr{U}^n$ and any alternative $x \in X$, define the utility range

$$RA(x, \mathbf{U}) = \max\{U_1(x), \ldots, U_n(x)\} - \min\{U_1(x), \ldots, U_n(x)\}.$$

Now consider the minimal range SEF e^{RA} defined by

$$x\, e^{RA}(\mathbf{U})\, y \quad \Leftrightarrow \quad RA(x, \mathbf{U}) \leq RA(y, \mathbf{U}),$$

for all $x, y \in X$ and all $\mathbf{U} \in \mathscr{U}^n$.

(i) Determine $e^{RA}(\mathbf{U})$ for the example given in the following table. (5 points)

$\mathbf{U}$	x	y	z
U_1	5	1	0
U_2	0	2	2
U_3	2	2	3

(ii) Construct an example where all agents strictly prefer x over y, but the SEF e^{RA} yields a strict preference for y over x. (3 points)

(c) Show that SEF e^{RA} is consistent with the utility functions being cardinally measurable and level-comparable (CM-LC), but not with the utility functions being cardinally measurable and unit-comparable (CM-UC), and also not with the utility functions being ordinally measurable and level-comparable (OM-LC). (12 points)