# Problem Set 12

## 1. Asset Markets

**Notation:**

- A feasible **ex-ante** allocation $(x_J, x_T)$
- Asset $k$ has **return vector** $r_k = (r_{0k}, r_{1k}) \in \mathbb{R}^2$
- **Asset prices** at $t=0$ are denoted by $q = (q_1, q_2) \in \mathbb{R}^K$
- A trading plan $z_i = (z_{1i}, \cdots, z_{Ki}) \in \mathbb{R}^K$ is called a **portfolio**. i.e., $z_{ki}$ corresponds to the quantity of asset $k$ demanded $(z_{ki} > 0)$ or supplied $(z_{ki} < 0)$ by consumer $i$ .

$$\text{two-person economy} = \begin{cases} \text{John} \\ \text{Tim} \end{cases}$$

$$\text{one single good: 'crops'}$$

$$\begin{cases} \text{state } s = 0: & \pi_0 \\ \text{state } s = 1: & 1-\pi_0 \end{cases}$$

$$\text{state-dependent endowments} = \begin{cases} \omega_J = (1,0) \\ \omega_T = (0,1) \end{cases}$$

$$\text{return structure:} \quad \begin{pmatrix} r_{01} \\ r_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} r_{02} \\ r_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha \end{pmatrix}$$

### 1(a)

$$\text{price of asset } k = \begin{cases} 1 & k=1 \\ q_2 & k=2 \end{cases}$$

Suppose John wants to increase consumption in $s=0$ ($dx_J = 1$). Hence John needs to buy $dz_{1J} = 1$ units of asset $k=1$

$$\begin{cases} q_1 z_{1J} + q_2 z_{2J} = 0 \\ z_{1J} = 1 \\ q_1 = 1 \end{cases} \implies z_{2J} = -\frac{1}{q_2}$$

John must sell $\frac{1}{q_2}$ units of asset $k=2$ . But selling $\frac{1}{q_2}$ units of asset $k=2$ means a sacrifice of $dx_{1J} = -\frac{1}{q_2}\alpha$ units of crops in state 1.

Opportunity cost is $\frac{\alpha}{q_2}$ .

### 1(b)

**John's problem:**

$$\max_{\substack{x_{0J}, x_{1J} \\ z_{1J}, z_{2J}}} \quad \pi_0 u(x_{0J}) + (1-\pi_0) u(x_{1J})$$

$$\text{s.t.} \quad \begin{cases} \text{Asset market:} & z_{1J} + q_2 z_{2J} = 0 \\ \text{State } s=0\text{:} & x_{0J} = 1 + z_{1J} \\ \text{State } s=1\text{:} & x_{1J} = \alpha z_{2J} \end{cases}$$

$$\mathcal{L} = \pi_0 u(x_{0J}) + (1-\pi_0) u(x_{1J}) - \lambda(z_{1J} + q_2 z_{2J}) - \mu(x_{0J} - 1 - z_{1J}) - \nu(x_{1J} - \alpha z_{2J})$$

FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{0J}} = \pi_0 u'(x_{0J}) - \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_{1J}} = (1-\pi_0) u'(x_{1J}) - \nu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z_{1J}} = -\lambda + \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z_{2J}} = -\lambda q_2 + \nu\alpha = 0 \end{cases} \implies \begin{cases} \pi_0 u'(x_{0J}) = \mu \\ (1-\pi_0) u'(x_{1J}) = \nu \\ \lambda = \mu \\ \lambda q_2 = \nu\alpha \end{cases} \implies \frac{\alpha}{q_2} = \frac{\pi_0}{1-\pi_0} \cdot \frac{u'(x_{0J})}{u'(x_{1J})}$$

**Tim's problem:**

$$\max_{\substack{x_{0T}, x_{1T} \\ z_{1T}, z_{2T}}} \quad \pi_0 u(x_{0T}) + (1-\pi_0) u(x_{1T})$$

$$\text{s.t.} \quad \begin{cases} \text{Asset market:} & z_{1T} + q_2 z_{2T} = 0 \\ \text{State } s=0\text{:} & x_{0T} = z_{1T} \\ \text{State } s=1\text{:} & x_{1T} = 1 + \alpha z_{2T} \end{cases}$$

$$\mathcal{L} = \pi_0 u(x_{0T}) + (1-\pi_0) u(x_{1T}) - \lambda(z_{1T} + q_2 z_{2T}) - \mu(x_{0T} - z_{1T}) - \nu(x_{1T} - 1 - \alpha z_{2T})$$

FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{0T}} = \pi_0 u'(x_{0T}) - \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_{1T}} = (1-\pi_0)u'(x_{1T}) - \nu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z_{1T}} = -\lambda + \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial z_{2T}} = -\lambda q_2 + \nu\alpha \end{cases} \implies \begin{cases} \pi_0 u'(x_{0T}) = \mu \\ (1-\pi_0)u'(x_{1T}) = \nu \\ \lambda = \mu \\ \lambda q_2 = \nu\alpha \end{cases} \implies \frac{\alpha}{q_2} = \frac{\pi_0}{1-\pi_0} \cdot \frac{u'(x_{0T})}{u'(x_{1T})}$$

**Market clearing:**

$$\begin{cases} \dfrac{\alpha}{q_2} = \dfrac{\pi_0}{1-\pi_0} \cdot \dfrac{u'(x_{0J})}{u'(x_{1J})} \\ \dfrac{\alpha}{q_2} = \dfrac{\pi_0}{1-\pi_0} \cdot \dfrac{u'(x_{0T})}{u'(x_{1T})} \end{cases} \implies \frac{u'(x_{0J})}{u'(x_{1J})} = \frac{u'(x_{0T})}{u'(x_{1T})} \implies \frac{u'(x_{0J})}{u'(x_{1J})} = \frac{u'(1-x_{0J})}{u'(1-x_{1J})} \implies \begin{cases} x_{0J} = x_{1J} \\ x_{0T} = x_{1T} \end{cases}$$

*Proof by contradiction*

Suppose that $x_{0J} > x_{1J}$:

$$x_{0J} > x_{1T} \implies \frac{u'(x_{0J})}{u'(x_{1J})} < 1 \quad \text{by concavity of } u(\cdot) \implies \frac{u'(1-x_{0J})}{u'(1-x_{1J})} < 1 \implies 1 - x_{0J} > 1 - x_{1J} \implies x_{1J} > x_{0J} \quad \text{(contradictio}$$

$$\begin{cases} \dfrac{\alpha}{q_2} = \dfrac{\pi_0}{1-\pi_0} \cdot \dfrac{u'(x_{0J})}{u'(x_{1J})} \\ x_{0J} = x_{1J} \end{cases} \implies q_2 = \alpha \cdot \frac{1-\pi_0}{\pi_0}$$

$$\begin{cases} x_{0J} = 1 + z_{1J} \\ x_{1J} = \alpha z_{2J} \\ x_{0J} = x_{1J} \end{cases} \implies z_{2J} = \frac{1+z_{1J}}{\alpha}$$

$$\text{plug} \quad \begin{cases} q_1 = 1 \\ q_2 = \alpha \cdot \frac{1-\pi_0}{\pi_0} \\ z_{2J} = \frac{1+z_{1J}}{\alpha} \end{cases} \quad \text{into} \quad q_1 z_{1J} + q_2 z_{2J} = 0 \quad \implies \quad z_{1J} = -(1-\pi_0)$$

$$\text{plug} \quad z_{1J} = -(1-\pi_0) \quad \text{into} \quad \begin{cases} z_{2J} = \frac{1+z_{1J}}{\alpha} \\ x_{0J} = 1 + z_{1J} \\ x_{0J} = x_{1J} \end{cases} \implies \begin{cases} z_{2J} = \frac{\pi_0}{\alpha} \\ x_{0J} = x_{1J} = \pi_0 \end{cases}$$

$$\text{by market clearing:} \quad x_{0T}^* = x_{1T}^* = 1 - \pi_0$$

$$\textbf{Radner equilibrium:} \quad (x_J^*, x_T^*, q_2^*) = (\pi_0, 1-\pi_0, \alpha \cdot \frac{1-\pi_0}{\pi_0})$$

## 1(c)

$$q_2^* = \alpha \cdot \frac{1-\pi_0}{\pi_0} \implies q_2 \text{ is proportional to } \alpha \text{ (i.e., } q_2 \text{ is linear in } \alpha)$$

The higher the return of the asset in state 1, the higher the price $q_2$

# 2. Incomplete Markets

$$\text{return structure:} \quad \begin{pmatrix} r_{01} \\ r_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} r_{02} \\ r_{12} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$u_i(x) = \ln(x) \quad \text{for } i \in \{J, T\}$$

**Note:**

- Any equilibrium must be interior i.e., $(x_{0J}, x_{1J}, x_{0T}, x_{1T} > 0)$ because otherwise the expected utility is not well-defined.
- If $q_1 = 0$ , people will buy asset 1 as much as possible. There is no market clearing.

**John's problem:**

$$\max_{\substack{x_{0J}, x_{1J} \\ z_{1J}, z_{2J}}} \quad \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J})$$

$$\text{s.t.} \quad \begin{cases} q_1 z_{1J} + q_2 z_{2J} = 0 & \text{asset market} \\ x_{0J} = 1 + z_{1J} + 2z_{2J} & \text{state } s = 0 \\ x_{1J} = z_{1J} + 2z_{2J} & \text{state } s = 1 \end{cases}$$

$$\mathcal{L} = \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J}) - \lambda(q_1 z_{1J} + q_2 z_{2J}) - \mu(x_{0J} - 1 - z_{1J} - 2z_{2J}) - \nu(x_{1J} - z_{1J} - 2z_{2J})$$

FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{0J}} = \dfrac{\pi_0}{x_{0J}} - \mu = 0 & \implies \dfrac{\pi_0}{x_{0J}} = \mu \\ \dfrac{\partial \mathcal{L}}{\partial x_{1J}} = \dfrac{1-\pi_0}{x_{1J}} - \nu = 0 & \implies \dfrac{1-\pi_0}{x_{1J}} = \nu \\ \dfrac{\partial \mathcal{L}}{\partial z_{1J}} = -\lambda q_1 + \mu + \nu = 0 & \implies \lambda q_1 = \mu + \nu \\ \dfrac{\partial \mathcal{L}}{\partial z_{2J}} = -\lambda q_2 + 2\mu + 2\nu = 0 & \implies \lambda q_2 = 2\mu + 2\nu \end{cases} \implies \frac{q_2}{q_1} = 2$$

$$\text{asset market condition: } q_1 z_{1J} + q_2 z_{2J} = 0 \implies z_{1J} + 2z_{2J} = 0 \implies z_{1J} = -2z_{2J}$$

$$\text{state budget constraint: } \begin{cases} x_{0J} = 1 + z_{1J} + 2z_{2J} \\ x_{1J} = z_{1J} + 2z_{2J} \end{cases} \implies \begin{cases} x_{0J} = 1 \\ x_{1J} = 0 \end{cases} \text{ violation of interiority equilibrium}$$

There can be no equilibrium since the market is incomplete. There are only two assets, which are linearly dependent in two states of the world $2r_1 = r_2$

## 3. Asset Markets with Aggregate Risk

$$\text{endowments} = \begin{cases} \omega_J = (1,0) \\ \omega_T = (0,2) \end{cases}$$

$$\text{Arrow security return structure:} \quad r_1 = \begin{pmatrix} r_{01} \\ r_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad r_2 = \begin{pmatrix} r_{02} \\ r_{12} \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

### 3(a)

$$\omega_J + \omega_T = (1,0) + (0,2) = (1,2)$$

There is aggregate uncertainty about total endowment of crops in the economy.

### 3(b)

**Pareto set:**

$$\max_{\substack{x_{0J},x_{1J} \\ x_{0T},x_{1T}}} \quad \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J})$$

$$\text{s.t.} \quad \begin{cases} \pi_0 \ln(x_{0T}) + (1-\pi_0)\ln(x_{1T}) \geq \bar{u}_T \\ x_{0J} + x_{0T} = 1 \\ x_{1J} + x_{1T} = 2 \end{cases}$$

$$\mathcal{L} = \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J}) - \lambda(\bar{u}_T - \pi_0 \ln(1-x_{0J}) - (1-\pi_0)\ln(2-x_{1J}))$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{0J}} = \dfrac{\pi_0}{x_{0J}} - \lambda \dfrac{\pi_0}{1-x_{0J}} = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_{1J}} = \dfrac{1-\pi_0}{x_{1J}} - \lambda \dfrac{1-\pi_0}{2-x_{1J}} = 0 \end{cases} \implies \begin{cases} \lambda = \dfrac{1-x_{0J}}{x_{0J}} \\ \lambda = \dfrac{2-x_{1J}}{x_{1J}} \end{cases} \implies x_{1J} = 2x_{0J}$$

$$\begin{cases} x_{0T} = 1 - x_{0J} \\ x_{1T} = 2 - x_{0T} \end{cases} \implies x_{1T} = 2x_{0T}$$

$$\textbf{Pareto set:} \quad \begin{cases} x_{1J} = 2x_{0J} \\ x_{1T} = 2x_{0T} \end{cases}$$

### 3(c)

**John's problem:**

$$\max_{\substack{x_{0J},x_{1J} \\ z_{1J},z_{2J}}} \quad \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J})$$

$$\text{s.t.} \quad \begin{cases} q_1 z_{1J} + q_2 z_{2J} = 0 & \text{asset market} \\ x_{0J} = 1 + z_{1J} & \text{state } s = 0 \\ x_{1J} = z_{2J} & \text{state } s = 1 \end{cases}$$

$$\mathcal{L} = \pi_0 \ln(x_{0J}) + (1-\pi_0)\ln(x_{1J}) - \lambda(q_1 z_{1J} + q_2 z_{2J}) - \mu(x_{0J} - 1 - z_{1J}) - \nu(x_{1J} - z_{2J})$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{0J}} = \dfrac{\pi_0}{x_{0J}} - \mu = 0 & \implies \dfrac{\pi_0}{x_{0J}} = \mu \\ \dfrac{\partial \mathcal{L}}{\partial x_{1J}} = \dfrac{1-\pi_0}{x_{1J}} - \nu = 0 & \implies \dfrac{1-\pi_0}{x_{1J}} = \nu \\ \dfrac{\partial \mathcal{L}}{\partial z_{1J}} = -\lambda q_1 + \mu = 0 & \implies \lambda q_1 = \mu \\ \dfrac{\partial \mathcal{L}}{\partial z_{2J}} = -\lambda q_2 + \nu = 0 & \implies \lambda q_2 = \nu \end{cases} \implies \frac{q_1}{q_2} = \frac{\pi_0}{1-\pi_0} \cdot \frac{x_{1J}}{x_{0J}}$$

**Tim's problem:**

$$\frac{q_1}{q_2} = \frac{\pi_0}{1-\pi_0} \cdot \frac{x_{1T}}{x_{0T}}$$

**Market clearing:**

$$\begin{cases} x_{0J} + x_{0T} = 1 \\ x_{1J} + x_{1T} = 2 \end{cases} \implies \frac{x_{1J}}{x_{0J}} = \frac{x_{1T}}{x_{0T}} = \frac{2 - x_{1J}}{1 - x_{0J}} \implies \frac{1 - x_{0J}}{x_{0J}} = \frac{2 - x_{1J}}{x_{1J}} \implies x_{1J} = 2x_{0J}$$

$$\text{Normalization:} \quad q_1 = 1 \implies q_2 = \frac{1-\pi_0}{2\pi_0}$$

$$\begin{cases} q_1 z_{1J} + q_2 z_{2J} = 0 \\ x_{0J} = 1 + z_{1J} \\ x_{1J} = z_{2J} \end{cases} \implies \begin{cases} z_{1J} + \dfrac{1-\pi_0}{2\pi_0} z_{2J} = 0 \\ \dfrac{z_{2J}}{1+z_{1J}} = \dfrac{x_{1J}}{x_{0J}} = 2 \end{cases} \implies \begin{cases} z_{1J} = -(1-\pi_0) \\ z_{2J} = 2\pi_0 \end{cases} \implies \begin{cases} x_{0J} = \pi_0 \\ x_{1J} = 2\pi_0 \end{cases} \implies \begin{cases} x_{0T} = 1 - \pi_0 \\ x_{1T} = 2(1-\pi_0) \end{cases}$$

$$\textbf{Radner equilibrium:} \quad (x_{0J}^*, x_{1J}^*, x_{0T}^*, x_T^*; q_2^*) = \left(\pi_0, 2\pi_0, 1-\pi_0, 2(1-\pi_0), \frac{1-\pi_0}{2\pi_0}\right)$$

Radner equilibrium is a PO

**Reason:** With complete asset markets, the FWT must hold. RE is a PO.

## 3(d)

$$r_1 = \begin{pmatrix} r_{01} \\ r_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad r_2 = \begin{pmatrix} r_{02} \\ r_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$r_1' = \begin{pmatrix} r_{01} \\ r_{11} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad r_2' = \begin{pmatrix} r_{02} \\ r_{12} \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\begin{cases} r_1' = r_1 + r_2 \\ r_2' = r_1 + 2r_2 \end{cases} \implies \begin{cases} q_1' = q_1 + q_2 \\ q_2' = q_1 + 2q_2 \end{cases}$$

$$q_1' = 1 \implies q_2' = \frac{1+2q_2}{1+q_2} = \frac{1 + \frac{1-\pi_0}{\pi_0}}{1 + \frac{1-\pi_0}{2\pi_0}} = \frac{2}{1+\pi_0}$$