Problem Set 9

Exercise 1: Intertemporal optimality in the TnT model

$$U(C_1, C_2) = u(C_1) + \beta u(C_2)$$

with CRRA utility function

$$u(C_t) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

$$\sigma \text{ is a risk coefficient, constant intertemporal elasticity substitution}$$

$$\begin{cases} \theta & \text{intratemporal elasticity of substitution} \\ \theta \to \infty & \text{perfect substitution} \\ \theta \to 0 & \text{perfect complement} \end{cases}$$

Consumption bundle is given by

$$C_t = \left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}$$

Budget constraints

$$\begin{aligned} C_1^T + \frac{P_1^N}{P_1^T}C_1^N &= Y_1^T + \frac{P_1^N}{P_1^T}Y_1^N + rB_0 - CA_1 \\ C_2^T + \frac{P_2^N}{P_2^T}C_2^N &= Y_2^T + \frac{P_2^N}{P_2^T}Y_2^N + (1+r)(B_0 + CA_1) \end{aligned}$$

1(a)

A price index P_t exists such that

$$\underbrace{P_t}_{\text{price index}} \cdot \underbrace{C_t}_{\text{consumption bundle}} = \underbrace{P_t^T C_t^T + P_t^N C_t^N}_{\text{sum of nominal expenditure}}$$

Find P_t that satisfies the equation and P_t is defined as the minimal expenditure for the consumption of 1 unit of C_t

$$\begin{aligned} \min_{C_t} \quad & P_t^T C_t^T + P_t^N C_t^N \\ \text{s.t.} \quad & C_t = 1 \quad \text{and} \quad C_t = \left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} \end{aligned}$$

$$\mathcal{L} = P_t^T C_t^T + P_t^N C_t^N - \lambda(C_t - 1)$$

where the Lagrangian multiplier λ is the shadow price of increasing C by 1 unit. If C increases by 1, budget increases by λ.

i.e., I will be charged λ to increase the consumption of C because that is the amount up to which I am willing to pay. As such, λ is the minimal expenditure for consumption of an additional unit of C. Hence, $\lambda = P_t$

$$
\begin{aligned}
\mathcal{L} &= P_t^T C_t^T + P_t^N C_t^N - P_t(C_t - 1) \\
&= P_t^T C_t^T + P_t^N C_t^N - P_t\left(\left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}} - 1\right)
\end{aligned}
$$

FOC w.r.t C_t^T :

$$
\frac{\partial \mathcal{L}}{\partial C_t^T} = P_t^T - P_t\left(\frac{\theta}{\theta-1}\left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}(1-\gamma)^{\frac{1}{\theta}}\frac{\theta-1}{\theta}(C_t^T)^{-\frac{1}{\theta}}\right) = 0
$$

$$
\begin{aligned}
\frac{P_t^T}{P_t} &= \left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}(1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{-\frac{1}{\theta}} \\
\left(\frac{P_t^T}{P_t}\right)^{\theta} &= C_t(1-\gamma)(C_t^T)^{-1} \\
C_t^T &= (1-\gamma)\left(\frac{P_t^T}{P_t}\right)^{-\theta} C_t
\end{aligned}
$$

FOC w.r.t. C_t^N :

$$
\frac{\partial \mathcal{L}}{\partial C_t^N} = P_t^N - P_t\left(\frac{\theta}{\theta-1}\left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{1}{\theta-1}}\gamma^{\frac{1}{\theta}}\frac{\theta-1}{\theta}(C_t^N)^{-\frac{1}{\theta}}\right) = 0
$$

$$
\left(\frac{P_t^N}{P_t}\right)^{\theta} = \left[\gamma^{\frac{1}{\theta}}(C_t^N)^{\frac{\theta-1}{\theta}} + (1-\gamma)^{\frac{1}{\theta}}(C_t^T)^{\frac{\theta-1}{\theta}}\right]^{\frac{\theta}{\theta-1}}\gamma(C_t^N)^{-1}
$$

$$
C_t^N = \gamma\left(\frac{P_t^N}{P_t}\right)^{-\theta} C_t
$$

$$
\begin{cases}
C_t^T = \left(\frac{P_t^T}{P_t}\right)^{-\theta}(1-\gamma)C_t \\
C_t^N = \left(\frac{P_t^N}{P_t}\right)^{-\theta}\gamma C_t
\end{cases}
$$

$$
\begin{aligned}
P_t C_t &= P_t^T C_t^T + P_t^N C_t^N \\
P_t &= P_t^T\left(\frac{P_t^T}{P_t}\right)^{-\theta}(1-\gamma) + P_t^N\left(\frac{P_t^N}{P_t}\right)^{-\theta}\gamma \\
P_t^{1-\theta} &= (P_t^T)^{1-\theta}(1-\gamma) + (P_t^N)^{1-\theta}\gamma \\
P_t &= \left((P_t^T)^{1-\theta}(1-\gamma) + (P_t^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}}
\end{aligned}
$$

We found price index P_t such that $P_t C_t = P_t^T C_t^T + P_t^N C_t^N$

1(b)

$$
\begin{cases}
\sigma & \text{elasticity of intertemporal substitution} \\
\theta & \text{elasticity of intratemporal substitution}
\end{cases}
$$

- Elasticity of Intertemporal Substitution

  By how much I am going to increase consumption today if consumption tomorrow becomes 1% more expensive relative to today (see PS2, ex2).

- Elasticity of Intratemporal Substitution

  How much domestic goods I am going yo consume if the price of the tradable good increase by 1% relative to the domestic good (see PS8, ex1).

1(c)

Given $P_1^T = P_2^T = 1, B_0 = 0$, solve household's optimization problem

$$\begin{aligned}\max_{C_1,C_2} \quad & U(C_1,C_2) = \frac{C_1^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta\frac{C_2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \\ \text{s.t.} \quad & C_1^T + P_1^N C_1^N = Y_1^T + P_1^N Y_1^N - CA_1 \\ & C_2^T + P_2^N C_2^N = Y_2^T + P_2^N Y_2^N + (1+r)CA_1\end{aligned}$$

Lifetime budget constraint

$$\underbrace{P_1C_1 + \frac{P_2C_2}{1+r}}_{\text{present value of consumption}} = \underbrace{Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r}}_{\text{present value of lifetime endowment}}$$

$$C_2 = \frac{1+r}{P_2}\left(Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} - P_1C_1\right)$$

$$\begin{aligned}\max_{C_1} \quad & \frac{C_1^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \beta\frac{C_2^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \\ \text{s.t.} \quad & C_2 = \frac{1+r}{P_2}\left(Y_1^T + P_1^N Y_1^N + \frac{Y_2^T + P_2^N Y_2^N}{1+r} - P_1C_1\right)\end{aligned}$$

$$\text{FOC}: \quad C_1^{-\frac{1}{\sigma}} + \beta C_2^{-\frac{1}{\sigma}}\left(-(1+r)\frac{P_1}{P_2}\right) = 0 \implies C_1 = \left(\beta(1+r)\frac{P_1}{P_2}\right)^{-\sigma} C_2$$

$$C_2 = \left(\beta(1+r)\frac{P_1}{P_2}\right)^{\sigma} C_1 \quad \text{intertemporal optimality condition}$$

Note that non-tradable goods must be consumed

$$\begin{cases} C_1^N = Y_1^N \\ C_2^N = Y_2^N \end{cases}$$

Constraints become

$$\begin{cases} C_1^T = Y_1^T - CA_1 \\ C_2^T = Y_2^T + (1+r)CA_1 \end{cases} \implies C_1^T + \frac{C_2^T}{1+r} = Y_1^T + \frac{Y_2^T}{1+r}$$

$$\underbrace{C_1^T + \frac{C_2^T}{1+r}}_{\text{discounted lifetime } C^T} = \underbrace{Y_1^T + \frac{Y_2^T}{1+r}}_{\text{discounted endowment}}$$

In 1(a), we derived

$$C_t^T = \left(\frac{P_t^T}{P_t}\right)^{-\theta}(1-\gamma)C_t \implies C_t^T = P_t^{\theta}(1-\gamma)C_t \implies \begin{cases} C_1^T = P_1^{\theta}(1-\gamma)C_1 \\ C_2^T = P_2^{\theta}(1-\gamma)C_2 \end{cases}$$

$$P_1^{\theta}(1-\gamma)C_1 + \frac{P_2^{\theta}(1-\gamma)C_2}{1+r} = Y_1^T + \frac{Y_2^T}{1+r}$$

$$C_2 = \frac{1+r}{P_2^{\theta}(1-\gamma)}\left(Y_1^T + \frac{Y_2^T}{1+r} - P_1^{\theta}(1-\gamma)C_1\right) \quad \text{intratemporal optimality condition}$$

$$\begin{cases} C_2 = \left(\beta(1+r)\dfrac{P_1}{P_2}\right)^{\sigma} C_1 \\ C_2 = \dfrac{1+r}{P_2^{\theta}(1-\gamma)}\left(Y_1^T + \dfrac{Y_2^T}{1+r} - P_1^{\theta}(1-\gamma)C_1\right) \end{cases}$$

$$\left(\beta(1+r)\frac{P_1}{P_2}\right)^{\sigma} P_1^{\theta}(1-\gamma)C_1 = \left(\frac{P_1}{P_2}\right)^{\theta}(1+r)\left(Y_1^T + \frac{Y_2^T}{1+r}\right) - \left(\frac{P_1}{P_2}\right)^{\theta}(1+r)P_1^{\theta}(1-\gamma)C_1$$

$$\left(\beta(1+r)\frac{P_1}{P_2}\right)^{\sigma} C_1^T = \left(\frac{P_1}{P_2}\right)^{\theta}(1+r)\left(Y_1^T + \frac{Y_2^T}{1+r}\right) - \left(\frac{P_1}{P_2}\right)^{\theta}(1+r)C_1^T$$

$$\beta^{\sigma}(1+r)^{\sigma-1}\frac{P_1}{P_2}^{\sigma-\theta} C_1^T = Y_1^T + \frac{Y_2^T}{1+r} - C_1^T$$

$$C_1^T = \frac{Y_1^T + \dfrac{Y_2^T}{1+r}}{1 + \beta^{\sigma}(1+r)^{\sigma-1}\dfrac{P_1}{P_2}^{\sigma-\theta}}$$

$$\begin{aligned} CA_1 &= Y_1^T - C_1^T \\ &= Y_1^T - \frac{Y_1^T + \dfrac{Y_2^T}{1+r}}{1 + \beta^{\sigma}(1+r)^{\sigma-1}\dfrac{P_1}{P_2}^{\sigma-\theta}} \end{aligned}$$

1(d)

$$\begin{cases} \sigma > \theta \\ P^T \text{ is constant over time} \\ P_2 > P_1 \end{cases}$$

- **Intertemporal effect**

$$C_2 = \left(\beta(1+r)\frac{P_1}{P_2}\right)^{\sigma} C_1 \quad \text{intertemporal optimality condition}$$

  Price of consumption tomorrow increases, hence I reduce consumption tomorrow and consume more today (of both tradable and non-tradable goods). According to $CA = Y^T - C^T$, current account gets smaller.

- **Intratemporal effect**

$$P_t = \left((P_t^T)^{1-\theta}(1-\gamma) + (P_t^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}}$$

$$\begin{cases} P_1 = \left((P^T)^{1-\theta}(1-\gamma) + (P_1^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}} \\ P_2 = \left((P^T)^{1-\theta}(1-\gamma) + (P_2^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}} \end{cases}$$

$$P_2 > P_1$$

$$\left((P^T)^{1-\theta}(1-\gamma) + (P_2^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}} > \left((P^T)^{1-\theta}(1-\gamma) + (P_1^N)^{1-\theta}\gamma\right)^{\frac{1}{1-\theta}}$$

$$P_2^N > P_1^N$$

$$\frac{P^T}{P_2^N} < \frac{P^T}{P_1^N}$$

  Tradable relative to non-tradable is more expensive today than that tomorrow. Hence, I consume less of C_1^T and more of C_1^N. CA increases.

Intertemporal effect dominates because $\sigma > \theta$