Exam
Social Choice Theory
Spring 2019

Instructions:

- The duration of the exam is 90 minutes.
  Please answer all three problems.
  You can achieve up to 90 points.
- Please solve and answer all problems on the answer sheet that is distributed separately.
  Write your name and your student ID number on the answer sheet.
  Please have your student identity card ready.
- You must document the solution process and provide sufficient arguments for your solution.
- This question sheet has 4 pages (including cover page). Please check for completeness.
- You are not allowed to use any auxiliary equipment.
  Switch off all electronic devices.

Points:

- Problem 1 (May's Theorem): 30 points
- Problem 2 (Manipulability): 30 points
- Problem 3 (Social Evaluation Functions): 30 points

Good Luck!

Problem 1: May's Theorem (30 points)

Assume that there are n voters, $N = \{1, \dots, n\}$, and only two alternatives, $X = \{x, y\}$. Then voter i's preference can be written as $\alpha_i \in \{-1, 0, +1\}$, where $\alpha_i = +1$ denotes a strict preference for x over y, $\alpha_i = -1$ denotes a strict preference for y over x, and $\alpha_i = 0$ denotes indifference. Preference profiles are given by $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \{-1, 0, +1\}^n$.

(a) Consider the following voting method. For any preference profile $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$, we first calculate the average

$$\overline{\alpha}(\boldsymbol{\alpha}) = \frac{1}{n} \sum_{i=1}^{n} \alpha_i.$$

We then define

$$f(\boldsymbol{\alpha}) = \begin{cases} +1 & \text{if } \overline{\alpha}(\boldsymbol{\alpha}) > +\tilde{\alpha} \\ 0 & \text{if } -\tilde{\alpha} \le \overline{\alpha}(\boldsymbol{\alpha}) \le +\tilde{\alpha} \\ -1 & \text{if } \overline{\alpha}(\boldsymbol{\alpha}) < -\tilde{\alpha} \end{cases}$$

where $\tilde{\alpha} \in [0, 1]$ is a fixed parameter.

Apply this method to the profile $\boldsymbol{\alpha} = (+1, 0, -1, +1, +1)$. In particular, show how the result depends on the value of the parameter $\tilde{\alpha}$. (8 points)

(b) State May's Theorem. Give a precise definition of the underlying axioms. (10 points)

(c) Which of May's axioms are satisfied and which are violated by the method described in part (a) of this problem? In particular, show how the answer depends on the value of the parameter $\tilde{\alpha}$. (12 points)

Problem 2: Manipulability (30 points)

Let $X = \{x, y, z\}$ be the set of alternatives and $N = \{1, 2\}$ the set of voters. Assume that both voters have strict preferences, i.e., treat $\mathscr{A} = \mathscr{P}^2$ as the universal domain.

The following table specifies an SCF on this domain. For instance, the field with row yzx and column xyz contains y, which means that alternative y is selected if voter 1 has preference yP_1zP_1x and voter 2 has preference xP_2yP_2z. The other fields are interpreted analogously.

When voter 1 has preference zP_1yP_1x and voter 2 has preference zP_2yP_2x, the table does not yet specify which alternative is selected. This is indicated by the symbol $*$ in the table. In the following, you are asked to consider all three possible cases, where $*$ is replaced by either x, y or z.

$R_1 \backslash R_2$	xyz	xzy	yzx	yxz	zxy	zyx
xyz	x	x	x	x	x	x
xzy	x	x	x	x	x	x
yzx	y	y	y	y	y	y
yxz	y	y	y	y	y	y
zxy	z	z	z	z	z	z
zyx	z	z	z	z	z	$*$

(a) Which of Arrow's axioms for SCFs are satisfied and which are violated by this method, and how does this depend on whether $*$ is replaced by x, y or z? (15 points)

(b) Which of Gibbard-Satterthwaite's axioms are satisfied and which are violated? Is the SCF surjective? Again, how does this depend on whether $*$ is replaced by x, y or z? (10 points)

(c) For the case of two voters (and an arbitrary number of alternatives), consider the following voting method. First, both voters submit their preferences. Then one of the two voters is selected at random, each with probability $1/2$, and a top-ranked alternative of the selected voter is chosen as the winner. Does this method contradict the Gibbard-Satterthwaite theorem? (5 points)

Problem 3: Social Evaluation Functions (30 points)

The following social evaluation function (SEF) reflects a concept commonly used to define poverty rates, but here applied to utilities rather than incomes. The SCF requires counting how many people have a utility below 50% of the median utility, and then to select the alternative for which this number is lowest.

Formally, let X be a set of alternatives and $N = \{1, 2, \ldots, n\}$ a set of citizens, where n is odd. For any profile of utility functions $\mathbf{U} = (U_1, \ldots, U_n) \in \mathscr{U}^n$ and any alternative $x \in X$, let $i_k(x, \mathbf{U}) \in N$ be the k'th best-off person in x under $\mathbf{U}$, i.e.,

$$U_{i_1(x,\mathbf{U})}(x) > U_{i_2(x,\mathbf{U})}(x) > \ldots > U_{i_n(x,\mathbf{U})}(x),$$

where you are allowed to abstract from ties. Define $m = (n+1)/2$. Then the median utility in x under $\mathbf{U}$ is $U_{i_m(x,\mathbf{U})}(x)$. The number of citizens whose utility is (weakly) smaller than 50% of the median utility is

$$\mathrm{NP}(x, \mathbf{U}) = |\{i \in N \mid \; U_i(x) \leq \frac{1}{2} U_{i_m(x,\mathbf{U})}(x)\}|.$$

The social evaluation function e^{NP} is then defined by

$$x \, e^{\mathrm{NP}}(\mathbf{U}) \, y \quad \Leftrightarrow \quad \mathrm{NP}(x, \mathbf{U}) \leq \mathrm{NP}(y, \mathbf{U}),$$

for all $x, y \in X$ and all $\mathbf{U} \in \mathscr{U}^n$.

(a) Determine $e^{\mathrm{NP}}(\mathbf{U})$ for the example given in the following table. (10 points)

$\mathbf{U}$	v	w	x	y	z
U_1	1	1	5	5	2
U_2	2	2	2	2	4
U_3	3	4	1	0	4
U_4	5	5	6	10	3
U_5	6	6	4	4	2

(b) With which of the seven information structures introduced in class is e^{NP} consistent? (20 points)