Problem Set 7

1. Bayesian Updating

Notation

$$
\begin{cases}
V > 0 & \text{non-fundable fixed cost} \\
X > V & \text{prize} \\
S & \text{the idea is a success} \\
F & \text{the idea is a failure} \\
\tau & \text{signal} \\
s & \text{signal cost}
\end{cases}
$$

$$
\text{acquire signal}
\begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement idea} \begin{cases}
\text{yes} & U(+,1) \\
\text{no} & \text{acquire signal} \begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement} \begin{cases} \text{yes} & U(+,+,1) \\ \text{no} & U(+,+,0) \end{cases} \\
- & \text{implement} \begin{cases} \text{yes} & U(+,-,1) \\ \text{no} & U(+,-,0) \end{cases}
\end{cases} \\
\text{no} & U(+,0)
\end{cases}
\end{cases} \\
- & \text{implement idea} \begin{cases}
\text{yes} & U(-,1) \\
\text{no} & \text{acquire signal} \begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement} \begin{cases} \text{yes} & U(-,+,1) \\ \text{no} & U(-,+,0) \end{cases} \\
- & \text{implement} \begin{cases} \text{yes} & U(-,-,1) \\ \text{no} & U(-,-,0) \end{cases}
\end{cases} \\
\text{no} & U(-,0)
\end{cases}
\end{cases}
\end{cases} \\
\text{no} & \text{implement idea} \begin{cases} \text{yes} & U(1) \\ \text{no} & U(0) \end{cases}
\end{cases}
$$

$$
\begin{cases}
P(+|S) = \theta \\
P(-|F) = \theta \\
X = 2F \\
\theta = \frac{4}{5} \\
P(S) = \frac{1}{2}
\end{cases}
$$

$$
\begin{cases} P(S) = \frac{1}{2} \\ P(F) = \frac{1}{2} \end{cases}
\begin{cases} P(+) = \frac{1}{2} \\ P(-) = \frac{1}{2} \end{cases}
\begin{cases} P(+|S) = \frac{4}{5} \\ P(-|S) = \frac{1}{5} \end{cases}
\begin{cases} P(+|F) = \frac{1}{5} \\ P(-|F) = \frac{4}{5} \end{cases}
$$

1(a)

Calculating posteriors

$$
\begin{aligned}
P(S|+) &= \frac{P(S,+)}{P(+)} \\
&= \frac{P(+|S) \cdot P(S)}{P(+|S) \cdot P(S) + P(+|F) \cdot P(F)} \\
&= \frac{\frac{1}{2}\theta}{\frac{1}{2}\theta + \frac{1}{2}(1-\theta)} \\
&= \theta = \frac{4}{5}
\end{aligned}
$$

$$
\begin{aligned}
P(S|-) &= \frac{P(S,-)}{P(-)} \\
&= \frac{P(-|S)\cdot P(S)}{P(-|S)\cdot P(S) + P(-|F)\cdot P(F)} \\
&= \frac{\frac{1}{2}(1-\theta)}{\frac{1}{2}(1-\theta)+\frac{1}{2}\theta} \\
&= 1-\theta = \frac{1}{5}
\end{aligned}
$$

[Bayes' rule with 3 events]

$$
\begin{aligned}
P(A|B,C) &= \frac{P(A,B,C)}{P(B,C)} \\
&= \frac{P(B|A,C)\cdot P(A,C)}{P(B,C)} \\
&= \frac{P(B|A,C)\cdot P(A|C)\cdot P(C)}{P(B,C)} \\
&= \frac{P(B|A,C)\cdot P(A|C)\cdot P(C)}{P(B|C)\cdot P(C)} \\
&= \frac{P(B|A,C)\cdot P(A|C)}{P(B|C)}
\end{aligned}
$$

$$
\begin{aligned}
P(S|+,+) &= \frac{P(+|S,+)\cdot P(S|+)}{P(+|S,+)\cdot P(S|+) + P(+|F,+)\cdot P(F|+)} \\
&= \frac{\theta^2}{\theta^2+(1-\theta)^2} \\
&= \frac{\frac{16}{25}}{\frac{16}{25}+\frac{1}{25}} \\
&= \frac{16}{17}
\end{aligned}
$$

$$
P(F|+,+) = 1 - P(S|+,+) = \frac{1}{17}
$$

[Conditional Independence]

$$
\begin{aligned}
P(S|+,-) &= \frac{P(-|S,+)\cdot P(+|S)}{P(-|+)} \\
&= \frac{P(-|S,+)\cdot P(+|S)}{P(-|S,+)\cdot P(+|S) + P(-|F,+)\cdot P(+|F)} \\
&= \frac{(1-\theta)\theta}{(1-\theta)\theta+\theta(1-\theta)} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
P(F|+,-) = 1 - P(S|+,-) = \frac{1}{2}
$$

$$
\begin{aligned}
P(S|-,+) &= \frac{P(+|S,-)\cdot P(-|S)}{P(+|-)} \\
&= \frac{P(+|S,-)\cdot P(-|S)}{P(+|S,-)\cdot P(-|S) + P(+|F,-)\cdot P(-|F)} \\
&= \frac{\theta(1-\theta)}{\theta(1-\theta)+(1-\theta)\theta} \\
&= \frac{1}{2}
\end{aligned}
$$

$$
P(F|-,+) = 1 - P(S|-,+) = \frac{1}{2}
$$

$$
\begin{aligned}
P(S|-,-) &= \frac{P(-|S,-)\cdot P(-|S)}{P(-|-)} \\
&= \frac{P(-|S,-)\cdot P(-|S)}{P(-|S,-)\cdot P(-|S) + P(-|F,-)\cdot P(-|F)} \\
&= \frac{(1-\theta)^2}{(1-\theta)^2+\theta^2} \\
&= \frac{1}{17}
\end{aligned}
$$

Decision tree

$$
\text{acquire signal}
\begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement idea} \begin{cases}
\text{yes} & U(+,1) \\
\text{no} & \text{acquire signal} \begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement} \begin{cases} \text{yes} & U(+,+,1) \\ \text{no} & U(+,+,0) \end{cases} \\
- & \text{implement} \begin{cases} \text{yes} & U(+,-,1) \\ \text{no} & U(+,-,0) \end{cases}
\end{cases} \\
\text{no} & U(+,0)
\end{cases}
\end{cases} \\
- & \text{implement idea} \begin{cases}
\text{yes} & U(-,1) \\
\text{no} & \text{acquire signal} \begin{cases}
\text{yes} & \begin{cases}
+ & \text{implement} \begin{cases} \text{yes} & U(-,+,1) \\ \text{no} & U(-,+,0) \end{cases} \\
- & \text{implement} \begin{cases} \text{yes} & U(-,-,1) \\ \text{no} & U(-,-,0) \end{cases}
\end{cases} \\
\text{no} & U(-,0)
\end{cases}
\end{cases}
\end{cases} \\
\text{no} & \text{implement idea} \begin{cases} \text{yes} & U(1) \\ \text{no} & U(0) \end{cases}
\end{cases}
$$

$$
\begin{aligned}
U(+,1) &= P(S|+)\times(V-s) + P(F|+)\times(-V-s) \\
&= \frac{4}{5}\times(V-s) + \frac{1}{5}\times(-V-s) \\
&= \frac{3}{5}V - s
\end{aligned}
$$

$$
\begin{aligned}
U(+,+,1) &= P(S|+,+)\times(V-2s) + P(F|+,+)\times(-V-2s) \\
&= \frac{16}{17}\times(V-2s) + \frac{1}{17}\times(-V-2s) \\
&= \frac{15}{17}V - 2s
\end{aligned}
$$

$$U(+,+,0) = -2s$$

$$
\begin{aligned}
U(+,-,1) &= P(S|+,-)\times(V-2s) + P(F|+,-)\times(-V-2s) \\
&= \frac{1}{2}\times(V-2s) + \frac{1}{2}\times(-V-2s) \\
&= -2s
\end{aligned}
$$

$$U(+,-,0) = -2s$$

$$U(+,0) = -s$$

$$
\begin{aligned}
U(-,1) &= P(S|-)\times(V-s) + P(F|-)\times(-V-s) \\
&= \frac{1}{5}\times(V-s) + \frac{4}{5}\times(-V-s) \\
&= -\frac{3}{5}V - s
\end{aligned}
$$

$$
\begin{aligned}
U(-,+,1) &= P(S|-,+)\times(V-2s) + P(F|-,+)\times(-V-2s) \\
&= \frac{1}{2}\times(V-2s) + \frac{1}{2}\times(-V-2s) \\
&= -2s
\end{aligned}
$$

$$U(-,+,0) = -2s$$

$$\begin{aligned} U(-,-,1) &= P(S|-,-) \times (V-2s) + P(F|-,-) \times (-V-2s) \\ &= \frac{1}{17} \times (V-2s) + \frac{16}{17} \times (-V-2s) \\ &= -\frac{15}{17}V - 2s \end{aligned}$$

$$U(-,-,0) = -2s$$

$$U(-,0) = -s$$

$$U(1) = P(S) \times V + P(F) \times (-V) = 0$$
$$U(0) = 0$$

$$\text{acquire signal} \begin{cases} \text{yes} & \begin{cases} + & \text{implement idea} \begin{cases} \text{yes} & \frac{3}{5}V - s \\ \text{no} & \text{acquire signal} \begin{cases} \text{yes} & \begin{cases} + & \text{implement} \begin{cases} \text{yes} & \frac{15}{17}V - 2s \\ \text{no} & -2s \end{cases} \\ - & \text{implement} \begin{cases} \text{yes} & -2s \\ \text{no} & -2s \end{cases} \end{cases} \\ \text{no} & -s \end{cases} \end{cases} \\ - & \text{implement idea} \begin{cases} \text{yes} & -\frac{3}{5}V - s \\ \text{no} & \text{acquire signal} \begin{cases} \text{yes} & \begin{cases} + & \text{implement} \begin{cases} \text{yes} & -2s \\ \text{no} & -2s \end{cases} \\ - & \text{implement} \begin{cases} \text{yes} & -\frac{15}{17}V - 2s \\ \text{no} & -2s \end{cases} \end{cases} \\ \text{no} & -s \end{cases} \end{cases} \end{cases} \\ \text{no} & \text{implement idea} \begin{cases} \text{yes} & 0 \\ \text{no} & 0 \end{cases} \end{cases}$$

1(b)

$$P(S|+) = \frac{1}{1 + \left(\frac{1-\theta}{\theta}\right)^c \left(\frac{1-p}{p}\right)}$$

$c = 1$

$$P(S|+) = \frac{1}{1 + \frac{1-p}{p}} = p \implies \text{no belief updating}$$

The entrepreneur is not willing to pay for a signal because the signal will be ignored anyways.

Optimal decision

- never acquire signal
- indifferent between implementing and not implementing

2. Optimal Information Acquisition

$$\begin{cases} \omega_0 & \text{project } j = 0 \text{ is successful} \\ \omega_1 & \text{project } j = 1 \text{ is successful} \end{cases}$$

$$\begin{cases} P(\omega_0) = \frac{1}{2} \\ P(\omega_1) = \frac{1}{2} \end{cases}$$

$$\text{utility} = \begin{cases} 1 & \text{project } j = 0 \text{ is successful} \\ 1 & \text{project } j = 1 \text{ is successful} \\ -c & \text{project } j = 0 \text{ fails} \\ 0 & \text{project } j = 1 \text{ fails} \end{cases}$$

2(a)

$$
\begin{aligned}
EU(S) &= P(S_0) \cdot [P(\omega_0|S_0) \cdot 1 + P(\omega_1|S_0) \cdot (-c)] + P(S_1) \cdot [P(\omega_1|S_1) \cdot 1 + P(\omega_0|S_1) \cdot 0] \\
&= P(S_0|\omega_0) \cdot P(\omega_0) \cdot 1 + P(S_0|\omega_1) \cdot P(\omega_1)(-c) + P(S_1|\omega_1) \cdot P(\omega_1) \cdot 1 + P(S_1|\omega_0) \cdot P(\omega_0) \cdot 0 \\
&= \frac{1}{2}[P(S_0|\omega_0) - c \cdot P(S_0|\omega_1) + P(S_1|\omega_1)] \\
&= \frac{1}{2}[P(S_0|\omega_0) - c \cdot (1 - P(S_1|\omega_1)) + P(S_1|\omega_1)] \\
&= \frac{1}{2}[P(S_0|\omega_0) + (1+c)P(S_1|\omega_1) - c]
\end{aligned}
$$

2(b)

$$
\begin{aligned}
EU(\text{invest in } j=0) &= P(\omega_0) \cdot 1 + P(\omega_1) \cdot (-c) \\
&= \frac{1}{2} \times 1 - \frac{1}{2} \cdot c \\
&= \frac{1}{2}(1-c) \\
EU(\text{invest in } j=1) &= P(\omega_0) \cdot 0 + P(\omega_1) \cdot 1 \\
&= \frac{1}{2}
\end{aligned}
$$

DM will always choose to invest in $j = 1$ project and gain a utility of $\frac{1}{2}$ when deciding according to the prior.

In order to make sure separating signals yield a higher utility than deciding according to the prior, we need to set

$$
\begin{aligned}
\frac{1}{2}[P(S_0|\omega_0) + (1+c)P(S_1|\omega_1) - c] &> \frac{1}{2} \\
P(S_1\omega_1) &\geq \frac{c - P(S_0|\omega_0)}{1+c}
\end{aligned}
$$

2(c)

$$
\begin{aligned}
\frac{\partial EU(S)}{\partial P(S_0|\omega_0)} &= \frac{1}{2} \\
\frac{\partial EU(S)}{\partial P(S_1|\omega_1)} &= \frac{1}{2}(1+c) > \frac{1}{2}
\end{aligned}
$$

Acquire information (direct attention) in $P(S_1|\omega_1)$

Intuition: you want to avoid the mismatch