Problem Set 4

1

Quasilinear preference

$$
\begin{gathered}
X=(-\infty, \infty) \times \mathbb{R}_{+}^{L-1} \\
(x_1, x_2, \cdots, x_L) \in(-\infty, \infty) \times \mathbb{R}_{+} \times \mathbb{R}_{+} \cdots \times \mathbb{R}_{+} \\
x_1 \in \mathbb{R} \quad x_1, x_2, \cdots, x_L \geq 0
\end{gathered}
$$

Proposition 1.6

$$
u(x)=x_1+g(x_2, \cdots, x_L)
$$

$$
u(x) \text{ is strictly increasing in } x_1 \quad \Longrightarrow \quad p \cdot x=w
$$

1(a)

$$
\begin{aligned}
\max \quad & x_1+g(x_2, \cdots, x_L) \\
\text{s.t.} \quad & \sum_{l=1}^L p_l x_l \leq w \quad x_l \geq 0 \; \forall \, l=2, \cdots, L
\end{aligned}
$$

$$
\begin{aligned}
\sum_{l=1}^L p_l x_l &= w \\
p_1 x_1+\sum_{l=2}^L p_l x_l &= w \\
x_1+\sum_{l=2}^L p_l x_l &= w \\
x_1 &= w-\sum_{l=2}^L p_l x_l
\end{aligned}
$$

Recall Karush-Kuhn-Tucker conditions:

$$
\begin{aligned}
\min \quad & f(x) \\
\text{s.t.} \quad & h(x) \leq 0
\end{aligned}
$$

$$
\begin{gathered}
\mathcal{L}=f(x)+\mu \cdot h(x) \\
\begin{cases}\mu \geq 0 \\ \mu \cdot h(x)=0\end{cases}
\end{gathered}
$$

$$
\begin{aligned}
\min \quad & (-x_1-g(x_2, \cdots, x_L)) \\
\text{s.t.} \quad & \begin{cases}x_1=w-\sum\limits_{l=2}^L p_l x_l \\ -x_l \leq 0 \quad \forall \, l=2, \cdots, L\end{cases}
\end{aligned}
$$

$$
\begin{aligned}
\min \quad & \sum_{l=2}^L p_l x_l-w-g(x_2, \cdots, x_L) \\
\text{s.t.} \quad & -x_l \leq 0 \quad \forall \, l=2, \cdots, L
\end{aligned}
$$

Use the Lagrangian function:

$$\mathcal{L}(x,\mu) = \sum_{l=2}^{L} p_l x_l - w - g(x_2,\cdots,x_L) + \sum_{l=2}^{L} \mu_l(-x_l)$$

FOC:

$$\frac{\partial \mathcal{L}}{\partial x_i} = p_i - \frac{\partial g(x_2,\cdots,x_L)}{\partial x_i} - \mu_i = 0 \quad \text{with} \quad \begin{cases} \mu_i \geq 0 \quad \forall\, i = 2,\cdots,L \\ \mu_i x_i = 0 \quad \forall\, i = 2,\cdots,L \end{cases}$$

Corner solution:

$$\mu_i \geq 0, x_i = 0$$

$$p_i = \frac{\partial g(x_2,\cdots,x_L)}{\partial x_i} + \mu_i > \frac{\partial g(x_2,\cdots,x_L)}{\partial x_i}$$

Interior solution:

$$\mu_i = 0$$

$$p_i = \frac{\partial g(x_2,\cdots,x_L)}{\partial x_i}$$

1(b)

$$p_i = \frac{\partial g(x_2,\cdots,x_L)}{\partial x_i} \implies x_i = x_i(p_i, x_2,\cdots,x_L)$$

The demands for $l = 2,\cdots,L$ depend on $p_2,\cdots,p_L$ that do not depend on w

Therefore, there is no wealth effect.

2

$$\textbf{WARP:} \quad p\cdot x(p',w') \leq w \ \wedge \ x(p',w') \neq x(p,w) \Longrightarrow p'\cdot x(p,w) > w'$$

2(a)

- $p \cdot x(p',w') \leq w$ means that the choice under (p',w') is affordable under (p,w)
- $x(p',w') \neq x(p,w)$ means that choices are different under (p,w) and (p',w')
- $p' \cdot x(p,w) > w'$ means that the choice under (p,w) cannot be affordable under (p',w')

$x(p,w)$ is preferred to $x(p',w')$. Whenever we choose $x(p',w')$, $x(p,w)$ must not be available.

2(b)

If $x(p,w)$ is from a UMP, WARP must hold.

Prove by contraction: $x(p,w)$ is from a UMP but WARP does not hold.

Main idea

$$\begin{cases} p\cdot x(p',w') \leq w \\ x(p,w) \neq x(p',w') \end{cases} \implies p'\cdot x(p,w) \leq w' \quad \text{WARP violation}$$

$$p \cdot x(p,w) \leq w$$
$$\Downarrow$$
$$x(p,w) \text{ and } x(p',w') \text{ are affordable under } p,w$$
$$\Downarrow$$
$$u(x(p,w)) > u(x(p',w'))$$

$$p' \cdot x(p',w') \leq w'$$
$$\Downarrow$$
$$x(p,w) \text{ and } x(p',w') \text{ are affordable under } p,w$$
$$\Downarrow$$
$$u(x(p',w')) > u(x(p,w))$$

$$(\neg \text{WARP} \implies \neg \text{UMP}) \iff (\text{UMP} \implies \text{WARP})$$

3

3(a)

$$u_1(x_1,x_2) = \min\{x_1, 2x_2\}$$
$$x_2 = \frac{1}{2}x_1$$
$$x_1, x_2 \text{ are perfect complement}$$

$$u_2(x_1,x_2) = \min\{\sqrt{x_1}, x_2\}$$
$$x_2 = \sqrt{x_1}$$
$$x_1, x_2 \text{ are perfect complement}$$

3(b)

$$\begin{aligned} u_1(\lambda x_1, \lambda x_2) &= \min\{\lambda x_1, \lambda 2x_2\} \\ &= \lambda \min\{x_1, 2x_2\} \\ &= \lambda u_1(x_1,x_2) \quad \forall\, x_1, x_2 \quad \forall\, \lambda > 0 \end{aligned}$$

u_1 is homogeneous of degree 1

Preferences are homothetic

Homotheticity

$$\forall\, x, y, x \sim y \implies \lambda x \sim \lambda y\, \forall \lambda > 0$$
$$u(x) = u(y) \implies u(\lambda x) = u(\lambda y)$$

$$x = (1,1)$$
$$u_2(1,1) = \min\{\sqrt{1}, 1\} = 1$$
$$y = (2,1)$$
$$u_2(2,1) = \min\{\sqrt{2}, 1\} = 1$$
$$\therefore x \sim y$$

$$\lambda = 2$$
$$\lambda x = (2,2)$$
$$u_2(2,2) = \min\{\sqrt{2}, 2\} = \sqrt{2}$$
$$\lambda y = (4,2)$$
$$u_2(4,2) = \min\{\sqrt{4}, 2\} = 2$$
$$\therefore \lambda x \nsim \lambda y$$

u_2 does not represent homothetic preferences

3(c)

Utility Maximization Problem

$$\begin{aligned} \max_{x_1, x_2} \quad & \min\{x_1, 2x_2\} \\ \text{s.t.} \quad & p_1 x_1 + p_2 x_2 \le w \end{aligned}$$

- u represents monotone preferences $\implies$ Walra's Law holds $\implies$ $p_1 x_1 + p_2 x_2 = w$
- Consumer consumes in a fixed ratio: $x_2 = \frac{1}{2} x_1$

$$\begin{cases} x_1 = 2x_2 \\ p_1 x_1 + p_2 x_2 = w \end{cases} \implies \begin{cases} x_1 = \frac{2w}{2p_1 + p_2} \\ x_2 = \frac{w}{2p_1 + p_2} \end{cases}$$