Problem Set 5

1

1(a)

$$
\begin{aligned}
P(L_1) &= 1 \\
P(L_2) &= \frac{2}{3}P(L_1) = \frac{2}{3} \\
P(c_1) &= P(c_1|L_1)P(L_1) + P(c_1|L_2)P(L_2) \\
&= \frac{1}{3} \times 1 + \frac{2}{3} \times \frac{1}{4} \\
&= \frac{1}{2} \\
P(c_2) &= P(c_2|L_2)P(L_2) = \frac{1}{3} \times \frac{2}{3} = \frac{2}{9} \\
P(c_3) &= P(c_3|L_2)P(L_2) = \frac{5}{12} \times \frac{2}{3} = \frac{5}{18}
\end{aligned}
$$

2(b)

Method 1

$$
\begin{aligned}
P(c_1) &= \frac{1}{3} \times \frac{1}{4} + \left(\frac{2}{3} \times \frac{1}{4}\right)\frac{1}{3} \times \frac{1}{4} + \left(\frac{2}{3} \times \frac{1}{4}\right)^2 \frac{1}{3} \times \frac{1}{4} + \cdots \\
&= \frac{1}{3} \times \frac{1}{4} \sum_{i=0}^{\infty} \left(\frac{2}{3} \times \frac{1}{4}\right)^i \\
&= \frac{1}{12} \sum_{i=0}^{\infty} \left(\frac{1}{6}\right)^i \\
&= \frac{1}{12} \times \frac{6}{5} \\
&= \frac{1}{10}
\end{aligned}
$$

$$
P(c_1) + P(c_2) = 1 \implies P(c_2) = 1 - P(c_1) = \frac{9}{10}
$$

$$
\begin{aligned}
P(L_2) &= \frac{1}{3} + \left(\frac{2}{3} \times \frac{1}{4}\right)\frac{1}{3} + \left(\frac{2}{3} \times \frac{1}{4}\right)^2 \frac{1}{3} + \cdots \\
&= \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{2}{3} \times \frac{1}{4}\right)^i \\
&= \frac{1}{3} \sum_{i=0}^{\infty} \left(\frac{1}{6}\right)^i \\
&= \frac{1}{3} \times \frac{6}{5} \\
&= \frac{2}{5}
\end{aligned}
$$

$$
\begin{aligned}
P(c_2) &= P(c_2|L_2)P(L_2) + \frac{2}{3}\times\frac{3}{4} + \left(\frac{2}{3}\times\frac{1}{4}\right)\frac{2}{3}\times\frac{3}{4} + \left(\frac{2}{3}\times\frac{1}{4}\right)^2\frac{2}{3}\times\frac{3}{4} + \cdots \\
&= \frac{3}{4}\times\frac{2}{5} + \frac{2}{3}\times\frac{3}{4}\sum_{i=0}^{\infty}\left(\frac{2}{3}\times\frac{1}{4}\right)^i \\
&= \frac{3}{10} + \frac{1}{2}\sum_{i=0}^{\infty}\left(\frac{1}{6}\right)^i \\
&= \frac{3}{10} + \frac{1}{2}\times\frac{6}{5} \\
&= \frac{9}{10}
\end{aligned}
$$

Method 2

$$
\begin{cases} n(L_1) = 1 + \frac{1}{4}n(L_3) \\ n(L_3) = \frac{2}{3}n(L_1) \end{cases} \implies \begin{cases} n(L_1) = \frac{6}{5} \\ n(L_3) = \frac{4}{5} \end{cases}
$$

$$
\begin{aligned}
n(L_2) &= \frac{1}{3}n(L_1) = \frac{2}{5} \\
n(c_1) &= \frac{1}{4}n(L_2) = \frac{1}{10} \\
n(c_2) &= \frac{3}{4}n(L_2) + \frac{3}{4}n(L_3) = \frac{9}{10}
\end{aligned}
$$

2

- $n = 4$
- vNM utility function: $U(L) = \sum p_l u_l$

2(a)

$$
\begin{aligned}
U(L_1) &= \frac{1}{2}u_1 + \frac{1}{2}u_4 \\
U(L_2) &= \frac{1}{2}u_2 + \frac{1}{2}u_3
\end{aligned}
$$

We know that

$$
\begin{gathered}
u_1 - u_2 > u_3 - u_4 \\
u_1 + u_4 > u_2 + u_3 \\
U(L_1) > U(L_2) \\
L_1 \succ L_2
\end{gathered}
$$

2(b)

It depends. **Strictly monotonic transformation** $f: U \to f(U)$

BUT $f(U)$ might not be a vNM utility function

Bernoulli utility u is not ordinal

Example:

$$
\begin{cases} u_1 = 1 \\ u_2 = 2 \\ u_3 = 3 \end{cases}
$$

$$
\begin{cases} L_1 = (\frac{1}{2}, 0, \frac{1}{2}) \\ L_2 = (0, 1, 0) \end{cases}
$$

$$\begin{cases} U(L_1) = \frac{1}{2}(u_1 + u_3) = 2 \\ U(L_2) = u_2 = 2 \end{cases} \implies L_1 \sim L_2$$

$$f(x) = x^2 \implies \begin{cases} f(U(L_1)) = U^2(L_1) = 4 \\ f(U(L_2)) = U^2(L_2) = 4 \end{cases}$$

$f(U(L)) = (\sum p_i u_i)^2$ is not a vNM utility function (not linear in probability).

$$\begin{cases} f(u_1) = u_1^2 = 1 \\ f(u_2) = u_2^2 = 4 \\ f(u_3) = u_3^2 = 9 \end{cases}$$

$$\begin{cases} \overline{U}(L_1) = \frac{1}{2}(f(u_1) + f(u_2)) = 5 \\ \overline{U}(L_2) = f(u_2) = 4 \end{cases} \implies L_1 \succ L_2$$

3

U is a vNM utility function

$$\tilde{U}(L) = \beta U(L) + \gamma$$

- $\tilde{U}$ is a vNM utility function

$$\begin{aligned} \tilde{U}(L) &= \beta U(L) + \gamma \\ &= \beta \sum_{l=1}^{L} p_l u_l + \gamma \quad \sum_{l=1}^{L} p_l = 1 \\ &= \sum_{l=1}^{L} p_l \underbrace{(\beta u_i - \gamma)}_{\tilde{u}_l} \\ &= \sum_{l=1}^{L} p_l \tilde{u}_l \end{aligned}$$

  $\tilde{u}$ is a vNM utility function

- $\tilde{U}$ represents the same preferences $\succeq$

$$\begin{gathered} L \succeq L' \\ U(L) \geq U(L') \\ \beta U(L) \geq \beta U(L') \\ \beta U(L) + \gamma \geq \beta U(L') + \gamma \\ \tilde{U}(L) \geq \tilde{U}(L') \end{gathered}$$

  U and $\tilde{U}$ represent the same preferences

4

$$\begin{aligned} L \sim L' &\iff L \succeq L' \wedge L' \succeq L \\ L \succeq L' &\iff \alpha L + (1-\alpha)L'' \succeq \alpha L' + (1-\alpha)L'' \\ L' \succeq L &\iff \alpha L' + (1-\alpha)L'' \succeq \alpha L + (1-\alpha)L'' \end{aligned}$$

$$\alpha L + (1-\alpha)L'' \sim L' + (1-\alpha)L''$$

5

5(a)

$$F_L(x) = P(L \le x) = \begin{cases} 0, & x < \frac{2}{3} \\ 1, & x \ge \frac{2}{3} \end{cases}$$

$$U(F_L) = \int_0^\infty u(x) dF_L(x) = 1 \times u(\frac{2}{3}) = \sqrt{\frac{2}{3}}$$

- Whenever F_L is a step function

$$\int u(x) dF_L(x) = \sum_{i=1}^{N} u(x_i) p(x_i)$$

- Whenever F_L is continuously differentiable: $F_L' = f_L$

$$\int u(x) dF_L(x) = \int u(x) f_L(x) dx$$

5(b)

$$F_L(x) = \int_0^x f_L(t) dt = x$$

$$F_L(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \le x \le 1 \\ 1, & x > 1 \end{cases}$$

$$U(L) = \int_0^1 u(x) dF_L(x) = \int_0^1 u(x) f_L(x) dx = \int_0^1 \sqrt{x} dx = \frac{2}{3} x^{\frac{3}{2}} |_0^1 = \frac{2}{3}$$

5(c)

$$c(F_L, \sqrt{\cdot}) = CE$$
$$u(CE) = U(L)$$
$$\sqrt{CE} = \int_0^1 u(x) dF_L(x) = \frac{2}{3}$$
$$CE = \left(\frac{2}{3}\right)^2 = \frac{4}{9}$$