Mock Exam

Question 1

1(a)

$$\text{budegt set:} \quad p_1 q_1 + p_2 q_2 \leq w \iff q_1 + q_2 \leq w$$

1(b)

Utility maximization:

$$\max_{q_1, q_2} \quad U = \int u(q_1 + q_2 z_2) f(z_2) dz_2$$
$$\text{s.t.} \quad p_1 q_1 + p_2 q_2 \leq w$$

DM always wants to use all his wealth to acquire assets $\implies p_1 q_1 + p_2 q_2 = w \implies q_1 = \dfrac{1}{p_1}(w - p_2 q_2)$

$$\mathcal{L} = \int u(q_1 + q_2 z_2) f(z_2) dz_2 - \lambda(p_1 q_1 + p_2 q_2 - w)$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial q_1} = \displaystyle\int u'(q_1 + q_2 z_2) f(z_2) dz_2 - \lambda p_1 = 0 \\ \dfrac{\partial \mathcal{L}}{\partial q_2} = \displaystyle\int u'(q_1 + q_2 z_2) z_2 f(z_2) dz_2 - \lambda p_2 = 0 \end{cases}$$

Assume $q_2 = 0$:

$$\begin{cases} u'(q_1) \displaystyle\int f(z_2) dz_2 = \lambda p_1 \\ u'(q_1) \displaystyle\int z_2 f(z_2) dz_2 = \lambda p_2 \end{cases} \implies \frac{p_1}{p_2} = \frac{\int f(z_2) dz_2}{\int z_2 f(z_2) dz_2} = 1$$

Zero demand for asset 2 always leads to $p_1 = p_2$. Whenever $p_1 > p_2$, the DM has a strictly positive demand for asset 2 since q_2 cannot be negative.

Question 2

2(a)

- Consumer $i = 1$ has a strictly **increasing** and strictly **concave** Bernoulli utility function, which means the marginal utility for consumer 1 is decreasing. Therefore, consumer $i = 1$ is risk-averse.
- Consumer $i = 2$ has a linear Bernoulli utility function $u_2(z) = z$, which means the marginal utility for consumer 2 is constant. Therefore, consumer $i = 2$ is risk-neural.

2(b)

Consumer $i = 1$:

$$\max_{x_{01}, x_{11} z_{11}, z_{21}} \quad \pi u_1(x_{01}) + (1 - \pi) u_1(x_{11})$$

$$\text{s.t.} \quad \begin{cases} \text{asset market:} & z_{11} + z_{21} q = 0 \\ \text{state } s = 0: & x_{01} = 1 + z_{11} + z_{21} \\ \text{state } s = 1: & x_{11} = z_{21} \end{cases}$$

$$\mathcal{L} = \pi u_1(x_{01}) + (1-\pi)u_1(x_{11}) - \lambda(z_{11} + z_{21}q) - \mu(x_{01} - 1 - z_{11} - z_{21}) - \nu(x_{11} - z_{21})$$

FOCs:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{01}} = \pi u_1'(x_{01}) - \mu = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial x_{11}} = (1-\pi)u_1'(x_{11}) - \nu = 0 \end{cases} \implies \dfrac{\pi}{1-\pi} \cdot \dfrac{u_1'(x_{01})}{u_1'(x_{11})} = \dfrac{\mu}{\nu}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial z_{11}} = -\lambda + \mu = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial z_{21}} = -\lambda q + \mu + \nu = 0 \end{cases} \implies \dfrac{1}{q-1} = \dfrac{\mu}{\nu}$$

$$\dfrac{\pi}{1-\pi} \cdot \dfrac{u_1'(x_{01})}{u_1'(x_{11})} = \dfrac{1}{q-1}$$

Consumer $i = 2$:

$$\max_{x_{02}, x_{12}z_{12}, z_{22}} \quad \pi x_{02} + (1-\pi)x_{12}$$

$$\text{s.t.} \quad \begin{cases} \text{asset market:} & z_{12} + z_{22}q = 0 \\ \text{state } s = 0: & x_{02} = z_{12} + z_{22} \\ \text{state } s = 1: & x_{12} = 1 + z_{22} \end{cases}$$

$$\mathcal{L} = \pi x_{02} + (1-\pi)x_{12} - \lambda(z_{12} + z_{22}q) - \mu(x_{02} - z_{12} - z_{22}) - \nu(x_{12} - 1 - z_{22})$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial x_{02}} = \pi - \mu = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial x_{12}} = 1 - \pi - \nu = 0 \end{cases} \implies \dfrac{\pi}{1-\pi} = \dfrac{\mu}{\nu}$$

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial z_{12}} = -\lambda + \mu = 0 \\[2mm] \dfrac{\partial \mathcal{L}}{\partial z_{22}} = -\lambda q + \mu + \nu = 0 \end{cases} \implies \dfrac{1}{q-1} = \dfrac{\mu}{\nu}$$

$$\dfrac{\pi}{1-\pi} = \dfrac{1}{q-1}$$

Market clearing:

$$\begin{cases} \dfrac{\pi}{1-\pi} \cdot \dfrac{u_1'(x_{01})}{u_1'(x_{11})} = \dfrac{1}{q-1} \\[3mm] \dfrac{\pi}{1-\pi} = \dfrac{1}{q-1} \end{cases} \implies \begin{cases} x_{01} = x_{11} \\[2mm] q = \dfrac{1}{\pi} \end{cases}$$

$$\begin{cases} x_{01} + x_{02} = \omega_{01} + \omega_{02} = 1 \\ x_{11} + x_{12} = \omega_{11} + \omega_{12} = 1 \end{cases} \implies \begin{cases} x_{02} = 1 - x_{01} \\ x_{12} = 1 - x_{11} \end{cases}$$

For consumer $i = 1$, plug $x_{01} = x_{11}$ back into budget constraint:

$$\begin{cases} z_{11} + z_{21}q = 0 \\ x_{01} = 1 + z_{11} + z_{21} \\ x_{11} = z_{21} \\ x_{01} = x_{11} \\ q = \dfrac{1}{\pi} \end{cases} \implies \begin{cases} x_{01}^* = \pi \\ x_{11}^* = \pi \end{cases} \implies \begin{cases} x_{02}^* = 1 - \pi \\ x_{12}^* = 1 - \pi \end{cases}$$

Radner equilibrium: $\quad (x_{01}^*, x_{11}^*, x_{02}^*, x_{12}^*; q) = \left(\pi, \pi, 1-\pi, 1-\pi, \dfrac{1}{\pi}\right)$

2(c)

In equilibrium, we have $x_{01} = x_{11}$ and $x_{02} = x_{12}$. Hence, the asset market, jointly with the state-wise spot market, yields a **(Pareto) efficient allocation of risk**, by making the ex-ante risk tradable.

$$\frac{u_1'(x_{01})}{u_1'(x_{11})}$$

In any PO, the uncertainty completely vanishes (perfect consumption smoothing). In particular, both consumers perfectly insure each other against the uncertainty (efficient allocation of risk).

Question 3

3(a)

$$\because \quad x_2 < 2$$
$$\therefore \quad \frac{\partial U}{\partial x_1} = (2 - x_2)^{-2} > 0$$

$$\because \quad x_1 > 1 \text{ and } x_2 < 2$$
$$\therefore \quad \frac{\partial U}{\partial x_2} = 2(x_1 - 1)(2 - x_2)^{-3} > 0$$

Therefore, $U(x_1, x_2)$ is monotonically increasing in x_1 and x_2.

We know that:

$$\text{strong monotonicity} \implies \text{local non-satiation}$$

Therefore, preferences represented by $U(x_1, x_2)$ verify local non-satiation on X.

3(b)

Utility maximization:

$$\max_{x_1, x_2} \quad (x_1 - 1)(2 - x_2)^{-2}$$
$$\text{s.t.} \quad \begin{cases} p_1 x_1 + p_2 x_2 \leq 1 \\ x_1 > 1 \\ 0 < x_2 < 2 \end{cases}$$

Log-utility transformation:

$$\max_{x_1, x_2} \quad \log(x_1 - 1) - 2\log(2 - x_2)$$
$$\text{s.t.} \quad \begin{cases} p_1 x_1 + p_2 x_2 \leq 1 \\ x_1 > 1 \\ 0 < x_2 < 2 \end{cases}$$

Lagrange:

$$\mathcal{L} = \log(x_1 - 1) - 2\log(2 - x_2) - \lambda(p_1 x_1 + p_2 x_2 - 1)$$

FOC:

$$\begin{cases} \frac{\partial \mathcal{L}}{\partial x_1} = \frac{1}{x_1 - 1} - \lambda p_1 = 0 \\ \frac{\partial \mathcal{L}}{\partial x_2} = \frac{2}{2 - x_2} - \lambda p_2 = 0 \end{cases} \implies \frac{2 - x_2}{2(x_1 - 1)} = \frac{p_1}{p_2}$$

$$(2 - x_2)^{-2} - \lambda p_1 = 0 \implies \lambda > 0 \implies \text{Walra's Law holds} \implies p_1 x_1 + p_2 x_2 = 1$$

$$\begin{cases} \text{FOC:} & 2p_1 x_1 + p_2 x_2 = 2(p_1 + p_2) \\ \text{BC:} & p_1 x_1 + p_2 x_2 = 1 \end{cases} \implies \begin{cases} x_1 = \dfrac{2(p_1 + p_2) - 1}{p_1} \\ x_2 = \dfrac{2(1 - p_1 - p_2)}{p_2} \end{cases}$$

3(c)

$$\begin{cases} x_1 > 1 \\ 0 < x_2 < 2 \end{cases} \implies \begin{cases} \dfrac{2(p_1 + p_2) - 1}{p_1} > 1 \\ 0 < \dfrac{2(1 - p_1 - p_2)}{p_2} < 2 \end{cases} \implies \begin{cases} p_1 + 2p_2 > 1 \\ p_1 + p_2 > 1 \end{cases}$$

3(d)

$$\frac{\partial x_1(p_1, p_2)}{\partial p_1} = \frac{1 - 2p_2}{p_1^2} \quad \text{the sign is ambiguous under the set of prices } (p_1, p_2)$$

$$\frac{\partial x_2(p_1, p_2)}{\partial p_2} = \frac{2(p_1 - 1)}{p_2^2} \quad \text{the sign is ambiguous under the set of prices } (p_1, p_2)$$

The Law of Demand is not satisfied for all prices.

Question 4

4(a)

$$p_1 x_1 + p_2 x_2 \le w$$

If both prices are not strictly positive in Walrasian equilibrium, the consumer can consume infinite quantity of corresponding goods.

$p_1 \le 0$ or $p_2 \le 0$ cannot be supported as a price in a Walrasian equilibrium because the product market would not clear.

4(b)

The consumer's utility function is independent of labor. As long as the wage is non-negative, the consumer would supply all her labor to to produce more consumption goods, which leads to a higher utility.

If this consumer would not supply all her labor to the market, the total wealth would reduce. This is not Pareto optimal. Consumer can simply supply all her labor and become better off without losing anything. By the FWT, any WE must belong to the Pareto set. Therefore, consumer must supply all her labor to the market in any WE.

4(c)

Consumer's problem (utility maximization):

$$\max_{x_1, x_2} \quad x_1 x_2$$

$$\text{s.t.} \quad \begin{cases} x_1 \le q_1 \\ x_2 \le q_2 \\ L_1 + L_2 \le 1 \end{cases}$$

Log-utility transformation:

$$\max_{L_1, L_2} \quad \ln\left(2\sqrt{L_1}\right) + \ln\left(\sqrt{L_2}\right)$$

$$\text{s.t.} \quad L_1 + L_2 = 1$$

$$\mathcal{L} = \ln 2 + \frac{1}{2}\ln\left(L_1\right) + \frac{1}{2}\ln\left(L_2\right) - \lambda(L_1 + L_2 - 1)$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial L_1} = \dfrac{1}{2L_1} - \lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial L_2} = \dfrac{1}{2L_2} - \lambda = 0 \end{cases} \implies L_1 = L_2$$

$$\begin{cases} L_1 = L_2 \\ L_1 + L_2 = 1 \end{cases} \implies \begin{cases} L_1 = \dfrac{1}{2} \\ L_2 = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} x_1 = q_1 = 2\sqrt{L_1} = \sqrt{2} \\ x_2 = q_2 = \sqrt{L_2} = \dfrac{\sqrt{2}}{2} \end{cases}$$

Firm's problem (profit maximization):

$$\max_{q_1, q_2} \quad p_1 q_1 + p_2 q_2 - L_1 - L_2$$

$$\max_{L_1, L_2} \quad 2p_1\sqrt{L_1} + p_2\sqrt{L_2} - L_1 - L_2$$

FOC:

$$\begin{cases} (L_1): \quad \dfrac{p_1}{\sqrt{L_1}} - 1 = 0 \\ (L_2): \quad \dfrac{p_2}{2\sqrt{L_2}} - 1 = 0 \end{cases} \implies \begin{cases} p_1 = \sqrt{L_1} = \dfrac{\sqrt{2}}{2} \\ p_2 = 2\sqrt{L_2} = \sqrt{2} \end{cases}$$

$$\textbf{Walrasian equilibrium allocation:} \quad (x_1^*, x_2^*, L_1^*, L_2^*) = \left(\sqrt{2}, \frac{\sqrt{2}}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

4(d)

$\begin{cases} q_1 = 2\sqrt{L_1} \\ q_2 = \sqrt{L_2} \end{cases}$ means that using the same amount of labor produces more consumption goods 1 than 2.

Therefore, it is more rewarding in terms of utility to produce more consumptions 1, $q_1 > q_2$. In order to ensure good 1 to be consumed, p_1 must decrease.