Problem Set 9

1. (**Profit Maximization**) This is a basic exercise about profit maximization (see Chapter 5, MWWG).

   Consider a firm with a *single-output production technology*, where $q \in \mathbb{R}_+$ denotes the quantity produced, and $f(x)$ is a production function, where $x \in \mathbb{R}_+^{L-1}$ are the inputs. The production possibility set of the firm is described by

   $$Y = \{(q, x) \in \mathbb{R}_+ \times \mathbb{R}_+^{L-1} : q \leq f(x)\},$$

   where $f : \mathbb{R}_+^{L-1} \to \mathbb{R}_+$ is a strictly concave C^2-function with $f(0) = 0$. Let $p > 0$ denote the price vector associated with (q, x).

   Remark: By convention, one denotes inputs by negative entries and outputs by positive entries. However, if it is clear which goods are inputs, as the present example, one commonly abolishes this convention, and simply also uses positive amounts for inputs.

   (a) Let $L = 2$ and $f(x) = \sqrt{x}$ (i.e., we have a single-input single-output case). Depict Y in a (q, x)-diagram. Intuitively describe the central properties of this production possibility set.

   (b) In general, the profit-maximization problem (PMP) of a firm is (see MWG Ch. 5)

   $$\max_{y \in Y} p \cdot x$$

   Let $L \geq 2$, set up the profit-maximization problem (PMP), and argue that necessarily $q = f(x)$ at any optimum ("no-waste" property of efficient production). (*Recall that we treat inputs also as positive amounts for simplicity here*)

   (c) Derive the first-order conditions that characterize the optimal production plan.

(d) Find the solution of the PMP for the case where $L = 2$ and $f(x) = \sqrt{x}$. Depict the optimum in the (q, x)-plane using *iso-profit lines* (these are described by equations of the form $\Pi = pq - wx$, where $w > 0$ is the cost of the input).

2. (**Exchange Economy**) Consider a two-consumer exchange economy with two consumption goods, and utility functions

$$u^1(x_{11}, x_{21}) = x_{11}^{\alpha}, x_{21}^{1-\alpha} \quad, \alpha \in (0,1)$$

$$u^2(x_{12}, x_{22}) = x_{12}^{\beta}, x_{22}^{1-\beta} \quad, \beta \in (0,1)$$

(a) Derive the Walrasian demand $x_1(p, w_1), x_2(p, w_1)$ for consumer $i = 1$ treating her income $w_1 > 0$ as exogenous. How does $x_1(p, w_1)$ depend on p_2?

(b) Derive the Walrasian demand for consumer 1 for endogenous income $w_1 = p_1\omega_{11} + p_2\omega_{21}.$, where $p_1, p_2 > 0$ and $\omega_{11}, \omega_{21} > 0$. How does $x_1(p, w_1)$ depend on p_2? Explain the difference to the last question.

(c) The endowments are $(\omega_{11}, \omega_{21}) = (1, 0)$ and $(\omega_{12}, \omega_{22}) = (0, 1)$. Normalize $p_1 = 1$, $p_2 = p$, . Derive the Walrasian equilibrium (p, x_1, x_2), and depict the equilibrium in an Edgeworth Box.

3. (**Exchange Economy**) Consider the same exchange economy as before, but suppose that the utility functions are

$$u^1(x_{11}, x_{21}) = \min\{2x_{11}, x_{21}\}$$

$$u^2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$$

(a) Show that there cannot be a Walrasian equilibrium with positive prices $p_1, p_2 > 0$ (a graphical argument in the Edgeworth Box suffices).

(b) Show that $(p_1, p_2) = (0, 1)$ is a WE price vector and depict all corresponding equilibrium allocations in an Edgeworth Box.