Problem Set 1

1. (Preference Relations)

If $\succeq$ is rational, then:

1. $\succ$ is **transitive**, but **not** reflexive ($x \succ x$ false $\forall x$).
2. $\sim$ is an **equivalence relation** (reflexive, transitive and symmetric).

Transitivity $(x \succ y \wedge y \succ z \overset{?}{\implies} x \succ z)$

$$\begin{aligned} x \succ y &\iff x \succeq y \wedge y \nsucceq x \\ y \succ z &\iff y \succeq z \wedge z \nsucceq y \end{aligned}$$

$x \succeq y \wedge y \succeq z \implies x \succeq z$ (by transitivity).

We know

$$x \succ z \iff x \succeq z \wedge z \nsucceq x$$

We need to prove that $z \nsucceq x$. Suppose $z \succeq x$

$$\begin{gathered} z \succeq x \wedge x \succeq y \implies z \succeq y \quad \text{by transitivity} \\ z \succeq y \wedge z \nsucceq y \quad \text{contradiction} \\ z \nsucceq x \\ x \succeq z \wedge z \nsucceq x \implies z \succ x \end{gathered}$$

$\succ$ is transitive.

Irreflexivity $(x \succ x, \forall x)$

$$x \succ x \iff x \succeq x \wedge x \nsucceq x \quad \text{not possible}$$

Equivalence relation $(x \sim y \wedge y \sim z \overset{?}{\implies} x \sim z)$

$$\begin{gathered} x \sim y \iff x \succeq y \wedge y \succeq x \quad \text{by reflexivity} \\ y \sim z \iff y \succeq z \wedge z \succeq y \quad \text{by reflexivity} \\ x \succeq y \wedge y \succeq z \implies x \succeq z \quad \text{by transitivity} \\ z \succeq y \wedge y \succeq x \implies z \succeq x \quad \text{by transitivity} \\ x \succeq z \wedge z \succeq x \implies x \sim z \end{gathered}$$

2. (Preference Relations)

3.(Indifference sets)

3(a)

$$I(x) \equiv \{y \in X : y \sim x\}$$

Prove:

$$I(x) \neq I(x') \implies I(x) \cap I(x') = \varnothing$$

3(b)

Prove:

$$\begin{aligned} \text{rationality} &\implies \text{no intersection} \\ \text{intersection} &\implies \text{irrational} \end{aligned}$$

Example:

$$\succeq = \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,c)\}$$

$$\begin{aligned} I(a) &= \{y \in \{a,b,c\} : y \sim a\} \\ &= \{y \in \{a,b,c\} : y \succeq a \wedge a \succeq y\} \\ &= \{a,b,c\} \end{aligned}$$

$$\begin{aligned} I(b) &= \{y \in \{a,b,c\} : y \sim b\} \\ &= \{y \in \{a,b,c\} : y \succeq b \wedge b \succeq y\} \\ &= \{a,b\} \end{aligned}$$

$$\begin{aligned} I(c) &= \{y \in \{a,b,c\} : y \sim c\} \\ &= \{y \in \{a,b,c\} : y \succeq c \wedge c \succeq y\} \\ &= \{a,c\} \end{aligned}$$

$$I(a) \cap I(b) \cap I(c) = \{a\}$$

Completeness is satisfied while transitivity is violated.

4. (WARP)

Let $X = \{x,y,z\}$ and

$$\mathscr{B}_1 = (\{x,y\},\{x,y,z\}) \quad \mathscr{B}_2 = \{\{x,y\},\{x,z\},\{y,z\}\}$$

Two choice structures:

$$(\mathscr{B}_1, C_1(\cdot)) \quad (\mathscr{B}_2, C_2(\cdot))$$

$$\begin{gathered} C_1(\{x,y\}) = \{x,y\}, \quad C_1(\{x,y,z\}) = \{y,z\} \\ C_2(\{x,y\}) = \{x\}, \quad C_2(\{x,z\}) = \{z\}, \quad C_2(\{y,z\}) = \{y\} \end{gathered}$$

4(a)

For $(\mathscr{B}_1, C_1(\cdot))$, set $B_1 = \{x,y\}$ and $B_2 = \{x,y,z\}$

$$
\begin{gathered}
C_1(B_1)=\{x,y\} \implies x\in C_1(B_1)\\
C_1(B_2)=\{y,z\} \implies y\in C_1(B_2)\\
x,y\in B_1\wedge x\in C_1(B_1), \exists B_2 \text{ with } x,y\in B_2 : y\in C_1(B_2) \text{ but } x\notin C_1(B_2)\\
x,y\in B_1,B_2 \quad x\in C_1(B_1)\wedge y\in C_1(B_2) \not\Rightarrow x\in C_1(B_2)
\end{gathered}
$$

Therefore, choice structure $(\mathscr{B}_1, C_1(\cdot))$ violates WARP.

For $(\mathscr{B}_2, C_2(\cdot)), \nexists x,y\in B,B'$, we cannot find a WARP violation.

4(b)

$$
C^*(B,\succeq)=\{x\in B : x\succeq y\,\forall\, y\in B\}
$$

$$
\begin{aligned}
\text{rational preference} &\implies \text{WARP}\\
\text{not WARP} &\implies \text{not rational preference}
\end{aligned}
$$

- Since choice structure $(\mathscr{B}_1, C_1(\cdot))$ does not satisfy WARP, it cannot be rationalized by a rational preference relation.
- For the choice structure $(\mathscr{B}_2, C_2(\cdot))$, we cannot know whether it can be rationalized given the information.

6. (Utility and Preferences)

6(a)

$$
\begin{gathered}
\text{The function } u : X\to\mathbb{R}^2_+ \text{ represents } \succeq \text{ on } X \text{ if for any } x,y\in X, \text{ then}\\
x\succeq y \iff u(x)\geq u(y)
\end{gathered}
$$

$$
\begin{aligned}
u_1(x,y)&=10(\ln x+\ln y)\\
u_2(x,y)&=\exp{(xy)}\\
u_3(x,y)&=\sqrt{2xy}-10
\end{aligned}
$$

i)

$$
\begin{aligned}
v_1(x,y)&=\exp\left(\frac{u_1(x,y)}{10}\right)\\
&=\exp\left(\frac{10(\ln x+\ln y)}{10}\right)\\
&=\exp{(\ln x+\ln y)}\\
&=xy\\
&=u(x,y)
\end{aligned}
$$

Note: domain in this case is a bit different from that given in this problem ($x,y\neq 0$)

$$
\begin{aligned}
v_2(x,y)&=\ln u_2(x,y)\\
&=\ln e^{xy}\\
&=xy\\
&=u(x,y)
\end{aligned}
$$

$$
\begin{aligned}
v_3(x, y) &= \frac{(u_3(x, y) + 10)^2}{2} \\
&= xy \\
&= u(x, y)
\end{aligned}
$$

6(c)

- Monotone but not strictly monotone when the bundle is on the slope.
- Convex but not strictly convex when the bundle is on the slope.