Problem Set 11

Exercise 1: Market Completeness and Risk Sharing

1(a)

$$\begin{cases} \mathcal{S} = 2 \\ \rho = 1 \\ \beta = 0.9 \end{cases} \quad \begin{cases} \pi(1) = \frac{1}{2} \\ \pi(2) = \frac{1}{2} \end{cases} \quad \begin{cases} Y_1 = 1 \\ Y_1^\star = 1 \end{cases} \quad \begin{cases} Y_2(1) = 1 \\ Y_2^\star(1) = 1 \end{cases} \quad \begin{cases} Y_2(2) = \frac{3}{2} \\ Y_2^\star(2) = \frac{1}{2} \end{cases}$$

(i)

$$\left(\frac{Y_2^W(s)}{Y_2^W(s')}\right)^{-\rho} \frac{\pi(s)}{\pi(s')} = \frac{p(s)}{p(s')}$$

$$Y_2^W(1) = Y_2(1) + Y_2^\star(1) = 1 + 1 = 2$$

$$Y_2^W(2) = Y_2(2) + Y_2^\star(2) = \frac{3}{2} + \frac{1}{2} = 2$$

$$\frac{p(1)}{p(2)} = \left(\frac{Y_2^W(1)}{Y_2^W(2)}\right)^{-1} \frac{\pi(1)}{\pi(2)} = 1$$

$$\text{From the lecture:} \quad \sum_{s=1}^{\mathcal{S}} p(s) = 1 \implies p(1) + p(2) = 1$$

$$\begin{cases} p(1) = p(2) \\ p(1) + p(2) = 1 \end{cases} \implies \begin{cases} p(1) = \frac{1}{2} \\ p(2) = \frac{1}{2} \end{cases}$$

(ii)

Euler equation:

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^\rho = \frac{p(s)}{1+r}$$

From PS10:

$$Y_1^W = Y_1 + Y_1^\star = 1 + 1 = 2$$

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^\rho = \frac{p(s)}{1+r} \implies \beta\pi(s)\left(\frac{Y_1^W}{Y_2^W(s)}\right)^\rho = \frac{p(s)}{1+r}$$

$$\beta\pi(1)\left(\frac{Y_1^W}{Y_2^W(1)}\right)^\rho = \frac{p(1)}{1+r}$$

$$0.9 \times \frac{1}{2}\left(\frac{2}{2}\right)^1 = \frac{\frac{1}{2}}{1+r}$$

$$r = \frac{1}{9}$$

(iii)

$$\begin{aligned} CA_1 &= \frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\left[\frac{p(1)Y_2(1)+p(2)Y_2(2)}{1+r}\right] \\ &= \frac{0.9}{1+0.9}\times 1 - \frac{1}{1+0.9}\left[\frac{\frac{1}{2}\times 1+\frac{1}{2}\times\frac{3}{2}}{1+\frac{1}{9}}\right] \\ &= -\frac{9}{76} \quad (-0.1184) \end{aligned}$$

Home country is relatively richer than Foreign country in period 2, hence it will run a current deficit by selling A.D. securities.

(iv)

$$\begin{cases} C_1 = Y_1 - CA_1 = 1-(-0.1184) = 1.1184 \\ C_1^\star = Y_1^\star + CA_1 = 1 - 0.1184 = 0.8816 \end{cases}$$

Euler equation:

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^\rho = \frac{p(s)}{1+r} \iff C_2(s) = \beta\pi(s)\frac{1+r}{p(s)}C_1$$

$$C_2(s) = \beta\pi(s)\frac{1+r}{p(s)}C_1 = 0.9\times\frac{1}{2}\times\frac{1+\frac{1}{9}}{\frac{1}{2}}\times 1.1184 = 1.1184 \quad \forall s$$

$$C_2^\star(s) = \beta\pi(s)\frac{1+r}{p(s)}C_1^\star = 0.9\times\frac{1}{2}\times\frac{1+\frac{1}{9}}{\frac{1}{2}}\times 0.8816 = 0.8816 \quad \forall s$$

If prices are actuarially fair (which was the case since $Y_2^W(1) = Y_2^W(2)$) , agents fully insure themselves.

(v)

$$\tilde{p}(s) = \frac{p(s)}{1+r} \implies \begin{cases} \tilde{p}(1) = 0.45 \\ \tilde{p}(2) = 0.45 \end{cases}$$

$$P^k = \boldsymbol{E}(MX^k) = \boldsymbol{E}(M)\boldsymbol{E}(X^k) + Cov(M, X^k)$$

$$\begin{cases} M & \text{Stochastic Discount Factor} \\ X^k & \text{Payoff of A.D. Security} \end{cases}$$

$$\boldsymbol{E}(M(s)) = \boldsymbol{E}\left(\frac{\beta u'(C_2(s))}{u'(C_1)}\right) = \beta = \frac{1}{1+r} = 0.9$$

$$\begin{cases} \boldsymbol{E}(X^1) = \pi(1)\times 1 + \pi(2)\times 0 = \frac{1}{2} \\ \boldsymbol{E}(X^2) = \pi(1)\times 0 + \pi(2)\times 1 = \frac{1}{2} \end{cases}$$

$$P^k = \boldsymbol{E}(MX^k) = \boldsymbol{E}(M)\boldsymbol{E}(X^k) + Cov(M, X^k)$$

$$\begin{cases} \boldsymbol{E}(M(s)) = 0.9 \\ \boldsymbol{E}(X^k) = \frac{1}{2} \quad \forall k = 1,2 \end{cases} \implies \begin{cases} Cov(M, X^1) = 0 \\ Cov(M, X^2) = 0 \end{cases}$$

Scholastic discount factor and payoffs are uncorrelated because agents are fully insured.

(vi)

$$CA_1 = Y_1 - C_1 = \frac{p(1)}{1+r}B_2(1) + \frac{p(2)}{1+r}B_2(2)$$

$$\begin{aligned}
\frac{p(1)}{1+r}B_2(1) &= \pi(1)CA_1 + \frac{\pi(2)p(2)Y_2(1)}{1+r}\left(\frac{\pi(1)}{\pi(2)}\frac{Y_2(2)}{Y_2(1)} - \frac{p(1)}{p(2)}\right) \\
0.45B_2(1) &= \frac{1}{2} \times (-\frac{9}{76}) + \frac{\frac{1}{2} \times \frac{1}{2} \times 1}{1+\frac{1}{9}}\left(\frac{3}{2} - 1\right) \\
B_2(1) &= \frac{0.0533}{0.45} = 0.1184
\end{aligned}$$

$$\begin{aligned}
\frac{p(2)}{1+r}B_2(2) &= CA_1 - \frac{p(1)}{1+r}B_2(1) \\
0.45B_2(2) &= -\frac{9}{76} - 0.45 \times 0.1184 \\
B_2(2) &= -\frac{0.1717}{0.45} = -0.3816
\end{aligned}$$

Market clearing condition

$$\begin{cases} B_2(1) + B_2^\star(1) = 0 \\ B_2(2) + B_2^\star(2) = 0 \end{cases} \Longrightarrow \begin{cases} B_2^\star(1) = -0.1184 \\ B_2^\star(2) = 0.3816 \end{cases}$$

$$B_2 = \begin{pmatrix} 0.1184 \\ -0.3816 \end{pmatrix} \quad B_2^\star = \begin{pmatrix} -0.1184 \\ 0.3816 \end{pmatrix}$$

Home country is long A.D. security that pays in state 1 because it is relatively poorer in state 1 than 2. However, home country sells more than it buys ($CA < 0$).

1(b)

$$\begin{cases} \mathcal{S} = 2 \\ \rho = 1 \\ \beta = 0.9 \end{cases} \quad \begin{cases} \pi(1) = \frac{1}{2} \\ \pi(2) = \frac{1}{2} \end{cases} \quad \begin{cases} Y_1 = 1 \\ Y_1^\star = 1 \end{cases} \quad \begin{cases} Y_2(1) = \frac{1}{2} \\ Y_2^\star(1) = 1 \end{cases} \quad \begin{cases} Y_2(2) = 2 \\ Y_2^\star(2) = \frac{1}{2} \end{cases}$$

(i)

$$\begin{aligned}
Y_2^W(1) &= Y_2(1) + Y_2^\star(1) = \frac{1}{2} + 1 = \frac{3}{2} \\
Y_2^W(2) &= Y_2(2) + Y_2^\star(2) = 2 + \frac{1}{2} = \frac{5}{2}
\end{aligned}$$

$$\frac{p(1)}{p(2)} = \left(\frac{Y_2^W(1)}{Y_2^W(2)}\right)^{-1}\frac{\pi(1)}{\pi(2)} = \frac{5}{3}$$

$$\begin{cases} 3p(1) = 5p(2) \\ p(1) + p(2) = 1 \end{cases} \Longrightarrow \begin{cases} p(1) = \frac{5}{8} \\ p(2) = \frac{3}{8} \end{cases}$$

$Y_2^W(1) < Y_2^W(2)$, hence insuring consumption in state of the world 1 (when the endowment is scarce) is more expensive than in state of the world 2.

Note: prices are not actuarially fair (i.e., prices do not reflect the true probabilities of events), there is no full insurance.

(ii)

$$Y_1^W = Y_1 + Y_1^\star = 2$$

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^\rho = \frac{p(s)}{1+r} \implies \beta\pi(s)\left(\frac{Y_1^W}{Y_2^W(s)}\right)^\rho = \frac{p(s)}{1+r}$$

$$\begin{aligned}
\beta\pi(1)\left(\frac{Y_1^W}{Y_2^W(1)}\right)^\rho &= \frac{p(1)}{1+r} \\
0.9 \times \frac{1}{2}\left(\frac{4}{3}\right)^1 &= \frac{\frac{5}{8}}{1+r} \\
r &= \frac{1}{24} \quad (0.417)
\end{aligned}$$

Home country is relatively richer than foreign country in period 2, hence, it will borrow from abroad in period 1 by selling A.D. securities (i.e., importing consumption goods).

(iii)

$$\begin{aligned}
CA_1 &= \frac{\beta}{1+\beta}Y_1 - \frac{1}{1+\beta}\left[\frac{p(1)Y_2(1) + p(2)Y_2(2)}{1+r}\right] \\
&= \frac{0.9}{1+0.9} \times 1 - \frac{1}{1+0.9}\left[\frac{\frac{5}{8} \times \frac{1}{2} + \frac{3}{8} \times 2}{1 + \frac{1}{24}}\right] \\
&= -\frac{6}{95} \quad (-0.0632)
\end{aligned}$$

(iv)

$$\begin{cases} C_1 = Y_1 - CA_1 = 1 - (-0.0632) = 1.0632 \\ C_1^\star = Y_1^\star + CA_1 = 1 - 0.0632 = 0.9368 \end{cases}$$

Euler equation:

$$\beta\pi(s)\left(\frac{C_1}{C_2(s)}\right)^\rho = \frac{p(s)}{1+r} \iff C_2(s) = \beta\pi(s)\frac{1+r}{p(s)}C_1$$

$$\begin{cases} C_2(1) = \beta\pi(1)\dfrac{1+r}{p(1)}C_1 = 0.9 \times \dfrac{1}{2} \times \dfrac{1+\frac{1}{24}}{\frac{5}{8}} \times \dfrac{101}{95} = 0.7974 \\ C_2(2) = \beta\pi(2)\dfrac{1+r}{p(2)}C_1 = 0.9 \times \dfrac{1}{2} \times \dfrac{1+\frac{1}{24}}{\frac{3}{8}} \times \dfrac{101}{95} = 1.3289 \end{cases}$$

$$\begin{cases} C_2^\star(1) = \beta\pi(1)\dfrac{1+r}{p(1)}C_1^\star = 0.9 \times \dfrac{1}{2} \times \dfrac{1+\frac{1}{24}}{\frac{5}{8}} \times \dfrac{89}{95} = 0.7026 \\ C_2^\star(2) = \beta\pi(2)\dfrac{1+r}{p(2)}C_1^\star = 0.9 \times \dfrac{1}{2} \times \dfrac{1+\frac{1}{24}}{\frac{3}{8}} \times \dfrac{89}{95} = 1.1711 \end{cases}$$

Full insurance is not possible. Consumption is higher when endowment is higher (s=2).

(v)

$$\tilde{p}(s) = \frac{p(s)}{1+r} \implies \begin{cases} \tilde{p}(1) = \dfrac{p(1)}{1+r} = \dfrac{\frac{5}{8}}{1+\frac{1}{24}} = 0.6 \\ \tilde{p}(2) = \dfrac{p(2)}{1+r} = \dfrac{\frac{3}{8}}{1+\frac{1}{24}} = 0.36 \end{cases}$$

$$\boldsymbol{E}(M(s)) = \boldsymbol{E}\left(\frac{\beta u'(C_2(s))}{u'(C_1)}\right) = \boldsymbol{E}\left(\frac{\beta C_1}{C_2(s)}\right)$$

$$\begin{cases} M(1) = \dfrac{\beta C_1}{C_2(1)} = \dfrac{0.9 \times 1.0632}{0.7974} = 1.2 \\ M(2) = \dfrac{\beta C_1}{C_2(2)} = \dfrac{0.9 \times 1.0632}{1.3289} = 0.72 \end{cases}$$

$$M = \begin{bmatrix} M(1) \\ M(2) \end{bmatrix} = \begin{bmatrix} 1.2 \\ 0.72 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{cases} \boldsymbol{E}(M) = \pi(1)M(1) + \pi(2)M(2) = \dfrac{1}{2} \times 1.2 + \dfrac{1}{2} \times 0.72 = 0.96 \\ \boldsymbol{E}(X^1) = \pi(1) \times 1 + \pi(2) \times 0 = \dfrac{1}{2} \\ \boldsymbol{E}(X^2) = \pi(1) \times 0 + \pi(2) \times 1 = \dfrac{1}{2} \end{cases}$$

$$\begin{cases} \boldsymbol{E}(M)\boldsymbol{E}(X^1) = 0.48 \\ \boldsymbol{E}(M)\boldsymbol{E}(X^2) = 0.48 \end{cases}$$

$$P^k = \boldsymbol{E}(MX^k) = \boldsymbol{E}(M)\boldsymbol{E}(X^k) + Cov(M, X^k)$$

$$\begin{cases} Cov(M, X^1) = \tilde{p}(1) - M(1) = 0.6 - 0.48 = 0.12 \\ Cov(M, X^2) = \tilde{p}(2) - M(2) = 0.36 - 0.48 = -0.12 \end{cases}$$

(vi)

$$\frac{p(1)}{1+r} B_2(1) = \pi(1) CA_1 + \frac{\pi(2)p(2)Y_2(1)}{1+r}\left(\frac{\pi(1)}{\pi(2)}\frac{Y_2(2)}{Y_2(1)} - \frac{p(1)}{p(2)}\right)$$

$$0.6 B_2(1) = \frac{1}{2} \times (-\frac{6}{95}) + \frac{\frac{1}{2} \times \frac{3}{8} \times \frac{1}{2}}{1 + \frac{1}{24}}\left(4 - \frac{5}{3}\right)$$

$$B_2(1) = \frac{0.1784}{0.6} = 0.2974$$

$$\frac{p(2)}{1+r} B_2(2) = CA_1 - \frac{p(1)}{1+r} B_2(1)$$

$$0.36 B_2(2) = -\frac{6}{95} - 0.6 \times 0.2974$$

$$B_2(2) = -\frac{0.2416}{0.36} = -0.6711$$

Market clearing condition:

$$\begin{cases} B_2(1) + B_2^\star(1) = 0 \\ B_2(2) + B_2^\star(2) = 0 \end{cases} \implies \begin{cases} B_2^\star(1) = -0.2974 \\ B_2^\star(2) = 0.6711 \end{cases}$$

$$B_2 = \begin{pmatrix} 0.2974 \\ -0.6711 \end{pmatrix} \quad B_2^\star = \begin{pmatrix} -0.2974 \\ 0.6711 \end{pmatrix}$$

When A.D. security offers a hedge against low endowment (i.e., in state of world 1), agents value the A.D. security more and are willing to pay a higher price.

	State 1	State 2
Asset 1	1	0
Asset 2	0	1

Since the Stochastic Discount Factor is high when consumption is low (you discount the future consumption at a higher rate because you value consumption today a lot), then

$$\begin{cases} Cov(M, X^1) > 0 \\ Cov(M, X^2) < 0 \end{cases}$$