Exercises
Social Choice Theory
University of Zurich
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Part 1: Preliminaries

Exercise 1.1

(a) Find an example of a binary relation that is quasi-transitive but not transitive.

(b) Find an example of a binary relation that is acyclical but not quasi-transitive.

(c) Show that, if R is transitive, then its induced I is also transitive.

Exercise 1.2 (Gaertner 2009, p. 16)

(a) Let $X = \{w, x, y, z\}$ and assume that $\succeq$ is a preference on X that satisfies $x \sim y$, $y \succ z$, and $z \succ w$. Determine the set of greatest elements $G(\{w, x, y, z\}, \succeq)$.

(b) Let $X = \{x, y, z\}$ and assume that R is a binary relation on X which is reflexive and complete, and satisfies xPy, yPz and zPx. Determine the sets $M(\{x, y, z\}, R)$ and $G(\{x, y, z\}, R)$.

Exercise 1.3
Assume $X = \{x, y, z\}$. Check whether properties α and β are satisfied:

(a) $C(\{x, y, z\}) = \{x\}, C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{x\}, C(\{y, z\}) = \{y\}$.

(b) $C(\{x, y, z\}) = \{z\}, C(\{x, y\}) = \{x\}, C(\{x, z\}) = \{x, z\}, C(\{y, z\}) = \{z\}$.

(c) $C(\{x, y, z\}) = \{y, z\}, C(\{x, y\}) = \{y\}, C(\{x, z\}) = \{z\}, C(\{y, z\}) = \{y, z\}$.

(d) $C(\{x, y, z\}) = \{x, y\}, C(\{x, y\}) = \{y\}, C(\{x, z\}) = \{x\}, C(\{y, z\}) = \{y\}$.

Exercise 1.4 (Gaertner 2009, p. 17)
A decision-maker chooses peanuts and apple juice when peanuts, mineral water, and apple juice are available. She chooses peanuts and beer when peanuts, mineral water, apple juice, and beer are available.

(a) Write this formally in terms of a choice function.

(b) Do these choices satisfy β? Discuss the plausibility of property β.

Part 2: The Problem of Social Choice

Exercise 2.1

(a) Consider the following example:

#	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
1	$z\ P\ x\ P\ y$

Apply pairwise majority voting. Is there a Condorcet winner?

(b) Give an example of a domain $\mathscr{A}$ on which pairwise majority voting is an SWF.

Exercise 2.2

Compare the methods PV, IR, PM, CO, and BC for the case when $m = 2$.

Exercise 2.3

Apply the methods PV, IR, PM, CO, BC, EM, and PE to the following example:

#	preferences
2	$x\ P\ w\ P\ y\ P\ z$
2	$y\ P\ w\ P\ z\ P\ x$
1	$w\ P\ z\ P\ x\ P\ y$

Exercise 2.4

(a) Give an example where a Condorcet winner exists but is not selected by IR.

(b) Can a Condorcet winner be Pareto dominated by some other alternative?

(c) For both EM and PE, give an example where a Condorcet winner exists but is not ranked as the unique best alternative.

Exercise 2.5

(a) Which of the scoring methods on slide 56 are equivalent to each other...

...for arbitrary values of m?

...for $m = 3$?

(b) Give an example of a scoring method that is equivalent to BC.

(c) Give an example of a scoring method that is not equivalent to BC but still allows voters to reveal their full preference ranking.

Exercise 2.6 ([GA], p. 118)
Apply the scoring methods given by $s^1 = (3, 2, 1, 0)$, $s^2 = (1, 3/4, 1/4, 0)$ and $s^3 = (1, 1/2, 1/4, 0)$ to the following example:

#	preferences
1	$x\ P\ z\ P\ v\ P\ y$
1	$y\ P\ z\ P\ v\ P\ x$
1	$v\ P\ z\ P\ y\ P\ x$
1	$x\ P\ y\ P\ v\ P\ z$

Part 3: Arrow's Theorem

Exercise 3.1
For each of the following methods, check which of Arrow's axioms are violated.

(a) Plurality voting (PV)

(b) Pairwise majority voting (PM)

(c) Copeland method (CO)

(d) Borda count (BC)

(e) Pareto efficiency method (EM)

(f) The preference of society coincides with the preference of individual $h \in N$.

Exercise 3.2
Consider instant-runoff voting (IR). Assume that ties are broken in alphabetical order, i.e. any tie is broken in favor of the alternative earlier in the alphabet. Which of Arrow's axioms for SCFs are violated?

Exercise 3.3

(a) For the case when $m = 2$, compare the Pareto efficiency method (EM) and the Pareto extension rule (PE). Are they both SWFs in this case?

(b) Which of May's axioms are violated by EM and PE?

Exercise 3.4 ([MC], p. 812)
Assume $m = 2$. Give an example of an SWF that satisfies all of May's axioms...

(a) ...except [N].

(b) ...except [PR].

(c) ...except [A].

Exercise 3.5

(a) Consider preference $zPxPyPwIvPu$.
Is it single-peaked with respect to $w > y > z > x > v > u$?
Is it single-peaked with respect to $v > w > y > z > x > u$?

(b) Consider preference $yPwIzPvIx$. Is is single-peaked with respect to some order?
Consider preference $yPwIzIvPx$. Is is single-peaked with respect to some order?

Exercise 3.6

(a) Consider the preference profile

#	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
1	$y\ P\ x\ P\ z$

Are these preferences single-peaked with respect to $y > x > z$?
Is there an order with respect to which they are all single-peaked?

(b) Consider the preference profile

#	preferences
1	$x\ P\ y\ P\ z$
1	$y\ P\ z\ P\ x$
1	$z\ P\ x\ P\ y$

Is there an order with respect to which these preferences are all single-peaked?

Exercise 3.7 (based on [GA], p. 54)
Consider the preference profile

#	preferences
2	$w\ P\ x\ P\ y\ P\ z$
2	$x\ P\ y\ P\ z\ P\ w$
1	$y\ P\ z\ P\ x\ P\ w$

(a) Is there an order with respect to which this profile is single-peaked?

(b) Apply pairwise majority voting. Is there a Condorcet winner/loser?

(c) Identify the median voter(s).

Part 4: Individual Rights

Exercise 4.1

Consider two individuals, $N = \{1, 2\}$, who can either go to the cinema C or the opera O. Allocations specify where each person goes, $X = \{CC, OO, CO, OC\}$.

(a) Formalize the rights system $\mathbf{D} = (D_1, D_2)$ which captures that the own choice where to go belongs to each individual's private sphere.

(b) Construct a preference profile $\mathbf{R}$ for which the outcome of any decision function that respects $\mathbf{D}$ and satisfies [P*] must be cyclical.

Part 5: Manipulability

Exercise 5.1 ([GA], p. 94)

Suppose we select the alternative that is top-ranked by Borda count. Which voters will want to manipulate in the following two preference profiles?

#	true preferences
1	$x\ P\ y\ P\ z\ P\ w$
1	$y\ P\ z\ P\ w\ P\ x$
1	$y\ P\ w\ P\ z\ P\ x$

#	true preferences
1	$y\ P\ z\ P\ x\ P\ w$
1	$z\ P\ y\ P\ x\ P\ w$
1	$w\ P\ z\ P\ y\ P\ x$

Exercise 5.2 ([GA], p. 94)

Assume $n = 2$ and $m = 3$. The following table completely specifies an SCF on the domain of strict preferences ($\mathscr{A} = \mathscr{P}^n$).

$R_1 \backslash R_2$	xyz	xzy	yxz	yzx	zxy	zyx
xyz	x	x	x	x	x	x
xzy	x	x	x	x	x	x
yxz	x	x	y	y	x	y
yzx	x	x	y	y	y	y
zxy	x	x	x	y	z	z
zyx	x	x	y	y	z	z

At which of these preference profiles is the SCF manipulable?

Exercise 5.3

Consider plurality voting with alphabetical tie-breaking, where the alternative earlier in the alphabet wins in case of ties. Show that this rule is manipulable.

Part 6: Distributive Justice

Exercise 6.1

For each of the following SEFs, check with which information structures it is consistent:

(a) Pareto efficiency method.

(b) The Nash SEF e^{NA} that selects the preference represented by

$$NA(x, \mathbf{U}) = \prod_{i=1}^{n} U_i(x), \quad \forall\, x \in X, \mathbf{U} \in \mathscr{U}^n.$$

(c) The isoelastic SEF e^{IE} that selects the preference represented by

$$IE(x, \mathbf{U}) = \frac{1}{1-\rho} \sum_{i=1}^{n} (U_i(x))^{1-\rho}, \quad \forall\, x \in X, \mathbf{U} \in \mathscr{U}^n,$$

where we assume $\rho > 0$ and $\rho \neq 1$.

Exercise 6.2

For each of the following information structures, give an example of an SEF that is consistent with it (and all stronger structures) but not with any other structure:

(a) CM-NC

(b) CM-LC