Problem Set 4

Exercise 1: Production Economy

A two-period economy with concave production function

$$Y_t = A_t K_t^{\alpha} \quad \text{with} \quad 0 < \alpha < 1$$

$$\begin{cases} Y_t & \text{the ouput level of the numeraire good} \\ K_t & \text{capital stock} \\ A_t & \text{the level of factor productivity in period } t \end{cases}$$

They dynamic evolution of capital

$$K_{t+1} = K_t + I_t$$

The period budget constraint of the representative household is

$$C_t + I_t + B_{t+1} = Y_t + B_t(1+r)$$

1(a)

$$\begin{cases} B_1 = 0 & \text{no initial foreign asset} \\ B_3 = 0 & \text{no foreign assest at the end of period 2} \end{cases}$$

$$\begin{cases} C_1 + I_1 + B_2 = Y_1 + B_1(1+r) \\ C_2 + I_2 + B_3 = Y_2 + B_2(1+r) \end{cases} \implies \begin{cases} C_1 + I_1 + B_2 = Y_1 \\ C_2 + I_2 = Y_2 + B_2(1+r) \end{cases} \implies \begin{cases} C_1 + B_2 = Y_1 - I_1 \\ \dfrac{C_2}{1+r} = \dfrac{Y_2 - I_2}{1+r} + B_2 \end{cases}$$

$$C_1 + \frac{C_2}{1+r} = Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}$$

$$\underbrace{C_1 + \frac{C_2}{1+r}}_{\text{present value of consumption}} = \underbrace{Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}}_{\text{present value of net output}}$$

1(b)

PPF in closed economy

$$NO_2 = PPF(NO_1)$$

$$\begin{cases} NO_t = Y_t - I_t \\ C_t = Y_t - I_t \end{cases} \implies NO_t = C_t$$

$$NO_2 = PPF(NO_1) \iff C_2 = PPF(C_1)$$

$$\begin{cases} C_1 = Y_1 - I_1 \\ C_2 = Y_2 - I_2 \end{cases} \implies \begin{cases} C_1 = A_1 K_1^{\alpha} - I_1 \\ C_2 = A_2 K_2^{\alpha} - I_2 \end{cases}$$

$$\begin{aligned} K_2 &= K_1 + I_1 \\ &= K_1 + A_1 K_1^{\alpha} - C_1 \end{aligned}$$

$$\begin{aligned} C_2 &= A_2 K_2^{\alpha} - I_2 \\ &= A_2 K_2^{\alpha} + K_2 \\ &= A_2(K_1 + A_1 K_1^{\alpha} - C_1)^{\alpha} + K_1 + A_1 K_1^{\alpha} - C_1 \\ &= PPF(C_1) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}PPF(C_1)}{\mathrm{d}C_1} &= -\alpha A_2(K_1 + A_1 K_1^{\alpha} - C_1)^{\alpha-1} - 1 \\ &= -\alpha A_2 K_2^{\alpha-1} - 1 \\ &= -F'(K_2) - 1 < 0 \end{aligned}$$

PPF is downward sloping

$$\begin{aligned}\frac{\mathrm{d}^2 PPF(C_1)}{\mathrm{d}C_1^2} &= \alpha(\alpha-1)A_2(K_1 + A_1K_1^\alpha - C_1)^{\alpha-2} \\ &= \alpha(\alpha-1)A_2K_2^{\alpha-2} \\ &= F''(K_2) < 0\end{aligned}$$

PPF is strictly concave for any $\alpha \in (0,1)$

1(c)

The profit maximization

$$\max_{K_2} \quad A_2K_2^\alpha - rK_2$$

FOC:

$$\alpha A_2 K_2^{\alpha-1} - r = 0 \implies K_2 = \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}$$

The output in period 2

$$Y_2 = A_2K_2^\alpha = A_2\left(\frac{\alpha A_2}{r}\right)^{\frac{\alpha}{1-\alpha}}$$

1(d)

$$I_1 = K_2 - K_1 = \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} - K_1$$

$$\begin{aligned}\frac{\partial I_1}{\partial r} &= \left(\frac{1}{1-\alpha}\right)\left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}-1}\left(-\frac{\alpha A_2}{r^2}\right) \\ &= -\frac{1}{1-\alpha}\left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}-1}\frac{\alpha A_2}{r^2} < 0\end{aligned}$$

Investment profile function is downward sloping.

$$\frac{\partial^2 I_1}{\partial r^2} = \frac{3-2\alpha}{(1-\alpha)^2}\left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}}\frac{\alpha A_2}{r^3} > 0$$

1(e)

$$\begin{aligned}\max_{C_1,C_2} \quad & \log(C_1) + \beta\log(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}\end{aligned}$$

Lagarangian function

$$\mathcal{L} = \log(C_1) + \beta\log(C_2) - \lambda\left(C_1 + \frac{C_2}{1+r} - (Y_1 - I_1) - \frac{Y_2 - I_2}{1+r}\right)$$

FOC

$$\begin{cases}\dfrac{\partial \mathcal{L}}{\partial C_1} = \dfrac{1}{C_1} - \lambda = 0 \\ \dfrac{\partial \mathcal{L}}{\partial C_2} = \dfrac{\beta}{C_2} - \dfrac{\lambda}{1+r} = 0\end{cases} \implies C_2 = \beta(1+r)C_1$$

Plug $C_2 = \beta(1+r)C_1$ into IBC

$$\begin{cases} C_1 = \frac{1}{1+\beta}\left(Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}\right) \\ C_2 = \frac{\beta}{1+\beta}(1+r)\left(Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}\right) \end{cases}$$

1(f)

$$\begin{aligned} S_1 &= Y_1 - C_1 \\ &= Y_1 - \frac{1}{1+\beta}\left(Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}\right) \\ &= \frac{\beta}{1+\beta}Y_1 + \frac{1}{1+\beta}I_1 - \frac{1}{1+\beta}\cdot\frac{Y_2 - I_2}{1+r} \\ &= \frac{\beta}{1+\beta}A_1K_1^\alpha + \frac{1}{1+\beta}I_1 - \frac{1}{1+\beta}\cdot\frac{A_2K_2^\alpha - I_2}{1+r} \end{aligned}$$

$$\frac{\partial S_1}{\partial r} = \frac{1}{1+\beta}\cdot\frac{A_2K_2^\alpha - I_2}{(1+r)^2} > 0$$

Savings schedule is upward sloping.

1(g)

(i)

Investment schedule

$$I_1 = K_2 - K_1 = \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} - K_1$$

$$\frac{\partial I_1}{\partial A_1} = 0 \quad I_1 \text{ is independent of } A_1$$

(ii)

Savings schedule

$$S_1 = \frac{\beta}{1+\beta}A_1K_1^\alpha + \frac{1}{1+\beta}I_1 - \frac{1}{1+\beta}\cdot\frac{A_2K_2^\alpha - I_2}{1+r}$$

$$\frac{\partial S_1}{\partial A_1} = \frac{\beta}{1+\beta}K_1^\alpha$$

$$A_1\uparrow \implies Y_1\uparrow \implies C_1\uparrow (\Delta Y_1 > \Delta C_1 \text{ smooth consumption}) \implies S_1\uparrow$$

<u>Exercise 2:</u> Open Economy with Production - Two-Country Model

$$\text{lifetime utility} = \begin{cases} U(C_1, C_2) = u(C_1) + \beta u(C_2) & \text{domestic country} \\ U(C_1^*, C_2^*) = u(C_1^*) + \beta u(C_2^*) & \text{foreign country} \end{cases}$$

$$\text{where} \quad u(C_t) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}}$$

$$\text{production function} = \begin{cases} Y_t = A_tF(K_t) = A_tK_t^\alpha & \text{domestric country} \\ Y_t^* = A_t^*F(K_t^*) = A_t^*(K_t^*)^\alpha & \text{foreign country} \end{cases} \quad \text{where} \quad \alpha \in (0,1)$$

$$\text{capital accumnulation} = \begin{cases} K_{t+1} = K_t + I_t & \text{domestic country} \\ K_{t+1}^* = K_t^* + I_t^* & \text{foreign country} \end{cases}$$

Note:

- There is no capital depreciation.
- There is no initial holdings of foreign assets

$$\text{household budget constraint} = \begin{cases} C_t + I_t + B_{t+1} = Y_t + B_t(1+r) & \text{domestic country} \\ C_t^* + I_1^* + B_{t+1}^* = Y_t^* + B_t^*(1+r) & \text{foreign country} \end{cases}$$

$$\text{aggregate saving} = \begin{cases} S_t = Y_t - C_t & \text{domestic country} \\ S_t^* = Y_t^* - C_t^* & \text{foreign country} \end{cases}$$

2(a)

Production

$$\begin{cases} NO_1 = Y_1 - I_1 \\ NO_2 = Y_2 - I_2 \end{cases} \implies \begin{cases} NO_1 = A_1K_1^\alpha - I_1 \\ NO_2 = A_2K_2^\alpha - I_2 \end{cases} \implies \begin{cases} NO_1 = A_1K_1^\alpha - I_1 \\ NO_2 = A_2K_2^\alpha + K_2 \end{cases} \implies \begin{cases} NO_1 = A_1K_1^\alpha - I_1 \\ NO_2 = A_2K_2^\alpha + K_1 + I_1 \end{cases}$$

$$\begin{cases} K_2 = K_1 + I_1 \\ K_3 = K_2 + I_2 \\ K_3 = 0 \end{cases} \implies \begin{cases} K_2 = K_1 + I_1 \\ K_2 = -I_2 \end{cases} \implies K_2 = K_1 + A_1K_1^\alpha - NO_1$$

$$\begin{cases} NO_1 = A_1K_1^\alpha - I_1 \\ NO_2 = A_2K_2^\alpha + K_1 + I_1 \end{cases} \implies NO_1 + NO_2 = A_1K_1^\alpha + A_2K_2^\alpha + K_1$$

$$NO_2 = A_2(K_1 + A_1K_1^\alpha - NO_1)^\alpha + K_1 + A_1K_1^\alpha - NO_1$$

$$\begin{aligned} \frac{\partial NO_2}{\partial NO_1} &= -\alpha A_2(K_1 + A_1K_1^\alpha - NO_1)^{\alpha-1} - 1 \\ &= -A_2F'(K_2) - 1 \end{aligned}$$

Utility

$$C_1 + \frac{C_2}{1+r} = Y_1 - I_1 + \frac{Y_2 - I_2}{1+r} \implies C_2 = -(1+r)C_1 + (1+r)\left(Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}\right)$$

$$\frac{\partial C_2}{\partial C_1} = -(1+r)$$

In open economy

$$\frac{\partial NO_2}{\partial NO_1} = \frac{\partial C_2}{\partial C_1} \implies r = A_2F'(K_2)$$

$$\begin{aligned} r &= A_2F'(K_2) \\ &= \alpha A_2K_2^{\alpha-1} \\ &= \alpha A_2(K_1 + I_1)^{\alpha-1} \end{aligned}$$

$$r = \alpha A_2(K_1 + I_1)^{\alpha-1} \implies (K_1 + I_1)^{1-\alpha} = \frac{\alpha A_2}{r} \implies I_2 = \left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} - K_1$$

$$\begin{aligned} \frac{\mathrm{d}I_1}{\mathrm{d}r} &= -\frac{1}{1-\alpha}(\alpha A_2)^{\frac{1}{1-\alpha}} r^{-\frac{1}{1-\alpha}-1} \\ &= -\frac{1}{1-\alpha}\left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} r^{-1} < 0 \\ \frac{\mathrm{d}^2I_1}{\mathrm{d}r^2} &= \frac{1}{(1-\alpha)^2}\left(\frac{\alpha A_2}{r}\right)^{\frac{1}{1-\alpha}} r^{-2} > 0 \end{aligned}$$

2(b)

$$S_1 = Y_1 - C_1 = A_1K_1^\alpha - C_1$$

$$\frac{\mathrm{d}S_1}{\mathrm{d}r} = -\frac{\mathrm{d}C_1}{\mathrm{d}r}$$

$$\begin{cases} C_1 + I_1 + B_2 = Y_1 + B_1(1+r) \\ C_2 + I_2 + B_3 = Y_2 + B_2(1+r) \\ B_1 = 0 \quad \text{no initial foreign assets} \\ B_3 = 0 \quad \text{no foreign assets at the end of period 2} \end{cases} \implies \begin{cases} C_1 + B_2 = Y_1 - I_1 \\ \dfrac{C_2}{1+r} = \dfrac{Y_2 - I_2}{1+r} + B_2 \end{cases}$$

$$C_1 + \frac{C_2}{1+r} = Y_1 - I_1 + \frac{Y_2 - I_2}{1+r}$$

$$C_1 + \frac{C_2}{1+r} = NO_1 + \frac{NO_2}{1+r} \implies C_2 = (1+r)(NO_1 - C_1) + NO_2$$

$$\begin{aligned} C_2 &= (1+r)(NO_1 - C_1) + NO_2 \\ &= (1+r)(Y_1 - I_1 - C_1) + Y_2 - I_2 \\ &= (1+r)(A_1F(K_1) - I_1 - C_1) + A_2F(K_2) + K_2 \\ &= (1+r)(A_1F(K_1) - I_1 - C_1) + A_2F(K_1 + I_1) + K_1 + I_1 \\ &= IBC(C_1 | r, A_1, A_2, K_1) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}C_2}{\mathrm{d}r} &= A_1F(K_1) - I_1 - C_1 - (1+r)\left(\frac{\mathrm{d}I_1}{\mathrm{d}r} + \frac{\mathrm{d}C_1}{\mathrm{d}r}\right) + A_2F'(K_2)\frac{\mathrm{d}I_1}{\mathrm{d}r} + \frac{\mathrm{d}I_1}{\mathrm{d}r} \\ \frac{\mathrm{d}C_2}{\mathrm{d}r} &= A_1F(K_1) - I_1 - C_1 - (1+r)\left(\frac{\mathrm{d}I_1}{\mathrm{d}r} + \frac{\mathrm{d}C_1}{\mathrm{d}r}\right) + (1+r)\frac{\mathrm{d}I_1}{\mathrm{d}r} \\ \frac{\mathrm{d}C_2}{\mathrm{d}r} &= A_1F(K_1) - I_1 - C_1 - (1+r)\frac{\mathrm{d}C_1}{\mathrm{d}r} \end{aligned}$$

$$\text{Euler equation:} \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r} \implies u'(C_1) = \beta(1+r)u'(C_2)$$

$$\begin{aligned} u'(C_1) &= \beta(1+r)u'(IBC(\cdot)) \\ \frac{\mathrm{d}}{\mathrm{d}r}u'(C_1) &= \frac{\mathrm{d}}{\mathrm{d}r}(\beta(1+r)u'(IBC(\cdot))) \\ u''(C_1)\frac{\mathrm{d}C_1}{\mathrm{d}r} &= \beta u'(IBC(\cdot)) + \beta(1+r)u''(IBC(\cdot))\frac{\mathrm{d}IBC(\cdot)}{\mathrm{d}r} \\ u''(C_1)\frac{\mathrm{d}C_1}{\mathrm{d}r} &= \beta u'(C_2) + \beta(1+r)u''(C_2)\frac{\mathrm{d}C_2}{\mathrm{d}r} \\ u''(C_1)\frac{\mathrm{d}C_1}{\mathrm{d}r} &= \beta u'(C_2) + \beta(1+r)u''(C_2)(A_1F(K_1) - I_1 - C_1) - \beta(1+r)^2u''(C_2)\frac{\mathrm{d}C_1}{\mathrm{d}r} \\ (u''(C_1) + \beta(1+r)^2u''(C_2))\frac{\mathrm{d}C_1}{\mathrm{d}r} &= \beta u'(C_2) + \beta(1+r)u''(C_2)(A_1F(K_1) - I_1 - C_1) \\ \frac{\mathrm{d}C_1}{\mathrm{d}r} &= \frac{\beta u'(C_2) + \beta(1+r)u''(C_2)(A_1F(K_1) - I_1 - C_1)}{u''(C_1) + \beta(1+r)^2u''(C_2)} \end{aligned}$$

$$\textbf{Euler euqtaion:} \quad \frac{\beta u'(C_2)}{u'(C_1)} = \frac{1}{1+r} \implies \frac{\beta C_2^{-\frac{1}{\sigma}}}{C_1^{-\frac{1}{\sigma}}} = \frac{1}{1+r} \implies \left(\frac{C_1}{C_2}\right)^{-\frac{1}{\sigma}} = \beta(1+r)$$

$$\textbf{isoelastic utility:} \quad u(C_t) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} \implies \begin{cases} u'(C_t) = C_t^{-\frac{1}{\sigma}} \\ u''(C_t) = -\frac{1}{\sigma}C_t^{-\frac{1}{\sigma}-1} \end{cases}$$

$$\begin{aligned} \frac{u'(C_2)}{u''(C_2)} &= \frac{C_2^{-\frac{1}{\sigma}}}{-\frac{1}{\sigma}C_2^{-\frac{1}{\sigma}-1}} = -\sigma C_2 \\ \frac{u''(C_1)}{u''(C_2)} &= \frac{-\frac{1}{\sigma}C_1^{-\frac{1}{\sigma}-1}}{-\frac{1}{\sigma}C_2^{-\frac{1}{\sigma}-1}} = \left(\frac{C_1}{C_2}\right)^{-\frac{1}{\sigma}-1} = \beta(1+r)\frac{C_2}{C_1} \end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}C_1}{\mathrm{d}r} &= \frac{\beta u'(C_2) + \beta(1+r)u''(C_2)(A_1F(K_1) - I_1 - C_1)}{u''(C_1) + \beta(1+r)^2 u''(C_2)} \\
&= \frac{\beta \frac{u'(C_2)}{u''(C_2)} + \beta(1+r)(A_1F(K_1) - I_1 - C_1)}{\frac{u''(C_1)}{u''(C_2)} + \beta(1+r)^2} \\
&= \frac{-\beta\sigma C_2 + \beta(1+r)(A_1F(K_1) - I_1 - C_1)}{\beta(1+r)\frac{C_2}{C_1} + \beta(1+r)^2} \\
&= \frac{A_1F(K_1) - I_1 - C_1 - \frac{\sigma C_2}{1+r}}{\frac{C_2}{C_1} + 1 + r} \\
&= \frac{CA_1 - \frac{\sigma C_2}{1+r}}{\frac{C_2}{C_1} + 1 + r}
\end{aligned}
$$

2(c)

$$
\begin{aligned}
\underbrace{Y_1 - C_1}_{S_1} + \underbrace{Y_1^* - C_1^*}_{S_1^*} &= I_1 + I_1^* \\
S_1 + S_1^* &= I_1 + I_1^* \\
S_1 - I_1 &= -(S_1^* - I_1^*) \\
CA_1 &= -CA_1^*
\end{aligned}
$$