Problem Set 3

1. (Walrasian Demand Function)

1(a)

$$\begin{aligned} p \cdot x &= w \\ p \cdot x(p,w) &= w \\ x(p,w) &= (p^{-1} \cdot w)^T \\ x(p,w) &= w \cdot (p^{-1})^T \end{aligned}$$

$$x(p,w) = \begin{pmatrix} x_1(p,w) \\ x_2(p,w) \\ \vdots \\ x_L(p,w) \end{pmatrix}$$

$$D_w x(p,w) = \nabla_w x(p,w) = p^{-1}$$

$$p \cdot D_w x(p,w) = p \cdot p^{-1} = 1$$

$$D_w x(p,w) = \frac{\partial x(p,w)}{\partial w}$$

Walras' Law :

$$p \cdot x(p,w) = w$$

Differentiate both sides with respect to w :

$$p \cdot D_w x(p,w) = 1$$

2(b)

$$\text{At least one good must be a normal good}$$
$$\Downarrow$$
$$\exists\, l : \frac{\partial x_l(p,w)}{\partial w} \geq 0$$

$$\text{negation:} \quad \forall\, l : \frac{\partial x_l(p,w)}{\partial w} < 0$$

$$\neg \exists\, l : \frac{\partial x_l(p,w)}{\partial w} \geq 0 \equiv \forall\, l : \frac{\partial x_l(p,w)}{\partial w} < 0$$

Assume that there is no such good $\forall\, l : \frac{\partial x_l(p,w)}{\partial w} < 0$.

$$p \cdot D_w x(p,w) = \underbrace{\sum_{l=1}^{L} p_l}_{\text{nonnegative}} \cdot \underbrace{\frac{\partial x_l(p,w)}{\partial w}}_{\text{negative}} \leq 0$$

In **2(a)** , we derived $p \cdot D_w x(p,w) = 1$. We have a contradiction.

Therefore, there must be at least one normal good.

2. (UMP and strictly convex preferences)

$$\begin{gathered} \succeq \text{ are strictly convex} \\ \Updownarrow \\ u(\cdot) \text{ is strictly quasi-concave} \\ \Downarrow \\ x(p,w) \text{ is single-valued} \end{gathered}$$

Prove by contradiction

Assume $\exists\, p, w : x_1, x_2 \in x(p,w), x_1 \neq x_2 \text{ but } u(x_1) = u(x_2)$

$$u(\lambda x_1 + (1-\lambda)x_2) > \min\{u(x_1), u(x_2)\} \quad \text{(by strict quasi-concavity)}$$

Check if $\lambda x_1 + (1-\lambda)x_2$ is affordable under p, w

$$p \cdot [\lambda x_1 + (1-\lambda)x_2] = \lambda \underbrace{p \cdot x_1}_{\leq w} + (1-\lambda) \cdot \underbrace{p \cdot x_2}_{\leq w} \leq w$$

I can always find a linear combination of x_1 and x_2 within the budget constraint such that this linear combination yields a higher utility.

Therefore, x_1, x_2 could never have been optimal (contradiction).

$$\forall\, p, w : x(p,w) \text{ is single-valued}$$

3. (Kuhn-Tucker)

Constant Elasticity of Substitution (CES)

$$u(x) = \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}}$$

3(a)

Convexity

$$\frac{\partial u(x)}{\partial x_l} = \frac{1}{\sigma}\left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}-1} \sigma \cdot x_l^{\sigma-1} = \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}-1} x_l^{\sigma-1}$$

$$\frac{\partial^2 u(x)}{\partial x_l^2} = (\frac{1}{\sigma}-1)\left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}-2} (\sigma-1)x_l^{\sigma-2} = -\frac{(1-\sigma)^2}{\sigma}\left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}-2} x_l^{\sigma-2} < 0$$

Homotheticity

$$u(\lambda x) = \left(\sum_{l=1}^{L} (\lambda x_l)^{\sigma}\right)^{\frac{1}{\sigma}} = \left(\lambda^{\sigma}\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}} = \lambda\left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}} = \lambda u(x)$$

Solution:

$$\text{strictly convex} \Longleftrightarrow \text{strictly quasiconcave}$$

We need to show that $u(\cdot)$ is strictly quasiconcave.

Let us do a monotonic transformation

$$f(x) = x^\sigma$$
$$v(x) = f(u(x)) = (u(x))^\sigma = \sum_{l=1}^{L} x_l^\sigma$$

$v(x)$ represents the same preferences as $u(x)$

We therefore need to show that $v(\cdot)$ is strictly quasiconcave.

$$\begin{gathered} x^\sigma \text{ is strictly concave } (\sigma \in (0,1)) \\ \Downarrow \\ \text{the sum of } x^\sigma \text{ is also strictly concave} \\ \Downarrow \\ v(x) \text{ is strictly concave} \\ \Downarrow \\ v(x) \text{ is strictly quasiconcave} \\ \Downarrow \\ u(x) \text{ is strictly quasiconcave} \\ \Downarrow \\ u(x) \text{ is strictly convex} \end{gathered}$$

3(b)

$$\begin{array}{rl} \max & u(x) \\ \text{s.t.} & p \cdot x \leq w, x_l \geq 0 \quad \forall l = 1, 2, \cdots, L \end{array}$$

$$L = \left(\sum_{l=1}^{L} x_l^\sigma \right)^{\frac{1}{\sigma}} - \lambda(p \cdot x - w) + \eta_1 \cdot x_1 + \cdots \eta_L \cdot x_L$$

FOC:

$$\left(\sum_{l=1}^{L} x_l^\sigma \right)^{\frac{1}{\sigma}-1} x_l^{\sigma-1} \leq \lambda p_l$$

$$x_l \left[\left(\sum_{l=1}^{L} x_l^\sigma \right)^{\frac{1}{\sigma}-1} x_l^{\sigma-1} - \lambda p_l \right] = 0$$

$$\lambda(p \cdot x - w) = 0$$

At the interior optimum: $x_l > 0 : \nabla u(x) = \lambda p$

$$\left(\sum_{l=1}^{L} x_l^\sigma \right)^{\frac{1}{\sigma}-1} x_l^{\sigma-1} = \lambda p_l$$

$$p \cdot x = w$$

Solution

$$\max \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}} \quad \text{s.t.} \quad p \cdot x \leq w$$

Monotonic transformation:

$$u(x) = \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}}$$

$$f(x) = \frac{x^{\sigma}}{\sigma}$$

$$v(x) = f(u(x)) = \frac{\sum_{l=1}^{L} x_l^{\sigma}}{\sigma} = \sum_{l=1}^{L} \frac{x_l^{\sigma}}{\sigma}$$

$$u(x) \text{ is strictly increasing in each argument}$$

$$\Downarrow$$

$$\text{Walras' Law holds} \quad p \cdot x = w$$

Lagrangian function:

$$L(x, \lambda) = \sum_{l=1}^{L} \frac{x_l^{\sigma}}{\sigma} - \lambda(\sum_{l=1}^{L} p_l \cdot x_l - w)$$

FOC:

$$\begin{cases} \dfrac{\partial L}{\partial x_i} = x_i^{\sigma-1} - \lambda p_i = 0 \\ \dfrac{\partial L}{\partial x_j} = x_j^{\sigma-1} - \lambda p_j = 0 \end{cases} \implies \begin{cases} x_i^{\sigma-1} = \lambda p_i \\ x_j^{\sigma-1} = \lambda p_j \end{cases} \implies x_i = \left(\frac{p_i}{p_j}\right)^{\frac{1}{\sigma-1}} x_j$$

Use budget constraint to pin down the demand function

$$p \cdot x = w$$

$$\sum_{i=1}^{L} p_i \cdot x_i = w$$

$$\sum_{i=1}^{L} p_i \cdot \left(\frac{p_i}{p_j}\right)^{\frac{1}{\sigma-1}} x_j = w$$

$$\sum_{i=1}^{L} p_i^{\frac{\sigma}{\sigma-1}} \cdot \frac{x_j}{p_j^{\frac{1}{\sigma-1}}} = w$$

$$\text{demand function:} \quad x_j = \frac{w \cdot p_j^{\frac{1}{\sigma-1}}}{\sum_{i=1}^{L} p_i^{\frac{\sigma}{\sigma-1}}}$$

As $\sigma \to 1$

$$\lim_{\sigma \to 1} \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}} = \sum_{l=1}^{L} x_l^{\sigma}$$

As $\sigma \to 0$, goods become perfect substitutes

$$f(x) = \ln x$$

$$v(x) = f(u(x)) = \ln \left(\sum_{l=1}^{L} x_l^{\sigma}\right)^{\frac{1}{\sigma}} = \frac{1}{\sigma} \ln \left(\sum_{l=1}^{L} x_l^{\sigma}\right)$$

$$
\begin{aligned}
\lim_{\sigma\to 0} v(x) &= \lim_{\sigma\to 0} \frac{1}{\sigma}\ln\left(\sum_{l=1}^{L} x_l^\sigma\right) \\
&= \lim_{\sigma\to 0} \frac{\sum_{l=1}^{L} x_l^\sigma \ln x_l}{\sum_{l=1}^{L} x_l^\sigma} \\
&= \frac{1}{L}\sum_{l=1}^{L} \ln x_l
\end{aligned}
$$

$$
\lim_{\sigma\to -\infty} u(x) = \min_x\{x_i\} \quad \text{(leontief utility function)}
$$

3(c)

$$
u(x_1, x_2) = (x_1^\sigma + x_2^\sigma)^{1/\sigma} \quad x_1, x_2 \geq 0
$$

The Kuhn-Tucker conditions:

$$
\begin{cases}
(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_1^{\sigma-1} \leq \lambda p_1 \\
(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_2^{\sigma-1} \leq \lambda p_2 \\
x_1[(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_1^{\sigma-1} - \lambda p_1] = 0 \\
x_2[(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_2^{\sigma-1} - \lambda p_2] = 0 \\
\lambda(p_1 x_1 + p_2 x_2 - w) = 0
\end{cases}
$$

$x_1, x_2 > 0$:

$$
\begin{cases}
(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_1^{\sigma-1} - \lambda p_1 = 0 \\
(x_1^\sigma + x_2^\sigma)^{\frac{1}{\sigma}-1} x_2^{\sigma-1} - \lambda p_2 = 0
\end{cases}
\implies \frac{\lambda p_1}{x_1^{\sigma-1}} = \frac{\lambda p_2}{x_2^{\sigma-1}}
$$

Solution

$$
x_j = \frac{w \cdot p_j^{\frac{1}{\sigma-1}}}{\sum_{i=1}^{L} p_i^{\frac{\sigma}{\sigma-1}}}
$$

$$
x_1 = \frac{w \cdot p_1^{\frac{1}{\sigma-1}}}{p_1^{\frac{\sigma}{\sigma-1}} + p_2^{\frac{\sigma}{\sigma-1}}}
$$

$$
\frac{\partial x_1(p, w)}{p_2} = (-1) \cdot w \cdot p_1^{\frac{1}{\sigma-1}} \cdot \frac{1}{(p_1^{\frac{\sigma}{\sigma-1}} + p_2^{\frac{\sigma}{\sigma-1}})^2} \cdot \frac{\sigma}{\sigma - 1} \cdot p_2^{\frac{\sigma}{\sigma-1}-1} > 0
$$

x_1, x_2 are gross substitutes.

3(d)

$x_1(p, w), x_2(p, w)$ are linear in w

$$
\frac{x_1(p, w)}{x_2(p, w)} \text{ does not depend on } w
$$

4. (Multiple Optima)

4(a)

$$u(x) = \begin{cases} x, & x \in [0,1) \\ 1, & x \geq 1 \end{cases}$$

Continuity

$$\begin{cases} f(x) = x \text{ is continuous} \\ f(x) = 1 \text{ is continuous} \end{cases}$$

$$\begin{cases} \lim\limits_{x \to 1-} u(x) = \lim\limits_{x \to 1-} x = 1 \\ \lim\limits_{x \to 1+} u(x) = \lim\limits_{x \to 1+} 1 = 1 \end{cases} \implies \lim_{x \to 1} u(x) = 1 \quad u(x) \text{ is continuous at } x = 1$$

Convexity (Quasiconcavity)

$$f(x) = x \text{ is convex}$$
$$f(x) = 1 \text{ is convex}$$

Check $y < 1 < x$

$$u(\lambda x + (1-\lambda)y) \geq u(\lambda y + (1-\lambda)y) = u(y) \implies \text{quasicavity} \implies \text{convex}$$

LNS

$$x > 1, \nexists\, \varepsilon > 0, x' \in [x-\varepsilon, x+\varepsilon] \text{ such that } u(x') > u(x) = 1$$

$\succeq$ is not locally non-satiated.

4(b)

$$\max u(x) \quad \text{s.t.} \quad p \cdot x \leq w \implies x(p,w) = \begin{cases} \dfrac{w}{p}, & \dfrac{w}{p} < 1 \\ [1, \dfrac{w}{p}], & \dfrac{w}{p} \geq 1 \end{cases}$$

Zero-homothetic

$$x(p,w) = \begin{cases} \dfrac{\lambda w}{\lambda p}, & \dfrac{\lambda w}{\lambda p} < 1 \\ [1, \dfrac{\lambda w}{\lambda p}], & \dfrac{\lambda w}{\lambda p} \geq 1 \end{cases}$$

5. (Homothetic Preferences)

5(a)

$$x_1 p_1 + x_2 p_2 = w$$

$$MRS(x_1, x_2) = \frac{p_1}{p_2}$$

$$MRS(x_1, x_2) = MRS(\lambda x_1, \lambda x_2)$$

5(b)

Prove

$$x(p, \lambda w) = \lambda x(p, w)$$

Prove by contradiction

Assume $x \neq \lambda x'$ and $x \succ \lambda x^*$, x^* is optimal

$$x \succ \lambda x^* \implies u(x) > u(\lambda x^*)$$
$$u(x) > \lambda u(x^*)$$
$$\frac{1}{\lambda} u(x) > u(x^*)$$
$$u(\frac{x}{\lambda}) > u(x^*)$$
$$\frac{x}{\lambda} \succeq x^*$$

$$p \cdot \frac{x}{\lambda} = p \cdot \lambda x^* \cdot \frac{1}{\lambda} = w \quad \frac{x^*}{\lambda} \text{ is feasible under } p, w$$

We find a bundle $\dfrac{x^*}{\lambda}$ that yields higher utility within the budget constraint. Therefore, x^* could not have been optimal.