Exam
Social Choice Theory
Spring 2019
Solution

Problem 1: May's Theorem

(a) For $\boldsymbol{\alpha} = (+1, 0, -1, +1, +1)$ we obtain $\overline{\alpha}(\boldsymbol{\alpha}) = 2/5$.

Hence it follows that $f(\boldsymbol{\alpha}) = +1$ (a strict preference for x over y) if $\tilde{\alpha} < 2/5$, and $f(\boldsymbol{\alpha}) = 0$ (an indifference between x and y) if $\tilde{\alpha} \geq 2/5$.

(b) See lecture slides.

(c) [U] This axiom is always satisfied, because the method is applicable to all $\boldsymbol{\alpha}$ and always delivers a preference $f(\boldsymbol{\alpha}) \in \{-1, 0, +1\}$.

[N] It holds that $\overline{\alpha}(-\boldsymbol{\alpha}) = -\overline{\alpha}(\boldsymbol{\alpha})$.

In case $f(\boldsymbol{\alpha}) = +1$, which is equivalent to $\overline{\alpha}(\boldsymbol{\alpha}) > +\tilde{\alpha}$, we get $\overline{\alpha}(-\boldsymbol{\alpha}) = -\overline{\alpha}(\boldsymbol{\alpha}) < -\tilde{\alpha}$, which implies $f(-\boldsymbol{\alpha}) = -1$.

In case $f(\boldsymbol{\alpha}) = -1$, we get $f(-\boldsymbol{\alpha}) = +1$ by an analogous argument.

In the remaining case $f(\boldsymbol{\alpha}) = 0$, which is equivalent to $-\tilde{\alpha} \leq \overline{\alpha}(\boldsymbol{\alpha}) \leq +\tilde{\alpha}$, we obtain $+\tilde{\alpha} \geq \overline{\alpha}(-\boldsymbol{\alpha}) \geq -\tilde{\alpha}$ and hence $f(-\boldsymbol{\alpha}) = 0$.

Therefore, the axiom is always satisfied.

[PR] Note first that $\overline{\alpha}(\boldsymbol{\alpha})$ takes values in increments of $1/n$, i.e., around zero the possible values of $\overline{\alpha}(\boldsymbol{\alpha})$ are $\ldots, -2/n, -1/n, 0, +1/n, +2/n, \ldots$

Now consider some $\boldsymbol{\alpha}$ such that $f(\boldsymbol{\alpha}) = 0$, which is equivalent to $-\tilde{\alpha} \leq \overline{\alpha}(\boldsymbol{\alpha}) \leq +\tilde{\alpha}$.

If $\tilde{\alpha} < 1/n$, the only case in which this can hold is when $\overline{\alpha}(\boldsymbol{\alpha}) = 0$. If support for x increases towards $\boldsymbol{\alpha}'$ (in the sense of the axiom), we then obtain $\overline{\alpha}(\boldsymbol{\alpha}') \geq 1/n$ and $f(\boldsymbol{\alpha}') = +1$. Therefore, the axiom is satisfied when $\tilde{\alpha} < 1/n$. Note that the method is equivalent to majority voting in that case.

If $\tilde{\alpha} \geq 1/n$, consider profile $\boldsymbol{\alpha} = (-1, 0, \ldots, 0)$ for which $\overline{\alpha}(\boldsymbol{\alpha}) = -1/n$ and thus $f(\boldsymbol{\alpha}) = 0$. For $\boldsymbol{\alpha}' = (0, 0, \ldots, 0)$ we then obtain $\overline{\alpha}(\boldsymbol{\alpha}') = 0$ and thus $f(\boldsymbol{\alpha}') = 0$, in contradiction to the axiom. Therefore, the axiom is violated when $\tilde{\alpha} \geq 1/n$.

[A] This axiom is always satisfied, because only the sum $\sum_{i=1}^{n} \alpha_i$ matters.

Problem 2: Manipulability

(a) [$\bar{\text{U}}$] This axiom is satisfied (for $\mathscr{A} = \mathscr{P}^2$) by definition of the method, no matter how $*$ is replaced.

[$\bar{\text{M}}$] If $* = z$, the axiom is satisfied. We only need to check preference changes of voter 1, because voter 2 does not matter in that case. Since the winning alternative is always the top-ranked alternative of voter 1, maintaining position means that it remains top-ranked, and hence it remains the winner.

If $* = x$, the axiom is violated. Start from $R_1 = zyx$ and $R_2 = zyx$, so that x wins. When voter 1's preference changes to $R_1' = zxy$, x maintains its position but no longer wins.

If $* = y$, the axiom is violated. Start from $R_1 = zyx$ and $R_2 = zyx$, so that y wins. When voter 2's preference changes to $R_2' = yzx$, y maintains its position but no longer wins.

[$\bar{\text{P}}$] If $* = z$, the axiom is satisfied. Since the winning alternative is always the top-ranked alternative of voter 1 in that case, it cannot be strongly Pareto-dominated.

If $* = x$ or $* = y$, the axiom is violated, because when $R_1 = zyx$ and $R_2 = zyx$, the winner is strongly Pareto-dominated by z.

[$\bar{\text{D}}$] If $* = z$, the axiom is violated because voter 1 is a dictator.

If $* = x$ or $* = y$, the axiom is satisfied, because voter 1 can no longer enforce her top-ranked alternative when $R_1 = zyx$ and $R_2 = zyx$.

(b) The method is surjective no matter how $*$ is specified.

[$\bar{\text{U}}$] See part (a).

[$\bar{\text{D}}$] See part (a).

[$\bar{\text{S}}$] If $* = z$, the axiom is satisfied, because a dictatorship is strategy-proof.

If $* = x$ or $* = y$, the axiom is violated, because when $R_1 = zyx$ and $R_2 = zyx$, voter 1 can benefit from unilaterally misreporting to have preference $R_1' = zxy$ instead.

(c) This method does not contradict the Gibbard-Satterthwaite theorem. It is not an SCF $c : \mathscr{A} \to X$, because its outcome is not deterministic. Hence the Gibbard-Satterthwaite theorem does not apply.

The method is a stochastic SCF $\tilde{c} : \mathscr{A} \to \Delta X$, which generates a distribution of winning alternatives. It is a random dictatorship.

Problem 3: Social Evaluation Functions

(a) The following table computes NP for each alternative in the example:

	v	w	x	y	z
U_1	1	1	5	5	2
U_2	2	2	2	2	4
U_3	3	4	1	0	4
U_4	5	5	6	10	3
U_5	6	6	4	4	2
$U_{i_m}/2$	3/2	2	2	2	3/2
NP	1	2	2	2	0

Hence we obtain $z \; e_P^{\text{NP}}(\mathbf{U}) \; v \; e_P^{\text{NP}}(\mathbf{U}) \; w \; e_I^{\text{NP}}(\mathbf{U}) \; x \; e_I^{\text{NP}}(\mathbf{U}) \; y$.

(b)
- The SEF is not consistent with RM-NC (and hence not with CM-NC and OM-NC). In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by multiplying citizen 1's utility by 1/2 and leaving everything else unchanged. We obtain $e^{\text{NP}}(\mathbf{U}) \neq e^{\text{NP}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	1	2
U_2	3	3
U_3	4	4
$U_{i_m}/2$	3/2	3/2
NP	1	0

$\mathbf{U}'$	x	y
U'_1	1/2	1
U'_2	3	3
U'_3	4	4
$U'_{i_m}/2$	3/2	3/2
NP	1	1

- The SEF is not consistent with CM-LC (and hence not with CM-UC and OM-LC). In the following example, $\mathbf{U}'$ is obtained from $\mathbf{U}$ by adding 3 to every citizen's utility. We obtain $e^{\text{NP}}(\mathbf{U}) \neq e^{\text{NP}}(\mathbf{U}')$.

$\mathbf{U}$	x	y
U_1	1	2
U_2	3	3
U_3	4	4
$U_{i_m}/2$	3/2	3/2
NP	1	0

$\mathbf{U}'$	x	y
U'_1	4	5
U'_2	6	6
U'_3	7	7
$U'_{i_m}/2$	3	3
NP	0	0

- The SCF is consistent with RM-LC. Suppose $\mathbf{U}'$ is obtained from $\mathbf{U}$ by a common positive linear transformation $\varphi(u) = \beta u$, where $\beta > 0$. For each alternative x, we then obtain $i_m(x, \mathbf{U}) = i_m(x, \mathbf{U}')$. This implies $U'_{i_m(x,\mathbf{U}')}(x) = \beta U_{i_m(x,\mathbf{U})}(x)$. The condition

$$U_i(x) \leq \frac{1}{2} U_{i_m(x,\mathbf{U})}(x)$$

in the definition of NP is therefore equivalent to

$$U'_i(x) \leq \frac{1}{2} U'_{i_m(x,\mathbf{U}')}(x),$$

which implies $\text{NP}(x, \mathbf{U}) = \text{NP}(x, \mathbf{U}')$.