Problem Set 11

1. Partial Equilibrium

$$\begin{cases} u = m + \ln x & \text{utility of a single consumer} \\ \omega = 1 & \text{endowment of numeraire} \\ N & \text{number of firms} \\ x & \text{quantity of goods procuced by } N \text{ firms} \\ z_j & \text{amount of numeraire used by firm } j \text{ in the production} \\ q_j = \sqrt{z_j} & \text{technology} \\ p & \text{price of the consumption goods} \end{cases}$$

- **Consumer problem:**

$$\begin{aligned} \max_{m,x} \quad & m + \ln x \\ \text{s.t.} \quad & m + p \cdot x \leq 1 \cdot \omega + p \cdot 0 + \Pi \end{aligned}$$

  Constraint must hold with equality due to monotone preference

$$\max_{x \geq 0} \quad 1 + \Pi - p \cdot x + \ln x$$

$$\text{FOC:} \quad -p + \frac{1}{x} = 0 \implies x = \frac{1}{p} \quad \text{demand is independent of wealth}$$

- **Firm problem:**

$$\text{Cost of producing quantity } q_j\text{:} \quad c(q) = q_j^2$$

$$\Pi = N \cdot (p \cdot q_j - q_j^2)$$

$$\max_{q_j} \quad p \cdot q_j - q_j^2$$

$$\text{FOC:} \quad p - 2q_j = 0 \implies q_j = \frac{p}{2}$$

- **Market clearing:**

$$\text{Aggregate production:} \quad q = \sum_{j=1}^{N} q_j = \sum_{j=1}^{N} \frac{p}{2} = N \cdot \frac{p}{2}$$

$$\begin{aligned} \text{demand} &= \text{supply} \\ x &= q \\ \frac{1}{p} &= N \cdot \frac{p}{2} \\ p^* &= \sqrt{\frac{2}{N}} \end{aligned}$$

$$x^* = \frac{1}{p^*} = \sqrt{\frac{N}{2}}$$
$$q_j^* = \frac{p^*}{2} = \frac{1}{\sqrt{2N}}$$
$$z_j^* = (q_j^*)^2 = \frac{1}{2N}$$
$$\Pi^* = N \cdot (p^* \cdot q_j^* - (q_j^*)^2) = N\left(\sqrt{\frac{2}{N}} \cdot \frac{1}{\sqrt{2N}} - \frac{1}{2N}\right) = \frac{1}{2}$$
$$m^* = \Pi^* + \omega - p^* \cdot x^* = \frac{1}{2} + 1 - \sqrt{\frac{2}{N}} \cdot \sqrt{\frac{N}{2}} = \frac{1}{2}$$

2. Partial Equilibrium: Pareto Set

- There are $I = 2$ consumers with utility $u_i = m_i + \phi_1(x)$, where $\phi_i' > 0$ and $\phi_i'' < 0$.
- $J = 1$ firm uses the numeraire good to produce output y_i with cost function $c_j(y_j) \geq 0$, where $c_j' c_j'' > 0$.
- $c_j(y_j)$ corresponds to the quantity of input used by firm j to produce $y_j \geq 0$ units of output.
- The aggregate endowment of the production good (numeraire) is $\omega > 0$.

2(a)

$$\max_{m_1, x_1, m_2, x_2} \quad m_1 + \phi_1(x_1)$$
$$\text{s.t.} \quad \begin{cases} m_2 + \phi_2(x_2) \geq u_2 & (\lambda) \\ x_1 + x_2 \leq y_j & (\mu) \\ m_1 + m_2 + c_j(y_j) \leq \omega & (\eta) \end{cases}$$

$$\mathcal{L} = m_1 + \phi_1(x_1) - \lambda(u_2 - m_2 - \phi_2(x_2)) - \mu(x_1 + x_2 - y_j) - \eta(m_1 + m_2 + c_j(y_j) - \omega)$$

FOC:

$$\begin{cases} \dfrac{\partial \mathcal{L}}{\partial m_1} = 1 - \eta = 0 \\ \dfrac{\partial \mathcal{L}}{\partial m_2} = \lambda - \eta = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_1} = \phi_1'(x_1) - \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial x_2} = \phi_2'(x_2) - \mu = 0 \\ \dfrac{\partial \mathcal{L}}{\partial y_i} = \mu - \eta c_j'(y_j) = 0 \\ \dfrac{\partial \mathcal{L}}{\partial \mu} = x_1 + x_2 - y_j = 0 \quad \text{interior solution} \end{cases} \implies \begin{cases} 1 = \eta \\ \lambda = \eta \\ \phi_1'(x_1) = \mu \\ \phi_2'(x_2) = \mu \\ \mu = \eta c_j'(y_j) \\ x_1 + x_2 = y_j \end{cases} \implies \begin{cases} \phi_1'(x_1) = \phi_2'(x_2) = c_j'(y_j) \\ x_1 + x_2 = y_j \end{cases}$$

2(b)

$$\max_{\substack{m_1, x_1 \\ m_2, x_2}} \quad W(u_1, u_2) \quad \text{s.t.} \quad \begin{cases} x_1 + x_2 \leq y_j & (\mu) \\ m_1 + m_2 + c_j(y_j) \leq \omega & (\eta) \end{cases}$$

$$\begin{cases} u_1 = m_1 + \phi_1(x_1) \\ u_2 = m_2 + \phi_2(x_2) \end{cases} \implies W(m_1 + \phi_1(x_1), m_2 + \phi_2(x_2))$$

$$\mathcal{L} = W(m_1 + \phi_1(x_1), m_2 + \phi_2(x_2)) - \mu(x_1 + x_2 - y_j) - \eta(m_1 + m_2 + c_j(y_j) - \omega)$$

FOC:

$$
\begin{aligned}
(m_1): \quad & \frac{\partial \mathcal{L}}{\partial m_1} = \frac{\partial W}{\partial u_1} - \eta = 0 \\
(m_2): \quad & \frac{\partial \mathcal{L}}{\partial m_2} = \frac{\partial W}{\partial u_2} - \eta = 0 \\
(x_1): \quad & \frac{\partial \mathcal{L}}{\partial x_1} = \frac{\partial W}{\partial u_1}\phi_1'(x_1) - \mu = 0 \\
(x_2): \quad & \frac{\partial \mathcal{L}}{\partial x_2} = \frac{\partial W}{\partial u_2}\phi_2'(x_2) - \mu = 0 \\
(y_j): \quad & \frac{\partial \mathcal{L}}{\partial y_j} = \mu - \eta c_j'(y_j) = 0 \\
(\mu): \quad & x_1 + x_2 - y_j = 0
\end{aligned}
$$

$$
\begin{cases}
\frac{\partial W}{\partial u_1} - \eta = 0 \\
\frac{\partial W}{\partial u_2} - \eta = 0
\end{cases}
\implies \frac{\partial W}{\partial u_1} = \frac{\partial W}{\partial u_2}
$$

$$
\begin{cases}
\frac{\partial W}{\partial u_1}\phi_1'(x_1) - \mu = 0 \\
\frac{\partial W}{\partial u_2}\phi_2'(x_2) - \mu = 0
\end{cases}
\implies \frac{\partial W}{\partial u_1}\phi_1'(x_1) = \frac{\partial W}{\partial u_2}\phi_2'(x_2) \implies \phi_1'(x_1) = \phi_2'(x_2)
$$

$$
c_j'(y_j) = \frac{\mu}{\eta} = \phi_1'(x_1) = \phi_2'(x_2)
$$

The optimal allocation is a Pareto optimum.

3. Partial Equilibrium: Unconditional Basic Income

- There are $I > 1$ consumers with utility $u_i = m_i + \alpha_i \ln x_i$, where $\alpha_i > 0$ is a parameter.
- Each consumer holds one unit of the numeraire $(\omega_i = 1)$.
- There is a single firm with cost function $C(y) = y^2$.
- $p > 0$ denotes the price of consumption good.
- The firm raises a revenue py .
- Total tax income is $T = \tau py, \tau \in [0, 1]$.
- Each consumer receives an income $\frac{T}{I}$.

3(a)

Consumer problem:

$$
\begin{aligned}
\max_{x_i, m_i} \quad & m_i + \alpha_i \ln x_i \\
\text{s.t.} \quad & m_i + p \cdot x_i \le \omega_i + \theta_i \Pi + \frac{T}{I}
\end{aligned}
$$

$$
\mathcal{L} = m_i + \alpha_i \ln x_i - \lambda(m_i + p \cdot x_i - 1 - \theta_i \Pi - \frac{T}{I})
$$

FOC:

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial m_i} &= 1 - \lambda = 0 \implies \lambda = 1 \\
\frac{\partial \mathcal{L}}{\partial x_i} &= \frac{\alpha_i}{x_i} - \lambda p = 0 \implies x_i = \frac{\alpha_i}{p}
\end{aligned}
$$

Firm problem:

$$\max_{y} \quad py - C(y) - T$$

$$\mathcal{L} = (1-\tau)py - y^2$$

$$\frac{\partial \mathcal{L}}{\partial y} = (1-\tau)p - 2y = 0 \implies y = \frac{1}{2}(1-\tau)p$$

Market clearing:

$$\begin{aligned} y &= \sum_{i=1}^{I} x_i \\ \frac{1}{2}(1-\tau)p &= \frac{\sum_{i=1}^{I} \alpha_i}{p} \\ p^2 &= \frac{2\sum_{i=1}^{I} \alpha_i}{1-\tau} \\ p^* &= \sqrt{\frac{2\sum_{i=1}^{I} \alpha_i}{1-\tau}} \end{aligned}$$

$$x_i^* = \frac{\alpha_i}{p^*} = \alpha_i \sqrt{\frac{1-\tau}{2\sum_{i=1}^{I} \alpha_i}}$$

$$y^* = \frac{1}{2}(1-\tau)p^* = \sqrt{\frac{1-\tau}{2}\sum_{i=1}^{I} \alpha_i}$$

$$\begin{aligned} \Pi^* &= (1-\tau)p^*y^* - (y^*)^2 \\ &= (1-\tau)\sqrt{\frac{2\sum_{i=1}^{I} \alpha_i}{1-\tau}}\sqrt{\frac{1-\tau}{2}\sum_{i=1}^{I} \alpha_i} - \frac{1-\tau}{2}\sum_{i=1}^{I} \alpha_i \\ &= \frac{1-\tau}{2}\sum_{i=1}^{I} \alpha_i \end{aligned}$$

$$T^* = \tau p y = \tau \sum_{i=1}^{I} \alpha_i$$

3(b)

$$\begin{aligned} x_i(\tau): &\quad \text{equilibrium consumption of consumer } i \\ X(\tau): &\quad \text{aggregate consumption} \\ s_i^*(\tau) = \frac{x_i(\tau)}{X(\tau)}: &\quad \text{relative consumption inequality} \\ x_i(\tau) - x_{i'}(\tau): &\quad \text{absolute consumption inequality} \end{aligned}$$

In equilibrium, we have

$$x_i(\tau) = \alpha_i \sqrt{\frac{1-\tau}{2\sum_i \alpha_i}}$$

$$X(\tau) = \sum_i x_i(\tau) = \sum_i \alpha_i \sqrt{\frac{1-\tau}{2\sum_i \alpha_i}}$$

Relative consumption inequality

$$s_i^*(\tau) = \frac{x_i(\tau)}{X(\tau)} = \frac{\alpha_i}{\sum_i \alpha_i} \quad \text{indepedent of } \tau$$

Absolute consumption inequality

$$x_i(\tau) - x_{i'}(\tau) = (\alpha_i - \alpha_{i'})\sqrt{\frac{1-\tau}{2\sum_i \alpha_i}} \quad \text{decreasing in } \tau \text{ if } \alpha_i > \alpha_{i'}$$

Explanation:

Price increase due to a higher tax leads to less consumption and more so for consumption intense consumers while relative consumption levels remain the same.

3(c)

$$\text{Equilibrium income:} \quad w_i(\tau) = 1 + \theta_i \Pi + \frac{T}{I} \quad \text{where } \theta \in [0,1]$$

$$\begin{aligned} w_i(\tau) &= 1 + \theta_i \Pi + \frac{T}{I} \\ &= 1 + \theta_i \frac{1-\tau}{2} \sum_{i=1}^{I} \alpha_i + \frac{\tau}{I} \sum_{i=1}^{I} \alpha_i \\ &= 1 + \frac{1}{2}\theta_i \sum_{i=1}^{I} \alpha_i + \tau \sum_{i=1}^{I} \alpha_i \left(\frac{1}{I} - \frac{\theta_i}{2}\right) \end{aligned}$$

Absolute income inequality:

$$w_i(\tau) - w_{i'}(\tau) = \frac{1}{2}(\theta_i - \theta_{i'}) \sum_{i=1}^{I} \alpha_i - \frac{\tau}{2}(\theta_i - \theta_{i'}) \sum_{i=1}^{I} \alpha_i = \frac{1-\tau}{2}(\theta_i - \theta_{i'}) \sum_{i=1}^{I} \alpha_i$$

Relative income inequality:

$$\begin{aligned} \sum_{i=1}^{I} w_i(\tau) &= \sum_{i=1}^{I} \left(1 + \theta_i \frac{1-\tau}{2} \sum_{i=1}^{I} \alpha_i + \frac{\tau}{I} \sum_{i=1}^{I} \alpha_i\right) \\ &= I + \frac{1-\tau}{2} \sum_{i=1}^{I} \alpha_i + \tau \sum_{i=1}^{I} \alpha_i \\ &= I + \frac{1+\tau}{2} \sum_{i=1}^{I} \alpha_i \quad \text{increasing in } \tau \end{aligned}$$

$$w_i(\tau) = 1 + \theta_i \frac{1-\tau}{2} \sum_{i=1}^{I} \alpha_i + \frac{\tau}{I} \sum_{i=1}^{I} \alpha_i \quad \text{can be decreasing in } \tau \text{ for high } \theta_i$$

$$\frac{w_i(\tau)}{\sum_{i=1}^{I} w_i(\tau)} \quad \text{decreasing in } \tau \text{ for high } \theta_i$$

$$\text{less income inequality in relative terms (depending on firm share } \theta_i)$$

Do all consumers benefit?

$$
\begin{aligned}
m_i &= \omega_i + \theta_i \Pi + \frac{T}{I} - p \cdot x_i \\
&= 1 + \theta_i \frac{1-\tau}{2} \sum_{i=1}^{I} \alpha_i + \frac{\tau}{I} \sum_{i=1}^{I} \alpha_i - \sum_{i=1}^{I} \alpha_i \\
&= 1 + \left(\theta_i \frac{1-\tau}{2} + \frac{\tau}{I} - 1 \right) \sum_{i=1}^{I} \alpha_i
\end{aligned}
$$

$$
\begin{aligned}
u_i &= m_i + \alpha_i \ln(x_i) \\
&= 1 + \left(\theta_i \frac{1-\tau}{2} + \frac{\tau}{I} - 1 \right) \sum_{i=1}^{I} \alpha_i + \alpha_i \ln \left(\alpha_i \sqrt{\frac{1-\tau}{2 \sum_i \alpha_i}} \right)
\end{aligned}
$$

$$
\frac{\partial u_i}{\partial \tau} = \left(\frac{1}{I} - \frac{\theta_i}{2} \right) \sum_{i=1}^{I} \alpha_i - \frac{\alpha_i}{2(1-\tau)}
$$

$$
\frac{\partial u_i}{\partial \tau} = \underbrace{\left(\frac{1}{I} - \frac{\theta_i}{2} \right) \sum_{i=1}^{I} \alpha_i}_{\Delta W} - \frac{\alpha_i}{2(1-\tau)}
$$

Income can decrease for high θ_1-types. Not everyone benefits from unconditional basic income.

(d)

$$
\begin{aligned}
U = \sum_{i=1}^{I} u_i &= I + \left(\frac{1-\tau}{2} + \tau - I \right) \sum_{i=1}^{I} \alpha_i + \sum_{i=1}^{I} \alpha_i \ln \left(\alpha_i \sqrt{\frac{1-\tau}{2 \sum_i \alpha_i}} \right) \\
&= I + \left(\frac{1+\tau}{2} - I \right) \sum_{i=1}^{I} \alpha_i + \sum_{i=1}^{I} \alpha_i \ln \left(\alpha_i \sqrt{\frac{1-\tau}{2 \sum_i \alpha_i}} \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial U}{\partial \tau} &= \frac{1}{2} \sum_{i=1}^{I} \alpha_i + \frac{1}{2(1-\tau)} \sum_{i=1}^{I} \alpha_i \\
&= \left(\frac{1}{2} - \frac{1}{2(1-\tau)} \right) \sum_{i=1}^{I} \alpha_i \quad \text{Total welfare decreases in } \tau
\end{aligned}
$$

Reason:

Higher price due to higher tax leads to lower consumption of all consumers which dominates the income effect.