Problem Set 2

Exercise 1: Basic Model

A small open economy inhabited by a representative household that maximizes a two-period lifetime utility function

$$U(C_1, C_2) = u(C_1) + \beta u(C_2)$$

subject to the intertemporal budget constraint

$$C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

1(a)

$$\underbrace{C_1 + \frac{C_2}{1+r}}_{\text{present value of lifetime consumption}} \quad = \quad \underbrace{Y_1 + \frac{Y_2}{1+r}}_{\text{present value of lifetime income}}$$

Intuition: the present value of consumption in two periods equals the present value of income in two periods.

1(b)

The representative household's optimization problem

$$\begin{aligned} \max_{C_1, C_2} \quad & \ln(C_1) + \beta \ln(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

$$\mathcal{L} = \ln(C_1) + \beta \ln(C_2) - \lambda \left(C_1 + \frac{C_2}{1+r} - Y_1 - \frac{Y_2}{1+r} \right)$$

FOCs:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial C_1} &= \frac{1}{C_1} - \lambda = 0 \\ \frac{\partial \mathcal{L}}{\partial C_2} &= \frac{\beta}{C_2} - \frac{\lambda}{1+r} = 0 \end{aligned} \implies \beta \frac{C_1}{C_2} = \frac{1}{1+r}$$

We know that

$$\overline{W} = Y_1 + \frac{Y_2}{1+r}$$

$$\text{plug} \quad \begin{cases} \beta \frac{C_1}{C_2} = \frac{1}{1+r} \\ \overline{W} = Y_1 + \frac{Y_2}{1+r} \end{cases} \quad \text{into the IBC} \quad C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r}$$

We obtain

$$\begin{aligned} C_1 &= \frac{1}{1+\beta} \overline{W} \\ C_2 &= \frac{\beta}{1+\beta} (1+r) \overline{W} \end{aligned}$$

1(c)

Primary current account in the first period

$$
\begin{aligned}
PCA_1 &= Y_1 - C_1 \\
&= Y_1 - \frac{1}{1+\beta}\overline{W} \\
&= Y_1 - \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right)
\end{aligned}
$$

$$
\begin{aligned}
PCA_2 &= Y_2 - C_2 \\
&= Y_2 - \frac{\beta}{1+\beta}(1+r)\overline{W} \\
&= Y_2 - \frac{\beta}{1+\beta}(1+r)\left(Y_1 + \frac{Y_2}{1+r}\right)
\end{aligned}
$$

$$PCA_1 + \frac{PCA_2}{1+r} = Y_1 - \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right) + \frac{Y_2}{1+r} - \frac{\beta}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right) = 0$$

$$PCA_1 = -\frac{PCA_2}{1+r}$$

1(d)

$$
\begin{aligned}
PCA_1 &= Y_1 - \frac{1}{1+\beta}\left(Y_1 + \frac{Y_2}{1+r}\right) \\
&= \frac{\beta}{1+\beta}Y_1 - \frac{1}{(1+\beta)(1+r)}Y_2
\end{aligned}
$$

Take the partial derivative of CA_1 with respect to Y_1

$$\frac{\partial PCA_1}{\partial Y_1} = \frac{\beta}{1+\beta} \in (0,1)$$

The primary current account today will react in the same direction as the income changes.

1(e)

From 1(b), we derived

$$\beta\frac{C_1}{C_2} = \frac{1}{1+r}$$

$$\begin{cases} \beta\frac{C_1}{C_2} = \frac{1}{1+r} \\ \beta = \frac{1}{1+r} \end{cases} \implies C_1 = C_2$$

This implies complete consumption smoothing.

If $\beta > \frac{1}{1+r}$,

$$\begin{cases} \beta\frac{C_1}{C_2} = \frac{1}{1+r} \\ \beta > \frac{1}{1+r} \end{cases} \implies C_1 < C_2$$

β measures how patient the agent is.

Exercise 2: Elasticity of Intertemporal Substitution (EIS)

2(a)

$$U = U(C_1, C_2) = u(C_1) + \beta u(C_2)$$

$$\begin{cases} U_1 = \dfrac{\partial U(C_1, C_2)}{\partial C_1} = u'(C_1) \\ U_2 = \dfrac{\partial U(C_1, C_2)}{\partial C_2} = \beta u'(C_2) \end{cases}$$

The inverse of the curvature of the utility function

$$\sigma(C) = -\frac{d\left(\frac{C_2}{C_1}\right)/\frac{C_2}{C_1}}{d\left(\frac{U_2}{U_1}\right)/\frac{U_2}{U_1}} = -\frac{d\log\left(\frac{C_2}{C_1}\right)}{d\log\left(\frac{U_2}{U_1}\right)} = -\frac{d\log\left(\frac{C_2}{C_1}\right)}{d\log\left(\frac{\beta u'(C_2)}{u'(C_1)}\right)}$$

Elasticity of Intertemporal Substitution

Assumptions:

- The utility function is homothetic
- The utility function is time separable
- The change only comes from C_2 while C_1 remains the same (consider C_1 as a constant)

$$\begin{aligned} \sigma(C_2) &= -\frac{d\left(\frac{C_2}{C_1}\right)/\frac{C_2}{C_1}}{d\left(\frac{U_2}{U_1}\right)/\frac{U_2}{U_1}} \\ &= -\frac{dC_2/C_2}{dU_2/U_2} \\ &= -\frac{dC_2}{dU_2} \cdot \frac{U_2}{C_2} \\ &= -\frac{1}{\beta u''(C_2)} \cdot \frac{\beta u'(C_2)}{C_2} \\ &= -\frac{u'(C_2)}{C_2 \cdot u''(C_2)} \end{aligned}$$

$$\sigma(C) = \sigma = -\frac{u'(C)}{C \cdot u''(C)}$$

Relative Risk Aversion (RRA)

$$\begin{cases} R(c) & \text{Relative risk aversion} \\ A(c) & \text{Absolute risk aversion} \end{cases}$$

$$R(c) = cA(c) = -\frac{C \cdot u''(C)}{u'(C)}$$

2(b)

$$U(C_1, C_2) = u(C_1) + \beta u(C_2)$$

$$u(C_t) = \begin{cases} \frac{C_t^{1-\rho} - 1}{1-\rho} & \rho \neq 1 \\ \ln(C_t) & \rho = 1 \end{cases}$$

$$u'(C_t) = \begin{cases} C_t^{-\rho} & \rho \neq 1 \\ (C_t)^{-1} & \rho = 1 \end{cases} \implies u'(C_t) = (C_t)^{-\rho} \quad \forall \rho$$

$$u''(C_t) = -\rho(C_t)^{-\rho-1}$$

$$\sigma(C_t) = -\frac{u'(C_t)}{C_t \cdot u''(C_t)} = -\frac{(C_t)^{-\rho}}{-C_t \cdot \rho(C_t)^{-\rho-1}} = \frac{1}{\rho}$$

2(c)

Households are utility maximizing:

$$\begin{aligned} \max_{C_1, C_2} \quad & u(C_1) + \beta u(C_2) \\ \text{s.t.} \quad & C_1 + \frac{C_2}{1+r} = Y_1 + \frac{Y_2}{1+r} \end{aligned}$$

$$\begin{aligned} \beta \frac{u'(C_2)}{u'(C_1)} &= \frac{1}{1+r} \\ \ln(\beta) + \ln\left(u'(C_2)\right) - \ln\left(u'(C_1)\right) &= -\ln(1+r) \\ d\left(\ln(\beta) + \ln\left(u'(C_2)\right) - \ln\left(u'(C_1)\right)\right) &= -d\left(\ln(1+r)\right) \\ \frac{u''(C_2)}{u'(C_2)} dC_2 - \frac{u''(C_1)}{u'(C_1)} dC_1 &= -d\left(\ln(1+r)\right) \\ -\frac{1}{\sigma} \cdot \frac{dC_2}{C_2} + \frac{1}{\sigma} \cdot \frac{dC_1}{C_1} &= -d\left(\ln(1+r)\right) \\ \frac{dC_2}{C_2} - \frac{dC_1}{C_1} &= \sigma \cdot d\left(\ln(1+r)\right) \\ d\ln(C_2) - d\ln(C_1) &= \sigma \cdot d\left(\ln(1+r)\right) \\ d\ln\left(\frac{C_2}{C_1}\right) &= \sigma \cdot d\left(\ln(1+r)\right) \\ d\ln\left(\frac{C_2 - C_1}{C_1} + 1\right) &= \sigma \cdot d\left(\ln(1+r)\right) \end{aligned}$$

The EIS gives the percentage change in consumption over time for one percentage change in r .

2(d)

Utility discount rate