Business Cycles: Empirics and Theory

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1 Introduction

Some jargons:

- ar = annual rate
- na = not adjusted
- sa = seasonally adjusted (Swiss case: sa = sport-event adjusted)
- q/q sa quarterly seasonally adjusted growth
- q/q csa quarterly seasonally and calendar adjusted growth
- q/q saar = quarterly seasonally growth expressed at an annual rate

$$\text{q/q saar} = (1 + \text{q/q sa})^4 - 1$$

- y/y = growth relative to the same quarter of the previous year (usually calculated using na series)

Some examples:

- $Y_{\text{Q2 2018}}$ = level of real GDP in Q2 2018
- $4 \times Y_{\text{Q2 2018}}$ = level of real GDP in Q2 2018, expressed at an annual rate
- $Y^{sa}_{\text{Q2 2018}}$ = seasonally adjusted level of real GDP in Q2 2018
- q/q growth in Q2 2018 (sa) $= \left(\dfrac{Y^{sa}_{\text{Q2 2018}}}{Y^{sa}_{\text{Q1 2018}}}\right)^4 - 1$
- y/y growth $= \dfrac{Y_{\text{Q2 2018}}}{Y_{\text{Q2 2017}}} - 1$
- annual growth $= \dfrac{Y_{\text{Q1 2018}} + Y_{\text{Q2 2018}} + Y_{\text{Q3 2018}} + Y_{\text{Q4 2018}}}{Y_{\text{Q1 2017}} + Y_{\text{Q2 2017}} + Y_{\text{Q3 2017}} + Y_{\text{Q4 2017}}} - 1$

Some notations:

- X_t = level in period t, e.g., GDP during Q1 2018
- $x_t = \log(X_t)$ = natural logarithm of X_t
- $\Delta x_t = x_t - x_{t-1}$
- The growth rate of X_t

$$\frac{X_t - X_{t-1}}{X_{t-1}} \simeq \log \frac{X_t}{X_{t-1}} = \Delta x_t$$

Some useful approximations:

- y/y growth = 4 quarter moving average of q/q ar growth = sum of q/q growth rates = $\Delta x_t + \Delta x_{t-1} + \Delta x_{t-2} + \Delta x_{t-3}$
- Some definitions to write down approximations to annual growth rates
  - $g_{t,q}^{q/q} \equiv$ q/q growth rate in qth quarter of the year t
  - $g_{t,q}^{y/y} \equiv$ y/y growth rate in qth quarter of the year t
  - $g_{t,q}^{A} \equiv$ annual growth rate in year t
- The following approximations hold
  - $g_t^A \simeq \frac{1}{4}(g_{t,4}^{q/q} + 2g_{t,3}^{q/q} + 3g_{t,2}^{q/q} + 4g_{t,1}^{q/q} + 3g_{t-1,4}^{q/q} + 2g_{t-1,3}^{q/q} + g_{t-1,2}^{q/q})$
  - $g_t^A \simeq \frac{1}{4}(g_{t,4}^{y/y} + g_{t,3}^{y/y} + g_{t,2}^{y/y} + g_{t,1}^{y/y})$

Important implications of approximations to annual growth rate

- Annual growth is not the average of the quarterly growth rates of the respective year.
- Annual growth depends also on quarterly growth in the previous year.
- In fact, the previous year's Q4 growth rate is three times as important for this year's annual growth than this year's Q4 growth rate.
- Once you know growth up to Q2, you can already make a very precise forecast for annual growth (you already know 13/16 of the annual growth rate).
- Similar effect for y/y growth rates (you always know already 3/4 of the next y/y growth rate).

2 Detrending

Linear trend

$$\log(Y_t) = \beta_0 + \beta_1 t + \varepsilon_t$$

- β_1 is the trend growth rate.
- ε_t are the deviations from the trend and therewith the cycle.

Time-varying trend

$$c_t \equiv \frac{Y_t - Z_t}{Z_t}$$

- Z_t: the "trend level" of GDP.
- c_t: the percentage deviation of GDP from trend by c_t.

Using log-approximations:

$$c_t = \frac{Y_t}{Z_t} - 1 \simeq \log \frac{Y_t}{Z_t} = y_t - z_t$$

$$y_t = \log(Y_t) = z_t + c_t$$

i.e., log-level of GDP consists of a trend z_t and a cyclical component c_t that fluctuates around the trend.

How to govern the degree of time variation in trend growth
Trend growth rate:

$$\frac{Z_t}{Z_{t-1}} - 1 \simeq \log \frac{Z_t}{Z_{t-1}} = z_t - z_{t-1}$$

The change of the trend growth rate:

$$v_t = (z_t - z_{t-1}) - (z_{t-1} - z_{t-2})$$

If we allow for a lot of time-variation in the trend growth rate, we can choose v_t such that most of the observed changes in $\log(Y_t)$ is attributable to changes in z_t and only little to cyclical component c_t and vice versa.

HP filter

$$\begin{aligned} L &= \sum_t c_t^2 + \lambda \sum_t v_t^2 \\ &= \sum_t (\log(Y_t) - z_t)^2 + \lambda \sum_t ((z_t - z_{t-1}) - (z_{t-1} - z_{t-2}))^2 \end{aligned}$$

$$\hat{z}_t = \underset{z_t}{\arg\min} \quad L$$

For $\lambda \to 0$:

$$\min_{z_t} L = \min_{z_t} \sum_t (\log(Y_t) - z_t)^2$$

- The time trend z_t that minimizes the loss function is simply $\log(Y_t) = z_t$.
- There is no cycle!

For $\lambda \to \infty$:

$$\min_{z_t} L = \min_{z_t} \sum_t ((z_t - z_{t-1}) - (z_{t-1} - z_{t-2}))^2$$

- The loss function is minimized with $z_t - z_{t-1} = z_{t+j} - z_{t+j-1} \quad \forall t, j$.
- Trend growth is constant, i.e., linear time trend.

Problems with HP filters:

- Instability of the HP trend at the margin.
- HP filter cannot separate demand from supply shocks (i.e., a slowdown in trend growth is identified during a long-lasting recession).
- What is the right λ for HP filter.

Production function approach
Cobb-Douglas production function:

$$Y = AK^{\alpha}L^{1-\alpha}$$

- Measure K and L (which is not trivial) and solve for A.
- Filter A, K, L to get their potential levels.

- Plug these back into production function to get potential GDP.

Another way of writing the HP filter
Assume that the change in the trend is a random variable:

$$\Delta z_t - \Delta z_{t-1} = v_t, v_t \sim \mathcal{N}(0, \sigma^2)$$

$$(z_t - z_{t-1}) - (z_{t-1} - z_{t-2}) = v_t$$

$$\underbrace{\begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}}_{Z_t} = \underbrace{\begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} z_{t-1} \\ z_{t-2} \end{bmatrix}}_{Z_{t-1}} + \underbrace{\begin{bmatrix} \sigma & 0 \\ 0 & 0 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \varepsilon_t^{\Delta z} \\ 0 \end{bmatrix}}_{w_t}$$

$$Z_t = FZ_{t-1} + Qw_t, w_t \sim \mathcal{N}(0, 1)$$

We do not observe the trend Z_t, but we observe data that is a function of the unobserved trend:

$$\log(Y_t) = z_t + c_t = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{H} \underbrace{\begin{bmatrix} z_t \\ z_{t-1} \end{bmatrix}}_{Z_t} + \underbrace{\begin{bmatrix} \sigma & 0 \end{bmatrix}}_{R} \underbrace{\begin{bmatrix} \lambda\varepsilon_t^c \\ 0 \end{bmatrix}}_{e_t}$$

$$\log(Y_t) = HZ_t + Re_t, e_t \sim \mathcal{N}(0, 1)$$

Note: variance of c_t is λ times larger than variance of v_t.

The state-space form
<u>Measurement equation</u> with standard normal i.i.d. shocks e_t:

$$\log(Y_t) = X_t = HZ_t + Re_t$$

<u>Transition equation</u> with standard normal i.i.d. shocks w_t:

$$Z_t = FZ_{t-1} + Qw_t$$

Note:

- Whenever, we can write something in state-space form, we can use *Kalman filter* to estimate the unobserved states Z_t given the parameter matrices H, R, F, Q and the data X_t.
- *Kalman filter* can easily handle missing data.

3 Recessions v.s. Expansions

Recessions

- Periods with contracting economic activity.
- Popular rule of thumb: two consecutive quarters of negative real GDP growth.

Regime switching model (Hamilton filter)

- Assume that there are two regimes $S_t = 1, 2$
- Assume that the economy stochastically switches between the two regimes with the following probabilities

		$t=1$	
		$S_1=1$	$S_1=2$
$t=0$	$S_0=1$	p_{11}	$1-p_{11}$
	$S_0=2$	$1-p_{22}$	p_{22}

Regimes differ in average growth:

$$\Delta y_t = \begin{cases} c_1 + \varepsilon_t & S_t = 1 \\ c_2 + \varepsilon_t & S_t = 2 \end{cases}, \varepsilon_t \sim \mathcal{N}(0, \sigma^2)$$

Comparison
Linear view:

- Business cycles are fluctuations around a trend.
- Most modern macro models (DSGEs) implicitly have this view.

Non-linear view:

- Business cycles are transitions between two (or more) distinct regimes.
- While there are fluctuations within a regime, what really matters is the transition between the regimes.
- The public and high-level policy makers often take this view.

One of the best forecasting indicators for recessions: the yield curve

- Yield curve slope is historically one of the best indicators for upcoming recessions.
- All recessions were preceded by an inverted (negatively sloped) yield curves, yield curve was rarely inverted without a recession following.

Why is an inverted yield curve a good recession predictor?
Assume you have 100 USD to invest for two years
There are two options:

- Option 1: buy a 2-year treasury bond with a yield of i_t^{2y}, get the principal payment $2(100(1+i_t^{2y})^2)$ after 2 years.
- Option 2: buy a 1-year treasury bond with a yield of i_t^{1y}, get the principal payment $100(1+i_t^{1y})$ after 1 year and reinvest it in a new 1-year treasury bond with an expected yield $\mathbf{E}\left[(1+i_{t+1}^{1y})\right]$. You expect to have $100(1+i_t^{1y})\mathbf{E}\left[(1+i_{t+1}^{1y})\right]$ after 2 years.

Except for the uncertainty about i_{t+1}^{1y}, the two options are very similar, therefore the return has to be similar

$$100(1+i_t^{2y})^2 \approx 100(1+i_t^{1y})\mathbf{E}\left[(1+i_{t+1}^{1y})\right]$$

Take logs of both sides:

$$2\log(1+i_t^{2y}) \approx \log(1+i_t^{1y}) + \log\left(1+\mathbf{E}\left[i_{t+1}^{1y}\right]\right)$$

Use the approximation equation $\log(1+x) = x$:

$$i_t^{2y} \approx \frac{1}{2}\left(i_t^{1y} + \mathbf{E}\left[i_{t+1}^{1y}\right]\right)$$

Similarly, for 10-year yield:

$$i_t^{10y} \approx \frac{1}{10}\sum_{i=0}^{9} \mathbf{E}_t\left[i_{t+i}^{1y}\right]$$

- Short-term yields are mostly driven by monetary policy ($i_t^{1y} \approx$ policy rate).
- An inverted yield curve implies that the policy rate is currently higher than what is expected in the future.
- In other words, market participants expect the central bank to lower the policy rate.
- Central banks typically only lower policy rates in response to economic weakness.
- Word of caution: the risk-premium may lead to an inverted yield curve, without market participants expecting falling policy rates

$$i_t^{10y} = \frac{1}{10}\sum_{i=0}^{9} \mathbf{E}_t i_{t+i}^{1y} + \text{risk premium}_t$$

Recession nowcasting is feasible!
Good rule of thumb: a recession has started if 3-month change of the 3-month moving average of seasonally adjusted unemployment rate rises above 0.3 pp.
Important references: baseline recession probabilities
NBER recession

- Time span: from Q1 1950 until Q2 2021 (286 quarters in total)
- # of expansion quarters = 247
- # of recession quarters = 39
- Unconditional recession probability $= \frac{39}{286} = 14\%$
- # of expansion quarters that are followed by a recession = 11
- Conditional of being in an expansion quarter:
  - probability of a recession starting next quarter $= \frac{11}{247} = 4.5\%$
  - probability of staying in an expansion $= 100 - 4.5 = 95.5\%$
- Conditional on being in an expansion in t:
  - probability of next recession starting sometimes during next 2 quarters ($t+1$ and $t+2$) $= 1 - 0.955 \times 0.955$
  - probability of next recession starting sometimes during next n quarters (between $t+1$ and $t+n$) $= 1 - 0.955^n$

4 Welfare Costs of Business Cycles

HP decomposition: $\log C_t = z_t + x_t$ (or equally: $C_t = Z_t X_t$), where z_t is the trend level of consumption and x_t fluctuations around this trend.
The Lucas calculation - approximating the utility function

$$U(C) \simeq U(Z) + U'(Z)(ZX - Z) + \frac{1}{2}U''(Z)(ZX - Z)^2$$

Calculating expected utility:

$$\mathbf{E}[U(C)] \simeq U(Z) + U'(Z)\mathbf{E}[(ZX - Z)] + \frac{1}{2}U''(Z)\mathbf{E}\left[(ZX - Z)^2\right]$$

Rewriting:

$$\mathbf{E}[U(C)] \simeq U(Z) + \frac{1}{2}U''(Z)\mathbf{E}\left[(X - 1)^2\right] Z^2$$

Utility function:

$$U(C) = \frac{C_{1-\sigma}}{1-\sigma}$$

Hence,

$$\mathbf{E}[U(C)] \simeq \frac{Z^{1-\sigma}}{1-\sigma} - \frac{\sigma}{2}Z^{-\sigma-1}\mathbf{E}\left[(\underbrace{X-1}_{\simeq \log X = x})^2\right] Z^2$$

$$\mathbf{E}[U(C)] \simeq \frac{Z_{1-\sigma}}{1-\sigma} - \frac{\sigma}{2}Z^{1-\sigma}\mathbf{E}[x^2]$$

- $\mathbf{E}[x^2]$ is the variance of log-difference ($\simeq$ percentage deviation) of consumption from its trend level
- The higher the variance, the lower expected utility
- The higher risk aversion, *sigma*, the more costly are fluctuations

Certainty equivalence
Let us scale trend level of consumption by δ and offer δZ with certainty to the representative consumer

$$\frac{(\delta Z)^{1-\sigma}}{1-\sigma} = \frac{Z_{1-\sigma}}{1-\sigma} - \frac{\sigma}{2}Z^{1-\sigma}\mathbf{E}[x^2]$$

Rearrange:

$$\delta = \left(1 - (1-\sigma)\frac{\sigma}{2}\mathbf{E}[x^2]\right)^{\frac{1}{1-\sigma}}$$

The resulting estimate for the costs of economic fluctuations is tiny, but why should we care about business cycles?

- Observed fluctuations are the ones resulting despite macroeconomic stabilization policies (might be much larger without stabilization)
- Cost of business cycles are not equally distributed over population (some become unemployed, some not; not everyone becomes 4% unemployed, i.e. can work 4% fewer hours)
- Direct costs of unemployment
- Persistent effects of economic fluctuations

5 Comovement

Principal Component Analysis

$$y_{it} = \lambda_i f_t + \varepsilon_{it}$$

- y_{it}: the standardized value of the indicator i at time t
- f_t: the underlying force that drives the common variation in the dataset
- λ_i: the "loading" of indicator i on f_t

Assume that we know the loadings λ_i:

$$y_t = \begin{bmatrix} y_{1t} \\ \vdots \\ y_{Nt} \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} f_t + \begin{bmatrix} \varepsilon_{1t} \\ \vdots \\ \varepsilon_{Nt} \end{bmatrix}$$

For given loadings λ, we can obtain f_t by simply running a regression in each time period t

$$\hat{f}_t = (\lambda'\lambda)^{-1}\lambda' y_t$$

$$\hat{\varepsilon}_t = y_t - \lambda(\lambda'\lambda)^{-1}\lambda' y_t$$

The sum of squared errors:

$$\sum_t \hat{\varepsilon}_t' \hat{\varepsilon}_t = \sum_t \left(y_t - \lambda(\lambda'\lambda)^{-1}\lambda' y_t\right)' \left(y_t - \lambda(\lambda'\lambda)^{-1}\lambda' y_t\right)$$

6 Swiss GDP

How to estimate Swiss GDP

- Annual GDP adds up the value added according to firm survey
- Value added is estimated from the production side
- Demand side: inventories as residual
- Income side: profits as residual
- First estimate published 8 months after the end of reference year
- i.e. first estimate for 2021 will be published at the end of August 2022

Quarterly GDP is an intra- and extra-polation of annual GDP

- Regress annual value added of a sector y_t^A on indicators

$$y_t^A = \beta_0 + \beta_1 x_t^A + \varepsilon_t$$

7 Introduction to Theory

IS curve: goods & services market equilibrium
Aggregate demand equals investment, private consumption and government spending (closed economy):

$$Y^D = I + C + G$$

Aggregate demand depends on aggregate income Y^I, interest rate, r, and other factors, v_D

$$Y^D = f(\underset{+}{Y^I}, \underset{-}{r}) + v_D$$

In equilibrium:

$$\underbrace{\text{aggregate demand}}_{Y^D} = \underbrace{\text{aggregate income}}_{Y^I} = \underbrace{\text{aggregate production}}_{Y^P}$$

LM curve: money market
Monetary policy rule (MP-rule)

$$r = \rho + \theta_\pi \pi + v_{MP}$$

- π: inflation
- v_{MP}: monetary policy shock

The IS-LM model provides the equilibrium combination of interest rate and inflation for a given demand and monetary policy shock.

- The IS-LM model describes goods and money market equilibrium in the short-term
- Inflation is assumed to be constant

AS-AD model:

- The AS-AD model is an extension of the IS-LM framework
- It describes the relationship between aggregate output and prices

AD curve: aggregate demand

- The AD curve is defined by the IS-LM equilibrium
- For given inflation rate, π, the MP rule describes, what interest rate, r, should be chosen by the central bank
- From IS curve, we can get the resulting aggregate demand, Y
- The AD curve describes the resulting aggregate demand for each inflation rate
- The monetary policy shock and the demand shock shift the AD curve

AS curve

- The inflation rate follows from the pricing decisions of producers
- The AS curve describes the change in prices (i.e., inflation)
- Prices $\rightarrow$ aggregate output of Y produced by firms
- If firms produce a lot relative to some benchmark level of output ($\bar{Y}$), they tend to raise prices

$$\pi = \bar{\pi} + \kappa(Y - \bar{Y})$$

Assessing the Economic Effects of the Coronavirus with the AS-AD Model
Split the economy into 2 groups:

- directly affected sectors: businesses that are directly affected by containment measures (e.g., restaurants)
- non-affected sectors: all other businesses (e.g., universities can switch to online education, insurances to home office)

8 New Keynesian IS Curve

Household utility maximization problem:

$$\max_{\{C_{t+j},B_{t+j},N_{t+j}\}_{j=0}^{\infty}} \quad \mathbb{E}_t\left[\sum_{j=0}^{\infty}\beta^j Z_{t+j}\left(\frac{C_{t+j}^{1-\sigma}}{1-\sigma}-\frac{N_{t+j}^{1+\phi}}{1+\phi}\right)\right]$$
$$\text{s.t.} \quad P_{t+j}C_{t+j}+Q_{t+j}B_{t+j}\leq B_{t-1+j}+W_{t+j}N_{t+j}+D_{t+j}$$

- Z_t: preference shifter
- Q_t: Bond price in t, pays 1 in $t+1$
- B_t: quantity of bonds purchased in t
- N_t: labor supply
- W_t: hourly wage payment
- P_t: the price of a consumption bundle C_t
- D_t: dividends paid by firms to households

Optimal consumption:

$$C_t^{-\sigma}=\mathbb{E}_t\left[\beta(1+i_t)\frac{Z_{t+1}}{Z_t}\frac{1}{1+\pi_{t+1}}C_{t+1}^{-\sigma}\right] \quad \text{Euler equation}$$

- β low: impatient, higher consumption today.
- $\frac{Z_{t+1}}{Z_t}$ high: assign a lot to today's utility relative to tomorrow.
- i_t high: less consumption and more saving today.
- π_{t+1} high: tomorrow's prices high, more consumption today.
- σ low: households substitute consumption today more strongly with consumption tomorrow when the interest rate is raised (σ equals the inverse elasticity of substitution).

Optimal labor:

$$\frac{N_t^{\phi}}{C_t^{-\sigma}}=\frac{W_t}{P_t}$$

Log-linear form of Euler equation (using first-order Taylor approximation and defining $\log\beta\equiv\rho$):

$$c_t\simeq\mathbb{E}_tc_{t+1}-\frac{1}{\sigma}(i_t-\mathbb{E}_t\pi_{t+1}-\rho)+\frac{1-\rho_z}{\sigma}z_t \quad \text{NKIS}$$

9 New Keynesian Phillips Curve

Optimal allocation of spending across varieties:

$$\max_{C_t(i)} \quad C_t = \left(\int_{i=0}^{1} C_t(i)^{\frac{\varepsilon-1}{\varepsilon}} \mathrm{d}i \right)^{\frac{\varepsilon}{\varepsilon-1}}$$
$$\text{s.t.} \quad \int_{i=0}^{1} P_t(i) C_t(i) \mathrm{d}i \leq S_t$$

Optimal consumption of good i:

$$C_t(i) = C_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon}$$

where

$$P_t \equiv \left(\int_{i=0}^{1} P_t(i)^{1-\varepsilon} \mathrm{d}i \right)^{\frac{1}{1-\varepsilon}}$$

Good market clearing: $Y_t(i) = C_t(i)$, $Y_t = C_t$

$$Y_t(i) = Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} \quad \text{Demand function}$$

$$Y_t(i) = A_t L_t(i) \quad \text{Production function}$$

Profit maximization under flexible prices:

$$\max_{P_t(i)} \quad \Pi_t(i) = P_t(i) Y_t(i) - W_t L_t(i)$$
$$\max_{P_t(i)} \quad P_t(i) Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon} - \frac{W_t}{A_t} Y_t \left(\frac{P_t(i)}{P_t} \right)^{-\varepsilon}$$

Profit-maximizing price:

$$P_t^*(i) = \frac{\varepsilon}{\varepsilon - 1} \frac{W_t}{A_t}$$

Aggregate production under flexible prices:

$$Y_t = A_t L_t$$

Labor market equilibrium: $L_t = N_t$, $Y_t = C_t$

$$\begin{cases} \dfrac{N_t^{\varphi}}{C_t^{-\sigma}} = \dfrac{W_t}{P_t} \\ P_t^*(i) = \dfrac{\varepsilon}{\varepsilon - 1} \dfrac{W_t}{A_t} \\ Y_t = A_t L_t \end{cases} \implies \begin{cases} L_t = \left(A_t^{1-\sigma} \dfrac{\varepsilon - 1}{\varepsilon} \right)^{\frac{1}{\varphi+\sigma}} \\ Y_t = A_t^{\frac{1+\varphi}{\varphi+\sigma}} \left(\dfrac{\varepsilon - 1}{\varepsilon} \right)^{\frac{1}{\varphi+\sigma}} \end{cases}$$

Key insight: Equilibrium production does not depend on prices.
Profit maximization under sticky prices:

$$\max_{P_i(i)} \quad PV = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \frac{\theta^j}{R_{t+j}} \left(P_t(i) Y_{t+j} \left(\frac{P_t(i)}{P_{t+j}} \right)^{-\varepsilon} - \frac{W_{t+j}}{A_{t+j}} Y_{t+j} \left(\frac{P_t(i)}{P_{t+j}} \right)^{-\varepsilon} \right) \right]$$

where

$$R_{t+j} = \prod_{\tau=1}^{j} (1 + i_{t+\tau} - \mathbb{E}_t \pi_{t+\tau})$$

Optimal price under sticky prices:

$$O_t = \mathbb{E}_t \left[\sum_{j=0}^{\infty} \omega_j P^*_{t+j} \right]$$

where

$$\omega_j = \frac{\frac{\theta^j}{R_{t+j}} Y_{t+j} P^{\varepsilon}_{t+j}}{\sum_{j=0}^{\infty} \frac{\theta^j}{R_{t+j}} Y_{t+j} P^{\varepsilon}_{t+j}}$$

- θ^j: probability that price is still in place in $t+j$
- $\frac{1}{R_{t+j}}$: the present value of the profit that is generated in $t+j$
- $Y_{t+j} P^{\varepsilon}_{t+j}$: how many units a firm can sell for a given price in $t+j$

Log-linear approximation of the optimal price:

$$o_t \simeq \sum_{j=0}^{\infty} \frac{(\beta\theta)^j}{\sum_{k=0}^{\infty} (\beta\theta)^k} \mathbb{E}_t p^*_{t+j}$$

$$\frac{\frac{\theta^j}{R_{t+j}} Y_{t+j} P^{\varepsilon}_{t+j}}{\sum_{j=0}^{\infty} \frac{\theta^j}{R_{t+j}} Y_{t+j} P^{\varepsilon}_{t+j}} \implies \frac{(\beta\theta)^j}{\sum_{k=0}^{\infty} (\beta\theta)^k}$$

- In steady state $Y_{t+j} P^{\varepsilon}_{t+j}$ is constant and therefore cancels out
- In steady state we have $i = -\log \beta = \log \frac{1}{\beta} \simeq \frac{1}{\beta} - 1$ and $\pi = 0$, hence $1 + i = \frac{1}{\beta}$

$$R_{t+j} = \prod_{\tau=1}^{j} (1 + i - \pi) = \prod_{\tau=1}^{j} \frac{1}{\beta} = \left(\frac{1}{\beta} \right)^j \Longleftrightarrow \frac{1}{R_{t+j}} = \beta^j$$

- θ^j: probability that price is still in place after j periods

$$\begin{aligned} o_t &\simeq \sum_{j=0}^{\infty} \frac{(\beta\theta)^j}{\sum_{k=0}^{\infty} (\beta\theta)^k} \mathbb{E}_t p^*_{t+j} \\ &= (1 - \beta\theta) \sum_{k=0}^{\infty} (\beta\theta)^k \mathbb{E}_t p^*_{t+j} \\ &= (1 - \beta\theta) p^*_t + \underbrace{(1 - \beta\theta) \sum_{k=1}^{\infty} (\beta\theta)^k \mathbb{E}_t p^*_{t+j}}_{\beta\theta \mathbb{E}_t o_{t+1}} \\ &= (1 - \beta\theta) p^*_t + \beta\theta \mathbb{E}_t o_{t+1} \end{aligned}$$

- The optimal price today is a weighted average of today's flexible price and tomorrow's optimal price
- Higher discounting ($\beta \downarrow$) and more often firms adjust ($\theta \downarrow$), the larger weight on today's flexible price

Aggregate price dynamics:
The price index in t is a weighted average between the price chosen by the adjusting firms (o_t) and last period's price index (p_{t-1})

$$\begin{aligned} p_t &\simeq (1 - \theta) o_t + \theta p_{t-1} \\ \underbrace{p_t - p_{t-1}}_{\pi_t} &\simeq (1 - \theta)(o_t - p_{t-1}) \\ \pi_t &\simeq (1 - \theta)(o_t - p_{t-1}) \end{aligned}$$

Intuition: inflation is driven by

- $1-\theta$: how many adjusting firms reset prices
- $o_t - p_{t-1}$: how much adjusting firms change prices

Iterate above equation one period forward:

$$\mathbb{E}_t \pi_{t+1} = (1-\theta)(\mathbb{E}_t o_{t+1} - p_t)$$

$$\begin{cases} o_t = (1-\beta\theta)p_t^* + \beta\theta\mathbb{E}_t o_{t+1} \\ \pi_t = (1-\theta)(o_t - p_{t-1}) \\ \mathbb{E}_t\pi_{t+1} = (1-\theta)(\mathbb{E}_t o_{t+1} - p_t) \end{cases} \implies \pi_t = \frac{1-\theta}{\theta}(1-\beta\theta)(p_t^* - p_t) + \beta\mathbb{E}_t\pi_{t+1}$$

Inflation is high whenever

- expected inflation is high (comes from adjusting firms choosing high o_t because they expect high $\mathbb{E}_t o_{t+1}$)
- current flexible prices are high relative to the price level $(p_t^* - p_t)$ (comes from adjusting firms choosing high o_t because p_t^* is high)

Labor market equilibrium: $L_t = N_t$, $Y_t = C_t$

$$\begin{cases} \dfrac{N_t^{\varphi}}{C_t^{-\sigma}} = \dfrac{W_t}{P_t} \\ P_t^*(i) = \dfrac{\varepsilon}{\varepsilon-1}\dfrac{W_t}{A_t} \end{cases} \implies \begin{cases} \dfrac{P_t^*}{P_t} = \dfrac{\varepsilon}{\varepsilon-1}A_t^{-\varphi-1}Y_t^{\varphi+\sigma} \\ 1 = \dfrac{\varepsilon}{\varepsilon-1}A_t^{-\varphi-1}(Y_t^n)^{\varphi+\sigma} \end{cases} \implies \frac{P_t^*}{P_t} = \left(\frac{Y_t}{Y_t^n}\right)^{\varphi+\sigma}$$

Log-linearize above equation:

$$p^* - p_t = (\varphi+\sigma)\underbrace{(y_t - y_t^n)}_{\widetilde{y}_t}$$

$$\pi_t = \kappa\widetilde{y}_t + \beta\mathbb{E}_t\pi_{t+1} \quad \text{NKPK}$$

where

$$\kappa = \frac{1-\theta}{\theta}(1-\beta\theta)(\varphi+\sigma)$$