Exercises
Social Choice Theory

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Part 1: Preliminaries

Exercise 1.1

(a)

Let $X = \{x, y, z\}$. A binary relation R on the set X is as follows:

$$R = \{(x, y), (y, z), (z, y)\}$$

The asymmetric part is:

$$P = \{(x, y)\}$$

- The induced asymmetric part $P = \{(x, y)\}$ is transitive, hence R is quasi-transitive.
- $(x, y) \in R$ and $(y, z) \in R$ while $(x, z) \notin R$, a contradiction to transitivity.

Therefore $R = \{(x, y), (y, z), (z, y)\}$ is quasi-transitive but not transitive.

(b)

Let $X = \{x, y, z\}$. A binary relation R on the set X is as follows:

$$R = \{(x, y), (y, z)\}$$

Its asymmetric part P is:

$$P = \{(x, y), (y, z)\}$$

- Since $(x, z) \notin P$, P is not transitive and R is not quasi-transitive.
- Since $(z, x) \notin P$, P has no cycle and R is acyclical.

(c)

Need to show:

$$xIy \wedge yIz \implies xIz$$

Proof:

We can rewrite induced I into binary relation:

$$\begin{aligned} xIy &\implies [xRy \wedge yRx] \\ yIz &\implies [yRz \wedge zRy] \end{aligned}$$

$$\begin{aligned} xRy \wedge yRz &\implies xRz \quad \text{by transitivity} \\ yRx \wedge zRy &\implies zRx \quad \text{by transitivity} \end{aligned}$$

$$xRz \wedge zRx \implies xIz$$

Exercise 1.2

(a)

$\succeq$ is a preference

$$x \sim y$$
$$y \succ z$$
$$z \succ w$$

$$\begin{aligned} G(\{w, x, y, z\}) &= \{x, y\} \\ M(\{w, x, y, z\}) &= \{x, y\} \end{aligned}$$

$$R = \{(x, y), (y, x), (y, z), (z, w)\}$$

(b)

$$M(\{x, y, z\}, R) = \{\} = G(\{x, y, z\})$$

Exercise 1.3

(a)

We know that x is chosen in the larger set (i.e. $C(\{x, y, z\}) = \{x\}$), and we need to check whether x is still chosen from all the possible subset containing x.

$$\begin{aligned} C(\{x, y\}) &= \{x\} \\ C(\{x, z\}) &= \{x\} \end{aligned}$$

Hence, property α is satisfied.

We cannot find any two alternatives that are generate by the choice function. Hence, we cannot find a counter example that violates property β (i.e. β is satisfied).

(b)

We know that z is chosen from the larger set (i.e. $C(\{x, y, z\}) = \{z\}$), and we need to check whether z is still chosen from all the possible subset containing z.

Base relation:

$$C(\{x, z\}) = \{x, z\}$$
$$C(\{y, z\}) = \{z\}$$

Preference:

$$x \sim z \succ y$$

Hence, property α is satisfied.

We see that $\{x, z\}$ is chosen from the smaller set $\{x, z\}$ (i.e. $C(\{x, z\}) = \{x, z\}$), and z is chosen from the larger set $\{x, y, z\}$ while x is not, hence a violation of property β.

(c)

We know that $\{y, z\}$ is chosen from the larger set $\{x, y, z\}$, and we need to check whether $\{y, z\}$ is still chosen from all the possible subset containing $\{y, z\}$.

We see:

$$C(\{y, z\}) = \{y, z\}$$

Hence, property α is satisfied.

We see that $\{y, z\}$ is chosen from the smaller set $\{y, z\}$, and $\{y, z\}$ is chosen from the larger set containing both y and z. Hence, property β is satisfied.

Exercise 1.4

Shorthands for choices:

- P: peatnuts
- A: apple juice
- M: mineral water
- B: beer

(a)

$$X = \{P, A, M, B\}$$

$$C(\{P, A, M\}) = \{P, A\}$$

$$C(\{P, A, M, B\}) = \{P, B\}$$

β is violated?

Better translation of the information:

$$X = \{PA, PM, PB\}$$

$$C(\{PA, PM\}) = \{PA\}$$

$$C(\{PA, PM, PB\}) = \{PB\}$$

β is satisfied.

Part 2: The Problem of Social Choice

Exercise 2.1

(a)

#	preferences
1	$x \ P \ y \ P \ z$
1	$y \ P \ z \ P \ x$
1	$z \ P \ x \ P \ y$

Pairwise comparison	Votes	Social preference
x vs.y	2 vs.1	xPy
x vs.z	1 vs.2	zPx
y vs.z	2 vs.1	yPz

This leads to an outcome that is not transitive and not acyclical.

There is no Condorcet winner (loser) as each alternative loses in the pairwise voting at once.

(b)

Fix any $R^* \in R$, let $\mathscr{A} = \{(R^*, R^*, \cdots, R^*)\}$. In other words, all individuals share the same preference. In this example, the pairwise majority voting is an SWF.

Exercise 2.2

When $m = 2$, PV, IR, PM, CO, and BC are identical. They all collapse to the majority voting method.

Exercise 2.3

#	preferences
2	$x \ P \ w \ P \ y \ P \ z$
2	$y \ P \ w \ P \ z \ P \ x$
1	$w \ P \ z \ P \ x \ P \ y$

PV:

In plurality voting, we only consider the top-ranked alternatives given by voters.

$$x \, I \, y \, P \, w \, P \, z$$

IR:

- Stage 1: z is eliminated
- Stage 2: w is eliminated
- Stage 3: x wins with 3 votes (majority)

PM:

pairwise comparison	votes	binary relation
$w : x$	$3 : 2$	wPx
$w : y$	$3 : 2$	wPy
$w : z$	$5 : 0$	wPz
$x : y$	$3 : 2$	xPy
$x : z$	$2 : 3$	zPx
$y : z$	$4 : 1$	yPz

The pairwise majority voting method delivers a Condorcet winner w but there exists a cycle in x, y, z.

CO:

	w	x	y	z	$\sum$
w		$+1$	$+1$	$+1$	3
x	-1		$+1$	-1	-1
y	-1	-1		$+1$	-1
x	-1	$+1$	-1		-1

Copeland method delivers the social preference $w \, P \, x \, I \, y \, I \, z$.

BC:

#	preferences	w	x	y	z
2	$x \, P \, w \, P \, y \, P \, z$	2	3	1	0
2	$y \, P \, w \, P \, z \, P \, x$	2	0	3	1
1	$w \, P \, z \, P \, x \, P \, y$	3	1	0	2
		11	7	8	4

Borda Count method delivers the social preference $wPyPxPz$

EM:

There exists at least one individual who ranks x, y, w as his/her top preference, hence the Pareto efficient set is $X^E = \{w, x, y\}$. All individuals strictly prefer w over z, hence z is strictly dominated by w. $X^I = \{z\}$.

Pareto Efficient Method delivers the social preference $wIxIyPz$.

PE:

pairwise comparison	Pareto extension rule
$w : x$	wIx
$w : y$	wIy
$w : z$	wPz
$x : y$	xIy
$x : z$	zIx
$y : z$	yIz

Pareto Extension Rule delivers an outcome that violates transitivity but is quasi-transitive. This is SDF but not SWF.

Exercise 2.4

(a)

#	preferences
2	$x\ P\ z\ P\ y$
2	$y\ P\ z\ P\ x$
1	$z\ P\ y\ P\ x$

Applying IR:

- Stage 1: z is eliminated.
- Stage 2: y wins with 3 votes (majority).

pairwise comparison	votes	binary relation
$x : y$	$2 : 3$	yPx
$x : z$	$2 : 3$	zPx
$y : z$	$2 : 3$	zPy

- z wins all pairwise comparison hence a Condorcet winner.
- IR is not a Condorcet method.

(b)

Proof by contradiction:

No, a Condorcet winner is Pareto efficient, otherwise it would lose at least one pairwise vote.

(c)

We can construct the following example resulting x to be a Condorcet winner:

#	preferences
2	$x\ P\ y$
1	$y\ P\ x$

There is no Pareto dominance between x and y so EM delivers social indifference between x and y (i.e. xIy).

Same argument for PE.

Exercise 2.5

(a)

For arbitrary values of m:

- Anti-plurality voting and rejection voting are equivalent.
- Nameless example I and nameless example II are equivalent.

For $m = 3$:

- Anti-plurality voting and rejection voting are equivalent.
- Borda count, nameless example I, and nameless example II are equivalent.

(b)

$$s = (m, m-1, \cdots, 1)$$

(c)

$$s = (m^2, (m-1)^2, \cdots, 1^2)$$

Exercise 2.6

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1	3	0	2
1	$y\,P\,z\,P\,v\,P\,x$	1	0	3	2
1	$v\,P\,z\,P\,y\,P\,x$	3	0	1	2
1	$x\,P\,y\,P\,v\,P\,z$	1	3	2	0
		6	6	6	6

s^1 delivers social preference $vIxIyIz$

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1/4	1	0	3/4
1	$y\,P\,z\,P\,v\,P\,x$	1/4	0	1	3/4
1	$v\,P\,z\,P\,y\,P\,x$	1	0	1/4	3/4
1	$x\,P\,y\,P\,v\,P\,z$	1/4	1	3/4	0
		7/4	8/4	8/4	9/4

s^2 delivers social preference $zPxIyPv$

#	preferences	v	x	y	z
1	$x\,P\,z\,P\,v\,P\,y$	1/4	1	0	1/2
1	$y\,P\,z\,P\,v\,P\,x$	1/4	0	1	1/2
1	$v\,P\,z\,P\,y\,P\,x$	1	0	1/4	1/2
1	$x\,P\,y\,P\,v\,P\,z$	1/4	1	1/2	0
		7/4	8/4	7/4	6/4

s^3 delivers social preference $xPvIyPz$