PS4

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Names are listed in alphabetical order

Instrumental Variables

1. Bias of the IV estimator

$$y_i = \beta_0 + \beta_1 x_i + u_i$$

1(a)

$$\begin{aligned}
\hat{\beta}_{IV} &= \frac{\widehat{Cov(y_i, z_i)}}{\widehat{Cov(x_i, z_i)}} \\
&= \frac{\sum_{i=1}^{n}(z_i - \overline{z})(y_i - \overline{y})}{\sum_{i=1}^{n}(z_i - \overline{z})(x_i - \overline{x})} \\
&= \frac{\sum_{i=1}^{n}(z_i y_i - z_i\overline{y} - y_i\overline{z} + \overline{yz})}{\sum_{i=1}^{n}(z_i x_i - z_i\overline{x} - x_i\overline{z} + \overline{xz})} \\
&= \frac{\sum_{i=1}^{n}(z_i y_i - y_i\overline{z}) - \overline{y}\sum_{i=1}^{n} z_i + n\overline{yz}}{\sum_{i=1}^{n}(z_i x_i - x_i\overline{z}) - \overline{x}\sum_{i=1}^{n} z_i + n\overline{xz}} \\
&= \frac{\sum_{i=1}^{n}(z_i y_i - y_i\overline{z}) - n\overline{yz} + n\overline{yz}}{\sum_{i=1}^{n}(z_i x_i - x_i\overline{z}) - n\overline{xz} + n\overline{xz}} \\
&= \frac{\sum_{i=1}^{n}(z_i - \overline{z})y_i}{\sum_{i=1}^{n}(z_i - \overline{z})x_i}
\end{aligned}$$

1(b)

$$\begin{aligned}
\hat{\beta}_{IV} &= \frac{\sum_{i=1}^{n}(z_i - \overline{z})y_i}{\sum_{i=1}^{n}(z_i - \overline{z})x_i} \\
&= \frac{\sum_{i=1}^{n}(z_i - \overline{z})(\beta_0 + \beta_1 x_i + u_i)}{\sum_{i=1}^{n}(z_i - \overline{z})x_i} \\
&= \frac{\beta_0\sum_{i=1}^{n}(z_i - \overline{z}) + \beta_1\sum_{i=1}^{n}(z_i - \overline{z})x_i + \sum_{i=1}^{n}(z_i - \overline{z})u_i}{\sum_{i=1}^{n}(z_i - \overline{z})x_i} \\
&= \beta_1 + \frac{\sum_{i=1}^{n}(z_i - \overline{z})u_i}{\sum_{i=1}^{n}(z_i - \overline{z})x_i}
\end{aligned}$$

1(c)

$$\begin{aligned}
p\lim(\hat{\beta}_{IV} - \beta_1) &= \frac{p\lim \sum_{i=1}^{n}(z_i - \overline{z})u_i}{p\lim \sum_{i=1}^{n}(z_i - \overline{z})x_i} \\
&= \frac{p\lim \sum_{i=1}^{n}(z_i - \overline{z})(u_i - \overline{u})}{p\lim \sum_{i=1}^{n}(z_i - \overline{z})(x_i - \overline{x})} \\
&= \frac{p\lim \frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})(u_i - \overline{u})}{p\lim \frac{1}{n}\sum_{i=1}^{n}(z_i - \overline{z})(x_i - \overline{x})} \\
&\approx \frac{Cov(z_i, u_i)}{Cov(z_i, x_i)} = 0
\end{aligned}$$

$\hat{\beta}_{IV}$ is a consistent estimator of β_1.

1(d)

In a small sample, $\hat{\beta}_{IV}$ is biased. But as the sample increases, β_{IV} will probability converge to the β_1. Therefore, in a large sample, IV estimator will never affect the consistency of the true estimator regardless of whether there exists an endogenous problem.

2. Derivation of the Wald estimator

2(a)

From question 1, we know that

$$\begin{aligned}
\delta^W = \hat{\beta}_1 &= \frac{Cov(y_i, z_i)}{Cov(d_i, z_i)} \\
&= \frac{\mathbb{E}(y_i|z_i = 1) - \mathbb{E}(y_i|z_i = 1)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 1)} \\
&= \frac{\mathbb{E}(\beta_1 + \beta_1 d_i + u_i|z_i = 1) - \mathbb{E}(\beta_1 + \beta_1 d_i + u_i|z_i = 0)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} \\
&= \frac{\beta_1\mathbb{E}(d_i|z_i = 1) + \mathbb{E}(u_i|z_i = 1) - \beta_1\mathbb{E}(d_i|z_i = 0) - \mathbb{E}(u_i|z_i = 0)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} \\
&= \frac{\beta_1(\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)) + \mathbb{E}(u_i|z_i = 1) - \mathbb{E}(u_i|z_i = 0)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} \\
&= \beta_1 + \frac{\mathbb{E}(u_i|z_i = 1) - \mathbb{E}(u_i|z_i = 0)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)}
\end{aligned}$$

2(b)

In order to identify β_1 using the instrument, we need

$$\frac{\mathbb{E}(u_i|z_i = 1) - \mathbb{E}(u_i|z_i = 0)}{\mathbb{E}(d_i|z_i = 1) - \mathbb{E}(d_i|z_i = 0)} = 0 \iff \mathbb{E}(u_i|z_i = 1) = \mathbb{E}(u_i|z_i = 0)$$

Assumptions

- SUTVA (Stable Unit Treatment Value Assumption): outcomes of the ith individual are independent of other individuals' outcome

- Exclusion restriction: $\mathbb{E}(y_i|z=1,d) = \mathbb{E}(y_i|z=0,d) \quad \forall i = 0,1$
- Instrument assumption: $\mathbb{E}(d|z=1) \neq \mathbb{E}(d|z=0)$
- Monotonicity assumption: $d_i(z_i=1) \geq d_i(z_i=0) \quad \forall i$

Only instrument assumption can be tested empirically. The vadility of other assumptions must be assessd on a case-by-case basis.

3. Self selection revisited

$$
\begin{aligned}
D_i &= \mathbf{1}(Y_{1i} - Y_{0i} > 0) \\
&= \mathbf{1}(\beta_1 + u_{1i} - u_{0i} > 0)
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{\text{ATE}} &= \mathbb{E}(Y_{1i} - Y_{0i}) \\
&= \mathbb{E}[(\beta_0 + \beta_1 + u_{1i}) - (\beta_0 + u_{0i})] \\
&= \mathbb{E}(\beta_1 + u_{1i} - u_{0i}) \\
&= \mathbb{E}(\beta_1) + \mathbb{E}(u_{1i}) + \mathbb{E}(u_{0i}) \\
&= \beta_1 > 0
\end{aligned}
$$

$$
\begin{aligned}
\Delta^{\text{ATT}} &= \mathbb{E}(Y_{1i} - Y_{0i}|D=1) \\
&= \mathbb{E}(\beta_1 + u_{1i} - u_{0i}|D=1) \\
&= \beta_1 + \mathbb{E}(u_{1i} - u_{0i}|D=1) \\
&= \Delta^{\text{ATE}} + \mathbb{E}(u_{1i} - u_{0i}|D=1) \\
\mathbb{E}(u_{1i} - u_{0i}|D=1) &= \mathbb{E}(u_{1i} - u_{0i}|\beta_1 + u_{1i} - u_{0i} > 0) \\
&= \mathbb{E}(u_{1i} - u_{0i}|u_{1i} - u_{0i} > -\beta_1) > 0
\end{aligned}
$$

ATT is larger than ATE.

3(b)

$$
\begin{aligned}
\Delta^{\text{naive}} &= \mathbb{E}(Y_{1i}|D=1) - \mathbb{E}(Y_{0i}|D=0) \\
&= \mathbb{E}(Y_{1i}|D=1) - \mathbb{E}(Y_{0i}|D=0) - \mathbb{E}(Y_{0i}|D=1) + \mathbb{E}(Y_{0i}|D=1) \\
&= \underbrace{\mathbb{E}(Y_{1i}|D=1) - \mathbb{E}(Y_{0i}|D=1)}_{\Delta^{\text{ATT}}} + \underbrace{\mathbb{E}(Y_{0i}|D=1) - \mathbb{E}(Y_{0i}|D=0)}_{\text{selection bias}}
\end{aligned}
$$

$$
\begin{aligned}
\text{selection bias} &= \mathbb{E}(Y_{0i}|D=1) - \mathbb{E}(Y_{0i}|D=0) \\
&= \mathbb{E}(\beta_0 + u_{0i}|\beta_1 + u_{1i} - u_{0i} > 0) - \mathbb{E}(\beta_0 + u_{0i}|u_{1i} - u_{0i} \leq 0) \\
&= \mathbb{E}(u_{0i}|u_{1i} - u_{0i} > -\beta_1) - \mathbb{E}(u_{0i}|u_{1i} - u_{0i} \leq 0) \\
&= \mathbb{E}(u_{0i}|u_{1i} - u_{0i} > -\beta_1) - \mathbb{E}(u_{0i}|u_{1i} - u_{0i} > 0) \quad \text{symmetric distribution} \\
&> 0
\end{aligned}
$$

If individuals can self-select themselves into the program, the naive estimator will be larger since the selection bias is positive ($\mathbb{E}(Y_{0i}|D=1) > \mathbb{E}(Y_{0i}|D=0)$)

3(c)

$$\begin{aligned}
\text{LATE} &= \frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(D|Z=1)-\mathbb{E}(D|Z=0)} \\
&= \frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(D_{1i}|Z=1)-\mathbb{E}(D_{0i}|Z=0)} \\
&= \frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(\mathbf{1}(Y_{1i}-Y_{0i}+1>0))-\mathbb{E}(\mathbf{1}(Y_{1i}-Y_{0i}>0))} \\
&= \frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(\mathbf{1}(\beta_1+u_{1i}-u_{0i}+1>0))-\mathbb{E}(\mathbf{1}(\beta_1+u_{1i}-u_{0i}>0))} \\
&= \frac{\mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0)}{\mathbb{E}(\beta_1+u_{1i}-u_{0i}+1)-\mathbb{E}(\beta_1+u_{1i}-u_{0i})} \\
&= \mathbb{E}(Y|Z=1)-\mathbb{E}(Y|Z=0) \\
&= \text{ITT}
\end{aligned}$$

3(d)

$$\text{LATE} = \frac{\text{ITT}}{\text{proportion of compliers}} = \text{ITT}$$

We can derive that the proportion of compliers in the randomization experiment is 1. In other words, the instrument is randomly assigned with perfect compliance. Therefore, LATE=ATE.

4. Application: Angrist's (1990) study on military service

4(a)

$$\ln(\widehat{\text{earnings}}_i) = 6.4364 \underbrace{-0.0255}_{\text{treatment effect}} \cdot\text{veteran}_i + \underbrace{\beta \cdot x_i}_{\text{other omitted control variables}}$$

Earnings are not only determined by the veteran status. There are some omitted variables.

4(b)

	$Z=0$	$Z=1$
$D=0$	5,928	1,875
$D=1$	1,400	863

Due to monotonicity:

- In the observed $Z=0$ group, the individuals who received treatment ($D=1$) must be always-takers.

$$p_A = \frac{\sum_i D_i(Z_i=0)}{\sum_i Z_i=0} = \frac{1400}{5928+1400} = 0.191$$

- In the observed $Z=1$ group, the individuals who did not receive treatment ($D=0$) must be never-takers.

$$p_N = \frac{\sum_i D_i(Z_i=1)}{\sum_i Z_i=1} = \frac{1875}{1875+863} = 0.685$$

Due to randomization:

- The proportions of compliers, always-takers, and never-takers are the same between $Z = 0$ and $Z = 1$ group.

$$p_C = 1 - p_A - p_N = 0.124$$

Note:

- N denotes **never takers**
- C denotes **compliers**
- A denotes **always takers**

4(c)

	$Z = 0$	$Z = 1$
$D = 0$	$\widehat{\mathbb{E}(Y)} = 6.4472$	$\widehat{\mathbb{E}(Y)} = 6.4028$
$D = 1$	$\widehat{\mathbb{E}(Y)} = 6.4076$	$\widehat{\mathbb{E}(Y)} = 6.4289$

- Average potential outcome for always-takers $\mathbb{E}(Y_1|Z = 1, A) = 6.4028$
- Average potential outcome for never-takers $\mathbb{E}(Y_0|Z = 0, N) = 6.4076$

In $Z = 0$ group:

$$\frac{p_C}{p_N + p_C} \times \mathbb{E}(Y_0|Z = 0, C) + \frac{p_N}{p_N + p_C} \times \mathbb{E}(Y_0|Z = 0, N) = 6.4472 \implies \mathbb{E}(Y_0|Z = 0, C) = 6.666$$

In $Z = 1$ group:

$$\frac{p_C}{p_A + p_C} \times \mathbb{E}(Y_1|Z = 1, C) + \frac{p_A}{p_A + p_C} \times \mathbb{E}(Y_1|Z = 1, A) = 6.4289 \implies \mathbb{E}(Y_1|Z = 1, C) = 6.4691$$

- Average potential outcome for untreated compliers $\mathbb{E}(Y_0|Z = 0, C) = 6.666$
- Average potential outcome for treated compliers $\mathbb{E}(Y_1|Z = 1, C) = 6.4691$

4(d)

$$\begin{aligned}
\delta &= \frac{\mathbb{E}(Y|Z = 1) - \mathbb{E}(Y|Z = 0)}{\mathbb{E}(Y|Z = 1) - \mathbb{E}(Y|Z = 0)} \\
&= \frac{p_C[\mathbb{E}(Y_1|Z = 1, C) - \mathbb{E}(Y_0|Z = 0, C)]}{p_C} \\
&= \mathbb{E}(Y_1|Z = 1, C) - \mathbb{E}(Y_0|Z = 0, C) \\
&= 6.4691 - 6.666 \\
&= -0.1969
\end{aligned}$$

5. IV in action

5(a)

```
# load relevant libraries
library(haven) # read dta file
library(lattice) # density plot
library(stargazer) # print summary statistics
library(AER) # iv regression
```

```
d.mort <- read_dta('mortality.dta')
```

```
# outcome of interest
summary(d.mort$before67dead)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
## 0.00000 0.00000 0.00000 0.07311 0.00000 1.00000
```

```
# treatment
summary(d.mort$dist65_ageATend4emp)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  -4.833   5.000   6.333   6.499   8.000  11.000
```

```
# instrument
summary(d.mort$Zd_during)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##  0.0000  0.0000  0.0000  0.4787  1.0000  1.0000
```

5(b)

```
model.ols1 <- lm(before67dead ~ dist65_ageATend4emp + czeit1yATage50 +
                   czeit2yATage50 + czeit5yATage50 + czeit10yATage50 +
                   czeit25yATage50 + I(czeit1yATage50^2) + I(czeit2yATage50^2) +
                   I(czeit5yATage50^2) + I(czeit10yATage50^2) +
                   I(czeit25yATage50^2) + as.factor(halfyearOFbirth) +
                   as.factor(nutsATage50), data=d.mort)

model.ols2 <- lm(before67dead ~ dist65_ageATend4emp, data=d.mort)

stargazer(model.ols1, model.ols2, keep.stat='n', header=F,
          keep='dist65_ageATend4emp', font.size='small',
          column.labels=c('Control', 'Non-control'), digits=4,
          title='Comparison between control and non-control')
```

Table 3: Comparison between control and non-control

	Dependent variable:	
	before67dead	
	Control	Non-control
	(1)	(2)
dist65_ageATend4emp	0.0046^{*}	0.0049^{**}
	(0.0026)	(0.0024)
Observations	2,298	2,298
Note:	*p<0.1; **p<0.05; ***p<0.01	

5(c)

As we can see, the coefficient on the treatment slightly increases from column 1 (with control variables) to column 2 (without control variables).

Significance

- With control variables, *p-value* is smaller than 10
- Without control variables, *p-value* is smaller than 5

We can reject the null hypothesis in both cases but we are more confident to reject $\beta_1 = 0$ with control variables.

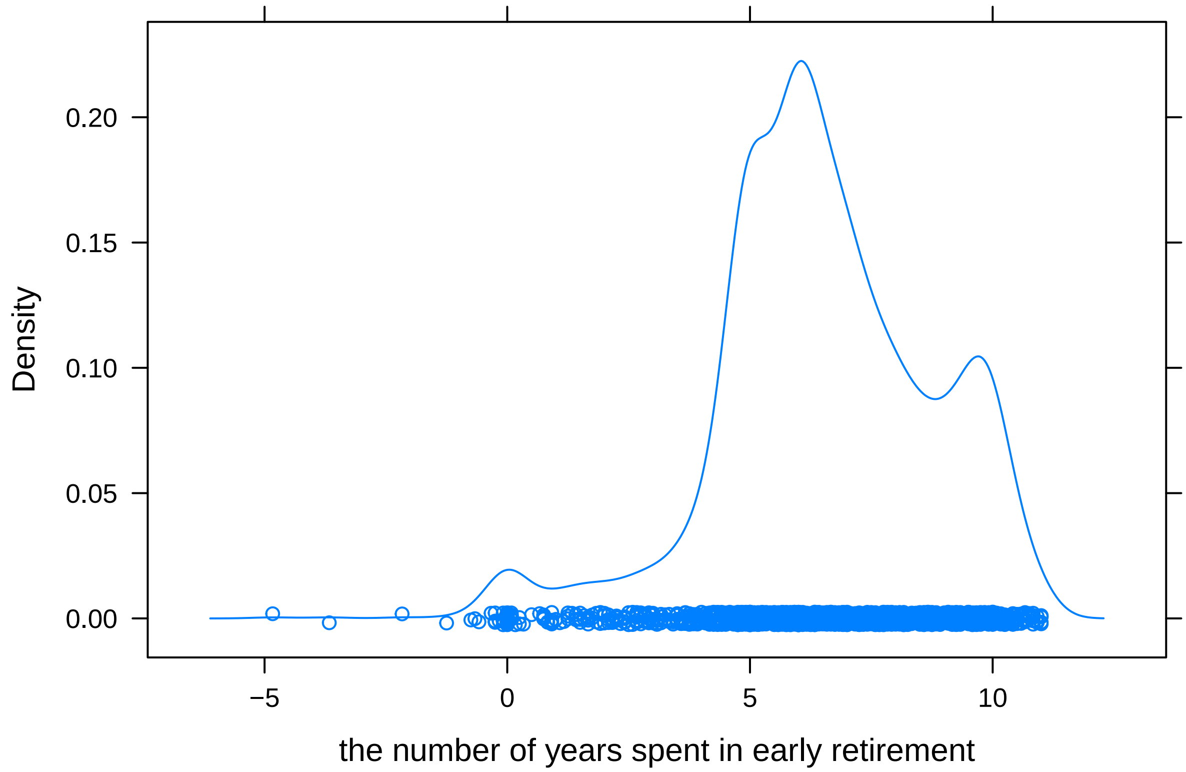
5(d)

Omitted-variable bias

Health status. If people are in a bad physical condition, they are more likely to spend less years in their early retirement or even die before retirement. Therefore, the estimator for β_1 is biased upwards and we expect a positive bias.

5(e)

```
densityplot(~ d.mort$dist65_ageATend4emp, auto.key = TRUE,
            xlab = 'the number of years spent in early retirement',
            data = data.frame(treatment=d.mort$dist65_ageATend4emp,
                              instrument=d.mort$Zd_during))
```



5(f)

```
# first stage regression
iv.1st.stage <- ivreg(dist65_ageATend4emp ~ Zd_during*as.factor(halfyearOFbirth) +
                        czeit1yATage50 + czeit2yATage50 + czeit5yATage50 +
                        czeit10yATage50 + czeit25yATage50 + I(czeit1yATage50^2) +
                        I(czeit2yATage50^2) + I(czeit5yATage50^2) + I(czeit10yATage50^2) +
                        I(czeit25yATage50^2) + as.factor(halfyearOFbirth) +
                        as.factor(nutsATage50), data=d.mort)

stargazer(iv.1st.stage, keep='Zd_during', keep.stat='n', header=F,
          font.size='small', title='First stage regression', no.space=T)
```

5(g)

```
# second stage regression
iv.2nd.stage <- lm(before67dead ~ predict(iv.1st.stage) + czeit1yATage50 +
                     czeit2yATage50 + czeit5yATage50 + czeit10yATage50 +
                     czeit25yATage50 + I(czeit1yATage50^2) + I(czeit2yATage50^2) +
                     I(czeit5yATage50^2) + I(czeit10yATage50^2) +
                     I(czeit25yATage50^2) + as.factor(halfyearOFbirth) +
```

Table 4: First stage regression

	Dependent variable:
	dist65_ageATend4emp
Zd_during	0.664*
	(0.345)
Zd_during:as.factor(halfyearOFbirth)-50	−0.448
	(0.492)
Zd_during:as.factor(halfyearOFbirth)-49	−0.070
	(0.505)
Zd_during:as.factor(halfyearOFbirth)-48	0.179
	(0.506)
Zd_during:as.factor(halfyearOFbirth)-47	−0.280
	(0.492)
Zd_during:as.factor(halfyearOFbirth)-46	0.424
	(0.471)
Zd_during:as.factor(halfyearOFbirth)-45	0.253
	(0.509)
Zd_during:as.factor(halfyearOFbirth)-44	0.525
	(0.504)
Zd_during:as.factor(halfyearOFbirth)-43	0.558
	(0.495)
Zd_during:as.factor(halfyearOFbirth)-42	0.150
	(0.460)
Zd_during:as.factor(halfyearOFbirth)-41	1.229***
	(0.450)
Zd_during:as.factor(halfyearOFbirth)-40	0.681
	(0.451)
Zd_during:as.factor(halfyearOFbirth)-39	0.608
	(0.463)
Zd_during:as.factor(halfyearOFbirth)-38	0.524
	(0.479)
Zd_during:as.factor(halfyearOFbirth)-37	0.345
	(0.459)
Observations	2,298
Note:	*p<0.1; **p<0.05; ***p<0.01

```r
                     as.factor(nutsATage50), data=d.mort)

# iv regression
model.iv <- ivreg(before67dead ~ dist65_ageATend4emp + czeit1yATage50 +
                     czeit2yATage50 + czeit5yATage50 + czeit10yATage50 +
                     czeit25yATage50 + I(czeit1yATage50^2) + I(czeit2yATage50^2) +
                    I(czeit5yATage50^2) + I(czeit10yATage50^2) + I(czeit25yATage50^2) +
                    as.factor(halfyearOFbirth) + as.factor(nutsATage50) |
                    Zd_during*as.factor(halfyearOFbirth) + czeit1yATage50 +
                    czeit2yATage50 + czeit5yATage50 + czeit10yATage50 +
                    czeit25yATage50 + I(czeit1yATage50^2) + I(czeit2yATage50^2) +
                    I(czeit5yATage50^2) + I(czeit10yATage50^2) + I(czeit25yATage50^2) +
                    as.factor(halfyearOFbirth) + as.factor(nutsATage50), data=d.mort)

stargazer(iv.2nd.stage, model.iv, font.size='small', header=F,
          keep.stat=c('n', 'f'), title='Comparsion between 2SLS and ivreg',
          keep=c('iv.1st.stage', 'dist65_ageATend4emp'), digits=4)
```

Table 5: Comparsion between 2SLS and ivreg

	Dependent variable:	
	before67dead	
	OLS	*instrumental variable*
	(1)	(2)
predict(iv.1st.stage)	−0.0143	
	(0.0109)	
dist65_ageATend4emp		−0.0143
		(0.0110)
Observations	2,298	2,298
F Statistic	1.3992* (df = 32; 2265)	
Note:	*p<0.1; **p<0.05; ***p<0.01	

As we can see, **2SLS** and **ivreg** yield exactly the same estimate but with different standard errors.

5(h)

```r
stargazer(model.ols1, model.iv, font.size='small', header=F,
          keep.stat=c('n', 'f'), keep='dist65_ageATend4emp',
          title='Comparison between OLS and 2SLS results', digits=4)
```

As expected, from column(1) to column(2), we see a decrease in the coefficient on *dist65_ageATend4emp*, which verifies our statement in 5(d) - a positive bias in the OLS estimator.

Table 6: Comparison between OLS and 2SLS results

	Dependent variable:	
	before67dead	
	OLS	*instrumental variable*
	(1)	(2)
dist65_ageATend4emp	0.0046*	−0.0143
	(0.0026)	(0.0110)
Observations	2,298	2,298
F Statistic	1.4431* (df = 32; 2265)	
Note:		*p<0.1; **p<0.05; ***p<0.01