CR-P0 方法求解二维 Navier-Stokes 方程

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1 理论推导

1.1 问题描述

对于 Navier-Stokes 问题

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = f, \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 = h, \quad \text{in } \Omega \\ \mathbf{u} = g = 0, \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

其中，取 $\nu = 1$。这里函数 $h(\mathbf{x}, t)$ 为预留函数接口，方便后续处理随机项。

第一步，先求解如下方程

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = f, \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 = h, \quad \text{in } \Omega \\ \mathbf{u} = g = 0, \quad \text{on } \partial\Omega \end{cases} \tag{2}$$

其中

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \begin{pmatrix} u_1\partial_x + u_2\partial_y \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1\partial_x u_1 + u_2\partial_y u_1 \\ u_1\partial_x u_2 + u_2\partial_y u_2 \end{pmatrix}$$

对于此非线性问题，需要使用牛顿迭代方法，即找到 $u^{(l)} \in H^1(\Omega) \times H^1(\Omega)$，$p^{(l)} \in L^2(\Omega)$ 使得

$$\begin{cases} c(\mathbf{u}^{(l)}, \mathbf{u}^{(l-1)}, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l)}, \mathbf{v}) + a(\mathbf{u}^{(l)}, \mathbf{v}) + b(\mathbf{v}, p^{(l)}) \\ \qquad = (f, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l-1)}, \mathbf{v}), \\ b(\mathbf{u}^{(l)}, q) = 0 = (h, q). \end{cases} \tag{3}$$

其中

$$a(\mathbf{u},\mathbf{v})=\nu(\nabla\mathbf{u},\nabla\mathbf{v}),\quad b(\mathbf{v},p)=-(\mathrm{div}\ \mathbf{v},p)$$

和用 CRP0 方法求解 Stokes 方程中的算子一样，而

$$c(\mathbf{w},\mathbf{u},\mathbf{v})=((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}).$$

1.2 推导 $c(\mathbf{u}_h^{(l-1)},\mathbf{u}_h^{(l)},\mathbf{v}_h)$

为了方便描述，这里假设牛顿迭代前一步得到的为 $\mathbf{u}_h^0$, 现在要求的是 $\mathbf{u}_h$，则

$$\begin{aligned}
&c(\mathbf{u}_h^0,\mathbf{u}_h,\mathbf{v}_h)\\
&=\iint_K(\vec{u}_h^0\cdot\nabla)\vec{u}_h\cdot\vec{v}_h dxdy\\
&=\iint_K(u_{h1}^0\partial_x+u_{h2}^0\partial_y)\vec{u}_h\cdot\vec{v}_h dxdy\\
&=\iint_K\begin{pmatrix}u_{h1}^0\partial_x u_1+u_{h2}^0\partial_y u_1\\ u_{h1}^0\partial_x u_2+u_{h2}^0\partial_y u_2\end{pmatrix}\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}dxdy\\
&=\iint_K\begin{pmatrix}\vec{u}_h^0\cdot(\nabla u_1)\\ \vec{u}_h^0\cdot(\nabla u_2)\end{pmatrix}\cdot\begin{pmatrix}v_1\\ v_2\end{pmatrix}dxdy\\
&=\iint_K(\vec{u}_h^0\cdot(\nabla u_1))v_1+(\vec{u}_h^0\cdot(\nabla u_2))v_2 dxdy.
\end{aligned}$$

其中

$$\vec{u}_h=\begin{pmatrix}\phi_1&\phi_2&\phi_3&0&0&0\\0&0&0&\phi_1&\phi_2&\phi_3\end{pmatrix}\begin{pmatrix}u_{11}^0\\u_{12}^0\\u_{13}^0\\u_{21}^0\\u_{22}^0\\u_{23}^0\end{pmatrix},$$

$$\nabla u_i=\begin{pmatrix}\nabla\phi_1&\nabla\phi_2&\nabla\phi_3\end{pmatrix}_{2\times 3}\begin{pmatrix}u_{i1}\\u_{i2}\\u_{i3}\end{pmatrix},i=1,2$$

$$v_i=\begin{pmatrix}v_{i1}&v_{i2}&v_{i3}\end{pmatrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix}\quad i=1,2.$$