CR-P0 方法求解二维 Navier-Stokes 方程

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1 理论推导

1.1 问题描述

对于 Navier-Stokes 问题

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = f, \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 = h, \quad \text{in } \Omega \\ \mathbf{u} = g = 0, \quad \text{on } \partial\Omega \end{cases} \tag{1}$$

其中，取 $\nu = 1$。这里函数 $h(\mathbf{x}, t)$ 为预留函数接口，方便后续处理随机项。

第一步，先求解如下方程

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = f, \quad \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 = h, \quad \text{in } \Omega \\ \mathbf{u} = g = 0, \quad \text{on } \partial\Omega \end{cases} \tag{2}$$

其中

$$\nabla = \begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix},$$

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \begin{pmatrix} u_1\partial_x + u_2\partial_y \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_1\partial_x u_1 + u_2\partial_y u_1 \\ u_1\partial_x u_2 + u_2\partial_y u_2 \end{pmatrix}$$

对于此非线性问题，需要使用牛顿迭代方法，即找到 $u^{(l)} \in H^1(\Omega)\times H^1(\Omega)$，$p^{(l)} \in L^2(\Omega)$ 使得

$$\begin{cases} c(\mathbf{u}^{(l)}, \mathbf{u}^{(l-1)}, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l)}, \mathbf{v}) + a(\mathbf{u}^{(l)}, \mathbf{v}) + b(\mathbf{v}, p^{(l)}) \\ \qquad = (f, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l-1)}, \mathbf{v}), \\ b(\mathbf{u}^{(l)}, q) = 0 = (h, q). \end{cases} \tag{3}$$

其中

$$a(\mathbf{u},\mathbf{v}) = \nu(\nabla\mathbf{u},\nabla\mathbf{v}),\quad b(\mathbf{v},p) = -(\text{div }\mathbf{v},p)$$

和用 CRP0 方法求解 Stokes 方程中的算子一样，而

$$c(\mathbf{w},\mathbf{u},\mathbf{v}) = ((\mathbf{w}\cdot\nabla)\mathbf{u},\mathbf{v}).$$

1.2 常用等式

常用向量展开:

$$\vec{u}_h = \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \\ u_{21}^0 \\ u_{22}^0 \\ u_{23}^0 \end{pmatrix},$$

$$\nabla u_i = \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix}_{2\times 3} \begin{pmatrix} u_{i1} \\ u_{i2} \\ u_{i3} \end{pmatrix}, i=1,2$$

$$v_i = \begin{pmatrix} v_{i1} & v_{i2} & v_{i3} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \quad i=1,2.$$

常用重心坐标函数积分:

$$\int_K \boldsymbol{\lambda}^\alpha(\boldsymbol{x})d\boldsymbol{x} = \frac{\alpha!n!}{(|\alpha|+n)!}|K|. \tag{4}$$

其中 α 是多重指标，$\alpha = (\alpha_1,\alpha_2,\cdots,\alpha_k)$。$\alpha$ 的长度 $|\alpha| = \sum_{i=1}^k \alpha_i$，$\alpha! = \alpha_1!\alpha_2!\cdots\alpha_k!$。对于一个给定的向量 $\boldsymbol{x} = (x_1,x_2,\ldots,x_k)$，$\boldsymbol{x}^\alpha = x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_k^{\alpha_k}$。最后设 $\boldsymbol{\lambda} = (\lambda_1,\lambda_2,\cdots,\lambda_{n+1})$ 表示重心坐标向量，n 表示空间的维数。

对于维数为 2 的情况，有

$$\begin{aligned} \iint_K \lambda_i^2 dxdy &= \frac{2!2!}{(2+2)!}|K| \\ &= \frac{4}{4!}|K| \\ &= \frac{1}{6}|K| \end{aligned}$$

$$\begin{aligned}\iint_K \lambda_i dxdy &= \frac{2!}{(1+2)!}|K| \\ &= \frac{2}{3!}|K| \\ &= \frac{1}{3}|K|\end{aligned}$$

当 $i \neq j$ 时

$$\begin{aligned}\iint_K \lambda_i\lambda_j dxdy &= \frac{2!}{(2+2)!}|K| \\ &= \frac{2}{4!}|K| \\ &= \frac{1}{12}|K|\end{aligned}$$

下面推导在三角形单元上基函数积分 $\iint_K \phi_i\phi_j dxdy$，由上述重心坐标函数积分得

当 $i = j$ 时

$$\begin{aligned}\iint_K \phi_i\phi_j dxdy &= \iint_K \phi_i^2 dxdy \\ &= 4\iint_K (\lambda_i - \frac{1}{2})^2 dxdy \\ &= 4\iint_K (\lambda_i^2 - \lambda_i + \frac{1}{4}) dxdy \\ &= 4|K|(\frac{1}{6} - \frac{1}{3} + \frac{1}{4}) \\ &= 4|K|(\frac{2}{12} - \frac{4}{12} + \frac{3}{12}) \\ &= \frac{|K|}{3},\end{aligned}$$

当 $i \neq j$ 时

$$\begin{aligned}\iint_K \phi_i\phi_j dxdy &= 4\iint_K (\lambda_i - \frac{1}{2})(\lambda_j - \frac{1}{2})dxdy \\ &= 4\iint_K (\lambda_i\lambda_j - \frac{1}{2}\lambda_i - \frac{1}{2}\lambda_j + \frac{1}{4})dxdy \\ &= 4|K|(\frac{1}{12} - \frac{1}{2}\times\frac{1}{3} - \frac{1}{2}\times\frac{1}{3} + \frac{1}{4}) \\ &= 4|K|(\frac{1}{12} - \frac{2}{12} - \frac{2}{12} + \frac{3}{12}) \\ &= 0.\end{aligned}$$

1.3 推导 $c(\mathbf{u}_h^{(l)}, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h)$，矩阵 NA

为了方便描述，这里假设牛顿迭代前一步得到的为 $\mathbf{u}_h^0$, 现在要求的是 $\mathbf{u}_h$，则

$$\begin{aligned}&c(\mathbf{u}_h^l, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h) \\ &= \iint_K (\vec{u}\cdot\nabla)\vec{u}^0 v dxdy \\ &= \iint_K (u_1\partial x + u_2\partial_y)\begin{pmatrix} u_1^0 \\ u_2^0 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy \\ &= \iint_K \begin{pmatrix} u_1\partial x u_1^0 + u_2\partial_y u_1^0 \\ u_1\partial x u_2^0 + u_2\partial_y u_2^0 \end{pmatrix}\cdot\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy \\ &= \iint_K (\vec{u}\cdot(\nabla u_1^0))v_1 + (\vec{u}\cdot(\nabla u_2^0))v_2 dxdy\end{aligned}$$

展开得到

$$
\begin{aligned}
&c(\mathbf{u}_h^l, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h) \\
&= \iint_K \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \\
&\cdot (\nabla\phi_1 u_{11}^0 + \nabla\phi_1 u_{12}^0 + \nabla\phi_1 u_{13}^0) \begin{pmatrix} v_{11} & v_{12} & v_{13} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \\
&+ \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \\ u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \\
&\cdot (\nabla\phi_1 u_{21}^0 + \nabla\phi_1 u_{22}^0 + \nabla\phi_1 u_{23}^0) \begin{pmatrix} v_{21} & v_{22} & v_{23} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} dxdy \\
&= \begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \iint_K \\
&\begin{bmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 & u_{12}^0 & u_{13}^0 \end{pmatrix} \begin{pmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \\ \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \begin{pmatrix} u_{21}^0 & u_{22}^0 & u_{23}^0 \end{pmatrix} \begin{pmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \end{pmatrix} \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \end{bmatrix} dxdy\vec{u}
\end{aligned}
$$

记

$$
sumUvec1 = \begin{pmatrix} u_{11}^0 & u_{12}^0 & u_{13}^0 \end{pmatrix} \begin{pmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \end{pmatrix}
$$

$$sumUvec2 = \begin{pmatrix} u_{21}^0 & u_{22}^0 & u_{23}^0 \end{pmatrix} \begin{pmatrix} \nabla\phi_1 \\ \nabla\phi_2 \\ \nabla\phi_3 \end{pmatrix}$$

我们可以知道矩阵 $sumUvec1$ 及 $sumUvec2$ 均为 1×2 矩阵，记 $sumUvec1(i)$ 及 $sumUvec2(i)$ 为矩阵的第一行第 i 列元素。下面计算推算矩阵 NA：

$$NA$$
$$=\iint_K$$
$$\left[\begin{matrix}\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix}\begin{pmatrix}sumUvec1(1) & sumUvec1(2)\end{pmatrix}\begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0\\0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3\end{pmatrix}\\\begin{pmatrix}\phi_1\\\phi_2\\\phi_3\end{pmatrix}\begin{pmatrix}sumUvec2(1) & sumUvec2(2)\end{pmatrix}\begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0\\0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3\end{pmatrix}\end{matrix}\right]$$
$$dxdy$$
$$=\iint_K$$
$$\left[\begin{matrix}\begin{pmatrix}\phi_1 sumUvec1(1) & \phi_1 sumUvec1(2)\\\phi_2 sumUvec1(1) & \phi_2 sumUvec1(2)\\\phi_3 sumUvec1(1) & \phi_3 sumUvec1(2)\end{pmatrix}\begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0\\0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3\end{pmatrix}\\\begin{pmatrix}\phi_1 sumUvec2(1) & \phi_1 sumUvec2(2)\\\phi_2 sumUvec2(1) & \phi_2 sumUvec2(2)\\\phi_3 sumUvec2(1) & \phi_3 sumUvec2(2)\end{pmatrix}\begin{pmatrix}\phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0\\0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3\end{pmatrix}\end{matrix}\right]$$
$$dxdy$$
$$=\frac{|K|}{3}\times$$
$$\begin{bmatrix}Uvec1(1) & 0 & 0 & Uvec1(2) & 0 & 0\\0 & Uvec1(1) & 0 & 0 & Uvec1(2) & 0\\0 & 0 & Uvec1(1) & 0 & 0 & Uvec1(2)\\Uvec2(1) & 0 & 0 & Uvec2(2) & 0 & 0\\0 & Uvec2(1) & 0 & 0 & Uvec2(2) & 0\\0 & 0 & Uvec2(1) & 0 & 0 & Uvec2(2)\end{bmatrix}$$

生成 NA 矩阵的 Matlab 代码如下:

```
%% 计算左端矩阵c(u^(l), u^(l), v): NA
NA1 = sparse(Nu,Nu);
NA2 = sparse(Nu,Nu);
```

```
NA3 = sparse(Nu,Nu);
NA4 = sparse(Nu,Nu);
sumUvec1 = zeros(NT,1,2);
sumUvec2 = zeros(NT,1,2);
test2 = sumUvec1(:,:,1);
for i = 1:NT
    sumUvec1(i,:)=reshape(u0vec(i,1:3),1,3)*reshape(Dlambda(i,:),3,2);
    sumUvec2(i,:)=reshape(u0vec(i,4:6),1,3)*reshape(Dlambda(i,:),3,2);
end
for i = 1:3
    for j = 1:3
        % 局部节点到全局节点
        ii = double(elem2dof(:,i));
        jj = double(elem2dof(:,j));
        if(i==j)
            NA1 = NA1 + sparse(ii,jj,sumUvec1(:,:,1).*area(:)/3,Nu,Nu);
            NA2 = NA2 + sparse(ii,jj,sumUvec1(:,:,2).*area(:)/3,Nu,Nu);
            NA3 = NA3 + sparse(ii,jj,sumUvec2(:,:,1).*area(:)/3,Nu,Nu);
            NA4 = NA4 + sparse(ii,jj,sumUvec2(:,:,2).*area(:)/3,Nu,Nu);
        end
    end
end
NA = [NA1 NA2; NA3 NA4];
clear NA1 NA2 NA3 NA4
```

1.4 推导 $c(\mathbf{u}_h^{(l-1)}, \mathbf{u}_h^{(l)}, \mathbf{v}_h)$，矩阵 NB

$$
\begin{aligned}
&c(\mathbf{u}_h^0, \mathbf{u}_h, \mathbf{v}_h)\\
&= \iint_K (\vec{u}_h^0 \cdot \nabla)\vec{u}_h \cdot \vec{v}_h dxdy\\
&= \iint_K (u_{h1}^0 \partial_x + u_{h2}^0 \partial_y)\vec{u}_h \cdot \vec{v}_h dxdy\\
&= \iint_K \begin{pmatrix} u_{h1}^0 \partial_x u_1 + u_{h2}^0 \partial_y u_1 \\ u_{h1}^0 \partial_x u_2 + u_{h2}^0 \partial_y u_2 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy\\
&= \iint_K \begin{pmatrix} \vec{u}_h^0 \cdot (\nabla u_1) \\ \vec{u}_h^0 \cdot (\nabla u_2) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy\\
&= \iint_K (\vec{u}_h^0 \cdot (\nabla u_1)) v_1 + (\vec{u}_h^0 \cdot (\nabla u_2)) v_2 dxdy.
\end{aligned}
$$

因此

$$
\begin{aligned}
&c(\boldsymbol{u}_h^{(l-1)}, \boldsymbol{u}_h^{(l)}, \boldsymbol{v}_h) \\
&= \iint_K (\vec{u}_h^0 \cdot (\nabla u_1)) v_1 + (\vec{u}_h^0 \cdot (\nabla u_2)) v_2 dxdy \\
&= \iint_K \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \\ u_{21}^0 \\ u_{21}^0 \\ u_{21}^0 \end{pmatrix} \\
&\cdot \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix} \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \\
&+ \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \\ u_{21}^0 \\ u_{21}^0 \\ u_{21}^0 \end{pmatrix} \\
&\cdot \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix} \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} \begin{pmatrix} v_{21} & v_{22} & v_{23} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} dxdy \\
&= \begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \begin{pmatrix} NB1 & O_{3\times 3} \\ O_{3\times 3} & NB1 \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}
\end{aligned}
$$

其中

$NB1$

$$= \iint_K \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0\phi_1 + u_{12}^0\phi_2 + u_{13}^0\phi_3 & u_{21}^0\phi_1 + u_{22}^0\phi_2 + u_{23}^0\phi_3 \end{pmatrix} dxdy$$

$$\begin{pmatrix} \nabla\phi_1 & \nabla\phi_1 & \nabla\phi_1 \end{pmatrix}$$

$$= \frac{|K|}{3} \begin{pmatrix} u_{11}^0 & u_{21}^0 \\ u_{12}^0 & u_{22}^0 \\ u_{13}^0 & u_{23}^0 \end{pmatrix} \begin{pmatrix} \nabla\phi_1 & \nabla\phi_1 & \nabla\phi_1 \end{pmatrix}$$

$$= \frac{|K|}{3} \begin{pmatrix} u_{11}^0 & u_{21}^0 \\ u_{12}^0 & u_{22}^0 \\ u_{13}^0 & u_{23}^0 \end{pmatrix} \begin{pmatrix} \frac{\partial\phi_1}{\partial x} & \frac{\partial\phi_2}{\partial x} & \frac{\partial\phi_3}{\partial x} \\ \frac{\partial\phi_1}{\partial y} & \frac{\partial\phi_2}{\partial y} & \frac{\partial\phi_3}{\partial y} \end{pmatrix}$$

因此

$$NB1_{ij} = \frac{|K|}{3}(u_{1i}^0\frac{\partial\phi_j}{\partial x} + u_{2i}^0\frac{\partial\phi_j}{\partial y})$$

设

$$newU0vec = \begin{pmatrix} u_{11}^0 & u_{21}^0 \\ u_{12}^0 & u_{22}^0 \\ u_{13}^0 & u_{23}^0 \end{pmatrix}$$

生成矩阵 NA 的 Matlab 代码如下:

```
%% 计算左端矩阵c(u^(l-1), u^(l), v): NB
NB = sparse(Nu,Nu);
newU0vec = reshape(u0vec,NT,3,2);
for i = 1 : 3
    for j = 1 : 3
        % 局部节点到全局节点
        ii = double(elem2dof(:,i));
        jj = double(elem2dof(:,j));
        NBij = (newU0vec(:,i,1).*Dlambda(:,1,j) + ...
            newU0vec(:,i,2).*Dlambda(:,2,j)).*area(:)/3;
        NB = NB + sparse(ii,jj,NBij,Nu,Nu);
    end
end
clear NBij
NB = blkdiag(NB,NB);
```

1.5 推导 $c(\mathbf{u}_h^{(l-1)}, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h)$，矩阵 NF

$$
\begin{aligned}
& c(\mathbf{u}_h^0, \mathbf{u}_h^0, \mathbf{v}_h) \\
& = \iint_K (\vec{u}_h^0 \cdot \nabla)\vec{u}_h^0 \cdot \vec{v}_h dxdy \\
& = \iint_K (u_{h1}^0 \partial_x + u_{h2}^0 \partial_y)\vec{u}_h^0 \cdot \vec{v}_h dxdy \\
& = \iint_K \begin{pmatrix} u_{h1}^0 \partial_x u_1^0 + u_{h2}^0 \partial_y u_1^0 \\ u_{h1}^0 \partial_x u_2^0 + u_{h2}^0 \partial_y u_2^0 \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy \\
& = \iint_K \begin{pmatrix} \vec{u}_h^0 \cdot (\nabla u_1^0) \\ \vec{u}_h^0 \cdot (\nabla u_2^0) \end{pmatrix} \cdot \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} dxdy \\
& = \iint_K (\vec{u}_h^0 \cdot (\nabla u_1^0))v_1 + (\vec{u}_h^0 \cdot (\nabla u_2^0))v_2 dxdy.
\end{aligned}
$$

因此

$$
\begin{aligned}
&c(\boldsymbol{u}_h^{(l-1)}, \boldsymbol{u}_h^{(l-1)}, \boldsymbol{v}_h) \\
&= \iint_K (\vec{u}_h^0 \cdot (\nabla u_1^0))v_1 + (\vec{u}_h^0 \cdot (\nabla u_2^0))v_2 dxdy \\
&= \iint_K \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \\ u_{21}^0 \\ u_{21}^0 \\ u_{21}^0 \end{pmatrix} \\
&\cdot \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} & v_{13} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \\
&+ \begin{pmatrix} \phi_1 & \phi_2 & \phi_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0 \\ u_{12}^0 \\ u_{13}^0 \\ u_{21}^0 \\ u_{21}^0 \\ u_{21}^0 \end{pmatrix} \\
&\cdot \begin{pmatrix} \nabla\phi_1 & \nabla\phi_2 & \nabla\phi_3 \end{pmatrix} \begin{pmatrix} u_{21}^0 \\ u_{22}^0 \\ u_{23}^0 \end{pmatrix} \begin{pmatrix} v_{21} & v_{22} & v_{23} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} dxdy \\
&= \begin{pmatrix} v_1^T & v_2^T \end{pmatrix} \begin{pmatrix} NF1_{3\times1} \\ NF2_{3\times1} \end{pmatrix} \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}
\end{aligned}
$$

其中

$NF1$

$$
\begin{aligned}
&= \iint_K \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} \begin{pmatrix} u_{11}^0\phi_1 + u_{12}^0\phi_2 + u_{13}^0\phi_3 & u_{21}^0\phi_1 + u_{22}^0\phi_2 + u_{23}^0\phi_3 \end{pmatrix} dxdy[sumUvec1]^T \\
&= \frac{|K|}{3} \begin{pmatrix} u_{11}^0 & u_{21}^0 \\ u_{12}^0 & u_{22}^0 \\ u_{13}^0 & u_{23}^0 \end{pmatrix} [sumUvec1]^T \\
&= \frac{|K|}{3} \mathbf{newU0vec}[sumUvec1]^T
\end{aligned}
$$

因此

$$
NF = \frac{|K|}{3} \begin{pmatrix} \mathbf{newU0vec}[sumUvec1]^T \\ \mathbf{newU0vec}[sumUvec2]^T \end{pmatrix}
$$

生成矩阵 NF 的 Matlab 代码如下:

```
%% 计算右端矩阵c(u^(l-1), u^(l-1), v): NF
nf1 = zeros(NT,3,1);
nf2 = zeros(NT,3,1);
for i = 1 : NT
    nf1(i,:,:) = ...
        reshape(newU0vec(i,:,:),3,2)*reshape(sumUvec1(i,:,:),2,1)...
    .*area(i)/3;
    nf2(i,:,:) = ...
        reshape(newU0vec(i,:,:),3,2)*reshape(sumUvec2(i,:,:),2,1)...
    .*area(i)/3;
end
NF1 = accumarray(elem2dof(:),[nf1(:,1); nf1(:,2); nf1(:,3)],[Nu 1]);
NF2 = accumarray(elem2dof(:),[nf2(:,1); nf2(:,2); nf2(:,3)],[Nu 1]);
NF = [NF1;NF2];
```