有限元：1960 R.W. Clough

1943

20世纪60年代 冯康 提出有限元法

基于变分原理的有限差分法 辛几何算法

自适应有限元法

有限差分法 ① 网格限制

② 对第二、三类边界条件处理复杂，可能降阶

1. Sobolev空间初步

1.1 一维区间上的Sobolev空间

$$L^2(I) = \{ f 是 I 上的可测函数，\int_I |f(x)|^2 dx < \infty \}$$ 完备的Hilbert空间

$$(f, g) = \int_I f(x) g(x) dx$$

$$\|f\| = \left(\int_I |f(x)|^2 dx \right)^{1/2}$$

$$-u'' = f$$

弱导数：

$f(x) \in L^2(I)$，若有 $g(x) \in L^2(I)$，满足：

$$\int_I g(x)\varphi(x)dx = -\int_I f(x)\varphi'(x)dx \quad \forall \varphi(x) \in C_0^\infty(I)$$

则称 $g(x)$ 为 $f(x)$ 的一阶弱导数，记作 $f'(x) = g(x)$.

* $f(x) = |x|$， $I = (-1, 1)$

$$\int_{-1}^{1} f(x)\varphi'(x)dx = \int_{-1}^{1} |x|\varphi'(x)dx$$

$$= \int_{-1}^{0} |x|\varphi'(x)dx + \int_0^1 |x|\varphi'(x)dx$$

$$= \int_{-1}^{0} -x\varphi'(x)dx + \int_{0}^{1} x\varphi'(x)\,dx$$

$$= -\int_{-1}^{0} x\,d\varphi(x) + \int_{0}^{1} x\,d\varphi(x)$$

$$= -\,x\varphi(x)\Big|_{-1}^{0} + \int_{-1}^{0}\varphi(x)dx + x\varphi(x)\Big|_{0}^{1} - \int_{0}^{1}\varphi(x)dx$$

$$= \int_{-1}^{0}\varphi(x)dx - \int_{0}^{1}\varphi(x)dx$$

$$= -\int_{-1}^{1} g(x)\varphi(x)dx \qquad g(x) = \begin{cases} -1, & -1<x<0 \\ 1, & 0<x<1 \end{cases}$$

$$\|g(x)\|^2 = \int_{-1}^{1}|g(x)|^2dx = 2<\infty$$

$\therefore g(x)\in L^2(I)$

* 一个函数的广义导数不唯一，可在一个零测集上不相等.

* $f(x) = \begin{cases} 0 & -1\le x\le 0 \\ 1 & 0<x\le 1 \end{cases}$

$$\int_{-1}^{1} g(x)\varphi(x)dx = -\int_{-1}^{1} f(x)\varphi'(x)dx = -\int_{0}^{1}\varphi'(x)dx = -\varphi(x)\Big|_{0}^{1} = \varphi(0)$$

$g(x)=\delta(x)$ — Dirac函数 $\qquad \delta(x)\in H^{-1}(I)$

若 $g(x)=\delta(x)\in L^2(I)$

$$|\varphi(0)| = \left|\int_{-1}^{1} g(x)\varphi(x)dx\right| = |(g,\varphi)| \le \|g\|\|\varphi\|$$

对 $\forall\, 0<\varepsilon<1$，取

$$\varphi(x) = \varphi_\varepsilon(x) = \begin{cases} e^{-\frac{1}{1-(x/\varepsilon)^2}}, & |x|<\varepsilon \\ 0, & |x|\ge\varepsilon \end{cases} \in C_0^\infty(I)$$

$\varphi_\varepsilon(0) = e^{-1}$

$$\|\varphi_\varepsilon\|^2 = \int_{-\varepsilon}^{\varepsilon} e^{-\frac{2}{1-(x/\varepsilon)^2}}dx = 2\int_0^{\varepsilon} e^{-\frac{2}{1-(x/\varepsilon)^2}}dx$$

$$\overset{x=t\varepsilon}{=\!=} 2\varepsilon\int_0^1 e^{-\frac{2}{1-t^2}}dt \to 0\ (\varepsilon\to 0)$$

$e^{-1} = |\varphi_\varepsilon(0)| \le \|g\|\,\|\varphi_\varepsilon\| \to 0\ (\varepsilon\to 0)$ 矛盾 □

阶梯函数没有广义导数.

* $H^1(I) = \{ f\in L^2(I) : f'\in L^2(I)\}$

Sobolev空间

$f, g\in H^1(I)$

$$(f,g) = \int_I (f(x)g(x) + f'(x)g'(x))\,dx$$

$$\|f\|^2 = \int_I (|f(x)|^2 + |f'(x)|^2)\,dx$$

* $H_E^1(I) = \{ f(x)\in H^1(I) : f(a)=0\} \subset H^1(I)$ $\quad I=(a,b)$

$H_0^1(I) = \{ f(x)\in H^1(I) : f(a)=f(b)=0\} \subset H^1(I)$

* 对 $\forall m\in\mathbb{N}$, 可定义 $f(x)\in L^2(I)$ 的 m 阶广义导数 $g(x)\in L^2(I)$ 满足:

$$\int_I g(x)\varphi(x)\,dx = (-1)^m\int_I f(x)\varphi^{(m)}(x)\,dx \quad \forall\varphi\in C_0^\infty(I)$$

$H^m(I) = \{ f\in L^2(I) : f^{(\ell)}(x)\in L^2(I),\ \forall \ell\le m\}$

$f, g\in H^m(I)$

$$(f,g) = \int_I \Big(\sum_{\ell=0}^m f^{(\ell)}(x)g^{(\ell)}(x)\Big)dx$$

$$\|f\|^2 = \int_I \Big(\sum_{\ell=0}^m |f^{(\ell)}(x)|^2\Big)dx$$

1.2 二维区域上的Sobolev空间

$\Omega\subset\mathbb{R}^2$ 有界凸区域

$\alpha = (\alpha_1,\alpha_2)$ $\quad \alpha_i\ (i=1,2)$ 为非负整数.

对 $\varphi\in C_0^\infty(\Omega)$

$$D^{\alpha}\varphi=\frac{\partial^{\alpha_1+\alpha_2}\varphi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}\qquad \alpha=(2,0)\quad D^{\alpha}\varphi=\frac{\partial^2\varphi}{\partial x_1^2}\qquad \alpha=(1,1)\quad D^{\alpha}\varphi=\frac{\partial^2\varphi}{\partial x_1\partial x_2}$$

$|\alpha|=\alpha_1+\alpha_2$

$f(x)\in L^2(\Omega)$，若存在 $g(x)\in L^2(\Omega)$，满足：

$$\int_\Omega f(x)D^{\alpha}\varphi(x)dx=(-1)^{|\alpha|}\int_\Omega g(x)\varphi(x)dx\qquad \forall\varphi(x)\in C_0^\infty(\Omega)$$

则称 $g(x)$ 为 $f(x)$ 的 $|\alpha|$ 阶广义导数.

记为 $D^{\alpha}f(x)=g(x)$

$$H^m(\Omega)=\{f(x)\in L^2(\Omega):D^{\alpha}f(x)\in L^2(\Omega),\ \forall|\alpha|\le m\}$$

$f,g\in H^m(\Omega)$

$$(f,g)=\sum_{|\alpha|\le m}(D^{\alpha}f(x),D^{\alpha}g(x))$$

$$\|f\|^2=\sum_{|\alpha|\le m}\|D^{\alpha}f(x)\|^2$$

$$H^1(\Omega)=\{f(x)\in L^2(\Omega):\frac{\partial f}{\partial x_1},\frac{\partial f}{\partial x_2}\in L^2(\Omega)\}$$

$$H_0^1(\Omega)=\{f(x)\in H^1(\Omega):f(x)|_{\partial\Omega}=0\}$$

2. 变分学基本引理

定理：设 $f\in L^2(\Omega)$，

$$\int_\Omega f(x)\varphi(x)dx=0\qquad\forall\varphi(x)\in C_0^\infty(\Omega)$$

则 $f(x)$ 在 Ω 上几乎处处为零.

$$L^2(\Omega)=\overline{C^\infty(\Omega)}^{\|\cdot\|_{L^2}}\qquad L^2(\Omega)=\overline{C(\Omega)}^{\|\cdot\|_{L^2}}$$

证明：只就 $f(x)\in C(\bar\Omega)$ 时证

$f(x)\equiv0$ in Ω

若 $f(x)\not\equiv0$，不妨设 $f(x)$ 于 $x_0\in\Omega$ 处不等于0.

例如 $f(x_0)>0$，由 $f(x)$ 连续知 $\exists\eta>0$，s.t.

$|x-x_0|<\eta$ 时，有 $f(x)>0$，取

$$\varphi(x) = \begin{cases} \exp\left(-\dfrac{1}{\eta^2-|x-x_0|^2}\right) & |x-x_0|<\eta \\ 0 & \text{其它} \end{cases} \in C_0^\infty(\Omega)$$

易知

$$\int_\Omega f(x)\varphi(x)dx = \int_{|x-x_0|<\eta} f(x)\exp\left(-\frac{1}{\eta^2-|x-x_0|^2}\right)dx > 0$$

与已知条件矛盾，所以 $f(x)\equiv 0$ in Ω.

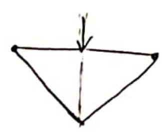
3. 两点边值问题

$$\begin{cases} -Tu'' = f(x) & \text{in } I=(0,1) \\ u(0)=0,\ u'(1)=0 \end{cases} \quad \text{胡克定律} \qquad (1)$$

3.1 极小势能原理

$$\underbrace{J(u)}_{\text{势能}} = \underbrace{\tfrac{1}{2}(-Tu'', u)}_{\text{内力所做的功}} - \underbrace{(f,u)}_{\text{外力所做的功}}$$

取 $T=1$



$$(-u'', u) = \int_0^1 -u''u\,dx$$

$$= -\int_0^1 u\,du'$$

$$= -uu'\Big|_0^1 + \int_0^1 u'u'dx$$

$$= \int_0^1 u'u'dx$$

$$= (u', u')$$

$$J(u) = \frac{1}{2}(u',u') - (f,u) = \frac{1}{2}a(u,u) - (f,u) \qquad a(u,u) = (u',u')$$

$$J(u_*) = \min_{u\in H_E^1(I)} J(u) \qquad (2)$$

u_* 称为 (1) 的弱解.

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$$(1)\ \begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \qquad u \in H_0^1(\Omega)$$

$$\int_\Omega -\Delta u\, v\, d\Omega = \int_\Omega f v\, d\Omega$$

$$\Rightarrow \int_\Omega \nabla u \cdot \nabla v\, d\Omega = \int_\Omega f v\, d\Omega$$

记 $a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, d\Omega$

$$= \int_\Omega \left(\frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} \right) d\Omega$$

求 $u \in H_0^1(\Omega)$. s.t.

$$(2) \quad a(u,v) = (f,v) \quad \forall v \in H_0^1(\Omega)$$

若 $f \in L^2(\Omega)$，且 (2) 解 $u \in H_0^1(\Omega) \cap H^2(\Omega)$，则 u 是(1)的解.

证. $\because u \in H_0^1(\Omega) \cap H^2(\Omega)$

$$\therefore a(u,v) = \int_\Omega \nabla u \cdot \nabla v\, d\Omega$$

$$= -\int_\Omega \Delta u \cdot v\, d\Omega + \underbrace{\int_{\partial\Omega} \frac{\partial u}{\partial n} \cdot v\, ds}_{=0}$$

$$= \int_\Omega f v\, d\Omega \quad \forall v \in H_0^1(\Omega)$$

$$\cancel{\Rightarrow \int_\Omega (-\Delta u}$$

$$\Rightarrow \int_\Omega (-\Delta u - f) v\, d\Omega = 0 \quad \forall v \in H_0^1(\Omega)$$

$$\Rightarrow \int_\Omega \underbrace{(-\Delta u - f)}_{\in L^2(\Omega)} v\, d\Omega = 0 \quad \forall v \in C_0^\infty(\Omega)$$

由变分原理 $\Rightarrow -\Delta u - f = 0$

几乎处处为0 在 $L^2(\Omega)$

$$\Rightarrow -\Delta u = f \quad \text{in } \Omega \quad L^2(\Omega)$$

□

(1) $a(u,v)$ 双线性形式

(2) $a(u,v)=a(v,u)\quad \forall u,v\in H_0^1(\Omega)$ 对称性

(3) $a(u,u)\geq \gamma\|u\|_1^2\quad \forall u\in H_0^1(\Omega)$ 强制性

$\boxed{\text{Poincaré 不等式}}$

$$\|u\|_1^2=\int_\Omega(u^2+|\nabla u|^2)d\Omega$$

（可以把 $a(u,v)$ 看成一个内积，诱导的范数 $|u|_1^2=\int_\Omega|\nabla u|^2d\Omega$，则 $\|u\|_1^2$ 与 $|u|_1^2$ 等价）

(4) $|a(u,v)|=|(\nabla u,\nabla v)|\leq\|\nabla u\|\|\nabla v\|\leq\|u\|_1\|v\|_1$ 双线性性的连续性

Ritz-Galerkin 方法

用 U 表示 Sobolev 空间，求：

$$u\in U\quad s.t.$$

(3) $a(u,v)=(f,v)\quad\forall v\in U$

如何近似求解 (3).

取 $U_N\subset U$ 有限维子空间，取 $\phi_1,\phi_2,\cdots,\phi_N$ 作为 U_N 的一组基函数，用方程：

求 $u_N\in U_N$ s.t.

(4) $a(u_N,v)=(f,v)\quad\forall v\in U_N$

来近似 (3)

$$u_N=\sum_{i=1}^N c_i\phi_i$$

求 $c_i\in\mathbb{R}\ (i=1,2,\cdots,N)$ s.t.

$$a\left(\sum_{i=1}^N c_i\phi_i,v\right)=(f,v)\quad\forall v\in U_N$$

$$\Leftrightarrow\sum_{i=1}^N a(\phi_i,v)c_i=(f,v)\quad\forall v\in U_N$$

$$\Leftrightarrow\sum_{i=1}^N \underbrace{a(\phi_i,\phi_j)}_{\text{已知的常数}}c_i=(f,\phi_j)\quad j=1,2,\cdots,N$$

$$\Leftrightarrow \underbrace{\begin{pmatrix} a(\phi_1,\phi_1) & \cdots & a(\phi_N,\phi_1) \\ a(\phi_1,\phi_2) & \cdots & a(\phi_N,\phi_2) \\ \vdots & \ddots & \vdots \\ a(\phi_1,\phi_N) & \cdots & a(\phi_N,\phi_N) \end{pmatrix}}_{A} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_N) \end{pmatrix}$$

$\forall C=(c_1,\cdots,c_N)^T \in \mathbb{R}^N$ 非零

$$C^TAC = a\left(\sum_{i=1}^N c_i\phi_i, \sum_{i=1}^N c_i\phi_i\right)$$

强制性

$$\geq r\left\|\sum_{i=1}^N c_i\phi_i\right\|_1^2 > 0$$

误差估计

$$\begin{cases} a(u,v)=(f,v) & \forall v\in U_N\subset U \\ a(u_N,v)=(f,v) & \forall v\in U_N \end{cases}$$

$$a(u-u_N,v)=0 \quad \forall v\in U_N$$

$$\|u-u_N\|_1^2 \leq \frac{1}{\gamma}a(u-u_N,u-u_N)$$ 强制性

$$=\frac{1}{\gamma}a(u-u_N, u-v+v-u_N) \quad \forall v\in U_N$$

$$\cancel{=\frac{1}{\gamma}a(u,}$$

$$=\frac{1}{\gamma}a(u-u_N,u-v)+\frac{1}{\gamma}a(u-u_N,\underbrace{v-u_N}_{\in U_N})$$

$$=\frac{1}{\gamma}a(u-u_N,u-v)$$

$$\leq\frac{1}{\gamma}\|u-u_N\|_1\|u-v\|_1$$

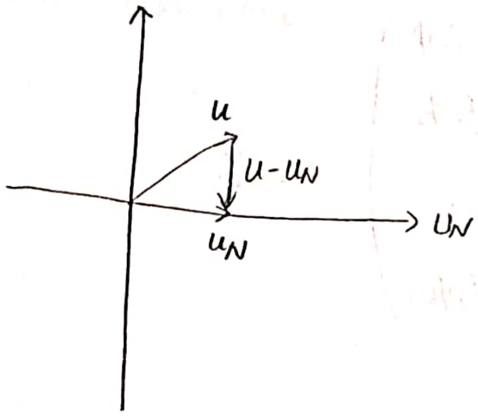
若$\phi_1,\cdots,\phi_N$选 Fourier 级数，为谱方法

$\Rightarrow$ $\cancel{\|u-u_N\|\leq\frac{1}{\gamma}\|u-v\|_1}$

Δ $\|u-u_N\|\leq\frac{1}{\gamma}\underbrace{\inf_{v\in U_N}\|u-v\|_1}$ (céa引理) 最佳逼近

U与U_N集合之间的距离

只要U_N在U中稠密，$u\Rightarrow u_N$



有限元

(1) 把问题转化成变分形式

(2) 对计算区域作网格剖分

(3) 构造基函数或单元形状函数 ☆

(4) 形成有限元方程 ☆

(5) 有限元方程的求解

(6) 收敛性以及误差估计

$$\begin{cases} -u''=f(x) & 0<x<1 \\ u(0)=0,\ \underline{u'(1)=0} & \end{cases}$$

→ 证明过程中自然

有限元程序　期末程序

最好用面向对象来写

x_i: $i=0,1,2,\cdots,N$ 节点

$h_i = x_i - x_{i-1}$

$I_i=(x_{i-1},x_i)$ $i=1,2,\cdots,N$ 单元

$u(x)$　　$u_I(x)=\sum_{i=0}^{N} u(x_i)\varphi_i(x)$

构造基函数: φ_i, $i=0,1,2,\cdots$　　hat function

原则: (1) $\sum_{i=0}^{N}\varphi_i=1$　(2) $\cancel{\varphi_i(x_i)=1}$ $\varphi_i(x_j)=\delta_{ij}$　(3) φ_i与φ_j的支集尽量不相交

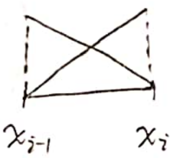
$$\varphi_0(x)=\begin{cases} \dfrac{x_1-x}{x_1-x_0}, & x_0\le x\le x_1 \\ 0, & 其它 \end{cases}$$

$$\phi_i(x)=\begin{cases}\dfrac{x-x_{i-1}}{x_i-x_{i-1}}, & x_{i-1}\le x<x_i \\ \dfrac{x_{i+1}-x}{x_{i+1}-x_i}, & x_i\le x\le x_{i+1} \\ 0, & \text{其它}\end{cases} \qquad i=1,2,\cdots,N-1$$

$$\phi_N(x)=\begin{cases}\dfrac{x-x_{N-1}}{x_N-x_{N-1}}, & x_{N-1}\le x\le x_N \\ 0, & \text{其它}\end{cases}$$

$$u(x)\approx u_I(x)=\sum_{i=0}^{N}u(x_i)\phi_i(x)$$

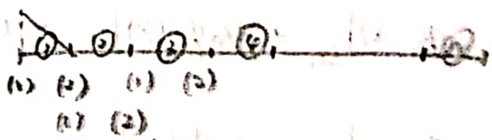
$$\cancel{u_I(x)=\frac{x-x_{i-1}}{x_i-x_{i-1}}u(x_{i-1})}$$



$$u_I(x)=\frac{x_i-x}{x_i-x_{i-1}}u(x_{i-1})+\frac{x-x_{i-1}}{x_i-x_{i-1}}u(x_i) \qquad x\in I_i$$

称为单元形状函数.

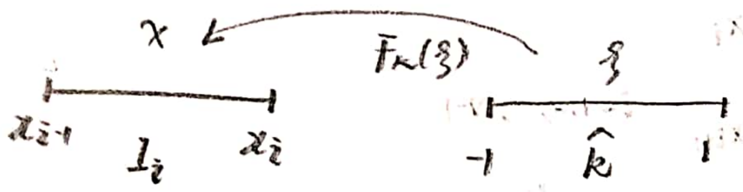
局部编号与整体编号的对应



单元	①	②
1	0	1
2	1	2
3	2	3
4	3	4
5	4	5
⋮		
N	N-1	N

网格、坐标

节点	x_i
0	
1	
2	
⋮	
N	



参考单元

$$x = F_k(\xi) = \frac{x_i - x_{i-1}}{2}\xi + \frac{x_i + x_{i-1}}{2}$$

$$N_1(\xi) = \tfrac{1}{2}(1-\xi) \qquad N_2(\xi) = \tfrac{1}{2}(1+\xi)$$

$$\cancel{a(\phi_i,\phi_j) = \int_0^1 \phi_i' \phi_j' \, dx}$$

$$\cancel{= \sum_{n=1}^{N} \int_{I_n} \phi_i' \phi_j' \, dx = \sum_{n=1}^{N} \frac{h_i}{2} \int_{-1}^{1}}$$

$$a(u,v) = \int_0^1 u'(x) v'(x) \, dx$$

$$= \sum_{n=1}^{N} \int_{I_n} u'(x) v'(x) \, dx$$

$$= \sum_{n=1}^{N} \frac{h_n}{2} \int_{-1}^{1} u'(x) v'(x) \, d\xi = \sum_{n=1}^{N} (v_{n-1}, v_n) \frac{2}{h_n} \int_{-1}^{1} \begin{pmatrix} N_1'(\xi) \\ N_2'(\xi) \end{pmatrix} (N_1'(\xi), N_2'(\xi)) \, d\xi \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$$

$$\cancel{= \sum_{n=1}^{N} \frac{2}{h_n} \frac{h_n}{2}}$$

$$= \sum_{n=1}^{N} (v_{n-1}, v_n) \underbrace{\frac{2}{h_n} \int_{-1}^{1} \begin{bmatrix} N_1'(\xi)^2 & N_1'(\xi) N_2'(\xi) \\ N_1'(\xi) N_2'(\xi) & N_2'(\xi)^2 \end{bmatrix} d\xi}_{\text{单元刚度矩阵}} \begin{bmatrix} u_{n-1} \\ u_n \end{bmatrix}$$

$$\begin{bmatrix} a_{11}^{(n)} & a_{12}^{(n)} \\ a_{21}^{(n)} & a_{22}^{(n)} \end{bmatrix}$$

$$u'(x)\big|_{I_n} = u_{n-1} \phi_{n-1}'(x)\big|_{I_n} + u_n \phi_n'(x)\big|_{I_n} = (\phi_{n-1}', \phi_n')\big|_{I_n} \begin{pmatrix} u_{n-1} \\ u_n \end{pmatrix}$$

$$v'(x)\big|_{I_n} = v_{n-1} \phi_{n-1}'(x)\big|_{I_n} + v_n \phi_n'(x)\big|_{I_n}$$

$$= (v_{n-1}, v_n) \begin{pmatrix} \phi_{n-1}'(x) \\ \phi_n'(x) \end{pmatrix}\Bigg|_{I_n}$$

$$(\phi_{n-1}'(x), \phi_n'(x))\big|_{I_n} = (N_1'(\xi), N_2'(\xi)) \frac{d\xi}{dx}$$

$$= \frac{2}{h_n} (N_1'(\xi), N_2'(\xi))$$

$$(v_1, v_2, \cdots, v_n)\ \begin{array}{c} \\ n-1 \\ n \\ \\ \end{array}\begin{array}{c} \begin{array}{cc} n-1 & n \end{array} \\ \begin{pmatrix} & & & \\ & \bullet & \bullet & \\ & \bullet & \bullet & \\ & & & \end{pmatrix}_{(N+1)\times(N+1)} \end{array} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \\ u_n \\ \vdots \\ u_N \end{pmatrix} = a(u,v)$$

总体刚度矩阵.

$$(f, v) = \sum_{n=1}^{N} \int_{I_n} f(x) v(x)\, dx$$

$$= \sum_{n=1}^{N} (v_{n-1}\ \ v_n) \int_{I_n} \begin{pmatrix} \varphi_{n-1}(x) \\ \varphi_n(x) \end{pmatrix} \cancel{v(x)}\, f(x)\, dx$$

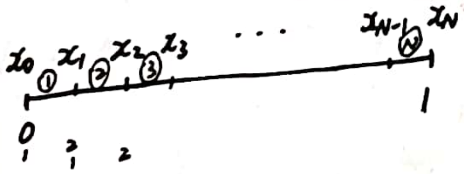
$$= \sum_{n=1}^{N} (v_{n-1}, v_n) \underbrace{\frac{h_n}{2} \int_{-1}^{1} \begin{pmatrix} N_1(\xi) \\ N_2(\xi) \end{pmatrix} f(x(\xi))\, d\xi}_{\text{单元载荷向量}}$$

$$\begin{cases} -u'' = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

$\Rightarrow$ 求 $u \in H_0^1(0,1)$ s.t.

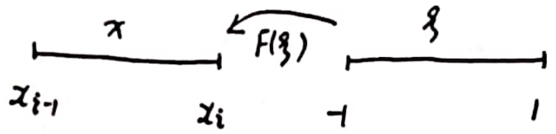
$$a(u, v) = (f, v) \qquad \forall v \in H_0^1(0,1)$$

其中 $a(u,v) = \int_0^1 u'(x) v'(x)\, dx$



$I_i = (x_{i-1}, x_i)$

$h_i = x_i - x_{i-1}$



$N_1(\xi) = \frac{1}{2}(1-\xi)$

$N_2(\xi) = \frac{1}{2}(1+\xi)$

$\therefore$ $F(\xi) = \frac{h_i}{2}\xi + \frac{x_{i-1}+x_i}{2}$

定 ~~单元~~

定 $U_h \subset H_0^1(0,1)$

$u_h(x)\big|_{I_i} = u_{i-1}\varphi_{i-1}\big|_{I_i} + u_i\varphi_i\big|_{I_i} = u_{i-1}N_1(\xi) + u_i N_2(\xi)$

$= (N_1 \ \ N_2)\begin{pmatrix} u_{i-1} \\ u_i \end{pmatrix}$

求 $u_h \in U_h$ s.t. $a(u_h, v_h) = (f, v_h) \quad \forall v_h \in U_h$

则 $a(u_h, v_h) = \int_0^1 u_h'(x) v_h'(x)\,dx = \sum_{i=1}^{N}\int_{x_{i-1}}^{x_i} u_h'(x)v_h'(x)\,dx$

$$= \sum_{i=1}^{N}\int_{-1}^{1}(v_{i-1} \ \ v_i)\begin{pmatrix} N_1(\xi) \\ N_2(\xi)\end{pmatrix}'(N_1(\xi) \ \ N_2(\xi))'\begin{pmatrix} u_{i-1} \\ u_i\end{pmatrix}\frac{h_i}{2}\,d\xi$$

$$= \sum_{i=1}^{N}\int_{-1}^{1}\frac{h_i}{2}(v_{i-1} \ \ v_i)\begin{pmatrix} N_1'(\xi)\frac{d\xi}{dx} \\ N_2'(\xi)\frac{d\xi}{dx}\end{pmatrix}\left(N_1'(\xi)\frac{d\xi}{dx} \ \ N_2'(\xi)\frac{d\xi}{dx}\right)d\xi\begin{pmatrix} u_{i-1} \\ u_i\end{pmatrix}$$

$$= \sum_{i=1}^{N}\frac{2}{h_i}(v_{i-1} \ \ v_i)\int_{-1}^{1}\begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2}\end{pmatrix}\left(-\frac{1}{2} \ \ \frac{1}{2}\right)d\xi\begin{pmatrix} u_{i-1} \\ u_i\end{pmatrix}$$

$$= \sum_{i=1}^{N}\frac{2}{h_i}(v_{i-1} \ \ v_i)\begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}\begin{pmatrix} u_{i-1} \\ u_i\end{pmatrix}$$

$$= \sum_{i=1}^{N}(v_{i-1} \ \ v_i)\underbrace{\begin{pmatrix} \frac{1}{h_i} & -\frac{1}{h_i} \\ -\frac{1}{h_i} & \frac{1}{h_i}\end{pmatrix}}_{\text{单元刚度矩阵}}\begin{pmatrix} u_{i-1} \\ u_i\end{pmatrix}$$

$$
= (v_0, v_1, \cdots, v_N) \sum_{i=1}^{N} \underbrace{\begin{array}{c} \begin{pmatrix} & & & & \\ & \frac{1}{h_i} & -\frac{1}{h_i} & \cdots & \\ & -\frac{1}{h_i} & \frac{1}{h_i} & \cdots & \\ & \vdots & \vdots & & \\ & & & & \end{pmatrix}_{(N+1)\times(N+1)} \begin{array}{l} \\ \cdots i-1 \\ \cdots i \\ \\ \\ \end{array} \\ \begin{array}{cc} i-1 & i \end{array} \end{array}}_{\text{总体刚度矩阵 } A} \underbrace{\begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix}}_{x}
$$

$$
(f, v_h) = \int_0^1 f(x)\, v_h(x)\, dx = \sum_{i=1}^{N} \int_{x_{i-1}}^{x_i} f(x)\, v_h(x)\, dx
$$

$$
= \sum_{i=1}^{N} \frac{h_i}{2} \int_{-1}^{1} f(F(\xi))\,(v_{i-1} \ \ v_i) \begin{pmatrix} N_1(\xi) \\ N_2(\xi) \end{pmatrix} d\xi
$$

$$
= \sum_{i=1}^{N} (v_{i-1} \ \ v_i) \underbrace{\boxed{\frac{h_i}{2} \int_{-1}^{1} f(F(\xi)) \begin{pmatrix} N_1(\xi) \\ N_2(\xi) \end{pmatrix}} d\xi}_{\text{单元载荷向量}}
$$

$$
= (v_0, v_1, \cdots, v_N) \underbrace{\sum_{i=1}^{N} \frac{h_i}{2} \int_{-1}^{1} \begin{array}{c} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ f(F(\xi))N_1(\xi) \\ f(F(\xi))N_2(\xi) \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(N+1)\times 1} \end{array} \begin{array}{l} \\ \\ \\ \cdots i \\ \cdots i-1 \\ \\ \\ \\ \end{array} d\xi}_{\text{载荷向量}}
$$

b

$Ax = b$

边界条件的处理

$$
\begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0N} \\ a_{10} & a_{11} & \cdots & a_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N0} & a_{N1} & \cdots & a_{NN} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \end{pmatrix}
$$

扫描全能王 创建

①
$$a_{00}u_0 + a_{01}u_1 + \cdots + a_{0,N-1}u_{N-1} + a_{0,N}u_N = b_0$$

$$a_{01}u_1 + \cdots + a_{0,N-1}u_{N-1} = b_0 - a_{00}u_0 - a_{0,N}u_N$$
$$\vdots$$
$$a_{N1}u_1 + \cdots + a_{N,N-1}u_{N-1} = b_N - a_{N0}u_0 - a_{NN}u_N$$

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1N-1} \\ a_{21} & a_{22} & \cdots & a_{2,N-1} \\ \cdots & - & & - \\ a_{N-1,1} & a_{N-1,2} & \cdots & a_{N-1,N-1} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_{N-1} \end{pmatrix} - \begin{pmatrix} a_{1,0} \\ a_{2,0} \\ \vdots \\ a_{N-1,0} \end{pmatrix} u_0 - \begin{pmatrix} a_{1,N} \\ a_{2,N} \\ \vdots \\ a_{N-1,N} \end{pmatrix} u_N$$

②
$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & a_{11} & \cdots & 0 \\ \vdots & \vdots & & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} u_0 \\ b_1 \\ \vdots \\ b_{N-1} \\ u_N \end{pmatrix} - \begin{pmatrix} 0 \\ a_{1,0} \\ \vdots \\ a_{N-1,0} \\ 0 \end{pmatrix} u_0 - \begin{pmatrix} 0 \\ a_{1,N} \\ \vdots \\ a_{N-1,N} \\ 0 \end{pmatrix} u_N$$

$f=1$

$$\frac{h_i}{2}\int_{-1}^{1} \begin{pmatrix} \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) \end{pmatrix} d\xi = \begin{pmatrix} \frac{h_i}{2} \\ \frac{h_i}{2} \end{pmatrix}$$

稀疏矩阵 3维 主对角、次对角

扫描全能王 创建

二维

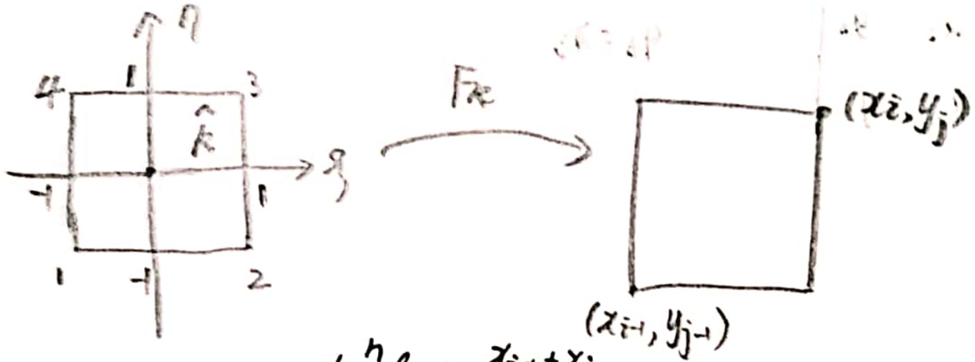
$$\begin{cases} -\Delta u = f(x,y) & (x,y)\in\Omega \\ u=0 & \text{on } \partial\Omega \end{cases}$$

求 $u\in H_0^1(\Omega)$ s.t.

$$a(u,v)=(f,v) \quad \forall v\in H_0^1(\Omega)$$

$$a(u,v)=\iint_\Omega \nabla u\cdot\nabla v\,dxdy$$

hat function



$$\begin{pmatrix} x \\ y \end{pmatrix} = F_K(\xi,\eta) = \begin{pmatrix} \frac{h}{2}\xi + \frac{x_{i-1}+x_i}{2} \\ \frac{h}{2}\eta + \frac{y_{j-1}+y_j}{2} \end{pmatrix}$$

仿射变换

双线性有限元

$$1$$
$$\xi \quad \eta$$
$$\xi^2 \quad \xi\eta \quad \eta^2$$
$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$

0次

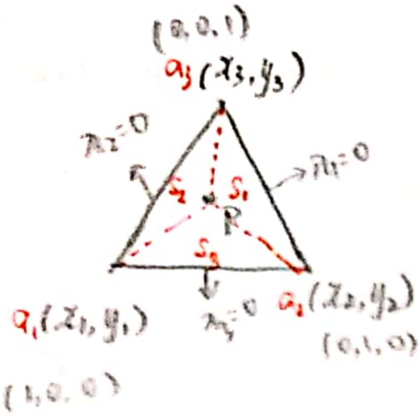
1次

2次多项式

$$N_1(\xi,\eta)=\frac{1}{4}(1-\xi)(1-\eta) \qquad N_4(\xi,\eta)=\frac{1}{4}(1-\xi)(1+\eta)$$

$$N_2(\xi,\eta)=\frac{1}{4}(1+\xi)(1-\eta)$$

$$N_3(\xi,\eta)=\frac{1}{4}(1+\xi)(1+\eta)$$



面积坐标

$P(x,y)$

$$\lambda_i = \frac{S_i}{S} \quad i=1,2,3$$

称$(\lambda_1,\lambda_2,\lambda_3)$为P点的面积坐标.

性质 (1) $0 \le \lambda_i \le 1 \quad i=1,2,3$

(2) $\lambda_1+\lambda_2+\lambda_3=1$.

(3) λ_i是x,y的一次多项式.

$$S = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$$S_1 = \frac{1}{2}\begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

$\varphi_1=\lambda_1$

$\varphi_2=\lambda_2$

$\varphi_3=\lambda_3$