Discontinuous Galerkin (DG) Method 与 Finite Element Method (FEM)、Spectral Element Method (SEM) 理论及统一编程框架

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1 多项式基函数 Polynomials

Modal and Nodal

$$\begin{aligned} u(\boldsymbol{r}) &= \sum_{n=1}^{Np} \hat{u}_n P_{n-1}(\boldsymbol{r}) \\ &= \sum_{i=1}^{Np} u(\boldsymbol{\xi}_i) l_i(\boldsymbol{r}) \end{aligned} \tag{1}$$

其中, $\boldsymbol{r}$ 表示物理量的位置坐标, $\boldsymbol{\xi}_i$ 表示第 i 个节点插值函数 l_i 插值点的位置坐标. u 代表物理量的值, n 为多项式的阶数, P 为多项式

Modal(模式):

$$u(\boldsymbol{r}) = \sum_{n=1}^{Np} \hat{u}_n P_{n-1}(\boldsymbol{r}) \tag{2}$$

$$u(\boldsymbol{r}) = \left[\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_{Np}\right] \begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix}$$

Nodal(节点):

$$u(\boldsymbol{r}) = \sum_{i=1}^{Np} u(\boldsymbol{\xi}_i) l_i(\boldsymbol{r}) \tag{3}$$

$$u(\boldsymbol{r}) = \left[u(\boldsymbol{\xi}_1), u(\boldsymbol{\xi}_2), \cdots, u(\boldsymbol{\xi}_{Np})\right] \begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix}$$

令 $\boldsymbol{r} = \boldsymbol{\xi}_i$, 根据 Eq.(2), 于是有:

$$u(\boldsymbol{\xi}_i) = \sum_{n=1}^{Np} \hat{u}_n P_{n-1}(\boldsymbol{\xi}_i) \tag{4}$$

$$u(\boldsymbol{\xi}_i) = \left[\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_{Np}\right] \begin{bmatrix} P_0(\boldsymbol{\xi}_i) \\ P_1(\boldsymbol{\xi}_i) \\ \vdots \\ P_{Np-1}(\boldsymbol{\xi}_i) \end{bmatrix}$$

将代入 Eq.(4) 到 Eq.(3), 有

$$
\begin{aligned}
u(\boldsymbol{r}) &= \left[u(\boldsymbol{\xi}_1), u(\boldsymbol{\xi}_2), \cdots, u(\boldsymbol{\xi}_{Np})\right] \begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix} \\
&= \left[\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_{Np}\right] \begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_0(\boldsymbol{\xi}_2) & \cdots & P_0(\boldsymbol{\xi}_{Np}) \\ P_1(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_2) & \cdots & P_1(\boldsymbol{\xi}_{Np}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{Np}(\boldsymbol{\xi}_1) & P_{Np}(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix} \begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix}
\end{aligned}
$$

而且, 由 Eq.(2)

$$
u(\boldsymbol{r}) = \left[\hat{u}_1, \hat{u}_2, \cdots, \hat{u}_{Np}\right] \begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix}
$$

因此, 容易得到:

$$
\begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_0(\boldsymbol{\xi}_2) & \cdots & P_0(\boldsymbol{\xi}_{Np}) \\ P_1(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_2) & \cdots & P_1(\boldsymbol{\xi}_{Np}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{Np}(\boldsymbol{\xi}_1) & P_{Np}(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix} \begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix} = \begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix} \tag{5}
$$

我们可以将 Eq.(5) 表示成

$$
V^T \boldsymbol{l}(\boldsymbol{r}) = \boldsymbol{P}(\boldsymbol{r}) \tag{6}
$$

$$
\begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix} = \begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_0(\boldsymbol{\xi}_2) & \cdots & P_0(\boldsymbol{\xi}_{Np}) \\ P_1(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_2) & \cdots & P_1(\boldsymbol{\xi}_{Np}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{Np}(\boldsymbol{\xi}_1) & P_{Np}(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix}^{-1} \begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix} \tag{7}
$$

质量矩阵 (Mass Matrix)

$$
\begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix} \begin{bmatrix} l_1(\boldsymbol{r}) l_2(\boldsymbol{r}) \cdots l_{Np}(\boldsymbol{r}) \end{bmatrix}
= \begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_0(\boldsymbol{\xi}_2) & \cdots & P_0(\boldsymbol{\xi}_{Np}) \\ P_1(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_2) & \cdots & P_1(\boldsymbol{\xi}_{Np}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{Np}(\boldsymbol{\xi}_1) & P_{Np}(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix}^{-1}
\begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix} \begin{bmatrix} P_0(\boldsymbol{r}) P_1(\boldsymbol{r}) \cdots P_{Np-1}(\boldsymbol{r}) \end{bmatrix}
\begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_1) & \cdots & P_{Np}(\boldsymbol{\xi}_1) \\ P_0(\boldsymbol{\xi}_2) & P_1(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(\boldsymbol{\xi}_{Np}) & P_1(\boldsymbol{\xi}_{Np}) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix}^{-1} \tag{8}
$$

一般来说, 我们选用的基函数 $\boldsymbol{P_{n-1}(r)}$ 多为正交基函数, 那么则有:

$$
\begin{bmatrix} P_0(\boldsymbol{r}) \\ P_1(\boldsymbol{r}) \\ \vdots \\ P_{Np-1}(\boldsymbol{r}) \end{bmatrix} \begin{bmatrix} P_0(\boldsymbol{r}) P_1(\boldsymbol{r}) \cdots P_{Np-1}(\boldsymbol{r}) \end{bmatrix} = \boldsymbol{E} \tag{9}
$$

于是, 有

$$
\begin{bmatrix} l_1(\boldsymbol{r}) \\ l_2(\boldsymbol{r}) \\ \vdots \\ l_{Np}(\boldsymbol{r}) \end{bmatrix} \begin{bmatrix} l_1(\boldsymbol{r}) l_2(\boldsymbol{r}) \cdots l_{Np}(\boldsymbol{r}) \end{bmatrix}
= \begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_0(\boldsymbol{\xi}_2) & \cdots & P_0(\boldsymbol{\xi}_{Np}) \\ P_1(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_2) & \cdots & P_1(\boldsymbol{\xi}_{Np}) \\ \vdots & \vdots & \ddots & \vdots \\ P_{Np}(\boldsymbol{\xi}_1) & P_{Np}(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix}^{-1}
\begin{bmatrix} P_0(\boldsymbol{\xi}_1) & P_1(\boldsymbol{\xi}_1) & \cdots & P_{Np}(\boldsymbol{\xi}_1) \\ P_0(\boldsymbol{\xi}_2) & P_1(\boldsymbol{\xi}_2) & \cdots & P_{Np}(\boldsymbol{\xi}_2) \\ \vdots & \vdots & \ddots & \vdots \\ P_0(\boldsymbol{\xi}_{Np}) & P_1(\boldsymbol{\xi}_{Np}) & \cdots & P_{Np}(\boldsymbol{\xi}_{Np}) \end{bmatrix}^{-1} \tag{10}
$$

2 间断 Galerkin 原理

守恒型方程:

$$\frac{\partial u(\boldsymbol{x},t)}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{f}(\boldsymbol{x},t) = 0 \tag{11}$$

Integration by parts(分部积分)

$$\int_{D_k} \phi\boldsymbol{\nabla}\cdot\boldsymbol{f}d\Omega = \int_{\partial\boldsymbol{x}} \phi\boldsymbol{n}\cdot\boldsymbol{f}dS - \int_{D_k} \boldsymbol{\nabla}\phi\cdot\boldsymbol{f}d\Omega \tag{12}$$

Galerkin 变分

$$\int_{D_k} \frac{\partial u(\boldsymbol{x},t)}{\partial t}\phi(\boldsymbol{x})d\Omega + \int_{D_k} \boldsymbol{\nabla}\cdot\boldsymbol{f}(\boldsymbol{x},t)\phi(\boldsymbol{x})d\Omega = 0 \tag{13}$$

分部积分

$$\int_{D_k} \frac{\partial u^k}{\partial t}\phi d\Omega = \int_{D_k} \boldsymbol{f}^k\cdot\boldsymbol{\nabla}\phi^k d\Omega - \int_{\partial\boldsymbol{x}} \boldsymbol{n}\cdot\boldsymbol{f}^*\phi_k dS \tag{14}$$

插值多项式:

$$u^k = \sum_{n=1}^{Np} C_i^k(t)\phi_i^k \tag{15}$$

$$\int_{D_k} \frac{\partial \sum\limits_{n=1}^{Np} C_i^k(t)\phi_i^k}{\partial t}\phi_j^k d\Omega = \int_{D_k} \boldsymbol{f}^k\cdot\boldsymbol{\nabla}\phi_j^k d\Omega - \int_{\partial\boldsymbol{x}} \boldsymbol{n}\cdot\boldsymbol{f}^*\phi_j^k dS \tag{16}$$

2.1 质量矩阵 (Mass Matrix)

$$\begin{aligned}
&\int_{D_k} \frac{\partial u^k}{\partial t}\boldsymbol{\phi}^k d\Omega \\
&= \left[\tfrac{\partial C_1^k}{\partial t}, \tfrac{\partial C_t^k}{\partial t}, \cdots, \tfrac{\partial C_{Np}^k}{\partial t}\right]\begin{bmatrix}\phi_1^k \\ \phi_2^k \\ \vdots \\ \phi_{Np}^k\end{bmatrix} \\
&= \left[\tfrac{\partial C_1^k}{\partial t}, \tfrac{\partial C_t^k}{\partial t}, \cdots, \tfrac{\partial C_{Np}^k}{\partial t}\right]\begin{bmatrix}\int_{D_k}\phi_1^k\phi_1^k d\Omega & \int_{D_k}\phi_1^k\phi_2^k d\Omega & \cdots & \int_{D_k}\phi_1^k\phi_{Np}^k d\Omega \\ \int_{D_k}\phi_2^k\phi_1^k d\Omega & \int_{D_k}\phi_2^k\phi_2^k d\Omega & \cdots & \int_{D_k}\phi_2^k\phi_{Np}^k d\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_{D_k}\phi_{Np}^k\phi_1^k d\Omega & \int_{D_k}\phi_{Np}^k\phi_2^k d\Omega & \cdots & \int_{D_k}\phi_{Np}^k\phi_{Np}^k d\Omega\end{bmatrix}
\end{aligned} \tag{17}$$

因此, 质量矩阵为

$$\boldsymbol{M}_{ij} = \int_{D_k} \phi_i^k\phi_j^k d\Omega \tag{18}$$

$$\begin{aligned}
\boldsymbol{M}_{ij} &= \int_{D_k} \phi_i^k(\boldsymbol{x})\phi_j^k(\boldsymbol{x})d\Omega \\
&= \int_I \phi_i(\boldsymbol{r})\phi_j(\boldsymbol{r})J^k d\sigma
\end{aligned} \tag{19}$$

2.2 刚度矩阵 (Stiff Matrix)

2.2.1 线性方程

以 $f = au$ 为例子, 只讨论 x 方向, 其他方向完全一致:

$$
\begin{aligned}
\int_{D_k} f^k \frac{\partial \phi^k}{\partial x} d\Omega &= a\left[C_1^k, C_t^k, \cdot, C_{Np}^k\right] \begin{bmatrix} \phi_1^k \\ \phi_2^k \\ \vdots \\ \phi_{Np}^k \end{bmatrix} \left[\frac{\partial \phi_1^k}{\partial x}, \frac{\partial \phi_2^k}{\partial x}, \cdots, \frac{\partial \phi_{Np}^k}{\partial x}\right] \\
&= a\left[C_1^k, C_t^k, \cdots, C_{Np}^k\right] \begin{bmatrix} \int_{D_k} \phi_1^k \frac{\partial \phi_1^k}{\partial x} d\Omega & \int_{D_k} \phi_1^k \frac{\partial \phi_2^k}{\partial x} d\Omega & \cdots & \int_{D_k} \phi_1^k \frac{\partial \phi_{Np}^k}{\partial x} d\Omega \\ \int_{D_k} \phi_2^k \frac{\partial \phi_1^k}{\partial x} d\Omega & \int_{D_k} \phi_2^k \frac{\partial \phi_2^k}{\partial x} d\Omega & \cdots & \int_{D_k} \phi_2^k \frac{\partial \phi_{Np}^k}{\partial x} d\Omega \\ \vdots & \vdots & \ddots & \vdots \\ \int_{D_k} \phi_{Np}^k \frac{\partial \phi_1^k}{\partial x} d\Omega & \int_{D_k} \phi_{Np}^k \frac{\partial \phi_2^k}{\partial x} d\Omega & \cdots & \int_{D_k} \phi_{Np}^k \frac{\partial \phi_{Np}^k}{\partial x} d\Omega \end{bmatrix}
\end{aligned} \tag{20}
$$

因此, 刚度矩阵

$$
\boldsymbol{S}_{ij} = \int_{D_k} \phi_i^k \frac{\partial \phi_j^k}{\partial x} d\Omega \tag{21}
$$

由 Eq.(33) 可得

$$
\begin{aligned}
\boldsymbol{S}_{ij} &= \int_{D_k} \phi_i^k \frac{\partial \phi_j^k}{\partial x} d\Omega \\
&= \int_I \phi_i(\boldsymbol{r}) \left(\frac{\partial \phi_j(\boldsymbol{r})}{\partial r} \frac{\partial r}{\partial x} + \frac{\partial \phi_j(\boldsymbol{r})}{\partial s} \frac{\partial s}{\partial x} + \frac{\partial \phi_j(\boldsymbol{r})}{\partial t} \frac{\partial t}{\partial x} \right) J^k d\sigma
\end{aligned} \tag{22}
$$

这里我们根据 Eq.(41)-2D 和 Eq.(44)-3D 分别推导 $2D$ 和 $3D$ 刚度矩阵的沿着各个方向的链式 (Chain Rule) 展开:

二维:

$$
\begin{aligned}
\boldsymbol{S}_{ij} &= \int_{D_k} \phi_i^k \frac{\partial \phi_j^k}{\partial x} d\Omega \\
&= \int_I \phi_i(r,s) \left(\frac{\partial \phi_j(r,s)}{\partial r} r_x + \frac{\partial \phi_j(r,s)}{\partial s} s_x \right) J^k d\sigma \\
&= \int_I \phi_i(r,s) \left(\frac{\partial \phi_j(r,s)}{\partial r} \frac{y_s}{J^k} + \frac{\partial \phi_j(r,s)}{\partial s} (-\frac{y_r}{J^k}) \right) J^k d\sigma \\
&= \int_I \phi_i(r,s) \left(\frac{\partial \phi_j(r,s)}{\partial r} y_s - \frac{\partial \phi_j(r,s)}{\partial s} y_r \right) d\sigma
\end{aligned} \tag{23}
$$

$$
\begin{aligned}
\boldsymbol{S}_{ij} &= \int_{D_k} \phi_i^k \frac{\partial \phi_j^k}{\partial y} d\Omega \\
&= \int_I \phi_i(r,s) \left(\frac{\partial \phi_j(r,s)}{\partial r} y_s - \frac{\partial \phi_j(r,s)}{\partial s} y_r \right) d\sigma
\end{aligned} \tag{24}
$$

三维:

2.2.2 非线性方程

当然, 如果是非线性, 那么 C_1^k 就不能提出来, 每个时间迭代都需要计算 $\boldsymbol{f}$ 与 $\boldsymbol{\phi}$ 相乘获取的新函数在高斯点的值, 然后用高斯权重求取高斯积分.

2.3 通量积分 (Flux Integral)

首先, 我们考虑最普世的 2D、3D 的 Surface Integral 方法. 具体内容参考了 Youtube 上的视频, 如 Mu Prime Math2D、Mu Prime Math3D. 当然也参考了一些网上的资料, 如 Paul's Online Notes2D、Paul's Online Notes3D、Khan Academy. 其中, Khan Academy 的视频和文章给出了详细的物理意义.

2.3.1 二维 (2D)

对于 2 维问题, 就是 Line Integral, 积分公式如下:

矢量积分:

$$\int_C \boldsymbol{f}(\boldsymbol{x}) \cdot d\boldsymbol{x} = \int_a^b \boldsymbol{f}(\boldsymbol{x}(t)) \cdot \boldsymbol{x}^{'}(t) dt \tag{25}$$

标量积分:

$$\int_C f ds = \int_a^b f(\boldsymbol{x}(t)) \left|\boldsymbol{x}^{'}(t)\right| dt \tag{26}$$

其中, $\boldsymbol{x}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, $\left|\boldsymbol{x}^{'}(t)\right| = \sqrt{x^{'}(t)^2 + y^{'}(t)^2}$.

2.3.2 三维 (3D)

对于三维问题, 就是典型的 Surface Integral.

矢量面积分:

$$\begin{aligned}\int_S \boldsymbol{f} \cdot d\boldsymbol{S} &= \int_S \boldsymbol{f} \cdot \boldsymbol{n} dS \\ &= \int_S \boldsymbol{f} \cdot \frac{\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}}{\left|\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}\right|} \cdot \left|\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}\right| d\xi d\eta \\ &= \int_S \boldsymbol{f} \cdot \frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta} d\xi d\eta\end{aligned} \tag{27}$$

由于, $\boldsymbol{n} = \frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}$, $dS = \left|\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}\right| d\xi d\eta$

标量面积分:

$$\begin{aligned}A = \int_S dS &= \int_T \left\|\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}\right\| d\xi d\eta \\ \int_S f dS = \int_S dS &= \int_T f \cdot \left\|\frac{\partial \boldsymbol{x}}{\partial \xi} \times \frac{\partial \boldsymbol{x}}{\partial \eta}\right\| d\xi d\eta\end{aligned} \tag{28}$$

2.3.3 对应实际问题: 以线性方程为例, 求解边质量矩阵

二维 2D 问题

$$\int_{S_k} \boldsymbol{n} \cdot \boldsymbol{f}^k \phi^k(\boldsymbol{x}) ds = \int_C (a \cdot n_x + b \cdot n_y) u^k(\boldsymbol{x}(t)) \phi^k(\boldsymbol{x}) \left|\boldsymbol{x}^{'k}(t)\right| dt \tag{29}$$

对比质量矩阵的求解方法, 很容易就可以得 2D 边质量矩阵 (通量矩阵):

$$\begin{aligned}\boldsymbol{F}_{ij}^k &= \int_C \phi_i^k(x(t), y(t)) \phi_j^k(x(t), y(t)) \left|\boldsymbol{x}^{'k}(t)\right| dt \\ &= \int_C \phi_i(r(t), s(t)) \phi_j(r(t), s(t)) \left|\boldsymbol{x}^{'k}(r(t), s(t))\right| dt\end{aligned} \tag{30}$$

这里, 我们以最简单的直边元为例, 其中包括三角形元和四边形元, 我们只选择一条边积分, 另外的边积分完全一样. 我们选择的边为$(x_1, y_1) \to (x_2, y_2)$

方法一, 直接用边长求解:

线段函数的表达式 2D, 可以写成:

$$\boldsymbol{r} = \begin{bmatrix} r(t) \\ s(t) \end{bmatrix} = \begin{bmatrix} (1-t) \cdot r_1 + t \cdot r_2 \\ (1-t) \cdot s_1 + t \cdot s_2 \end{bmatrix}$$

$$\boldsymbol{F}_{ij}^k = \int_{(r_1,s_1)\to(r_2,s_2)} \phi_i(r(t), s(t))\phi_j(r(t), s(t))\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)}dt \tag{31}$$

然后, 直接代入关于 t 的一维高斯积分点 tg, 就有对应的 $r(tg)$ 及 $s(tg)$. 根据一维 Gauss Quadrature, 于是其中一条边的积分方式为:

$$\boldsymbol{F}_{ij}^k = J^{1D} \sum_n^{Ng} \phi_i(r(tg_n), s(tg_n))\phi_j(r(tg_n), s(tg_n))$$

其中, $J^{1D} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ 为直边元其中一条边的长度.

三维 3D 问题

$$\boldsymbol{F}_{ij}^k = \int_C \phi_i(\boldsymbol{r})\phi_j(\boldsymbol{r}) \cdot \left\| \frac{\partial \boldsymbol{x}^k}{\partial \xi} \times \frac{\partial \boldsymbol{x}^k}{\partial \eta} \right\| d\xi d\eta \tag{32}$$

与 2D 的处理相似, 我们任取一个面, 研究其积分方式, 其他面处理完全一样.

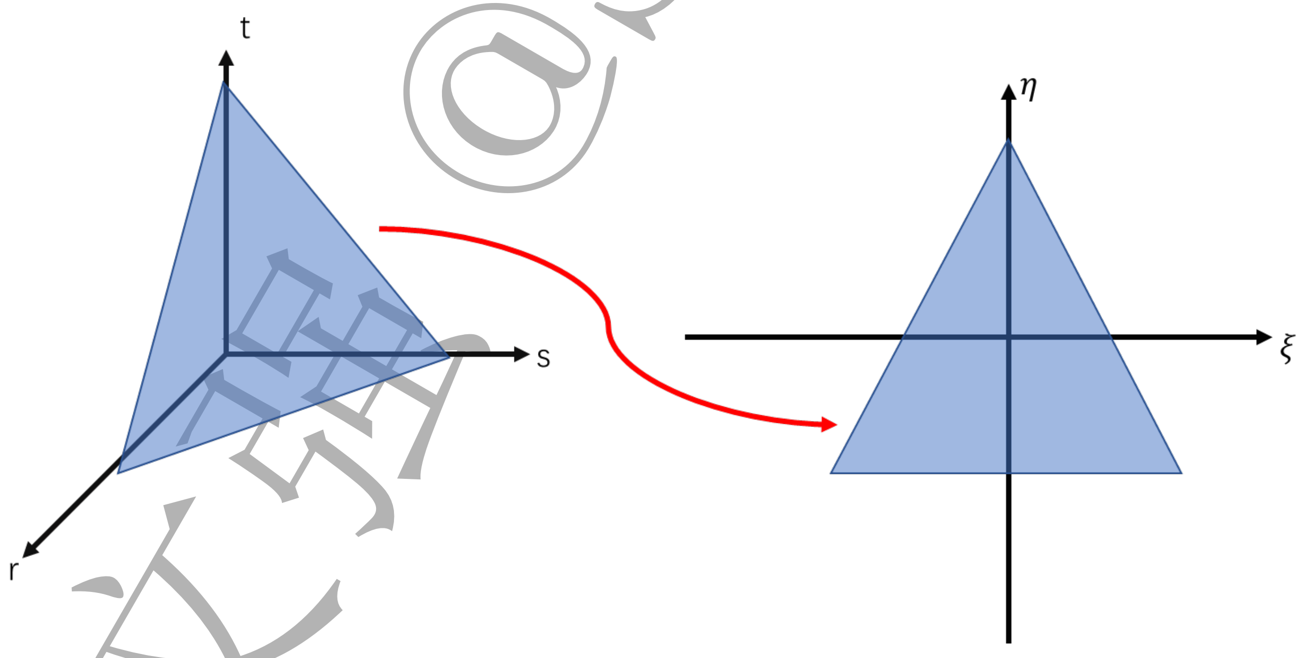


图 1: 三维面积分

如图2.3.3, 我们可以根据右边 2D 三角形的高斯点 (ξ_G, η_G) 确定左边 3D 中对应的位置, 通过几何关系可以求解

$$\boldsymbol{r}_G = \begin{bmatrix} r(\xi_G, \eta_G) \\ s(\xi_G, \eta_G) \\ t(\xi_G, \eta_G) \end{bmatrix}$$

进而,

$$\boldsymbol{F}_{ij}^{k}=J^{2D}\sum_{n}^{Ng}\phi_i(\boldsymbol{r}_G)\phi_j(\boldsymbol{r}_G)$$

其中, $J^{2D}=\left\|\frac{\partial \boldsymbol{x}^k}{\partial\xi}\times\frac{\partial \boldsymbol{x}^k}{\partial\eta}\right\|$ 为其中一个面的面积.

2.4 仿射变换 (Jacobian Transform)

对于任意 N 维的链式求导法则 (Chain Rule), 以 x 方向为例子

$$\begin{aligned}\frac{\partial\phi(\boldsymbol{x})}{\partial x}&=\boldsymbol{\nabla}_r\phi(\boldsymbol{r})\cdot\frac{\partial\boldsymbol{r}}{\partial x}\\&=\left[\frac{\partial\phi(\boldsymbol{r})}{\partial r_1},\frac{\partial\phi(\boldsymbol{r})}{\partial r_2},\cdots,\frac{\partial\phi(\boldsymbol{r})}{\partial r_N}\right]\cdot\left[\frac{\partial r_1}{\partial x},\frac{\partial x}{\partial x},\cdots,\frac{\partial r_N}{\partial x}\right]\\&=\sum_{i=1}^{N}\frac{\partial\phi(\boldsymbol{r})}{\partial r_i}\frac{\partial r_i}{\partial x}\end{aligned}\tag{33}$$

$$\begin{aligned}\boldsymbol{J}&=\boldsymbol{\nabla}\boldsymbol{f}\\&=\begin{bmatrix}\frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \cdots, & \frac{\partial f_1}{\partial x_N}\\ \frac{\partial f_2}{\partial x_1}, & \frac{\partial f_2}{\partial x_2}, & \cdots, & \frac{\partial f_2}{\partial x_N}\\ \vdots, & \vdots, & \ddots, & \vdots\\ \frac{\partial f_N}{\partial x_1}, & \frac{\partial f_N}{\partial x_2}, & \cdots, & \frac{\partial f_N}{\partial x_N}\end{bmatrix}\end{aligned}\tag{34}$$

$$J=\begin{vmatrix}\frac{\partial f_1}{\partial x_1}, & \frac{\partial f_1}{\partial x_2}, & \cdots, & \frac{\partial f_1}{\partial x_N}\\ \frac{\partial f_2}{\partial x_1}, & \frac{\partial f_2}{\partial x_2}, & \cdots, & \frac{\partial f_2}{\partial x_N}\\ \vdots, & \vdots, & \ddots, & \vdots\\ \frac{\partial f_N}{\partial x_1}, & \frac{\partial f_N}{\partial x_2}, & \cdots, & \frac{\partial f_N}{\partial x_N}\end{vmatrix}\tag{35}$$

对于有限元的 Jacobian 为

$$\boldsymbol{J}=\begin{bmatrix}\frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s}, & \frac{\partial x}{\partial t}\\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s}, & \frac{\partial y}{\partial t}\\ \frac{\partial z}{\partial r}, & \frac{\partial z}{\partial s}, & \frac{\partial z}{\partial t}\end{bmatrix}\tag{36}$$

$$\begin{bmatrix}\frac{\partial x}{\partial x}, & \frac{\partial x}{\partial y}, & \frac{\partial x}{\partial z}\\ \frac{\partial y}{\partial x}, & \frac{\partial y}{\partial y}, & \frac{\partial y}{\partial z}\\ \frac{\partial z}{\partial x}, & \frac{\partial z}{\partial y}, & \frac{\partial z}{\partial z}\end{bmatrix}=\begin{bmatrix}1, & 0, & 0\\ 0, & 1, & 0\\ 0, & 0, & 1\end{bmatrix}\tag{37}$$

$$\Updownarrow$$

$$
\begin{aligned}
&\begin{bmatrix}
\frac{\partial x}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial x}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial x}{\partial t}\frac{\partial t}{\partial x} & \frac{\partial x}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial x}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial x}{\partial t}\frac{\partial t}{\partial y} & \frac{\partial x}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial x}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial x}{\partial t}\frac{\partial t}{\partial z} \\
\frac{\partial y}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial y}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial x} & \frac{\partial y}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial y}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial y} & \frac{\partial y}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial y}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial y}{\partial t}\frac{\partial t}{\partial z} \\
\frac{\partial z}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial z}{\partial s}\frac{\partial s}{\partial x}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial x} & \frac{\partial z}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial z}{\partial s}\frac{\partial s}{\partial y}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial y} & \frac{\partial z}{\partial r}\frac{\partial r}{\partial z}+\frac{\partial z}{\partial s}\frac{\partial s}{\partial z}+\frac{\partial z}{\partial t}\frac{\partial t}{\partial z}
\end{bmatrix} \\
=&\begin{bmatrix}
\frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s}, & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s}, & \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial r}, & \frac{\partial z}{\partial s}, & \frac{\partial z}{\partial t}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial r}{\partial x}, & \frac{\partial r}{\partial y}, & \frac{\partial r}{\partial z} \\
\frac{\partial s}{\partial x}, & \frac{\partial s}{\partial y}, & \frac{\partial s}{\partial z} \\
\frac{\partial t}{\partial x}, & \frac{\partial t}{\partial y}, & \frac{\partial t}{\partial z}
\end{bmatrix} \\
=&\boldsymbol{J}\boldsymbol{J}^{-1} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{r}}\frac{\partial \boldsymbol{r}}{\partial \boldsymbol{x}} \\
=&\begin{bmatrix}
1, & 0, & 0 \\
0, & 1, & 0 \\
0, & 0, & 1
\end{bmatrix}
\end{aligned}
\tag{38}
$$

二维 Jacobian Matrix:

$$
\boldsymbol{J} = \begin{bmatrix}
\frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s} \\
\frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s}
\end{bmatrix}
\tag{39}
$$

二维 Jacobian Inverse Matrix:

$$
\boldsymbol{J}^{-1} = \begin{bmatrix}
\frac{\partial r}{\partial x}, & \frac{\partial r}{\partial y} \\
\frac{\partial s}{\partial x}, & \frac{\partial s}{\partial y}
\end{bmatrix}
\tag{40}
$$

于是,

$$
r_x = \frac{y_s}{J}, r_y = -\frac{x_s}{J}, s_x = -\frac{y_r}{J}, s_y = \frac{x_r}{J}
\tag{41}
$$

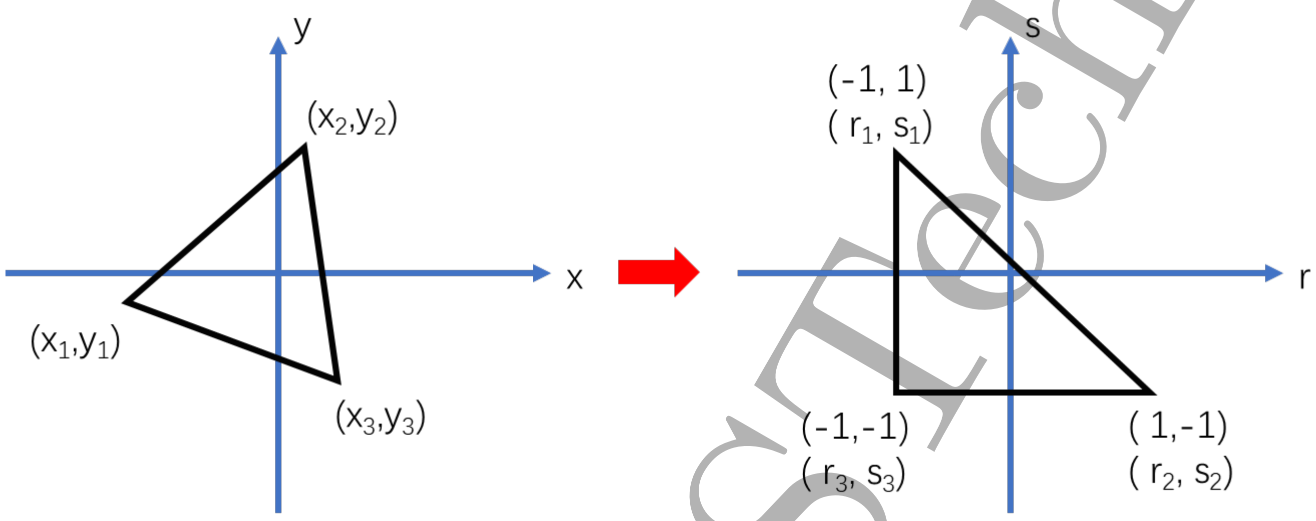
三维 Jacobian Matrix:

$$
\boldsymbol{J} = \begin{bmatrix}
\frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s}, & \frac{\partial x}{\partial t} \\
\frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s}, & \frac{\partial y}{\partial t} \\
\frac{\partial z}{\partial r}, & \frac{\partial z}{\partial s}, & \frac{\partial z}{\partial t}
\end{bmatrix}
\tag{42}
$$

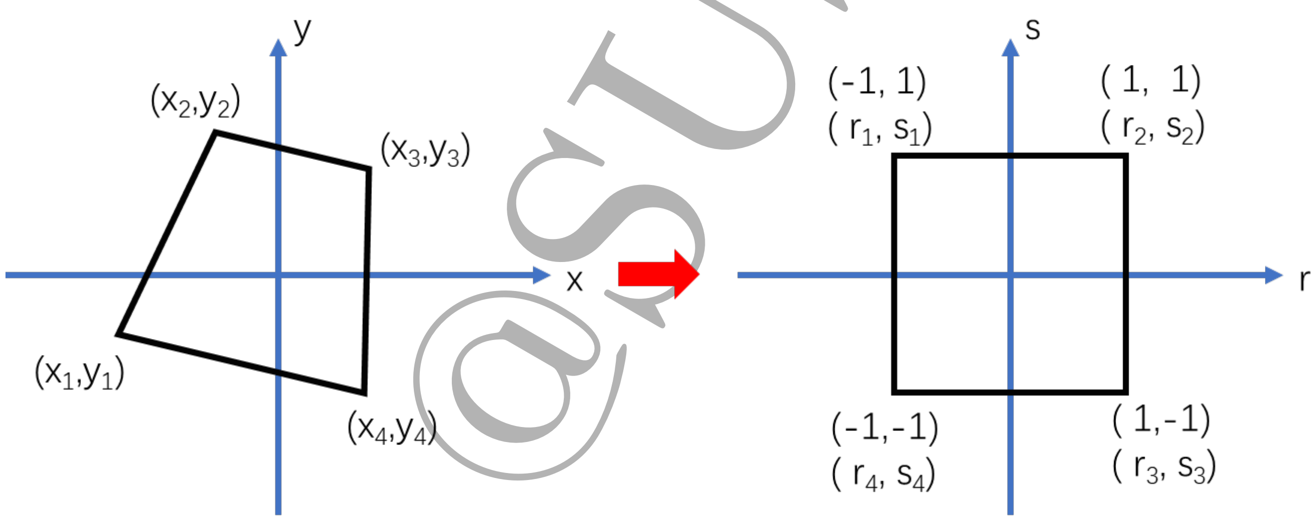
三维 Jacobian Inverse Matrix:

$$
\boldsymbol{J}^{-1} = \begin{bmatrix}
\frac{\partial r}{\partial x}, & \frac{\partial r}{\partial y}, & \frac{\partial r}{\partial z} \\
\frac{\partial s}{\partial x}, & \frac{\partial s}{\partial y}, & \frac{\partial s}{\partial z} \\
\frac{\partial t}{\partial x}, & \frac{\partial t}{\partial y}, & \frac{\partial t}{\partial z}
\end{bmatrix}
\tag{43}
$$

$$
\begin{aligned}
r_x &= \frac{y_s z_t - y_t z_s}{J}, \quad & r_y &= \frac{x_t z_s - x_s z_t}{J}, \quad & r_z &= \frac{x_s y_t - x_t y_s}{J}, \\
s_x &= \frac{y_t z_r - y_r z_t}{J}, \quad & s_y &= \frac{x_r z_t - x_t z_r}{J}, \quad & s_z &= \frac{x_t y_r - x_r y_t}{J}, \\
t_x &= \frac{y_r z_s - y_s z_r}{J}, \quad & t_y &= \frac{x_s z_r - x_r z_s}{J}, \quad & t_z &= \frac{x_r y_s - x_s y_r}{J},
\end{aligned}
\tag{44}
$$



(a) 二维三角元



(b) 二维四边元

形函数 (Shape Function)$N_i(\boldsymbol{r})$ 及 $\boldsymbol{x}$ 与 $\boldsymbol{r}$ 的映射

形函数和基函数有区别, 形函数顾名思义, 是有限元的单元的骨架, 决定着有限元单元的形状. 有限元有直边元和曲边元之分, 如果是曲边元, 需要用 ≥ 2 次的形函数. 如果是直边元, 直接选用一次形函数即可. 形函数有个很好的特性,

$$\begin{cases} N_i(\boldsymbol{r_i}) = 0 \\ N_i(\boldsymbol{r_j}) = 0, i \neq j \end{cases} \tag{45}$$

利用这个性质可以求取形函数的系数, 进而获取形函数表达式 $N_i(\boldsymbol{r})$. 后面我们以二维三角形及四边形单元做了计算演示. 假设我们采用的单元为任意的 M 边形, 那么,

$$\begin{aligned}\boldsymbol{x} &= \sum_i^M \boldsymbol{x}_i N_i(\boldsymbol{r}) \\ &= \begin{bmatrix} \sum_i^M x_i N_i(\boldsymbol{r}) \\ \sum_i^M y_i N_i(\boldsymbol{r}) \\ \sum_i^M z_i N_i(\boldsymbol{r}) \end{bmatrix}\end{aligned} \tag{46}$$

由 Eq.(46) 参数便可计算雅可比矩阵、雅可比逆矩阵、雅可比行列式:

$$\begin{aligned}\boldsymbol{J} = \begin{bmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s}, & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s}, & \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial r}, & \frac{\partial z}{\partial s}, & \frac{\partial z}{\partial t} \end{bmatrix} &= \begin{bmatrix} \sum_i^M x_i \frac{\partial N_i(\boldsymbol{r})}{\partial r}, & \sum_i^M x_i \frac{\partial N_i(\boldsymbol{r})}{\partial s}, & \sum_i^M x_i \frac{\partial N_i(\boldsymbol{r})}{\partial t} \\ \sum_i^M y_i \frac{\partial N_i(\boldsymbol{r})}{\partial r}, & \sum_i^M y_i \frac{\partial N_i(\boldsymbol{r})}{\partial s}, & \sum_i^M y_i \frac{\partial N_i(\boldsymbol{r})}{\partial t} \\ \sum_i^M z_i \frac{\partial N_i(\boldsymbol{r})}{\partial r}, & \sum_i^M z_i \frac{\partial N_i(\boldsymbol{r})}{\partial s}, & \sum_i^M z_i \frac{\partial N_i(\boldsymbol{r})}{\partial t} \end{bmatrix} \\ &= \begin{bmatrix} \sum_i^M x_i \frac{\partial N_i(r,s,t)}{\partial r}, & \sum_i^M x_i \frac{\partial N_i(r,s,t)}{\partial s}, & \sum_i^M x_i \frac{\partial N_i(r,s,t)}{\partial t} \\ \sum_i^M y_i \frac{\partial N_i(r,s,t)}{\partial r}, & \sum_i^M y_i \frac{\partial N_i(r,s,t)}{\partial s}, & \sum_i^M y_i \frac{\partial N_i(r,s,t)}{\partial t} \\ \sum_i^M z_i \frac{\partial N_i(r,s,t)}{\partial r}, & \sum_i^M z_i \frac{\partial N_i(r,s,t)}{\partial s}, & \sum_i^M z_i \frac{\partial N_i(r,s,t)}{\partial t} \end{bmatrix}\end{aligned} \tag{47}$$

以二维三角形为例

我们以二维直边元为例: 三角形单元形函数为线性函数, 四边形单元的为双线性函数.

三角形形函数为: $\begin{cases} N_i(r,s) = a_i \cdot r + b_i \cdot s + c_i \\ N_i(\boldsymbol{r}_j) = \delta_{ij} \end{cases}$

以图2(a)为例

$$\begin{cases} N_1(r_1,s_1) = a_1 \cdot r_1 + b_1 \cdot s_1 + c_1 = 1 \\ N_1(r_2,s_2) = a_1 \cdot r_2 + b_1 \cdot s_2 + c_1 = 0 \\ N_1(r_3,s_3) = a_1 \cdot r_3 + b_1 \cdot s_3 + c_1 = 0 \end{cases} \Longrightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$\begin{cases} N_2(r_1,s_1) = a_2 \cdot r_1 + b_2 \cdot s_1 + c_2 = 0 \\ N_2(r_2,s_2) = a_2 \cdot r_2 + b_2 \cdot s_2 + c_2 = 1 \\ N_2(r_3,s_3) = a_2 \cdot r_3 + b_2 \cdot s_3 + c_2 = 0 \end{cases} \Longrightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \end{bmatrix}$$

$$\begin{cases} N_3(r_1,s_1) = a_3 \cdot r_1 + b_3 \cdot s_1 + c_3 = 0 \\ N_3(r_2,s_2) = a_3 \cdot r_2 + b_3 \cdot s_2 + c_3 = 0 \\ N_3(r_3,s_3) = a_3 \cdot r_3 + b_3 \cdot s_3 + c_3 = 1 \end{cases} \Longrightarrow \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} -0.5 \\ -0.5 \\ 0 \end{bmatrix}$$

于是, 图2(a)三角形形函数

$$\begin{cases} N_1 = 0.5s + 0.5 \\ N_2 = 0.5r + 0.5 \\ N_3 = -0.5r - 0.5s \end{cases} \tag{48}$$

可根据 Eq.(47),

$$
\begin{aligned}
\boldsymbol{J} &= \begin{bmatrix} \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial s} \\ \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} x_1 \frac{\partial N_1}{\partial r} + x_2 \frac{\partial N_2}{\partial r} + x_3 \frac{\partial N_3}{\partial r}, & x_1 \frac{\partial N_1}{\partial s} + x_2 \frac{\partial N_2}{\partial s} + x_3 \frac{\partial N_3}{\partial s} \\ y_1 \frac{\partial N_1}{\partial r} + y_2 \frac{\partial N_2}{\partial r} + y_3 \frac{\partial N_3}{\partial r}, & y_1 \frac{\partial N_1}{\partial s} + y_2 \frac{\partial N_2}{\partial s} + y_3 \frac{\partial N_3}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} 0.5 * (x_2 - x_3), & 0.5 * (x_1 - x_3) \\ 0.5 * (y_2 - y_3), & 0.5 * (y_1 - y_3) \end{bmatrix}
\end{aligned}
\tag{49}
$$

以二维四边形为例

四边形形函数为双线性函数, 因此可以根据顶点的坐标直接写出来

$$
\begin{cases}
N_1 = \frac{1}{4}(r + r_1)(s + s_1) = \frac{1}{4}(r - 1)(s + 1) \\
N_2 = \frac{1}{4}(r + r_2)(s + s_2) = \frac{1}{4}(r + 1)(s + 1) \\
N_3 = \frac{1}{4}(r + r_3)(s + s_3) = \frac{1}{4}(r + 1)(s - 1) \\
N_4 = \frac{1}{4}(r + r_4)(s + s_4) = \frac{1}{4}(r - 1)(s - 1)
\end{cases}
$$

可根据 Eq.(47),

$$
\begin{aligned}
\boldsymbol{J} &= \begin{bmatrix} \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial s} \\ \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} x_1 \frac{\partial N_1}{\partial r} + x_2 \frac{\partial N_2}{\partial r} + x_3 \frac{\partial N_3}{\partial r} + x_4 \frac{\partial N_4}{\partial r}, & x_1 \frac{\partial N_1}{\partial s} + x_2 \frac{\partial N_2}{\partial s} + x_3 \frac{\partial N_3}{\partial s} + x_4 \frac{\partial N_4}{\partial s} \\ y_1 \frac{\partial N_1}{\partial r} + y_2 \frac{\partial N_2}{\partial r} + y_3 \frac{\partial N_3}{\partial r} + y_4 \frac{\partial N_4}{\partial r}, & y_1 \frac{\partial N_1}{\partial s} + y_2 \frac{\partial N_2}{\partial s} + y_3 \frac{\partial N_3}{\partial s} + y_4 \frac{\partial N_4}{\partial s} \end{bmatrix}
\end{aligned}
\tag{50}
$$

令 $X = \sum_i^4 x_i, Y = \sum_i^4 y_i$, 令 $\hat{X} = x_1 + x_2 - (x_3 + x_4), \hat{Y} = y_1 + y_2 - (y_3 + y_4)$, 根据 Eq.(50), 我们有

$$
\boldsymbol{J} = \begin{bmatrix} \frac{1}{4}(X \cdot s + \hat{X}), & \frac{1}{4}(X \cdot r + \hat{X}) \\ \frac{1}{4}(Y \cdot s + \hat{Y}), & \frac{1}{4}(Y \cdot r + \hat{Y}) \end{bmatrix}
$$

3 弹性波动力学方程 (Elastic Wave Equation)

3.1 弹性波动力学控制方程

$$\begin{cases} \rho\dfrac{\partial^2 \boldsymbol{U}}{\partial t^2} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{f} \\ \boldsymbol{\sigma} = \boldsymbol{c} : \boldsymbol{\varepsilon} \end{cases} \tag{51}$$

3.1.1 二维公式推导

$$\rho\frac{\partial^2 \boldsymbol{U}}{\partial t^2} = \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \tag{52}$$

$$\int_{D_k} \rho\frac{\partial^2 \boldsymbol{U}}{\partial t^2}\phi dD = \int_{D_k} \boldsymbol{\nabla}\cdot\boldsymbol{\sigma}\phi dD \tag{53}$$

分部积分:

$$\int_{D_k} \rho\frac{\partial^2 \boldsymbol{U}}{\partial t^2}\phi dD = \int_S \phi\boldsymbol{\sigma}\cdot\boldsymbol{n} dS - \int_{D_k} \boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\phi dD \tag{54}$$

其中,

$$\boldsymbol{\sigma} = \begin{bmatrix} (\lambda+2\mu)\frac{\partial U_x}{\partial x} + \lambda\frac{\partial U_z}{\partial z}, & \mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}) \\ \mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z}), & \lambda\frac{\partial U_x}{\partial x} + (\lambda+2\mu)\frac{\partial U_z}{\partial z} \end{bmatrix}$$

最简单的情况为, 自由地表应力条件, $\boldsymbol{\sigma}\cdot\boldsymbol{n} = \boldsymbol{0}$. 于是,

$$\int_{D_k} \rho\frac{\partial^2}{\partial t^2}\begin{bmatrix} Ux \\ Uz \end{bmatrix}\phi dD = -\int_{D_k} \begin{bmatrix} \left[(\lambda+2\mu)\frac{\partial U_z}{\partial x} + \lambda\frac{\partial U_z}{\partial z}\right]\frac{\partial \phi}{\partial x} + \left[\mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z})\right]\frac{\partial \phi}{\partial z} \\ \left[\mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z})\right]\frac{\partial \phi}{\partial x} + \left[\lambda\frac{\partial U_x}{\partial x} + (\lambda+2\mu)\frac{\partial U_z}{\partial z}\right]\frac{\partial \phi}{\partial z} \end{bmatrix} dD \tag{55}$$

$$\begin{cases} \displaystyle\int_{D_k} \rho\frac{\partial^2 U_x}{\partial t^2}\phi dD = -\int_{D_k} \left[(\lambda+2\mu)\frac{\partial U_x}{\partial x} + \lambda\frac{\partial U_z}{\partial z}\right]\frac{\partial \phi}{\partial x} + \left[\mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z})\right]\frac{\partial \phi}{\partial z} dD \\ \displaystyle\int_{D_k} \rho\frac{\partial^2 U_x}{\partial t^2}\phi dD = -\int_{D_k} \left[\mu(\frac{\partial U_z}{\partial x} + \frac{\partial U_x}{\partial z})\right]\frac{\partial \phi}{\partial x} + \left[\lambda\frac{\partial U_x}{\partial x} + (\lambda+2\mu)\frac{\partial U_z}{\partial z}\right]\frac{\partial \phi}{\partial z} dD \end{cases} \tag{56}$$

$$\begin{cases} \displaystyle\int_{D_k} \rho\frac{\partial^2 U_x}{\partial t^2}\phi dD = -\int_{D_k} \left[(\lambda+2\mu)\frac{\partial U_x}{\partial x}\frac{\partial \phi}{\partial x} + \lambda\frac{\partial U_z}{\partial z}\frac{\partial \phi}{\partial x}\right] + \left[\mu\frac{\partial U_z}{\partial x}\frac{\partial \phi}{\partial z} + \mu\frac{\partial U_x}{\partial z}\frac{\partial \phi}{\partial z}\right] dD \\ \displaystyle\int_{D_k} \rho\frac{\partial^2 U_z}{\partial t^2}\phi dD = -\int_{D_k} \left[\mu\frac{\partial U_z}{\partial x}\frac{\partial \phi}{\partial x} + \mu\frac{\partial U_x}{\partial z}\frac{\partial \phi}{\partial x}\right] + \left[\lambda\frac{\partial U_x}{\partial x}\frac{\partial \phi}{\partial z} + (\lambda+2\mu)\frac{\partial U_z}{\partial z}\frac{\partial \phi}{\partial z}\right] dD \end{cases} \tag{57}$$

3.2 三角元:

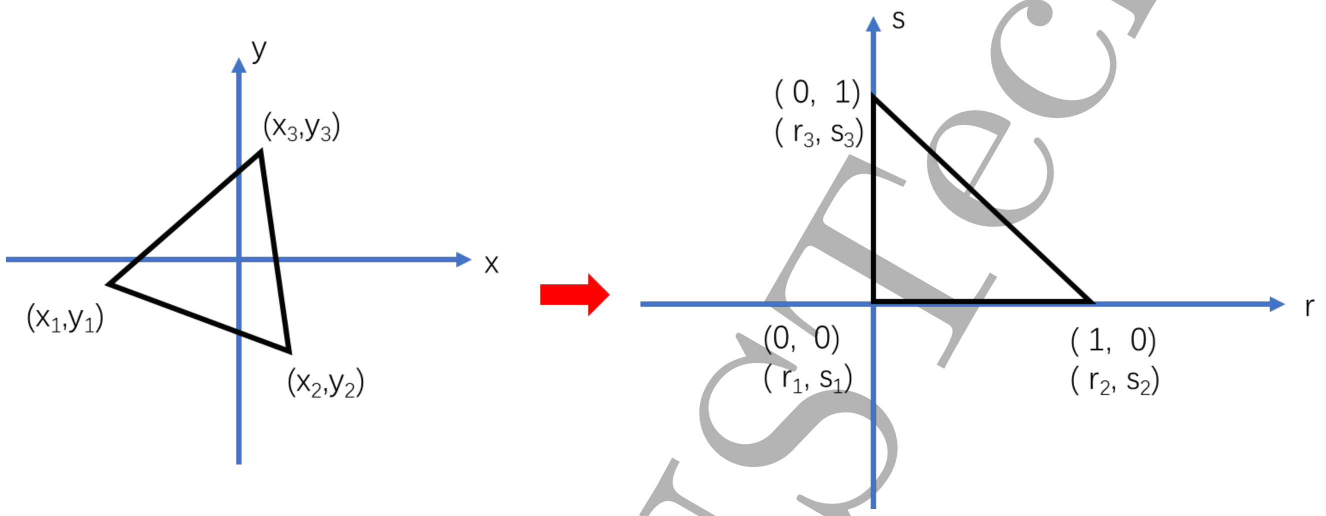


图 2: 二维三角元

我们首先讨论线性元:

三角形形函数为: $\begin{cases} N_i(r,s) = a_i \cdot r + b_i \cdot s + c_i \\ N_i(\boldsymbol{r}_j) = \delta_{ij} \end{cases}$

以图3.2为例

$$\begin{cases} N_1(r_1,s_1) = a_1 \cdot r_1 + b_1 \cdot s_1 + c_1 = 1 \\ N_1(r_2,s_2) = a_1 \cdot r_2 + b_1 \cdot s_2 + c_1 = 0 \\ N_1(r_3,s_3) = a_1 \cdot r_3 + b_1 \cdot s_3 + c_1 = 0 \end{cases} \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{cases} N_2(r_1,s_1) = a_2 \cdot r_1 + b_2 \cdot s_1 + c_2 = 0 \\ N_2(r_2,s_2) = a_2 \cdot r_2 + b_2 \cdot s_2 + c_2 = 1 \\ N_2(r_3,s_3) = a_2 \cdot r_3 + b_2 \cdot s_3 + c_2 = 0 \end{cases} \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{cases} N_3(r_1,s_1) = a_3 \cdot r_1 + b_3 \cdot s_1 + c_3 = 0 \\ N_3(r_2,s_2) = a_3 \cdot r_2 + b_3 \cdot s_2 + c_3 = 0 \\ N_3(r_3,s_3) = a_3 \cdot r_3 + b_3 \cdot s_3 + c_3 = 1 \end{cases} \Longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \Longrightarrow \begin{bmatrix} a_3 \\ b_3 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

于是, 图3.2三角形形函数

$$\begin{cases} N_1 = -r - s + 1 \\ N_2 = r \\ N_3 = s \end{cases} \tag{58}$$

对于 Jacobian 矩阵, 根据 Eq.(42)

$$
\begin{aligned}
\boldsymbol{J} &= \begin{bmatrix} \frac{\partial x}{\partial r}, & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial r}, & \frac{\partial y}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 x_i \frac{\partial N_i(r,s)}{\partial s} \\ \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial r}, & \sum_i^3 y_i \frac{\partial N_i(r,s)}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} x_1 \frac{\partial N_1}{\partial r} + x_2 \frac{\partial N_2}{\partial r} + x_3 \frac{\partial N_3}{\partial r}, & x_1 \frac{\partial N_1}{\partial s} + x_2 \frac{\partial N_2}{\partial s} + x_3 \frac{\partial N_3}{\partial s} \\ y_1 \frac{\partial N_1}{\partial r} + y_2 \frac{\partial N_2}{\partial r} + y_3 \frac{\partial N_3}{\partial r}, & y_1 \frac{\partial N_1}{\partial s} + y_2 \frac{\partial N_2}{\partial s} + y_3 \frac{\partial N_3}{\partial s} \end{bmatrix} \\
&= \begin{bmatrix} x_2 - x_1, & x_3 - x_1 \\ y_2 - y_1, & y_3 - y_1 \end{bmatrix}
\end{aligned} \tag{59}
$$

$$
\begin{aligned}
x_r &= x_2 - x_1, & x_s &= x_3 - x_1, & y_r &= y_2 - y_1, & y_s &= y_3 - y_1 \\
r_x &= \frac{y_s}{J}, & r_y &= -\frac{x_s}{J}, & s_x &= -\frac{y_r}{J}, & s_y &= \frac{x_r}{J}
\end{aligned} \tag{60}
$$

因此, $J = (y_3 - y_1)(x_2 - x_1) - (y_2 - y_1)(x_3 - x_1)$

由 Eq.(19) 质量矩阵

三角形高斯积分

$\boldsymbol{M}^e_{ij} = J^e \int_I \phi_i \phi_j dD$:

$$
\boldsymbol{M}^e = J^e \begin{bmatrix} \int_I \phi_1\phi_1 dD, & \int_I \phi_1\phi_2 dD, & \int_I \phi_1\phi_3 dD \\ \int_I \phi_1\phi_2 dD, & \int_I \phi_2\phi_2 dD, & \int_I \phi_2\phi_3 dD \\ \int_I \phi_1\phi_3 dD, & \int_I \phi_2\phi_3 dD, & \int_I \phi_3\phi_3 dD \end{bmatrix} \tag{61}
$$