6.4

題: $\hat{\theta}_1 = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}$, $\hat{\theta}_2 = \dfrac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}$

$\boxed{\bar{x} = \dfrac{x_1+x_2}{n}}$

$E(x_i) = \mu$, $E(\bar{x}) = \mu$

$V(x_i) = \sigma^2 = E(x_i^2) - \mu^2$; $V(\bar{x}) = \dfrac{\sigma^2}{n} = E(\bar{x}^2) - \mu^2$

$$E(\hat{\theta}_1) = E\left[\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n}\right] = \frac{1}{n}E\Big(\underbrace{\sum_{i=1}^{n}x_i^2}_{①} - \underbrace{n\bar{x}^2}_{②}\Big)$$

① $E(\Sigma x_i^2) = \Sigma E(x_i^2)$ $\because V(x) = E(x^2) - E^2(x)$

$\therefore E(x_i^2) = V(x_i) + E^2(x_i)$

$\Sigma E(x_i^2) = \Sigma[V(x_i) + E^2(x_i)] = n\sigma^2 + n\mu^2$

② $E(n\bar{x}^2) = nE(\bar{x}^2)$

$V(x) = E(x^2) - E^2(x)$ $E(\bar{x}^2) = \dfrac{\sigma^2}{n} + \mu^2$

$E(\bar{x}^2) = V(\bar{x}) + E^2(\bar{x})$ $nE(\bar{x}^2) = \sigma^2 + n\mu^2$

得 $\dfrac{1}{n}E\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right) = E(x_i^2) - nE(\bar{x}^2)$

$= n\sigma^2 + \cancel{n\mu^2} - (\sigma^2 + \cancel{n\mu^2}) = \dfrac{n-1}{n}\sigma^2$

偏誤估計量

$$E(\hat{\theta}_2) = E\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}\right) = \frac{1}{n-1}E\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right)$$

$$= \frac{1}{n-1}E\left(\sum_{i=1}^{n}x_i^2 - n\bar{x}^2\right)$$

$$= \frac{1}{n-1}(n\sigma^2 + n\mu^2 - \sigma^2 - n\mu^2) = \sigma^2$$

不偏估計量