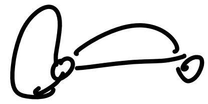
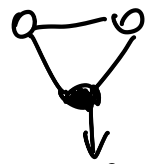
Graph Theory.

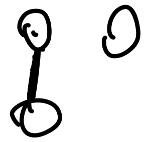
Graph $V(G)$, $E(G)$.



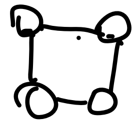
① rooted graph $(\underbrace{G, \rho}_{\text{root.}})$
(only one)



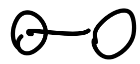
ρ



② <u>subgraph</u> H. $\underline{V(H) \subseteq V(G), \quad E(H) \subseteq E(G)}$



③ <u>induced subgraph</u> H. $\underline{V}(H) \subseteq V(G)$



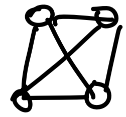
④ connected Graph.

⑤ degree $\deg_G(v)$

⑥ $v \sim w$ neighbours

$$\sum_{v \in V(G)} \deg_G(v) = 2\,|E(G)|.$$

Common types.



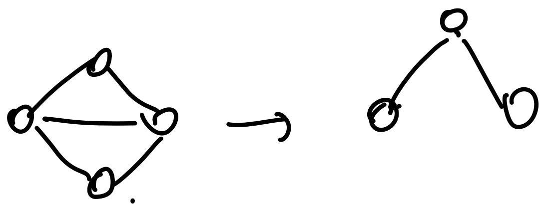
① complete graph K_n.

② empty graph.

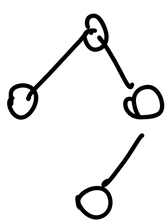
③ path

④ cycle.

Tree:



spanning tree: $V(T)=V(G)$, T is a tree.

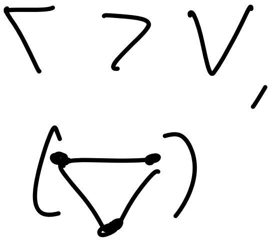


4^{4-2}

Cayley's formula, for K_n, # spanning tree $= n^{n-2}$

$n=1$,  $1 = 1^{1-2} = 1^{-1} = 1$

$n=2$, , $2^{2-2} = 1$

$n=3$, , $3^{3-2} = 3$

Uniform spanning tree.

Definition: T is a UST if it is uniformly distributed on $\mathcal{T}$ (the set of all spanning trees)

$$P(T = t) = \frac{1}{|\mathcal{T}|}.$$

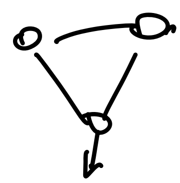
t ↓ a fixed UST

Random walk on a finite graph G.

① Markov chain

② $P_{v,w} = \dfrac{1}{\deg_G(v)}$, $\{v, w\} \in E(G)$

$P_{v,w} = 0$, other



start

Hitting time

$$\tau_v = \underline{\underline{\inf}} \{ n \geq 0, X_n = v \}$$

(第一次访问 vertice v 的时间)

Stopping time.

If τ is a stopping time, $\{\tau \leq t\} \in \mathcal{F}_t$

Cover time

$$t_{cov} = \sup_{v \in V(G)} T_v = \inf\{n \geq 0 : \{X_0, \dots, X_n\} = V(G)\}$$

State of G.

Recurrent (常返): SRW start from vertex v visits itself $\# \infty$

Transient (暂态): SRW start from vertex v visits itself $\# < \infty$

Finite G recurrent definitely. (positive recurrent)

Reversible Markov chain (时间可逆)

$\mathbb{Z} \setminus \mathbb{Z}^- = \{0, \dots\}$

expand to $\mathbb{Z}$ (eternal stationary version)

$$X_n = \begin{cases} X'_n & \text{if } n \geq 0 \\ X''_{-n} & \text{if } n < 0 \end{cases}$$

(Theorem ☆, exist P_{ij}. If it can find Q_{ji} and π_i (satisfy $\sum_i \pi_i = 1$), such that

$$\pi_i P_{ij} = \pi_j Q_{ji}.$$

Then, Q_{ij} is the transition probability of the reverisble chain and $\{\pi_i\}$ is the stationary probability)

SRW on finite connected graph G.

is reversible and $\pi_v \propto \deg_G(v)$.

proof: SRW is irreducible and positive recurrent.

detailed balance equation:

$$\underbrace{\pi_v \frac{1}{\deg_G(v)}}_{\text{long time proportion of } v \text{ to } w} = \underbrace{\pi_w \frac{1}{\deg_G(w)}}_{\text{long time propotion of } w \text{ to } v.}$$

$$\sum_v \pi_v = 1$$

By theorem ☆, π_v is the stationary probability. And $\sum_v \pi_v = \sum_v \pi_w \frac{\deg_G(v)}{\deg_G(w)}$

$$= \frac{\pi_w}{\deg_G(w)} \, 2|E(G)| = 1$$

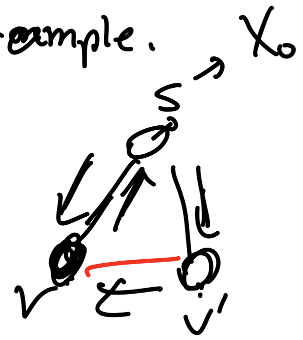
$$\Rightarrow \pi_w = \frac{\deg_G(w)}{2|E(G)|}, \quad \text{reversible.}$$

Aldous-Broder algorithm.

Connected, recurrent graph G, run a SRW starting at $X_0 \in V(G)$. Let T subgraph, $V(T) = V(G)$, including the edge along we <u>first</u> reach that vertex, i.e.

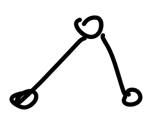
$$E(T) = \{\{X_{J_v - 1}, X_{J_v}\} : v \in V(G) \setminus \{X_0\}\}$$

Example.



v. $J_v = 1$ $\{X_{J_v - 1}, X_{J_v}\} = \{X_0, X_1\}$.
s, v

v'. $J_{v'} = 3$ $\{X_{J_{v'} - 1}, X_{J_{v'}}\} = \{X_0, X_3\}$.
s, v'



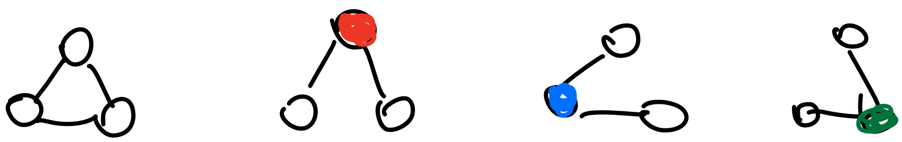
prove the random subgraph T generated by A-B algorithm is a UST on <u>finite</u> <u>connected</u> G.

Proof:

① T is a spanning tree.

② Uniform?

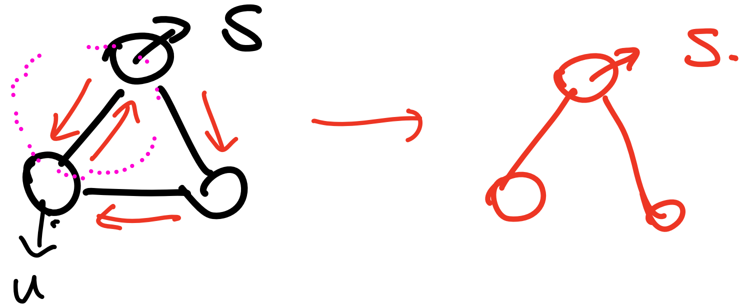
$\underline{R}$: a set of $\underline{\text{rooted spanning trees}}$ of G.



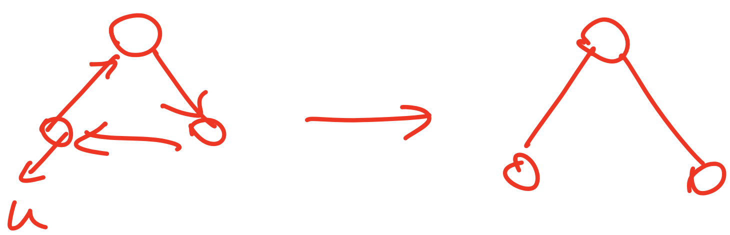
$(X_n)_{n \in \mathbb{Z}}$: SRW on G.

$\underline{(T_k, X_k)}$: $\underline{\text{rooted tree}}$ from A-B algorithm using SRW from time k onwards.

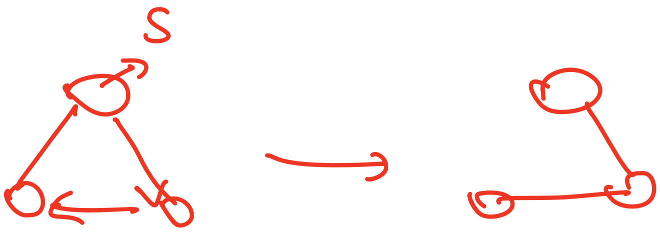
$k=0$. $X_0 = s$ (T_0, X_0)
$\downarrow$ $X_0, X_1, \dots$



$k=1$. $X_1 = u$. (T_1, u)
$\downarrow$



$k=2$. $X_2 = J$ (T_2, s)



$(T_n, X_n)_{n\in\mathbb{Z}}$ is an irreducible Markov chain with state space $\mathcal{R}$. X_{k+1} depends on T_k.

T_k is a measurable functional of $X_k, X_{k+1}, \ldots$

T_{k+1} depends on T_k and X_{k+1}.

A length of ℓ SRW generates (t,ρ), a length of m subsequent SRW generates (t',ρ'); path exists with positive probability

Define

$$q((t,\rho),(t',\rho')) = P(T_{-1}=t', X_{-1}=\rho' \mid T_0=t, X_0=\rho)$$

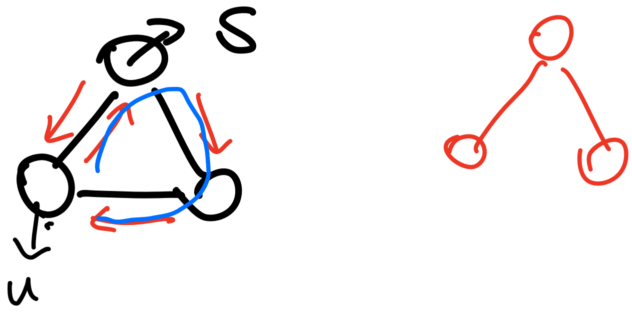
(后一步在 (t,ρ), 前一步在 (t',ρ'))

$\tilde{J}(T_0, X_0) = J_{X_0}$ on X_{-1}

$q((t,\rho),(t',\rho')) = \dfrac{1}{\deg_G(\rho)}$ for $\deg_G(\rho)$ values of (t',ρ')

$q((t,\rho),(t',\rho')) = 0$, $(t',\rho') \notin E(G)$

⊣



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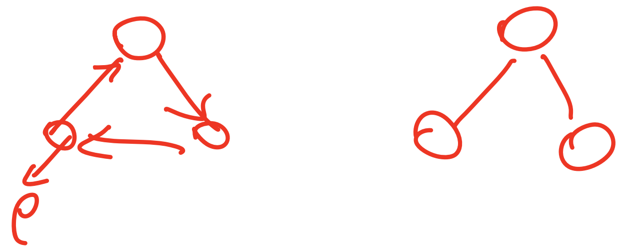
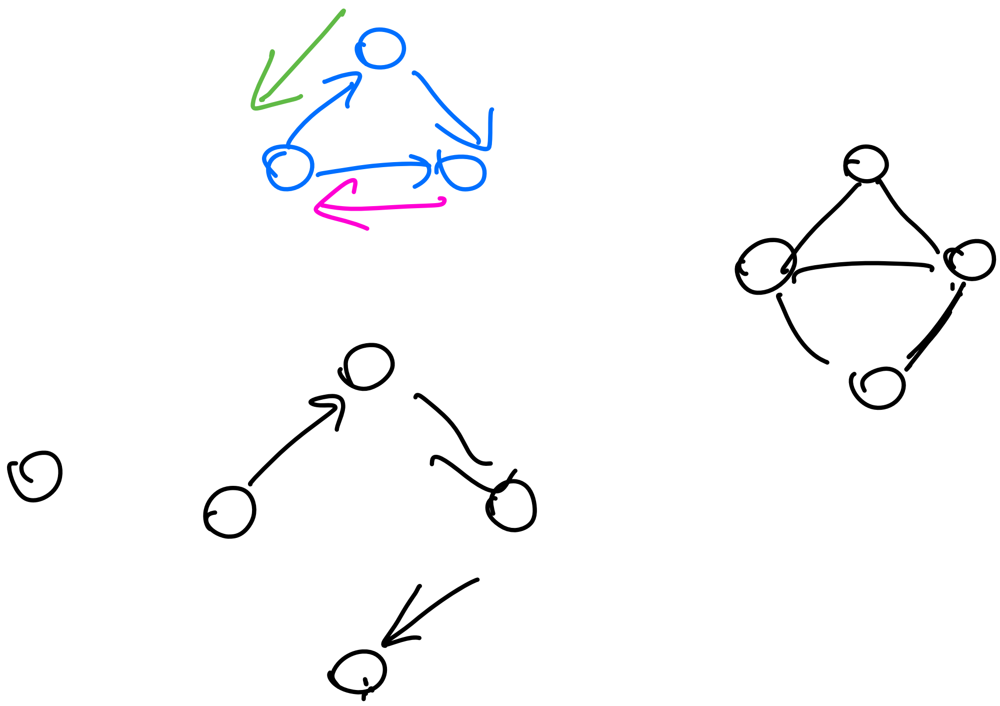
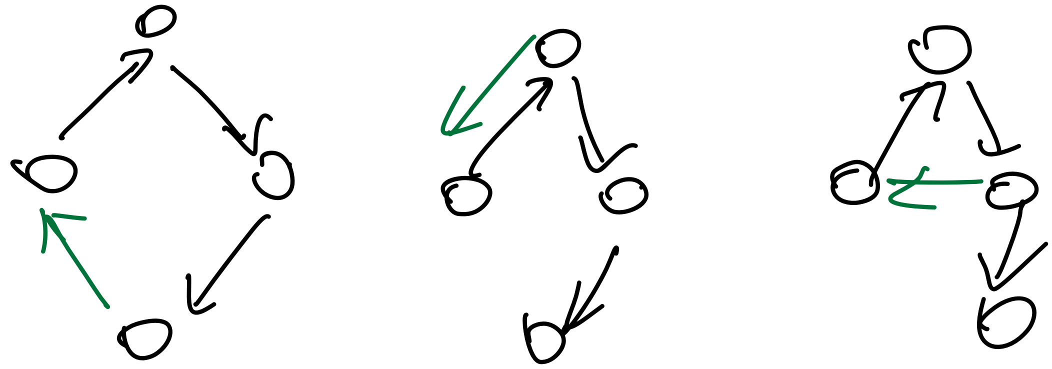




⊣



Fixed (t', ρ'), $\#(t, \rho) = \deg_G(\rho')$

$$\left\{\begin{aligned} &\sum_{(t,\rho)\in R} \underline{\underline{q((t,\rho),(t',\rho'))}}\ \underline{\underline{\deg_G(\rho)}} = \underline{\underline{\deg_G(\rho')}}. \\ &\qquad\qquad \underline{\#(t',\rho')} \qquad\qquad\qquad\qquad \underline{\#(t,\rho)} \\ &\qquad (\text{全部的 rooted tree}) \qquad\qquad (\text{全部的 rooted tree}) \\ &\sum_{(t',\rho')\in R} q((t,\rho),(t',\rho')) = 1 \end{aligned}\right.$$

$(\pi_j = \sum \pi_i p_{ij}) \Rightarrow \mathcal{J}_{(t,\rho)} \propto \deg_G(\rho)$

$\mathcal{J}_{(t,\rho)}$ is independent of t.

Wilson's algorithm.

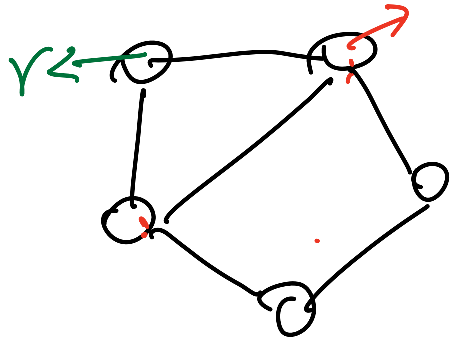
① 取一个顶点 r, initial tree $T=\{r\}$.

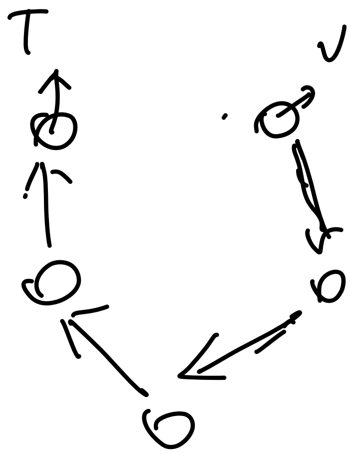
② 任取 $v\notin V(T)$, 做从 v 出发的SRW.

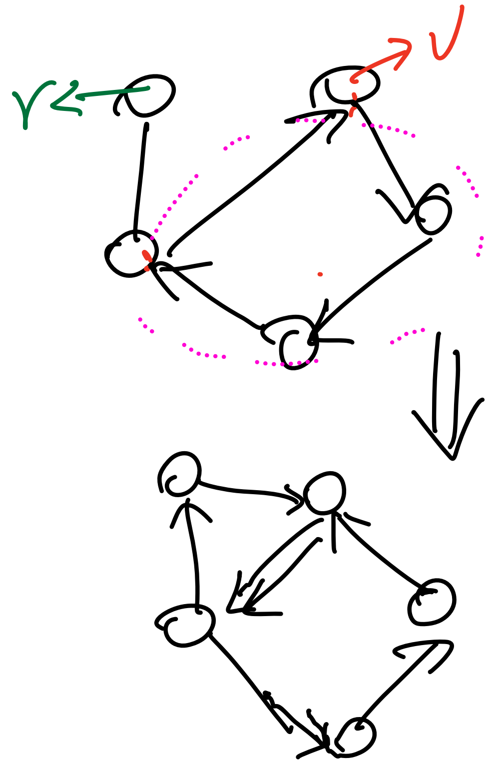
边走边去擦掉path中出现的loop. 每当走到一个之前访问过的顶点 x, 则两次访问 x 之间的路径擦掉. 直到与 T 相遇为止.

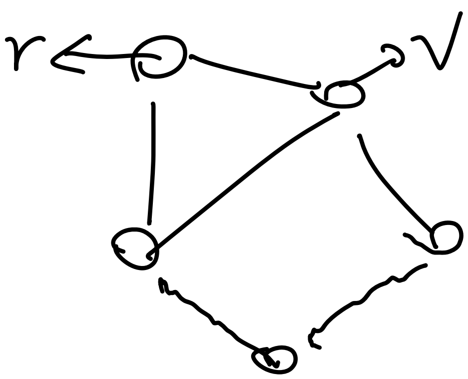
得到一条从 v 到 T 不含loop的path P, $T=T\cup P$.

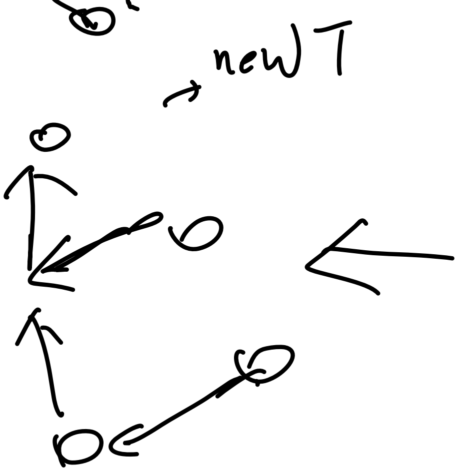
③ repeat until T 包含 all vertices.

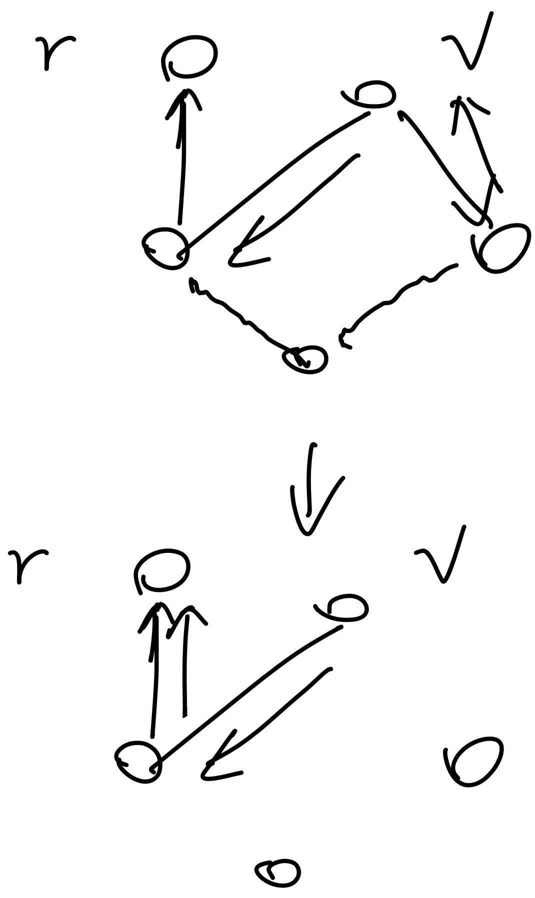

r
v

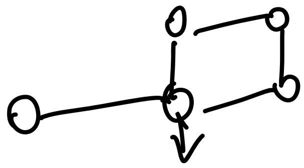

T
v


r
v


r
v


new T


r
v
r
v



intersect.