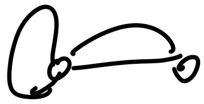
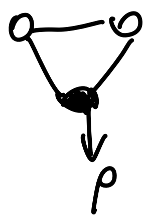
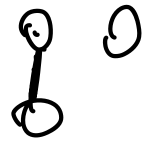
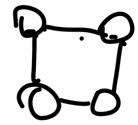
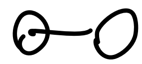
Graph Theory.

Graph $V(G)$, $E(G)$. 

① rooted graph (G, ρ) (only one) root.  

② subgraph H. $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ 

③ induced subgraph H. $V(H) \subseteq V(G)$ 

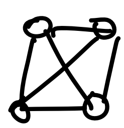
④ connected Graph.

⑤ degree $\deg_G(v)$

⑥ $v \sim w$ neighbours

$$\sum_{v \in V(G)} \deg_G(v) = 2\,|E(G)|.$$

Common types.



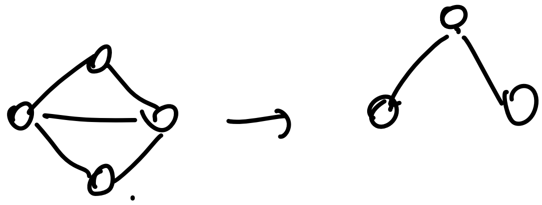
① complete graph K_n.

② empty graph.

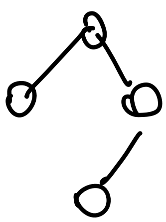
(3) path

(4) <u>cycle.</u>

Tree :



spanning tree : $V(T) = V(G)$, T is a tree.



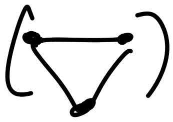
4^{4-2}

Cayley's formula, for K_n, # spanning tree $= n^{n-2}$

$n=1$, • $\quad 1 = 1^{1-2} = 1^{-1} = 1$

$n=2$,  , $2^{2-2} = 1$

$n=3$,  , $3^{3-2} = 3$



Uniform spanning tree.

Definition: T is a UST if it is uniformly distributed on $\mathcal{T}$ (the set of all spanning trees)

$$P(T = t) = \frac{1}{|\mathcal{T}|}.$$

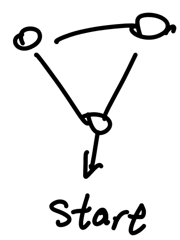
t: a fixed UST

Random walk on a finite graph G.

① Markov chain

② $P_{v,w} = \frac{1}{\deg_G(v)}$, $\{v,w\} \in E(G)$

$P_{v,w} = 0$, other



Hitting time

$$\tau_v = \underline{\underline{\inf}} \{ n \geq 0, X_n = v \}$$

(第一次访问 vertice v 的时间)

Stopping time.

If τ is a stopping time, $\{\tau \leq t\} \in \mathcal{F}_t$

Cover time

$$t_{cov} = \sup_{v \in V(G)} \tau_v = \inf\{n \geq 0 : \{X_0, \dots, X_n\} = V(G)\}$$

State of G.

<u>Recurrent</u> (常返): SRW start from vertex v visits itself $\# \infty$

Transient (暂态): SRW start from vertex v visits itself $\# < \infty$

Finite G recurrent definitely. (positive recurrent)

Reversible Markov chain (时间可逆)

$\mathbb{Z} \setminus \mathbb{Z}^- = \{0, \dots\}$

expand to $\mathbb{Z}$ (eternal stationary version)

$$X_n = \begin{cases} X'_n & \text{if } n \geq 0 \\ X''_{-n} & \text{if } n < 0 \end{cases}$$

(Theorem ☆, exist P_{ij}. If it can find Q_{ji} and π_i (satisfy $\sum_i \pi_i = 1$), such that

$$\pi_i P_{ij} = \pi_j Q_{ji}.$$

Then, Q_{ij} is the transition probability of the reverisble chain and $\{\pi_i\}$ is the stationary probability)

SRW on finite connected graph G.

is reversible and $\pi_v \propto \deg_G(v)$.

proof: SRW is irreducible and positive recurrent.

detailed balance equation:

$$\underbrace{\pi_v \frac{1}{\deg_G(v)}}_{\text{long time proportion of } v \text{ to } w} = \underbrace{\pi_w \frac{1}{\deg_G(w)}}_{\text{long time proportion of } w \text{ to } v.}$$

$$\sum_v \pi_v = 1$$

By theorem ☆, π_v is the stationary probability. And $\sum_v \pi_v = \sum_v \pi_w \frac{\deg_G(v)}{\deg_G(w)}$

$$= \frac{\pi_w}{\deg_G(w)} \, 2|E(G)| = 1$$

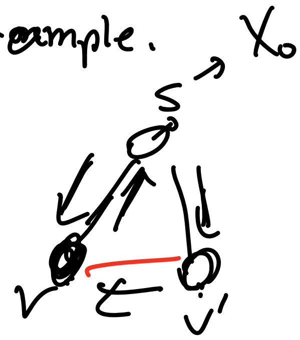
$$\Rightarrow \pi_w = \frac{\deg_G(w)}{2|E(G)|}, \quad \text{reversible.}$$

Aldous–Broder algorithm.

Connected, recurrent graph G, run a SRW starting at $X_0 \in V(G)$. Let T subgraph, $V(T) = V(G)$, including the edge along we <u>first</u> reach that vertex, i.e.

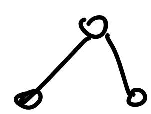
$$E(T) = \{\{X_{J_v-1}, X_{J_v}\} : v \in V(G) \setminus \{X_0\}\}$$

Example.



v. $J_v = 1$ $\{X_{J_v-1}, X_{J_v}\} = \{X_0, X_1\}$
s, v

v'. $J_{v'} = 3$ $\{X_{J_{v'}-1}, X_{J_{v'}}\} = \{X_0, X_3\}$
s, v'



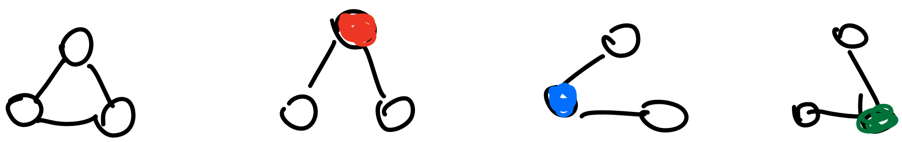
prove the random subgraph T generated by A-B algorithm is a UST on <u>finite</u> <u>connected</u> G.

Proof:

① T is a spanning tree.

② Uniform?

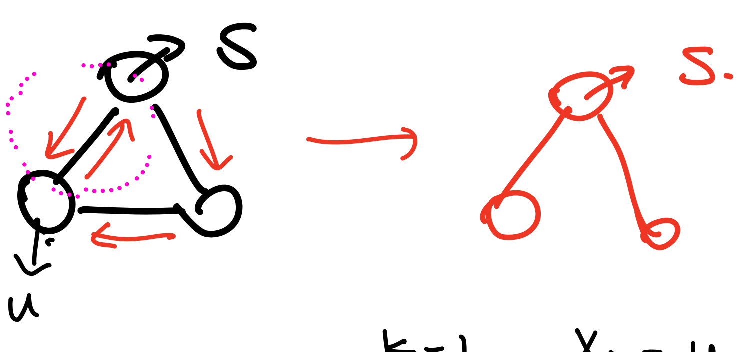
$\underline{R}$: a set of $\underline{\text{rooted spanning trees}}$ of G.



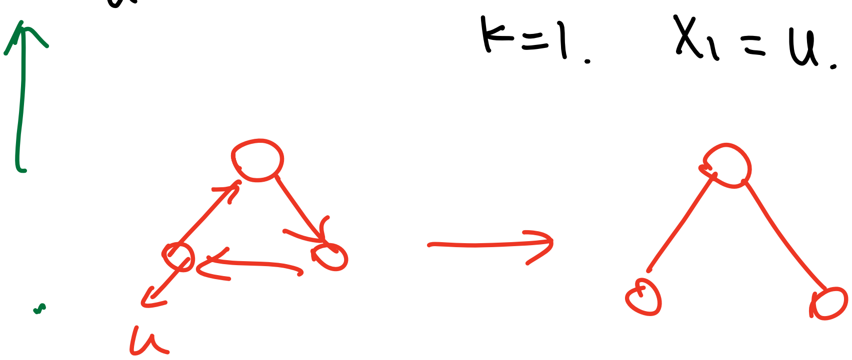
$(X_n)_{n\in\mathbb{Z}}$: SRW on G.

$\underline{(T_k, X_k)}$: $\underline{\text{rooted tree}}$ from A-B algorithm using SRW from time k onwards.

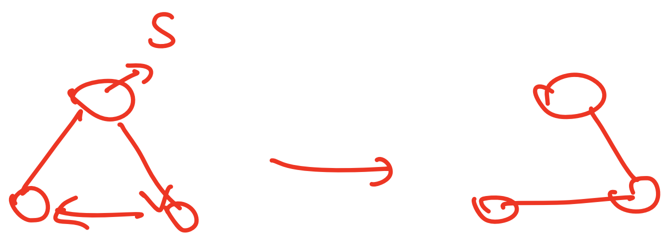
$k=0$. $X_0 = S$ (T_0, X_0) $\downarrow$ $X_0, X_1, \ldots$



$k=1$. $X_1 = u$. (T_1, u) $\downarrow$



$k=2.\quad X_2 = J \qquad (T_2, s)$



$(T_n, X_n)_{n\in\mathbb{Z}}$ is an irreducible Markov chain with state space $\mathcal{R}$. X_{k+1} depends on T_k.

T_k is a measurable functional of $X_k, X_{k+1}, \ldots$

T_{k+1} depends on T_k and X_{k+1}.

A length of ℓ SRW generates (t, ρ), a length of m subsequent SRW generates (t', ρ')

path exists with positive probability

Define

$$q((t,\rho),(t',\rho')) = P(T_{-1}=t', X_{-1}=\rho' \mid T_0 = t, X_0 = \rho)$$

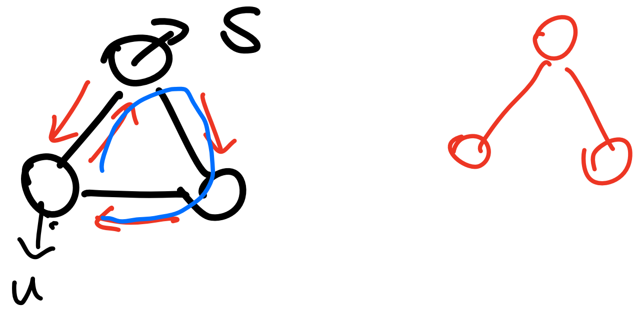
(后一步在 (t,ρ), 前一步在 (t',ρ'))

$\mathcal{F}(T_0, X_0) = \mathcal{F}_{X_0}$ on X_{-1}

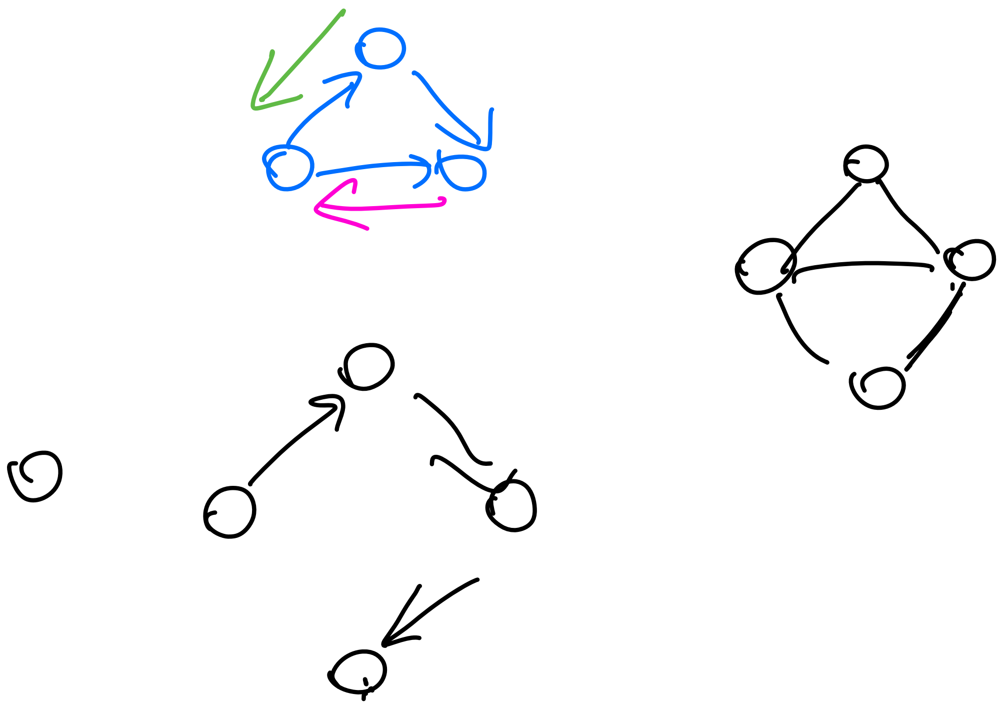
$q((t,\rho),(t',\rho')) = \dfrac{1}{\deg_G(\rho)}$ for $\deg_G(\rho)$ values of (t',ρ')

$q((t,\rho),(t',\rho')) = 0, \quad (t',\rho') \notin E(G)$

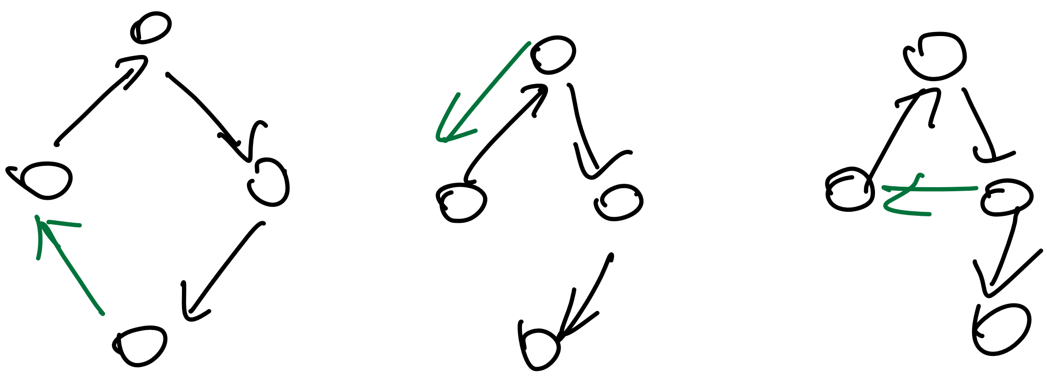
⊣



o



⊣



Fixed (t', ρ'), $\#(t, \rho) = \deg_G(\rho')$

$$\begin{cases} \displaystyle\sum_{(t,\rho)\in R} \underbrace{q\big((t,\rho),(t',\rho')\big)\, \deg_G(\rho)}_{\substack{\#(t',\rho') \\ (\text{全部的 rooted tree})}} = \underbrace{\deg_G(\rho')}_{\substack{\#(t,\rho) \\ (\text{全部的 rooted tree})}} \\[2em] \displaystyle\sum_{(t',\rho')\in R} q\big((t,\rho),(t',\rho')\big) = 1 \end{cases}$$

$\left(\pi_j = \sum \pi_i p_{ij}\right) \Rightarrow \pi_{(t,\rho)} \propto \deg_G(\rho)$

$\pi_{(t,\rho)}$ is independent of t.