Inspecting COVID-19 Using SAIRS Model with Immunization

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Background & Goals

1. Different mathematical modeling papers in fighting and controlling COVID-19
2. SAIR Model [5]
3. To build the SAIRS model with the immunization factor
4. To find out the equilibrium point, study what conditions could make desired equilibrium points stable, and explore real life implications
5. To find which parameter have the most impact on the outcome by tuning each of them

Model Explanation: Definitions

Population definitions

S	Susceptible
A	Asymptomatic
I	Identified Infected
R	Recovered and vaccinated

Parameters definitions

β_A	Contact rate between S and A
β_I	Contact rate between S and I
γ_A	Recover rate of A
γ_I	Recover rate of I
δ	Identification rate
ω	Vaccination rate
θ	Re-susceptible rate

Model Explanation: Key Assumptions

1. Total population is fixed with no vital dynamics => $S + A + I + R = 1$
2. Any person who is infected must first become asymptomatic before becoming infected (showing symptoms)
3. Some constant proportion of R will become susceptible again (going back to S) within unit of time.
4. The immunization rate, ω, is a constant per unit of time.
5. Only people in S will get vaccinated.

Model Explanation: SAIRS Model

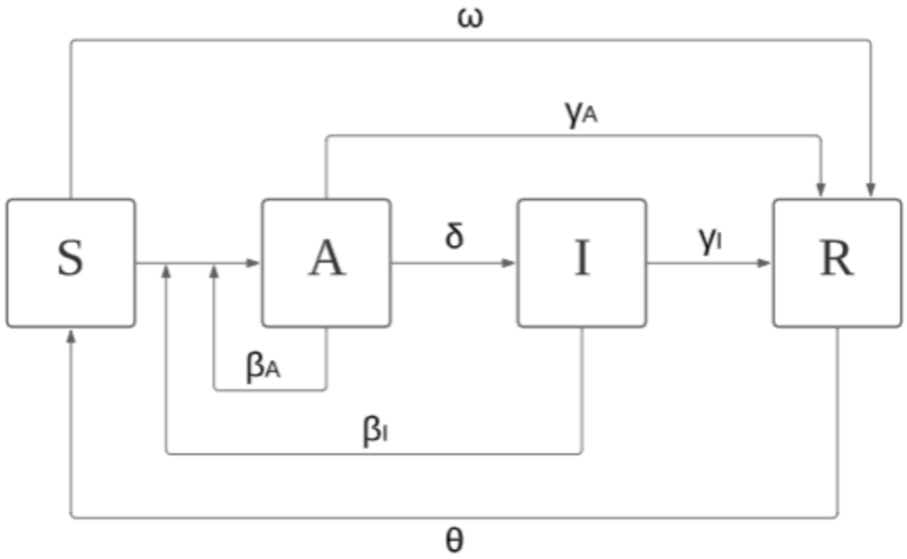


Fig. 1: SAIR Model flowchart

β_A	Contact rate between S and A
β_I	Contact rate between S and I
γ_A	Recover rate of A
γ_I	Recover rate of I
δ	Identification rate
ω	Vaccination rate
θ	Re-susceptible rate

$$\frac{dS}{dt} = -\beta_A \cdot A \cdot S - \beta_I \cdot I \cdot S - \omega \cdot S + \theta \cdot R \tag{1}$$

$$\frac{dA}{dt} = \beta_A \cdot A \cdot S + \beta_I \cdot I \cdot S - \gamma_A \cdot A - \delta \cdot A \tag{2}$$

$$\frac{dI}{dt} = \delta \cdot A - \gamma_I \cdot I \tag{3}$$

$$\frac{dR}{dt} = \gamma_A \cdot A + \gamma_I \cdot I + \omega \cdot S - \theta \cdot R \tag{4}$$

Also, with our assumptions, the overall population as the sum of the groups is:

$$S + A + I + R = 1 \tag{5}$$

Theoretical Investigation: Equilibrium Point 1

$$I \cdot (\frac{\beta_A \gamma_I}{\delta} S + \beta_I S - \frac{\gamma_A \gamma_I}{\delta} - \gamma_I) = 0$$

$$\begin{cases} S = \dfrac{\theta}{\omega + \theta} \\ A = 0 \\ I = 0 \\ R = \dfrac{\omega}{\omega + \theta} \end{cases}$$

End of the pandemic!

Reduced System and Linear Approximation Matrix

$$\frac{dS}{dt} = -\beta_A \cdot A \cdot S - \beta_I \cdot I \cdot S - \omega \cdot S + \theta \cdot (1 - S - A - I)$$

$$\frac{dA}{dt} = \beta_A \cdot A \cdot S + \beta_I \cdot I \cdot S - \gamma_A \cdot A - \delta \cdot A$$

$$\frac{dI}{dt} = \delta \cdot A - \gamma_I \cdot I$$

$$\begin{bmatrix} -\beta_A A - \beta_I I - \omega - \theta & -\beta_A S - \theta & -\beta_I S - \theta \\ \beta_A A + \beta_I I & \beta_A S - \gamma_A - \delta & \beta_I S \\ 0 & \delta & -\gamma_I \end{bmatrix}$$

Stability Analysis - Analytic Solution

Characteristic Polynomial:

$$(-\omega-\theta-\lambda)[\lambda^2+(\gamma_A+\gamma_I+\delta-\beta_A\frac{\theta}{\theta+\omega})\lambda+(\gamma_A\gamma_I+\delta\gamma_I-\gamma_I\beta_A\frac{\theta}{\theta+\omega}-\delta\beta_I\frac{\theta}{\theta+\omega})]$$

So if we let

$$\begin{cases} b = \gamma_A + \gamma_I + \delta - \beta_A \dfrac{\theta}{\theta+\omega} \\ c = \gamma_A\gamma_I + \delta\gamma_I - \gamma_I\beta_A \dfrac{\theta}{\theta+\omega} - \delta\beta_I \dfrac{\theta}{\theta+\omega} \end{cases}$$

then, we can get the symbolic representation of the eigenvalues of the resulting matrix:

$$\begin{cases} \lambda_1 = -\omega - \theta \text{ (always negative)} \\ \lambda_2 = \dfrac{-b+\sqrt{b^2-4c}}{2} \\ \lambda_3 = \dfrac{-b-\sqrt{b^2-4c}}{2} \end{cases}$$

Stable Condition

The other two eigenvalues follows:

$$\begin{cases} \lambda_2 + \lambda_3 = -b \\ \lambda_2 * \lambda_3 = c \end{cases}$$

So we must have b > 0 and c > 0 for all the eigenvalues to be negative. That is:

$$\frac{\theta}{\theta + \omega} < \frac{\gamma_A + \gamma_I + \delta}{\beta_A} \text{ and } \frac{\theta}{\theta + \omega} < \frac{\gamma_A \gamma_I + \delta \gamma_I}{\gamma_I \beta_A + \delta \beta_I}$$

Stable Condition

From the inequalities, the equilibrium point in general tends to be stable as:

$\boldsymbol{\gamma}\uparrow$: Improving medical level or inventing specialized medicine

$\boldsymbol{\beta}\downarrow$: Promoting protective measures such as wearing a mask and social distancing

$\boldsymbol{\omega}\uparrow$: Promoting vaccination

$\boldsymbol{\theta}\downarrow$: Encouraging people who are recovered or vaccinated to keep their awareness of continuous protection

$\boldsymbol{\delta}\uparrow$: Conducting COVID-19 tests more frequently

Stability Analysis: Numeric Solution

Empirical parameters from the referenced paper [2]

$\boldsymbol{\beta}_A$ = 0.28, $\boldsymbol{\beta}_I$ = 0.25, $\boldsymbol{\gamma}_A$ = 0.03, $\boldsymbol{\gamma}_I$ = 0.02 and $\boldsymbol{\delta}$ = 0.02

Our new parameters:

$\boldsymbol{\theta}$ = 0.001, $\boldsymbol{\omega}$ = 0.015

Eigenvectors			Eigenvalues		
v_1	v_2	v_3	λ_1	λ_2	λ_3
1	-0.1387	0.9422	-0.0160	0	0
0	-0.7733	-0.1776	0	-0.0450	0
0	0.6187	-0.2841	0	0	-0.0075

Theoretical Investigation: Equilibrium Point 2

Recall that: $I \cdot (\frac{\beta_A \gamma_I}{\delta} S + \beta_I S - \frac{\gamma_A \gamma_I}{\delta} - \gamma_I) = 0$

$$
\begin{cases}
S = \dfrac{\gamma_A \gamma_I + \delta \gamma_I}{\beta_A \gamma_I + \delta \beta_I} \\[2em]
A = \dfrac{-\gamma_I(\delta\gamma_I\omega - \beta_I\delta\theta - \beta_A\gamma_I\theta + \gamma_A\gamma_I\omega + \delta\gamma_I\theta + \gamma_A\gamma_I\theta)}{\beta_A\delta\gamma_I^2 + \beta_I\delta^2\gamma_I + \beta_A\gamma_A\gamma_I^2 + \beta_I\delta^2\theta + \beta_A\gamma_I^2\theta + \beta_A\delta\gamma_I\theta + \beta_I\delta\gamma_I\theta + \beta_I\delta\gamma_A\gamma_I} \\[2em]
I = \dfrac{-\delta(\delta\gamma_I\omega - \beta_I\delta\theta - \beta_A\gamma_I\theta + \gamma_A\gamma_I\omega + \delta\gamma_I\theta + \gamma_A\gamma_I\theta)}{\beta_A\delta\gamma_I^2 + \beta_I\delta^2\gamma_I + \beta_A\gamma_A\gamma_I^2 + \beta_I\delta^2\theta + \beta_A\gamma_I^2\theta + \beta_A\delta\gamma_I\theta + \beta_I\delta\gamma_I\theta + \beta_I\delta\gamma_A\gamma_I} \\[2em]
R = \dfrac{\begin{gathered}(\beta_A\delta\gamma_I^2 - \gamma_A^2\gamma_I^2 - \delta^2\gamma_I^2 + \beta_I\delta^2\gamma_I + \beta_A\gamma_A\gamma_I^2 - 2\delta\gamma_A\gamma_I^2 \\ + \delta\gamma_I^2\omega + \delta^2\gamma_I\omega + \gamma_A\gamma_I^2\omega + \delta\gamma_A\gamma_I\omega + \beta_I\delta\gamma_A\gamma_I)\end{gathered}}{\beta_A\delta\gamma_I^2 + \beta_I\delta^2\gamma_I + \beta_A\gamma_A\gamma_I^2 + \beta_I\delta^2\theta + \beta_A\gamma_I^2\theta + \beta_A\delta\gamma_I\theta + \beta_I\delta\gamma_I\theta + \beta_I\delta\gamma_A\gamma_I}
\end{cases}
$$

Theoretical Investigation: Stability 2

- The symbols are overly complicated!
- Use the same set of realistic parameters to calculate eigenvalues/eigenvectors instead
- $\boldsymbol{\beta}_A = 0.28$, $\boldsymbol{\beta}_I = 0.25$, $\boldsymbol{\gamma}_A = 0.03$, $\boldsymbol{\gamma}_I = 0.02$, $\boldsymbol{\delta} = 0.02$

  $\boldsymbol{\theta} = 0.001$ and $\boldsymbol{\omega} = 0.015$

Simulations: Equilibrium Points

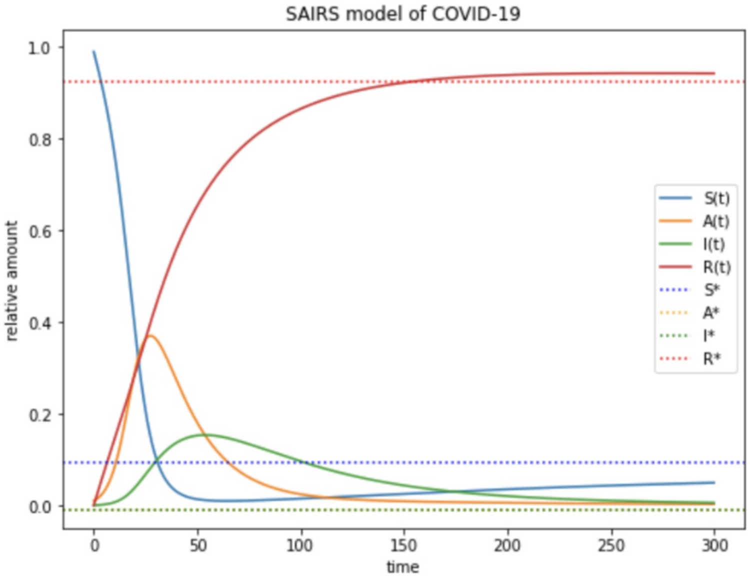


Fig. 2: SAIR Model with Equilibrium points

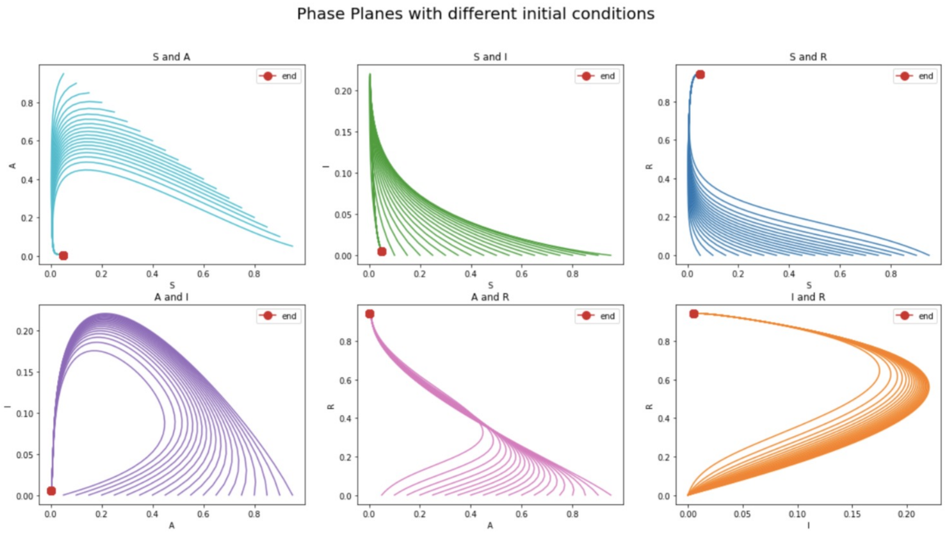
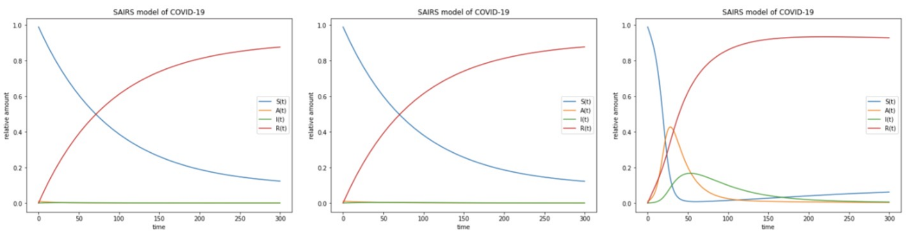


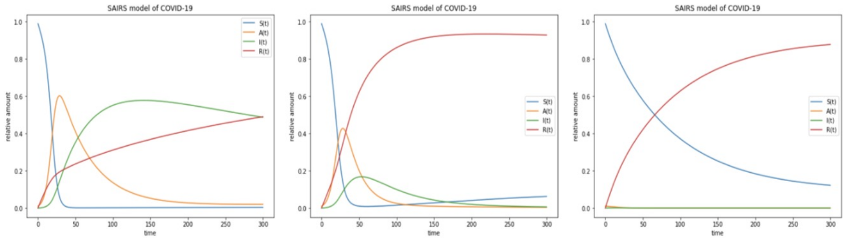
Fig. 3: 2D Phase Plane of SAIRS Model

Simulations: Parameters Tuning



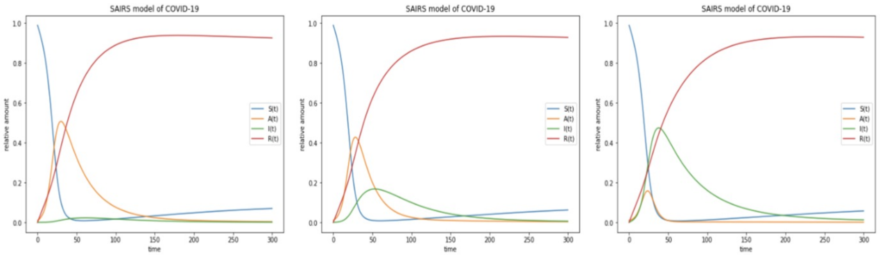
(a) $\beta_A = 0.0028$ (b) $\beta_A = 0.028$ (c) $\beta_A = 0.28$

Fig. 4: parameter change of β



(a) $\gamma_A = 0.003$ (b) $\gamma_A = 0.03$ (c) $\gamma_A = 0.3$

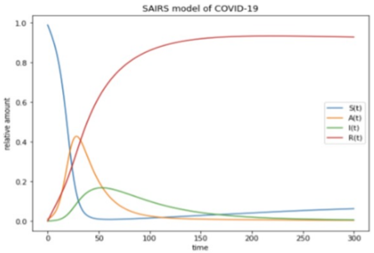
Fig. 5: parameter change of γ



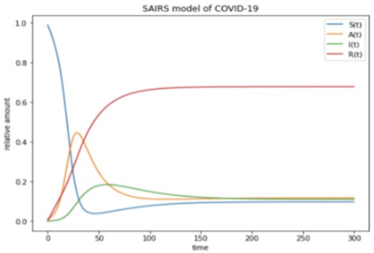
(a) $\delta = 0.002$ (b) $\delta = 0.02$ (c) $\delta = 0.2$

Fig. 6: parameter change of δ

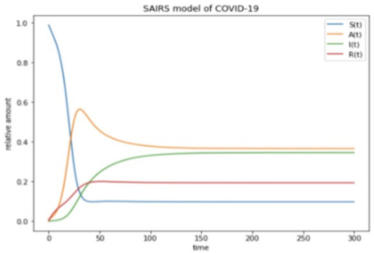
Simulations: Parameters Tuning



(a) $\theta = 0.001$

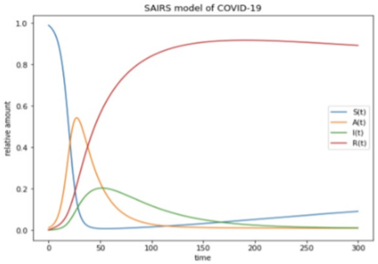


(b) $\theta = 0.01$

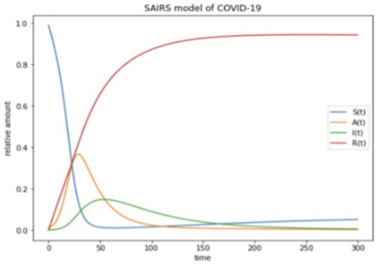


(c) $\theta = 0.1$

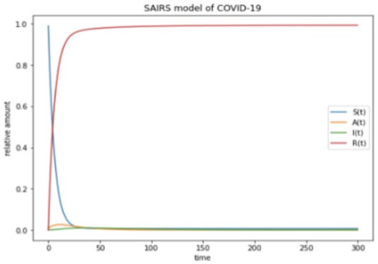
Fig. 7: parameter change of θ



(a) $\omega = 0.0015$



(b) $\omega = 0.015$



(c) $\omega = 0.15$

Fig. 8: parameter change of ω

Summary: Conclusion

- SAIRS Model with Immunization Rate
- Theoretical Investigation
    - Two discovered equilibrium points
    - Under the empirical parameters, one is stable and explainable; the other is unstable and unrealistic
- Simulations
    - Validate and visualize the model
    - θ and ω are crucial to our model

Summary: Future Studies

- Vaccinated people as a separate group
- Vital Dynamics
- Nonlinear relationship between the susceptible and infected group [5]

Reference

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