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STAT 243 Statistics Theory with Application I
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FINAL PROJECT — Normal Distribution

1 Introduction

Normal Distribution, also known as Gaussian Distribution, is a statistical distribution known for its bell-shaped curve [1]. This type of distribution is characterized by two parameters: mean (μ) and standard deviation (σ), where the mean and standard deviation measure the average and the spread of a set of observations, respectively. Essentially, it suggests there is a greater frequency of occurrences near the mean, and that frequency diminishes as one deviates in either direction from the mean.

1.1 Examples

Normal distribution is particularly significant due to its prevalence in nature. Below, we will discuss three examples of situations that could be modeled using Normal distribution [2].

- **SAT Scores**

  The distribution of test scores is a common situation that can be modeled using Normal distribution. This is especially true when we look at standardized exams like the SAT, as they are designed to conform to the characteristics of the Normal distribution. For example, the College Board publishes statistics on SAT scores every year, and the most recent one published in 2023 outlines the percentiles for total scores out of 1600 [3]. These percentile statistics resemble a Normal distribution.

- **Height**

  The distribution of height is often considered to be approximately normally distributed due to genetic and environmental variance [4]. Sourced from Our World in Data, a scientific online publication, **Figure 1** provides a visualization of the distribution of adult heights for men and women based on studies conducted across various populations [5]. The visualization shows a clear bell-shaped curve for both the distribution of female and male height with μ being 5ft 5in and 5ft 10in respectively for the two sexes.

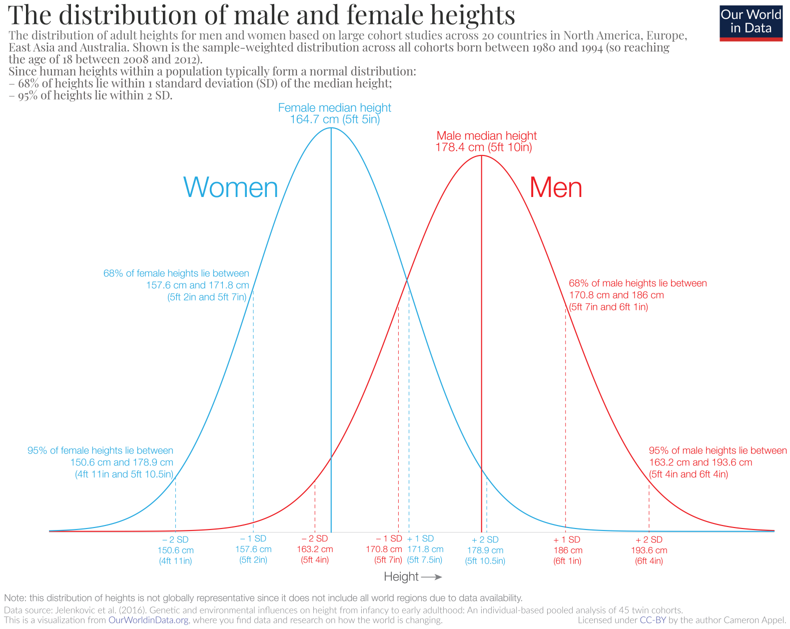

Figure 1: The distribution of male and female heights. Sourced from Our World in Data [5]

- **Birth Weight**

  Similar to human heights, the distribution of birth weight is also approximately Normal due to genetic and environmental variance. A study conducted in Norway recorded the birth weights for 405,676 live and still births from 1992 to 1998 [6]. **Figure 2** shows the distribution of birth weight recorded for those 405,676 individuals in kg. As one can see, the distribution resembles that of a bell-curve, except with an extended lower tail, possibly indicating instances of lower birth weights associated with premature births or specific health conditions. The majority of the babies in the study has a birth weight between 3-4kg, identifying the "typical birth weights."

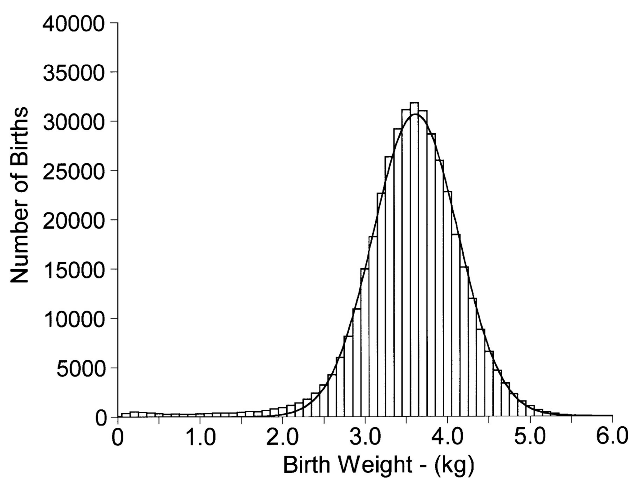

Figure 2: Distribution of birth weights for 405,676 live and still births, Norway, 1992–1998 [6]

1.2 Probability Density Functions

The probability density function (PDF) is a theoretical model for the frequency distribution of a continuous random variable [7]. The PDF, $f(x)$ of a continuous random variable X, has two key properties:

1. $f(x) \geq 0$ for all x, $-\infty < x < \infty$
2. $\int_{-\infty}^{\infty} f(x)\, dx = 1$

In the case of a Normal distribution with mean μ and variance σ^2, the PDF $f(x)$ is defined as:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \tag{1-1}$$

where $-\infty < x < \infty$, $-\infty < \mu < \infty$, and $0 < \sigma < \infty$.

Figure 3 shows a visualization of the PDF for a normally distributed random variable Y.

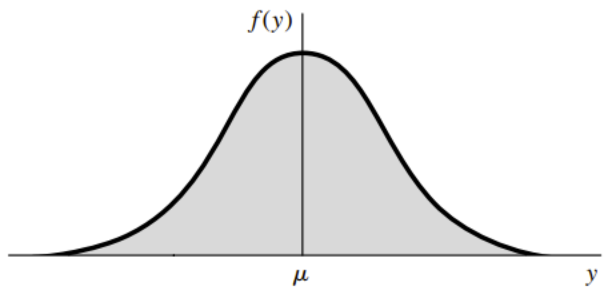

Figure 3: The normal probability density function [7]

However, the PDF of a Normal distribution is often not used because its integral does not have a closed-form; as a result, its evaluation requires additional integration techniques. Due to this, a standard Normal distribution is often used to compute areas under the distribution curve. This is a special case of Normal distribution where the mean (μ) is 0, and the variance (σ^2) is 1. In the case where X is a normally distributed random variable with mean μ and variance σ^2, then Z is a standard normally distributed random variable with mean 0 and variance 1, where $Z = \frac{X-\mu}{\sigma}$.

The PDF $f(z)$ of a standard Normal distribution is defined as:

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \tag{1-2}$$

where $-\infty < z < \infty$.

1.3 Moment Generating Function

From the parameters μ and σ, we are able to locate the center and understand the spread of a random variable Y. However, they do not provide a unique characterization of the distribution of Y. This is where moments and additional statistical measures come into play. Moments give us additional information to help distinguish different distributions that may share μ and σ. The moment-generating function (MGF) is a tool that helps us utilize the moments [7].

The moment-generating function $m(t)$ for a random variable Y is defined as:

$$m(t) = E(e^{tY}) \tag{1-3}$$

if there exists a positive constant a such that $m(t)$ is finite for $|t| \leq a$.

Using the MGF, the n-th moment of the random variable Y can be derived by taking the n-th derivative of m(t) with respect to t and then solving for $m(t)$ at $t = 0$.

In the case of a normally distributed random variable X with mean μ and variance σ^2, the MGF is defined as:

$$m(t) = e^{\mu t + \frac{t^2 \sigma^2}{2}} \tag{1-4}$$

Using the MGF for X from **Equation 1-4**, we can derive the mean and variance of X. The mean is equal to the first moment, and the variance can be derived using the second moment and the mean.

- **Mean**
  $$\begin{aligned} E(X) &= m'(t)\Big|_{t=0} \\ &= \mu + \sigma^2 t\Big|_{t=0} \\ &= \mu \end{aligned}$$
- **Variance**
  $$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= m''(t)\Big|_{t=0} - [\mu]^2 \\ &= \mu^2 + \sigma^2 - \mu^2 \\ &= \sigma^2 \end{aligned}$$

2 Properties of Normal PDF

In this section, we will explore and prove some properties of the normal distribution. Here, assume that $X \sim \mathcal{N}(\mu, \sigma^2)$ with PDF denoted by $f(x)$ and $Z \sim \mathcal{N}(0, 1)$ with PDF denoted by $f_z(z)$.

(a) The PDF (density) of a standard normal distribution Z, where $Z \sim \mathcal{N}(0, 1)$ is a valid PDF

As defined in **Section 1.2**, a valid PDF has two properties. Therefore, we will verify that the PDF of Z, $f(z)$, defined in **Equation 1-2** satisfies those two properties.

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$$

1. $f(z) \geq 0$ for all z, $-\infty < z < \infty$
   Proof:
   Consider $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$. We will show that $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} \geq 0$ for any real number z.
   i. The exponential function with base e is always positive with any real number power. Therefore, $e^{-\frac{1}{2}z^2} > 0$.
   ii. $\frac{1}{\sqrt{2\pi}}$ is a positive constant.
   iii. $f(z) = (+)(+) = (+) \geq 0$

2. $\int_{-\infty}^{\infty} f(z)\,dz = 1$
Proof:
$$\begin{aligned}\int_{-\infty}^{\infty} f(z)\,dz &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}\,dz\\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz, \text{ because } \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi} \text{ by Gaussian integral as shown below}\\ &= \frac{1}{\sqrt{2\pi}}\sqrt{2\pi}\\ &= 1\end{aligned}$$

Proof for Gaussian Integral [8]: Consider the square of the integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$

$$\begin{aligned}(\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz)^2 &= \int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz \int_{-\infty}^{\infty} e^{-\frac{1}{2}y^2}\,dy\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2+y^2)}\,dz\,dy\end{aligned}$$

Now, consider the function $e^{-\frac{1}{2}(z^2+y^2)}$ as a polar coordinate on the plane R^2 by defining $r = \sqrt{z^2+y^2}$, so we get $e^{-\frac{1}{2}(r^2)}$. And consider θ as the angle. Polar transformation is $z = r\cos(\theta)$ and $y = r\sin(\theta)$. So $dz\,dy = r\,dr\,d\theta$. Compute its integral:

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2+y^2)}\,dz\,dy &= \int_{0}^{\infty}\int_{0}^{2\pi} e^{-\frac{1}{2}(r^2)} r\,d\theta\,dr\\ &= 2\pi \int_{0}^{\infty} re^{-\frac{1}{2}(r^2)}\,dr\\ &= 2\pi \int_{0}^{\infty} e^{-u}\,du, \text{ perform u-sub where } u = \frac{1}{2}r^2\\ &= 2\pi\left(-e^{-u}\right)\Big|_{0}^{\infty}\\ &= 2\pi\left(-e^{-\frac{1}{2}(r^2)}\right)\Big|_{0}^{\infty}, \text{ plug } u \text{ back}\\ &= 2\pi[-0+1]\\ &= 2\pi\end{aligned}$$

We got that the square of the integral $\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz$ is 2π; therefore:

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}z^2}\,dz = \sqrt{2\pi}$$

(b) The height of the Normal Density curve (PDF), $f(x)$, is maximized at $x = \mu$

Proof:

In **Equation 1-1**, we found that the PDF of a normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

To find the max, we need to differentiate $f(x)$ with respect to x and solve for x after setting the derivative equal to zero.

1. Taking derivative of $f(x)$

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) \\ &= -\frac{x-\mu}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \end{aligned}$$

2. Setting $f'(x) = 0$

$$f'(x) = -\frac{x-\mu}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} = 0$$

3. Solve for x
By the properties of a valid PDF in **Section 1.2** and the fact that exponential function is > 0, $f'(x) = 0$ if $-\frac{x-\mu}{\sigma^2} = 0$.

So $x = \mu$ when $f'(x) = 0 \longrightarrow$ local maximum.

(c) $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$
Proof:

- $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ only consider the $e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$ portion of $f(x)$. If we ignore all the non-x variables as constant, we will see that the limit of $f(x)$ reflects solely on e^{-x^2}.
- $\lim_{x\to\infty} e^{-x^2} = 0$ and $\lim_{x\to-\infty} e^{-x^2} = 0$ since the power is going toward $-\infty$.
- Therefore, $\lim_{x\to\infty} f(x) = 0$ and $\lim_{x\to-\infty} f(x) = 0$.

(d) The normal density curve is symmetric about the mean: $f(\mu + x) = f(\mu - x)$ for all $-\infty < x < \infty$
Proof:

- Plugging in $(\mu + x)$ for x

$$\begin{aligned} f(\mu + x) &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{(\mu+x)-\mu}{\sigma}\right)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2}\right)} \end{aligned}$$

- Plugging in $(\mu - x)$ for x

$$\begin{aligned} f(\mu - x) &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{(\mu-x)-\mu}{\sigma}\right)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{-x}{\sigma}\right)^2} \\ &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{(-x)^2}{\sigma^2}\right)} \\ &= \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x^2}{\sigma^2}\right)} \end{aligned}$$

- Therefore, $f(\mu + x) = f(\mu - x)$, so the normal density curve is symmetric about the mean.

(e) The normal density has two points of inflection at $\mu \pm \sigma$

Proof:

Inflection points are where the function changes concavity. We would have to look at the second derivative with respect to x to find where the second derivative changes signs (when the derivative is equal to zero).

1. Finding the second derivative $f''(x)$

   In **Part 2.b.1**, we found $f'(x) = -\frac{x-\mu}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$. So:

$$\begin{aligned}
f''(x) &= \frac{d}{dx}(f'(x)) \\
&= \frac{d}{dx}\left(-\frac{x-\mu}{\sigma^2}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma}\ \frac{d}{dx}\left(-\frac{x-\mu}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right) \\
&= \frac{1}{\sqrt{2\pi}\sigma}\left[-\frac{1}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} + \left(-\frac{x-\mu}{\sigma^2}\right)\left(-\frac{x-\mu}{\sigma^2}\right)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right], \text{ by chain rule} \\
&= \frac{1}{\sqrt{2\pi}\sigma}\left[-\frac{1}{\sigma^2}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} + \left(\frac{x-\mu}{\sigma^2}\right)^2 e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right] \\
&= \left(\frac{(x-\mu)^2}{\sqrt{2\pi}\sigma^5} - \frac{1}{\sqrt{2\pi}\sigma^3}\right)e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}
\end{aligned}$$

2. Now set $f''(x)$ to 0 and solve for x:

   $f''(x) = 0$ if $\left(\frac{(x-\mu)^2}{\sqrt{2\pi}\sigma^5} - \frac{1}{\sqrt{2\pi}\sigma^3}\right) = 0$ as the exponential function is non-zero. So:

$$\begin{aligned}
\frac{(x-\mu)^2}{\sqrt{2\pi}\sigma^5} &= \frac{1}{\sqrt{2\pi}\sigma^3} \\
\frac{(x-\mu)^2}{\sqrt{2\pi}\sigma^5}\cdot\sqrt{2\pi}\sigma^5 &= \frac{1}{\sqrt{2\pi}\sigma^3}\cdot\sqrt{2\pi}\sigma^5 \\
(x-\mu)^2 &= \sigma^2 \\
x-\mu = \sigma &\ \textbf{or}\ x-\mu = -\sigma \\
x = \mu+\sigma &\ \textbf{or}\ x = \mu-\sigma \\
x &= \mu \pm \sigma
\end{aligned}$$

   Therefore, the points of inflections are $x = \mu \pm \sigma$

3 Bivariate Normal Distribution

Now we will explore the Bivariate Normal Distribution, which is a continuous bivariate distribution that generalizes the Normal distribution into two dimensions [9].

- **Joint PDF for a bivariate normal distribution**

  The bivariate normal density function for continuous random variables (X, Y) is a function of five parameters:

  - $\mu_1 = E(X)$

- $\sigma_1^2 = Var(X)$
- $\mu_2 = E(Y)$
- $\sigma_2^2 = Var(Y)$
- $\rho = Corr(X, Y)$, where $\rho \in (-1, 1)$

The joint PDF for a bivariate normal distribution for (X, Y) is defined as [10]:

$$f(x, y) = \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}, \ -\infty < x < \infty, \ -\infty < y < \infty \tag{3-1}$$

where

$$Q = \frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]$$

- **Marginal PDFs of X and Y**

Now we will look into the marginal PDFs of X and Y, which provides information about the probability distribution of individual variables. To compute the marginal PDF of a bivariate normal distribution variable, we integrate the joint PDF with respect to the other variable. Below, we will compute $f_X(x)$ and $f_Y(y)$, the marginal PDF of X and Y respectively [11].

- $f_X(x)$

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \tag{3-2}$$

- $f_Y(y)$

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2} \tag{3-3}$$

As shown in **Equations 3-2 and 3-3** above, both $f_X(x)$ and $f_Y(y)$ follow the PDF of a normal distribution shown in **Equation 1-1**.

- For $f_X(x)$:
  - mean $\mu = \mu_1 = E(X)$
  - variance $\sigma^2 = \sigma_1^2 = Var(X)$
- For $f_Y(y)$:
  - mean $\mu = \mu_2 = E(Y)$
  - variance $\sigma^2 = \sigma_2^2 = Var(Y)$

The integration to obtain $f_X(x)$ and $f_Y(y)$ is shown on the next two pages to ensure a good formatting of the submission.

To compute $f_X(x)$:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\ dy$$

$$= \int_{-\infty}^{\infty} \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ dy$$

plug in Q

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ dy$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}\ dy$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\boxed{\rho\left(\frac{x-\mu_1}{\sigma_1}\right)}\left(\frac{y-\mu_2}{\sigma_2}\right)\right]}\ dy$$

complete the square

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\underbrace{\left[\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}_{\text{perfect square}} - \rho^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)}\ dy$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ e^{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{x-\mu_1}{\sigma_1}\right)^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2}\ dy$$

$$= \frac{e^{-\frac{1}{2\cancel{(1-\rho^2)}}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\cancel{\left[\rho^2-1\right]}}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]^2}\ dy$$

Let $u = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{y-\mu_2}{\sigma_2} - \rho\left(\frac{x-\mu_1}{\sigma_1}\right)\right]$, then $du = \frac{1}{\sqrt{1-\rho^2}\ \sigma_2}\,dy$

subsitute u

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1\cancel{\sigma_2}\cancel{\sqrt{1-\rho^2}}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2)}\ \cancel{\sqrt{1-\rho^2}}\ \cancel{\sigma_2}\ du$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1} \int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du$$

Gaussian Integral proven in Section 2.a.2

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{2\pi\sigma_1}\left(\sqrt{2\pi}\right)$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}}$$

Similarly, we can compute $f_Y(y)$:

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\ dx$$

$$= \int_{-\infty}^{\infty} \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ dx$$

plug in Q

$$= \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ dx$$

$$= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}\ dx$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\boxed{\rho\left(\frac{y-\mu_2}{\sigma_2}\right)}\left(\frac{x-\mu_1}{\sigma_1}\right)\right]}\ dx$$

complete the square

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left(\underbrace{\left[\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{y-\mu_2}{\sigma_2}\right)\left(\frac{x-\mu_1}{\sigma_1}\right) + \rho^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}_{\text{perfect square}} - \rho^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)}\ dx$$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\ e^{\frac{\rho^2}{2(1-\rho^2)}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right]^2}\ dx$$

$$= \frac{e^{-\frac{1}{2\cancel{(1-\rho^2)}}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\left[\cancel{\rho^2-1}\right]}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\int_{-\infty}^{\infty} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x-\mu_1}{\sigma_1} - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right]^2}\ dx$$

Let $u = \frac{1}{\sqrt{1-\rho^2}}\left[\frac{x-\mu_1}{\sigma_1} - \rho\left(\frac{y-\mu_2}{\sigma_2}\right)\right]$, then $du = \frac{1}{\sqrt{1-\rho^2}\ \sigma_1}\ dx$

subsitute u

$$= \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\cancel{\sigma_1}\sigma_2\cancel{\sqrt{1-\rho^2}}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(u^2)}\ \cancel{\sqrt{1-\rho^2}}\ \cancel{\sigma_1}\ du$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\sigma_2}\int_{-\infty}^{\infty} e^{-\frac{1}{2}u^2}\ du$$

Gaussian Integral proven in Section 2.a.2

$$= \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{2\pi\sigma_2}\left(\sqrt{2\pi}\right)$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{\sigma_2\sqrt{2\pi}}$$

- **Conditional PDFs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$**

  Lastly, we will look at the Conditional PDFs $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ of a bivariate normal distribution. By the definition of conditional PDFs [10], we know that:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}, \text{ where } f_Y(y) > 0 \tag{3-4}$$

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \text{ where } f_X(x) > 0 \tag{3-5}$$

  In earlier parts, we have found $f(x,y)$ as **Equation 3-1**, $f_X(x)$ as **Equation 3-2**, and $f_Y(y)$ as **Equation 3-3**:

$$f(x,y) = \frac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}$$

$$f_X(x) = \frac{1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}$$

$$f_Y(y) = \frac{1}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}$$

  Using the three equations above, we are able to compute and obtain the following conditional PDFs:

  - $f_{X|Y}(x|y)$

$$f_{X|Y}(x|y) = \frac{e^{-\frac{1}{2}\left[\frac{x-\left(\mu_1+\frac{\rho\sigma_1(y-\mu_2)}{\sigma_2}\right)}{\sigma_1\sqrt{1-\rho^2}}\right]}}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{2\pi}} \tag{3-6}$$

  - $f_{Y|X}(y|x)$

$$f_{Y|X}(y|x) = \frac{e^{-\frac{1}{2}\left[\frac{y-\left(\mu_2+\frac{\rho\sigma_2(x-\mu_1)}{\sigma_1}\right)}{\sigma_2\sqrt{1-\rho^2}}\right]}}{\left(\sigma_2\sqrt{1-\rho^2}\right)\sqrt{2\pi}} \tag{3-7}$$

  As shown in **Equations 3-6 and 3-7** above, both $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ follow the PDF of a normal distribution as shown in **Equation 1-1**:

  - For $f_{X|Y}(x|y)$:
    - mean $\mu = \mu_1 + \frac{\rho\sigma_1(y-\mu_2)}{\sigma_2}$
    - variance $\sigma^2 = \sigma_1^2(1-\rho^2)$
  - For $f_{Y|X}(y|x)$:
    - mean $\mu = \mu_2 + \frac{\rho\sigma_2(x-\mu_1)}{\sigma_1}$
    - variance $\sigma^2 = \sigma_2^2(1-\rho^2)$

  The derivation of $f_{X|Y}(x|y)$ and $f_{Y|X}(y|x)$ is shown on the next two pages to ensure a good formatting of the submission.

To compute $f_{X|Y}(x|y)$:

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

$$= \frac{\dfrac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}}{\dfrac{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{\sigma_2\sqrt{2\pi}}}$$

$$= \frac{e^{-Q/2}}{\sqrt{2\pi}\,\sigma_1\cancel{\sigma_2}\sqrt{1-\rho^2}} \cdot \frac{\cancel{\sigma_2}\cancel{\sqrt{2\pi}}}{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}}{\sigma_1\sqrt{2\pi(1-\rho^2)}} \cdot \frac{1}{e^{-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \left(-\frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)}}{\sigma_1\sqrt{2\pi(1-\rho^2)}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] + \frac{1}{2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{1}{1-\rho^2}\cdot\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \left(\frac{y-\mu_1}{\sigma_2}\right)^2\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$\frac{1-1+\rho^2}{1-\rho^2} = \frac{\rho^2}{1-\rho^2}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\left[\cancel{\frac{1}{1-\rho^2} - 1}\right]\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{\rho^2}{1-\rho^2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

factor out $\frac{1}{1-\rho^2}$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\cdot\frac{\sigma_1}{\sigma_1}\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \rho^2\cdot\frac{\sigma_1^2}{\sigma_1^2}\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

factor out $\frac{1}{\sigma_1^2}$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left[(x-\mu_1)^2 - 2\rho\sigma_1\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_2} + \rho^2\sigma_1^2\left(\frac{y-\mu_2}{\sigma_2}\right)^2\right]}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

perfect square

$$= \frac{e^{-\frac{1}{2(1-\rho^2)\sigma_1^2}\left[(x-\mu_1) - \frac{\rho\sigma_1(y-\mu_2)}{\sigma_2}\right]^2}}{\sigma_1\sqrt{1-\rho^2}\sqrt{2\pi}}$$

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$$= \frac{e^{-\frac{1}{2}\left[\frac{x-\left(\mu_1+\frac{\rho\sigma_1(y-\mu_2)}{\sigma_2}\right)}{\left(\sigma_1\sqrt{1-\rho^2}\right)}\right]^2}}{\left(\sigma_1\sqrt{1-\rho^2}\right)\sqrt{2\pi}}$$

Similarly, we can compute $f_{Y|X}(y|x)$:

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

$$= \frac{\dfrac{e^{-Q/2}}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}}{\dfrac{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sigma_1\sqrt{2\pi}}}$$

$$= \frac{e^{-Q/2}}{\sqrt{2\pi}\,\cancel{\sigma_1}\sigma_2\sqrt{1-\rho^2}} \cdot \frac{\cancel{\sigma_1}\cancel{\sqrt{2\pi}}}{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right]}}{\sigma_2\sqrt{2\pi(1-\rho^2)}} \cdot \frac{1}{e^{-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \left(-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right)}}{\sigma_2\sqrt{2\pi(1-\rho^2)}}$$

$$= \frac{e^{-\frac{1}{2}\cdot\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] + \frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\left[\frac{(x-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}\right] - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{1}{1-\rho^2}\cdot\left(\frac{y-\mu_2}{\sigma_2}\right)^2 - \left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$\frac{1-1+\rho^2}{1-\rho^2} = \frac{\rho^2}{1-\rho^2}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \left(\frac{x-\mu_1}{\sigma_1}\right)^2\left[\frac{1}{1-\rho^2}\cancel{-1}\right]\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

$$= \frac{e^{-\frac{1}{2}\left[\frac{1}{1-\rho^2}\cdot\frac{(y-\mu_2)^2}{\sigma_2^2} - \frac{2\rho}{1-\rho^2}\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{\rho^2}{1-\rho^2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

factor out $\frac{1}{1-\rho^2}$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)}\left[\frac{(y-\mu_2)^2}{\sigma_2^2} - 2\rho\cdot\frac{\sigma_2}{\sigma_1}\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \rho^2\cdot\frac{\sigma_2^2}{\sigma_2^2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

factor out $\frac{1}{\sigma_2^2}$

$$= \frac{e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[(y-\mu_2)^2 - 2\rho\sigma_2\cdot\frac{(x-\mu_1)(y-\mu_2)}{\sigma_1} + \rho^2\sigma_2^2\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

perfect square

$$= \frac{e^{-\frac{1}{2(1-\rho^2)\sigma_2^2}\left[(y-\mu_2) - \frac{\rho\sigma_2(x-\mu_1)}{\sigma_1}\right]^2}}{\sigma_2\sqrt{1-\rho^2}\sqrt{2\pi}}$$

Forming the shape for PDF of Normal Distribution

$$= \frac{e^{-\frac{1}{2}\left[\frac{y-\left(\mu_2+\frac{\rho\sigma_2(x-\mu_1)}{\sigma_1}\right)}{\left(\sigma_2\sqrt{1-\rho^2}\right)}\right]^2}}{\left(\sigma_2\sqrt{1-\rho^2}\right)\sqrt{2\pi}}$$

4 Simulation

In this section, we will do some simulations to further study the Normal distribution.

(a) Let $X \sim \mathcal{N}(\mu = 1, \sigma^2 = 4)$. We will sample 1000 data points from X and plot the PDF of X.

To sample 1000 points, we will use the *NumPy* Python library as shown below:

```
import numpy as np
import matplotlib.pyplot as plt

# set the mean and variance
mean = 1
variance = 4

# generate 1000 data points
np.random.seed(1) # set the seed
normal_sample = np.random.normal(mean, np.sqrt(variance), 1000)
```

Now, we will use the *Matplotlib* Python library to plot the density of the simulated data using a histogram and also plot the PDF curve of X on top of the histogram:

```
# plot the simulated data
plt.hist(normal_sample, bins=50, density=True, edgecolor='black')

# plot the PDF of X~N(mu=1, var=4)
xmin, xmax = plt.xlim()
x = np.linspace(xmin, xmax)
pdf = np.exp(-0.5*((x-mean)/np.sqrt(variance))**2)/(np.sqrt(variance*2*np.pi))
plt.plot(x, pdf, linewidth=2)

plt.title('Density Curve of X')
plt.xlabel('X')
plt.ylabel('Density')
plt.show()
```

The output plot is shown below in **Figure 4**

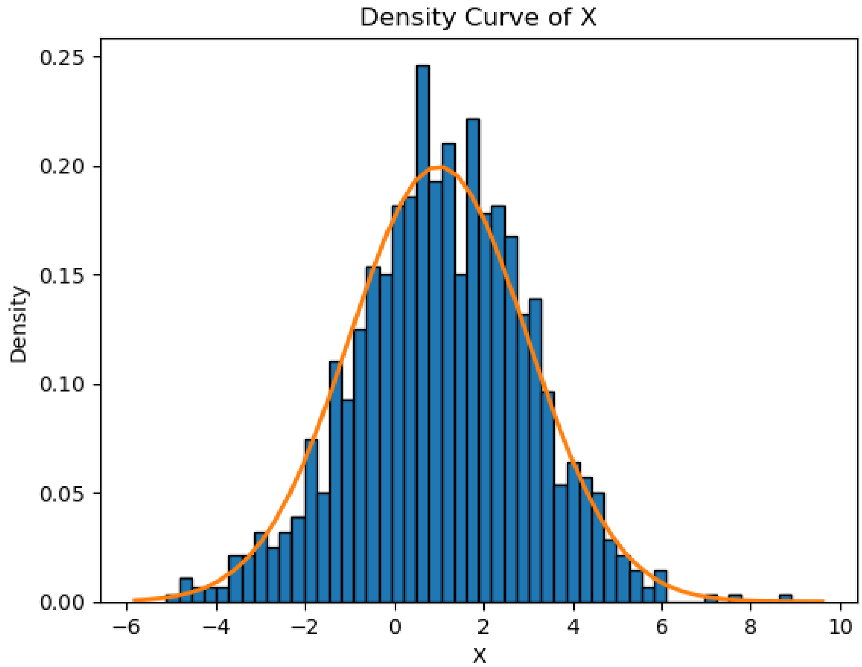

Figure 4: Density Curve of $X \sim \mathcal{N}(\mu = 1, \sigma^2 = 4)$

(b) Now let's find the probability that $P(3 < X < 5)$ for $X \sim \mathcal{N}(\mu = 1, \sigma^2 = 4)$, which is basically the area under the PDF curve between $X = 3$ and $X = 5$.

To do this, we will use the cumulative distribution function (CDF) to find the probability that X will have a value less than or equal to 3 and the probability that X will have a value less than or equal to 5, denoted as $F_X(3)$ and $F_X(5)$ respectively. $P(3 < X < 5)$ is equal to $F_X(5) - F_X(3)$.

We can calculate these values for X using the *SciPy* Python library as shown below:

```python
from scipy.stats import norm

# calculate the cdf
lower_bound = norm.cdf(3, mean, np.sqrt(variance))
upper_bound = norm.cdf(5, mean, np.sqrt(variance))

# P(3 < X < 5)
prob = upper_bound - lower_bound

print(f"P(3 < X < 5) ~= {prob:.5f}")
```

The output will return: $P(3 < X < 5) \approx 0.13591$

(c) Lastly, let's look at a Bivariate Normal distribution (X, Y) with $\mu_1 = 0, \mu_2 = 0, \sigma_1^2 = 1, \sigma_2^2 = 1, \rho = 0.75$.

To plot the bivariate density for (X, Y), we can use the *SciPy* Python library again. Specially, the *scipy.stats.multivariate_normal* package to define the bivariate normal distribution. The following parameters need to be defined:

- mean: an array-like object containing the mean of X and Y $\longrightarrow [\mu_1, \mu 2]$
- cov: the covariance matrix of the distribution $\longrightarrow Cov(X,Y)$

The *mean* parameter is already given as $[\mu_1, \mu 2] = [0, 0]$. On the other hand, we can compute $Cov(X,Y)$ using the definition of Correlation Coefficient (ρ) for bivariate distribution, which states:

$$\rho = \frac{Cov(X,Y)}{\sigma_x \cdot \sigma_y} \tag{4-1}$$

Since ρ, $\sigma_x = \sigma_1$, and $\sigma_y = \sigma_2$ are given, $Cov(X,Y)$ can be computed as follows:

$$\begin{aligned} Cov(X,Y) &= \rho \cdot \sigma_1 \cdot \sigma_2 \\ Cov(X,Y) &= 0.75 \cdot 1 \cdot 1 \\ Cov(X,Y) &= 0.75 \end{aligned}$$

After finding $Cov(X,Y)$, we can use it to define the covariance matrix:

$$\Sigma = \begin{bmatrix} Cov(X,X) & Cov(X,Y) \\ Cov(Y,X) & Cov(Y,Y) \end{bmatrix}$$

Plug in $Cov(X,Y) = Cov(Y,X) = 0.75$ as computed above. Also note $Cov(X,X) = \sigma_1^2 = 1$ and $Cov(Y,Y) = \sigma_2^2 = 1$ as the covariance of a random variable with itself is just its variance.

$$\Sigma = \begin{bmatrix} 1 & 0.75 \\ 0.75 & 1 \end{bmatrix}$$

Now, let's define the bivariate normal distribution and plot it:

```python
from scipy.stats import multivariate_normal

# create a bivariate normal distribution
bivariate_normal = multivariate_normal([0,0], [[1, 0.75], [0.75, 1]])

# create a grid for plotting
x, y = np.meshgrid(np.linspace(-3, 3, 100), np.linspace(-3, 3, 100))
pos = np.dstack((x, y))

# plot the bivariate normal distribution with rho = 0.75
fig = plt.figure()
ax = fig.add_subplot(111, projection='3d')

# plot the bivariate normal distribution with rho = 0.75
ax.plot_surface(x, y, bivariate_normal.pdf(pos), cmap='viridis')
ax.set_title('Bivariate Normal Distribution (rho = 0.75)')
ax.set_xlabel('X')
ax.set_ylabel('Y')

plt.show()
```

The output plot is shown below in **Figure 5**

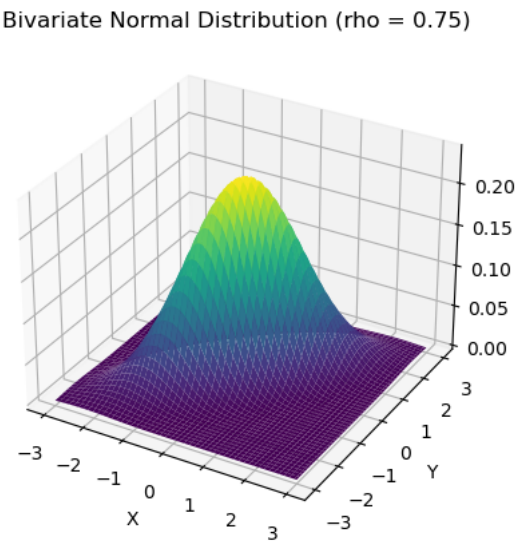

Figure 5: Bivariate Density for (X, Y) with $\rho = 0.75$

Now, let plot the same bivariate density but with $\rho = 0$. Following the same steps to compute the covariance matrix as demonstrated above, this means that the covariance matrix will change to:

$$\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

To plot the density, the code will be the same as earlier, except we are changing one line to re-define the bivariate normal distribution with the new covariance matrix:

```
bivariate_normal = multivariate_normal([0,0], [[1, 0], [0, 1]])
```

The plot should look similar to **Figure 6** below

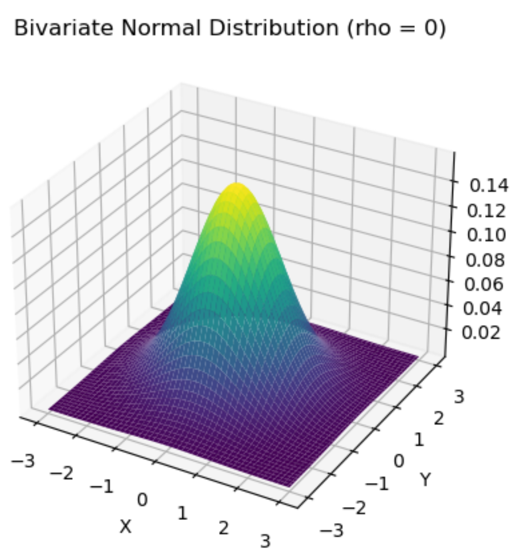

Figure 6: Bivariate Density for (X, Y) with $\rho = 0$

Comparing the two plots in **Figure 5** and **Figure 6**, we can see that they have different shapes. The density plot with $\rho = 0.75$ seems to be an ellipse along the line Y = X, while the density plot with $\rho = 0$ seems to be a circular shape. This makes sense as $\rho = 0.75$ indicates a strong positive correlation between X and Y, while $\rho = 0$ indicate no correlation between X and Y.

To see the shapes more clearly, we can also visualize them on a 2D plane:

```
# create the two distributions
rho_75 = multivariate_normal([0,0], [[1, 0.75], [0.75, 1]])
rho_0 = multivariate_normal([0,0], [[1, 0], [0, 1]])

# create a grid for plotting
x, y = np.meshgrid(np.linspace(-3, 3, 100), np.linspace(-3, 3, 100))
pos = np.dstack((x, y))

fig = plt.figure(figsize=(10, 5))

# rho = 0.75
ax1 = fig.add_subplot(121)
ax1.contourf(x, y, rho_75.pdf(pos), cmap='viridis')
ax1.set_title("Bivariate Normal Distribution (rho = 0.75)")
ax1.set_xlabel('X')
ax1.set_ylabel('Y')

# rho = 0
ax2 = fig.add_subplot(122)
ax2.contourf(x, y, rho_0.pdf(pos), cmap='viridis')
ax2.set_title("Bivariate Normal Distribution (rho = 0)")
ax2.set_xlabel('X')
ax2.set_ylabel('Y')

plt.show()
```

The output of the 2D visualization is shown below in **Figure 7**

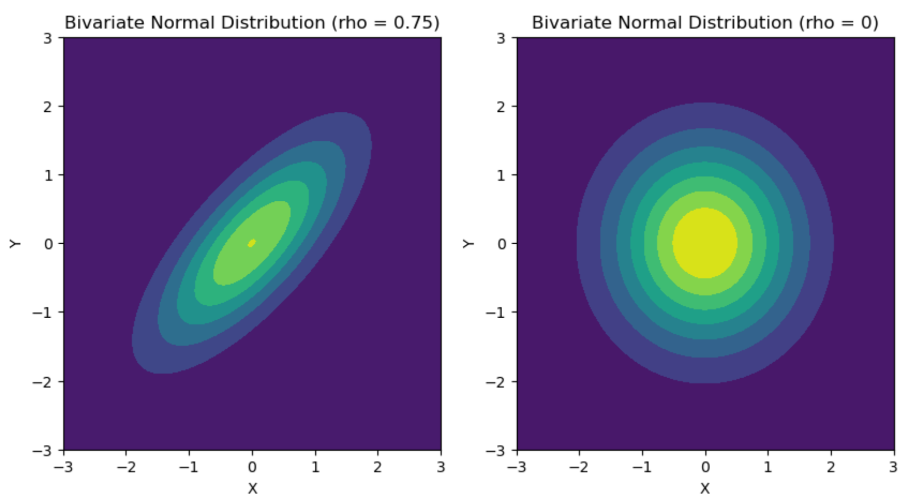

Figure 7: 2D Visualization of Bivariate Density for (X, Y)

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