Selected Answers to *"**Principles of Mathematical Analysis**"*

wentao

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1 The Real and Complex Number System

1. If r is rational($r \neq 0$) and x is irrational, prove that $r + x$ and rx are irrational.
   Proof: Suppose $r + x$ is rational, then$r + x = \frac{m}{n}, m, n \in \mathbb{Z}$, and m,n have no common factors. Then $m = n(r + x)$. Let $r = \frac{p}{q}, p, q \in \mathbb{Z}$, the former equation implies that $m = n(\frac{p}{q} + x)$,i.e.,$qm = n(p + qx)$,i.e.,$x = \frac{mq-np}{nq}$, which says that x can be written as the quotient of two integers. This is contradict to the assumption that x is irrational. The proof for the case rx is similar.

2. Prove that there is no rational number whose square is 12.
   Proof: Suppose on the contrary, there is a rational number p satisfies $p^2 = 12$, then let $p = \frac{m}{n}, m, n \in \mathbb{Z}$, m, n have no common factors, so $\frac{m^2}{n^2} = 12$, i.e., $m^2 = 12n^2$, which shows m^2 is even, m is even to. Suppose $m = 2k$, then $4k^2 = 12n^2$, i.e., $k^2 = 3n^2$, i.e., k^2 can be divided by 3, i.e., k can be divided by 3, so m can be divided by 3. Let $k = 3p$, then $k^2 = 9p^2$,i.e.,$9p^2 = 3n^2$,i.e.,$n^2 = 3p^2$, so n^2 can be divided by 3, i.e., n can be divided by 3, which is contradict to our choice of m, n.

3. Prove Proposition 1.15.

4. Let E be a nonempty subset of an ordered set; suppose α is a lower bound of E and β is an upper bound of E. Prove that $\alpha \leq \beta$.
   Proof: $\forall x \in E$, $\alpha \leq x$ and $x \leq \beta$, since E is also an ordered set, which implies that $\alpha \leq \beta$.

5. Let A be a nonempty set of real numbers which is bounded below. Let $-A$ be the set of all numbers $-x$, where $x \in A$. Prove that $\inf A = -\sup(-A)$.
   Proof: Suppose y is a lower bound of A, which means $\forall x \in A$, $y \leq x$, then $-x \leq -y$, $\forall -x \in -A$. In other words, $-A$ is bounded above, thus $z = \sup(-A)$ exists. What remains to prove is $\inf A = -z$. According

to the previous process, $-z$ is a lower bound of A, (*) and if $w > -z$, then $z > -w$, i.e. $-w$ is not an upper bound of $-A$, thus $\exists y = -x \in -A (x \in A)$, $y > -w$, i.e. $-y < w$, but $-y = -(-x) = x$, so $x < w$, which shows that w is not a lower bound of A. Combined with (*), we conclude that $-z$ is the greatest lower bound of A, i.e. $-z = \inf A$, thus $\inf A = -\sup(-A)$.

6. Fix $b > 1$.

   (a) If m, n, p, q are integers, $n > 0$, $q > 0$, and $r = m/n = p/q$, prove that $(b^m)^{1/n} = (b^p)^{1/q}$. Hence it makes sense to define $b^r = (b^m)^{1/n}$.
   Proof: There is unique positive real numbers r_1 and r_2, which satisfy $r_1 = (b^m)^{1/n}$ and $r_2 = (b^p)^{1/q}$, what we need to prove then becomes $r_1 = r_2$. For $r_1^n = b^m = b^{nr}$ and $r_2^q = b^p = b^{qr}$, thus $r_1^{nq} = b^{nrq} = r_2^{qn}$, i.e.$r_1^{nq} = r_2^{nq}$, which means $r_1 = r_2$ if we take $\frac{1}{nq}$ root from both sides.

   (b) Prove that $b^{r+s} = b^r b^s$ if r and s are rational.
   Proof: Suppose $r = \frac{m_1}{n_1}$ and $s = \frac{m_2}{n_2}$, then $r + s = \frac{m_1 n_2 + m_2 n_1}{n_1 n_2}$, thus $(b^{r+s})^{n_1 n_2} = b^{m_1 n_2 + m_2 n_1}$, and $(b^r b^s)^{n_1 n_2} = (b^r)^{n_1 n_2} (b^s)^{n_1 n_2} = b^{m_1 n_2} b^{m_2 n_1} = b^{m_1 n_2 + m_2 n_1}$, which shows $(b^{r+s})^{n_1 n_2} = (b^r b^s)^{n_1 n_2}$, i.e. $b^{r+s} = b^r b^s$.

   (c) If x is real, define $B(x)$ to be the set of all numbers b^t, where t is *rational* and $t \leq x$. Prove that $b^r = \sup B(r)$, when r is rational. Hence it makes sense to define $b^x = \sup B(x)$ for every real x.
   Proof: $B(r) = \{b^t | t \in \mathbb{Q} \wedge t \leq r\}$. It's easy to see that $b^t \leq b^r$, t,$r \in \mathbb{Q}$ and $t \leq r$. (Let $r = \frac{m}{n}$, $t = \frac{p}{q}$, then $(b^t)^{qn} = (b^{\frac{p}{q}})^{qn} = b^{pn}$. From $t \leq r$, we can obtain that $\frac{p}{q} \leq \frac{m}{n}$, which is equivalent to $pn \leq qm$. Thus, $b^{pn} \leq b^{qm}$, for $b > 1$, by assumption. This means that $(b^t)^{qn} \leq b^{qm}$, taking $\frac{1}{qn}$ root gives us $b^t \leq b^{\frac{m}{n}}$, i.e. $b^t \leq b^r$.)
   **Why can we do the operation of taking $\frac{1}{qn}$ root and don't affect the direction of the inequality? The identity

   $$b^n - a^n = (b - a)(b^{n-1} + b^{n-2}a + \cdots + a^{n-1}) \tag{1}$$

   tells us that $(b^t)^{qn} - b^{qm} = ((b^t)^n - b^m)T(q)$, where $T(q) > 0$, thus if $(b^t)^{qn} \leq b^{qm}$, then $(b^t)^n \leq b^m$. By rewriting b^m as $(b^{\frac{m}{n}})^n$ and using (1) again tells us that $b^t \leq b^{\frac{m}{n}} = b^r$
   Now, we know that b^r is an upper bound of $B(r)$. Note that $b^r \in B(r)$, so b^r must be the smallest upper bound of $B(x)$, otherwise there is an upper bound α of $B(r)$ satisfies $\alpha < b^r$, which is absurd because $b^r \in B(r)$.

   (d) Prove that $b^{x+y} = b^x b^y$ for all real x and y.
   Proof: By definition, $b^x = \sup B(x)$, $b^y = \sup B(y)$ and $b^{x+y} = \sup B(x + y)$.
   We need to prove that $b^x b^y$ is the supremum of $B(x+y)$. This can be obtained from $b^{x+y} = \sup B(x + y) = \sup\{b^u | u \in \mathbb{Q} \wedge u \leq x + y\} =$

$\sup\{b^{s+t}|s \in \mathbb{Q}, t \in \mathbb{Q} \wedge s \leq x, t \leq y\} = \sup\{b^s b^t|s \in \mathbb{Q}, t \in \mathbb{Q} \wedge s \leq x, t \leq y\} = \sup\{b^s|s \in \mathbb{Q} \wedge s \leq x\} \sup\{b^t|t \in \mathbb{Q} \wedge t \leq y\} = b^x b^y$.

7. Fix $b > 1$, $y > 0$, and prove that there is a unique real x such that $b^x = y$, by completing the following outline. (This x is called the *logarithm of y to the base b*.)

   (a) For any positive integer n, $b^n - 1 \geq n(b-1)$.
   Proof: $b^n - 1 = (b-1)(1 + b + \cdots + b^{n-1}) \geq (b-1)n$, for $b > 1$.

   (b) Hence $b - 1 \geq n(b^{1/n} - 1)$.
   Proof: Directly from (a).

   (c) If $t > 1$ and $n > (b-1)/(t-1)$, then $b^{1/n} < t$.
   Proof: $b^{1/n} \leq \frac{b-1}{n} + 1 < (t-1) + 1 = t$, i.e. $b^{1/n} < t$.

   (d) If w is such that $b^w < y$, then $b^{w+(1/n)} < y$ for sufficiently large n; to see this apply part(c) with $t = y \cdot b^{-w}$.
   Proof: Sufficiently large means $n > (b-1)/(y \cdot b^{-w} - 1)$.

   (e) If $b^w > y$, then $b^{w-(1/n)} > y$ for sufficiently large n.
   Proof: $b^w = \sup B(w) = \{b^r|r \in \mathbb{Q} \wedge r \leq w\}$, if $b^w > y$, then y is not an upper bound of $B(w)$, so there exists r, $r \in \mathbb{Q} \wedge r \leq w$, $b^r > y$, which also means $b^{r-(1/n)} > y$, for sufficiently large n. Thus, $b^{w-(1/n)} = \sup\{b^s|s \in \mathbb{Q} \wedge s \leq w - (1/n)\} \geq b^{r-(1/n)} > y$.

   (f) Let A be the set of all w such that $b^w < y$, and show that $x = \sup A$ satisfies $b^x = y$.
   Proof: Suppose, if, $b^x > y$, then by (e), $b^{x-(1/n)} > y$ for some sufficiently large n(which means x is not the *least* upper bound of A), which is contradict to the fact that $x = \sup A$; On the other hand, if, $b^x < y$, then by (d), $b^{x+(1/n)} < y$, for some sufficiently large n(which means x is not an upper bound of A), which is also contradict to the fact $x = \sup A$. Thus, $b^x = y$.

   (g) Prove that this x is unique.
   Proof: It's sufficient to show that if $x_1 \neq x_2$, then $b^{x_1} \neq b^{x_2}$. This is clearly from the definition of b^x which says that $b^x = \sup B(x)$. (To see this, suppose $x_1 > x_2$, then there must exist at least one $r \in \mathbb{Q}$ that $b^r \in B(x_1)$ but $b^r \notin B(x_2)$. The case that $x_1 < x_2$ is similar.

8. Prove that no order can be defined in the complex field that turns it into an ordered field. *Hint:* -1 is a square.
   Proof: According to Definition 1.17, An *ordered field* is a *field* F which is also an *ordered set*, such that (i) $x + y < x + z$ if x, y, $z \in F$ and $y < z$; (ii)$xy > 0$ if $x \in F$, $y \in F$, $x > 0$, and $y > 0$.
   Suppose $\mathbb{C}$ is an ordered field, then $x^2 > 0$, if $x > 0$.(Here 0 means $(0,0) \in \mathbb{C}$) If $x < 0$, then $-x > 0$, so $(-x)^2 > 0$ and $x^2 = (-x)^2 > 0$. We have show that $x^2 > 0$ if $x \neq 0$. But if we take $x = (0,1)$, then

$x^2 = (-1, 0) > (0, 0)$. On the other hand, $(1, 0)^2 = (1, 0) > (0, 0)(*)$, by (i) we have $(-1, 0) + (1, 0) > (0, 0) + (1, 0)$, which gives $(0, 0) > (1, 0)$, a contradiction with (*).

9. Suppose $z = a + bi$, $w = c + di$. Define $z < w$ if $a < c$, and also if $a = c$ but $b < d$. Prove that this turns the set of all complex numbers into an ordered set. (This type of order relation is called a *dictionary order*, or *lexicographic order*, for obvious reasons.) Does this ordered set have the least-upper-bound property?
Proof: According to Definition 1.6, An *ordered set* is a set S in which an order is defined. An *order* on S is a relation, denoted by $<$, with the following two properties: (i) If $x \in S$ and $y \in S$ then one and only one of the statements $x < y$, $x = y$, $y < x$ is true. (ii) If x, y, $z \in S$, if $x < y$ and $y < z$, then $x < z$.
To prove (i), it's easily to see that $w = z$ if and only if $a = b \wedge c = d$. To prove (ii), let $x = (a, b)$, $y = (c, d)$ and $z = (e, f)$, then $x < y$ means $a < c$ or $a = c \wedge b < d$, $y < z$ means $c < e$ or $c = e \wedge d < f$. Combinations of the four conditions will give either $a < e$ or $a = e \wedge b < f$, which implies $x < z$. So $\mathbb{C}$ turns to be an *ordered set* under this order definition.
This ordered set has the *least-upper-bound* property. Given any nonempty set S of $\mathbb{C}$. Let $A = \{a|z = (a, b) \in S\}$, $B = \{b|z = (a, b) \in S \wedge a = \sup A\}$. Then we can easily see that $\sup S = (\sup A, \sup B)$.

10. Suppose $z = a+bi$, $w = u+iv$, and $a = \left(\frac{|w|+u}{2}\right)^{1/2}$, $b = \left(\frac{|w|-u}{2}\right)^{1/2}$. Prove that $z^2 = w$ if $v \geq 0$ and that $(\bar{z}^2 = w$ if $v \leq 0$. Conclude that every complex number(with one exception!) has two complex square roots.
Proof:
   (a) If $v \geq 0$, $z^2 = z{\cdot}z = (a+bi)^2 = (a^2-b^2+i(2ab) = u+i|v| = u+iv = w$.
   (b) If $v < 0$, $(\bar{z})^2 = (a - bi)^2 = (a^2 - b^2) - i(2ab) = u - i|v| = u + iv = w$.
   (c) If $w = 0$, then $u = 0 \wedge v = 0$, which implies $a = 0 \wedge b = 0$, so $z = \bar{z} = 0$ is the unique square root of w. If $w \neq 0$, then according to the above two statements, either $x = z$ or $x = \bar{z}$ is a square root of w, i.e., $x^2 = w$. On the other hand we have known that $(-x)^2 = x^2$, so $-x$ is also a square root of w, and $x = -x$ if and only if $x = 0$. Thus we have shown that every complex number w will have two complex square roots if $w \neq 0$.

11. If z is a complex number, prove that there exists an $r \geq 0$ and a complex number w with $|w| = 1$ such that $z = rw$. Are w and r always uniquely determined by z?
Proof:Suppose $z = (a, b)$, then we take $r = |z| = \sqrt{a^2 + b^2} \geq 0$ and $w = (\frac{a}{r}, \frac{b}{r})$. Obviously, $z = rw$ holds, and $|w| = \sqrt{\frac{a^2}{r^2} + \frac{b^2}{r^2}} = \sqrt{\frac{a^2+b^2}{r^2}} = 1$. From the above definitions of r and w, we conclude that w and r always

uniquely determined by z. (In fact, if we take absolute value from both sides of $z = rw$, we can obtain $|z| = r|w| = r$. So r is uniquely defined and so is w).

12. If $z_1, \cdots, z_n$ are complex, prove that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.
Proof: We can prove this by induction on n.
(i)$n = 1$, this is the trivial case;
(ii)Suppose the inequality holds when $n = k$. When $n = k+1$, $|z_1 + z_2 + \cdots + z_k + z_{k+1}| = |(z_1 + z_2 + \cdots + z_k) + z_{k+1}| \leq |z_1 + z_2 + \cdots + z_k| + |z_{k+1}| \leq (|z_1| + |z_2| + \cdots + |z_k|) + |z_{k+1}| = |z_1| + |z_2| + \cdots + |z_k| + |z_{k+1}|$, which completes our proof.

13. If x, y are complex, prove that $||x| - |y|| \leq |x - y|$.
Proof: $|x| = |(x-y) + y| \leq |x-y| + |y| \Rightarrow |x| - |y| \leq |x-y|$, similarly, $|y| = |y - x + x| \leq |y - x| + |x| = |x - y| + |x| \Rightarrow |y| - |x| \leq |x - y| \Rightarrow |x| - |y| \geq -|x - y|$. Combining these two inequalities gives us the desired result.

14. If z is a complex number such that $|z| = 1$, that is, such that $z\bar{z} = 1$, compute $|1 + z|^2 + |1 - z|^2$.
Proof:$|1 + z|^2 + |1 - z|^2 = (1 + z)(1 + \bar{z}) + (1 - z)(1 - \bar{z}) = 1 + z + \bar{z} + z\bar{z} + 1 - z - \bar{z} + z\bar{z} = 2 + 2z\bar{z} = 2 + 2 = 4$.

15. Under what conditions does equality hold in the Schwarz inequality?
Proof: If equality hold in the Schwarz inequality, we have $AB = |C|^2$, i.e., $|\sum\limits_{j=1}^{n} a_j \bar{b}_j|^2 = \sum\limits_{j=1}^{n} |a_j|^2 \sum\limits_{j=1}^{n} |b_j|^2$.
From the proof of Theorem 1.35, this is equivalent to $|Ba_j - Cb_j| = 0, \forall j$, i.e., $Ba_j = Cb_j, \forall j$, i.e., $a_j \sum\limits_{k=1}^{n} |b_k|^2 = b_j \sum\limits_{k=1}^{n} a_k \bar{b}_k, \forall j$.

16. Suppose $k \geq 3$, $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^k$, $|\mathbf{x} - \mathbf{y}| = d > 0$, and $r > 0$. Prove:
    (a) If $2r > d$, there are infinitely many $\mathbf{z} \in \mathbb{R}^k$ such that $|\mathbf{z} - \mathbf{x}| = |\mathbf{z} - \mathbf{y}| = r$.
    Proof: If $2r > d$, then $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ can form a triangle in the $\mathbb{R}^k$. The orbits of $\mathbf{z}$ forms a circle in the $\mathbb{R}^k$, and it is obviously that the number of $\mathbf{z}$ is infinite.
    (b) If $2r = d$, there is exactly one such $\mathbf{z}$.
    Proof: If $2r = d$, then clearly $\mathbf{z} = \frac{\mathbf{x}+\mathbf{y}}{2}$ is the only satisfied point, which is the middle point of the line determined by the line with ends $\mathbf{x}$ and $\mathbf{y}$.
    (c) If $2r < d$, there is no such $\mathbf{z}$.
    Proof: This can be seen from the fact that $|\mathbf{x} - \mathbf{y}| \leq |\mathbf{x} - \mathbf{z}| + |\mathbf{z} - \mathbf{y}| = |\mathbf{z} - \mathbf{x}| + |\mathbf{z} - \mathbf{y}|$, which tells us that $d \leq 2r$.

How must these statements be modified if k is 2 or 1?
If $k = 2$, then in (a) there are two satisfied points z; (b), (c) still holds.
If $k = 1$, then in (a) there is no satisfied point z; (b), (c) still holds.

17. Prove that $|\mathbf{x}+\mathbf{y}|^2 + |\mathbf{x}-\mathbf{y}|^2 = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$, if $\mathbf{x} \in \mathbb{R}^k$ and $\mathbf{y} \in \mathbb{R}^k$. Interpret this geometrically, as a statement about parallelograms.
Proof: $|\mathbf{x}+\mathbf{y}|^2+|\mathbf{x}-\mathbf{y}|^2 = (\mathbf{x}+\mathbf{y})(\bar{\mathbf{x}}+\bar{\mathbf{y}})+(\mathbf{x}-\mathbf{y})(\bar{\mathbf{x}}-\bar{\mathbf{y}}) = 2\mathbf{x}\bar{\mathbf{x}}+2\mathbf{y}\bar{\mathbf{y}} = 2|\mathbf{x}|^2 + 2|\mathbf{y}|^2$. This is to say, the sum of the square of the two diagonals is twice of the sum of the square of the two edges of a parallelogram.

18. If $k \geq 2$ and $\mathbf{x} \in \mathbb{R}^k$, prove that there exists $\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{y} \neq \mathbf{0}$ but $\mathbf{x} \cdot \mathbf{y} = 0$. Is this also true if $k = 1$?
Proof: We classify $\mathbf{x}$ into the following cases:

   (a) $\mathbf{x} = \mathbf{0}$, this case is trivial because each $\mathbf{y} \neq \mathbf{0}$ satisfies $\mathbf{x} \cdot \mathbf{y} = 0$.

   (b) Now we suppose that $\mathbf{x} \neq \mathbf{0}$, then at least one of the coordinates of $\mathbf{x}$ is not 0.

      i. If there is at least one(but not all) 0 in the coordinates of $\mathbf{x}$, then suppose $x_i = 0$, let $\mathbf{y}$ be $y_i = 1$ and $y_j = 0$, $\forall j \neq i$, we can see that $\mathbf{y} \neq 0$ but $\mathbf{x} \cdot \mathbf{y} = 0$.

      ii. If all of the coordinates of $\mathbf{x}$ are not 0, then we can clarify k according to its oddity.
      When k is even, suppose $\mathbf{x} = (x_1, ..., x_{k/2}, x_{k/2+1}, ..., x_k)$, let $\mathbf{y} = (x_k, ..., x_{k/2+1}, -x_{k/2}, ..., -x_1)$, then $\mathbf{y} \neq \mathbf{0}$ and $\mathbf{x} \cdot \mathbf{y} = 0$.
      When k is odd, suppose $\mathbf{x} = (x_1, ..., x_{(k+1)/2-1}, x_{(k+1)/2}, x_{(k+1)/2+1}$ $, ..., x_k)$,let $\mathbf{y} = (x_k, ..., x_{(k+1)/2+1}, 0, -x_{(k+1)/2-1}, ..., -x_1)$, then $\mathbf{y} \neq 0$(because $k \geq 2$) and $\mathbf{x} \cdot \mathbf{y} = 0$. This completes our proof.

19. Suppose $\mathbf{a} \in \mathbb{R}^k$, $\mathbf{b} \in \mathbb{R}^k$. Find $\mathbf{c} \in \mathbb{R}^k$ and $r > 0$ such that $|\mathbf{x}-\mathbf{a}| = 2|\mathbf{x}-\mathbf{b}|$ if and only if $|\mathbf{x}-\mathbf{c}| = r$.
The solution is $3\mathbf{c} = 4\mathbf{b} - \mathbf{a}$, $3r = 2|\mathbf{b}-\mathbf{a}|$, but I doesn't know how to obtain it...

20. With reference to the Appendix, suppose that property (III) were omitted from the definition of a cut. Keep the same definitions of order and addition. Show that the resulting ordered set has the least-upper-bound property, that addition satisfies axioms (A1) to (A4)(with a slightly different zero-element!) but that (A5) fails.
Proof: First we prove the resulting ordered set $\boldsymbol{R}$ has the *least-upper-bound property.*
Let A is be a nonempty subset of $\boldsymbol{R}$, and assume that $\beta \in \boldsymbol{R}$ is an upper bound of A. Define γ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We shall prove that $\gamma \in \boldsymbol{R}$ and that $\gamma = \sup A$.

Since A is not empty, there exists an $\alpha_0 \in A$. This α_0 is not empty. Since $\alpha_0 \subset \gamma$, γ is not empty. Next, $\gamma \subset \beta$(since $\alpha \subset \beta$ for every αin A), and therefore $\gamma \neq \mathbb{Q}$. Thus γ satisfies property(I). To prove(II), pick $p \in \gamma$. Then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$.
Thus $\gamma \in \boldsymbol{R}$.
It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$.
Suppose $\delta < \gamma$. Then there is an $s \in \gamma$ and that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in A$. Hence $\delta < \alpha$, and δ is not an upper bound of A.
This gives the desired result: $\gamma = \sup A$.
Next, we will prove that the addition satisfies axioms (A1) to (A4):
(A1)We have to show that $\alpha + \beta$ is a cut. It is clear that $\alpha + \beta$ is a nonempty subset of $\mathbb{Q}$. Take $r' \notin \alpha$, $s' \notin \beta$. Then $r' + s' > r + s$ for all choices of $r \in \alpha$, $s \in \beta$. Thus $r' + s' \notin \alpha + \beta$. It follows that $\alpha + \beta$ has property(I).
Pick $p \in \alpha + \beta$. Then $p = r + s$, with $r \in \alpha$, $s \in \beta$. If $q < p$, then $q < r + s \Rightarrow q - s < r$, so $q - s \in \alpha$. Thus $q = (q - s) + s \in \alpha + \beta$ and (II) holds.
(A2)$\alpha + \beta$ is the set of all $r + s$, with $r \in \alpha$, $s \in \beta$. By the same definition, $\beta + \alpha$ is the set of all $s + r$. Since $r + s = s + r$ for all $r \in \mathbb{Q}$, $s \in \mathbb{Q}$, we have $\alpha + \beta = \beta + \alpha$.
(A3)As above, this follows from the associative law in $\mathbb{Q}$.
(A4)We have to modify the definition of 0^* to be the set of all negative rational numbers plus the number 0. (The reason will be clear if we look back to the proof of (A4) on page 18, which use property (III) that has been removed.)
If $r \in \alpha$ and $s \in 0^*$, then $r + s \leq r$, hence $r + s \in \alpha$. Thus $\alpha + 0^* \subset \alpha$. To obtain the opposite inclusion, pick $p \in \alpha$, then $p = p + 0 \in \alpha + 0^*$. Thus $\alpha \subset \alpha + 0^*$. We conclude that $\alpha + 0^* = \alpha$.
Finally, we will show that (A5) can no longer be held.
Suppose, on the contrary, $\forall \alpha$, $\alpha \in \boldsymbol{R}$, there is a $\beta \in \boldsymbol{R}$ satisfies $\alpha + \beta = 0^*$. Let α to be the set of all negative rational numbers. Clearly α is a cut, but we cannot find another cut β satisfies $\alpha + \beta = 0^*$. (This needs a little thinking...) This is why property(III) cannot be omitted from the definition of *cut*.

2 Basic Topology

1. Prove that the empty set is a subset of every set.
   Proof: If this is not true, then $\exists A$, $\emptyset \not\subseteq A$, which means there is at least one $x \in \emptyset$ but $x \notin A$. Obviously this cannot be held since $\emptyset$ has no elements.

2. A complex number z is said to be *algebraic* if there are integers $a_0, ..., a_n$, not all zero, such that $a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n = 0$. Prove that the set of all algebraic numbers is countable.

Proof: A simple proof will be:
(i) z is a root of the n-degree polynomial $a_0x^n+a_1x^{n-1}+\cdots+a_{n-1}x+a_n=0$, and we know the fact that each n-degree polynomial has n roots in the complex plane(*);
(ii)The set of all n-degree polynomials with integral coefficients is *countable*, so is the set of all polynomials with integral coefficients.
Combined (i) and (ii), we know that the set of all algebraic numbers is *countable*.
But if we don't know the fact (*), how to prove this? (I don't know at present...)

3. Prove that there exist real numbers which are not algebraic.
Proof: Suppose this is not the fact. Let A denote the set of all *algebraic* numbers, then $\mathbb{R} \subseteq A$. Since $\mathbb{R}$ is *uncountable*, so is A, which is contradict to the result of 2.

4. Is the set of all irrational real numbers countable?
Proof: The answer is obviously *No*. To see this, let $\mathbb{U}$ denote the set of all irrational real numbers. If $\mathbb{U}$ is *countable*, then $\mathbb{R} = \mathbb{Q} \cup \mathbb{U}$ is *countable*, which is contradict to the fact that $\mathbb{R}$ is *uncountable*.

5. Construct a bounded set of real numbers with exactly three limit points.
Proof: Let $A = \{\frac{1}{n} | n \in \mathbb{I}^+\}$, $B = \{2 + \frac{1}{n} | n \in \mathbb{I}^+\}$, $C = \{4 + \frac{1}{n} | n \in \mathbb{I}^+\}$ and $S = A \cup B \cup C$, then S is bounded, since $|x| < 6$, $\forall x \in S$ and S has exactly 3 limit points, namely, 0, 2, and 4.

6. Let E' be the set of all limit points of a set E. Prove that E' is closed. Prove that E and $\bar{E}$ have the same limit points. (Recall that $\bar{E} = E \cup E'$.) Do E and E' always have the same limit points?
Proof:
   (a) Let p be a limit point of E', then for every $r > 0$, there is a $q \in E'$ and $q \in N_r(p)$. Since $N_r(p)$ is *open*, there is a neighborhood N_q of q, $N_q \subset N_r(p)$ and since q is a limit point of E, there is a $s \in N_q$, $s \neq q$ and $s \in E$. Combining these facts, we get that for every $r > 0$, there is an $s \in N_r(p)$, $s \in E$, which is equivalent to say that p is a limit point of E, thus $p \in E'$ and E' is closed.
   (b) $\Rightarrow$: Suppose p is a limit point of E, since $E \subseteq \bar{E}$, p is also a limit point of $\bar{E}$.
   $\Leftarrow$: On the other hand, let p be a limit point of $\bar{E}$, then $\forall r > 0$, there is a $q \in \bar{E} \wedge q \in N_r(p)$. If $q \notin E$, then $q \in E'$ and q is a limit point of E. Since $N_r(p)$ is *open*, there is a neighborhood N_q of q, $N_q \subset N_r(p)$. Due to the fact that q is a limit point of E, there is an $s \in N_q$, $s \neq q \wedge s \in E$. This is to say, $\forall r > 0$, there is an $t \in E \wedge t \in N_r(p)$ (Here, t will either be q or s in the above statements). Thus p is a limit point of E.

(c) Obviously this is not the case. An easy example will be: $E = \{(x,y)|x^2+y^2 < 1, x \in \mathbb{R}, y \in \mathbb{R}\}$, thus $E' = \{(x,y)|x^2+y^2 = 1, x \in \mathbb{R}, y \in \mathbb{R}\}$ and $\bar{E} = \{(x,y)|x^2+y^2 \leq 1, x \in \mathbb{R}, y \in \mathbb{R}\}$. Thus the set of all limit points of E is $\bar{E}$ and the set of all limit points of E' is E' itself. Clearly $E' \subset \bar{E}$.

7. Let $A_1, A_2, A_3, ...$ be subsets of a metric space.

(a) If $B_n = \cup_{i=1}^n A_i$, prove that $\bar{B}_n = \cup_{i=1}^n \bar{A}_n$, for $n = 1, 2, 3, ...$

Proof:

$\Rightarrow$: Suppose $p \in \bar{B}_n$, then $p \in B_n$ or $p \in B_n'$. If $p \in B_n$, then $p \in A_i$, for some $1 \leq i \leq n$, thus $p \in \bar{A}_i$ and $p \in \cup_{i=1}^n \bar{A}_i$. If $p \in B_n'$, then p is a limit point of B_n and $\forall r > 0$, there is a $q \in N_r(p) \wedge q \in B_n$, i.e. $q \in \cup_{i=1}^n A_i$, since $A_i \in \bar{A}_i$, thus $q \in \cup_{i=1}^n \bar{A}_i$ and p is a limit point of $\cup_{i=1}^n \bar{A}_i$. We have known $\bar{A}_i$ is *closed*, so is $\cup_{i=1}^n \bar{A}_i$ since n is finite (Recall that a finite union of closed sets is also closed). Thus $p \in \cup_{i=1}^n \bar{A}_i$. This gives $\bar{B}_n \subseteq \cup_{i=1}^n \bar{A}_i$.

$\Leftarrow$: Suppose $p \in \cup_{i=1}^n \bar{A}_i$, then $p \in \bar{A}_i$ for some $1 \leq i \leq n$, i.e., $p \in A_i \cup A_i'$. If $p \in A_i$, then $p \in B_n$, thus $p \in \bar{B}_n$; On the other hand, if $p \in A_i'$, then p is a limit point of A_i, since $A_i \subseteq B_n$, p is also a limit point of B_n. Thus $p \in \bar{B}_n$. This gives $\cup_{i=1}^n \bar{A}_i \subseteq \bar{B}_n$.

(b) If $B = \cup_{i=1}^\infty A_i$, prove that $\bar{B} \supseteq \cup_{i=1}^\infty \bar{A}_i$.

Proof: Suppose $p \in \cup_{i=1}^\infty \bar{A}_i$, then $p \in \bar{A}_i$ for some $i \geq 1$, i.e., $p \in A_i \cup A_i'$. If $p \in A_i$, then $p \in B$; If $p \in A_i'$, p is a limit point of A_i. Since $A_i \subseteq B$, p is also a limit point of B, thus $p \in \bar{B}$. This gives $\cup_{i=1}^\infty \bar{A}_i \subseteq \bar{B}$.

Show, by an example, that this inclusion can be proper.

Let $A_i = (\frac{1}{i}, 2]$, then $\bar{A}_i = [\frac{1}{i}, 2]$, $B = \cup_{i=1}^n \bar{A}_i = (0, 2]$ and $\bar{B} = [0, 2]$. But $0 \notin \bar{A}_i$, $\forall i \geq 1$, thus $0 \notin \cup_{i=1}^\infty \bar{A}_i$ and $\cup_{i=1}^\infty \bar{A}_i \subset \bar{B}$.

8. Is every point of every open set $E \subset \mathbb{R}^2$ a limit point of E? Answer the same question for closed sets in $\mathbb{R}^2$.

Solution:

(i) The answer for *open* set is *Yes.* To see this, note that if $p \in E$ and E is open, $\exists r > 0$, $N_r(p) \subset E$. Let $N_{r'}(p)$ be an arbitrary neighborhood of p. If $r' \geq r$, then $N_r(p) \subseteq N_{r'}(p)$; If $r' < r$, we conclude that $N_{r'}(p)$ contains infinite number of points in E. Suppose on the contrary, this is not true. Then $\exists r' > 0$, $r' < r$, $N_{r'}(p)$ contains only finitely many points of E. Let these points be denoted as $p_1, p_2, ..., p_n$, and let $\delta = \min\{d(p, p_i)|1 \leq i \leq n\}$, then $\delta > 0$ and $N_\delta(p)$ contains no points of E other than p, thus $N_\delta(p) \nsubseteq E$. But $\delta <= r' < r \Rightarrow N_\delta(p) \subseteq N_{r'}(p) \subset N_r(p) \subseteq E$, which is a contradiction. We have show that every neighborhood of p contains infinitely many number of E and thus p is a limit point of E. (**Notes**: Since E is an open subset of $\mathbb{R}^2$, E must contain infinitely many points, according to Example 2.21. Thus, we only need to prove that every

neighborhood of a $p \in E$ contains infinitely many points of E. This is easy because every neighborhood of p is also an open subset of $\mathbb{R}^2$. This will be an much shorter proof instead of the above given one.)
(ii) The answer for *closed* set is obviously *No*. To see this, consider the set $E = \{(\frac{1}{n}, 0) | n \in \mathbb{I}^+\} \cup \{(0,0)\}$. Clearly E is closed, but only 0 is a limit point of E.

9. Let E° denote the set of all interior points of a set E. E° is called the *interior* of E.

   (a) Prove that E° is always open.
   Proof: Let $p \in E^\circ$, then p is an interior point of E, thus there is a $r > 0$ such that $N_r(p) \subseteq E$. Furthermore, let $q \in N_r(p)$, then since $N_r(p)$ is open, there is an neighborhood N_q of q, s.t., $N_q \subseteq N_r(p) \subseteq E$. Thus, q is an interior point of E and $q \in E^\circ$. This means $N_r(p) \subseteq E^\circ$ and therefore E° is open.

   (b) Prove that E is open if and only if $E^\circ = E$.
   Proof:
   $\Rightarrow$: Suppose that E is open. Let $p \in E^\circ$, then p is an interior point of E, since E is open, this gives $p \in E$ and therefore $E^\circ \subseteq E$; On the other hand, let $p \in E$, then p is an interior point of E since E is open, thus $p \in E^\circ$ and therefore $E \subseteq E^\circ$.
   $\Leftarrow$: Suppose $E^\circ = E$, let p be any point of E, then p is a point of E°. Thus p is an interior point of E and E is open.

   (c) If $G \subseteq E$ and G is open, prove that $G \subseteq E^\circ$.
   Proof: Let $p \in G$, then $p \in E$ since $G \subseteq E$. Furthermore, there is a neighborhood $N_G(p)$ of p, s.t., $N_G(p) \subseteq G$ since G is open. Let $N_E(p) = N_G(p) \cap E$, then $N_E(p)$ is also a neighborhood of p because $p \in N_E(p)$, and $N_E(p) \subseteq E$. Therefore, p is an interior point of E. Thus $p \in E^\circ$ and $G \subseteq E^\circ$.

   (d) Prove that the complement of E° is the closure of the complement of E.
   Proof: We need to prove that $(E^\circ)^c = \bar{E^c}$.
   $\Rightarrow$: Let $p \in (E^\circ)^c$, then $p \notin E^\circ$. Thus p is not an interior point of E, which is to say, $\forall r > 0$, there is a $q \in N_r(p) \wedge q \notin E$, i.e., $q \in E^c$. Therefore, p is a limit point of E^c and $p \in \bar{E^c}$. So $(E^\circ)^c \subseteq \bar{E^c}$.
   $\Leftarrow$: Let $p \in \bar{E^c}$, then $p \in E^c$ or $p \in (E^c)'$. If $p \in E^c$, $p \notin E$ and $p \notin E^\circ$, thus $p \in (E^\circ)^c$. If $p \in (E^c)'$, then p is a limit point of E^c. $\forall r > 0$, $\exists q \in N_r(p) \wedge q \in E^c$, i.e. $q \notin E$ and thus $q \notin E^\circ$. Therefore, $q \in (E^\circ)^c$ and p is a limit point of $(E^\circ)^c$. Since E° is open, $(E^\circ)^c$ is closed and $p \in (E^\circ)^c$. So $\bar{E^c} \subseteq (E^\circ)^c$.

   (e) Do E and $\bar{E}$ always have the same interiors?
   Solution: The answer will be *No*. To see this, let $E = \mathbb{Q}$, then $\bar{E} = \mathbb{R}$. Obviously, $E^\circ = \emptyset$, but $\bar{E}^\circ = \mathbb{R}$, if we let the whole space be $\mathbb{R}$.

(f) Do E and E° always have the same closures?
Solution: The answer is also No. To see this, let $E = \{x|x \in [0,1] \wedge x \in \mathbb{Q}\} \cup [2,3]$, then $\bar{E} = [0,1] \cup [2,3]$. But $E^\circ = (2,3)$ and thus $\bar{E^\circ} = [2,3]$. (The whole space is supposed to be $\mathbb{R}$.)

10. Let X be an infinite set. For $p \in X$ and $q \in X$, define

$$d(p,q) = \begin{cases} 1 & (\text{if } p \neq q) \\ 0 & (\text{if } p = q). \end{cases}$$

Prove that this is a metric. Which subsets of the resulting metric space are open? Which are closed? Which are compact?
Solution: First, we prove that d is a metric: (i) $d(p,q) > 0$, if $p \neq q$; $d(p,p) = 0$. (ii) $d(p,q) = d(q,p)$; (iii) $d(p,q) = 1$, $d(p,r) + d(r,q) = 2$, and thus $d(p,q) < d(p,r) + d(r,q)$. Therefore d is a metric.

(a) Every nonempty subset of X is open. To see this, let $S \subseteq X$ and S is not empty. Suppose $p \in S$, and some $r < 1$, then $N_r(p)$ contains the only point p. Clearly $N_r(p) \subseteq S$ and thus S is open. In fact, the empty set is trivially open, so every subset of X is open.

(b) Every nonempty subset of X is closed. To see this, let $S \subseteq X$, then S^c is also a subset of X and due to (a), S^c is open. Thus, S is closed. In fact, the empty set is trivially closed, so every subset of X is closed.

(c) Clearly, every finite subset of X is compact. But any infinite subset of X is not compact. To see this, let $S \subseteq X$ and S is infinite. Suppose $0 < r < 1$, then $\bigcup_{p \in S} N_r(p)$ is an open cover of S. Clearly there can be no finite subcover which can cover S, since each $N_r(p)$ contains only one point of S, namely, p. If there is one, then S will be finite, which is an contradiction.

11. For $x \in \mathbb{R}^1$ and $y \in \mathbb{R}^1$,define

(a) $d_1(x,y) = (x-y)^2$;
Solution: *Yes.*

(b) $d_2(x,y) = \sqrt{|x-y|}$;
Solution: *Yes.* The first two conditions are trivial. We next prove the triangular inequality: Let $x,y,z \in \mathbb{R}^1$, then $\sqrt{|x-y|} + \sqrt{|y-z|} - \sqrt{|x-z|} \geq \sqrt{|x-y|} + \sqrt{|y-z|} - \sqrt{|x-y| + |y-z|}$. Suppose $A = \sqrt{|x-y|} + \sqrt{|y-z|}$ and $B = \sqrt{|x-y| + |y-z|}$, then $A^2 - B^2 = 2\sqrt{|x-y}\sqrt{|y-z|} \geq 0$, i.e., $A^2 \geq B^2$, i.e., $A \geq B$, since $A, B \geq 0$.

(c) $d_3(x,y) = |x^2 - y^2|$;
Solution: *No.* e.g. $d(1,-1) = 0$.

(d) $d_4(x,y) = |x-2y|$;
Solution: *No.* e.g. $d(1,\frac{1}{2}) = 0$.

(e) $d_5(x,y) = \frac{|x-y|}{1+|x-y|}$.
Solution: *Yes.* The first two conditions are trivial again. Let's see the triangular inequality: Suppose $x,y,z \in \mathbb{R}^1$, and $a = |x-y|$, $b = |y-z|$, $c = |x-z|$, then $a,b,c \geq 0$. Thus $d(x,y) + d(y,z) - d(x,z) = \frac{a}{1+a} + \frac{b}{1+b} - \frac{c}{1+c} = \frac{a(1+b)(1+c)+b(1+c)(1+a)-c(1+a)(1+b)}{(1+a)(1+b)(1+c)} = \frac{a+b-c+2ab+abc}{(1+a)(1+b)(1+c)}$, since $a+b \geq c$, $d(x,y)+d(y,z)-d(x,z) \geq 0$, i.e., $d(x,y)+d(y,z) \geq d(x,z)$.

Determine, for each of these, whether it is a metric or not.

12. Let $K \subseteq \mathbb{R}^1$ consist of 0 and the numbers $1/n$, for $n = 1,2,3,....$ Prove that K is compact directly from the definition (without using the Heine-Borel theorem).
Proof: Suppose $\{G_\alpha\}$ is any open cover of K. Obviously, 0 is a limit point of K. Since $0 \in K$, $0 \in G_{\alpha_0}$ for some α_0. Furthermore, since G_{α_0} is open, there is a $r > 0$, $N_r(0) \subseteq G_{\alpha_0}$. Let $\frac{1}{N} < r$, $N \in \mathbb{N}$, then $N > \frac{1}{r}$ and when $n > N$, $n \in \mathbb{N}$, $\frac{1}{n} < \frac{1}{N} < r$, which implies $\frac{1}{n} \in N_r(0)$ and thus $\frac{1}{n} \in G_{\alpha_0}$. Let G_{α_i} denote the open set which covers the number $\frac{1}{i}$, where $1 \leq i \leq N$. Then $K \subseteq G_{\alpha_0} \cup (\bigcup\limits_{i=1}^{N} G_{\alpha_i})$ and therefore K is compact.

13. Construct a compact set of real numbers whose limit points form a countable set.
Solution: Let $A_k = \{\frac{1}{k+1}[1+\frac{1}{nk}] | n \geq 1 \wedge n \in \mathbb{N}\}$, $k \geq 1 \wedge k \in \mathbb{N}$, i.e., $A_1 = \{\frac{1}{2}[1+\frac{1}{n}], n \geq 1 \wedge n \in \mathbb{N}\}$, $A_2 = \{\frac{1}{3}[1+\frac{1}{2n}], n \geq 1 \wedge n \in \mathbb{N}\}$, and so on. Then $A_k \subseteq (\frac{1}{k+1}, \frac{1}{k}]$. Let $A = \{0\} \cup (\bigcup\limits_{k=1}^{\infty} A_k)$, then A is compact since A is closed and bounded (with Heine-Borel theorem). The set L=$\{0\} \cup \{\frac{1}{n} | n \geq 1 \wedge n \in \mathbb{N}\}$ contains all of the limit points of A and L is obviously countable.

14. Give an example of an open cover of the segment $(0,1)$ which has no finite subcover.
Solution: Let $G_n = (\frac{1}{n+1}, \frac{1}{n})$, where $n \geq 1 \wedge n \in \mathbb{N}$; let $P_n = (\frac{1}{n+1} - \frac{1}{2}(\frac{1}{n+1} - \frac{1}{n+2}), \frac{1}{n+1} + \frac{1}{2}(\frac{1}{n} - \frac{1}{n+1}))$, $n \geq 1 \wedge n \in \mathbb{N}$, i.e. $P_n = (\frac{2n+3}{2(n+1)(n+2)}, \frac{2n+1}{2n(n+1)})$. Clearly, $C = (\bigcup\limits_{n=1}^{\infty} G_n) \cup (\bigcup\limits_{n=1}^{\infty} P_n)$ is an open cover of $(0,1)$ but we can't find any finite subcover of C which also covers $(0,1)$.
Notes:
(i) Since $(0,1) \subset [0,1]$, then every open cover of $[0,1]$ is also an open cover of $(0,1)$, but the converse is not true. Thus the open cover that we need to seek must not be an open cover of $[0,1]$.

(ii) The only difference between $(0,1)$ and $[0,1]$ is the two end points 0 and 1, which makes $(0,1)$ is open, but $[0,1]$ is closed (thus compact). The reason that $[0,1]$ is compact but $(0,1)$ is not is clearly due to these two end points, since the part that cannot be covered by some finite open sets of a given open cover will be at the neighborhood of the end point. If the end point is included, this part can be covered (actually with one open set instead of the infinite open sets needed if the end point is not included), so is the whole interval.

15. Show that Theorem 2.36 and its Corollary become false(in $\mathbb{R}^1$, for example) if the word "compact" is replaced by "closed" or by "bounded".
Solution: Theorem 2.36 says that, If $\{K_\alpha\}$ is a collection of compact subsets of a metric space X such that the intersection of every *finite* subcollection of $\{K_\alpha\}$ is nonempty, then $\bigcap K_\alpha$ is nonempty.
(i) If the word "compact" is replaced by "closed", we have the following counterexample: Let $K_n = [n, +\infty)$, $n \in \mathbb{N}$, then K_n is closed, $K_n \supset K_{n+1}$ and thus the intersection of every finite subcollection of $\{K_n\}$ is nonempty. But $\bigcap K_n$ is empty. To see this, let $x > 0$ be any positive real number, then there is an integer n such that $n \leq x < n+1$, according to the *archimedean property* of $\mathbb{R}$. Thus, $x \in K_n$ but $x \notin K_{n+1}$, and $x \notin \bigcap K_n$. Therefore, $\bigcap K_n = \emptyset$.
(ii) If the word "compact" is replaced by "bounded", we also have the following counterexample: Let $K_n = (0, \frac{1}{n}]$, $n \geq 1 \wedge n \in \mathbb{N}$, then K_n is obviously bounded, $K_n \supset K_{n+1}$, and thus the intersection of every finite subcollection of $\{K_n\}$ is nonempty. But $\bigcap K_n$ is empty. To see this, let $x \in (0,1]$, then there is a positive integer N such that $Nx > 1$ according to the *archimedean property*, which implies $x > \frac{1}{N}$. Thus $x \notin (0, \frac{1}{N}]$, i.e., $x \notin K_N$, and $x \notin \bigcap K_n$. Therefore, $\bigcap K_n$ is empty.
Notes: Here we see that the *compactness* is essential for Theorem 2.36.

16. Regard $\mathbb{Q}$, the set of all rational numbers, as a metric space, with $d(p,q) = |p-q|$. Let E be the set of all $p \in \mathbb{Q}$ such that $2 < p^2 < 3$. Show that E is closed and bounded in $\mathbb{Q}$, but that E is not compact. Is E open in Q?
Proof: The fact that E is bounded is clear.
Next we show that E is closed. Suppose p is a limit point of E and $p \in \mathbb{Q}$ (without loss of generality, we assume that p is positive, the case that p is negative will be similar), then we need to show that $p \in E$. $\forall \epsilon > 0$, there is a $q \in E \wedge q > 0$, $d(p,q) = |p-q| < \epsilon$, since p is a limit point of E. This gives that $q - \epsilon < p < q + \epsilon$, i.e., $p + \epsilon > q$, i.e., $(p+\epsilon)^2 - 2 > q^2 - 2 > 0$, i.e., $p^2 + 2p\epsilon + \epsilon^2 > 2$. Due to the arbitrariness of ϵ, we conclude that $p^2 \geq 2$. Since $p \in \mathbb{Q}$, $p^2 \neq 2$, thus $p^2 > 2$. Similarly, we have $p - \epsilon < q$, i.e., $(p-\epsilon)^2 - 3 < q^2 - 3 < 0$, and thus $p^2 <= 3$. Since $p \in Q$, $p^2 \neq 3$, therefore $p^2 < 3$. Now we have proved that $2 < p^2 < 3$, thus $p \in E$ and E is closed.
Finally, we need to show that E is not compact in $\mathbb{Q}$. Suppose, on the contrary, E is compact in $\mathbb{Q}$, then according to Theorem 2.33, E is compact

in $\mathbb{R}$, which is obviously wrong since E is even not closed in $\mathbb{R}$.
It's easy to see that E is open in $\mathbb{Q}$.
Notes: Here we see an example of a set which is both closed and bounded but not compact. Now we are convinced why the premise of the Heine-Borel Theorem should be "in $\mathbb{R}^k$".

17. Let E be the set of all $x \in [0,1]$ whose decimal expansion contains only the digits 4 and 7. Is E countable? Is E dense in $[0,1]$? Is E compact? Is E perfect?
Solution:
    (a) E is uncountable. If, on the contrary, E is countable, let the elements of E be arranged as $x_1, x_2, \ldots$. We denote x_i as follows:
    $$x_1 = 0.x_{11}x_{12}...$$
    $$x_2 = 0.x_{21}x_{22}...$$
    $$......$$
    where $x_{ij} = 4$ or $x_{ij} = 7$, $i, j \geq 1 \wedge i, j \in \mathbb{N}$. Let$s = s_1 s_2...$ be defined as
    $$s_i = \begin{cases} 4 & \text{if } x_{ii} = 7 \\ 7 & \text{if } x_{ii} = 4 \end{cases}$$
    Then $s \in [0,1]$ and $s \in E$, but s has at least one digit different from each x_i, which gives $s \notin E$, a contradiction. Therefore, E must be uncountable.
    (b) E is not dense in [0,1]. This is easy to be seen since if $x \in E$, $x \geq 0.\dot{4}$, and every $y \in [0, 0.3]$ cannot be a point or a limit point of E.
    (c) E is closed. To see this, let p be any limit point of E, then we can conclude that $p \in E$. Thus, we need to show that $p \in [0,1]$ and p's decimal expansion contains only the digits 4 and 7. The fact that $p \in [0,1]$ is quite trivial. So next we will prove that p's decimal expansion contains only the digits 4 and 7. Suppose, on the contrary, this is not true. Let $p = 0.p_1 p_2 ....$, then there is a smallest $n \in \mathbb{I}^+$, such that $p_n \neq 4 \wedge p_n \neq 7$. Let $\delta = \min\{|q-p| | q \in E\}$, then it's clearly that $\delta > 0$ since $p \notin E$ and $|q - p|$ is a metric. Pick an r such that $0 < r < \delta$, then $N_r(p)$ contains no points $q \in E$ and thus p cannot be a limit point of E, a contradiction.
    (d) E is not perfect. e.g. $p = 0.44 \in E$, but its neighborhood $N_{0.001}(0.44)$ contains no points of E other than p. Thus p is not a limit point of E.

18. Is there a nonempty perfect set in $\mathbb{R}^1$ which contains no rational number?
Solution: *Yes.* We will construct a nonempty *perfect* set contained in $\mathbb{R}$ that contains no rational number.

We will begin with a *closed interval*, and then, imitating the construction of *Cantor set*, we will inductively delete each rational number in it together with an *open interval*. We will do it in such a way that the end points of the open intervals will never be deleted afterwards.

Let $E_0 = [a_0, b_0]$ for some *irrational* numbers a_0 and b_0. Let $\{q_1, q_2, q_3, ...\}$ be an enumeration of the rational numbers in $[a_0, b_0]$. For each q_i, we will define an open interval (a_i, b_i) and delete it.

Let a_1 and b_1 be two irrational numbers such that $a_0 < a_1 < q_1 < b_1 < b_0$. Define $E_1 = E_0 \backslash (a_1, b_1)$. Having defined $E_1, E_2, ..., E_n$, $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_n$, let's define a_{n+1} and b_{n+1}:

If $q_{n+1} \in \bigcup\limits_{k=1}^{n} (a_k, b_k)$, then there exists an $i \leq n$ such that $q_{n+1} \in (a_i, b_i)$. Let $a_{n+1} = a_i$ and $b_{n+1} = b_i$.

Otherwise let a_{n+1} and b_{n+1} be two irrational numbers such that $a_0 < a_{n+1} < q_{n+1} < b_{n+1} < b_0$, and which satisfy:

$$q_{n+1} - a_{n+1} < \min_{i=1,2,...,n} \{|q_{n+1} - b_i|\}$$

and

$$b_{n+1} - q_{n+1} < \min_{i=1,2,...,n} \{|a_i - q_{n+1}|\}$$

. Now define $E_{n+1} = E_n \backslash (a_{n+1}, b_{n+1})$. Note that by our choice of a_{n+1} and b_{n+1} *any of the previous end points are not removed from* E_n.

Let $E = \bigcap\limits_{n=1}^{\infty} E_n$. E is clearly nonempty, does not contain any rational number, and also it is compact, being an intersection of compact sets.

Now let us see that E does not have any isolated points. Let $x \in E$, and $\epsilon > 0$ be given. Choose a rational number q_k such that $x < q_k < x + \epsilon$. Then $q_k \in (a_k, b_k)$ and since $x \in E$ we must have $x < a_k$. which means $a_k \in (x, x + \epsilon)$, since $a_k < q_k$. But we know that $a_k \in E$, so we have shown that any point of E is a limit point, hence E is perfect.

Notes: There are two key points here:

(i) The essential idea is not only deleting the rational number but also a segment which contains it. This guarantees the result set will be closed, which is needed by the requirement of perfect sets. Furthermore, the set hence obtained will be compact since it is both closed and bounded in $\mathbb{R}^1$, and E is guaranteed to be *nonempty*, according to Theorem 2.36;

(ii) The two conditions for the choice of a_{n+1} and b_{n+1} is also important, which commits that we will not delete any previously chosen a_i and b_i by removing the segment (a_{n+1}, b_{n+1}). Furthermore, these two conditions are not so trivial and need a deep thought.

19. (a) If A and B are disjoint closed sets in some metric space X, prove that they are separated.

Proof: Since A and B are disjoint, $A \cap B = \emptyset$. On the other hand, A and B are closed, hence $A = \bar{A}$ and $B = \bar{B}$. Thus $A \cap \bar{B} = A \cap B = \emptyset$ and $\bar{A} \cap B = A \cap B = \emptyset$, i.e., A and B are separated.

(b) Prove the same for disjoint open sets.
Proof: If $A \cap \bar{B} \neq \emptyset$, then $\exists p \in A \cap \bar{B}$, i.e., $p \in A$ and $p \in \bar{B}$. Since $A \cap B = \emptyset$, $p \notin B$, thus $p \in B'$ and p is a limit point of B. Since $p \in A$ and A is open, there is a neighborhood N_p of p, s.t., $N_p \subseteq A$. Hence there is a $q \in N_p$ such that $q \neq p \wedge q \in B$ because p is a limit point of B. Therefore $q \in A$ and $q \in A \cap B$, which is contradict to the fact that $A \cap B$ is empty. We conclude that $A \cap \bar{B} = \emptyset$.
In just the same way, we can prove that $\bar{A} \cap B = \emptyset$. Hence A and B are separated.

(c) Fix $p \in X$, $\delta > 0$, define A to be the set of all $q \in X$ for which $d(p, q) < \delta$, define B similarly, with $>$ in place of $<$. Prove that A and B are separated.
Proof: Clearly A is open since A is a neighborhood of p. $B^c = \{q | q \in X \wedge d(p, q) \leq \delta\}$, we prove that B^c is closed. Let w be a limit point of B^c, then $\forall r > 0$, there is a $q \in B^c$ such that $d(w, q) < r$. Thus $d(p, w) \leq d(p, q) + d(q, w) < \delta + r$. Due to the arbitrariness of r, we conclude that $d(p, w) \leq \delta$ and thus $w \in B^c$. Hence B^c is closed and B is open. On the other hand, A and B are obviously disjoint. Therefore, A and B are separated according to the result of (b).

(d) Prove that every connected metric space with at least two points is uncountable.
Proof: Let X be a connected metric space with at least two points. Suppose, on the contrary, X is countable. Fix $p \in X$, Let $D = \{d(p, q) | q \in X \wedge q \neq p\}$, then D is not empty since X contains at least two points. Furthermore, D is at most countable and $D \subset (0, +\infty)$ since the latter is uncountable. ($[0, +\infty)$ is perfect and hence is uncountable according to Theorem 2.43. Thus $(0, +\infty)$ is clearly uncountable.) Hence there is a $\delta > 0$, $\delta \notin D$. Define $A = \{q | q \in X, q \neq p, d(p, q) < \delta\}$ and $B = \{q | q \in X, q \neq p, d(p, q) > \delta\}$, then $X = A \cup B$. Since A and B are separated according to (c), X is not connected, which is a contradiction.

20. Are closures and interiors of connected sets always connected? (Look at subsets of $\mathbb{R}^2$.)
Solution:

(a) Closures of connected sets always connected.
Proof: Let E be a nonempty connected set. Suppose, on the contrary $\bar{E}$ is not connected, then $\bar{E} = A \cup B$, $A, B \neq \emptyset$, $A \cap \bar{B} = \emptyset$, and $\bar{A} \cap B = \emptyset$. Since $E \subseteq \bar{E}$, $E \subseteq A \cup B$. Define $A_E = E \cap A$ and $B_E = E \cap B$, then $E = A_E \cup B_E$ (since $E = E \cap (A \cup B) = (E \cap A) \cup (E \cap B) = A_E \cup B_E$).
Next, we need to show that A_E and B_E are nonempty and separated. First if $A_E = \emptyset$, then $E = B_E = E \cap B$ and $E \subseteq B$. Thus $\bar{E} \subseteq \bar{B}$, and $A \cap \bar{E} \subseteq A \cap \bar{B}$. Since $A \cap \bar{B} = \emptyset$, $A \cap \bar{E} = \emptyset$, i.e., $A \cap (A \cup B) = \emptyset$,

i.e., $(A \cap A) \cup (A \cap B) = \emptyset$, i.e., $A \cup (A \cap B) = \emptyset$, i.e., $A = \emptyset$, which is contradict to our assumption. It's almost the same to show that B_E is nonempty. Next, let we prove that A_E and B_E are separated. Since $A_E = E \cap A$, $A_E \subseteq A$, then $\bar{A_E} \subseteq \bar{A}$, thus $\bar{A_E} \cap B_E \subseteq \bar{A} \cap B_E$. On the other hand, $B_E \subseteq B$, hence $\bar{A} \cap B_E \subseteq \bar{A} \cap B$, which implies $\bar{A_E} \cap B_E \subseteq \bar{A} \cap B$. But $\bar{A} \cap B = \emptyset$, thus $\bar{A_E} \cap B_E = \emptyset$. Similarly, we can prove that $A_E \cap \bar{B_E} = \emptyset$. Therefore, we conclude that E is separable, which is a contradiction to our assumption that E is connected.

Notes: The converse cannot be true.

(b) Interiors of connected sets are *not* always connected.
A counterexample is: take two closed disks in $\mathbb{R}^2$ that intersect in exactly one (boundary) point. e.g. $A = \{(x, y)|x^2 + y^2 \leq 1, (x, y) \in \mathbb{R}^2\}$, $B = \{(x, y)|(x-1)^2 + y^2 \leq 1, (x, y) \in \mathbb{R}^2\}$. Let $E = A \cup B$, then E is connected. But E° is two disjoint open disks, which is disconnected.
Notes: In $\mathbb{R}$, it is true: every connected set is some interval(closed, half closed etc) according to Theorem 2.47 and the interior is always an interval again(open interval, namely segment), so connected. This is why we need to *Look at subsets of* $\mathbb{R}^2$!

21. Let A and B be separated subsets of some $\mathbb{R}^k$, suppose $\mathbf{a} \in A$, $\mathbf{b} \in B$, and define

$$\mathbf{p}(t) = (1-t)\mathbf{a} + t\mathbf{b}$$

for $t \in \mathbb{R}^1$. Put $A_0 = \mathbf{p}^{-1}(A)$, $B_0 = \mathbf{p}^{-1}(B)$. [Thus $t \in A_0$ if and only if $\mathbf{p}(t) \in A$.]

(a) Prove that A_0 and B_0 are separated subsets of $\mathbb{R}^1$.
Proof: We need to show that $A_0 \cap \bar{B_0} = \emptyset$ and $B_0 \cap \bar{A_0} = \emptyset$. Suppose, $A_0 \cap \bar{B_0} \neq \emptyset$, then there is an $x \in A_0 \cap \bar{B_0}$, i.e., $x \in A_0$ and $x \in \bar{B_0}$. If $x \in B_0$, then $\mathbf{p}(x) \in A$ and $\mathbf{p}(x) \in B$, which means $\mathbf{p}(x) \in A \cap B$. Thus $A \cap B \neq \emptyset$, which is contradict to the fact that A and B are separated. If $x \in B_0'$, then x is a limit point of B_0. We show that $\mathbf{p}(x)$ is a limit point of B. To see this, let $r > 0$ be any given positive real number, so $N_r(\mathbf{p}(x))$ is a neighborhood of $\mathbf{p}(x)$ in $\mathbb{R}^k$. Denote $\delta = |\mathbf{b} - \mathbf{a}|$, $\delta > 0$ since $\mathbf{a} \neq \mathbf{b}$, and let $\varepsilon = \frac{r}{\delta} > 0$. Because x is a limit point of B_0, there is a $y \in B_0$ and $|y - x| < \varepsilon$. $y \in B_0$ thus $\mathbf{p}(y) \in B$, and $|\mathbf{p}(y) - \mathbf{p}(x)| = |((1-y)\mathbf{a} + y\mathbf{b}) - ((1-x)\mathbf{a} + x\mathbf{b})| = |(x-y)\mathbf{a} + (y-x)\mathbf{b}| = |(y-x)(\mathbf{b} - \mathbf{a})| = |y-x||\mathbf{b} - \mathbf{a}| < \varepsilon\delta = r$, which means $\mathbf{p}(y) \in N_r(\mathbf{p}(x))$. Therefore, $\mathbf{p}(x)$ is a limit point of B and $\mathbf{p}(x) \in \bar{B}$. Thus, $\mathbf{p}(x) \in A \cap \bar{B}$ and $A \cap \bar{B}$ is nonempty, which is again contradict to the fact that A and B are separated. Hence, $A_0 \cap \bar{B_0}$ must be empty. The proof of $\bar{A_0} \cap B_0 = \emptyset$ is similar and we omit it here.

(b) Prove that there exists $t_0 \in (0,1)$ such that $\mathbf{p}(t_0) \notin A \cup B$.
Proof: Let $C = A_0 \cap (0,1)$ and $D = B_0 \cap (0,1)$, then $C \subseteq A_0$ and $D \subseteq B_0$. Thus $\bar{C} \cap D \subseteq \bar{A_0} \cap D \subseteq \bar{A_0} \cap B_0 = \emptyset$, since A_0 and B_0 are separated according to (a). Therefore, $\bar{C} \cap D = \emptyset$. In the same reasoning process we can have $C \cap \bar{D} = \emptyset$ and hence C and D are separated. We next show that there exists $t_0 \in (0,1)$ that $t_0 \notin A_0 \cup B_0$. If this is not the truth, then $\forall t \in (0,1)$, $t \in A_0 \cup B_0$, i.e., $t \in A_0$ or $t \in B_0$. Hence $t \in A_0 \cap (0,1)$ or $t \in B_0 \cap (0,1)$, i.e., $t \in C$ or $t \in D$, i.e., $t \in C \cup D$, which gives $(0,1) \subseteq C \cup D$. On the other hand, we clearly have $C \subseteq (0,1)$ and $D \subseteq (0,1)$, thus $C \cup D \subseteq (0,1)$. Therefore, we get $C \cup D = (0,1)$, which means $(0,1)$ is not connected. But if we apply Theorem 2.47 on (0,1), we can conclude that $(0,1)$ is connected, which is a contradiction. So, there must exist $t_0 \in (0,1)$ that $t_0 \notin A_0 \cup B_0$, i.e., $t_0 \notin A_0$ and $t_0 \notin B_0$. Thus, $\mathbf{p}(t_0) \notin A$ and $\mathbf{p}(t_0) \notin B$, i.e., $\mathbf{p}(t_0) \notin A \cup B$.

(c) Prove that every convex subset of $\mathbb{R}^k$ is connected.
Proof: A subset E of $\mathbb{R}^k$ is convex if $\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in E$, whenever $\mathbf{x} \in E$, $\mathbf{y} \in E$, and $0 < \lambda < 1$. If E is not connected, then $E = A \cup B$, $A \neq \emptyset$, $B \neq \emptyset$ and A, B are separated. Let $\mathbf{a} \in A$, $\mathbf{b} \in B$, let $\mathbf{p}(\lambda) = (1-\lambda)\mathbf{a} + \lambda\mathbf{b}$, for $\lambda \in \mathbb{R}^1$. Note that these are the same conditions given in the premises, hence we can conclude that there exists $\lambda_0 \in (0,1)$ such that $\mathbf{p}(\lambda_0) \notin A \cup B$ according to (b). This is to say, when $\mathbf{a} \in E$, $\mathbf{b} \in E$, there is $\lambda_0 \in (0,1)$, $(1-\lambda_0)\mathbf{a} + \lambda_0\mathbf{b} \notin E$, which is contradict to the assumption that E is convex. Therefore, E must be connected.

22. A metric space is called *separable* if it contains a countable dense subset. Show that $\mathbb{R}^k$ is separable.
Proof: Let $\mathbb{Q}^k = \{\mathbf{x} = (x_1, ..., x_k) | \mathbf{x} \in \mathbb{R}^k \wedge x_i \in \mathbb{Q}, 1 \leq i \leq k\}$. Clearly, $\mathbb{Q}^k$ is a countable subset of $\mathbb{R}^k$ since $\mathbb{Q}$ is countable. Next, we show that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. To see this, let $\mathbf{x}$ be any point in $\mathbb{R}^k$ and $\mathbf{x} \notin \mathbb{Q}^k$. Denote $\mathbf{x} = (x_1, x_2, ..., x_k)$. Let $r > 0$ be any positive real number. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a $y_i \in \mathbb{Q}$ such that $|y_i - x_i| < \frac{r}{\sqrt{k}}$, for $1 \leq i \leq k$. Put $\mathbf{y} = (y_1, y_2, ..., y_k)$, then $\mathbf{y} \in \mathbb{Q}^k$ and $|\mathbf{y} - \mathbf{x}| = \sqrt{\sum_{i=1}^{k} |y_i - x_i|^2} < \sqrt{\frac{r^2}{k} \cdot k} = r$. Therefore, $\mathbf{x}$ is a limit point of $\mathbb{Q}^k$ and we conclude that $\mathbb{Q}^k$ is dense in $\mathbb{R}^k$. Hence $\mathbb{R}^k$ is a separable space.

23. A collection $\{V_\alpha\}$ of open subsets of X is said to be a *base* for X if the following is true: For every $x \in X$ and every open set $G \subseteq X$ such that $x \in G$, we have $x \in V_\alpha \subseteq G$ for some α. In other words, every open set in X is the union of a subcollection of $\{V_\alpha\}$.
Prove that every separable metric space has a *countable* base.

Proof: Let X be a separable metric space, then X contains a countable dense subset Y. Suppose the elements of Y can be arranged as $y_1, y_2, ...$, let $\mathcal{C} = \{V_\alpha\}$ be the collection of all neighborhoods with rational radius and center in Y, that is, $\mathcal{C} = \{N_r(y_i) | r > 0, r \in \mathbb{Q} \wedge y_i \in Y\}$. Clearly, $\mathcal{C}$ is a collection of open sets and $\mathcal{C}$ is countable since both Y and $\mathbb{Q}$ are countable. $\forall x \in X$, let G be an open set such that $G \subseteq X$ and $x \in G$. Since $x \in G$ and G is open, there is a $r > 0$ such that $N_r(x) \subseteq G$. If $x \in Y$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is a $r' \in \mathbb{Q}$ such that $0 < r' < r$, and since $N_{r'}(x) \in \mathcal{C}$, if we denote in another form that $\mathcal{C} = \{V_\alpha\}$, we are convinced ourselves that $x \in N_{r'}(x) = V_\alpha \subseteq N_r(x) \subseteq G$, for some α. On the other hand, if $x \notin Y$, there is a $y \in Y$ and $y \in N_\epsilon(x)$, $\epsilon \in \mathbb{Q} \wedge \epsilon < \frac{r}{2}$, for x is a limit point of Y since Y is a dense subset of X. Then $d(x, y) < \epsilon$, thus $x \in N_\epsilon(y)$ and $N_\epsilon(y) \in \mathcal{C}$. Furthermore, $\forall z \in N_\epsilon(y)$, $d(x, z) \leq d(x, y) + d(y, z) < \epsilon + \epsilon = 2\epsilon < r$, and hence $z \in N_r(x)$, which implies $N_\epsilon(y) \subseteq N_r(x)$. Combining these results together, we get $x \in N_\epsilon(y) \subseteq N_r(x) \subseteq G$ and $N_\epsilon(y) \in \mathcal{C}$. Therefore, $\mathcal{C}$ is a base for X.

24. Let X be a metric space in which every infinite subset has a limit point. Prove that X is separable.
Proof: Fix $\delta > 0$, and pick $x_1 \in X$. Having chosen $x_1, x_2, ..., x_j \in X$, choose $x_{j+1} \in X$, if possible, so that $d(x_i, x_{j+1}) \geq \delta$ for $i = 1, ..., j$. Next, we show that this process must stop after a finite number of steps. Suppose, if it's not, then we can obtain an infinite set $E = \{x_i\}_{i=1}^{\infty}$ such that $d(x_i, x_j) \geq \delta$, for all $i \neq j$ and $i, j \geq 1$. E can have no limit point in X. To see this, suppose p is a limit point of E, then $N_{\frac{\delta}{2}}(p)$ can have at most one point of E. This is because if $x \in E$ and $x \in N_{\frac{\delta}{2}}(p)$, $y \in E$ and $y \neq x$, then $d(y, p) \geq d(x, y) - d(x, p) > \delta - \frac{\delta}{2} = \frac{\delta}{2}$, which says $y \notin N_{\frac{\delta}{2}}(p)$. Thus, if we let $r = d(x, p)$ and pick a positive real number r' such that $0 < r' < r$, $N_{r'}(p)$ contains no point of E, which is absurd if p is a limit point of E. Therefore, E has no limit point in E, which is contradict to our assumption that "*every infinite subset has a limit point*". Hence, the previous process must stop after a finite number of steps and X can therefore be covered by finitely many neighborhoods of radius δ (since after finite number of steps, we cannot find any x in X, thus every x in X has been covered by neighborhoods with radius δ and centered in the points that have been selected).
Now, let's take $\delta = \frac{1}{n}$, $n = 1, 2, 3, ...$, and consider the centers of the corresponding neighborhoods, namely, the set $Y = \bigcup_{n=1}^{\infty} \{y | X \subseteq \bigcup N_{\frac{1}{n}}(y), y \in X\}$. We will prove that Y is a countable dense subset of X. The fact that Y is a countable subset of X is clear since every set in the above union is finite and the total number of sets is countable. Pick $p \in X$ and $p \notin Y$, and let $r > 0$ be an arbitrary positive real number, then there exists some $\delta = \frac{1}{n}$ for some sufficiently large positive integer n such that $\delta < r$, and $p \in N_\delta(y)$, for some $y \in Y$. Hence $d(p, y) < \delta < r$ and $y \in N_r(p)$, which

implies that p is a limit point of Y. Therefore Y is dense in X, so X is separable.

25. Prove that every compact metric space K has a countable base, and that K is therefore separable.
Proof: For every positive integer n, there are finitely many neighborhoods of radius $\frac{1}{n}$ whose union covers K ($\mathcal{C} = \{N_{\frac{1}{n}}(p)\}$, $p \in K$ forms an open cover of K, then there is a finite subcover of $\mathcal{C}$ which stills covers K since K is compact). Let $\mathcal{O}$ be the collection of all the finite subcovers which covers K when taking n to be $1, 2, 3, ...$, that is, $\mathcal{O} = \bigcup_{n=1}^{\infty} \{N_{\frac{1}{n}}(p) | K \subseteq \bigcup N_{\frac{1}{n}}(p)\}$.
We will show that $\mathcal{O}$ is a countable base of K.
$\forall x \in K$ and every open set G such that $G \subseteq K$ and $x \in G$, there is a $r > 0$ such that $N_r(x) \subseteq G$ since G is open. For this r, there is a sufficiently large n such that $0 < \frac{1}{n} < r$. If $N_{\frac{1}{n}}(x) \in \mathcal{O}$, then $x \in N_{\frac{1}{n}}(x) \subseteq N_r(x) \subseteq G$. If $N_{\frac{1}{n}}(x) \notin \mathcal{O}$, then let $\epsilon = \frac{1}{2n}$, and $x \in N_\epsilon(y)$ for some $y \in K$ such that $N_\epsilon(y) \in \mathcal{O}$. Thus, $d(x, y) < \epsilon$. Furthermore, $\forall z \in N_\epsilon(y)$, $d(z, x) \leq d(z, y) + d(y, x) < \epsilon + \epsilon = 2\epsilon = \frac{1}{n} < r$, hence $z \in N_r(x)$ and $N_\epsilon(y) \subseteq N_r(x)$. Taking these together, we have $x \in N_\epsilon(y) \subseteq N_r(x) \subseteq G$ and $N_\epsilon(y) \in \mathcal{O}$. Therefore, $\mathcal{O}$ is a base of K and $\mathcal{O}$ is countable since every set in the union is finite and the total number of sets is countable.
We have completed the proof that K has a countable base, and the result that K is separable is due to the fact that if K is compact, then every infinite subset of K has a limit point in K, by Theorem 2.37, and the result of Exercise 24.

26. Let X be a metric space in which every infinite subset has a limit point. Prove that X is compact.
Proof: By Exercise 23 and 24, X has a countable base. It follows that every open cover of X has a *countable* subcover $\{G_n\}$, $n = 1, 2, 3, ...$ If no finite subcollection of $\{G_n\}$ covers X, then the complement F_n of $G_1 \cup \cdots \cup G_n$ is nonempty for each n, but $\bigcap_{n=1}^{\infty} F_n$ is empty. Let $K_n = \bigcup_{i=1}^{n} G_i$, then $F_n = K_n^c$. Since $K_n \subseteq K_{n+1}$, $F_n \supseteq F_{n+1}$ but each F_n is not empty. Since $\bigcap_{n=1}^{\infty} F_n$ is empty , then $\forall x \in X$, $\exists N \in \mathbb{N}$, $x \in F_N$ but $x \notin F_{N+1}$.
If E is a set which contains a point from each F_n, then we obtain a infinite subset of X. We will show that E doesn't have a limit point. Suppose, on the contrary, if E has a limit point p, then there is an N such that $p \in F_N$ but $p \notin F_{N+1}$. In other words, p in K_N for some N (thus K_n for $n \geq N$) and P in G_α for some $1 \leq \alpha \leq N$. Since G_α is open, there is a neighborhood N_p of p such that $N_p \subseteq G_\alpha$. Therefore, $N_p \subseteq K_n$, for $n \geq N$, and $N_p \cap F_n = \emptyset$, for $n \geq N$. This is contradict to the fact that p is a limit point of E, since N_p can only contain points of F_n, $n < N$ and thus $N_p \cap E$ contains only finite number of points. Thus X must be covered by some finite subcover of $\{G_n\}$ and is compact.

27. Define a point p in a metric space X to be a *condensation point* of a set $E \subseteq X$ if every neighborhood of p contains uncountably many points of E.
Suppose $E \subseteq \mathbb{R}^k$, E is uncountable, and let P be the set of all condensation points of E. Prove that P is perfect and that at most countably many points of E are not in P. In other words, show that $P^c \cap E$ is at most countable.
Proof:
(i)Let x be a limit point of P, and let N_x be any neighborhood of x. Then there is a $y \in N_x$, $y \neq x$, such that y is a condensation point of E. Since N_x is open, there is a neighborhood N_y of y such that $N_y \subseteq N_x$. Therefore, N_x contains uncountably many points of E since N_y contains uncountably many points of E, and hence x is a condensation point of E. Thus, $x \in P$ and P is closed.
(ii) Let $\{V_n\}$ be a countable base of $\mathbb{R}^k$ since $\mathbb{R}^k$ is separable and every separable metric space has a countable base. Let W be the union of those V_n for which $E \cap V_n$ is at most countable and we will show that $P = W^c$.
$\Rightarrow$: Let x be any point of P, then x is a condensation point of E. If $x \in W$, then there is some $m \in \mathbb{N}$, $x \in V_m$ and $E \cap V_m$ is at most countable. Because V_m is open, there is a neighborhood U_x of x, such that $U_x \subseteq V_m$, and U_x contains uncountably many points of E since x is a condensation point of E. Hence V_m contains uncountably many points of E, which is a contradiction. So $x \notin W$, i.e., $x \in W^c$, and thus $P \subseteq W^c$.
$\Leftarrow$: On the other hand, let x be any point of W^c, then $x \notin W$. Let N_x be any neighborhood of x, then since N_x is open and $\{V_n\}$ is a countable base of $\mathbb{R}^k$, there is some m, $m \in \mathbb{N}$, such that $x \in V_m \subseteq N_x$. Since $x \notin W$, $x \notin V_i$ if $V_i \cap E$ is at most countable, hence $V_m \cap E$ must be uncountable. $V_m \subseteq N_x$ implies that $V_m \cap E \subseteq N_x \cap E$ and thus $N_x \cap E$ must be uncountable, which is equivalent to say that x is a condensation point of E. Therefore, $x \in P$ and $W^c \subseteq P$.
Now, we have completed the proof of $P = W^c$. Consider any point x in P, and let N_x be any neighborhood of x. Then there is some m such that $x \in V_m \subseteq N_x$. Since $x \notin W$, $V_m \cap E$ is uncountable and thus V_m is uncountable. Let $y \in V_m \cap W^c \subseteq N_x \cap W^c \subseteq N_x$ and $y \neq x$ (since W is at most countable, then $V_m \cap W$ is at most countable, thus $V_m \cap W^c$ must be uncountable because V_m is uncountable), then $y \in W^c$ and thus $y \in P$. This is to say, for every neighborhood N_x of x, there is a point $y \in N_x$ such that $y \neq x$ and $y \in P$. Therefore x is a limit point of P.
Combined with (i), P is perfect and since W is at most countable, $P = W^c$ implies that at most countably many points of E are not in P, namely, those points $x \in E$ and $x \in W$ ($P^c \cap E = W \cap E$ is at most countable).

28. Prove that every closed set in a separable metric space is the union of a (possibly empty) perfect set and a set which is at most countable. (*Corollary*: Every countable closed set in $\mathbb{R}^k$ has isolated points.)
Proof: Let E be a closed set and let P be the set of all condensation

points of E (P is possibly empty). Suppose $p \in P$, then every neighborhood of p contains uncountably many points of E and thus p is a limit point of E. Hence $p \in E$ since E is closed, so $P \subseteq E$.
Let $E = P \cup (E - P) = E \cup (E \cap P^c)$, and according to Exercise 27, then P is perfect and $E \cap P^c$ is at most countable, which completes our proof.
Proof of the Corollary:
Suppose, it is not true. Let E be a countable closed set in $\mathbb{R}^k$, and E has no isolated points. Then each point of E is a limit point of E and E is thus perfect since E is closed. According to Theorem 2.43, E is uncountable, which is a contradiction.

29. Prove that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments.
Proof: Let $\{V_\beta\}$ be the collection of open sets(segments) of $\mathbb{R}^1$ such that centered at every $p \in \mathbb{Q}$ and with rational radius. Then according to Exercise 22 and 23, we know that $\{V_\beta\}$ is a countable base of $\mathbb{R}^1$ and hence every open set O in $\mathbb{R}^1$ is the union of a subcollection $\{V_\alpha\}$ of $\{V_\beta\}$, i.e., $O = \bigcup_\alpha V_\alpha$. Note that $\{V_\alpha\}$ is at most countable.
Next, we show how to get an at most countable collection of *disjoint* segments by starting from $\{V_\alpha\}$. Let $E := \emptyset$, and let $\{V_\alpha\} = \{V_1, V_2, ...\}$ since $\{V_\alpha\}$ is at most countable. At step n, we add V_n to E according to the following rule:
(i) If $V_n \cap U_\gamma = \emptyset$, for every $U_\gamma \in E$, then add V_n directly into E;
(ii) Otherwise, let $\{U_\gamma\}$ be the collection of sets in E such that $U_\gamma \cap V_n \neq \emptyset$. If $V_n \subseteq U_\gamma$ for some γ, then we simply discard V_n, leaving E unchanged and move on to V_{n+1}. Otherwise, we first replace each U_γ by $V_n \cup U_\gamma$ and then we check to see whether there are any two of $V_n \cup U_\gamma$ intersected, and united them if any are found. Note that the union of two intersected segments which are centered at rational numbers p_1, p_2 and with rational radius r_1 and r_2 is still a segment centered at a rational number and with rational radius. To see this, let two segment be $V_1 = (a_1 = p_1 - r_1, b_1 = p_1 + r_1)$, $V_2 = (a_2 = p_2 - r_2, b_2 = p_2 + r_2)$, and let $a_1 < a_2 < b_1 < b_2$, without loss of generality. Then $V_1 \cup V_2 = (a_1, b_2)$, and $V_1 \cup V_2$ is centered at $\frac{a_1+b_2}{2}$, which is clearly a rational number; and with radius $\frac{b_2-a_1}{2}$, which is also a rational number. Hence $V_1 \cup V_2 \in \{V_\alpha\}$ if $V_1, V_2 \in \{V_\alpha\}$.
Therefore, we can convince us that the above construction is well defined and each step can be terminated in finite sub steps. The resulted collection E after each step n contains *disjoint* segments from $\{V_\alpha\}$, and $\bigcup_\gamma (U_\gamma \in E) = \bigcup_{i=1}^n V_i$ so we are sure that $O = E$ at last. Furthermore, the number of sets in E is less than or equal to n after each step n and thus E is at most countable.

30. Imitate the proof of Theorem 2.43 to obtain the following result:
If $\mathbb{R}^k = \bigcup_1^\infty F_n$, where each F_n is a closed subset of $\mathbb{R}^k$, then at least one F_n has a nonempty interior.

Equivalent statement: If G_n is a dense open subset of $\mathbb{R}^k$, for $n = 1, 2, 3, ...,$ then $\bigcap_1^\infty G_n$ is not empty (in fact, it is dense in $\mathbb{R}^k$).
(This is a special case of Baire's theorem; see Exercise 22, Chap. 3, for the general case.)
Proof: Suppose that, on the contrary, every F_n has an empty interior, which implies that $\forall p \in F_n$ and let N_p be any neighborhood of p, there is a $q \in N_p$, $q \neq p$ and $q \notin F_n$, i.e., $q \in F_n^c$. Thus p is a limit point of F_n^c. Because $\mathbb{R}^k = \bigcup_1^\infty F_n$, $F_n^c \subseteq \bigcup_{m \neq n} F_m$, and p is a limit point of $\bigcup_{m \neq n} F_m$.
Now, let x_1 be any point of F_1, then x_1 is a limit point of $\bigcup_{m=2}^\infty F_m$. Let V_1 be any neighborhood of x_1, then there is a point x_2 in V_1 such that $x_2 \notin F_1$ and $x_2 \in \bigcup_{m=2}^\infty F_m$. Without loss of generality, we assume $x_2 \in F_2$, and thus x_2 is a limit point of $\bigcup_{m \neq 2} F_m$. But since $x_2 \notin F_1$, x_2 is not a limit point of F_1 (because F_1 is closed), and thus x_2 must be a limit point of $\bigcup_{m=3}^\infty F_m$. On the other hand, x_2 is not a limit point of F_1 suggests that there is a neighborhood V_2^1 of x_2 such that $V_2^1 \cap F_1 = \emptyset$. If we let V_2^2 be a neighborhood of x_2 such that $x_1 \notin V_2^2$ and $\bar{V}_2^2 \subseteq V_1$, and denote $V_2 = V_2^1 \cap V_2^2$, then V_2 satisfies the following properties:
(i) $V_2 \cap F_1 = \emptyset$ and thus $x_1 \notin V_2$;
(ii) $\bar{V}_2 \subseteq V_1$;
(iii) x_2 is a limit point of $\bigcup_{m=3}^\infty F_m$.
The property (iii) allows us to continue the above construction steps.
Generally speaking, suppose x_n has been picked and V_n has been constructed. Then x_n is a limit point of $\bigcup_{m=n+1}^\infty F_m$ and there is a point x_{n+1} in V_n such that $x_{n+1} \notin F_n$ and $x_{n+1} \in \bigcup_{m=n+1}^\infty F_m$. Without loss of generality, we assume $x_{n+1} \in F_{n+1}$, and thus x_{n+1} is a limit point of $\bigcup_{m \neq n+1} F_m$. Since $x_{n+1} \notin F_n$, x_{n+1} is not a limit point of F_n(because F_n is closed) and there is a neighborhood V_{n+1}^1 of x_{n+1} such that $V_{n+1}^1 \cap F_n = \emptyset$. If we let V_{n+1}^2 be a neighborhood of x_{n+1} such that $x_n \notin V_{n+1}^2$ and $\bar{V}_{n+1}^2 \subseteq V_n$, and denote $V_{n+1} = V_{n+1}^1 \cap V_{n+1}^2$, then V_{n+1} satisfies the following properties:
(i) $V_{n+1} \cap F_n = \emptyset$ and thus $x_n \notin V_{n+1}$;
(ii) $\bar{V}_{n+1} \subseteq V_n$;.
(iii) According to our steps, x_{n+1} cannot be a limit point of F_i, $1 \leq i \leq n$, due to property (i) and (ii) (which implies $V_{n+1} \cap F_i = \emptyset$, for $1 \leq i \leq n$). Thus x_{n+1} must be a limit point of $\bigcup_{m=n+2}^\infty$.
By (iii), x_{n+1} satisfies our induction hypothesis, and the construction can proceed.
Since $x_n \in V_n$ and thus $x_n \in \bar{V}_n$, each $\bar{V}_n$ is nonempty. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Furthermore, by (ii), $\bar{V}_{n+1} \subseteq V_n \subseteq \bar{V}_n$, $\bigcap_{n=1}^\infty \bar{V}_n$ is nonempty, according to the Corollary of Theorem 2.36. Then there is some $x \in \bigcap_{n=1}^\infty \bar{V}_n$. But by (i), $(\bigcap_{n=1}^\infty \bar{V}_n) \cap F_m = \emptyset$, for every $m \in \mathbb{N}$, which means $x \notin F_m$, for every $m \in \mathbb{N}$. Thus $x \notin \bigcup_{n=1}^\infty F_n$, i.e., $x \notin \mathbb{R}^k$, which is absurd.
Therefore, we conclude that at least one F_n has a nonempty interior.

Proof of the equivalent statement:
Suppose $\bigcap_1^\infty G_n$ is empty, then $(\bigcap_1^\infty G_n)^c = \mathbb{R}^k$, i.e., $\mathbb{R}^k = \bigcup_1^\infty G_n^c$, and G_n^c is closed since G_n is open. Then according to the previous result, there is at least one G_N^c has a nonempty interior. $(G_N^c)^\circ \neq \emptyset$ means there is a $p \in (G_N^c)^\circ$, and thus there is a neighborhood N_p of p such that $N_p \subseteq G_N^c$, i.e., $N_p \cap G_n = \emptyset$ and thus p is not a limit point of G_n, which is contradict to the assumption that G_n is dense in $\mathbb{R}^k$.

3 Numerical sequences and series

1. Prove that convergence of $\{s_n\}$ implies convergence of $\{|s_n|\}$. Is the converse true?
   Proof: Suppose $\{s_n\}$ converges to s, then $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}$ such that $|s_n - s| < \varepsilon$, for all $n \geq N$. Since $||s_n| - |s|| \leq |s_n - s| < \varepsilon$, we know that $|s_n|$ converges to $|s|$.
   The converse is not true. e.g., $s_n = (-1)^n$.

2. Calculate $\lim\limits_{n\to\infty}(\sqrt{n^2+n} - n)$.
   Solution: $\lim\limits_{n\to\infty}(\sqrt{n^2-n}) = \lim\limits_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} = \lim\limits_{n\to\infty} \frac{1}{1+\sqrt{1+\frac{1}{n}}} = 1.$

3. If $s_1 = \sqrt{2}$, and $s_{n+1} = \sqrt{2+\sqrt{s_n}}, (n = 1, 2, 3, ...)$, prove that $\{s_n\}$ converges, and that $s_n < 2$ for $n = 1, 2, 3, ...$
   Proof: We first prove that $s_{n+1} > s_n$ by induction.
   (i)$n = 1$, $s_2 = \sqrt{2+\sqrt{s_1}} > \sqrt{2} = s_1$;
   (ii)Suppose the inequality holds when $n = k$, i.e., $s_{k+1} > s_k$. Let $n = k+1$, then $s_{k+2} - s_{k+1} = \sqrt{2+\sqrt{s_{k+1}}} - s_{k+1} = \sqrt{2+\sqrt{s_{k+1}}} - \sqrt{2+\sqrt{s_k}} = \frac{\sqrt{s_{k+1}}-\sqrt{s_k}}{\sqrt{2+\sqrt{s_{k+1}}}+\sqrt{2+\sqrt{s_k}}}$. Since $s_{k+1} > s_k$ by hypothesis, $\sqrt{s_{k+1}} > \sqrt{s_k}$ and thus $s_{k+2} > s_{k+1}$. Therefore, $s_{n+1} > s_n$ for all $n \in \mathbb{N}$. Similarly, by induction, we can show that $s_n < 2$ for all $n \in \mathbb{N}$. So, $\{s_n\}$ is monotonic and bounded, thus $\{s_n\}$ converges.

4. Find the upper and lower limits of the sequence $\{s_n\}$ defined by $s_1 = 0$; $s_{2m} = \frac{s_{2m-1}}{2}$; $s_{2m+1} = \frac{1}{2} + s_{2m}$.
   Solution: We can obtain that $s_{2m+1} = 1-(\frac{1}{2})^m$ and $s_{2m+2} = \frac{1}{2}(1-(\frac{1}{2})^m)$, for $m \geq 0$. Thus $\limsup\limits_{n\to\infty} s_n = 1$ and $\liminf\limits_{n\to\infty} = \frac{1}{2}$.

5. For any two real sequences $\{a_n\}$, $\{b_n\}$, prove that $\limsup\limits_{n\to\infty}(a_n + b_n) \leq \limsup\limits_{n\to\infty} a_n + \limsup\limits_{n\to\infty} b_n$, provided the sum on the right is not of the form $\infty - \infty$.
   Proof: Suppose, on the contrary, this is not true. For simplicity, let $A = \limsup\limits_{n\to\infty} a_n$, $B = \limsup\limits_{n\to\infty} b_n$, and $C = \limsup\limits_{n\to\infty}(a_n + b_n)$. Then, by

our assumption, $A + B < C$, i.e., $A < C - B$. Hence there is an S, $A < S < C - B$, and since $A = \limsup_{n\to\infty} a_n$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n < S$. So, when $n \geq N$, $a_n + b_n < S + b_n$ and thus $\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty}(S + b_n)$, according to Theorem 3.19. That is, $C \leq \limsup_{n\to\infty}(S + b_n) = S + \limsup_{n\to\infty} b_n = S + B < (C - B) + B = C$, which is absurd. Therefore, $C \leq A + B$.

6. Investigate the behavior(convergence or divergence) of $\sum a_n$ if

   (a) $a_n = \sqrt{n+1} - \sqrt{n}$;
   Solution: Let $s_n = \sum_{i=1}^{n} a_i = \sqrt{n+1} - 1$, then $s = \lim_{n\to\infty} s_n \to \infty$ and thus $\sum a_n$ diverges.

   (b) $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n}$;
   Solution: $a_n = \frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{n(\sqrt{n+1}+\sqrt{n})} < \frac{1}{2n\sqrt{n}} = \frac{1}{2n^{\frac{3}{2}}}$, then $\sum a_n < \frac{1}{2}\sum \frac{1}{n^{\frac{3}{2}}}$. Since $\sum \frac{1}{n^p}$ converges when $p > 1$, $\sum a_n$ converges by the comparison test.

   (c) $a_n = (\sqrt[n]{n} - 1)^n$;
   Solution: Since $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \limsup_{n\to\infty} |\sqrt[n]{n} - 1| = 0 < 1$, $\sum a_n$ converges due to the root test.

   (d) $a_n = \frac{1}{1+z^2}$, for complex values of z.
   Solution: When $|z| > 1$, $|\frac{a_{n+1}}{a_n}| = |\frac{z^n+1}{z^{n+1}+1}| = \frac{1}{|z|}\frac{|z^n+1|}{|z^n+\frac{1}{z}|}$. Since $\frac{|z^n+1|}{|z^n+\frac{1}{z}|} \leq \frac{|z|^n+1}{||z|^n-\frac{1}{|z|}|} = \frac{|z|^n+1}{|z|^n-\frac{1}{|z|}} < \frac{|z|^n+1}{|z|^n-1}$, $\limsup_{n\to\infty} |\frac{a_{n+1}}{a_n}| \leq \frac{1}{|z|} \limsup_{n\to\infty} \frac{|z|^n+1}{|z|^n-1} = \frac{1}{|z|} < 1$, and hence $\sum a_n$ converges due to the ratio test.
   When $|z| < 1$, let $z = |z|w$, where $w \in \mathbb{C}$ and $|w| = 1$, then $a_n = \frac{1}{1+z^n} = \frac{1}{1+|z|^n w^n}$. Since $|z| < 1$, $|z|^n \to 0$ when $n \to \infty$, thus $a_n \to 1$ when $n \to \infty$ and $\sum a_n$ diverges.
   Finally, when $|z| = 1$, $|a_n| = \frac{1}{|1+z^n|} \geq \frac{1}{1+|z|^n} = \frac{1}{2}$, thus a_n cannot $\to 0$ when $n \to \infty$ since otherwise $|a_n|$ should $\to 0$, too. Hence $\sum a_n$ diverges.
   Summarized, $\sum a_n$ converges when $|z| > 1$, and diverges when $|z| \leq 1$.

7. Prove that the convergence of $\sum a_n$ implies the convergence of $\sum \frac{\sqrt{a_n}}{n}$, if $a_n \geq 0$.
   Proof: $\forall \varepsilon > 0$, since $\sum a_n$ converges, there is an $N_1 \in \mathbb{N}$ such that $n > m > N_1$ implies $|\sum_{k=m}^{n} a_k| = \sum_{k=m}^{n} a_k < \varepsilon$, since $a_n \geq 0$. On the other hand, since $\sum \frac{1}{n^2}$ converges, there is an $N_2 \in \mathbb{N}$ such that $n > m > N_2$ implies $|\sum_{k=m}^{n} \frac{1}{k^2}| = \sum_{k=m}^{n} \frac{1}{k^2} < \varepsilon$. Then let $N = \max(N_1, N_2)$

and when $n > m > N$, we have $|\sum_{k=m}^{n} \frac{\sqrt{a_k}}{k}| \leq \sqrt{\sum_{k=m}^{n} (\sqrt{a_k})^2 \sum_{k=m}^{n} (\frac{1}{k})^2} = \sqrt{\sum_{k=m}^{n} a_k \sum_{k=m}^{n} \frac{1}{k^2}} < \sqrt{\varepsilon \cdot \varepsilon} = \varepsilon$, by the *Schwarz* inequality. Hence, $\sum \frac{\sqrt{a_n}}{n}$ converges.

8. If $\sum a_n$ converges, and if $\{b_n\}$ is monotonic and bounded, prove that $\sum a_n \sum b_n$ converges.
   Proof: $\sum a_n$ converges implies that $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $n > N$ and $p \in \mathbb{I}^+$ implies $|\sum_{k=n+1}^{n+p} a_k| < \epsilon$. Denote $A_m(n) = \sum_{k=1}^{m} a_{n+k}$ and put $A_0(n) = 0$, then $|A_m(n)| < \epsilon$ for every m. Using summation by parts, we obtain that $|\sum_{k=1}^{p} a_{n+k}b_{n+k}| = |\sum_{k=1}^{p} (A_k(n) - A_{k-1}(n))b_{n+k}| = |\sum_{k=1}^{p} A_k(n)b_{n+k} - \sum_{k=1}^{p} A_{k-1}(n)b_{n+k}| = |\sum_{k=1}^{p} A_k(n)b_{n+k} - \sum_{k=0}^{p-1} A_k(n)b_{n+(k+1)}| = |\sum_{k=1}^{p-1} A_k(n)(b_{n+k} - b_{n+(k+1)}) + A_p(n)b_{n+p} - A_0(n)b_{n+1}| = |\sum_{k=1}^{p-1} A_k(n)(b_{n+k} - b_{n+(k+1)}) + A_p(n)b_{n+p}|$ (*), since we put $A_0(n) = 0$. (*)$\leq |\sum_{k=1}^{p-1} A_k(n)(b_{n+k} - b_{n+(k+1)})| + |A_p(n)b_{n+p}| \leq \sum_{k=1}^{p-1} |A_k(n)| \cdot |(b_{n+k} - b_{n+(k+1)})| + |A_p(n)||b_{n+p}| < \epsilon(\sum_{k=1}^{p-1} |b_{n+k} - b_{n+(k+1)}| + |b_{n+p}|)$ (**). Since $\{b_n\}$ is monotonic, $\sum_{k=1}^{p-1} |b_{n+k} - b_{n+(k+1)}| = |b_{n+1} - b_{n+p}|$ and then (**)$=\epsilon(|b_{n+1} - b_{n+p}| + |b_{n+p}|) \leq \epsilon(|b_{n+1}| + |b_{n+p}| + |b_{n+p}|)$ (***). Since $\{b_n\}$ is bounded, $|b_n| \leq M$, for every $n \in \mathbb{N}$ and some M. Hence, (***)$\leq \epsilon \cdot 3M$, which gives $|\sum_{k=1}^{p} a_{n+k}b_{n+k}| < 3\epsilon M$. Therefore, $\sum a_n b_n$ converges.

9. Find the radius of convergence of each of the following power series:
   (a) $\sum n^3 z^n$: (**Solution**: $\alpha = 1, R = 1$);
   (b) $\sum \frac{2^n}{n!} z^n$: (**Solution**: $\alpha = 0, R = \infty$);
   (c) $\sum \frac{2^n}{n^2} z^n$: (**Solution**: $\alpha = 2, R = \frac{1}{2}$);
   (d) $\sum \frac{n^3}{3^n} z^n$: (**Solution**: $\alpha = \frac{1}{3}, R = 3$).

10. Suppose that the coefficients of the power series $\sum a_n z^n$ are integers, infinitely many of which are distinct from zero. Prove that the radius of convergence is at most 1.
    Proof: Let the radius of convergence is R, we first prove that power series converges *absolutely* in the interior of the disk with radius R and

with the center at the origin. In other words, $\sum a_n|z|^n$ converges when $|z| < R$. Put $c_n = a_n|z|^n$, and apply the root test: $\limsup_{n\to\infty} \sqrt[n]{|c_n|} = |z| \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{|z|}{R} < 1$ and thus $\sum c_n$ converges.
Next, suppose $R > 1$ and let $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|} = \frac{1}{R} < 1$. Pick a β such that $\alpha < \beta < 1$, then there is an $N \in \mathbb{N}$ such that $\sqrt[n]{|a_n|} < \beta$ for $n > N$, i.e., $|a_n| < \beta^n < 1$. Since by assumption, a_n are integers, $a_n = 0$ when $n > N$. Thus the only possible a_n which are distinct from zero are in the set $E = \{a_i | 1 \le i \le N\}$. But E is finite, which is contradict to our hypothesis that there are infinitely many of a_n which are distinct from zero. Therefore $R \le 1$.

11. Suppose $a_n > 0$, $s_n = a_1 + \cdots + a_n$, and $\sum a_n$ diverges.

(a) Prove that $\sum \frac{a_n}{1+a_n}$ diverges.
Proof: Note that $\frac{a_n}{1+a_n} > \frac{1}{2}a_n$ when $a_n < 1$. Suppose that, on the contrary, $\sum \frac{a_n}{1+a_n}$ converges. Then $\frac{a_n}{1+a_n} = \frac{1}{1+\frac{1}{a_n}} \to 0$ as $n \to \infty$, which gives $\frac{1}{a_n} \to \infty$ and thus $a_n \to 0$ as $n \to \infty$. Hence $\exists N \in \mathbb{N}$ such that $n > N$ implies $a_n < 1$, and as a result $\frac{a_n}{1+a_n} > \frac{1}{2}a_n$ when $n > N$. Since $\sum a_n$ diverges, so is $\sum \frac{a_n}{1+a_n}$, which gives a contradiction.

(b) Prove that $\frac{a_{N+1}}{s_{N+1}} + \cdots \frac{a_{N+k}}{s_{N+k}} \ge 1 - \frac{s_N}{s_{N+k}}$ and deduce that $\sum \frac{a_n}{s_n}$ diverges.
Proof: Since s_n is monotonically increasing, $\frac{a_{N+1}}{s_{N+1}} + \cdots \frac{a_{N+k}}{s_{N+k}} \ge \frac{a_{N+1}}{s_{N+k}} + \cdots \frac{a_{N+k}}{s_{N+k}} = \frac{a_{N+1}+\cdots+a_{N+k}}{s_{N+k}} = \frac{s_{N+k}-s_N}{s_{N+k}} = 1 - \frac{s_N}{s_{N+k}}$.
Suppose, on the contrary, $\sum \frac{a_n}{s_n}$ converges, then $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$, such that $|\frac{a_{N+1}}{s_{N+1}} + \cdots + \frac{a_{N+k}}{s_{N+k}}| < \epsilon$, $\forall k \in \mathbb{I}^+$. Hence $1 - \frac{s_N}{s_{N+k}} < \epsilon$, which gives $\frac{s_N}{s_{N+k}} > 1 - \epsilon$, i.e., $s_{N+k} < \frac{s_N}{1-\epsilon}$. Fix N and let $k \to \infty$, this gives $s_n < \frac{s_N}{1-\epsilon}$ as $n \to \infty$ and s_n is bounded. On the other hand s_n is monotonically increasing, hence s_n converges, which is contradict to the hypothesis that $\sum a_n$ diverges.

(c) Prove that $\frac{a_n}{s_n^2} \le \frac{1}{s_{n-1}} - \frac{1}{s_n}$ and deduce that $\sum \frac{a_n}{s_n^2}$ converges.
Proof: $\frac{a_n}{s_n^2} = \frac{s_n - s_{n-1}}{s_n^2} < \frac{s_n - s_{n-1}}{s_n s_{n-1}} = \frac{1}{s_{n-1}} - \frac{1}{s_n}$.
Then $\sum_{n=1}^{m} \frac{a_n}{s_n^2} < \frac{1}{s_1} + \sum_{n=2}^{m} (\frac{1}{s_{n-1}} - \frac{1}{s_n}) = \frac{1}{s_1} + (\frac{1}{s_1} - \frac{1}{s_m}) = \frac{2}{s_1} - \frac{1}{s_m}$,
Since $\sum a_n$ diverges, $s_n \to \infty$ as $n \to \infty$, and $\frac{1}{s_n} \to 0$ as $n \to \infty$. Hence, $\sum \frac{a_n}{s_n^2} \le \frac{2}{s_1} = \frac{2}{a_1}$ and is bounded. On the other hand, $\sum_{n=1}^{m} \frac{a_m}{s_m^2}$ increases monotonically and thus converges.

(d) What can be said about $\sum \frac{a_n}{1+na_n}$ and $\sum \frac{a_n}{1+n^2a_n}$?
Solution:
(i)$\sum \frac{a_n}{1+na_n}$ may converge or diverge. Let $a_n = 1$, then $\sum a_n$ diverges,

and $\sum \frac{a_n}{1+na_n} = \sum \frac{1}{n+1}$ diverges. On the other hand, let

$$a_n = \begin{cases} 1 & \text{if } n = 2^k \\ \frac{1}{n^2} & \text{otherwise} \end{cases}$$

, then $\sum a_m$ diverges, but

$$\frac{a_n}{1+na_n} = \begin{cases} \frac{1}{1+2^k} & \text{if } n = 2^k \\ \frac{1}{n^2+n} & \text{otherwise} \end{cases}$$

, converges to 0, in either case.
(ii)$\sum \frac{a_n}{1+n^2a_n}$ converges, since $\frac{a_n}{1+n^2a_n} = \frac{1}{n^2+\frac{1}{a_n}} < \frac{1}{n^2}$ and $\sum \frac{1}{n^2}$ converges.

12. Suppose $a_n > 0$ and $\sum a_n$ converges. Put $r_n = \sum\limits_{m=n}^{\infty} a_m$.

 (a) Prove that $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > 1 - \frac{r_n}{r_m}$ if $m < n$, and deduce that $\sum \frac{a_n}{r_n}$ diverges.
 Proof: Since $r_i > r_j$ if $i < j$, $\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n} > \frac{a_m + \cdots + a_n}{r_m} = \frac{r_m - r_{n+1}}{r_m} = 1 - \frac{r_{n+1}}{r_m} > 1 - \frac{r_n}{r_m}$.
 Suppose, on the contrary, $\sum \frac{a_n}{r_n}$ converges, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $n > m > N$ implies $|\frac{a_m}{r_m} + \cdots + \frac{a_n}{r_n}| < \epsilon$, and thus $\epsilon > 1 - \frac{r_n}{r_m}$, i.e., $\frac{r_n}{r_m} > 1 - \epsilon$, i.e., $r_n > (1-\epsilon)r_m$ (*). Fix this m, let $s = \sum\limits_{k=1}^{\infty} a_k$ since $\sum a_k$ converges, and let $s_n = \sum\limits_{k=1}^{n} a_k$. Then there is a $N' \in \mathbb{N}$ such that $|s_n - s| < (1-\epsilon)r_m$ if $n > N'$, i.e., $|r_{n+1}| = r_{n+1} < (1-\epsilon)r_m$(**). Now, let $K = \max(m, N'+1)$, and let $n > K$, then (*) and (**) tells us just contradict things. Therefore $\sum \frac{a_n}{r_n}$ must diverge.

 (b) Prove that $\frac{a_n}{\sqrt{r_n}} < 2(\sqrt{r_n} - \sqrt{r_{n+1}})$ and deduce that $\sum \frac{a_n}{\sqrt{r_n}}$ converges.
 Proof: $\frac{a_n}{\sqrt{r_n}} = \frac{r_n - r_{n+1}}{\sqrt{r_n}} = \frac{(\sqrt{r_n}+\sqrt{r_{n+1}})(\sqrt{r_n}-\sqrt{r_{n+1}})}{\sqrt{r_n}} < \frac{2\sqrt{r_n}(\sqrt{r_n}-\sqrt{r_{n+1}})}{\sqrt{r_n}} = 2(\sqrt{r_n} - \sqrt{r_{n+1}})$.
 Then $\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}} < 2\sum_{k=1}^{n}(\sqrt{r_n} - \sqrt{r_{n+1}}) = 2(\sqrt{r_1} - \sqrt{r_{n+1}})$. Because $r_n \to 0$ as $n \to \infty$, $\sum \frac{a_k}{\sqrt{r_k}} \leq 2\sqrt{r_1}$ as $n \to \infty$. Since $\sum_{k=1}^{n} \frac{a_k}{\sqrt{r_k}}$ increases monotonically, $\sum \frac{a_k}{\sqrt{r_k}}$ converges.

13. Prove that the Cauchy product of two absolutely convergent series converges absolutely.
 Proof: Let $\sum a_n$ and $\sum b_n$ be two absolutely convergent series and $\sum c_n$ is their Cauchy product. Denote $A = \sum_{n=0}^{\infty} |a_n|$, $B = \sum_{n=0}^{\infty} |b_n|$, then $C_n = \sum_{m=0}^{n} |c_m| = \sum_{m=0}^{n} |\sum_{k=0}^{m} a_k b_{m-k}| \leq \sum_{m=0}^{n}\sum_{k=0}^{m} |a_k||b_{m-k}| = \sum_{k=0}^{m} |a_k| \sum_{m=0}^{n} |b_{m-k}| = \sum_{k=0}^{n} |a_k| \sum_{m=0}^{n-k} |b_m| \leq \sum_{k=0}^{n} |a_k| \sum_{m=0}^{\infty} |b_m| = B\sum_{k=0}^{n} |a_k| \leq B\sum_{k=0}^{\infty} |a_k| = AB$ and thus C_n is bounded. On the other hand, C_n increases monotonically, thus C_n converges.

14. If $\{s_n\}$ is a complex sequence, define its arithmetic means σ_n by $\sigma_n = \frac{s_0+s_1+\cdots+s_n}{n+1}$, $(n = 0, 1, 2, ...)$.

(a) If $\lim s_n = s$, prove that $\lim \sigma_n = s$.
Proof: Since $\lim s_n = s$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $n > N$ implies $|s_n - s| < \epsilon$. $|\sigma_n - s| = |\frac{s_0+s_1+\cdots+s_n}{n+1} - s| = |\frac{(s_0-s)+(s_1-s)+\cdots+(s_n-s)}{n+1}| \leq \frac{|s_0-s|+|s_1-s|+\cdots+|s_n-s|}{n+1} = \frac{\sum_{k=0}^{N}|s_k-s|}{n+1} + \frac{\sum_{k=N+1}^{n}|s_k-s|}{n+1} < \frac{\sum_{k=0}^{N}|s_k-s|}{n+1} + \frac{(n-N)\epsilon}{n+1} < \frac{\sum_{k=0}^{N}|s_k-s|}{n+1} + \epsilon$. Pick $N' > N$, such that $\frac{\sum_{k=0}^{N}|s_k-s|}{n+1} < \epsilon$ when $n > N'$ and then $|\sigma_n - s| < \frac{\sum_{k=0}^{N}|s_k-s|}{n+1} + \epsilon < \epsilon + \epsilon = 2\epsilon$ when $n > N'$. Therefore $\lim \sigma_n = s$.

(b) Construct a sequence $\{s_n\}$ which does not converge, although $\lim \sigma_n = 0$.
Solution: $s_n = (-1)^n$.

(c) Can it happen that $s_n > 0$ for all n and that $\limsup s_n = \infty$, although $\lim \sigma_n = 0$?
Solution: *Yes.* See this example: $s_n = k$, if $n = 2^k$; and $s_n = \frac{1}{2^n}$, otherwise. Then $s_n > 0$ for all n and $\limsup s_n = \infty$. But suppose $2^k \leq n < 2^{k+1}$, then $\sigma_n = \frac{s_0+s_1+\cdots+s_n}{n+1} = \frac{\sum_{i=1}^{k} i + \sum_{m\neq 2^j}\frac{1}{2^m}}{n+1} < \frac{\frac{k(k+1)}{2}+\sum_{m=0}^{n}\frac{1}{2^m}}{n+1} < \frac{\frac{\log_2 n(\log_2 n+1)}{2}+\sum_{m=0}^{\infty}\frac{1}{2^m}}{n+1} = \frac{\frac{\log_2 n(\log_2 n+1)}{2}+2}{n+1}$
$= \frac{(\log_2 n)^2+\log_2 n+4}{2(n+1)} \to 0$ when $n \to \infty$.

(d) Put $a_n = s_n - s_{n-1}$, for $n \geq 1$. Show that $s_n - \sigma_n = \frac{1}{n+1}\sum\limits_{k=1}^{n} ka_k$.
Assume that $\lim(na_n) = 0$ and that $\{\sigma_n\}$ converges. Prove that $\{s_n\}$ converges.
Proof: $s_n - \sigma_n = s_n - \frac{s_0+s_1+\cdots+s_n}{n+1} = \frac{(n+1)s_n-(s_0+s_1+\cdots+s_n)}{n+1} = \frac{1}{n+1}(ns_n - \sum_{k=0}^{n-1} s_k) = \frac{1}{n+1}((s_n - s_{n-1}) + \cdots + (s_n - s_0)) = \frac{1}{n+1}(a_n + (a_n + a_{n-1}) + \cdots + \sum_{k=n}^{1} a_k) = \frac{1}{n+1}(na_n + (n-1)a_{n-1} + \cdots + 1 \cdot a_1) = \frac{1}{n+1}\sum_{k=1}^{n} ka_k$.
Suppose $\lim \sigma_n = \sigma$, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$, $n > N$ implies $|\sigma_n - \sigma| < \epsilon$. On the other hand, $\lim(na_n) = 0$ implies that $\exists N' \in \mathbb{N}$ s.t. $|na_n| < \epsilon$ when $n > N'$. Let $K = \max(N, N')$, and when $n > K$, $|s_n - \sigma| = |(s_n - \sigma_n) + (\sigma_n - \sigma)| \leq |s_n - \sigma_n| + |\sigma_n - \sigma| < |\frac{1}{n+1}\sum_{k=1}^{n} ka_k| + \epsilon = |\frac{1}{n+1}(\sum_{k=1}^{K} ka_k + \sum_{k=K+1}^{n} ka_k)| + \epsilon \leq \frac{1}{n+1}(|\sum_{k=1}^{K} ka_k| + |\sum_{k=K+1}^{n} ka_k|) + \epsilon \leq \frac{1}{n+1}(|\sum_{k=1}^{K} ka_k| + \sum_{k=K+1}^{n} |ka_k|) + \epsilon = \frac{1}{n+1}(|\sum_{k=1}^{K} ka_k| + (n-K)\epsilon) + \epsilon < \frac{1}{n+1}|\sum_{k=1}^{K} ka_k| + 2\epsilon$. Pick $K' > K$, such that $\frac{1}{n+1}|\sum_{k=1}^{K} ka_k| < \epsilon$ when $n > K'$, then $|s_n - \sigma| < \epsilon + 2\epsilon = 3\epsilon$ when $n > K'$. Therefore, $\lim s_n = \sigma$.

(e) Derive the last conclusion from a weaker hypothesis: Assume $M < \infty$, $|na_n| \leq M$ for all n, and $\lim \sigma_n = \sigma$. Prove that $\lim s_n = \sigma$.

Proof: If $m < n$, then $s_n - \sigma_n = \frac{m+1}{n-m}(\sigma_n - \sigma_m) + \frac{1}{n-m}\sum_{i=m+1}^{n}(s_n - s_i)$ (*). For these i, $|s_n - s_i| = |\sum_{k=i+1}^{n} a_k| \leq |\frac{\sum_{k=i+1}^{n} ka_k}{i+1}| \leq \frac{1}{i+1}\sum_{k=i+1}^{n}|ka_k| \leq \frac{(n-i)M}{i+1} \leq \frac{(n-m-1)M}{m+2}$.
Fix $\epsilon > 0$ and associate with each n the integer m such that satisfies $m \leq \frac{n-\epsilon}{1+\epsilon} < m+1$. Then $\frac{m+1}{n-m} \leq \frac{1}{\epsilon}$ and $|s_n - s_i| < M\epsilon$. Hence $\limsup_{n\to\infty}|s_n - \sigma| \leq M\epsilon$ by letting $m \to \infty$, thus $n \to \infty$ in (*). Since ϵ was arbitrary, $\lim s_n = \sigma$.

15. Definition 3.21 can be extended to the case in which the a_n lie in some fixed $\mathbb{R}^k$. Absolute convergence is defined as convergence of $\sum |\mathbf{a}_n|$. Show that Theorems 3.22, 3.23, 3.25(a), 3.33, 3.34, 3.42, 3.45, 3.47, and 3.55 are true in this more general setting. (Only slight modifications are required in any of the proofs.)
Proof: The proofs of these Theorem are very similar, only replace a_n by $\mathbf{a}_n$ and replace $|a_n|$ by $|\mathbf{a}_n|$.

16. Fix a positive number α. Choose $x_1 > \sqrt{\alpha}$, and define $x_2, x_3, x_4, ...,$ by the recursion formula $x_{n+1} = \frac{1}{2}(x_n + \frac{\alpha}{x_n})$.

    (a) Prove that $\{x_n\}$ decreases monotonically and that $\lim x_n = \sqrt{\alpha}$.
    Proof: $x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) - x_n = \frac{1}{2}(\frac{\alpha}{x_n} - x_n) = \frac{1}{2x_n}(\alpha - x_n^2)$.
    We prove that $x_n > \sqrt{\alpha}$, for every n, by induction.
    (i)$n = 1$, $x_1 > \sqrt{\alpha}$, which is trivial.
    (ii)Suppose when $n = k$, $x_k > \sqrt{\alpha}$. Let $n = k+1$, $x_{k+1} = \frac{1}{2}(x_k + \frac{\alpha}{x_k}) \geq \frac{1}{2} \cdot 2\sqrt{x_k \cdot \frac{\alpha}{x_k}} = \sqrt{\alpha}$ and the equality holds if and only if $x_k = \frac{\alpha}{x_k}$, namely, $x_k = \sqrt{\alpha}$. Since $x_k > \sqrt{\alpha}$ by hypothesis, we have $x_{k+1} > \sqrt{\alpha}$. Now we see $x_n > \sqrt{\alpha}$ for every n and thus $x_{n+1} - x_n < 0$, i.e., $x_{n+1} < x_n$. Hence, x_n decreases monotonically. Furthermore, x_n is bounded and thus x_n converges. Suppose $\lim x_n = x$, then $x = \lim x_n = \lim \frac{1}{2}(x_{n-1} + \frac{\alpha}{x_{n-1}}) = \frac{1}{2}(\lim x_{n-1} + \frac{\alpha}{\lim x_{n-1}}) = \frac{1}{2}(x + \frac{\alpha}{x})$, which gives $x = \sqrt{\alpha}$. Thus $\lim x_n = x = \alpha$.

    (b) Put $\epsilon_n = x_n - \sqrt{\alpha}$, and show that $\epsilon_{n+1} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$ so that, setting $\beta = 2\sqrt{\alpha}$, $\epsilon_{k+1} < \beta(\frac{\epsilon_1}{\beta})^{2^n}$, $(n = 1, 2, 3, ...)$.
    Proof: $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{1}{2}(x_n + \frac{\alpha}{x_n}) - \sqrt{\alpha} = \frac{x_n^2 - 2x_n\sqrt{\alpha} + \alpha}{2x_n} = \frac{(x_n - \sqrt{\alpha})^2}{2x_n} = \frac{\epsilon_n^2}{2x_n} < \frac{\epsilon_n^2}{2\sqrt{\alpha}}$, since $x_n > \sqrt{\alpha}$.
    Let $\beta = 2\sqrt{\alpha}$, then $\epsilon_{n+1} < \frac{\epsilon_n^2}{\beta} = \beta(\frac{\epsilon_n}{\beta})^2 < \beta(\frac{\beta(\frac{\epsilon_{n-1}}{\beta})^2}{\beta})^2 = \beta(\frac{\epsilon_{n-1}}{\beta})^{2^2} < \cdots < \beta(\frac{\epsilon_1}{\beta})^{2^n}$

    (c) This is a good algorithm for computing square roots, since the recursion formula is simple and the convergence is extremely rapid. For

example, if $\alpha = 3$ and $x_1 = 2$, show that $\epsilon_1/\beta < \frac{1}{10}$ and that therefore $\epsilon_5 < 4 \cdot 10^{-16}$, $\epsilon_6 < 4 \cdot 10^{-32}$.
Proof: $\alpha = 2 > \frac{25}{9}$, $\sqrt{\alpha} > \frac{5}{3}$, then $\frac{\epsilon_1}{\beta} = \frac{x_1-\sqrt{\alpha}}{2\sqrt{\alpha}} = \frac{2-\sqrt{\alpha}}{2\sqrt{\alpha}} = \frac{1}{\sqrt{\alpha}} - \frac{1}{2} < \frac{3}{5} - \frac{1}{2} = \frac{1}{10}$. Therefore, $\epsilon_5 < \beta(\frac{\epsilon_1}{\beta})^{16} < 4 \cdot 10^{-16}$, and $\epsilon_6 < \beta(\frac{\epsilon_1}{\beta})^{32} < 4 \cdot 10^{-32}$, since $\beta = 2\sqrt{\alpha} < 2\sqrt{4} = 4$.

17. Fix $\alpha > 1$. Take $x_1 > \sqrt{\alpha}$, and define $x_{n+1} = \frac{\alpha+x_n}{1+x_n} = x_n + \frac{\alpha - x_n^2}{1+x_n}$.

(a) Prove that $x_1 > x_3 > x_5 > \cdots$.

(b) Prove that $x_2 < x_4 < x_6 < \cdots$.
Proof for (a) and (b): We first prove that $x_{2n-1} > \sqrt{\alpha}$ and $x_{2n} < \sqrt{\alpha}$ by induction.
(i)$n = 1$, $x_1 > \sqrt{\alpha}$ by assumption. $x_2 - \sqrt{\alpha} = \frac{\alpha+x_1}{1+x_1} - \sqrt{\alpha} = \frac{\alpha+x_1-\sqrt{\alpha}-\sqrt{\alpha}x_1}{1+x_1} = \frac{(x_1-\sqrt{\alpha})(1-\sqrt{\alpha})}{1+x_1} < 0$, since $\alpha > 1$ by assumption. Thus the proposition holds.
(ii)Suppose when $n = k$, $x_{2k-1} > \sqrt{\alpha}$ and $x_{2k} < \sqrt{\alpha}$. Let $n = k+1$, then $x_{2k+1} - \sqrt{\alpha} = \frac{\alpha+x_{2k}}{1+x_{2k}} - \sqrt{\alpha} = \frac{(x_{2k}-\sqrt{\alpha})(1-\sqrt{\alpha})}{1+x_{2k}} > 0$, since $x_{2k} < 0$ by hypothesis and $\alpha > 1$ by assumption. $x_{2k+2} - \sqrt{\alpha} = \frac{(x_{2k+1}-\sqrt{\alpha})(1-\sqrt{\alpha})}{1+x_{2k+1}} < 0$, since $\alpha > 1$ by assumption.
Now, we see $x_{2n-1} > \sqrt{\alpha}$ and $x_{2n} < \sqrt{\alpha}$. Since $x_{2n+1} - x_{2n-1} = \frac{\alpha+x_{2n}}{1+x_{2n}} - x_{2n-1} = \frac{\alpha+\frac{\alpha+x_{2n-1}}{1+x_{2n-1}}}{1+\frac{\alpha+x_{2n-1}}{1+x_{2n=1}}} - x_{2n-1} = \frac{\alpha(1+x_{2n-1})+(\alpha+x_{2n-1})}{(1+x_{2n-1})+(\alpha+x_{2n-1})} - x_{2n-1} = \frac{2\alpha+(1+\alpha)x_{2n-1}}{2x_{2n-1}+(1+\alpha)} - x_{2n-1} = \frac{2(\alpha-x_{2n-1}^2)}{2x_{2n-1}+(1+\alpha)} < 0$, thus $x_{2n+1} < x_{2n-1}$. Similarly, we can show that $x_{2n+2} - x_{2n} = \frac{2(\alpha-x_{2n}^2)}{2x_{2n}+(1+\alpha)} > 0$ and thus $x_{2n+2} > x_{2n}$.

(c) Prove that $\lim x_n = \sqrt{\alpha}$.
Proof: Since $x_{2n+1} = \frac{2\alpha+(1+\alpha)x_{2n-1}}{2x_{2n-1}+(1+\alpha)}$, let $\limsup x_n = \lim x_{2n+1} = a$, then $a = \lim x_{2n+1} = \lim \frac{2\alpha+(1+\alpha)x_{2n-1}}{2x_{2n-1}+(1+\alpha)} = \frac{2\alpha+(1+\alpha)\lim x_{2n-1}}{2\lim x_{2n-1}+(1+\alpha)} = \frac{2\alpha+(1+\alpha)a}{2a+(1+\alpha)}$, which gives $a = \sqrt{\alpha}$. Similarly, since $x_{2n+2} = \frac{2\alpha+(1+\alpha)x_{2n}}{2x_{2n}+(1+\alpha)}$, let $\liminf x_n = \lim x_{2n+2} = b$, then $b = \lim x_{2n+2} = \lim \frac{2\alpha+(1+\alpha)x_{2n}}{2x_{2n}+(1+\alpha)} = \frac{2\alpha+(1+\alpha)\lim x_{2n}}{2\lim x_{2n}+(1+\alpha)} = \frac{2\alpha+(1+\alpha)b}{2b+(1+\alpha)}$, which gives $b = \sqrt{\alpha}$. Therefore, $\limsup x_n = \liminf x_n = \sqrt{\alpha}$ and $\lim x_n = \sqrt{\alpha}$.

(d) Compare the rapidity of convergence of this process with the one described in Exercise 16.
Solution: Again, let $\epsilon_n = x_n - \sqrt{\alpha}$. Then $\epsilon_{n+1} = x_{n+1} - \sqrt{\alpha} = \frac{\alpha+x_n}{1+x_n} - \sqrt{\alpha} = \frac{(x_n-\sqrt{\alpha})(1-\sqrt{\alpha})}{1+x_n} = \frac{\epsilon_n(1-\sqrt{\alpha})}{1+x_n}$. Since $x_2 \le x_n \le x_1$, $\frac{|1-\sqrt{\alpha}|}{1+x_1}|\epsilon_n| \le |\epsilon_{n+1}| \le \frac{|1-\sqrt{\alpha}|}{1+x_2}|\epsilon_n|$, thus the rapidity of convergence is slower than in Exercise 16.

18. Replace the recursion formula of Exercise 16 by $x_{n+1} = \frac{p-1}{p}x_n + \frac{\alpha}{p}x_n^{-p+1}$, where p is a fixed positive integer, and describe the behavior of the resulting sequences $\{x_n\}$.
Solution: Similar to Exercise 16, if $x_1 > \sqrt[p]{\alpha}$, then we can prove that $x_n > \sqrt[p]{\alpha}$ and $x_{n+1} < x_n$, i.e., $\{x_n\}$ decreases monotonically. Thus, $\lim x_n = \sqrt[p]{\alpha}$.

19. Associate to each sequence $a = \{\alpha_n\}$, in which α_n is 0 or 2, the real number $x(a) = \sum\limits_{n=1}^{\infty} \frac{\alpha_n}{3^n}$. Prove that the set of all $x(a)$ is precisely the Cantor set described in Sec. 2.44.
Proof: Suppose $W = \{x(a)|a = \{\alpha_n\}, \alpha_n = 0 \vee \alpha_n = 2\}$ and Let $E_\emptyset = [0,1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let $E_0 = [0, \frac{1}{3}]$, $E_2 = [\frac{2}{3}, 1]$. Remove the middle thirds of these intervals, and let $E_{00} = [0, \frac{1}{9}]$, $E_{02} = [\frac{2}{9}, \frac{3}{9}]$, $E_{20} = [\frac{6}{9}, \frac{7}{9}]$, $E_{22} = [\frac{8}{9}, 1]$. Continuing in this way, we obtain 2^n compact sets $\{E_{a_1 a_2 \cdots a_n}\}$ in step n, where $a_i = 0$ or $a_i = 2$ for $1 \le i \le n$. It's clearly to see that the Cantor set $E = (\bigcap_{n=1}^{\infty} E_n) \cap E_\emptyset$ and $E_n = \bigcup\limits_{\forall a_1 a_2 \cdots a_n} E_{a_1 a_2 \cdots a_n}$. Thus $E = \bigcap_{n=1}^{\infty} E_n = \bigcap_{n=1}^{\infty} \bigcup\limits_{\forall a_1 a_2 \cdots a_n} E_{a_1 a_2 \cdots a_n} = \bigcup\limits_{\forall a_1 a_2 \cdots} E_{a_1 a_2 \cdots}$. Furthermore, for every $E_{a_1 a_2 \cdots a_n}$, we have $E_{a_1} \supseteq E_{a_1 a_2} \supseteq \cdots \supseteq E_{a_1 a_2 \cdots a_n}$. Since $\lim\limits_{n\to\infty} \mathbf{diam} E_{a_1 a_2 \cdots a_n} = \lim\limits_{n\to\infty} \frac{1}{3^n} = 0$, there is exactly one point $x_{a_1 a_2 \cdots}$ in $E_{a_1 a_2 \cdots} = \bigcap\limits_{n=1}^{\infty} E_{a_1 a_2 \cdots a_n}$ and $x_{a_1 a_2 \cdots} = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$. To see this, we need to show that $x_{a_1 a_2 \cdots} = \sum\limits_{n=1}^{\infty} \frac{a_n}{3^n}$ lies in every $E_{a_1 a_2 \cdots a_n}$, $n = 1, 2, \cdots$. We prove this by induction.
(i)$n = 1$, $x_{a_1 a_2 \cdots} = \frac{a_1}{3} + \sum\limits_{n=2}^{\infty} \frac{a_n}{3^n}$. If $a_1 = 0$, $E_{a_1} = E_0 = [0, \frac{1}{3}]$, and $x_{a_1 a_2 \cdots} = 0 + \sum\limits_{n=2}^{\infty} \frac{a_n}{3^n} \le 0 + \sum\limits_{n=2}^{\infty} \frac{2}{3^n} = \frac{1}{3}$. Obviously, $x_{a_1 a_2 \cdots} \ge 0$, thus $x_{a_1 a_2 \cdots} \in E_0$; If $a_1 = 2$, $E_{a_1} = E_2 = [\frac{2}{3}, 1]$ and $x_{a_1 a_2 \cdots} = \frac{2}{3} + \sum\limits_{n=2}^{\infty} \frac{2}{3^n} \ge \frac{2}{3}$. Obviously, $x_{a_1 a_2 \cdots} \le \sum\limits_{n=1}^{\infty} \frac{2}{3^n} = 1$, thus $x_{a_1 a_2 \cdots} \in E_2$. Thus $x_{a_1 a_2 \cdots} \in E_{a_1}$.
(ii)Suppose when $n = k$, $x_{a_1 a_2 \cdots} \in E_{a_1 a_2 \cdots a_k}$ and $E_{a_1 a_2 \cdots a_k} = [a = \sum\limits_{n=1}^{k} \frac{a_k}{3^k}, a + \frac{1}{3^k}]$ (This can also be shown by induction easily). Let $n = k+1$, then $x_{a_1 a_2 \cdots} = \sum\limits_{n=1}^{k} \frac{a_k}{3^k} + \frac{a_{k+1}}{3^{k+1}} + \sum\limits_{n=k+2}^{\infty} \frac{a_n}{3^n} = a + \frac{a_{k+1}}{3^{k+1}} + \sum\limits_{n=k+2}^{\infty} \frac{a_n}{3^n}$. If $a_{k+1} = 0$, $E_{a_1 a_2 \cdots a_{k+1}} = [a, a + \frac{1}{3^{k+1}}]$, and $x_{a_1 a_2 \cdots} = a + \sum\limits_{n=k+2}^{\infty} \frac{a_n}{3^n} \le a + \sum\limits_{n=k+2}^{\infty} \frac{2}{3^n} = a + \frac{1}{3^{k+1}}$. Since $x_{a_1 a_2 \cdots} \ge a$, $x_{a_1 a_2 \cdots} \in E_{a_1 a_2 \cdots a_{k+1}}$. On the other hand, if $a_{k+1} = 2$, $E_{a_1 a_2 \cdots a_{k+1}} = [a + \frac{2}{3^{k+1}}, a + \frac{1}{3^k}]$ and $x_{a_1 a_2 \cdots} =$

$a + \frac{2}{3^{k+1}} + \sum\limits_{n=k+2}^{\infty} \frac{a_n}{3^n} \geq a + \frac{2}{3^{k+1}}$. Since $x_{a_1a_2\cdots} \leq a + \frac{2}{3^{k+1}} + \frac{1}{3^{k+1}} = a + \frac{1}{3^k}$, $x_{a_1a_2\cdots} \in E_{a_1a_2\cdots a_{k+1}}$.

Therefore, we have proved that $x_{a_1a_2\cdots} \in E_{a_1a_2\cdots a_n}$, for every n. So $x_{a_1a_2\cdots}$ is just the only point contained in $E_{a_1a_2\cdots}$. Since $E = \bigcup\limits_{\forall a_1a_2\cdots} E_{a_1a_2\cdots} = \{p|p \in E_{a_1a_2\cdots}\}$, we have shown that $x \in E$ implies $x \in W$ and thus $E \subseteq W$. On the other hand, for every $x \in W$, x has the form $x_{a_1a_2\cdots}$, thus $x \in E_{a_1a_2\cdots}$ and $x \in E$, namely, $W \subseteq E$. Hence $E = W$.

20. Suppose $\{p_n\}$ is a Cauchy sequence in a metric space X, and some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$. Prove that the full sequence $\{p_n\}$ converges to p.

Proof: Since $\{p_n\}$ is a Cauchy sequence, then $\forall\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > m > N$ implies $d(p_n, p_m) < \epsilon$. On the other hand, for this ϵ, since some subsequence $\{p_{n_i}\}$ converges to a point $p \in X$, there is an $N' \in \mathbb{N}$ such that $n_k > N'$ implies $d(p_{n_k}, p) < \epsilon$. Let n_k be the smallest integer such that $n_k > \max(N, N') + 1$, when $n > n_k$, $d(p_n, p) \leq d(p_n, p_{n_k}) + d(p_{n_k}, p) < \epsilon + \epsilon = 2\epsilon$. Therefore, $\{p_n\}$ converges to p.

21. Prove the following analogue of Theorem 3.10(b): If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supseteq E_{n+1}$, and if $\lim\limits_{n\to\infty} \mathrm{diam} E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.

Proof: Construct a sequence $\{p_n\}$ such that $p_n \in E_n$ for every n. Because E_n is bounded for every n, diam E_n is well-defined for every n. Since $\lim\limits_{n\to\infty}$ diam $E_n = 0$, $\forall\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies diam $E_n < \epsilon$, which is to say that $d(p_n, p_m) < \epsilon$ if $n > m > N$. Hence $\{p_n\}$ is a Cauchy sequence in X, and $\{p_n\}$ converges to a point $p \in X$ since X is complete. Thus $\forall\epsilon > 0$, there is an $N' \in \mathbb{N}$ such that $d(p_n, p) < \epsilon$ if $n > N'$. Hence p is a limit point of E_n when $n > N'$. Since $p_n \in E_n$, p is a limit point of E_n and thus is a limit point of E_m, where $1 \leq m \leq n$, because $E_n \subseteq E_m$. This is just to say, p is a limit point of E_n, for every $n \in \mathbb{N}$. Since E_n is closed for every n, $p \in E_n$ for every n, thus $p \in \bigcap_1^\infty E_n$ and $E = \bigcap_1^\infty E_n$ is not empty. The fact that $E = \bigcap_1^\infty E_n$ contains no more than one point is clear since otherwise diam $E > 0$ and thus diam $E_n \geq$ diam $E > 0$ since $E \subseteq E_n$. So $\lim\limits_{n\to\infty}$ diam $E_n > 0$, a contradiction. Therefore, E contains exactly one point, namely, p.

Note: One might notice that our choice of $\{x_n\}$ is not unique. We remark here that, no matter how we choose $\{x_n\}$, it will converge to the point p as long as $x_n \in E_n$ due to the condition $\lim\limits_{n\to\infty}$ diam $E_n = 0$.

22. Suppose X is a nonempty complete metric space, and $\{G_n\}$ is a sequence of dense open subsets of X. Prove Baire's theorem, namely, that $\bigcap_1^\infty G_n$

is not empty. (In fact, it is dense in X.)
Proof: Let $x_1 \in G_1$, since G_1 is open, there is a neighborhood V_1 of x_1 such that $V_1 \subseteq \bar{V}_1 \subseteq G_1$. (More specifically, we first pick a neighborhood $V_1' = N_r(x_1)$ of x_1 such that $V_1' \subseteq G_1$. Let s be some positive real number such that $0 < s < r$, then $\bar{N}_s(x_1) \subseteq N_r(x_1)$ and put $V_1 = N_s(x_1)$.)
Suppose V_n has been constructed, pick x_{n+1} in V_n such that $x_{n+1} \neq x_n$. Let V_{n+1} be a neighborhood of x_{n+1} satisfies the following conditions:
(i)$\bar{V}_{n+1} \subseteq V_n$;
(ii)$x_n \notin \bar{V}_{n+1}$.
(iii)$\bar{V}_{n+1} \subseteq G_{n+1}$;
(iv)diam $V_{n+1} < \frac{1}{2}$diam V_n.
Since G_{n+1} is dense in X, $x_{n+1} \in G_{n+1}$ or x_{n+1} is a limit point of G_{n+1}. If the former case happens, since G_{n+1} is open, we first pick a neighborhood V_{n+1}^1 of x_{n+1} such that $\bar{V}_{n+1}^1 \subseteq G_{n+1}$. Then we can pick a neighborhood V_{n+1}^2 of x_{n+1} satisfies (i) and (ii), since V_n is open and $x_{n+1} \in V_n$. Put $V_{n+1} \subseteq V_{n+1}^1 \cap V_{n+1}^2$ and let V_{n+1} satisfy condition (iv) gives us the desired neighborhood V_{n+1} satisfying all the four conditions. If the latter case happens, we first pick a neighborhood V_{n+1}^1 of x_{n+1} satisfying condition (i) and (ii), and then choose an $x_{n+1}' \in V_{n+1}^1 \cap G_{n+1}$. Since G_{n+1} is open, we can pick a neighborhood V_{n+1}^2 of x_{n+1}' satisfying condition (iii). Replace x_{n+1} by x_{n+1}' and let V_{n+1} be a neighborhood of x_{n+1} (the previous x_{n+1}') such that $V_{n+1} \subseteq V_{n+1}^1 \cap V_{n+1}^2$ and V_{n+1} satisfies (iv), we get the desired neighborhood V_{n+1}. So, in both cases, the above process can continue and we can obtain a sequence of $\{V_n\}$ satisfying conditions (i), (ii), (iii) and (iv).
Since $x_n \in V_n$, each $\bar{V}_n$ is nonempty. What's more, every $\bar{V}_n$ is closed and bounded. Furthermore, (i) tells us that $\bar{V}_{n+1} \subseteq \bar{V}_n$ and (iv) tells us that diam $\bar{V}_n$ = diam $V_n < \frac{1}{2^{n-1}}$diam V_1, for $n > 1$, thus $\lim_{n\to\infty}$ diam $\bar{V}_n = 0$. Hence the conditions in the premise of Exercise 21 are satisfied, $\bigcap_1^\infty \bar{V}_n$ contains exactly one point and thus is nonempty. By (iii), $\bar{V}_n \subseteq G_n$, so $\bigcap_1^\infty \bar{V}_n \subseteq \bigcap_1^\infty G_n$. Therefore, $\bigcap_1^\infty G_n$ is nonempty, too.

23. Suppose $\{p_n\}$ and $\{q_n\}$ are Cauchy sequences in a metric space X. Show that the sequence $\{d(p_n, q_n)\}$ converges.
Proof: $\forall \epsilon > 0$, since $\{p_n\}$ is a Cauchy sequence, there is an $N_1 \in \mathbb{N}$ such that $n > m > N_1$ implies $d(p_n, p_m) < \epsilon$; similarly, since $\{q_n\}$ is a Cauchy sequence, there is an $N_2 \in \mathbb{N}$ such that $n > m > N_2$ implies $d(q_n, q_m) < \epsilon$. Let $N = \max(N_1, N_2)$, $n > m > N$ implies that $d(p_n, q_n) - d(p_m, q_m) \leq (d(p_n, p_m) + d(p_m, q_m) + d(q_m, q_n)) - d(p_m, q_m) = d(p_n, p_m) + d(q_n, q_m) < 2\epsilon$. Similarly, $d(p_m, q_m) - d(p_n, q_n) < 2\epsilon$, and thus $|d(p_n, q_n) - d(p_m, q_m)| < 2\epsilon$. Therefore, $d(p_n, q_n)$ is a Cauchy sequence in $\mathbb{R}^1$ and hence converges since $\mathbb{R}^1$ is complete.

24. Let X be a metric space.

    (a) Call two Cauchy sequences $\{p_n\}$, $\{q_n\}$ in X *equivalent* if $\lim_{n\to\infty} d(p_n, q_n) =$

0. Prove that this is an equivalence relation.

Proof:

(i)Reflexivity: since $d(p_n, p_n) = 0$, $\lim_{n\to\infty} d(p_n, p_n) = 0$;

(ii)Symmetry: since $d(p_n, q_n) = d(q_n, p_n)$, $\lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = 0$;

(iii)transitivity: if $\lim_{n\to\infty} d(p_n, q_n) = 0$, $\lim_{n\to\infty} d(q_n, r_n) = 0$, then $0 \leq \lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = 0$.

Hence $\lim_{n\to\infty} d(p_n, r_n) = 0$.

Therefore this is an equivalence relation.

(b) Let X^* be the set of all equivalence classes so obtained. If $P \in X^*$, $Q \in X^*$, $\{p_n\} \in P$, $\{q_n\} \in Q$, define $\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n)$; by Exercise 23, this limit exists. Show that the number $\Delta(P, Q)$ is unchanged if $\{p_n\}$ and $\{q_n\}$ are replaced by equivalent sequences, and hence that Δ is a distance function in X^*.

Proof: Suppose $p_n \sim p'_n$ and $q_n \sim q'_n$.then

(i)$\Delta(P, Q') = \lim_{n\to\infty} d(p_n, q'_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, q'_n) = \Delta(P, Q) + 0 = \Delta(P, Q)$;

(ii)$\Delta(P', Q) = \lim_{n\to\infty} d(p'_n, q_n) \leq \lim_{n\to\infty} d(p'_n, p_n) + \lim_{n\to\infty} d(p_n, q_n) = 0 + \Delta(P, Q) = \Delta(P, Q)$;

(iii)$\Delta(P', Q') = \lim_{n\to\infty} d(p'_n, q'_n) \leq \lim_{n\to\infty} d(p'_n, p_n) + \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, q'_n) = 0 + \Delta(P, Q) + 0 = \Delta(P, Q)$.

The opposite direction of (i), (ii), (iii) can be proved similarly and hence we have $\Delta(P, Q) = \Delta(P', Q) = \Delta(P, Q') = \Delta(P', Q')$.

Next we show that Δ is a distance function in X^* by checking the three required conditions:

(i)$\Delta(P, P) = \lim_{n\to\infty} d(p_n, p'_n) = 0$;

(ii)$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n) = \lim_{n\to\infty} d(q_n, p_n) = \Delta(Q, P)$;

(iii)$\Delta(P, R) = \lim_{n\to\infty} d(p_n, r_n) \leq \lim_{n\to\infty} d(p_n, q_n) + \lim_{n\to\infty} d(q_n, r_n) = \Delta(P, Q) + \Delta(Q, R)$.

(c) Prove that the resulting metric space X^* is complete.

Proof: Let $\{P_n\}$ be a Cauchy sequence in X^*, and denote $P_k = \{p_{kn}\}$, for every $k \in \mathbb{N}$ ($\{p_{kn}\}$ is an arbitrary Cauchy sequence of X that belongs to P_k). We construct a sequence $\{s_n\}$ in X as follows: For each n, find the smallest $N \in \mathbb{N}$ such that $d(p_{nm}, p_{nk}) < \frac{1}{n}$ if $k > m > N$. Since P_n is a Cauchy sequence, this N must exists. Put $s_n = p_{n(N+1)}$.

Next, we show that $\{s_n\}$ is a Cauchy sequence in X. $\forall \epsilon > 0$, since $\{P_n\}$ is a Cauchy sequence, there is an $N_1 \in \mathbb{N}$ such that $n > m > N_1$ implies $\Delta(P_m, P_n) < \epsilon$, i.e., $\lim_{k\to\infty} d(p_{mk}, p_{nk}) < \epsilon$. Thus there is a $N_2 \in \mathbb{N}$ such that $d(p_{mm'}, p_{nn'}) < \epsilon$ if $n' > m' > N_2$. Choose an N_3 such that $\frac{1}{N_3} < \epsilon$, and put $N = \max(N_1, N_2, N_3)$. When $n' > m' >$

$n > m > N$, $d(s_m, p_{mm'}) < \frac{1}{m} < \frac{1}{N} < \epsilon$, and $d(s_n, p_{nn'}) < \frac{1}{n} < \frac{1}{N} < \epsilon$. Therefore, $d(s_n, s_m) \leq d(s_n, p_{nn'}) + d(p_{nn'}, p_{mm'}) + d(p_{mm'}, s_m) < 3\epsilon$, which means that $\{s_n\}$ is a Cauchy sequence.
Let P be the equivalence class containing $\{s_n\}$, then $P \in X^*$. Finally, we will show that $\{P_n\}$ converges to P.
$\forall \epsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that $d(s_n, s_m) < \epsilon$ if $n > m > N_1$ since we have proved that $\{s_n\}$ is a Cauchy sequence in X. Choose an N_2 such that $\frac{1}{N_2} < \epsilon$, then when $n' > m' > N_2$, $d(p_{m'n'}, s_{m'}) < \frac{1}{m'} < \frac{1}{N_2} < \epsilon$. Let $N = \max(N_1, N_2)$, then when $n' > m' > N$, $d(p_{m'n'}, s_{n'}) \leq d(p_{m'n'}, s_{m'}) + d(s_{m'}, s_{n'}) < 2\epsilon$. Fix m', and let $n' \to \infty$, we obtain that $\lim\limits_{n' \to \infty} d(p_{m'n'}, s_{n'}) < 2\epsilon$, i.e., $\Delta(P_{m'}, P) < 2\epsilon$. Putting these together, we get that $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $m' > N$ implies $\Delta(P_{m'}, P) < 2\epsilon$. Therefore, $\{P_n\}$ converges to P.

(d) For each $p \in X$, there is a Cauchy sequence all of whose terms are p; let P_p be the element of X^* which contains this sequence. Prove that $\Delta(P_p, P_q) = d(p, q)$ for all $p, q \in X$. In other words, the mapping φ defined by $\varphi(p) = P_p$ is an isometry (i.e., a distance-preserving mapping) of X into X^*.
Proof: $\Delta(P_p, P_q) = \lim\limits_{n \to \infty} d(p, q) = d(p, q)$.

(e) Prove that $\varphi(X)$ is dense in X^*, and that $\varphi(X) = X^*$ if X is complete. By (d), we may identify X and $\varphi(X)$ and thus regard X as embedded in the complete metric space X^*. We call X^* the *completion* of X.
Proof: Suppose $T \in X^*$ and $T \notin \varphi(X)$, then let $\{t_n\}$ be any Cauchy sequence in X that lies in T. We construct a sequence $\{T_n\}$ in $\varphi(X)$ such that $T_n = \varphi(t_n)$. $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > m > N$ implies $d(t_n, t_m) < \epsilon$. Hence $\Delta(T_m, T) = \lim\limits_{n \to \infty} d(t_m, t_n) < \epsilon$, i.e., $\{T_n\}$ converges to T and thus T is a limit point of $\varphi(X)$. Therefore, $\varphi(X)$ is dense in X^*.
If X is complete, $\forall T \in X^*$, choose any Cauchy sequence $\{t_n\}$ in X such that $\{t_n\} \in T$, then $\{t_n\}$ converges to some $t \in X$. We conclude that $\varphi(t) = T$, since $\Delta(\varphi(t), T) = \lim\limits_{n \to \infty} d(t, t_n) = 0$. Thus $T \in \varphi(X)$, and $X^* \subseteq \varphi(X)$. Clearly, we have $\varphi(X) \subseteq X^*$ and therefore $\varphi(X) = X^*$.

25. Let X be the metric space whose points are the rational number, with the metric $d(x, y) = |x - y|$. What is the completion of this space? (Compare Exercise 24.)
Proof: The completion of X is exactly the space $\mathbb{R}^1$ containing all the real numbers and with the same metric $d(x, y) = |x - y|$. This is another view of the relationship between $\mathbb{Q}$ and $\mathbb{R}$, namely, $\mathbb{R}$ is the *completion* of $\mathbb{Q}$, a great result!

4 Continuity

1. Suppose f is a real function defined on $\mathbb{R}^1$ which satisfies $\lim_{h\to 0}[f(x+h) - f(x-h)] = 0$, for every $x \in \mathbb{R}^1$. Does this imply that f is continuous?
   Solution: *No.* e.g., $f(x) = \begin{cases} 0, & x = 0 \\ 1, & \text{otherwise} \end{cases}$

2. If f is a continuous mapping of a metric space X into a metric space X into a metric space Y, prove that $f(\bar{E}) \subseteq \overline{f(E)}$ for every set $E \subseteq X$. ($\bar{E}$ denotes the closure of E.) Show, by an example, that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$.
   Proof: $\forall y \in f(\bar{E})$, $\exists x \in \bar{E}$ such that $f(x) = y$. If $x \in E$, then $y = f(x) \in f(E)$ and thus $y \in \overline{f(E)}$. If $x \notin E$, then x is a limit point of E. Given any $\epsilon > 0$, since f is continuous, there is a $\delta > 0$, s.t., $d_X(z,x) < \delta$ implies $d_Y(f(z), f(x)) < \epsilon$. Because x is a limit point of E, there must exists a $x_0 \in E$ such that $d_X(x_0, x) < \delta$, and hence $d_Y(f(x_0), f(x)) < \epsilon$, which means $y = f(x)$ is a limit point of $f(E)$. Therefore, $y \in \overline{f(E)}$ and $f(\bar{E}) \subseteq \overline{f(E)}$.
   Note that $f(\bar{E})$ can be a proper subset of $\overline{f(E)}$. e.g., $E = X = Y = \mathbb{R}^1$, $f(x) = \frac{x^2}{1+x^2}$, then $\bar{E} = E$ and $f(\bar{E}) = f(E) = [0,1)$, but $\overline{f(E)} = [0,1]$.

3. Let f be a continuous real function on a metric space X. Let $Z(f)$ (the *zero set* of f) be the set of all $p \in X$ at which $f(p) = 0$. Prove that $Z(f)$ is closed.
   Proof: Let q be a limit point of $Z(f)$, we need to show that $f(q) = 0$. To see this, suppose, on the contrary, $f(q) \neq 0$. Without loss of generality, we assume that $f(q) > 0$. Since f is continuous, there is a neighborhood N_q of q such that $f(p) > 0$ for all $p \in N_q$. Otherwise, for all neighborhoods V of q, there is at least one point $p \in V$ such that $f(p) \leq 0$. Then we can obtain a sequence $\{p_n\}$, where $|p_n - q| < \frac{1}{n}$, namely, $p_n \to q$ when $n \to \infty$, and $f(p_n) \leq 0$. But then $\lim_{n\to\infty} f(p_n) \leq 0$ and since f is continuous, $\lim_{x\to q} f(x) = f(q)$, which means $\lim_{n\to\infty} f(p_n) = f(q)$, by Theorem 4.2. Thus $f(q) \leq 0$, contradict with our assumption that $f(q) > 0$.
   The existence of N_q tells us that no point p in N_q satisfies $f(p) = 0$, which is again a contradiction since q is a limit point of $Z(f)$. The case $f(q) < 0$ can be examined similarly. Therefore, we conclude that $f(q) = 0$. So $q \in Z(f)$ and $Z(f)$ is closed.

4. Let f and g be continuous mappings of a metric space X into a metric space Y, and let E be a dense subset of X. Prove that $f(E)$ is dense in $f(X)$. If $g(p) = f(p)$ for all $p \in E$, prove that $g(p) = f(p)$ for all $p \in X$. (In other words, a continuous mapping is determined by its values on a dense subset of its domain.)
   Proof: Suppose that $q \in f(X)$ but $q \notin f(E)$, then there is a $p \in X$ but $p \notin E$ such that $q = f(p)$. Since E is dense in X, p is a limit point of E. Pick any $\epsilon > 0$, let $V = \{y | d_Y(y,q) < \epsilon\}$ be a neighborhood of q, then V is

open. Since f is continuous, $f^{-1}(V)$ is also open and $p \in f^{-1}(V)$. Hence there is a neighborhood V_p of p such that $V_p \subseteq f^{-1}(V)$. Since p is a limit point of E, there is a point $x \in E$ such that $x \in V_p$. Thus $x \in f^{-1}(V)$, i.e., $f(x) \in V$ and q is a limit point of $f(E)$. Therefore $f(E)$ is dense in $f(X)$.
Let $h(x) = g(x) - f(x)$, for all $x \in X$. Then $h(p) = 0$, if $p \in E$, which gives $E \subseteq Z(h)$. By Exercise 3, $Z(h)$ is closed, thus $\bar{E} \subseteq Z(h)$, according to Theorem 2.27. Furthermore, since E is dense in X, every point x in X but not in E is a limit point of E and thus $x \in \bar{E}$. Therefore $X \subseteq \bar{E} \subseteq Z(h)$. On the other hand, it is clear that $Z(h) \subseteq X$. Hence $Z(h) = X$, i.e., $h(p) = 0$ for all $p \in X$, i.e., $f(p) = g(p)$ for all $p \in X$.

5. If f is a real continuous function defined on a closed set $E \subseteq \mathbb{R}^1$, prove that there exist continuous real functions g on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. (Such functions g are called *continuous extensions* of f from E to $\mathbb{R}^1$.) Show that the result becomes false if the word "closed" is omitted. Extend the result to vector-valued functions.
Proof: Let the graph of g be a straight line on each of the segments which constitute the complement of E. Then g is continuous on $\mathbb{R}^1$ and $g(x) = f(x)$ for all $x \in E$.
Remark: Remember that Exercise 29, Chap 2 tells us that every open set in $\mathbb{R}^1$ is the union of an at most countable collection of disjoint segments. Since E is closed, E^c is open.
If E is not closed, then the result becomes false. e.g, let $E = (0, 1)$ and $f(x) = \frac{1}{x}$. Clearly, $f(x)$ is continuous on E. But we cannot extend it to $\mathbb{R}^1$, since $\lim_{x \to 0} f(x) = \infty$.
The result can be extended to vector-valued functions, namely:
If $f : \mathbb{R}^1 \to \mathbb{R}^k$ is a vector-valued continuous function defined on a dense subset $E \subseteq \mathbb{R}^1$, then there exist continuous vector-valued functions $g : \mathbb{R}^1 \to \mathbb{R}^k$ on $\mathbb{R}^1$ such that $g(x) = f(x)$ for all $x \in E$. The proof of this is the same, and the only difference is the straight lines should be interpreted in $\mathbb{R}^k$, not in $\mathbb{R}^1$.

6. If f is defined on E, the *graph* of f is the set of points $(x, f(x))$, for $x \in E$. In particular, if E is a set of real numbers, and f is real-valued, the graph of f is a subset of the plane.
Suppose E is compact, and prove that f is continuous on E if and only if its graph is compact.
Proof:
$\Rightarrow$: Suppose E is compact and f is continuous on E, and let G be the graph of f. Define $g : E \to G$, such that $g(x) = (x, f(x))$, for all $x \in E$, then g is one-to-one and onto (namely, $G = g(E)$). Fix a $p \in E$, since f is continuous on E, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|x - p| < \delta$ implies $|f(x) - f(p)| < \epsilon$. Pick r such that $r = \min(\delta, \epsilon)$, then when $|x - p| < r \leq \epsilon$, $|f(x) - f(p)| < \epsilon$ and thus $|g(x) - g(p)| = \sqrt{|x-p|^2 + |f(x) - f(p)|^2} < \sqrt{r^2 + \epsilon^2} < \sqrt{2}\epsilon$. Hence g is continuous and

therefore, G is compact, since E is compact.
$\Leftarrow$: Recall that a function $f : X \to Y$ is continuous if and only if $f^{-1}(C)$ is closed in X for every closed set C in Y.
Suppose E is compact. Let F be any closed subset of $f(E)$, G be the graph of f (G is compact by assumption), and define $\phi : G \to E$ such that $\phi(x, f(x)) = x$ for every $(x, f(x)) \in G$. Then ϕ is continuous, by Example 4.11. Notice that $f^{-1}(F) = \phi((E \times F) \cap G)$. Since E is compact, E is closed, and since F is closed, $E \times F$ is closed. Thus $(E \times F) \cap G$ is compact, since G is compact. Then $\phi((E \times F) \cap G)$ is compact, since ϕ is continuous. So $f^{-1}(F)$ is compact, thus is closed. Therefore, f must be continuous.

7. If $E \subseteq X$ and if f is a function defined on X, the *restriction* of f to E is the function g whose domain of definition is E, such that $g(p) = f(p)$ for $p \in E$. Define f and g on $\mathbb{R}^2$ by: $f(0,0) = g(0,0) = 0$, $f(x,y) = \frac{xy^2}{x^2+y^4}$, $g(x,y) = \frac{xy^2}{x^2+y^6}$ if $(x,y) \neq 0$. Prove that f is bounded on $\mathbb{R}^2$, that g is unbounded in every neighborhood of $(0,0)$, and that f is not continuous at $(0,0)$; nevertheless, the restriction of both f and g to every straight line in $\mathbb{R}^2$ are continuous!
Proof: $|f(x,y)| = \frac{|x|y^2}{x^2+y^4} \leq \frac{x^2+y^4}{2(x^2+y^4)} = \frac{1}{2}$.
Given any $r > 0$ and $M > 0$, suppose V_r is a neighborhood of $(0,0)$, namely, $\sqrt{x^2+y^2} < r$, for any $(x,y) \in V_r$, then there is a y, $0 < y < \frac{1}{2}$(if $r \geq 1$), or $0 < y < \frac{r}{2}$(if $r < 1$) and $\frac{1}{2y} > M$. Let $x = y^3$, then $x^2 + y^2 = y^6 + y^2 = y^2(y^4+1) < \frac{17}{64} < 1 \leq r^2$(if $r \leq 1$), or $< \frac{r^6+r^2}{64} < \frac{r^2}{32} < r^2$(if $r < 1$). Hence, $(x,y) \in V_r$, and $g(x,y) = \frac{1}{2y} > M$. So g is unbounded in every neighborhood of $(0,0)$.
Let $\{(x_n, y_n)\}$ be a sequence such that $\lim_{n\to\infty} y_n = 0$, $y_n \neq 0$ and $x_n = y_n^2$, for every n, then $\lim_{n\to\infty}(x_n, y_n) = (0,0)$, but $f(x_n, y_n) = \frac{1}{2}$, for every n. Thus $\lim_{n\to\infty} f(x_n, y_n) = \frac{1}{2} \neq f(0,0) = 0$, and therefore f is not continuous at $(0,0)$.
We classify the straight lines in $\mathbb{R}^2$ into the following cases:
(i)$x = a$, then $f(x,y) = \frac{ay^2}{a^2+y^2}$. If $a = 0$, $f(x,y) = 0$, and f is continuous, and if $a \neq 0$, $f(x,y)$ is continuous too, since ay^2 and $a^2 + y^2$ are both continuous. g is continuous can be proved similarly.
(ii)$y = b$, then $f(x,y) = \frac{b^2x}{x^2+b^2}$, and f is continuous, with the similar proof of (i). g is continuous can be proved by the same process.
(iii)The most general case comes as $y = ax + b$, where $a \neq 0$. Then $f(x,y) = \frac{x(ax+b)^2}{x^2+(ax+b)^4} = \frac{P_1(x)}{P_2(x)}$, where $P_1(x)$ and $P_2(x)$ are x' polynomials. Since $P_1(x)$ and $P_2(x)$ are continuous, by Example 4.11, $f(x,y)$ is continuous, too. g is continuous can be proved similarly.

8. Let f be a real uniformly continuous function on the bounded set E in $\mathbb{R}^1$. Prove that f is bounded on E.

Show that the conclusion is false if boundedness of E is omitted from the hypothesis.

Proof: Suppose that, on the contrary, f is not bounded. Then for any $M > 0$, there is an $x \in E$ such that $|f(x)| > M$. Let $M = n$, we can therefore obtain a sequence $\{x_n\}$ such that $|f(x_n)| > n$, and thus $|f(x_n)| \to \infty$, as $n \to \infty$. Since E is bounded, so is x_n. Then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges, by Theorem 3.6(b). Denote $\{x_{n_k}\}$ as $\{y_k\}$, and since $\{y_k\}$ converges in $\mathbb{R}^1$, $\{y_k\}$ is a Cauchy sequence. Given any $\epsilon > 0$, there is a $\delta > 0$ such that $|p-q| < \delta$ implies $|f(p)-f(q)| < \epsilon$, for any $p, q \in E$, since f is uniformly continuous on E. For this δ, since $\{y_k\}$ is a Cauchy sequence, there exists an N such that $|y_n - y_m| < \delta$ if $n > m > N$. Thus $|f(y_n) - f(y_m)| < \epsilon$, and we have $|f(y_n)| \leq |f(y_m)| + |f(y_n) - f(y_m)| < |f(y_m)| + \epsilon$. Fix m and let $n \to \infty$, then $\lim\limits_{n\to\infty} |f(y_n)| < |f(y_m)| + \epsilon$, which is contradict to the way we choose $\{x_n\}$ (Pick an integer N' such that $N' > |f(y_m)| + \epsilon$, then $|f(x_n)| > N'$, when $n > N'$. Pick a K such that $n_K > N'$, then $|f(x_{n_K})| > N' > |f(y_m)| + \epsilon$). Therefore, f must be bounded.

Remark: If E is not bounded, the conclusion will be false. e.g., Let $E = [0, +\infty)$, and $f(x) = x$, for all $x \in E$. Clearly, f is uniformly continuous (just take $\delta = \epsilon$ in the definition of *uniformly continuous*). But again clearly, f is unbounded on E.

9. Show that the requirement in the definition of uniform continuity can be rephrased as follows, in terms of diameters of sets: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\mathrm{diam} f(E) < \varepsilon$ for all $E \subseteq X$ with $\mathrm{diam} E < \delta$.

   Proof: The requirement in the definition of uniform continuity says that: To every $\varepsilon > 0$ there exists a $\delta > 0$ such that $d_X(p, q) < \delta$ implies $d_Y(f(p), f(q)) < \epsilon$, for all $p, q \in E$. Since $\mathrm{diam} E = \sup\{d_X(p, q) | p, q \in E\}$ and $\mathrm{diam} f(E) = \sup\{d_Y(f(p), f(q)) | f(p), f(q) \in f(E)\}$, $\mathrm{diam} E < \delta$ and $\mathrm{diam} f(E) < \varepsilon$, for all $E \subseteq X$.

10. Complete the details of the following alternative proof of Theorem 4.19: If f is not uniformly continuous, then for some $\epsilon > 0$ there are sequences $\{p_n\}$, $\{q_n\}$ in X such that $d_X(p_n, q_n) \to 0$ but $d_Y(f(p_n), f(q_n)) > \epsilon$. Use Theorem 2.37 to obtain a contradiction.

    Proof: Since X is compact, then $E_{p_n} = \{p_n\}$ is a infinite subset of E and by Theorem 2.37, E_{p_n} has a limit point p in X. Similarly, $E_{q_n} = \{q_n\}$ has a limit point q in X. So, we can obtain a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ and subsequence $\{q_{n_k}\}$ of $\{q_n\}$ (also, see Theorem 3.6), such that $\{p_{n_k}\}$ converges to p and $\{q_{n_k}\}$ converges to q. Therefore, $d_X(p, q) \leq d_X(p, p_{n_k}) + d_X(p_{n_k}, q_{n_k}) + d_X(q_{n_k}, q) \to 0$, when $n_k \to \infty$. Hence $d_X(p, q) = 0$ and $p = q$. On the other hand, $d_Y(f(p), f(q)) \geq d_Y(f(p_{n_k}), f(q_{n_k})) - d_Y(f(p), f(p_{n_k})) - d_Y(f(q), f(q_{n_k})) \geq \epsilon$, when $n_k \to \infty$, which is absurd if $p = q$.

11. Suppose f is a uniformly continuous mapping of a metric space X into a metric space Y and prove that $\{f(x_n)\}$ is a Cauchy sequence in Y for

every Cauchy sequence $\{x_n\}$ in X. Use this result to give an alternative proof of the theorem stated in Exercise 13.

Proof: Given any $\epsilon > 0$, since f is uniformly continuous, there is a δ such that $d_X(p,q) < \delta$ implies $d_Y(f(p), f(q)) < \epsilon$. Since $\{x_n\}$ is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that $n > m > N$ implies $d_X(x_n, x_m) < \delta$. Thus, $d_Y(f(x_n), f(x_m)) < \epsilon$ and $\{f(x_n)\}$ is a Cauchy sequence.

12. A uniformly continuous function of a uniformly continuous function is uniformly continuous.

State this more precisely and prove it.

Statement: Let f be a uniformly continuous mapping of a metric space X into a metric space Y, and g be a uniformly continuous mapping from Y into a metric space Z. Prove that the mapping $h = g \circ f$ from X into Z is also continuous.

Proof: Given any $\epsilon > 0$, since g is uniformly continuous, there is a $\theta > 0$ such that $d_Y(y_1, y_2) < \theta$ implies that $d_Z(g(y_1), g(y_2)) < \epsilon$. For this θ, since f is uniformly continuous, there is a $\delta > 0$ such that $d_X(x_1, x_2) < \delta$ implies that $d_Y(y_1, y_2) = d_Y(f(x_1), f(x_2)) < \theta$, and thus $d_Z(g(y_1), g(y_2)) = d_Z(g(f(x_1)), g(f(x_2))) = d_Z(h(x_1), h(x_2)) < \epsilon$. Therefore, h is uniformly continuous.

13. Let E be a dense subset of a metric space X, and let f be a uniformly continuous *real* function defined on E. Prove that f has a continuous extension from E to X(see Exercise 5 for terminology).(*Uniqueness* follows from Exercise 4.)

Proof: For each $p \in X$ and each positive integer n, let $V_n(p)$ be the set of all $q \in E$ with $d(p,q) < 1/n$. Since f is uniformly continuous on E, $\forall \epsilon > 0$, there is a $\delta > 0$ such that $\text{diam} S < \delta$ implies that $\text{diam} f(S) < \epsilon$, for every $S \subseteq E$. Therefore, for this δ, if we pick an $N \in \mathbb{N}$ such that $2/N < \delta$, then when $n > N$, $\text{diam} V_n(p) < 2/n < \delta$ and thus $\text{diam} f(V_n(p)) < \epsilon$. This is equivalent to say that $\lim\limits_{n\to\infty} \text{diam} f(V_n(p)) = 0$, and thus $\lim\limits_{n\to\infty} \text{diam}\overline{f(V_n(p))} = 0$, since $\text{diam}\overline{f(V_n(p))} = \text{diam} f(V_n(p))$. Because f is real, $\overline{f(V_n(p))}$ is closed and bounded, and thus compact. Furthermore, since $V_{n+1}(p) \subseteq V_n(p)$, $f(V_{n+1}(p)) \subseteq f(V_n(p))$ and $\overline{f(V_{n+1}(p))} \subseteq \overline{f(V_n(p))}$. Therefore, $\bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$ contains exactly one point, and we define this as $g(p)$.

Next, we shall prove that the function g so defined on X is the desired extension of f.

First, if $p \in E$, then since $f(p) \in f(V_n(p))$, for every n, we have $f(p) \in \bigcap_{n=1}^{\infty} \overline{f(V_n(p))}$ and thus $g(p) = f(p)$.

Then, let p be any point of X. Given any $\epsilon > 0$, we have shown that there is an $N \in \mathbb{N}$ such that if $\text{diam} V_n(p) < 2/N$(namely, $n > N$), then $\text{diam} f(V_n(p)) < \epsilon$. Let $r = \frac{2}{N+1}$, then when $d_X(q,p) < r/2$, for any $q \in X$, since E is dense in X, q is either a point of E, or is a limit point of E. If q is a point of E, $q \in V_{N+1}(p)$ and $g(q) = f(q) \in f(V_{N+1}(p))$, then $|g(q) - g(p)| < \epsilon$, since $\text{diam} f(V_{N+1}(p)) < \epsilon$ and $g(p) \in f(V_n(p))$,

for every n. If q is a limit point of E, then $g(q) \in f(V_n(q))$, for every n. Pick an $x \in E$ such that $d_X(x,q) < r/2$(this can be done since q is a limit point of E), then $d_X(x,p) \leq d_X(x,q) + d_X(p,q) < r/2 + r/2 = r$. Therefore, $|g(x) - g(q)| < \epsilon$(since $d_X(x,q) < r/2$ implies that $x \in V_{N+1}(q)$ and diam$V_{N+1}(q) < \frac{2}{N+1}$ implies diam$f(V_{N+1}(q)) < \epsilon$, because f is uniformly continuous on E) and $|g(x) - g(p)| < \epsilon$. Hence $|g(q) - g(p)| \leq |g(q) - g(x)| + |g(x) - g(p)| < 2\epsilon$ and thus g is continuous.
So g is the desired extension of f.
Remark1: Could the range space $\mathbb{R}^1$ be replaced by $\mathbb{R}^k$? By any compact metric space? By any complete metric space? By any metric space?
The case of $\mathbb{R}^k$ and compact metric space both hold, since the key steps involving "f is real" in the previous proof is that:
In $\mathbb{R}^1$, closedness and boundedness imply compactness, thus $\overline{V_n(p)}$ is compact and Theorem 3.10(b) can be applied.
Note that if R^1 is replaced by $\mathbb{R}^k$, nothing is new, since the Heine-Borel Theorem tells us in $\mathbb{R}^k$, closedness with boundedness is equivalent to compactness, and the proof process is the same. If $\mathbb{R}^1$ is replaced by compact metric space , the compactness also results, since closed subsets of compact space are compact.
If $\mathbb{R}^1$ is replaced by any complete metric space, recall Exercise 21 of Chapter 3: If $\{E_n\}$ is a sequence of closed nonempty and bounded sets in a *complete* metric space X, if $E_n \supseteq E_{n+1}$, and if $\lim\limits_{n\to\infty} \text{diam}E_n = 0$, then $\bigcap_1^\infty E_n$ consists of exactly one point.
We thus can conclude that if $\mathbb{R}^1$ is replaced by any complete metric space, the result also holds.
If $\mathbb{R}^1$ is replaced by any metric space, the result cannot always hold, since $\bigcap_1^n E_n$ may be empty. For example, consider the metric space $(\mathbb{Q}, d)$, where $d(p,q) = |p-q|$, for any $p, q \in \mathbb{Q}$. $\mathbb{Q}$ is not complete, by the remark under Definition 3.12; and closedness and boundedness not implies compactness, by Exercise 16 of Chapter 2.
Remark2: We can also use the result of Exercise 11 to give an alternative proof. This proof may be easier, and the cases stated in **Remark1** will be more clear under this circumstance. Next, we give out this alternative proof:
Suppose $p \in X$, if $p \in E$, we define $g(p) = f(p)$ and if $p \notin E$, since E is dense in X, p is a limit point of E. Let $\{p_n\}$ be any sequence of E which converges to p, then $\{p_n\}$ is a Cauchy sequence in E, since f is uniformly continuous on E, $\{f(p_n)\}$ is a Cauchy sequence in $\mathbb{R}^1$. Then $\{f(p_n)\}$ will converge to some $q \in \mathbb{R}^1$, since $\mathbb{R}^1$ is complete. We define $g(p) = q$. Firstly, we will show that g is well-defined. That is, if $s_n \to p$ and $t_n \to p$, then $\lim\limits_{n\to\infty} f(s_n) = \lim\limits_{n\to\infty} f(t_n) = q$. Given any $\epsilon > 0$, since f is uniformly continuous, there is a $\delta > 0$ such that $d_X(a,b) < \delta$ implies that $|f(a) - f(b)| < \epsilon$, for every $a, b \in E$. Since $s_n \to p$, there is an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $d_X(s_n, p) < \delta/2$ and since $t_n \to p$, there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $d_X(t_n, p) < \delta/2$. Let $N = \max(N_1, N_2)$,

when $n > N$, we have $d_X(s_n, t_n) \le d_X(s_n, p) + d_X(t_n, p) < \delta$. Hence, $|f(s_n) - f(t_n)| < \epsilon$. Since $f(s_n) \to q$, there is an N_3 such that when $n > N_3$, $|f(s_n) - q| < \epsilon$. Let $N' = \max(N, N_3)$, and when $n > N'$, $|f(t_n) - q| \le |f(s_n) - q| + |f(s_n) - f(t_n)| < 2\epsilon$. Therefore, $f(t_n) \to q$.
Now, we need to prove that g is continuous on X.
Let $p \in X$,
(i) If $p \in E$, then $g(p) = f(p)$. Given any $\epsilon > 0$, there is a $\delta > 0$, $d_X(a, b) < \delta$ implies $|f(a) - f(b)| < \epsilon/2$, for any $a, b \in E$. Let $x \in X$, such that $d_X(x, p) < \delta/2$. If $x \in E$, then $g(x) = f(x)$ and $|g(x) - g(p)| < \epsilon/2 < \epsilon$; if $x \notin E$, then x is a limit point of E. Suppose $s_n \to x$, $s_n \in E$, then $g(x) = \lim\limits_{n\to\infty} f(s_n) = \lim\limits_{n\to\infty} g(s_n)$. Since $s_n \to x$, there is an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $d_X(s_n, x) < \delta/2$. Thus $d_X(s_n, p) \le d_X(s_n, x) + d_X(x, p) < \delta$, and $|g(s_n) - g(p)| = |f(s_n) - f(p)| < \epsilon/2$. On the other hand, since $g(s_n) \to g(x)$, there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|g(s_n) - g(x)| \le \epsilon/2$. Let $N = \max(N_1, N_2)$, then when $n > N$, $|g(x) - g(p)| \le |g(x) - g(s_n)| + |g(s_n) - g(p)| < \epsilon$. Therefore, g is continuous at p.
(ii) If $p \notin E$, then p is a limit point of E. Suppose $t_n \to p$, $t_n \in E$, then $g(p) = \lim\limits_{n\to\infty} f(t_n) = \lim\limits_{n\to\infty} g(t_n)$. Given any $\epsilon > 0$, since f is uniformly continuous on E, there is a $\delta > 0$ such that $d_X(a, b) < \delta$ implies $|f(a) - f(b)| < \epsilon/3$. Let $x \in X$, such that $d_X(x, p) < \delta/3$. Then if $x \in E$, $g(x) = f(x)$, and since $t_n \to p$, there is an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $d_X(t_n, p) < \delta/3$. Thus $d_X(t_n, x) \le d_X(t_n, p) + d_X(x, p) < \delta$ and $|g(t_n) - g(x)| = |f(t_n) - f(x)| < \epsilon/3$. Since $g(t_n) \to g(p)$, there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|g(t_n) - g(p)| < \epsilon/3$. Let $N = \max(N_1, N_2)$, then when $n > N$, $|g(x) - g(p)| \le |g(x) - g(t_n)| + |g(t_n) - g(x)| < \epsilon$. If $x \notin E$, then x is a limit point of E. Suppose $s_n \to x$, $s_n \in E$, then $g(x) = \lim\limits_{n\to\infty} f(s_n) = \lim\limits_{n\to\infty} g(s_n)$. Thus there is an N_1 such that $n > N_1$ implies that $d_X(s_n, x) < \delta/3$. Since $t_n \to p$, there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies that $d_X(t_n, p) < \delta/3$. Let $N = \max(N_1, N_2)$, then when $n > N$, $d_X(s_n, t_n) \le d_X(s_n, x) + d_X(x, p) + d_X(t_n, p) < \delta$, and thus $|g(s_n) - g(t_n)| = |f(s_n) - f(t_n)| < \epsilon/3$. Since $g(s_n) \to g(x)$, there is an $N_3 \in \mathbb{N}$ such that $n > N_3$ implies $|g(s_n) - g(x)| < \epsilon/3$; and sine $g(t_n) \to g(p)$, there is an $N_4 \in \mathbb{N}$ such that $n > N_4$ implies $|g(t_n) - g(p)| < \epsilon/3$. Let $N' = \max(N, N_3, N_4)$, then when $n > N'$, $|g(x) - g(p)| \le |g(x) - g(s_n)| + |g(s_n) - g(t_n)| + |g(t_n) - g(p)| < \epsilon$. Therefore, g is continuous at p.
Combining (i) and (ii), we have shown that g is continuous at every point $p \in X$.
Notes: The key point in the above process involving "f is real" is that $\mathbb{R}^1$ is complete (thus every Cauchy sequence in $\mathbb{R}^1$ converges). Thus, if $\mathbb{R}^1$ is replaced by $\mathbb{R}^k$, or by any compact metric space, or by any complete metric space, the result also holds, according to Theorem 3.11.

14. Let $I = [0, 1]$ be the closed unit interval. Suppose f is a continuous

mapping of I into I. Prove that $f(x) = x$ for at least one $x \in I$.
Proof: Suppose that, on the contrary, $f(x) \neq x$, for all the $x \in I$. Define $g(x) = f(x) - x$, for $x \in I$, then g is continuous, too. Since $f(I) \subseteq I$, $g(0) = f(0) \neq 0$, thus $g(0) > 0$; on the other hand, $g(1) = f(1) - 1 < 0$, since $f(1) \neq 1$, by our assumption. Because g is continuous, and $g(0) > 0 > g(1)$, there is a $x_0 \in I$ such that $g(x_0) = 0$, by Theorem 4.23. Namely, $f(x_0) = x_0$, which is contradict to our assumption. Therefore, $f(x) = x$ for at least one $x \in I$.

15. Call a mapping of X into Y *open* if $f(V)$ is an open set in Y whenever V is an open set in X.
Prove that every continuous open mapping of $\mathbb{R}^1$ into $\mathbb{R}^1$ is monotonic.
Proof: Suppose that, on the contrary, some $f : \mathbb{R}^1 \to \mathbb{R}^1$ is continuous open but not monotonic. Then there is an $x \in \mathbb{R}^1$ and a $\delta > 0$ such that $(f(t) - f(x))(f(s) - f(x)) \geq 0$, for every $t \in (x - \delta, x)$ and $s \in (x, x + \delta)$. (More specifically, since f is open, f cannot be constant and furthermore, f must have infinitely many different values. Since f is not monotonic, there is $x_1 < x_2 < x_3$ such that $(f(x_2) - f(x_1))(f(x_2) - f(x_3)) > 0$. That is, either $f(x_2) > f(x_1)$ and $f(x_2) > f(x_3)$, or $f(x_2) < f(x_1)$ and $f(x_2) < f(x_3)$. Without loss of generality, let x_2 be the first case. Denote $E = [x_1, x_3]$, then since f is continuous, there is a $p \in E$, such that $f(p) = \inf_{x \in E} f(x)$. Because $f(x_2) > f(x_1)$ and $f(x_2) > f(x_3)$, $p \neq x_1$ and $p \neq x_3$. Hence $p \in (x_1, x_3)$, and since (x_1, x_3) is open, there is a $\delta > 0$ such that $(p-\delta, p+\delta) \subseteq (x_1, x_3) \subseteq E$. Obviously, $(f(t)-f(p))(f(s)-f(p)) \geq 0$ (since $f(t) \geq f(p)$ and $f(s) \geq f(p)$), for every $t \in (p - \delta, p)$ and $s \in (p, p + \delta)$. The second case of x_2 can be dealt with similarly.)
Let $V = (x - \delta, x + \delta)$, so V is open. But then $f(V)$ cannot be open, since $f(x) \in f(V)$ and $f(x)$ is not an interior point of $f(V)$. (Otherwise, there is an $\epsilon > 0$ such that $(f(x) - \epsilon, f(x) + \epsilon) \subseteq f(V)$. Since f is continuous, there is a $\delta' > 0$ such that $|t - x| < \delta'$ implies $|f(t) - f(x)| < \epsilon$. Let $r = \min(\delta, \delta')$, then when $|t - x| < r$, either $f(t) \geq f(x)$, for every $t \in (x - r, x + r)$, or $f(t) \leq f(x)$, for every $t \in (x - r, x + r)$. If it is the first case, $f(x) - \epsilon/2 \notin f(V)$ and if it is the second case, $f(x) + \epsilon/2 \notin f(V)$, either gives us a contradiction.)
Remark: We can also show that every continuous open function must be *injective*, and every continuous injective function is strictly monotonic.

16. Let $[x]$ denote the largest integer contained in x, that is, $[x]$ is the integer such that $x - 1 < [x] \leq x$; and let $(x) = x - [x]$ denote the fractional part of x. What discontinuities do the functions $[x]$ and (x) have?
Solution: Clearly, both $[x]$ and (x) are discontinuous at each integer-valued point.

17. Let f be a real function defined on (a, b). Prove that the set of points at which f has a simple discontinuity is at most countable.

Proof:
(i) Let E be the set on which $f(x-) < f(x+)$. With each point of E, associate a triple (p, q, r) of rational numbers such that

(a) $f(x-) < p < f(x+)$;

(b) $a < q < t < x$ implies $f(t) < p$;

(c) $x < t < r < b$ implies $f(t) > p$.

The set of all such triples is countable. We next show that each triple is associated with at most one point of E. Suppose that, on the contrary, there are two point of E are associated with one triple (p, q, r). Let these two points be denoted as x_1 and x_2, without loss of generality, we let $x_1 < x_2$. Then, there is a $t \in (a, b)$ such that $x_1 < t < x_2$, we have the following results:

(a) $f(x_1-) < p < f(x_1+)$, $f(x_2-) < p < f(x_2+)$;

(b) $a < q < t < x_2$ implies $f(t) < p$;

(c) $x_1 < t < r < b$ implies $f(t) > p$.

Obviously, (b) and (c) are contradict, and thus each triple is associated with at most one point of E. Therefore, E is at most countable.
(ii) Let F be the set on which $f(x-) > f(x+)$, with nearly the same procedure as (i), we can prove that F is at most countable (namely, with changes in (a) by $f(x-) > p > f(x+)$, in (b) by $f(t) > p$ and in (c) by $f(t) < p$).
(iii) Let G be the set on which $f(x-) = f(x+) < f(x)$, then we can let the above conditions (a), (b) and (c) be replaced as:

(a) $f(x) = p$;

(b) $a < q < t < x$ implies $f(t) < p$;

(c) $x < t < r < b$ implies $f(t) < p$;

Then if x_1 and x_2 are associated with the same triple (p, q, r), we have $f(x_1) = f(x_2)$ by (a). But (b) tells us $f(x_1) < p$, since $a < q < x_1 < x_2$ and (c) tells us $f(x_2) < p$, since $x_1 < x_2 < r < b$. Both give us contradictions. Thus, each triple is associated with at most one point of G and G is at most countable.
(iv) Let H be the set on which $f(x-) = f(x+) > f(x)$, with the same procedure as (iii), we can prove that H is at most countable (namely, change conditions (b) and (c) by $f(t) > p$).
Combining the results of (i), (ii), (iii), and (iv), we conclude that the set of points at which f has a simple discontinuity is at most countable.

18. Every rational x can be written in the form $x = m/n$, where $n > 0$, and m and n are integers without any common divisors. When $x = 0$, we take

$n = 1$.Consider the function f defined on $\mathbb{R}^1$ by

$$f(x) = \begin{cases} 0 & x \text{ irrational}, \\ \frac{1}{n} & (x = \frac{m}{n}). \end{cases}$$

Prove that f is continuous at every irrational point, and that f has a simple discontinuity at every rational point.

Proof: Fix an irrational number p. Given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $1/N < \epsilon$. Let x be any real number, suppose $x = [x] + (x)$, and $p = [p] + (p)$, we pick an N' so large that if $|x - p| < 1/N'$, then $[x] = [p]$. Let $M = \max(N, N')$, when $|x - p| < 1/M$, we have $|x - p| < 1/M < 1/N$, and $[x] = [p]$ so that $|x - p| = |(x) - (p)| < 1/M$. Since $|(x)| < 1$, if we let $(x) = \frac{m}{n}$, then m can only be one of the $2n - 1$ values $0, \pm 1, ..., \pm(n-1)$. Next, we remove from the neighborhood $V_{1/M}(p)$ of p those rational numbers with divisor smaller than or equal to M. Let $E = \{|q - p| | q \text{ is removed from } V_{1/M}(p)\}$, then E is finite, by our above statements. Thus $\delta = \min E > 0$, $\delta \leq 1/M$, and $V_\delta(p)$ contains only irrational numbers and those rational numbers with divisor larger than M. Let $x \in V_\delta(p)$ (namely, $|x - p| < \delta$), if x is irrational, $f(x) = 0$ and $|f(x) - f(p)| = 0 < \epsilon$; and if x is rational, let $x = m/n$, we have $|f(x) - f(p)| = |f(x)| = 1/n < 1/M < 1/N < \epsilon$ and therefore f is continuous at x.

Let $p = \frac{m}{n}$ be any rational number, then given any $\epsilon > 0$, if we apply the skill in the previous proof process, we can obtain a neighborhood $V_\delta(p)$(except p) which contains only irrational numbers and rational numbers with divisor larger than M, where $1/M < \epsilon$. Thus $0 \leq f(x) < \epsilon$, when $x \in V_\delta(p)$ and $x \neq p$, and we have $0 \leq f(p+) < \epsilon$, $0 \leq f(p-) < \epsilon$. Since ϵ is arbitrary, we have $f(p+) = 0$ and $f(p-) = 0$, but $f(p) = 1/n \neq 0$. Therefore, f has a simple discontinuity at every rational point.

19. Suppose f is a real function with domain $\mathbb{R}^1$ which has the intermediate value property: If $f(a) < c < f(b)$, then $f(x) = c$ for some x between a and b.

    Suppose also, for every rational r, that the set of all x with $f(x) = r$ is closed. Prove that f is continuous.

    Proof: If $x_n \to x_0$ but $f(x_n) > r > f(x_0)$ for some r and all n, then $f(t_n) = r$ for some t_n between x_0 and x_n, since f has the intermediate value property. Thus $t_n \to x_0$. But then x_0 is a limit point of the set E of all x with $f(x) = r$, and by assumption, E is closed. Therefore, $x_0 \in E$ and $f(x_0) = r$, which is contradict to the assumption that $f(x_0) < r$. Hence, f must be continuous.

20. If E is a nonempty set of a metric space X, define the distance from $x \in X$ to E by $\rho_E(x) = \inf_{z \in E} d(x, z)$.

    (a) Prove that $\rho_E(x) = 0$ if and only if $x \in \bar{E}$.

    Proof:

$\Rightarrow$: Suppose that $\rho_E(x) = 0$, and $x \notin \bar{E}$. Then $x \in \bar{E}^c$. Since $\bar{E}$ is closed, $\bar{E}^c$ is open. Thus there exists an $r > 0$, such that $d(y, x) < r$ implies $y \in \bar{E}^c$ and $y \notin \bar{E}$. Then $d(z, x) \geq r$, for every $z \in E$ and $\inf_{z \in E} d(x, z) \geq r > 0$, namely, $\rho_E(x) > 0$, a contradiction.
$\Leftarrow$: Suppose that $x \in \bar{E}$. If $x \in E$, then since $d(x, x) = 0$, $\rho_E(x)$ is trivially 0; If $x \notin E$, then x is a limit point of E, thus given any $\epsilon > 0$, there is a $z \in E$ such that $d(z, x) < \epsilon$. We conclude that $\rho_E(x) = \inf_{z \in E} d(x, z) = 0$, since every $\epsilon > 0$ is not a lower bound of $d(x, z)$.

(b) Prove that ρ_E is a uniformly continuous function on X, by showing that $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$ for all $x \in X$, $y \in X$.
Proof: Fix x, y, we have $\rho_E(x) \leq d(x, z) \leq d(x, y) + d(y, z)$, for every $z \in E$, thus $\rho_E(x) \leq d(x, y) + \rho_E(y)$ and $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Similarly, we have $\rho_E(y) - \rho_E(x) \leq d(x, y)$ and therefore $|\rho_E(x) - \rho_E(y)| \leq d(x, y)$. Hence ρ_E is uniformly continuous.

21. Suppose K and F are disjoint sets in a metric space X, K is compact, F is closed. Prove that there exists $\delta > 0$ such that $d(p, q) > \delta$ if $p \in K$, $q \in F$.
Show that the conclusion may fail for two disjoint closed sets if neither is compact.
Proof: First, we will show that $\rho_F(p) \neq 0$, for every $p \in K$. Suppose that, on the contrary, there is a $q \in K$ such that $\rho_F(q) = 0$. Then by Exercise 20(a), $q \in \bar{F}$. Since F is closed, $\bar{F} = F$ and $q \in F$, which is not possible because K and F are disjoint, by assumption. Thus, for every $p \in K$, $\rho_F(p) > 0$. Furthermore, by Exercise 20(b), we have ρ_F is continuous on K, and since K is compact, let $m = \inf_{p \in K} \rho_F(p)$, then there exists a point $q \in K$, such that $\rho_F(q) = m$, by Theorem 4.15. Since $\rho_F(q) > 0$, we have $m > 0$. Pick any δ such that $0 < \delta < m$, we have $d(p, q) \geq \rho_F(p) \geq m > \delta$, for every $p \in K$, $q \in F$.
If two disjoint sets are both closed but neither is compact, then the conclusion may fail. For example, let E be the set of all positive integers, and let F be the set of $\{n + \frac{1}{n}\}$, $n \geq 2$, then both E and F are closed (in fact, E and F both have no limit points). Clearly, since $\lim_{n \to \infty} d(p_n, q_n) = \lim_{n \to \infty} \frac{1}{n} = 0$, for $p_n \in E$, $q_n \in F$, no $\delta > 0$ can be found such that $d(p, q) > \delta$ for every $p \in E$, $q \in F$.

22. Let A and B be disjoint nonempty closed sets in a metric space X, and define

$$f(p) = \frac{\rho_A(p)}{\rho_A(p) + \rho_B(p)} (p \in X).$$

Show that f is a continuous function on X whose range lies in $[0, 1]$, that $f(p) = 0$ precisely on A and $f(p) = 1$ precisely on B. This establishes a converse of Exercise 3: Every closed set $A \subseteq X$ is $Z(f)$ for some continuous

real f on X. Setting $V = f^{-1}([0, \frac{1}{2}))$, $W = f^{-1}((\frac{1}{2}, 1])$, show that V and W are open and disjoint, and that $A \subseteq V$, $B \subseteq W$. (Thus pairs of disjoint closed sets in a metric space can be covered by pairs of disjoint open sets. This property of metric spaces is called *normality.*)
Proof:
(i) We first prove that $\rho_A(p) + \rho_B(p) \neq 0$, for every $p \in X$. Suppose that, on the contrary, there is a $q \in X$ such that $\rho_A(q) + \rho_B(q) = 0$, then $\rho_A(q) = 0$ and $\rho_B(q) = 0$. Thus $q \in \bar{A}$ and $q \in \bar{B}$, by Exercise 20(a). Then $q \in A$ and $q \in B$, since both A and B are closed, which is contradict to our assumption that A and B are disjoint.
The fact that f is continuously then comes directly from Exercise 20(b) and Theorem 4.9. Since $0 \leq \rho_A(p) \leq \rho_A(p) + \rho_B(p)$, for every $p \in X$, it's clear that $f(X) \subseteq [0, 1]$.
$f(p) = 0 \Leftrightarrow \rho_A(p) = 0 \Leftrightarrow p \in \bar{A} \Leftrightarrow p \in A$.
$f(p) = 1 \Leftrightarrow \rho_A(p) = \rho_A(p) + \rho_B(p) \Leftrightarrow \rho_B(p) = 0 \Leftrightarrow p \in \bar{B} \Leftrightarrow p \in B$.
(ii) Fix any $p \in V$, then $f(p) \in [0, \frac{1}{2})$, which gives $0 \leq \rho_A(p) < \rho_B(p)$.
If $\rho_A(p) \neq 0$, we pick an $r \in \mathbb{R}^1$ such that $0 < \rho_A(p) < r < \rho_B(p)$, then since ρ_A is uniformly continuous on X (by Exercise 20(b)), there is a $\delta_1 > 0$ such that $d_X(x, p) < \delta_1$ implies that $0 < \rho_A(x) < r$; and since ρ_B is uniformly continuous on X, there is a $\delta_2 > 0$ such that $d_X(x, p) < \delta_2$ implies that $\rho_B(x) > r$. Let $\delta = \min(\delta_1, \delta_2)$, then when $d_X(x, p) < \delta$, we have $0 < \rho_A(x) < r < \rho_B(x)$ and thus $0 < f(x) < \frac{1}{2}$. So $f(x) \in (0, \frac{1}{2})$ and $x \in V$.
If $\rho_A(p) = 0$ ($f(p) = 0$), since f is continuous on X, there is a $\delta > 0$ such that $d_X(x, p) < \delta$ implies $|f(x) - f(p)| < \frac{1}{2}$, namely $0 \leq f(x) < \frac{1}{2}$. Thus, $x \in V$.
Therefore, V is open. In just the same way, we can prove that W is open (this case the condition becomes $0 \leq \rho_B(p) < \rho_A(p)$. Next, we will prove that V and W are disjoint. If $V \cap W \neq \emptyset$, there is a $p \in X$ such that $p \in V$ and $p \in W$, which gives $f(p) \in [0, \frac{1}{2})$ and $f(p) \in (\frac{1}{2}, 1]$, a contradiction. Therefore, V and W are disjoint.
The fact $A \subseteq V$ and $B \subseteq W$ are clear, since $x \in A$ implies $f(x) = 0$ and $x \in B$ implies $f(x) = 1$.

23. A real-valued function f defined in (a, b) is said to be *convex* if $f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$ whenever $a < x < b$, $a < y < b$, $0 < \lambda < 1$. Prove that every convex function is continuous. Prove that every increasing convex function of a convex function is convex. (For example, if f is convex, so is e^f.)
If f is convex in (a, b) and if $a < s < t < u < b$, show that

$$\frac{f(t) - f(s)}{t - s} \leq \frac{f(u) - f(s)}{u - s} \leq \frac{f(u) - f(t)}{u - t}$$

.

Proof: Suppose that, on the contrary, f is not continuous at some $p \in (a, b)$. Then for this p, without loss of generality, there is some sequence

$\{x_n\}$ such that $x_n \to p$ when $n \to \infty$, but $f(x_n) > r > f(p)$ (and thus if we let $r < r_1 < f(x_n)$, $r > r_2 > f(p)$ and $\epsilon = r_1 - r_2 > 0$, then $f(x_n) - f(p) > r_1 - r_2 = \epsilon > 0$), for some r and all n. Then $x_n = \lambda_n p + (1-\lambda_n)x_1$, $0 < \lambda_n < 1$, for $n > 1$. $f(x_n) - f(p) = f(\lambda_n p + (1-\lambda_n)x_1) - f(p) \leq \lambda_n f(p) + (1-\lambda_n)f(x_1) - f(p) = (1-\lambda_n)(f(x_1) - f(p))$, which gives $\epsilon < f(x_n) - f(p) \leq (1-\lambda_n)(f(x_1) - f(p))$. Notice that when $x_n \to p$, $\lambda_n \to 1$ and $1 - \lambda_n \to 0$. Hence $f(x_n) - f(p) \to 0$ when $n \to \infty$, namely, $\epsilon \leq 0$, which is absurd. The case $f(x_n) < r < f(p)$ will be similar, by showing $0 \geq \epsilon$ to get a contradiction. Therefore, f must be continuous.(**Remark**: Recall the method used in Exercise 19 again.)

Suppose f is convex, and g is increasing convex. Let $h(x) = g(f(x))$, then $h(\lambda x + (1-\lambda)y) = g(f(\lambda x + (1-\lambda)y)) \leq g(\lambda f(x) + (1-\lambda)f(y)) \leq \lambda g(f(x)) + (1-\lambda)g(f(y)) = \lambda h(x) + (1-\lambda)h(y)$. Therefore, h is convex.

We have $t = \frac{u-t}{u-s}s + \frac{t-s}{u-s}u$, and since $\frac{u-t}{u-s} + \frac{t-s}{u-s} = 1$, if we let $\lambda(t) = \frac{u-t}{u-s}$, then $0 < \lambda(t) < 1$ for any t since $s < t < u$, and $t = \lambda(t)s + (1-\lambda(t))u$. Since f is convex, we have $f(t) \leq \lambda(t)f(s) + (1-\lambda(t))f(u)$, which gives us $(u-s)f(t) \leq (u-t)f(s) + (t-s)f(u)$. Then we have $(u-s)f(t) \leq (u-s)f(s) - (t-s)f(s) + (t-s)f(u) \Rightarrow (u-s)(f(t) - f(s)) \leq (t-s)(f(u) - f(s)) \Rightarrow \frac{f(t)-f(s)}{t-s} \leq \frac{f(u)-f(s)}{u-s}$, and $(u-s)f(t) \leq (u-t)f(s) + (u-s)f(u) - (u-t)f(u) \Rightarrow (u-t)(f(u) - f(s)) \leq (u-s)(f(u) - f(t)) \Rightarrow \frac{f(u)-f(s)}{u-s} \leq \frac{f(u)-f(t)}{u-t}$. Combining these two result gives us the desired inequality.

24. Assume that f is a continuous real function defined in (a, b) such that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$ for all $x, y \in (a, b)$. Prove that f is convex.
Proof: First, we will prove that if $\lambda = \frac{1}{2^n}$, $n = 1, 2, \ldots$, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. We can prove this by induction:
(i) $n = 1$, $\lambda = \frac{1}{2}$, then it is trivial that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$.
(ii) Suppose that when $n = k$, $\lambda = \frac{1}{2^k}$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. When $n = k+1$, we have $f(\frac{1}{2^{k+1}}x + (1 - \frac{1}{2^{k+1}})y) = f(\frac{(\frac{x}{2^k} + (1-\frac{1}{2^k})y) + y}{2}) \leq \frac{f(\frac{x}{2^k} + (1-\frac{1}{2^k})y) + f(y)}{2} \leq \frac{1}{2}[(\frac{1}{2^k}f(x) + (1 - \frac{1}{2^k})f(y)) + f(y)]$(by hypothesis)$= \frac{1}{2^{k+1}}f(x) + (1 - \frac{1}{2^{k+1}})f(y)$.
Furthermore, we can similarly prove that if $\lambda = \sum_{i=1}^{n} \frac{a_i}{2^i}$, where $a_i = 0$ or $a_i = 1$, then $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$ (*):
(i) $n = 1$, $\lambda = \frac{a_1}{2}$. Since $\lambda > 0$, $a_1 = 1$ and $\lambda = \frac{1}{2}$, then it is trivial that $f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$.
(ii) Suppose that when $n = k$, $\lambda = \sum_{i=1}^{k} \frac{a_i}{2^i}$, $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$. When $n = k+1$, $\lambda = \sum_{i=1}^{k+1} \frac{a_i}{2^i}$.
If $\sum_{i=1}^{k+1} \frac{a_i}{2^{i-1}} < 1$, we have

$f(\sum_{i=1}^{k+1}\frac{a_i}{2^i}x+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^i})y) = f(\frac{[\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}}x+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}})y]+y}{2})$.

Since $\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} < 1$, we must have $a_1 = 0$, then

$\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} = \sum_{i=0}^{k}\frac{a_{i+1}}{2^i} = \sum_{i=1}^{k}\frac{a_{i+1}}{2^i}$ and hence

$f(\frac{[\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}}x+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}})y]+y}{2}) = f(\frac{[\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}x+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})y]+y}{2})$

$\le \frac{f(\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}x+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})y)+f(y)}{2} \le \frac{\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}f(x)+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})f(y)+f(y)}{2}$

(by hypothesis)$= \sum_{i=1}^{k}\frac{a_{i+1}}{2^{i+1}}f(x)+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^{i+1}})f(y) = \sum_{i=2}^{k+1}\frac{a_i}{2^i}f(x)+(1-\sum_{i=2}^{k+1}\frac{a_i}{2^i})f(y) = \sum_{i=1}^{k+1}\frac{a_i}{2^i}f(x)+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^i})f(y)$, since $a_1 = 0$.

If $\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} > 1$, we have

$f(\sum_{i=1}^{k+1}\frac{a_i}{2^i}x+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^i})y) = f(\frac{[(\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}}-1)x+(2-\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}})y]+x}{2})$.

Since $\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} > 1$, we must have $a_1 = 1$, then $\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} = \sum_{i=0}^{k}\frac{a_{i+1}}{2^i} = 1+\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}$ (notice that $\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}} < 2$) and hence

$f(\frac{[(\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}}-1)x+(2-\sum_{i=1}^{k+1}\frac{a_i}{2^{i-1}})y]+x}{2}) = f(\frac{[\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}x+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})y]+x}{2})$

$\le \frac{f(\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}x+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})y)+f(x)}{2} \le \frac{\sum_{i=1}^{k}\frac{a_{i+1}}{2^i}f(x)+(1-\sum_{i=1}^{k}\frac{a_{i+1}}{2^i})f(y)+f(x)}{2}$

(by hypothesis)$= (\sum_{i=1}^{k}\frac{a_{i+1}}{2^{i+1}}+\frac{1}{2})f(x)+(\frac{1}{2}-\sum_{i=1}^{k}\frac{a_{i+1}}{2^{i+1}})f(y) = (\sum_{i=2}^{k+1}\frac{a_i}{2^i}+\frac{1}{2})f(x)+(\frac{1}{2}-\sum_{i=2}^{k+1}\frac{a_i}{2^i})f(y) = \sum_{i=1}^{k+1}\frac{a_i}{2^i}f(x)+(1-\sum_{i=1}^{k+1}\frac{a_i}{2^i})f(y)$, since $a_1 = 1$.

Now, suppose that f is not convex, then there is some $x_0, y_0 \in (a,b)$ and some $0 < \lambda_0 < 1$, such that $f(\lambda_0 x_0+(1-\lambda_0)y_0) > \lambda_0 f(x_0)+(1-\lambda_0)f(y_0)$. Let $g(\lambda) = f(\lambda x_0 + (1-\lambda)y_0) - (\lambda f(x_0) + (1-\lambda)f(y_0))$, then we have $g(\lambda_0) > 0$ and $g(\lambda) \le 0$, for every $\lambda = \sum_{i=1}^{n}\frac{a_i}{2^i}$, $a_i = 0$ or $a_i = 1$, $n = 1, 2, ...$ Since $f(p)$ is continuous, and $p = h(\lambda) = \lambda x_0 + (1-\lambda)y_0$ is continuous, $g(\lambda)$ is continuous. Let $0 < r < g(\lambda_0)$ and $\epsilon = g(\lambda_0) - r > 0$, then there is a $\delta > 0$ such that $|\lambda - \lambda_0| < \delta$ implies $|g(\lambda) - g(\lambda_0)| < \epsilon$, and thus $g(\lambda) > g(\lambda_0) - \epsilon = r > 0$. But, for this δ, we can pick an $N \in \mathbb{N}$ such that $n \ge N$ implies $\frac{1}{2^n} < \delta$. Then choose $\lambda_1 = \sum_{i=1}^{n}\frac{a_i}{2^i}$ such that $n > N$ and $|\lambda_1 - \lambda_0| < \frac{1}{2^N} < \delta$, we have $g(\lambda_1) > 0$, which is a contradiction.

Therefore, f must be convex.
Remark: Actually, every $0 < \lambda < 1$ can be represented as the form $\lambda = \sum_{i=1}^{\infty} \frac{a_i}{2^i}$, where $a_i = 0$ or $a_i = 1$, namely, the *binary* representation of λ. Hence, the result (*) can be generalized in the case when $n \to \infty$.

25. If $A \subseteq \mathbb{R}^k$ and $B \subseteq \mathbb{R}^k$, define $A + B$ to be the set of all sums $\vec{x} + \vec{y}$ with $\vec{x} \in A$, $\vec{y} \in B$.

    (a) If K is compact and C is closed in $\mathbb{R}^k$, prove that $K + C$ is closed.
    Proof: Take $\vec{z} \notin K + C$, put $F = \vec{z} - C$, the set of all $\vec{z} - \vec{y}$ with $\vec{y} \in C$. Then K and F are disjoint.(Otherwise, there is some $\vec{x} \in K$ and $\vec{x} \in F$. $\vec{x} \in F$ implies that $\vec{z} - \vec{x} \in C$, and since $\vec{x} \in K$, $\vec{z} = (\vec{z} - \vec{x}) + \vec{x} \in C + K$, contradicting to our choice of $\vec{z}$.) Since C is closed, F is closed. (Let $\vec{p}$ be a limit point of F, given any $\epsilon > 0$, then there is a $\vec{x}$ in F such that $|\vec{x} - \vec{p}| < \epsilon$. Since $\vec{x} \in F$, $\vec{x} = \vec{z} - \vec{y}$, for some $\vec{y} \in C$. Thus $|(\vec{z} - \vec{p}) - \vec{y}| < \epsilon$, which shows that $\vec{z} - \vec{p}$ is a limit point of C. Then $\vec{z} - \vec{p} \in C$, since C is closed. Therefore, $\vec{p} \in F$ and F is closed.) Because K is compact, by Exercise 21, there exists a $\delta > 0$ such that $|\vec{p} - \vec{q}| > \delta$ if $\vec{p} \in K$, $\vec{q} \in F$. Now let V be the open ball with center $\vec{z}$ and radius δ, we will prove that V does not intersect $K + C$.
    If, on the contrary, there is some $\vec{p}$ such that $\vec{p} \in V$ and $\vec{p} \in K + C$. Then $|\vec{p} - \vec{z}| < \delta$, and since $\vec{p} \in K + C$, $\vec{p} = \vec{a} + \vec{b}$, with $\vec{a} \in K$ and $\vec{b} \in C$. Thus $|\vec{a} + \vec{b} - \vec{z}| < \delta$, which gives $|\vec{a} - (\vec{z} - \vec{b})| < \delta$. Since $\vec{b} \in C$, we have $\vec{z} - \vec{b} \in F$ and obtain a contradiction to our choice of δ. Therefore, V does not intersect $K + C$ and we have shown that $(K + C)^c$ is open. Hence $K + C$ is closed.

    (b) Let α be an irrational real number. Let C_1 be the set of all integers, let C_2 be the set of all $n\alpha$ with $n \in C_1$. Show that C_1 and C_2 are closed subsets of $\mathbb{R}^1$ whose sum $C_1 + C_2$ is *not* closed, by showing that $C_1 + C_2$ is a countable dense subset of $\mathbb{R}^1$.
    Proof: The fact that C_1 and C_2 are closed is clear since neither C_1 nor C_2 has a limit point in $\mathbb{R}^1$. The fact that $C_1 + C_2$ is countable is also obvious, since both C_1 and C_2 are countable.
    Now, we need to prove that $C_1 + C_2$ is dense in $\mathbb{R}^1$. First, we will show that given any $\epsilon > 0$, there is m, n such that $|m\alpha + n| < \epsilon \Leftrightarrow |(m\alpha)| < \epsilon$. Since there is some N such that $n > N$ implies $1/n < 1/N < \epsilon$, we can divide the interval $[0, 1)$ into N segments, namely, $[0, 1/N)$, $[1/N, 2/N)$,..., $[(N-1)/N, 1)$. Since $(m_1\alpha) \neq (m_2\alpha)$, if $m_1 \neq m_2$, there is at least two integers $m_1, m_2 \in \{1, 2, ..., N+1\}$ such that $(m_1\alpha)$ and $(m_2\alpha)$ belong to the same segment. Let $m' = m_1 - m_2$, then $|(m'\alpha)| = |(m_1\alpha) - (m_2\alpha)| < 1/N < \epsilon$.
    Next, we let $\delta = (m'\alpha) = m'\alpha + k > 0$, without loss of generality, for some integer k. Suppose $p > 0$ be any positive real number (the

case $p < 0$ will be similar), according to the archimedean property of $\mathbb{R}^1$, there is an integer m_3 such that $m_3\delta > p$ and there is an integer m_4 such that $m_4(1/p) > 1/\delta$, namely, $p < m_4\delta$. Thus $m_3\delta < p < m_4\delta$, and we can find a m_5 such that $m_5\delta \le p < (m_5+1)\delta$. Let $\beta = m_5\delta = m_5 m'\alpha + m_5 k = m\alpha + n$(namely, $m = m_5 m'$ and $n = m_5 k$), then $|\beta - p| < \delta < \epsilon$. Since $\beta \in C_1 + C_2$, we have shown that $C_1 + C_2$ is dense. The fact that $C_1 + C_2$ is not closed then is clear, since otherwise every number of $\mathbb{R}^1$ is a limit point of $C_1 + C_2$ and thus $C_1 + C_2 = \mathbb{R}^1$, which is contradict to the fact that $C_1 + C_2$ is countable.

26. Suppose X, Y, Z are metric spaces, and Y is compact. Let f map X into Y, let g be a continuous one-to-one mapping of Y into Z, and put $h(x) = g(f(x))$ for $x \in X$.
Prove that f is uniformly continuous if h is uniformly continuous.
Prove also that f is continuous if h is continuous.
Show (by modifying Example 4.21, or by finding a different example) that the compactness of Y cannot be omitted from the hypotheses, even when X and Z are compact.
Proof:
(i) Since Y is compact and g is continuous, $g(Y)$ is compact. On the other hand, g is one-to-one and continuous implies $g^{-1} : g(Y) \to Y$ is continuous, by Theorem 4.17, and thus g^{-1} is uniformly continuous. Since $f(x) = g^{-1}(h(x))$, for every $x \in X$, f is therefore uniformly continuous if h is uniformly continuous, by Exercise 12.
(ii) g^{-1} is continuous, thus if h is continuous, then $h = g^{-1}f$ is also continuous, by Theorem 4.7.
(iii)As in Example 4.21, let $X = [0, 2\pi]$, $Y = [0, 2\pi)$ and Z be the unit circle on the plane. Suppose $f : X \to Y$ such that $f(x) = x$, for $x \in [0, 2\pi)$, and $f(2\pi) = 0$; $g : Y \to Z$ such that $g(y) = (\cos y, \sin y)$, for any $y \in [0, 2\pi)$; $h : X \to Z$, $h(x) = (\cos x, \sin x)$, for any $x \in [0, 2\pi)$, and $h(2\pi) = (1, 0)$.
We can easily check that $h(x) = g(f(x))$, for every $x \in X$. Furthermore, $h(x)$ is uniformly continuous (since $|h(x) - h(y)| = \sqrt{(\cos x - \cos y)^2 + (\sin x - \sin y)^2} = \sqrt{2(1 - (\cos x \cos y + \sin x \sin y))} = \sqrt{2(1 - \cos(x - y))} = 2\sqrt{(\sin \frac{x-y}{2})^2} = 2|\sin \frac{x-y}{2}| \le 2\frac{|x-y|}{2} = |x - y|$, and h is continuous at 2π). But clearly, f is not continuous, even both X and Z are compact.

5 Differentiation

1. Let f be defined for all real x, and suppose that $|f(x) - f(y)| \le (x - y)^2$ for all real x and y. Prove that f is constant.
Proof: Fix any real x, and let $\phi(t) = \frac{f(t)-f(x)}{t-x}$. Then we have $0 \le |\phi(t)| =$

$\frac{|f(t)-f(x)|}{|t-x|} \leq \frac{(t-x)^2}{|t-x|} = |t-x|$, and hence $0 \leq \lim_{t \to x} |\phi(t)| \leq \lim_{t \to x} |t-x| = 0$. Therefore, $\lim_{t \to x} |\phi(t)| = 0$ and thus $\lim_{t \to x} \phi(t) = 0$. This is equivalent to say, $f'(x) = \lim_{t \to x} \phi(t) = 0$, for any real x. So f must be constant, by Theorem 5.11(b).

2. Suppose $f'(x) > 0$ in (a, b). Prove that f is strictly increasing in (a, b), and let g be its inverse function. Prove that g is differentiable and that

$$g'(f(x)) = \frac{1}{f'(x)} \qquad (a < x < b).$$

Proof: Suppose that $x_1, x_2 \in (a, b)$ and $x_1 < x_2$, then $f(x_1) - f(x_2) = f'(\theta)(x_1 - x_2)$, where $\theta \in (x_1, x_2)$. Since $f'(\theta) > 0$ and $x_1 < x_2$, we have $f(x_1) < f(x_2)$, therefore f is strictly increasing in (a, b).
Fix $y = f(x)$, and let $s = f(t)$. Define $\phi(s) = \frac{g(s)-g(y)}{s-y} = \frac{g(f(t))-g(f(x))}{f(t)-f(x)} = \frac{t-x}{f(t)-f(x)} = \frac{1}{\frac{f(t)-f(x)}{t-x}}$. Since f is differentiable, f is continuous; furthermore, f is strictly increasing and thus is one-to-one. Hence $s \to y$ implies $t \to x$ and then $\lim_{s \to y} \phi(s) = \lim_{t \to x} \frac{1}{\frac{f(t)-f(x)}{t-x}} = \frac{1}{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}} = \frac{1}{f'(x)}$. Therefore, g is differentiable, and $g'(f(x)) = g'(y) = \lim_{s \to y} \phi(s) = \frac{1}{f'(x)}$.

3. Suppose g is a real function on $\mathbb{R}^1$, with bounded derivative (say $|g'| \leq M$). Fix $\epsilon > 0$, and define $f(x) = x + \epsilon g(x)$. Prove that f is one-to-one if ϵ is small enough. (A set of admissible values of ϵ can be determined which depends only on M.)
Proof: Suppose that $x_1, x_2 \in \mathbb{R}^1$ and $x_1 < x_2$, then $f(x_1) = x_1 + \epsilon g(x_1)$, $f(x_2) = x_2 + \epsilon g(x_2)$ and $f(x_1) - f(x_2) = (x_1 - x_2) + \epsilon(g(x_1) - g(x_2)) = (x_1 - x_2) + \epsilon g'(\theta)(x_1 - x_2) = (x_1 - x_2)(1 + \epsilon g'(\theta))$, where $\theta \in (x_1, x_2)$. Since $|g'| \leq M$, if $M = 0$, then $g' = 0$ so $f(x_1) - f(x_2) = x_1 - x_2$ and for any $\epsilon > 0$, $f(x_1) \neq f(x_2)$, if $x_1 \neq x_2$. Thus f is one-to-one. Now consider the case $M > 0$, we have $-M \leq g' \leq M$, and $1 - \epsilon M \leq 1 + \epsilon g'(\theta) < 1 + \epsilon M$. We can pick ϵ such that $0 < \epsilon < \frac{1}{M}$, then $1 - \epsilon M > 0$. Since $x_1 \neq x_2$, $(x_1 - x_2)(1 + \epsilon g'(\theta)) \neq 0$ so $f(x_1) \neq f(x_2)$ and thus f is one-to-one, when $0 < \epsilon < \frac{1}{M}$.

4. If

$$C_0 + \frac{C_1}{2} + \cdots + \frac{C_{n-1}}{n} + \frac{C_n}{n+1} = 0.$$

where $C_0, ..., C_n$ are real constants, prove that the equation $C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n = 0$ has at least one real root between 0 and 1.
Proof: Define $f(x) = C_0 x + \frac{C_1}{2} x^2 + \cdots + \frac{C_{n-1}}{n} x^n + \frac{C_n}{n+1} x^{n+1}$, and let $g(x) = C_0 + C_1 x + \cdots + C_{n-1} x^{n-1} + C_n x^n$. Then we have $f(0) = f(1) = 0$, and $f'(x) = g(x)$. Since $f(1) - f(0) = f'(\theta)(1 - 0) = f'(\theta) = g(\theta)$, where $\theta \in (0, 1)$, and since $f(1) - f(0) = 0$, we have $g(\theta) = 0$, for some $\theta \in (0, 1)$, which is exactly the required conclusion.

5. Suppose f is defined and differentiable for every $x > 0$, and $f'(x) \to 0$ as $x \to +\infty$. Put $g(x) = f(x+1) - f(x)$. Prove that $g(x) \to 0$ as $x \to +\infty$.
Proof: $g(x) = f(x+1) - f(x) = f'(\theta)((x+1) - x) = f'(\theta)$, where $\theta \in (x, x+1)$. When $x \to +\infty$, $\theta \to +\infty$, too, and thus $f'(\theta) \to 0$. Therefore, $g(x) = f'(\theta) \to 0$.

6. Suppose

   (a) f is continuous for $x \geq 0$,

   (b) $f'(x)$ exists for $x > 0$,

   (c) $f(0) = 0$,

   (d) f' is monotonically increasing.

   Put
$$g(x) = \frac{f(x)}{x} \qquad (x > 0)$$
and prove that g is monotonically increasing.
Proof: Let $h(x) = x$, then both f and h are continuous real functions on $[0, +\infty)$ which are differentiable in $(0, +\infty)$. Given any $x > 0$, by the *generalized mean value theorem*, we have $(f(x) - f(0))h'(\theta) = (h(x) - h(0))f'(\theta)$, where $\theta \in (0, x)$. Since $f(0) = 0$, this gives $f(x) = xf'(\theta)$, i.e., $f'(\theta) = \frac{f(x)}{x}$. Since f' is monotonically increasing and $\theta < x$, we have $f'(x) > f'(\theta) = \frac{f(x)}{x}$, i.e., $xf'(x) > f(x)$ for any $x > 0$. Then $g'(x) = \frac{f'(x)x - f(x)}{x^2} > 0$ and hence g is monotonically increasing, by Exercise 2.

7. Suppose $f'(x), g'(x)$ exist, $g'(x) \neq 0$, and $f(x) = g(x) = 0$. Prove that
$$\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}.$$
(This holds also for complex functions.)
Proof: We have
$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t)-f(x)}{g(t)-g(x)} = \lim_{t \to x} \frac{\frac{f(t)-f(x)}{t-x}}{\frac{g(t)-g(x)}{t-x}} = \frac{\lim_{t \to x} \frac{f(t)-f(x)}{t-x}}{\lim_{t \to x} \frac{g(t)-g(x)}{t-x}} = \frac{f'(x)}{g'(x)}.$

8. Suppose f' is continuous on $[a, b]$ and $\epsilon > 0$. Prove that there exists $\delta > 0$ such that
$$\left|\frac{f(t) - f(x)}{t - x} - f'(x)\right| < \epsilon$$
whenever $0 < |t - x| < \delta$, $a \leq x \leq b$, $a \leq t \leq b$. (This could be expressed by saying that f is *uniformly differentiable* on $[a, b]$ if f' is continuous on $[a, b]$.) Does this hold for vector-valued functions too?
Proof: Since $[a, b]$ is compact and f' is continuous, f' is uniformly continuous on $[a, b]$. Then given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |t - x| < \delta$ implies $|f'(t) - f'(x)| < \epsilon$, for any $t, x \in [a, b]$. We have $|\frac{f(t)-f(x)}{t-x} - f'(x)| = |\frac{f'(\theta)(t-x)}{t-x} - f'(x)| = |f'(\theta) - f'(x)|$, where $\theta \in (x, t)$(if

$t > x$) or $\theta \in (t, x)$(if $x > t$). Anyway, we have $0 < |\theta - x| < |t - x| < \delta$, and thus $|f'(\theta) - f'(x)| < \epsilon$, namely, $|\frac{f(t)-f(x)}{t-x} - f'(x)| < \epsilon$.
In the case of vector-valued functions, the above result also holds. To see this, suppose that $\mathbf{f} : \mathbb{R}^1 \to \mathbb{R}^k$, $k \geq 2$ and let $\mathbf{f} = (f_1, f_2, ..., f_k)$, then $\mathbf{f}' = (f_1', f_2', ..., f_k')$. Similar as the previous proof, since $\mathbf{f}'$ is continuous on $[a, b]$, $\mathbf{f}'$ is uniformly continuous on $[a, b]$. Hence given any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |t - x| < \delta$ implies $|\mathbf{f}'(t) - \mathbf{f}'(x)| < \frac{\epsilon}{\sqrt{k}}$, for any $t, x \in [a, b]$. Since $\frac{\mathbf{f}(t)-\mathbf{f}(x)}{t-x} = (\frac{f_1(t)-f_1(x)}{t-x}, \frac{f_2(t)-f_2(x)}{t-x}, ..., \frac{f_k(t)-f_k(x)}{t-x}) = (f_1'(\theta_1), f_2'(\theta_2), ..., f_k'(\theta_k))$, where $\theta_i \in (x, t)$ if $t > x$, or $\theta_i \in (t, x)$ if $t < x$, for $1 \leq i \leq k$. Then we have $|f_i'(\theta_i) - f_i'(x)| \leq |\mathbf{f}'(\theta_i) - \mathbf{f}'(x)| < \frac{\epsilon}{\sqrt{k}}$, for $1 \leq i \leq k$, and thus $|\frac{\mathbf{f}(t)-\mathbf{f}(x)}{t-x} - \mathbf{f}'(x)| = |(f_1'(\theta_1), f_2'(\theta_2), ..., f_k'(\theta_k)) - \mathbf{f}'(x)| = \sqrt{(f_1'(\theta_1) - f_1'(x))^2 + \cdots + (f_k'(\theta_k) - f_k'(x))^2} < \sqrt{\frac{\epsilon^2}{k} \cdot k} = \epsilon$.

9. Let f be a continuous real function on $\mathbb{R}^1$, of which it is known that $f'(x)$ exists for all $x \neq 0$ and that $f'(x) \to 3$ as $x \to 0$. Does it follow that $f'(0)$ exists?
Proof: $f'(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{x\to 0} \frac{f'(\theta)(x-0)}{x-0} = \lim\limits_{x\to 0} f'(\theta)$, where $\theta \in (0, x)$, if $x > 0$, or $\theta \in (x, 0)$, if $x < 0$. Anyway, $x \to 0$ implies $\theta \to 0$, and thus $f'(\theta) \to 3$, which gives $f'(0) = \lim\limits_{\theta\to 0} f'(\theta) = 3$.

10. Suppose f and g are complex differentiable functions on $(0, 1)$, $f(x) \to 0$, $g(x) \to 0$, $f'(x) \to A$, $g'(x) \to B$ as $x \to 0$, where A and B are complex numbers, $B \neq 0$. Prove that
$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{A}{B}.$$
Compare with Example 5.8.
Proof: We have $\frac{f(x)}{g(x)} = \{\frac{f(x)}{x} - A\} \cdot \frac{x}{g(x)} + A \cdot \frac{x}{g(x)}$. If we apply Theorem 5.13 to the real and imaginary parts of $\frac{f(x)}{x}$ and $\frac{g(x)}{x}$, we can obtain that $\lim\limits_{x\to 0} \frac{f(x)}{x} = A$ and $\lim\limits_{x\to 0} \frac{g(x)}{x} = B$. Hence $\lim\limits_{x\to 0} \frac{f(x)}{g(x)} = \lim\limits_{x\to 0}(\{\frac{f(x)}{x} - A\} \cdot \frac{x}{g(x)}) + \lim\limits_{x\to 0} A \cdot \frac{x}{g(x)} = (A - A) \cdot \frac{1}{B} + A \cdot \frac{1}{B} = \frac{A}{B}$.

11. Suppose f is defined in a neighborhood of x, and suppose $f''(x)$ exists. Show that
$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x).$$
Show by example that the limit may exist even if $f''(x)$ does not.
Proof: The limit on the left side satisfies the hypothesis of Theorem 5.13, and thus we have $\lim\limits_{h\to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = \lim\limits_{h\to 0} \frac{f'(x+h)+f'(x-h)(-1)}{2h} = \lim\limits_{h\to 0} \frac{f'(x+h)-f'(x-h)}{2h} = \frac{1}{2} \lim\limits_{h\to 0}\{\frac{f'(x+h)-f'(x)}{h} + \frac{f'(x)-f'(x-h)}{h}\} = \frac{1}{2}\{\lim\limits_{h\to 0} \frac{f'(x+h)-f'(x)}{h} + \lim\limits_{h\to 0} \frac{f'(x-h)-f'(x)}{-h}\} = \frac{1}{2}(f''(x) + f''(x)) = f''(x)$.

12. If $f(x) = |x|^3$, compute $f'(x), f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exist.
Proof:
(i) When $x > 0$, $f(x) = x^3$ and $f'(x) = \lim\limits_{t\to x} \frac{t^3-x^3}{t-x} = \lim\limits_{t\to x} \frac{(t-x)(t^2+tx+x^2)}{t-x} = \lim\limits_{t\to x} t^2+tx+x^2 = 3x^2$; when $x < 0$, $f(x) = -x^3$ and $f'(x) = \lim\limits_{t\to x} \frac{-t^3+x^3}{t-x} = -\lim\limits_{t\to x} t^2 + tx + x^2 = -3x^2$; and when $x = 0$, $f'(0+) = \lim\limits_{t\to 0} \frac{t^3-0}{t-0} = 0$, $f'(0-) = \lim\limits_{t\to 0} \frac{-t^3-0}{t-0} = 0$, hence $f'(0) = 0$.
(ii) When $x > 0$, $f''(x) = \lim\limits_{t\to x} \frac{3t^2-3x^2}{t-x} = \lim\limits_{t\to x} 3(t+x) = 6x$; when $x < 0$, $f''(x) = \lim\limits_{t\to x} \frac{-3t^2+3x^2}{t-x} = \lim\limits_{t\to x}(-3)(t+x) = -6x$; and when $x = 0$, $f''(0+) = \lim\limits_{t\to 0} \frac{3t^2-0}{t-0} = 0$, $f''(0-) = \lim\limits_{t\to 0} \frac{-3t^2-0}{t-0} = 0$, hence $f''(0) = 0$.
(iii) $f^{(3)}(0+) = \lim\limits_{t\to 0} \frac{6t-0}{t-0} = 6$, and $f^{(3)}(0-) = \lim\limits_{t\to 0} \frac{-6t-0}{t-0} = -6$, hence $f^{(3)}(0)$ does not exist.

13. Suppose a and c are real number, $c > 0$, and f is defined on $[-1, 1]$ by

$$f(x) = \begin{cases} x^a \sin\left(|x|^{-c}\right) & (\text{if } x \neq 0). \\ 0 & (\text{if } x = 0). \end{cases}$$

Prove the following statements:

(a) f is continuous if and only if $a > 0$.
Proof: Since $f(x)$ is continuous when $x \neq 0$, we only need to show that $f(x)$ is continuous at 0. We have $\lim\limits_{x\to 0} x^a \sin(|x|^{-c}) = 0$ if and only if $a > 0$.

(b) $f'(0)$ exists if and only if $a > 1$.
Proof: We have $f'(0) = \lim\limits_{x\to 0} \frac{x^a \sin(|x|^{-c})-0}{x-0} = \lim\limits_{x\to 0} x^{a-1} \sin(|x|^{-c})$, and when $a > 1$, $f'(0) = 0$; when $a \leq 1$, $f'(0)$ does not exist.

(c) f' is bounded if and only if $a \geq 1 + c$.
Proof: We have $f'(x) = ax^{a-1} \sin(x^{-c}) - cx^a \cos(x^{-c})x^{-c-1} = ax^{a-1} \sin(x^{-c}) - cx^{a-c-1} \cos(x^{-c})$, when $x > 0$, and
$f'(x) = ax^{a-1} \sin((-x)^{-c}) + cx^a \cos((-x)^{-c})(-x)^{-c-1} =$
$ax^{a-1} \sin((-x)^{-c}) + c(-1)^{-c-1}x^{a-c-1} \cos(x^{-c})$, when $x < 0$. When $a \geq c+1$, $|f'(x)| \leq |a||x|^{a-1} + |c||x|^{a-c-1} \leq |a| + |c|$, in both cases; and when $a < c+1$, $f'(x) \to \infty$ as $x \to 0$, in both cases.

(d) f' is continuous if and only if $a > 1 + c$.
Proof: Clearly, $f'(x)$ is continuous at any point $x \neq 0$. When $a \leq 0$, $f'(0)$ does not exist. According to (c), when $1 < a < 1 + c$, $f'(x) \to \infty$ as $x \to 0$, but $f'(0) = 0$ according to (c), hence f' is not continuous at 0. When $a = 1 + c$, $f'(x) = ax^{a-1} \sin(x^{-c}) - c\cos(x^{-c}) = (c+1)x^c \sin(x^{-c}) - c\cos(x^{-c})$ if $x > 0$, and $\lim\limits_{x\to 0+} f'(x) = \lim\limits_{x\to 0+} [(c+1)x^c \sin(x^{-c}) - c\cos(x^{-c})] = -c \lim\limits_{x\to 0+} \cos(x^{-c})$ and this

limit does not exist; similarly, if $x < 0$, $f'(x) = (c+1)x^c \sin((-x)^{-c}) + c(-1)^{-c-1}\cos(x^{-c})$, and $\lim_{x\to 0-} f'(x)$ does not exist, by the same reason. Hence f' is not continuous at 0. When $a > 1+c$, we have $f'(x) = ax^{a-1}\sin(x^{-c}) - cx^{a-c-1}\cos(x^{-c})$, if $x > 0$, and $\lim_{x\to 0+} f'(x) = 0$; $f'(x) = ax^{a-1}\sin((-x)^{-c}) + c(-1)^{-c-1}x^{a-c-1}\cos(x^{-c})$, if $x < 0$, and $\lim_{x\to 0-} f'(x) = 0$. Hence f' is continuous at 0.
Therefore, we can conclude that f' is continuous if and only if $a > 1 + c$.

(e) $f''(0)$ exists if and only if $a > 2 + c$.
Proof: $f''(0+) = \lim_{x\to 0+} \frac{f'(x)-f'(0)}{x-0}$
$= \lim_{x\to 0+} ax^{a-2}\sin(x^{-c}) - cx^{a-c-2}\cos(x^{-c}) = 0$, when $a > 2 + c$, and does not exist when $a < 2 + c$;
$f''(0-) = \lim_{x\to 0-} \frac{f'(x)-f'(0)}{x-0}$
$= \lim_{x\to 0-} ax^{a-2}\sin((-x)^{-c}) + c(-1)^{-c-1}x^{a-c-2}\cos(x^{-c}) = 0$, when $a > 2 + c$, and does not exist when $a < 2 + c$.
Hence, we have $f''(0) = 0$, when $a > 2 + c$, and $f''(0)$ does not exist when $a \le 2 + c$.

(f) f'' is bounded if and only if $a >= 2 + 2c$.
Proof: $f''(x) = a(a-1)x^{a-2}\sin(x^{-c}) - acx^{a-c-2}\cos(x^{-c}) - c(a-c-1)x^{a-c-2}\cos(x^{-c}) - c^2x^{a-2c-2}\sin(x^{-c})$, when $x > 0$; and $f''(x) = a(a-1)x^{a-2}\sin((-x)^{-c}) + ac(-1)^{-c-1}x^{a-c-2}\cos(x^{-c}) + c(a-c-1)(-1)^{-c-1}x^{a-c-2}\cos(x^{-c}) + c^2(-1)^{-c-1}x^{a-2c-2}\sin(x^{-c})$, when $x < 0$. We have $|f''(x)| \le |a(a-1)||x^{a-2}| + |ac||x^{a-c-2}| + |c(a-c-1)||x^{a-c-2}| + |c^2||x^{a-2c-2}| \le |a(a-1)| + |ac| + |c(a-c-1)| + |c^2|$, when $a \ge 2c + 2$, and $f''(x) \to \infty$, when $a < 2c + 2$ and $x \to 0$, in both cases.

(g) f'' is continuous if and only if $a > 2 + 2c$.
Proof: $f''(x)$ is continuous at every $x \ne 0$. When $a \le c + 2$, $f''(0)$ does not exist; when $c+2 < a < 2+2c$, $f''(x) \to \infty$ when $x \to 0$, but $f''(0) = 0$; when $a = 2 + 2c$, $\lim_{x\to 0+} f''(x)$ and $\lim_{x\to 0-} f''(x)$ does not exist; and when $a > 2 + 2c$, $\lim_{x\to 0+} f''(x) = \lim_{x\to 0-} f''(x) = 0 = f''(0)$, hence $f''(x)$ is continuous at 0.
Therefore, f'' is continuous if and only if $a > 2 + 2c$.

14. Let f be a differentiable real function defined in (a, b). Prove that f is convex if and only if f' is monotonically increasing. Assume next that $f''(x)$ exists for every $x \in (a, b)$, and prove that f is convex if and only if $f''(x) \ge 0$ for all $x \in (a, b)$.
Proof:
(i)⇒: Suppose f is convex, then $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$, for any $x, y \in (a, b)$ and $0 < \lambda < 1$. Let $x_1, x_2 \in (a, b)$, and $x_1 < x_2$. Since

$f'(x_1)$ exists, $f'(x_1) = \lim_{t\to x_1+} \frac{f(t)-f(x_1)}{t-x_1} = \lim_{\lambda\to 1} \frac{f(\lambda x_1+(1-\lambda)x_2)-f(x_1)}{\lambda x_1+(1-\lambda)x_2-x_1} \le \lim_{\lambda\to 1} \frac{\lambda f(x_1)+(1-\lambda)f(x_2)-f(x_1)}{(\lambda-1)(x_1-x_2)} = \lim_{\lambda\to 1} \frac{(\lambda-1)(f(x_1)-f(x_2))}{(\lambda-1)(x_1-x_2)} = \lim_{\lambda\to 1} \frac{f(x_1)-f(x_2)}{x_1-x_2} = \frac{f(x_1)-f(x_2)}{x_1-x_2}$;
and since $f'(x_2)$ exists, $f'(x_2) = \lim_{t\to x_2-} \frac{f(t)-f(x_2)}{t-x_2} = -\lim_{t\to x_2} \frac{f(t)-f(x_2)}{x_2-t} = -\lim_{\lambda\to 0} \frac{f(\lambda x_1+(1-\lambda)x_2)-f(x_2)}{x_2-(\lambda x_1+(1-\lambda)x_2)} \ge -\lim_{\lambda\to 0} \frac{\lambda f(x_1)+(1-\lambda)f(x_2)-f(x_2)}{x_2-(\lambda x_1+(1-\lambda)x_2)}$
$= -\lim_{\lambda\to 0} \frac{\lambda(f(x_1)-f(x_2))}{\lambda(x_2-x_1)} = \lim_{\lambda\to 0} \frac{f(x_1)-f(x_2)}{x_1-x_2} = \frac{f(x_1)-f(x_2)}{x_1-x_2}$. Therefore, $f'(x_1) \le f'(x_2)$ and f' is monotonically increasing.
$\Leftarrow$: Suppose f' is monotonically increasing, and let x, y be any two number in (a,b) such that $x < y$. For every $0 < \lambda < 1$, denote $z = \lambda x + (1-\lambda)y$, hence $z \in (x,y)$. Then $f(\lambda x + (1-\lambda)y) - (\lambda f(x) + (1-\lambda)f(y)) = f(z) - (\lambda f(x) + (1-\lambda)f(y)) = \lambda(f(z) - f(x)) + (1-\lambda)(f(z) - f(y)) = \lambda f'(\theta)(z-x) + (1-\lambda)f'(\phi)(z-y) = \lambda f'(\theta)(\lambda x + (1-\lambda)y - x) + (1-\lambda)f'(\phi)(\lambda x + (1-\lambda)y - y) = \lambda(1-\lambda)(y-x)f'(\theta) + (1-\lambda)\lambda(x-y)f'(\phi) = \lambda(1-\lambda)(y-x)(f'(\theta) - f'(\phi))$, where $\theta \in (x,z)$, and $\phi \in (z,y)$. Since then $\theta < \phi$ and f' is monotonically increasing, we have $f'(\theta) \le f'(\phi)$, which gives $f(\lambda x + (1-\lambda)y) - (\lambda f(x) + (1-\lambda)f(y)) \le 0$ and thus $f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda)f(y)$. Therefore, f is convex.
(ii) Suppose $f''(x)$ exists for every $x \in (a,b)$.
$\Rightarrow$: If f is convex, then by (i), f' is monotonically increasing. Fix any $x \in (a,b)$, we have $f''(x) = \lim_{t\to x} \frac{f'(t)-f'(x)}{t-x} \ge 0$. (Since $t < x$ implies $f'(t) \le f'(x)$ and $t > x$ implies $f'(t) \ge f'(x)$.)
$\Leftarrow$: If $f''(x) \ge 0$ for all $x \in (a,b)$, we have that f' is monotonically increasing, by Theorem 5.11, and thus f is convex by (i).

15. Suppose $a \in \mathbb{R}^1$, f is a twice-differentiable real function on (a,∞), and M_0, M_1, M_2 are the least upper bounds of $|f(x)|$, $|f'(x)|$, $|f''(x)|$, respectively, on (a,∞). Prove that $M_1^2 \le 4M_0M_2$.
Does $M_1^2 \le 4M_0M_2$ hold for vector-valued functions too?
Proof: If $h > 0$, Taylor's theorem shows that $f'(x) = \frac{1}{2h}[f(x+2h) - f(x)] - hf''(\xi)$ for some $\xi \in (x, x+2h)$. Hence $|f'(x)| \le hM_2 + \frac{M_0}{h}$, and $|f'(x)|^2 \le h^2M_2^2 + \frac{M_0^2}{h^2} + 2M_0M_2$. Since x and h are both arbitrary, we have $M_1^2 \le \inf h^2M_2^2 + \frac{M_0^2}{h^2} + 2M_0M_2$, which gives $M_1^2 \le 4M_0M_2$.
To show that $M_1^2 = 4M_0M_2$ can actually happen, take $a = -1$, define

$$f(x) \begin{cases} 2x^2 - 1 & (-1 < x < 0), \\ \frac{x^2-1}{x^2+1} & (0 \le x < \infty), \end{cases}$$

We will show that $M_0 = 1$, $M_1 = 4$, $M_2 = 4$.
First, we have $|f(x)| \le 1$ when $-1 < x < 0$, and $f(x) = 1 - \frac{2}{x^2+1}$ is strictly decreasing when $0 \le x < \infty$, thus $M_1 = 1$.
Then, $f'(x) = 4x$ when $-1 < x < 0$, and $f'(x) = \frac{4x}{(x^2+1)^2}$ when $0 < x < \infty$. Since $f'(0-) = \lim_{t\to 0-} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0-} 2t = 0$ and $f'(0+) =$

$\lim_{t\to 0+} \frac{f(t)-f(0)}{t-0} = \lim_{t\to 0+} \frac{2t}{t^2+1} = 0$, thus $f'(0) = 0$. (Actually, since f'' exists, f' must be continuous.) Therefore, $|f'(x)| \le 4$, when $-1 < x < 0$, and $f'(x) = \frac{4}{x^3+2x+\frac{1}{x}}$ when $0 < x < \infty$. Thus $f'(x) \to 0$ as $x \to \infty$ and since $f'(0) = 0$ and $f'(x) > 0$ when $0 < x < \infty$, there must be a point $x_0 \in (0, \infty)$ at which $f'(x)$ obtains its maximum (since $f'(x)$ is continuous), or equivalently, $g(x) = x^3 + 2x + \frac{1}{x}$ obtains its minimum at x_0. Hence $g'(x_0) = 0$, which gives $3x_0^2 + 2 - \frac{1}{x_0^2} = 0$, i.e., $3x_0^4 + 2x_0^2 - 1 = 0$, i.e., $(x_0^2 + 1)(3x_0^2 - 1) = 0$ and thus $x_0^2 = \frac{1}{3}$. Therefore, $x_0 = \frac{1}{\sqrt{3}}$ and $\sup f'(x) = f'(x_0) = \frac{3\sqrt{3}}{4}$ when $0 \le x < \infty$. So we have $M_1 = 4$.
Next, $f''(x) = 4$, when $-1 < x < 0$, and $f''(x) = \frac{4(1-3x^2)}{(x^2+1)^3}$ is strictly decreasing when $0 < x < \infty$, thus $|f''(x)| \le 4$. $f''(0-) = \lim_{t\to 0-} \frac{f'(t)-f'(0)}{t-0} = 4$, and $f''(0+) = \lim_{t\to 0+} \frac{f'(t)-f'(0)}{t-0} = \lim_{t\to 0+} \frac{4}{(t^2+1)^2} = 4$, then $f''(0) = 4$. Therefore, we have $M_2 = 4$.
In the case of vector-valued functions, $M_1^2 \le 4M_0M_2$ also holds, which will be clear after we prove Exercise 20.
By the result of Exercise 20, we have $|\mathbf{f}(x+2h) - \mathbf{f}(x) - \mathbf{f}'(x) \cdot 2h| \le \frac{|\mathbf{f}^{(2)}(\xi)|}{2} \cdot (2h)^2$, for some $\xi \in (x, x+2h)$. Since $|\mathbf{f}'(x) \cdot 2h| - |\mathbf{f}(x+2h) - \mathbf{f}(x)| \le |\mathbf{f}'(x) \cdot 2h - (\mathbf{f}(x+2h) - \mathbf{f}(x))| = |\mathbf{f}(x+2h) - \mathbf{f}(x) - \mathbf{f}'(x) \cdot 2h| \le \frac{|\mathbf{f}^{(2)}(\xi)|}{2} \cdot (2h)^2$, we have $|\mathbf{f}'(x) \cdot 2h| \le |\mathbf{f}(x+2h) - \mathbf{f}(x)| + \frac{|\mathbf{f}^{(2)}(\xi)|}{2} \cdot (2h)^2$, and hence $|\mathbf{f}'(x)| \le \frac{1}{2h}|\mathbf{f}(x+2h) - \mathbf{f}(x)| + h|\mathbf{f}^{(2)}(\xi)| \le \frac{1}{2h}(|\mathbf{f}(x+2h)| + |\mathbf{f}(x)|) + h|\mathbf{f}^{(2)}(\xi)| \le \frac{1}{2h} \cdot 2M_0 + hM_2 = \frac{M_0}{h} + hM_2$. The following proof is the same.

16. Suppose f is twice-differentiable on $(0, \infty)$, f'' is bounded on $(0, \infty)$, and $f(x) \to 0$ as $x \to \infty$. Prove that $f'(x) \to 0$ as $x \to \infty$.
Proof: Since f'' is bounded on $(0, \infty)$, we have $M_2 = \sup |f''| \le M$, for some positive M. Let $a \to \infty$ in Exercise 15, we have $M_0 = \sup |f| \to 0$, since $f(x) \to 0$ as $x \to \infty$. Then $0 \le M_1^2 \le 4M_0M_2 \le 4MM_0 \to 0$ as $x \to \infty$, and thus $M_1 = \sup |f'| \to 0$ as $x \to \infty$. Therefore, $f'(x) \to 0$ as $x \to \infty$.

17. Suppose f is a real, three times differentiable function on $[-1, 1]$, such that $f(-1) = 0$, $f(0) = 0$, $f(1) = 1$, $f'(0) = 0$. Prove that $f^{(3)}(x) \ge 3$ for some $x \in (-1, 1)$. Note that equality holds for $\frac{1}{2}(x^3 + x^2)$.
Proof: By Theorem 5.15(Taylor's Theorem), we have $f(\beta) = f(\alpha) + f'(\alpha)(\beta - \alpha) + \frac{f''(\alpha)}{2}(\beta - \alpha)^2 + \frac{f^{(3)}(x)}{6}(\beta - \alpha)^3$, where $\alpha, \beta \in [-1, 1]$ and x is between α and β. Let $\beta = 1$, $\alpha = 0$, then $\beta - \alpha = 1$, we have $f(1) = f(0) + f'(0) + \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6} = \frac{f''(0)}{2} + \frac{f^{(3)}(s)}{6}$, for some $s \in (0, 1)$. Let $\beta = -1$, $\alpha = 0$, then $\beta - \alpha = -1$, we have $f(-1) = f(0) + f'(0) + \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6} = \frac{f''(0)}{2} - \frac{f^{(3)}(t)}{6}$, for some $t \in (-1, 0)$. Since $f(1) = 1$, $f(-1) = 0$, we have $f^{(3)}(s) + f^{(3)}(t) = 6$, then either $f^{(3)}(s) \ge 3$ or $f^{(3)}(t) \ge 3$, which gives the desired result.

18. Suppose f is a real function on $[a, b]$, n is a positive integer, and $f^{(n-1)}$ exists for every $t \in [a, b]$. Let α, β, and P be as in Taylor's theorem(5.15). Define

$$Q(t) = \frac{f(t) - f(\beta)}{t - \beta}$$

for $t \in [a, b]$, $t \neq \beta$, differentiate $f(t) - f(\beta) = (t - \beta)Q(t)$ $n - 1$ times at $t = \alpha$, and derive the following version of Taylor's theorem:

$$f(\beta) = P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n.$$

Proof: We have $f(t) = f(\beta)+(t-\beta)Q(t)$, then $f'(t) = Q(t)+(t-\beta)Q'(t)$, $f''(t) = 2Q'(t) + (t - \beta)Q''(t)$, and in general, $f^{(i)}(t) = iQ^{(i-1)}(t) + (t - \beta)Q^{(i)}(t)$, for $1 \leq i \leq n - 1$. Thus, we have $f^{(i)}(\alpha) = iQ^{(i-1)}(\alpha) + (\alpha - \beta)Q^{(i)}(\alpha)$. Hence $P(\beta) + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n = \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta-\alpha)^k+(\frac{f^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^{n-1}+\frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta-\alpha)^n) = \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + (\frac{(n-1)Q^{(n-2)}(\alpha)+(\alpha-\beta)Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^{n-1} + \frac{Q^{(n-1)}(\alpha)}{(n-1)!}(\beta - \alpha)^n) = \sum_{k=0}^{n-2} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{Q^{(n-2)}(\alpha)}{(n-2)!}(\beta - \alpha)^{n-1} = \cdots = f(\alpha)+Q(\alpha)(\beta-\alpha) = f(\alpha)-(f(\alpha)-f(\beta)) = f(\beta)$, which gives the desired result.

19. Suppose f is defined in $(-1, 1)$ and $f'(0)$ exists. Suppose $-1 < \alpha_n < \beta_n < 1$, $\alpha_n \to 0$, and $\beta_n \to 0$ as $n \to \infty$. Define the difference quotients

$$D_n = \frac{f(\beta_n) - f(\alpha_n)}{\beta_n - \alpha_n}.$$

Prove the following statements:

(a) If $\alpha_n < 0 < \beta_n$, then $\lim D_n = f'(0)$.
Proof: Given any $\epsilon > 0$. We have $|D_n - f'(0)| = |\frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} - f'(0)| = |\frac{f(\beta_n)-f(0)}{\beta_n-\alpha_n} + \frac{f(0)-f(\alpha_n)}{\beta_n-\alpha_n} - f'(0)| = |\frac{f(\beta_n)-f(0)}{\beta_n-0} \cdot \frac{\beta_n}{\beta_n-\alpha_n} + \frac{f(\alpha_n)-f(0)}{\alpha_n-0} \cdot \frac{-\alpha_n}{\beta_n-\alpha_n} - f'(0)| = |(\frac{f(\beta_n)-f(0)}{\beta_n-0} - f'(0)) \cdot \frac{\beta_n}{\beta_n-\alpha_n} + (\frac{f(\alpha_n)-f(0)}{\alpha_n-0} - f'(0)) \cdot \frac{-\alpha_n}{\beta_n-\alpha_n}| \leq |\frac{f(\beta_n)-f(0)}{\beta_n-0} - f'(0)| \cdot |\frac{\beta_n}{\beta_n-\alpha_n}| + |\frac{f(\alpha_n)-f(0)}{\alpha_n-0} - f'(0)| \cdot |\frac{\alpha_n}{\beta_n-\alpha_n}|$. Since $f'(0) = \lim_{t\to 0} \frac{f(t)-f(0)}{t\to 0}$, and $\alpha_n \to 0$, $\beta_n \to 0$ as $n \to \infty$, $\lim_{n\to\infty} \frac{f(\alpha_n)-f(0)}{\alpha_n-0} = f'(0)$, and $\lim_{n\to\infty} \frac{f(\beta_n)-f(0)}{\beta_n-0} = f'(0)$. Thus there is an $N_1 \in \mathbb{N}$ such that $n > N_1$ implies $|\frac{f(\alpha_n)-f(0)}{\alpha_n-0} - f'(0)| < \epsilon$, and there is an $N_2 \in \mathbb{N}$ such that $n > N_2$ implies $|\frac{f(\beta_n)-f(0)}{\beta_n-0} - f'(0)| < \epsilon$. Let $N = \max(N_1, N_2)$, then when $n > N$, $|D_n - f'(0)| < (|\frac{\beta_n}{\beta_n-\alpha_n}| +$

$|\frac{\alpha_n}{\beta_n-\alpha_n}|)\epsilon$.
If $\alpha_n < 0 < \beta_n$, then $\beta_n - \alpha_n > 0$ and $|D_n - f'(0)| < (|\frac{\beta_n}{\beta_n-\alpha_n}| + |\frac{\alpha_n}{\beta_n-\alpha_n}|)\epsilon = (\frac{\beta_n}{\beta_n-\alpha_n} - \frac{\alpha_n}{\beta_n-\alpha_n})\epsilon = \epsilon$, when $n > N$. Therefore, $\lim_{n\to\infty} D_n = f'(0)$.

(b) If $0 < \alpha_n < \beta_n$ and $\{\frac{\beta_n}{\beta_n-\alpha_n}\}$ is bounded, then $\lim D_n = f'(0)$.
Proof: As in (a), if $0 < \alpha_n < \beta_n$ and $|\frac{\beta_n}{\beta_n-\alpha_n}| \le M$, for some $M > 0$, then $\beta_n - \alpha_n > 0$ and $|D_n - f'(0)| < (|\frac{\beta_n}{\beta_n-\alpha_n}| + |\frac{\alpha_n}{\beta_n-\alpha_n}|)\epsilon = \frac{\beta_n+\alpha_n}{\beta_n-\alpha_n}\epsilon = (2\frac{\beta_n}{\beta_n-\alpha_n} - 1)\epsilon \le (2M+1)\epsilon$ (since $2\frac{\beta_n}{\beta_n-\alpha_n} - 1 \le |2\frac{\beta_n}{\beta_n-\alpha_n} - 1| \le 2|\frac{\beta_n}{\beta_n-\alpha_n}| + 1 \le 2M+1$). Therefore, $\lim_{n\to\infty} D_n = f'(0)$.

(c) If f' is continuous in $(-1,1)$, then $\lim D_n = f'(0)$.
Give an example in which f is differentiable in $(-1,1)$(but f' is not continuous at 0) and in which α_n, β_n tend to 0 in such a way that $\lim D_n$ exists but is different from $f'(0)$.
Proof: We have $\lim_{n\to\infty} D_n = \lim_{n\to\infty} \frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} = \lim_{n\to\infty} f'(\xi_n)$, where ξ_n is between α_n and β_n. Since $\alpha_n \to 0$ and $\beta_n \to 0$, as $n \to \infty$, we know that $\xi_n \to 0$ as $n \to \infty$. Since f' is continuous, we conclude that $\lim_{n\to\infty} f'(\xi_n) = f'(0)$, which gives $\lim_{n\to\infty} D_n = f'(0)$.
Let f be the same function defined in Example 5.6(b), that is

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \ne 0 \\ 0 & x = 0 \end{cases}$$

Then by Example 5.6(b), f is differentiable in (-1,1), but f' is not continuous at 0. Let $\alpha_n = \frac{1}{2n\pi+\pi/2}$, $\beta_n = \frac{1}{2n\pi}$, then $-1 < \alpha_n < \beta_n \le \frac{1}{2\pi} < 1$, and $\alpha_n \to 0$, $\beta_n \to 0$, as $n \to \infty$. We have $D_n = \frac{f(\beta_n)-f(\alpha_n)}{\beta_n-\alpha_n} = \frac{(\frac{1}{2n\pi})^2\sin(2n\pi) - (\frac{1}{2n\pi+\pi/2})^2\sin(2n\pi+\pi/2)}{\frac{1}{2n\pi+\pi/2} - \frac{1}{2n\pi}} = -\frac{4}{2\pi+\frac{\pi}{2n}}$ and hence $\lim_{n\to\infty} D_n = -\lim_{n\to\infty} \frac{4}{2\pi+\frac{\pi}{2n}} = -\frac{2}{\pi}$. But $f'(0) = 0$, by Example 5.6(b).

20. Formulate and prove an inequality which follows from Taylor's theorem and which remains valid for vector-valued functions.
Proof: Taylor's Theorem states the following facts: Suppose f is a real function on $[a,b]$, n is a positive integer, $f^{(n-1)}$ is continuous on $[a,b]$, $f^{(n)}(t)$ exists for every $t \in (a,b)$. Let α, β be distinct points of $[a,b]$, and define

$$P(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k.$$

Then there exists a point x between α and β such that

$$f(\beta) = P(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n.$$

Thus, we have $|f(\beta) - P(\beta)| = |\frac{f^{(n)}(x)}{n!}(\beta - \alpha)^n| = |\frac{f^{(n)}(x)}{n!}||\beta - \alpha|^n \leq \frac{\sup |f^{(n)}(t)|}{n!}|\beta - \alpha|^n$, where t is between α and β.
This inequality also holds in the case of vector-valued functions. Specifically, suppose $\mathbf{f}$ is a continuous mapping of $[a, b]$ into $\mathbb{R}^k$, n is a positive integer, $\mathbf{f}^{(n-1)}$ is continuous on $[a, b]$, $\mathbf{f}^{(n)}(t)$ exists for every $t \in (a, b)$. Let α, β be distinct points of $[a, b]$, and define

$$\mathbf{P}(t) = \sum_{k=0}^{n-1} \frac{\mathbf{f}^{(k)}(\alpha)}{k!}(t - \alpha)^k.$$

Then there exists a point x between α and β such that

$$|\mathbf{f}(\beta) - \mathbf{P}(\beta)| \leq \frac{|\mathbf{f}^{(n)}(x)|}{n!}|\beta - \alpha|^n.$$

Now, we will give our proof.
Put $\mathbf{z} = \mathbf{f}(\beta) - \mathbf{P}(\beta)$, and define

$$\phi(t) = \mathbf{z} \cdot \mathbf{f}(t) \qquad (a \leq t \leq b).$$

Then ϕ is a real valued continuous function on $[a, b]$, $\phi^{(n-1)}$ is continuous on $[a, b]$, $\phi^{(n)}(t)$ exists for every $t \in (a, b)$. By Taylor's Theorem, we have

$$\phi(\beta) = \sum_{k=0}^{n-1} \frac{\phi^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{\phi^{(n)}(x)}{n!}(\beta - \alpha)^n,$$

for some x between α and β. Since $\phi(t) = \mathbf{z} \cdot \mathbf{f}(t)$, $\phi^{(k)}(t) = \mathbf{z} \cdot \mathbf{f}^{(k)}(t)$ and in particular, $\phi^{(k)}(\alpha) = \mathbf{z} \cdot \mathbf{f}^{(k)}(\alpha)$. Then we have

$$\phi(\beta) = \sum_{k=0}^{n-1} \frac{\mathbf{z} \cdot \mathbf{f}^{(k)}(\alpha)}{k!}(\beta - \alpha)^k + \frac{\mathbf{z} \cdot \mathbf{f}^{(n)}(x)}{n!}(\beta - \alpha)^n,$$

Since

$$P(\beta) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(\beta - \alpha)^k,$$

we have

$$\phi(\beta) = \mathbf{z} \cdot P(\beta) + \frac{\mathbf{z} \cdot \mathbf{f}^{(n)}(x)}{n!}(\beta - \alpha)^n,$$

and hence

$$\phi(\beta) - \mathbf{z} \cdot P(\beta) = \frac{\mathbf{z} \cdot \mathbf{f}^{(n)}(x)}{n!}(\beta - \alpha)^n.$$

On the other hand

$$\phi(\beta) - \mathbf{z} \cdot P(\beta) = \mathbf{z} \cdot f(\beta) - \mathbf{z} \cdot P(\beta) = \mathbf{z} \cdot (f(\beta) - P(\beta)) = \mathbf{z} \cdot \mathbf{z} = |\mathbf{z}|^2,$$

thus we have

$$|\mathbf{z}|^2 = \frac{\mathbf{z}\cdot\mathbf{f}^{(n)}(x)}{n!}(\beta-\alpha)^n = |\frac{\mathbf{z}\cdot\mathbf{f}^{(n)}(x)}{n!}(\beta-\alpha)^n| \leq |\mathbf{z}|\cdot\frac{|\mathbf{f}^{(n)}(x)|}{n!}|\beta-\alpha|^n,$$

by Schwarz inequality. Therefore,

$$|\mathbf{z}| \leq \frac{|\mathbf{f}^{(n)}(x)|}{n!}|\beta-\alpha|^n, i.e., |\mathbf{f}(\beta)-\mathbf{P}(\beta)| \leq \frac{|\mathbf{f}^{(n)}(x)|}{n!}|\beta-\alpha|^n,$$

which is the desired result.

21. Let E be a closed subset of $\mathbb{R}^1$. We saw in Exercise 22, Chap. 4, that there is a real continuous function f on $\mathbb{R}^1$ whose zero set is E. Is it possible, for each closed set E, to find such an f which is differentiable on $\mathbb{R}^1$, or one which is n times differentiable, or even one which has derivatives of all orders on $\mathbb{R}^1$?
Solution:

22. Suppose f is a real function on $(-\infty,+\infty)$. Call x a *fixed point* of f if $f(x)=x$.

    (a) If f is differentiable and $f'(t)\neq 1$ for every real t, prove that f has at most one fixed point.
    Proof: Suppose that, on the contrary, f has more than one fixed point. Let x_1, x_2 be two different fixed points of f, $x_1<x_2$, and define $g(x)=f(x)-x$. Then $g(x_1)=g(x_2)=0$, and since $g(x_1)-g(x_2)=g'(\xi)(x_1-x_2)$, for some $\xi\in(x_1,x_2)$, we have $g'(\xi)=0$, namely, $f'(\xi)=1$, which is contradict to the assumption that $f'(t)\neq 1$ for every real t.

    (b) Show that the function f defined by

    $$f(t)=t+(1+e^t)^{-1}$$

    has no fixed point, although $0<f'(t)<1$ for all real t.
    Proof: Since $e^t>0$, we have $f(t)>t$, for every real t, and thus f has no fixed point. $f'(t)=\frac{e^{2t}+e^t+1}{e^{2t}+2e^t+1}$, and it's clear that $0<f'(t)<1$, for all real t.

    (c) However, if there is a constant $A<1$ such that $|f'(t)|\leq A$ for all real t, prove that a fixed point x of f exists, and that $x=\lim x_n$, where x_i is an arbitrary real number and $x_{n+1}=f(x_n)$ for $n=1,2,3,....$
    Proof: We first show that by starting from an arbitrary real number x_1 and apply $x_{n+1}=f(x_n)$, the resulted sequence $\{x_n\}$ converges. To see this, suppose $n\in\mathbb{N}$, $m\in\mathbb{N}$, and $n>m$. Then $|x_n-x_{n-1}|=|f(x_{n-1})-f(x_{n-2})|=|f'(\xi)(x_{n-1}-x_{n-2})|=|f'(\xi)||x_{n-1}-x_{n-2}|\leq A|x_{n-1}-x_{n-2}|$, where ξ is between x_{n-1} and x_{n-2}. Hence $|x_n-x_{n-1}|\leq A|x_{n-1}-x_{n-2}|\leq\cdots\leq A^{n-2}|x_2-x_1|$, and we have $|x_n-x_m|\leq|x_n-x_{n-1}|+|x_{n-1}-x_{n-2}|+\cdots+|x_{m+1}-x_m|\leq(A^{n-2}+$

$A^{n-3} + \cdots + A^{m-1})|x_2 - x_1| = A^{m-1}(1 + A + \cdots + A^{n-m-1})|x_2 - x_1| < A^{m-1}(1 + A + \cdots)|x_2 - x_1| = A^{m-1}\frac{|x_2-x_1|}{1-A}$. Since $A < 1$, $A^{m-1} \to 0$ when $m \to \infty$. Then given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > m > N$ implies $A^{m-1} < \frac{(1-A)\epsilon}{|x_2-x_1|}$ and thus $|x_n - x_m| < A^{m-1}\frac{|x_2-x_1|}{1-A} < \epsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence in $\mathbb{R}^1$, then $\{x_n\}$ converges to some $x \in \mathbb{R}^1$ since $\mathbb{R}^1$ is complete.
Next, we will show that $x = \lim_{n\to\infty} x_n$ is a fixed point of f, that is, $f(x) = x$. Since $x_{n+1} = f(x_n)$, we have $x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} f(x_n) = f(x)$, for f is continuous.

(d) Show that the process described in (c) can be visualized by the zig-zag path

$$(x_1, x_2) \to (x_2, x_2 \to (x_2, x_3) \to (x_3, x_3) \to (x_3, x_4) \to \cdots .$$

23. The function f defined by $f(x) = \frac{x^3+1}{3}$ has three fixed points, say α, β, γ, where $-2 < \alpha < -1$, $0 < \beta < 1$, $1 < \gamma < 2$. For arbitrarily chosen x_1, define $\{x_n\}$ by setting $x_{n+1} = f(x_n)$.

(a) If $x_1 < \alpha$, prove that $x_n \to -\infty$ as $n \to \infty$.
Proof: We have $f'(x) = x^2$, for every $x \in \mathbb{R}^1$. If $x_1 < \alpha < -1$, we have $x_2 - \alpha = f(x_1) - f(\alpha) = f'(\xi_1)(x_1 - \alpha) = \xi_1^2(x_1 - \alpha)$, for some $\xi_1 \in (x_1, \alpha)$ and thus $|x_2 - \alpha| = |\xi_1^2||x_1 - \alpha| > \alpha^2|x_1 - \alpha|$. In general, we have $x_n - \alpha = f(x_{n-1}) - f(\alpha) = f'(\xi_{n-1})(x_{n-1} - \alpha) = \xi_{n-1}^2(x_{n-1} - \alpha)$, for some $\xi_{n-1} \in (x_{n-1}, \alpha)$ and hence $|x_n - \alpha| = \xi_{n-1}^2|x_{n-1} - \alpha| = \cdots = \xi_{n-1}^2 \cdot \cdots \cdot \xi_1^2|x_1 - \alpha|$, where $\xi_k \in (x_k, \alpha)$, $k = 1, 2, ..., n-1$. Therefore, $\xi_k^2 > \alpha^2 > 1$, for each k, and $|x_n - \alpha| > \alpha^{2(n-1)}|x_1 - \alpha|$. Furthermore, by induction we known that $x_n < \alpha$ for each n ($x_n - \alpha = \xi_{n-1}^2(x_{n-1} - \alpha)$, and if $x_{n-1} < \alpha$, then $x_n < \alpha$), and $x_n < x_{n-1}$(since $x_n - x_{n-1} = (x_n - \alpha) - (x_{n-1} - \alpha) = (\xi_{n-1}^2 - 1)(x_{n-1} - \alpha) < 0$). Hence, we have $\alpha - x_n > \alpha^{2(n-1)}|x_1 - \alpha|$, which gives $x_n < -\alpha^{2(n-1)}|x_1 - \alpha| + \alpha$. Since $\alpha < -1$, $\alpha^2 > 1$, $\alpha^{2(n-1)} \to +\infty$ as $n \to \infty$, thus $x_n \to -\infty$ as $n \to \infty$.

(b) If $\alpha < x_1 < \gamma$, prove that $x_n \to \beta$ as $n \to \infty$.
Proof:
(i) First, we prove that if $-1 < x_1 < 1$, then $x_n \to \beta$ as $n \to \infty$. We have $x_n - \beta = f(x_{n-1}) - f(\beta) = f'(\xi_{n-1})(x_{n-1} - \beta) = \xi_{n-1}^2(x_{n-1} - \beta)$, then $|x_n - \beta| = \xi_{n-1}^2|x_{n-1} - \beta| = \cdots = \xi_{n-1}^2 \cdot \cdots \cdot \xi_1^2|x_1 - \beta|$, where ξ_i is between x_i and β. Furthermore, since $x_n - \beta = \xi_{n-1}^2(x_{n-1} - \beta)$, we have $x_n - x_{n-1} = (x_n - \beta) - (x_{n-1} - \beta) = (\xi_{n-1}^2 - 1)(x_{n-1} - \beta)$. If $-1 < x_1 < \beta$, we can show that $-1 < x_n < \beta$ and $x_n > x_{n-1}$ by induction, then let $M = \max(x_1^2, \beta^2) < 1$, we have $|x_n - \beta| = \xi_{n-1}^2 \cdot \cdots \cdot \xi_1^2|x_1 - \beta| < M^{n-1}|x_1 - \beta|$. Since $M < 1$, $M^{n-1} \to 0$ as $n \to \infty$, then given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $M^{n-1} < \frac{\epsilon}{|x_1-\beta|}$, and hence $|x_n - \beta| < \epsilon$. Therefore,

$\lim_{n\to\infty} x_n = \beta$.
If $\beta < x_1 < 1$, we can also show by induction that $\beta < x_n < 1$ and $x_n < x_{n-1}$. then $|x_n - \beta| = \xi_{n-1}^2 \cdot \cdots \cdot \xi_1^2 |x_1 - \beta| < x_1^{2(n-1)}|x_1 - \beta|$. Since $x_1 < 1$, by the same reason, we have $\lim_{n\to\infty} x_n = \beta$.
(ii) Next, if $x_1 = -1$, $x_2 = f(-1) = 0 \in (-1, 1)$, then we can apply (i) for x_2. If $x_1 = 1$, $x_2 = f(1) = \frac{2}{3}$, and then we can also apply (i) for x_2. Thus now, we can conclude that if $-1 \le x_1 \le 1$, then $x_n \to \beta$ as $n \to \infty$.
(iii) Finally, if $\alpha < x_1 < -1$, we will show that there must exist an $N \in \mathbb{N}$ such that $n > N$ implies $x_n \ge -1$. Otherwise $x_n < -1$ for all n. Since f is strictly monotonically increasing, by induction, we can easily show that $\alpha < x_n < -1$. What's more, $x_n - x_{n-1} = (\xi_{n-1}^2 - 1)(x_{n-1} - \alpha) > 0$ implies that $x_n > x_{n-1}$, since $\xi_{n-1} \in (\alpha, x_{n-1})$ and thus $\xi_{n-1}^2 > 1$. Then, $\{x_n\}$ is monotonically increasing and bounded, hence $\{x_n\}$ must converge to some δ, by Theorem 3.14, and clearly, $\delta \ne \alpha, \beta, \gamma$. By Exercise 22, we can show that δ is also a fixed point of f, which is absurd. Therefore, there must be an N such that $n > N$ implies $x_n \ge -1$. Since $x_{N+1} = f(x_N)$ and $x_N < -1$, we have $x_{N+1} = f(x_N) < f(-1) = 0$. Then we can apply (i) to x_{N+1}.
If $1 < x_1 < \gamma$, in a similar way, we can find an N such that $n > N$ implies $x_n \le 1$. Since $x_{N+1} = f(x_N) > f(1) = \frac{2}{3}$, we can then apply (i) to x_{N+1}.

(c) If $\gamma < x_1$, prove that $x_n \to +\infty$ as $n \to \infty$.
Proof: The proof is very similar as (a). We have $x_n - \gamma = f(x_{n-1}) - f(\gamma) = f'(\xi_{n-1})(x_{n-1} - \gamma)$, and thus $|x_n - \gamma| = \xi_{n-1}^2 \cdot \cdots \cdot \xi_1^2 |x_{n-1} - \gamma|$, where $\xi_k \in (\gamma, x_k)$. Since $x_n - x_{n-1} = (\xi_{n-1}^2 - 1)(x_{n-1} - \gamma) > 0$, we have $x_n > x_{n-1}$. Hence, $|x_n - \gamma| > \gamma^{2(n-1)}|x_{n-1} - \gamma|$, and thus $x_n > \gamma + \gamma^{2(n-1)}|x_{n-1} - \gamma|$. Clearly, $x_n \to \infty$ as $n \to \infty$.

Thus β can be located by this method, but α and γ cannot.

24. The process described in part (c) of Exercise 22 can of course also be applied to functions that map $(0, \infty)$ to $(0, \infty)$.
Fix some $\alpha > 1$, and put

$$f(x) = \frac{1}{2}(x + \frac{\alpha}{x}), \qquad g(x) = \frac{\alpha + x}{1 + x}.$$

Both f and g have $\sqrt{\alpha}$ as their only fixed point in $(0, \infty)$. Try to explain, on the basis of properties of f and g, why the convergence in Exercise 16, Chap. 3, is so much more rapid than it is in Exercise 17. (Compare f' and g', draw the zig-zags suggested in Exercise 22.)
Do the same when $0 < \alpha < 1$.
Solution: We have $f'(x) = \frac{1}{2}(1 - \frac{\alpha}{x^2})$, and thus $f'(\sqrt{\alpha}) = 0$. But $g'(x) = \frac{1-\alpha}{(1+x)^2}$ and $g'(\sqrt{\alpha}) = \frac{1-\sqrt{\alpha}}{1+\sqrt{\alpha}}$.
Suppose $\epsilon_k = x_k - \sqrt{\alpha}$, then $x_{k+1} = f(x_k) = f(\sqrt{\alpha} + \epsilon_k) = f(\sqrt{\alpha}) +$

$f'(\sqrt{\alpha})\epsilon_k + \frac{f''(\xi_k)}{2}\epsilon_k^2 = \sqrt{\alpha} + \frac{f''(\xi_k)}{2}\epsilon_k^2$, for some $\xi_k \in (\sqrt{\alpha}, x_k)$. Then $\epsilon_{k+1} = x_{k+1} - \sqrt{\alpha} = \frac{f''(\xi_k)}{2}\epsilon_k^2 \leq M\epsilon_k^2$, where $M = \sup|f''(x)|/2 = 1/2\sqrt{\alpha}$ and $x \in (\sqrt{\alpha}, +\infty)$. (note that $f''(x) = \frac{\alpha}{x^3}$) This can explain why $x_{n+1} = f(x_n)$ converges much more rapid then $x_{n+1} = g(x_n)$.

25. Suppose f is twice differentiable on $[a,b]$, $f(a) < 0$, $f(b) > 0$, $f'(x) \geq \delta > 0$, and $0 \leq f''(x) \leq M$ for all $x \in [a,b]$. Let ξ be the unique point in (a,b) at which $f(\xi) = 0$.
Complete the details in the following outline of *Newton's method* for computing ξ.

(a) Choose $x_1 \in (\xi, b)$, and define $\{x_n\}$ by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n}.$$

Interpret this geometrically, in terms of a tangent to the graph of f.

(b) Prove that $x_{n+1} < x_n$ and that $\lim_{n\to\infty} x_n = \xi$.

Proof: Since $x_{n+1} - x_n = -\frac{f(x_n)}{f'(x_n)}$, $f'(x_n) > 0$ and $f(x_n) > 0$ since $x_n \in (\xi, b)$(we will prove this immediately), if $f(x_n) < 0$, then there is a $\alpha \in (x_n, b)$ such that $f(\alpha) = 0$, which is contradict to the hypothesis that ξ is the unique point in (a,b) at which $f(\xi) = 0$. Hence, $x_{n+1} - x_n < 0$, which gives $x_{n+1} < x_n$.
Now we prove that $x_n \in (\xi, b)$, for each x_n. We prove this by induction.
(i) When $n = 1$, it's trivial that $x_1 \in (\xi, b)$.
(ii) Suppose $x_k \in (\xi, b)$, when $n = k+1$, let $g(x) = x - \frac{f(x)}{f'(x)}$, then $g(\xi) = \xi$ and $g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$. $x_{k+1} - \xi = g(x_k) - g(\xi) = g'(\theta_k)(x_k - \xi)$, where $\theta_k \in (\xi, x_k)$. Thus $f(\theta_k) > 0$ and $x_{k+1} > \xi$. $x_{k+1} - b = g(x_k) - b = (x_k - b) - \frac{f(x_k)}{f'(x_k)}$, since $f(x_k) > 0$ (because $x_k \in (\xi, b)$), $x_{k+1} - b < x_k - b < 0$. Therefore, $x_{k+1} \in (\xi, b)$.
Next, we prove that $\lim_{n\to\infty} x_n = \xi$. First, since x_n is bounded and monotonically decreasing, x_n converges. Let $x = \lim_{n\to\infty} x_n$, then $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty}(x_n - \frac{f(x_n)}{f'(x_n)})$, which gives $\lim_{n\to\infty} f(x_n) = 0$. Since f is continuous, $\lim_{n\to\infty} f(x_n) = f(x)$, and then $f(x) = 0$. Since $x \in (a,b)$ and ξ is the unique point in (a,b) at which $f(\xi) = 0$, we conclude that $x = \xi$, namely, $\lim_{n\to\infty} x_n = \xi$.

(c) Use Taylor's theorem to show that

$$x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$$

for some $t_n \in (\xi, x_n)$.

Proof: By Taylor's Theorem, $f(\xi) = f(x_n) + f'(x_n)(\xi - x_n) +$

$\frac{f''(t_n)}{2}(\xi - x_n)^2$, which gives $0 = f(x_n) + f'(x_n)(\xi - x_n) + \frac{f''(t_n)}{2}(\xi - x_n)^2$, i.e., $x_n - \frac{f(x_n)}{f'(x_n)} = \xi + \frac{f''(t_n)}{2f'(x_n)}(\xi - x_n)^2$, i.e., $x_{n+1} - \xi = \frac{f''(t_n)}{2f'(x_n)}(x_n - \xi)^2$.

(d) If $A = M/2\delta$, deduce that

$$0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}.$$

(Compare with Exercises 16 and 18, Chap. 3.)

Proof: Since $0 < \delta \le f'(x)$ and $0 \le f''(x) \le M$, we have $\frac{f''(t_n)}{2f'(x_n)} \le \frac{M}{2\delta} = A$. Then by (d), we have $x_{n+1} - \xi \le A(x_n - \xi)^2 \le A \cdot (A(x_{n-1} - \xi)^2)^2 = A \cdot A^2(x_{n-1} - \xi)^{2^2} \le A \cdot A^2 \cdot \cdots \cdot A^{2^{n-1}}(x_1 - \xi)^{2^n} = A^{2^n - 1}(x_1 - \xi)^{2^n} = \frac{1}{A}[A(x_1 - \xi)]^{2^n}$. On the other hand, we have $x_{n+1} \le \xi$. Hence $0 \le x_{n+1} - \xi \le \frac{1}{A}[A(x_1 - \xi)]^{2^n}$.

(e) Show that Newton's method amounts to finding a fixed point of the function g defined by

$$g(x) = x - \frac{f(x)}{f'(x)}.$$

How does $g'(x)$ behave for x near ξ?

Proof: Clearly, $g(\xi) = \xi - \frac{f(\xi)}{f'(\xi)} = \xi$, and ξ is a fixed point of g. Since $g'(x) = \frac{f(x)f''(x)}{(f'(x))^2}$, we have $g'(\xi) = 0$, and $|g'(x)| \le \frac{M}{\delta^2}|f(x)|$. Since f is continuous, we have $f(x) \to 0$, when $x \to \xi$, then $g'(x) \to 0$, when $x \to \xi$. (namely, g' is continuous at 0)

(f) Put $f(x) = x^{1/3}$ on $(-\infty, \infty)$ and try Newton's method. What happens?

Solution: f is monotonically increasing on $(-\infty, \infty)$ and $f(x) = 0$ if and only if $x = 0$. Since $f'(x) = \frac{1}{3}x^{-2/3} \to \infty$ as $x \to 0$, the hypothesis of applying Newton's method does not hold. Furthermore, we can compute that $g(x) = -2x$, and hence if $x_1 \ne 0$, $x_{n+1} = g(x_n) = \cdots = (-2)^n x_1$ will diverge when $n \to \infty$.

26. Suppose f is differentiable on $[a, b]$, $f(a) = 0$, and there is a real number A such that $|f'(x)| \le |f(x)|$ on $[a, b]$. Prove that $f(x) = 0$ for all $x \in [a, b]$.
Proof: Fix $x_0 \in [a, b]$, let $M_0 = \sup |f(x)|$, $M_1 = \sup |f'(x)|$ for $a \le x \le x_0$. For any such x, we have $|f(x)| \le M_1(x_0 - a) \le A(x_0 - a)M_0$. Hence $M_0 = 0$ if $A(x_0 - a) < 1$. That is, $f = 0$ on $[a, x_0]$.
Now, if we let $x_0 = a + \frac{1}{2A}$, then clearly $A(x_0 - a) = \frac{1}{2} < 1$, hence $f = 0$ on $[a, x_0]$. If we replace a by x_0, then $[a, b]$ became $[x_0, b]$, $f(x_0) = 0$, and $|f'(x)| \le A|f(x)|$ on $[x_0, b]$. We can now proceed on by choose $x_1 = x_0 + \frac{1}{2A} = a + 2\frac{1}{2A}$ and show that $f = 0$ on $[x_0, x_1]$, and so on. Since $x_n = a + (n+1)\frac{1}{2A}$, there is an N such that $b - x_N < \frac{1}{A}$ and thus we can finally stop at $[x_N, b]$ and show that $f = 0$ on $[x_N, b]$. Therefore, $f(x) = 0$ for all $x \in [a, b]$.

27. Let ϕ be a real function defined on a rectangle R in the plane, given by $a \leq x \leq b$, $\alpha \leq y \leq \beta$. A *solution* of the initial-value problem

$$y' = \phi(x, y), \qquad y(a) = c \qquad (\alpha \leq c \leq \beta)$$

is, by definition, a differentiable function f on $[a, b]$ such that $f(a) = c$, $\alpha \leq f(x) \leq \beta$, and

$$f'(x) = \phi(x, f(x)) \qquad (a \leq b).$$

Prove that such a problem has at most one solution if there is a constant A such that

$$|\phi(x, y_2) - \phi(x, y_i)| \leq A|y_2 - y_1|$$

whenever $(x, y_1) \in R$ and $(x, y_2) \in R$.
Proof: Suppose that, on the contrary, there are two solutions f_1 and f_2 corresponding to the same problem, then $f_1'(x) = \phi(x, f_1(x))$, $f_1(a) = c$, and $f_2'(x) = \phi(x, f_2(x))$, $f_2(a) = c$. Let $g(x) = f_1(x) - f_2(x)$, then g is differentiable on $[a, b]$, $g(a) = f_1(a) - f_2(a) = 0$, and $|g'(x)| = |f_1'(x) - f_2'(x)| = |\phi(x, f_1(x)) - \phi(x, f_2(x))| \leq A|f_1(x) - f_2(x)| = A|g(x)|$ on $[a, b]$. By Exercise 22, $g(x) = 0$ for all $x \in [a, b]$. That is, $f_1(x) = f_2(x)$, for all $x \in [a, b]$.
Note: Note that this uniqueness theorem does not hold for the initial-value problem

$$y' = y^{1/2}, \qquad y(0) = 0,$$

which has two solutions: $f(x) = 0$ and $f(x) = x^2/4$.

28. Formulate and prove an analogous uniqueness theorem for systems of differential equations of the form

$$y_j' = \phi_j(x, y_1, ..., y_k), \qquad y_j(a) = c_j \qquad (j = 1, ..., k).$$

Note that this can be rewritten in the form $\mathbf{y}' = \phi(x, \mathbf{y})$, $\quad \mathbf{y}(a) = \mathbf{c}$ where $\mathbf{y} = (y_1, ..., y_k)$ range over a k-cell, ϕ is the mapping of a $(k+1)$-cell into the Euclidean k-space whose components are the functions $\phi_1, ..., \phi_k$, and $\mathbf{c}$ is the vector $(c_1, ..., c_k)$. Use Exercise 26, for vector-valued functions.
Proof: In the case of vector-valued functions, we have a similar result as Exercise 26. Specifically, suppose $\mathbf{f}$ is differentiable on $[a, b]$, $\mathbf{f}(a) = \mathbf{0}$, and there is a real number A such that $|\mathbf{f}'(x)| \leq A|\mathbf{f}(x)|$ on $[a, b]$. Then we have $\mathbf{f}(x) = 0$ for all $x \in [a, b]$. The proof is the same as in the real-value case, by using Theorem 5.19 in place of the mean value theorem.
Now, we can formulate an analogous uniqueness theorem for the above systems of differential equations by stating that such a problem has at most one solution if there is a constant A such that $|\phi(x, \mathbf{y}_2) - \phi(x, \mathbf{y}_1)| \leq A|\mathbf{y}_2 - \mathbf{y}_1|$. The proof is also the same as in Exercise 27, and with real-valued functions replaced by vector-valued functions.

29. Specialize Exercise 28 by considering the system

$$y_j' = y_{j+1} \qquad (j = 1, ..., k-1),$$

$$y_k' = f(x) - \sum_{j=1} k g_j(x) y_j,$$

where $f, g_1, ..., g_k$ are continuous real functions on $[a, b]$, and derive a uniqueness theorem for solutions of the equation

$$y^{(k)} + g_k(x)y^{(k-1)} + \cdots + g_2(x)y' + g_1(x)y = f(x),$$

subject to initial conditions

$$y(a) = c_1, \qquad y'(a) = c_2, \qquad ..., \qquad y^{(k-1)}(a) = c_k.$$

Proof: If we have that the above differentiable equation system has at most one solution, then we can conclude that the give equation has at most one solution, since every solution of the former is the solution of the latter, and vice versa. Then the uniqueness theorem follows, by Exercise 28, that there is a constant A such that $|\phi(x, \mathbf{y}_1) - \phi(x, \mathbf{y}_2)| \le A|\mathbf{y}_1 - \mathbf{y}_2|$. Or, equivalently, $|\mathbf{y}_1' - \mathbf{y}_2'| \le A|\mathbf{y}_1 - \mathbf{y}_2|$.
We have $|\mathbf{y}_1' - \mathbf{y}_2'| = \sqrt{(y_1' - y_2')^2 + (y_1^{(2)} - y_2^{(2)})^2 + \cdots + (y_1^{(k)} - y_2^{(k)})^2} =$
$\sqrt{\sum_{j=0}^{k-1}(y_1^{(j)} - y_2^{(j)})^2 + [(y_1^{(k)} - y_2^{(k)})^2 - (y_1^{(0)} - y_2^{(0)})^2]} =$
$\sqrt{\sum_{j=0}^{k-1}(y_1^{(j)} - y_2^{(j)})^2 + [(\sum_{j=1}^{k} g_j(x)(y_1^{(j-1)} - y_2^{(j-1)}))^2 - (y_1^{(0)} - y_2^{(0)})^2]} \le$
$\sqrt{\sum_{j=0}^{k-1}(y_1^{(j)} - y_2^{(j)})^2 + [(\sum_{j=1}^{k} g_j^2(x) \sum_{j=1}^{k}(y_1^{(j-1)} - y_2^{(j-1)}))^2 - (y_1^{(0)} - y_2^{(0)})^2]}$
(by Schwarz inequality)$\le \sqrt{(1 + \sum_{j=1}^{k} g_j^2(x)) \sum_{j=0}^{k-1}(y_1^{(j)} - y_2^{(j)})^2}$
$= \sqrt{(1 + \sum_{j=1}^{k} g_j^2(x))}|\mathbf{y}_1 - \mathbf{y}_2|$
$\le \sqrt{(1 + \sum_{j=1}^{k} M_i^2)}|\mathbf{y}_1 - \mathbf{y}_2|$, where $M_i = \sup g_i(x)$, $1 \le i \le k$ and $x \in [a, b]$. (Note that since g_i is continuous, g_i is bounded on $[a, b]$, and g_i can achieve its maximum and minimum on $[a, b]$.) Let $A = \sqrt{(1 + \sum_{j=1}^{k} M_i^2)}$, then we have $|\mathbf{y}_1' - \mathbf{y}_2'| \le A|\mathbf{y}_1 - \mathbf{y}_2|$, and then Exercise 28 applies.

6 The Riemann-Stieltjes integral

1. Suppose α increases on $[a,b]$, $a \le x_0 \le b$, α is continuous at x_0, $f(x_0) = 1$, and $f(x) = 0$ if $x \ne x_0$. Prove that $f \in \mathscr{R}(\alpha)$ and that $\int f d\alpha = 0$.
Proof: Given any $\epsilon > 0$, since α is continuous at x_0, there exists a $\delta > 0$ such that $|\alpha(t) - \alpha(x_0)| < \epsilon$, if $|t - x_0| < \delta$. Let P be a partition of $[a,b]$ such that $\Delta x_i < \delta$, for every i, and suppose $x_0 \in [x_{j-1}, x_j]$, for some j. Then we have $U(P,f,\alpha) = M_j \Delta\alpha_j = \alpha(x_j) - \alpha(x_{j-1})$ and $L(P,f,\alpha) = m_j \Delta\alpha_j = 0$. Hence $U(P,f,\alpha) - L(P,f,\alpha) = \alpha(x_j) - \alpha(x_{j-1}) < \epsilon$, and thus $f \in \mathscr{R}(\alpha)$. Furthermore, since for every $\epsilon > 0$ we can find a partition P of $[a,b]$ such that $U(P,f,\alpha) < \epsilon$, and since we have $0 = L(P,f,\alpha) \le U(P,f,\alpha)$, we conclude that $\inf U(P,f,\alpha) = 0$, namely, $\overline{\int} f d\alpha = 0$. Clearly, $\underline{\int} f d\alpha = \sup L(P,f,\alpha) = 0$, we thus have $\int f d\alpha = 0$.

2. Suppose $f \ge 0$, f is continuous on $[a,b]$, and $\int_a^b f(x)dx = 0$. Prove that $f(x) = 0$ for all $x \in [a,b]$. (Compare this with Exercise 1.)
Proof: Suppose that, on the contrary, $f(y) > 0$ at some $y \in [a,b]$. Let $f(y) = M > 0$, and let r be some positive real number such that $0 < r < M$, since f is continuous on $[a,b]$, there exists a $\delta > 0$ such that $|f(t) - f(y)| < M - r$, if $|t - y| < \delta$, which gives $f(t) - f(y) > r - M$ and thus $f(t) > r - M + M = r > 0$.
Now let P be a partition of $[a,b]$ such that $\Delta x_i < \delta$, for every i, and suppose $y \in [x_{j-1}, x_j]$ for some j. Then we have $\int_a^b f(x)dx \ge L(P,f) \ge m_j \Delta x_j > 0$, which is contradict to the assumption that $\int_a^b f dx = 0$.

3. Define three functions β_1, β_2, β_3 as follows: $\beta_j(x) = 0$ if $x < 0$, $\beta_j = 1$ if $x > 0$ for $j = 1,2,3$; and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$. Let f be a bounded function on $[-1,1]$.

   (a) Prove that $f \in \mathscr{R}(\beta_1)$ if and only if $f(0+) = f(0)$ and that then $\int f d\beta_1 = f(0)$.
   Proof: Suppose $f(0+) = f(0)$. Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = -1, x_1 = 0 < x_2 < x_3 = 1$. Then $U(P,f,\beta_1) = M_2$, $L(P,f,\beta_1) = m_2$. If $f(0+) = f(0)$, then M_2 and m_2 converges to $f(0)$ as $x_2 \to 0$. Thus $f \in \mathscr{R}(\beta_1)$, and $\int f d\beta_1 = f(0)$.
   On the other hand, if $f \in \mathscr{R}(\beta_1)$, then given any $\epsilon > 0$, there exists a partition P of $[-1,1]$ such that $U(P,f,\beta_1) - L(P,f,\beta_1) < \epsilon$. Suppose $0 \in [x_{j-1}, x_j]$, for some j. Then $U(P,f,\beta_1) - L(P,f,\beta_1) = M_j - m_j < \epsilon$ (since f is bounded, $M_j = \sup f(x)$, $m_j = \inf f(x)$, $x \in [x_{j-1}, x_j]$ must exist). Pick a $\delta > 0$ such that $[0,\delta] \subseteq [x_{j-1}, x_j]$, then we have $|f(t) - f(0)| \le M_j - m_j < \epsilon$, if $0 < t < \delta$. Hence we have $f(0+) = \lim_{x \to 0+} f(x) = f(0)$.

   (b) State and prove a similar result for β_2.
   Proof: For β_2, we have $f \in \mathscr{R}(\beta_2)$ if and only if $f(0-) = f(0)$. The proof is similar:

Suppose $f(0-) = f(0)$. Consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = -1 < x_1 < x_2 = 0, x_3 = 1$. Then $U(P, f, \beta_2) = M_2$, $L(P, f, \beta_2) = m_2$. If $f(0-) = f(0)$, then M_2 and m_2 converges to $f(0)$ as $x_1 \to 0$. Thus $f \in \mathscr{R}(\beta_2)$, and $\int f d\beta_2 = f(0)$.
On the other hand, if $f \in \mathscr{R}(\beta_2)$, then given any $\epsilon > 0$, there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_2) - L(P, f, \beta_2) < \epsilon$. Suppose $0 \in [x_{j-1}, x_j]$, for some j. Then $U(P, f, \beta_2) - L(P, f, \beta_2) = M_j - m_j < \epsilon$. Pick a $\delta > 0$ such that $[-\delta, 0] \subseteq [x_{j-1}, x_j]$, then we have $|f(t) - f(0)| \le M_j - m_j < \epsilon$, if $-\delta < t < 0$. Hence we have $f(0-) = \lim_{x \to 0-} f(x) = f(0)$.

(c) Prove that $f \in \mathscr{R}(\beta_3)$ if and only if f is continuous at 0.
Proof: Suppose f is continuous at 0. Let $0 < \delta < 1$, consider partitions $P = \{x_0, x_1, x_2, x_3\}$, where $x_0 = -1, x_1 = -\delta, x_2 = \delta, x_3 = 1$. Then $U(P, f, \beta_3) = M_2$, $L(P, f, \beta_3) = m_2$. If f is continuous at 0, then M_2 and m_2 converges to $f(0)$ as $\delta \to 0$. Thus $f \in \mathscr{R}(\beta_3)$, and $\int f d\beta_3 = f(0)$.
On the other hand, if $f \in \mathscr{R}(\beta_3)$, then given any $\epsilon > 0$, there exists a partition P of $[-1, 1]$ such that $U(P, f, \beta_3) - L(P, f, \beta_3) < \epsilon$.
If $0 \notin P$, then $0 \in (x_{j-1}, x_j)$, for some j, and $U(P, f, \beta_3) - L(P, f, \beta_3) = M_j - m_j < \epsilon$. Pick a δ such that $[-\delta, \delta] \subseteq [x_{j-1}, x_j]$, then we have $|f(t) - f(0)| \le M_j - m_j < \epsilon$, if $-\delta < t < \delta$. Hence f is continuous at 0.
If $0 \in P$, suppose $x_j = 0$, for some $0 < j < n$. Then $U(P, f, \beta_3) - L(P, f, \beta_3) = (M_j - m_j) \cdot \frac{1}{2} + (M_{j+1} - m_{j+1}) \cdot \frac{1}{2} < \epsilon$, which gives $M_j - m_j < 2\epsilon$ and $M_{j+1} - m_j < 2\epsilon$. Now pick a $\delta > 0$ such that $[-\delta, \delta] \subseteq [x_{j-1}, x_{j+1}]$, then we have $|f(t) - f(0)| < 2\epsilon$, if $-\delta < t < \delta$. (Note that if $-\delta < t < 0$, $|f(t) - f(0)| \le M_j - m_j < 2\epsilon$, and if $0 < t < \delta$, $|f(t) - f(0)| \le M_{j+1} - m_{j+1} < 2\epsilon$.) Hence, f is continuous at 0.

(d) If f is continuous at 0 prove that

$$\int f d\beta_1 = \int f d\beta_2 = \int f d\beta_3 = f(0).$$

Proof: If f is continuous at 0, then $f(0) = f(0+) = f(0-)$. By (a), (b), (c), we have the desired result.

4. If $f(x) = 0$ for all irrational x, $f(x) = 1$ for all rational x, prove that $f \notin \mathscr{R}$ on $[a, b]$ for any $a < b$.

   Proof: Given any partition P of $[a, b]$, we have $U(P, f) = \sum_{i=1}^{n} 1 \cdot \Delta x_i = b - a$, and $L(P, f) = \sum_{i=1}^{n} 0 \cdot \Delta x_i = 0$. Hence $U(P, f) - L(P, f) = b - a$ and $f \notin \mathscr{R}$.

5. Suppose f is a bounded real function on $[a, b]$, and $f^2 \in \mathscr{R}$ on $[a, b]$. Does it follow that $f \in \mathscr{R}$? Does the answer change if we assume that $f^3 \in \mathscr{R}$?

Solution: Even if $f^2 \in \mathscr{R}$ on $[a,b]$, we cannot conclude that $f \in \mathscr{R}$ on $[a,b]$. For example, let $f(x) = -1$ for all irrational x, $f(x) = 1$ for all rational x, similarly as Exercise 4, we have $f \notin \mathscr{R}$. Clearly, f is bounded and $f^2(x) = 1$, for every x. Thus $f^2 \in \mathscr{R}$, and $\int_a^b f^2(x)dx = b - a$.
If $f^3 \in \mathscr{R}$, however, we can conclude that $f \in \mathscr{R}$, too. Since f is bounded on $[a,b]$, then f^3 is bounded on $[a,b]$. Suppose $m \le f^3 \le M$, $\phi = y^{\frac{1}{3}}$ is continuous (and one-to-one) on $[m,M]$. Since $f^3 \in \mathscr{R}$, by Theorem 6.11, we have $f = \phi(f^3) \in \mathscr{R}$ on $[a,b]$.

6. Let P be the Cantor set constructed in Sec. 2.44. Let f be a bounded real function on $[0,1]$ which is continuous at every point outside P. Prove that $f \in \mathscr{R}$ on $[0,1]$.
Proof: Note that P can be covered by finitely many disjoint segments whose total length can be as small as desired. (But how to prove this formally?) Now, let $\epsilon > 0$ be given, put $M = \sup|f(x)|$, and let the finitely many disjoint segments which cover P be (u_j, v_j) such that the sum of the corresponding differences $v_j - u_j$ is less than ϵ.
Remove the segments (u_j, v_j) from $[0,1]$ (if $(u_j, v_j) \not\subseteq [0,1]$, remove their intersection part). The remaining set K is compact, since $K = (\bigcap (u_j, v_j)^c) \cap [0,1]$ is an intersection of closed sets, thus K is a closed subset of $[0,1]$ and $[0,1]$ is compact. Since f is continuous on K, f is uniformly continuous on K, and there exists a $\delta > 0$ such that $|f(s) - f(t)| < \epsilon$ if $s \in K$, $t \in K$, $|s - t| < \delta$.
Now form a partition $P' = \{x_0, x_1, ..., x_n\}$ of $[0,1]$, as follows: If $u_j \in [0,1]$, $u_j \in P'$. If $v_j \in [0,1]$, $v_j \in P'$. (But note that only u_1 and v_n will not in $[0,1]$.) No point of any segment (u_j, v_j) occurs in P'. If x_{i-1} is not one of the u_j, then $\Delta x_i < \delta$.
Note that $M_i - m_i \le 2M$ for every i, and that $M_i - m_i \le \epsilon$ unless x_{i-1} is one of the u_j. Hence we have

$$U(P', f) - L(P', f) \le 2M\epsilon + \epsilon(1 - 0) = (2M + 1)\epsilon.$$

Since ϵ is arbitrary, Theorem 6.6 shows that $f \in \mathscr{R}$.

7. Suppose f is a real function on $(0,1]$ and $f \in \mathscr{R}$ on $[c,1]$ for every $c > 0$. Define

$$\int_0^1 f(x)dx = \lim_{c \to 0} \int_c^1 f(x)dx$$

if this limit exists (and is finite).

(a) If $f \in \mathscr{R}$ on $[0,1]$, show that this definition of the integral agrees with the old one.
Proof: Given any $\epsilon > 0$, if we can prove that there exists an $r(0 < r < 1)$ such that $0 < c < r$ implies $|\int_c^1 f(x)dx - \int_0^1 f(x)dx| < \epsilon$, we are done.
Firstly, let's show that if $f \in \mathscr{R}$ on $[0,1]$, f is bounded on $[0,1]$. Suppose that f is unbounded, let P be any partition of $[0,1]$. Then

there is at least one $[x_{j-1}, x_j]$ such that f is unbounded on $[x_{j-1}, x_j]$. No matter which $\epsilon > 0$ is given, we can put $\delta = x_j - x_{j-1}$ and find two points $s, t \in [x_{j-1}, x_j]$ such that $f(s) - f(t) > \frac{\epsilon}{\delta}$. Then $U(P, f) - L(P, f) > (f(s) - f(t))\delta > \epsilon$, and hence $f \notin \mathscr{R}$, a contradiction.

Next, Let P be any partition of $[0, 1]$ such that $c \in P$, and let P_c be the partition of $[c, 1]$ with respect to P. Suppose $x_j = c$, then $U(P, f) = \sum_{i=1}^{n} M_i \Delta x_i = \sum_{i=1}^{j} M_i \Delta x_i + U(P_c, f)$. Since f is bounded on $[0, 1]$, we have $|f(x)| \leq M$, for every $x \in [0, 1]$. Then $|U(P, f) - U(P_c, f)| = |\sum_{i=1}^{j} M_i \Delta x_i| \leq M |\sum_{i=1}^{j} \Delta x_i| = M(x_j - x_0) = Mc$. Given any $\epsilon > 0$, we can pick an r $(0 < r < 1)$ such that $r < \epsilon/M$. Then when $0 < c < r$, we have $Mc < Mr < \epsilon$, which gives $|U(P, f) - U(P_c, f)| < \epsilon$.

Since $f \in \mathscr{R}$ on $[0, 1]$, we have $\int_0^1 f(x)dx = \overline{\int_0^1} f(x)dx = \inf U(P_{[0,1]}, f)$, and $\int_c^1 f(x)dx = \overline{\int_c^1} f(x)dx = \inf U(P_{[c,1]}, f)$. Then for the above ϵ, we can choose a partition P_1 of $[0, 1]$ such that $U(P_1, f) < \int_0^1 f(x)dx + \epsilon$. If $c \in P_1$, we are done and put $P' = P_1$; otherwise, let $P_2 = P_1 \cup \{c\}$, then P_2 is a refinement of P_1 and thus $U(P_2, f) \leq U(P_1, f) < \int_0^1 f(x)dx + \epsilon$, we put $P' = P_2$. Now we find a partition P' of $[0, 1]$ such that $c \in P'$ and $\int_0^1 f(x)dx \leq U(P', f) < \int_0^1 f(x)dx + \epsilon$.
Similarly, for this ϵ and $0 < c < r$, we can find a partition P'_c of $[c, 1]$ such that $\int_c^1 f(x)dx \leq U(P'_c, f) < \int_c^1 f(x)dx + \epsilon$. Now let $P^{(2)} = P'_c \cup \{0\}$, then $P^{(2)}$ is a partition of $[0, 1]$ and $c \in P^{(2)}$. Let P^* be the common refinement of P' and $P^{(2)}$ (thus $c \in P^*$) and let P^*_c be the partition on $[c, 1]$ with respect to P^*, we then have $\int_0^1 f(x)dx \leq U(P^*, f) \leq U(P', f) < \int_0^1 f(x)dx + \epsilon$ (i.e., $|U(P^*, f) - \int_0^1 f(x)dx| < \epsilon$), $\int_c^1 f(x)dx \leq U(P^*_c, f) \leq U(P'_c, f) < \int_c^1 f(x)dx + \epsilon$, (i.e., $|\int_c^1 f(x)dx - U(P^*_c, f)| < \epsilon$) and $|U(P^*_c, f) - U(P^*, f)| < \epsilon$. Therefore, $|\int_c^1 f(x)dx - \int_0^1 f(x)dx| \leq |\int_c^1 f(x)dx - U(P^*_c, f)| + |U(P^*_c, f) - U(P^*, f)| + |U(P^*, f) - \int_0^1 f(x)dx| < 3\epsilon$, which completes our proof.

(b) Construct a function f such that the above limit exists, although it fails to exist with $|f|$ in place of f.
Solution: An example is $f(x) = \frac{1}{x} \sin \frac{1}{x}$. Since we can find from many mathematical analysis textbooks that $\int_1^\infty \frac{\sin x}{x} dx$ converges but $\int_1^\infty |\frac{\sin x}{x}| dx$ diverges. If we replace x by $\frac{1}{t}$, then $\int_1^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{1}{t} \sin \frac{1}{t} dt$ and $\int_1^\infty |\frac{\sin x}{x}| dx = \int_0^1 |\frac{1}{t} \sin \frac{1}{t}| dt$. (But how to prove this with the knowledge we have learned from this book is still not clear

to me...)

8. Suppose $f \in \mathscr{R}$ on $[a,b]$ for every $b > a$ where a is fixed. Define

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx$$

if this limit exists (and is finite). In that case, we say that the integral on the left *converges*. If it also converges after f has been replaced by $|f|$, it is said to converge *absolutely*.
Assume that $f(x) \geq 0$ and that f decreases monotonically on $[1,\infty)$. Prove that $\int_1^\infty f(x)dx$ converges if and only if $\sum\limits_{n=1}^{\infty} f(n)$ converges. (This is the so-called "integral test" for convergence of series.)
Proof:
$\Rightarrow$: Suppose $\int_1^\infty f(x)dx$ converges. Let $\{y_n\}$ be a real sequence such that $y_n = \int_1^n f(x)dx$, then $\{y_n\}$ converges. Thus given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > m > N$ implies $|y_n - y_m| < \epsilon$, which is equivalent to say $|\int_m^n f(x)dx| < \epsilon$. Since $f(x) \geq 0$, $\int_m^n f(x)dx \geq 0$, and hence $\int_m^n f(x)dx < \epsilon$. Now form a partition $P = \{x_0 = m, x_1 = m+1, ..., x_{n-m} = n\}$ of $[m,n]$, then $U(P,f) = \sum\limits_{k=1}^{n-m} M_k \Delta x_k = \sum\limits_{k=1}^{n-m} M_k = \sum\limits_{k=1}^{n-m} f(x_{k-1}) = \sum\limits_{k=1}^{n-m} f(m+k-1) = \sum\limits_{k=m}^{n-1} f(k)$ and similarly, $L(P,f) = \sum\limits_{k=1}^{n-m} f(x_k) = \sum\limits_{k=1}^{n-m} f(m+k) = \sum\limits_{k=m+1}^{n} f(k)$, since $f(x)$ decreases monotonically.
Note that we have $L(P,f) \leq \underline{\int_m^n} f(x)dx = \int_m^n f(x)dx$, and thus $\sum\limits_{k=m+1}^{n} f(k) \leq \int_m^n f(x)dx < \epsilon$. Since $f(x) \geq 0$, $|\sum\limits_{k=m+1}^{n} f(k)| = \sum\limits_{k=m+1}^{n} f(k) < \epsilon$, then $\sum\limits_{n=1}^{\infty} f(n)$ converges, by Cauchy criterion.
$\Leftarrow$: Suppose $\sum\limits_{n=1}^{\infty} f(n)$ converges. Then given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > m > N$ implies $|\sum\limits_{k=m+1}^{n} f(k)| < \epsilon$, which gives $\sum\limits_{k=m+1}^{n} f(k) < \epsilon$ since $f(x) \geq 0$. Form the same sequence $\{y_n\}$ as the previous proof such that $y_n = \int_1^n f(x)dx$. Let $N' = N+1$ and $n' > m' > N'$, then $|y_{n'} - y_{m'}| = |\int_{m'}^{n'} f(x)dx| = \int_{m'}^{n'} f(x)dx$. Now form a partition $P' = \{x_0 = m', x_1 = m'+1, ..., x_{n'-m'} = n'\}$ of $[m',n']$, then similarly as the previous proof, we have $U(P',f) = \sum\limits_{k=m'}^{n'-1} f(k) = \sum\limits_{k=(m'-1)+1}^{n'-1} f(k)$. Since $n' > m' > N' = N+1$, $n'-1 > m'-1 > N'-1 = N$, hence

$\sum\limits_{k=(m'-1)+1}^{n'-1} f(k) < \epsilon$, namely, $U(P', f) < \epsilon$. Therefore $|y_{n'} - y_{m'}| = |\int_{m'}^{n'} f(x)dx| = \int_{m'}^{n'} f(x)dx \le U(P', f) < \epsilon$, and by Cauchy criterion, $\{y_n\}$ converges.
Note that for any $b > 1$, we can find an $n \in \mathbb{N}$ such that $n \le b \le n+1$, and thus $\int_1^n f(x)dx \le \int_1^b f(x)dx \le \int_1^{n+1} f(x)dx$. Therefore, $\int_1^\infty f(x)dx = \lim\limits_{b\to\infty} \int_1^b f(x)dx = \lim\limits_{n\to\infty} \int_1^n f(x)dx = \lim\limits_{n\to\infty} y_n$. Since $\lim\limits_{n\to\infty} y_n$ exists, $\int_1^\infty f(x)dx$ converges.

9. Show that integration by parts can sometimes be applied to the "improper" integrals defined in Exercise 7 and 8. (State appropriate hypotheses, formulate a theorem, and prove it.) For instance show that

$$\int_0^\infty \frac{\cos x}{1+x}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx.$$

Show that one of these integrals converges *absolutely*, but that the other does not.
Proof:
(i) Suppose F and G are differentiable functions on $[a, \infty)$, $F' = f \in \mathscr{R}$, and $G' = g \in \mathscr{R}$, on $[a, b]$ for every $b > a$. What's more, $\lim\limits_{b\to\infty} F(b)G(b)$, $\int_a^\infty F(x)g(x)dx$ and $\int_a^\infty f(x)G(x)dx$ exists. Then

$$\int_a^\infty F(x)g(x)dx = \lim_{b\to\infty} F(b)G(b) - F(a)G(a) - \int_a^\infty f(x)G(x)dx.$$

To see this, similarly as in the proof of Theorem 6.22, let $H(x) = F(x)G(x)$, and we have $\int_a^b H'(x)dx = H(b)-H(a)$, namely $\int_a^b F(x)g(x)dx = F(b)G(b) - F(a)G(a) - \int_a^b f(x)G(x)dx$, for every $b > a$. Let $b \to \infty$, we have the desired result.
(ii) $\int_0^\infty \frac{\cos x}{1+x}dx = \lim\limits_{b\to\infty} \frac{\sin b}{1+b} - \frac{\sin 0}{1+0} - \int_0^\infty \frac{\sin x}{-(1+x)^2}dx = \int_0^\infty \frac{\sin x}{(1+x)^2}dx$.
(iii) We can show that $\int_0^\infty \frac{\sin x}{(1+x)^2}dx$ converges absolutely, but $\int_0^\infty \frac{\cos x}{1+x}dx$ does not. To see this, define $y_n = \int_0^n \frac{|\sin x|}{(1+x)^2}dx$, then we have $y_n \le y_{n+1}$, and $0 \le y_n \le \int_0^\infty \frac{1}{(1+x)^2}dx = 1$. Hence $\{y_n\}$ converges, so is $\int_0^\infty \frac{|\sin x|}{(1+x)^2}dx$. But $\int_0^\infty \frac{|\cos x|}{1+x}$ does not converge. Since $0 \le |\cos x| \le 1$, we have $|\cos x| \ge (\cos x)^2 = \frac{1+\cos 2x}{2}$. Then $\int_0^\infty \frac{|\cos x|}{1+x} \ge \int_0^\infty \frac{1}{2(1+x)}dx + \int_0^\infty \frac{\cos 2x}{2(1+x)}dx$. Since $\int_0^\infty \frac{1}{2(1+x)}dx$ diverges (i.e., $\to \infty$), and $|\int_0^\infty \frac{\cos 2x}{2(1+x)}dx| = \frac{1}{4}|\int_0^\infty \frac{\sin 2x}{(1+x)^2}dx| \le \frac{1}{4}\int_0^\infty \frac{|\sin 2x|}{(1+x)^2} \le \frac{1}{4}\int_0^\infty \frac{1}{(1+x)^2} = \frac{1}{4}$. Hence $\int_0^\infty \frac{\cos 2x}{2(1+x)}dx$ is bounded, and therefore, $\int_0^\infty \frac{|\cos x|}{1+x}$ diverges. $(\to \infty)$

10. Let p and q be positive real numbers such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Prove the following statements.

(a) If $u \geq 0$ and $v \geq 0$, then

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}.$$

Equality holds if and only if $u^p = v^q$.

Proof: Consider the function $f(x) = \ln x$, $x \in (0, \infty)$. We have $f'(x) = \frac{1}{x} > 0$, and $f''(x) = -\frac{1}{x^2} < 0$. Then we know that $f(x)$ is monotonically increasing, and $g(x) = -\ln x$ is convex.

Hence, $-\ln uv = -(\ln u + \ln v) = -(\frac{1}{p}\ln u^p + \frac{1}{q}\ln v^q) \geq -\ln(\frac{1}{p}u^p + 1qv^q)$ (since $\frac{1}{p} + \frac{1}{q} = 1$, and $-\ln x$ is convex). Therefore, $-\ln uv \geq -\ln(\frac{1}{p}u^p + \frac{1}{q}v^q)$ and thus $\ln uv \leq \ln(\frac{1}{p}u^p + \frac{1}{q}v^q)$. Due to the monotone property of $\ln x$, we have $uv \leq \frac{1}{p}u^p + \frac{1}{q}v^q$. Clearly, equality holds if and only if $u^p = v^q$.

(b) If $f \in \mathscr{R}(\alpha)$, $g \in \mathscr{R}(\alpha)$, $f \geq 0$, $g \geq 0$, and

$$\int_a^b f^p d\alpha = 1 = \int_a^b g^q d\alpha,$$

then

$$\int_a^b fg d\alpha \leq 1.$$

Proof: By (a), we have $fg \leq \frac{1}{p}u^p + \frac{1}{q}v^q$, and hence, $\int_a^b fg d\alpha \leq \int_a^b (\frac{1}{p}u^p + \frac{1}{q}v^q) d\alpha = \int_a^b \frac{1}{p}u^p d\alpha + \int_a^b \frac{1}{q}v^q) d\alpha = \frac{1}{p} \cdot 1 + \frac{1}{q} \cdot 1 = \frac{1}{p} + \frac{1}{q} = 1$.

(c) If f and g are complex functions in $\mathscr{R}(\alpha)$, then

$$|\int_a^b fg d\alpha| \leq \{\int_a^b |f|^p d\alpha\}^{1/p} \{\int_a^b |g|^q d\alpha\}^{1/q}.$$

This is *Holder's inequality*. When $p = q = 2$ it is usually called the Schwarz inequality. (Note that Theorem 1.35 is very special case of this.)

Proof: We have that

$$\frac{|\int_a^b fg d\alpha|}{\{\int_a^b |f|^p d\alpha\}^{1/p} \{\int_a^b |g|^q d\alpha\}^{1/q}} \leq \frac{\int_a^b |fg| d\alpha}{\{\int_a^b |f|^p d\alpha\}^{1/p} \{\int_a^b |g|^q d\alpha\}^{1/q}},$$

and

$$\frac{\int_a^b |fg| d\alpha}{\{\int_a^b |f|^p d\alpha\}^{1/p} \{\int_a^b |g|^q d\alpha\}^{1/q}} = \frac{\int_a^b |f||g| d\alpha}{\{\int_a^b |f|^p d\alpha\}^{1/p} \{\int_a^b |g|^q d\alpha\}^{1/q}},$$

which equals

$$\int_a^b (\frac{|f|^p}{\int_a^b |f|^p d\alpha})^{1/p} (\frac{|g|^q}{\int_a^b |g|^q d\alpha})^{1/q} d\alpha,$$

and by (a), we have

$$\int_a^b \left(\frac{|f|^p}{\int_a^b |f|^p d\alpha}\right)^{1/p} \left(\frac{|g|^q}{\int_a^b |g|^q d\alpha}\right)^{1/q} d\alpha \leq \int_a^b \left[\frac{1}{p}\left(\frac{|f|^p}{\int_a^b |f|^p d\alpha}\right) + \frac{1}{q}\left(\frac{|g|^q}{\int_a^b |g|^q d\alpha}\right)\right] d\alpha,$$

and

$$\int_a^b \left[\frac{1}{p}\left(\frac{|f|^p}{\int_a^b |f|^p d\alpha}\right) + \frac{1}{q}\left(\frac{|g|^q}{\int_a^b |g|^q d\alpha}\right)\right] d\alpha = \frac{1}{p}\frac{\int_a^b |f|^p d\alpha}{\int_a^b |f|^p d\alpha} + \frac{1}{q}\frac{\int_a^b |g|^q d\alpha}{\int_a^b |g|^q d\alpha},$$

and

$$\frac{1}{p}\frac{\int_a^b |f|^p d\alpha}{\int_a^b |f|^p d\alpha} + \frac{1}{q}\frac{\int_a^b |g|^q d\alpha}{\int_a^b |g|^q d\alpha} = \frac{1}{p} + \frac{1}{q} = 1,$$

which gives

$$\frac{|\int_a^b fg d\alpha|}{\{\int_a^b |f|^p d\alpha\}^{1/p}\{\int_a^b |g|^q d\alpha\}^{1/q}} \leq 1.$$

Therefore,

$$|\int_a^b fg d\alpha| \leq \{\int_a^b |f|^p d\alpha\}^{1/p}\{\int_a^b |g|^q d\alpha\}^{1/q}.$$

(d) Show that Holder's inequality is also true for the "improper" integrals described in Exercise 7 and 8.

Proof: As the case of Exercise 7, we have

$$|\int_c^1 fg d\alpha| \leq \{\int_c^1 |f|^p d\alpha\}^{1/p}\{\int_c^1 |g|^q d\alpha\}^{1/q},$$

for every $c > 0$, and thus we have

$$\lim_{c\to 0} |\int_c^1 fg d\alpha| \leq \lim_{c\to 0}\{\int_c^1 |f|^p d\alpha\}^{1/p} \lim_{c\to 0}\{\int_c^1 |g|^q d\alpha\}^{1/q},$$

which gives

$$|\int_0^1 fg d\alpha| \leq \{\int_0^1 |f|^p d\alpha\}^{1/p}\{\int_0^1 |g|^q d\alpha\}^{1/q}.$$

Similarly, as the case of Exercise 8, we have

$$|\int_a^b fg d\alpha| \leq \{\int_a^b |f|^p d\alpha\}^{1/p}\{\int_a^b |g|^q d\alpha\}^{1/q},$$

for every $b > a$, and thus we have

$$\lim_{b\to\infty} |\int_a^b fg d\alpha| \leq \lim_{b\to\infty}\{\int_a^b |f|^p d\alpha\}^{1/p} \lim_{b\to\infty}\{\int_a^b |g|^q d\alpha\}^{1/q},$$

which gives

$$|\int_a^\infty fg d\alpha| \leq \{\int_a^\infty |f|^p d\alpha\}^{1/p}\{\int_a^\infty |g|^q d\alpha\}^{1/q}.$$

11. Let α be a fixed increasing function on $[a, b]$. For $u \in \mathscr{R}(\alpha)$, define

$$||u||_2 = \left\{ \int_a^b |u|^2 d\alpha \right\}^{1/2}.$$

Suppose f, g, $h \in \mathscr{R}(\alpha)$, and prove the triangle inequality

$$||f - h||_2 \leq ||f - g||_2 + ||g - h||_2$$

as a consequence of the Schwarz inequality, as in the proof of Theorem 1.37.

Proof: First we prove that if $u \in \mathscr{R}$, $v \in \mathscr{R}$, than $||u+v||_2 \leq ||u||_2 + ||v||_2$. We have $||u + v||_2 = \left\{ \int_a^b |u + v|^2 d\alpha \right\}^{1/2}$, and hence $||u + v||_2^2 = \int_a^b |u + v|^2 d\alpha \leq \int_a^b (|u| + |v|)^2 d\alpha$(by Schwarz inequality) $= \int_a^b |u|^2 d\alpha + \int_a^b |v|^2 d\alpha + 2\int_a^b |u||v| d\alpha \leq \int_a^b |u|^2 d\alpha + \int_a^b |v|^2 d\alpha + 2(\int_a^b |u|^2 d\alpha)^{1/2} (\int_a^b |v|^2 d\alpha)^{1/2}$ (by Holder's inequality)$= ((\int_a^b |u|^2 d\alpha)^{1/2} + (\int_a^b |v|^2 d\alpha)^{1/2})^2 = (||u||_2 + ||v||_2)^2$. Therefore, $||u + v|| \leq ||u||_2 + ||v||_2$.

If we replace $u = f - g$, $v = g - h$, then we have the desired result.

12. With the notations of Exercise 11, suppose $f \in \mathscr{R}(\alpha)$ and $\epsilon > 0$. Prove that there exists a continuous function g on $[a, b]$ such that $||f - g||_2 < \epsilon$.

Proof: Let $P = \{x_0, ..., x_n\}$ be any partition of $[a, b]$, define

$$g(t) = \frac{x_i - t}{\Delta x_i} f(x_{i-1}) + \frac{t - x_{i-1}}{\Delta x_i} f(x_i)$$

if $x_{i-1} \leq t \leq x_i$.

g is clearly continuous since $g(x_i+) = g(x_i-) = f(x_i) = g(x_i)$. Then we have

$$|f(t) - g(t)| = |\frac{x_i - t}{\Delta x_i}(f(t) - f(x_{i-1})) + \frac{t - x_{i-1}}{\Delta x_i}(f(t) - f(x_i))|,$$

which gives that

$$|f(t)-g(t)| \leq |\frac{x_i - t}{\Delta x_i}| \cdot |f(t)-f(x_{i-1})| + |\frac{t - x_{i-1}}{\Delta x_i}| \cdot |f(t)-f(x_i)| \leq M_i - m_i,$$

for $x_{i-1} \leq t \leq x_i$.

Since $f \in \mathscr{R}(\alpha)$, f is bounded, we can suppose that $|f| \leq M$ on $[a, b]$. Furthermore, given any $\epsilon > 0$, we can choose a partition P such that $U(P, f, \alpha) - L(P, f, \alpha) < \frac{\epsilon^2}{2M}$, namely, $\sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i < \frac{\epsilon^2}{2M}$. Hence, we have that

$$||f - g||_2 = \left\{ \int_a^b |f - g|^2 d\alpha \right\}^{1/2} = \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f - g|^2 d\alpha \right\}^{1/2}$$

$$\leq \left\{ \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (M_i - m_i)^2 d\alpha \right\}^{1/2} \leq \left\{ 2M \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} (M_i - m_i) d\alpha \right\}^{1/2}$$

$$= \left\{ 2M \sum_{i=1}^{n} (M_i - m_i)\Delta\alpha_i \right\}^{1/2} < \left\{ 2M \frac{\epsilon^2}{2M} \right\}^{1/2} = \epsilon,$$

which gives the desired result.

13. Define

$$f(x) = \int_x^{x+1} \sin(t^2) dt.$$

(a) Prove that $|f(x)| < 1/x$ if $x > 0$.
Proof: Put $t^2 = u$ (namely, $t = \sqrt{u}$), we have $du = 2tdt = 2\sqrt{u}dt$, or equivalently, $dt = \frac{du}{\sqrt{u}}$. Hence $f(x) = \int_{x^2}^{(x+1)^2} \frac{\sin u}{\sqrt{u}} du$, and if we integrate by parts, we have that $f(x) = -(\frac{\cos((x+1)^2)}{2(x+1)} - \frac{\cos(x^2)}{2x}) - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$.
Then $|f(x)| \leq |\frac{\cos(x^2)}{2x}| + |\frac{\cos((x+1)^2)}{2(x+1)}| + |\int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du| < \frac{1}{2x} + \frac{1}{2(x+1)} + \int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} du = \frac{1}{2x} + \frac{1}{2(x+1)} - (\frac{1}{2(x+1)} - \frac{1}{2x}) = \frac{1}{x}$, if $x > 0$.

(b) Prove that

$$2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$$

where $|r(x)| < c/x$ and c is a constant.
Proof: As in (a), we have $f(x) = \frac{\cos(x^2)}{2x} - \frac{\cos((x+1)^2)}{2(x+1)} - \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$, and thus $2xf(x) = \cos(x^2) - \cos[(x+1)^2] + r(x)$, where $r(x) = \frac{\cos[(x+1)^2]}{x+1} - 2x \int_{x^2}^{(x+1)^2} \frac{\cos u}{4u^{3/2}} du$. Hence $|r(x)| \leq \frac{|\cos[(x+1)^2]|}{x+1} + 2x|\int_{x^2}^{(x+1)^2} \frac{|\cos u|}{4u^{3/2}} du| \leq \frac{1}{x+1} + 2x|\int_{x^2}^{(x+1)^2} \frac{1}{4u^{3/2}} \leq \frac{1}{x+1} + 2x|\frac{1}{2(x+1)} - \frac{1}{2x}| = \frac{1}{x+1} + 2x(\frac{1}{2x} - \frac{1}{2(x+1)}) = \frac{2}{x+1} < \frac{2}{x}$.

(c) Find the upper and lower limits of $xf(x)$, as $x \to \infty$.

(d) Does $\int_0^\infty \sin(t^2) dt$ converge?

14. Deal similarly with

$$f(x) = \int_x^{x+1} \sin(e^t) dt.$$

Show that

$$e^x |f(x)| < 2$$

and that

$$e^x f(x) = \cos(e^x) - e^{-t} \cos(e^{x+1}) + r(x),$$

where $|r(x)| < Ce^{-x}$, for some constant C.
Proof: Similarly, put $u = e^t$, and then $du = e^t dt = udt$, namely, $dt = \frac{du}{u}$.

We thus have $f(x) = \int_{e^x}^{e^{x+1}} \frac{\sin u}{u} du = \frac{\cos e^x}{e^x} - \frac{\cos e^{x+1}}{e^{x+1}} - \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$.
Therefore, $e^x|f(x)| = |\cos e^x| + \frac{|\cos e^{x+1}|}{e} + e^x|\int_{e^x}^{e^{x+1}} \frac{|\cos u|}{u^2} du| < 1 + \frac{1}{e} + e^x|\int_{e^x}^{e^{x+1}} \frac{1}{u^2} du| = 1 + \frac{1}{e} + e^x(\frac{1}{e^x} - \frac{1}{e^{x+1}}) = 2$.
$e^x f(x) = \cos(e^x) - \frac{\cos(e^{x+1})}{e} - e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$, which gives
$r(x) = -e^x \int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du$. Hence $|r(x)| = e^x|\int_{e^x}^{e^{x+1}} \frac{\cos u}{u^2} du|$
$< e^x \frac{1}{e^{2x}} |\int_{e^x}^{e^{x+1}} \cos u du| = e^{-x}|\sin(e^{x+1}) - \sin(e^x)| \leq 2e^{-x}$, which gives the desired result.

15. Suppose f is a real, continuously differentiable function on $[a, b]$, $f(a) = f(b) = 0$, and

$$\int_a^b f^2(x)dx = 1.$$

Prove that

$$\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$$

and that

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx > \frac{1}{4}.$$

Proof: We have

$$\int_a^b xf(x)f'(x)dx = bf^2(b) - af^2(a) - \int_a^b f(x)(f(x) + xf'(x))dx$$

$$= 0-(\int_a^b f^2(x)dx + \int_a^b xf(x)f'(x)dx) = -\int_a^b f^2(x)dx - \int_a^b xf(x)f'(x)dx,$$

which gives that

$$\int_a^b xf(x)f'(x)dx = -\frac{1}{2}\int_a^b f^2(x)dx = -\frac{1}{2}.$$

By Holder's inequality, we have

$$|\int_a^b xf(x)f'(x)dx|^2 \leq \int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx,$$

which gives

$$\int_a^b [f'(x)]^2 dx \cdot \int_a^b x^2 f^2(x)dx \geq \frac{1}{4}.$$

Since the equality cannot hold in this case, we have the desired result. (Note that if the equality hold, then we have that $\frac{f'^2(x)}{\int_a^b f'^2(x)dx} = \frac{(xf(x))^2}{\int_a^b (xf(x))^2 dx}$.
Equivalently, we have $|f'(x)| = M|xf(x)|$, where $M = \sqrt{\frac{\int_a^b f'^2(x)dx}{\int_a^b (xf(x))^2 dx}}$.

Since $\int_a^b xf(x)f'(x)dx = -\frac{1}{2}$, we have $f'(x) = -Mxf(x) = Cxf(x)$ $(C = -M)$, namely, $\frac{df(x)}{dx} = Cxf(x)$, i.e., $\frac{df(x)}{f(x)} = Cxdx$. Solving this equation gives us that $\ln f(x) = \frac{1}{2}Cx^2 + K'$, namely, $f(x) = Ke^{(1/2)Cx^2}$, where $K = e^{K'} > 0$. But since $f(a) = f(b) = 0$, we have $K = 0$, a contradiction.)

16. For $1 < s < \infty$, define

$$\xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

(This is Riemann's zeta function, of great importance in the study of the distribution of prime numbers.) Prove that

(a) $\xi(s) = s\int_1^\infty \frac{[x]}{x^{s+1}}dx$.
Proof: Let's compute the difference between the integral over $[1, N]$ and the Nth Partial sum of the series that defines $\xi(s)$. This gives that

$$|s\int_1^N \frac{[x]}{x^{s+1}}dx - \sum_{n=1}^{N} \frac{1}{n^s}| = |s\sum_{n=1}^{N-1} \int_n^{n+1} \frac{[x]}{x^{s+1}}dx - \sum_{n=1}^{N} \frac{1}{n^s}|$$

$$= |s\sum_{n=1}^{N-1} n\int_n^{n+1} \frac{1}{x^{s+1}}dx - \sum_{n=1}^{N} \frac{1}{n^s}| = |\sum_{n=1}^{N-1} (\frac{n}{n^s} - \frac{n}{(n+1)^s}) - \sum_{n=1}^{N} \frac{1}{n^s}|$$

$$= |\sum_{n=1}^{N-1} (\frac{n-1}{n^s} - \frac{n}{(n+1)^s}) - \frac{1}{N^s}| = |0 - \frac{N-1}{N^s} - \frac{1}{N^s}| = \frac{1}{N^{s-1}}.$$

Let $N \to \infty$ and we have

$$|\lim_{N\to\infty} s\int_1^N \frac{[x]}{x^{s+1}}dx - \lim_{n\to\infty} \sum_{n=1}^{N} \frac{1}{n^s}| = 0,$$

which gives

$$s\int_1^\infty \frac{[x]}{x^{s+1}}dx = \sum_{n=1}^{\infty} \frac{1}{n^s}, i.e., \xi(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = s\int_1^\infty \frac{[x]}{x^{s+1}}dx.$$

(b)

$$\xi(s) = \frac{s}{s-1} - s\int_1^\infty \frac{x - [x]}{x^{s+1}}dx,$$

where $[x]$ denotes the greatest integer $\le x$. Prove that the integral in (b) converges for all $s > 0$.
Proof: We have

$$\frac{s}{s-1} - s\int_1^\infty \frac{x-[x]}{x^{s+1}}dx = \frac{s}{s-1} - s\int_1^\infty \frac{1}{x^s}dx + s\int_1^\infty \frac{[x]}{x^{s+1}}dx$$

$$= \frac{s}{s-1} - s \cdot \frac{1}{s-1} + \xi(s) = \xi(s).$$

Furthermore,

$$0 \leq \int_1^\infty \frac{x-[x]}{x^{s+1}} dx \leq \int_1^\infty \frac{1}{x^{s+1}} dx = \frac{1}{s}.$$

Let

$$y_n = \int_1^n \frac{x-[x]}{x^{s+1}} dx,$$

and we have $0 \leq y_n \leq \frac{1}{s}$, for every n. Then $\{y_n\}$ is bounded and since $y_n < y_{n+1}$, $\{y_n\}$ converges. Therefore, $\int_1^\infty \frac{x-[x]}{x^{s+1}} dx$ converges.

17. Suppose α increases monotonically on $[a, b]$, g is continuous, and $g(x) = G'(x)$ for $a \leq x \leq b$. Prove that

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b G d\alpha.$$

Proof: Take g real, without loss of generality. Given $P = \{x_0, x_1, ..., x_n\}$, choose $t_i \in (x_{i-1}, x_i)$ so that $g(t_i)\Delta x_i = G(x_i) - G(x_{i-1})$. Then we have that

$$\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i = \sum_{i=1}^n \alpha(x_i)(G(x_i) - G(x_{i-1}))$$

$$= \sum_{i=1}^n [\alpha(x_i)G(x_i) - \alpha(x_{i-1})G(x_{i-1}) + \alpha(x_{i-1})G(x_{i-1}) - \alpha(x_i)G(x_{i-1})]$$

$$= \sum_{i=1}^n [\alpha(x_i)G(x_i) - \alpha(x_{i-1})G(x_{i-1})] + \sum_{i=1}^n [\alpha(x_{i-1})G(x_{i-1}) - \alpha(x_i)G(x_{i-1})]$$

$$= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})[\alpha(x_{i-1}) - \alpha(x_i)]$$

$$= G(b)\alpha(b) - G(a)\alpha(a) - \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i,$$

and equivalently,

$$\sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i + \sum_{i=1}^n G(x_{i-1})\Delta\alpha_i = G(b)\alpha(b) - G(a)\alpha(a).$$

Since

$$L(P, g\alpha) \leq \sum_{i=1}^n \alpha(x_i)g(t_i)\Delta x_i \leq U(P, g\alpha)$$

and

$$L(P,G,\alpha) \le \sum_{i=1}^{n} G(x_{i-1})\Delta\alpha_i \le U(P,G,\alpha),$$

which gives that

$$L(P,g\alpha) + L(P,G,\alpha) \le G(b)\alpha(b) - G(a)\alpha(a) \le U(P,g\alpha) + U(P,G,\alpha).$$

Notice that P is arbitrary, we thus obtain that

$$\int_a^b \alpha(x)g(x)dx + \int_a^b Gd\alpha \le G(b)\alpha(b) - G(a)\alpha(a) \le \int_a^b \alpha(x)g(x)dx + \int_a^b Gd\alpha,$$

namely,

$$\int_a^b \alpha(x)g(x)dx + \int_a^b Gd\alpha = G(b)\alpha(b) - G(a)\alpha(a),$$

which is the same to say that

$$\int_a^b \alpha(x)g(x)dx = G(b)\alpha(b) - G(a)\alpha(a) - \int_a^b Gd\alpha.$$

18. Let γ_1, γ_2, γ_3 be curves in the complex plane, defined on $[0, 2\pi]$ by

$$\gamma_1(t) = e^{it}, \quad \gamma_2(t) = e^{2it}, \quad \gamma_3(t) = e^{2\pi it\sin(1/t)}.$$

Show that these three curves have the same range, that γ_1 and γ_2 are rectifiable, that the length of γ_1 is 2π, that the length of γ_2 is 4π, and that γ_3 is not rectifiable.
Proof:
(i) Clearly, γ_1, γ_2 and γ_3 all have the unit circle on the complex plane as their range.
(ii) Since $\gamma_1'(t) = ie^{it}$ and $\gamma_2'(t) = 2ie^{2it}$, both of which is continuous on $[0.2\pi]$. By Theorem 6.27, we have that γ_1 and γ_2 are rectifiable. And thus $\Lambda(\gamma_1) = \int_0^{2\pi} |ie^{it}|dt = 2\pi$, $\Lambda(\gamma_2) = \int_0^{2\pi} |2ie^{2it}|dt = 4\pi$.

19. Let γ_1 be a curve in $\mathbb{R}^k$, defined on $[a, b]$; let ϕ be a continuous 1-1 mapping of $[c, d]$ onto $[a, b]$, such that $\phi(c) = a$; and define $\gamma_2(s) = \gamma_1(\phi(s))$. Prove that γ_2 is an arc, a closes curve, or a rectifiable curve if and only if the same is true of γ_1. Prove that γ_2 and γ_1 have the same length.
Proof:
(i) Since $\gamma_2(s) = \gamma_1(\phi(s))$, and ϕ is one-to-one, it's clear that γ_1 is one-to-one if and only if γ_2 is one-to-one. That is, γ_2 is an arc if and only if γ_1 is an arc.
(ii) First we prove that if $\phi(c) = a$, then $\phi(d) = b$. Suppose that on the contrary, this is not the case. Then there must be an $s_0 \in [c, d]$, $s_0 \ne c, d$, such that $\phi(s_0) = b$. Hence we have $\phi(c) < \phi(s_0)$ and $\phi(d) < \phi(s_0)$.

Take a λ such that $\max(\phi(c), \phi(d)) < \lambda < \phi(s_0)$, then $\phi(c) < \lambda < \phi(s_0)$, $\phi(d) < \lambda < \phi(s_0)$. Since ϕ is continuous on $[c, d]$, we have that there is an $s_1 \in (c, s_0)$ such that $\phi(s_1) = \lambda$; and similarly, there is an $s_2 \in (s_0, d)$ such that $\phi(s_2) = \lambda$. This is contradict to the fact that ϕ is one-to-one.
If γ_1 is closed, we have that $\gamma_1(a) = \gamma_1(b)$. Then we have $\gamma_2(c) = \gamma_1(\phi(c)) = \gamma_1(a) = \gamma_1(b) = \gamma_1(\phi(d)) = \gamma_2(d)$, hence γ_2 is closed. And if γ_2 is closed, we have that $\gamma_2(c) = \gamma_2(d)$, then we have $\gamma_1(a) = \gamma_1(\phi(c)) = \gamma_2(c) = \gamma_2(d) = \gamma_1(\phi(d)) = \gamma_1(b)$, hence γ_1 is closed.
(iii)If γ_1 is rectifiable, $\Lambda(\gamma_2) = \sup \Lambda(P_{\gamma_2}, \gamma_2) = \sup \sum_{i=1}^{n} |\gamma_2(x_i) - \gamma_2(x_{i-1})| = \sup \sum_{i=1}^{n} |\gamma_1(\phi(x_i)) - \gamma_1(\phi(x_{i-1}))| = \sup \Lambda(P_{\gamma_1}, \gamma_1) = \Lambda(\gamma_1)$, hence γ_2 is rectifiable. And if γ_2 is rectifiable, $\Lambda(\gamma_1) = \sup \Lambda(P_{\gamma_1}, \gamma_1) = \sup \sum_{i=1}^{n} |\gamma_1(x_i) - \gamma_1(x_{i-1})| = \sup \sum_{i=1}^{n} |\gamma_2(\phi^{-1}(x_i)) - \gamma_2(\phi^{-1}(x_{i-1}))| = \sup \Lambda(P_{\gamma_2}, \gamma_2) = \Lambda(\gamma_2)$, hence γ_1 is rectifiable.
The fact that γ_2 and γ_1 have the same length is clear from the above proof process.

7 Sequences and series of functions

1. Prove that every uniformly convergent sequences of bounded functions is uniformly bounded.
   Proof: Suppose $\{f_n(x)\}$ converges to $f(x)$ uniformly for all $x \in E$, and $\{f_n(x)\}$ is bounded, for every n. Then we can pick an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < 1$, for all $x \in E$. which gives that $|f(x)| < |f_{N+1}(x)| + 1$. Let $|f_{N+1}(x)| \leq M$, then $|f(x)| < M + 1$. Furthermore, we also have $|f_n(x)| < |f(x)| + 1$, for all $n > N$, which gives $|f_n(x)| < M + 2$, for all $n > N$. Suppose $|f_i(x)| \leq M_i$, for $1 \leq i \leq N$, and let $M' = \max\{M_1, M_2, ..., M_N, M + 2\}$, then we have $|f_n(x)| \leq M'$, for all n and $x \in E$. Hence $\{f_n(x)\}$ is uniformly bounded.

2. If $\{f_n\}$ and $\{g_n\}$ converge uniformly on a set E, prove that $\{f_n + g_n\}$ converges uniformly on E. If, in addition, $\{f_n\}$ and $\{g_n\}$ are sequences of bounded functions, prove that $\{f_n g_n\}$ converges uniformly on E.
   Proof:(i) Since $\{f_n\}$ and $\{g_n\}$ converge uniformly on E, there exist N_1, $N_2 \in \mathbb{N}$ such that $n > m > N_1$ implies $|f_n - f_m| < \epsilon/2$ and $n > m > N_2$ implies $|g_n - g_m| < \epsilon/2$. Let $N = \max\{N_1, N_2\})$, then when $n > m > N$, we have $|(f_n + g_n) - (f_m + g_m)| = |(f_n - f_m) + (g_n - g_m)| \leq |f_n - f_m| + |g_n - g_m| < \epsilon$, and thus $\{f_n + g_n\}$ converges uniformly on E, by Cauchy's criterion.
   (ii) Since $\{f_n\}$ and $\{g_n\}$ converge uniformly on E and both of which are bounded, by Exercise 1, $\{f_n\}$ and $\{g_n\}$ are uniformly bounded. Let $|f_n| \leq M_f$ and $|g_n| \leq M_g$, by (i) there is an $N \in \mathbb{N}$ such that $n > m > N$

implies that $|f_n - f_m| < \epsilon/2M_g$ and $|g_n - g_m| < \epsilon/2M_f$. Hence we have $|f_n g_n - f_m g_m| = |f_n(g_n - g_m) + g_m(f_n - f_m)| \le |f_n||g_n - g_m| + |g_m||f_n - f_m| < \epsilon$, when $n > m > N$. Therefore, $\{f_n g_n\}$ converges on E uniformly.

3. Construct sequences $\{f_n\}$, $\{g_n\}$ which converge uniformly on some set E, but such that $\{f_n g_n\}$ does not converge uniformly on E (of course, $\{f_n g_n\}$ must converge on E).
Solution: Let $f_n(x) = x + \frac{\sin x}{n}$, $g_n(x) = \frac{1}{x} + \frac{\sin x}{n}$, $x \in (0, +\infty)$. Then f_n and g_n converge uniformly to x and $\frac{1}{x}$ (since $\sup|f_n - x| = \sup|\frac{\sin x}{n}| = |\frac{1}{n}| \to 0$, and $\sup|g_n - \frac{1}{x}| = \sup|\frac{\sin x}{n}| = |\frac{1}{n}| \to 0$, when $n \to \infty$). But $f_n g_n = 1 + \frac{(\sin x)^2}{n^2} + (x + \frac{1}{x})\frac{\sin x}{n}$ does not converge uniformly to 1, since $\sup|f_n g_n - 1| = \sup|\frac{(\sin x)^2}{n^2} + (x + \frac{1}{x})\frac{\sin x}{n}| = |\frac{1}{n^2} + (2kn\pi + \frac{1}{2kn\pi})\frac{1}{n}| = 2k\pi$ $(k > 1)$, when $n \to \infty$.

4. Consider
$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x}.$$
For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous wherever the series converges? Is f bounded?
Solution:
(i) When $x \ge 1$, we have $1 + n^2 x \ge 1 + n^2 > n^2$, which gives $\frac{1}{|1+n^2x|} = \frac{1}{1+n^2x} < \frac{1}{n^2}$ and hence $|f(x)| = |\sum_{n=1}^{\infty} \frac{1}{1+n^2x}| \le \sum_{n=1}^{\infty} \frac{1}{|1+n^2x|} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, $f(x)$ converges absolutely.
When $0 < x < 1$, we have $1 + n^2 x > n^2 x$, which gives $\frac{1}{|1+n^2x|} < \frac{1}{n^2x}$. Hence, $|f(x)| = \sum_{n=1}^{\infty} \frac{1}{1+n^2x} < \sum_{n=1}^{\infty} \frac{1}{n^2x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2}$, which shows $f(x)$ converges absolutely.
When $x = 0$, we have $f(x) = \sum_{n=1}^{\infty} 1$. Clearly, $f(x)$ diverges, so does $|f(x)|$.
When $x < 0$, $|1 + n^2 x| \ge ||x|n^2 - 1|$, and when n is sufficiently large (suppose $n > N$), we have $|x|n^2 - 1 > 0$. Hence, $|f(x)| = |\sum_{n=1}^{\infty} \frac{1}{1+n^2x}| \le \sum_{n=1}^{\infty} \frac{1}{|1+n^2x|} = \sum_{n=1}^{N} \frac{1}{|1+n^2x|} + \sum_{n=N+1}^{\infty} \frac{1}{|1+n^2x|} \le \sum_{n=1}^{N} \frac{1}{|1+n^2x|} + \sum_{n=N+1}^{\infty} \frac{1}{|x|n^2-1}$, which gives that $f(x)$ converges absolutely.
In summary, when $x \ne 0$, $f(x)$ converges absolutely, and when $x = 0$, $f(x)$ diverges.

(ii)When $x \in [a, +\infty)$, where $a > 0$, by picking any $r \in (0, a)$, we have that $1 + n^2 x > n^2 x > n^2 r$, which gives $\frac{1}{1+n^2x} < \frac{1}{n^2 r}$. Since $\sum_{n=1}^{\infty} \frac{1}{n^2 r} = \frac{1}{r}\sum_{n=1}^{\infty} \frac{1}{n^2}$

converges, $f(x)$ converges uniformly on $[a,+\infty)$, by Theorem 7.10.
Similarly, when $x \in (-\infty, b]$, where $b < 0$, by (i) we have $|1+n^2x| \geq |x|n^2 - 1 > 0$, when n is sufficiently large (suppose $n > N$). Pick any $r \in (b, 0)$, we have $|1+n^2x| \geq |r|n^2 - 1$, and thus $\frac{1}{|1+n^2x|} \leq \frac{1}{|r|n^2-1}$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{|r|n^2-1}$ converges, $\sum\limits_{n=N+1}^{\infty} \frac{1}{|r|n^2-1}$ also converges, we have $\sum\limits_{n=N+1}^{\infty} \frac{1}{|1+n^2x|}$ converges uniformly, and therefore $\sum\limits_{n=1}^{\infty} \frac{1}{|1+n^2x|}$ converges uniformly.
In summary, $f(x)$ converges uniformly on intervals such as $(-\infty, b]$ $(b < 0)$ or $[a, +\infty)$ $(a > 0)$. Conversely, $f(x)$ fails to converge uniformly on such intervals that $(0, a]$ $(a > 0)$, $[b, 0)$ $(b < 0)$, and $[b, a]$ $(b < 0$ and $a > 0)$.

(iii) By (i), $f(x)$ converges if and only if $x \neq 0$. For any $x_0 \neq 0$, we can construct an interval such that $x_0 \in [a, +\infty)$ $(a > 0$, if $x_0 > 0)$ or $x_0 \in (-\infty, b]$ $(b < 0$, if $x_0 < 0)$. By (ii), in either case, $f(x)$ converges uniformly on the constructed interval, then according to Theorem 7.12, $f(x)$ is continuous on the interval. Hence, in particular, f is continuous at x_0.

(iv) Clearly, $f(x)$ cannot be bounded, since $f(0)$ does not converge.

5. Let

$$f_n(x) = \begin{cases} 0 & (x < \frac{1}{n+1}), \\ \sin^2 \frac{\pi}{x} & (\frac{1}{n+1} \leq x \leq \frac{1}{n}), \\ 0 & (\frac{1}{n} < x). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.
Proof: Fix any $x_0 \in \mathbb{R}$, if $x_0 \leq 0$ or $x_0 > 1$, then clearly $f_n(x_0) = 0$, for every n; if $x_0 \in (0, 1]$, then there exist a unique N such that $\frac{1}{N+1} \leq x_0 \leq \frac{1}{N}$, and thus when $n > N$, we have $f_n(x_0) = 0$. Hence, $\lim\limits_{n\to\infty} f_n(x_0) = 0$, which gives that $f(x) = \lim\limits_{n\to\infty} f_n(x) = 0$. Clearly, $f(x)$ is continuous. To see that the convergence is not uniform, let $M_n = \sup|f_n(x) - f(x)| = \sup|f_n(x)| = \sup|\sin^2 \frac{\pi}{x}| = \sup|\frac{1-\cos 2\pi/x}{2}| = 1$, since $\frac{1}{n+1} \leq x \leq \frac{1}{n}$ implies $2n\pi \leq \frac{2\pi}{x} \leq 2(n+1)\pi$. Then $M_n = 1$ when $n \to \infty$, and thus the convergence is not uniform, by Theorem 7.9.
Let $g(x) = \sum f_n(x)$, from the above statements we have known that for any x_0, $g(x_0) = \sin^2 \frac{\pi}{x_0}$ if $x_0 \in (0, 1]$, and $g(x_0) = 0$ otherwise. Clearly, $\sum f_n(x_0)$ converges absolutely. But f_n cannot converge uniformly, since $g(\frac{2}{2n+1}) = f_n(\frac{2}{2n+1}) = 1$, for every n. This means, no matter which N chosen, we can pick $x = \frac{2}{2(N+1)+1}$, so that $|\sum\limits_{n=N+1}^{N+p} f_n(x)| = f_{N+1}(x) = 1$, contradicting the Cauchy's criterion.

6. Prove that the series
$$\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$$
converges uniformly in every bounded interval, but does not converge absolutely for any value of x.
Proof: Suppose $a \le x \le b$, then we can assume that $|x| \le M$, for some M. Since $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2} = \sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2} + \sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$. By Theorem 3.43 (Leibnitz), $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges. What's more, $\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2} \le M^2\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$, and since $|(-1)^n\frac{1}{n^2}| = \frac{1}{n^2}$ and $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, we have $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2}$ converges. Therefore, $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$ converges uniformly, by Theorem 7.10. (More specifically, $\sum_{n=1}^{\infty}(-1)^n\frac{x^2}{n^2}$ converges uniformly by Theorem 7.10, since $|(-1)^n\frac{x^2}{n^2}| = \frac{x^2}{n^2} \le M^2\frac{1}{n^2}$, and $\sum\frac{1}{n^2}$ converges. So does $\sum_{n=1}^{\infty}(-1)^n\frac{x^2+n}{n^2}$.)
Clearly, $|(-1)^n\frac{x^2+n}{n^2}| = \frac{x^2+n}{n^2} \ge \frac{n}{n^2} = \frac{1}{n}$, and hence $\sum|(-1)^n\frac{x^2+n}{n^2}|$ diverges, for any value of x, since $\sum\frac{1}{n}$ diverges.

7. For $n = 1, 2, 3, ...,$ x real, put
$$f_n(x) = \frac{x}{1+nx^2}.$$
Show that $\{f_n\}$ converges uniformly to a function f, and that the equation
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
is correct if $x \ne 0$, but false if $x = 0$.
Proof: Clearly, $f(x) = \lim_{n\to\infty} f_n(x) = 0$, for any real x. Put $M_n = \sup|f_n(x) - f(x)| = \sup|\frac{x}{1+nx^2} - 0| = \sup|\frac{x}{1+nx^2}| = \sup|\frac{1}{1/x+nx}| = \frac{1}{2\sqrt{n}}$ (if and only if $x = \sqrt{n}$ when the supremum is achieved), and we have $M_n \to 0$ when $n \to \infty$. Therefore, f_n converges uniformly, by Theorem 7.9.
$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} = \frac{2}{(1+nx^2)^2} - \frac{1}{1+nx^2}$, and hence $\lim_{n\to\infty} f_n'(x) = 0 = f'(x)$, when $x \ne 0$. But $f_n'(0) = 2 - 1 = 1 \ne 0 = f'(0)$.

8. If
$$I(x) = \begin{cases} 0 & (x \le 0), \\ 1 & (x > 0), \end{cases}$$
if $\{x_n\}$ is a sequence of distinct points of (a, b), and if $\sum|c_n|$ converges, prove that the series
$$f(x) = \sum_{n=1}^{\infty} c_n I(x - x_n) \quad (a \le x \le b)$$

converges uniformly, and that f is continuous for every $x \neq x_n$.

Proof: Since $|c_n I(x - x_n)| \leq |c_n|$ and $\sum |c_n|$ converges, by Theorem 7.10, $\sum c_n I(x - x_n)$ converges uniformly.

Let $f_N(x) = \sum_{n=1}^{N} c_n I(x - x_n)$, we have $\lim_{t \to x} f_N(t) = \sum_{x_n < x} c_n = f_N(x)$, if $x \neq x_n$, and hence $f_N(x)$ is continuous, for every $x \neq x_n$. Therefore, $f(x)$ is continuous if $x \neq x_n$, by Theorem 7.12. (Clearly, when $x = x_n$, $f(x)$ cannot be continuous, and only one side continuousness can be committed.)

9. Let $\{f_n\}$ be a sequence of continuous functions which converges uniformly to a function on a set E. Prove that

$$\lim_{n \to \infty} f_n(x_n) = f(x)$$

for every sequence of points $x_n \in E$ such that $x_n \to x$, and $x \in E$. Is the converse of this true?

Proof: Since f_n converges uniformly to f on E, then given any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon/2$, for all $x \in E$. In particular, $|f_n(x_n) - f(x_n)| < \epsilon/2$, when $n > N$. On the other hand, since every f_n is continuous on E, f is continuous on E, by Theorem 7.12. Hence there is an $M \in \mathbb{N}$ such that $n > M$ implies $|f(x_n) - f(x)| < \epsilon/2$. Let $N' = \max(N, M)$, then when $n > N'$, we have $|f_n(x_n) - f(x)| = |f_n(x_n) - f(x_n) + f(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \epsilon$, which is to say $\lim_{n \to \infty} f_n(x_n) = f(x)$.

The converse should be expressed as: Let $\{f_n\}$ be a sequence of continuous functions which converges to a function f, and if, for every sequence of points $x_n \in E$ such that $x_n \to x$ and $x \in E$, we have $\lim_{n \to \infty} f_n(x_n) = f(x)$, then $\{f_n\}$ converges to f uniformly.

This cannot be true, and Exercise 5 serves as a counterexample. By Exercise 5, we have that $f(x) = 0$, for every x. If $x_n \to x$, then if we suppose that $\frac{1}{N+1} \leq x \leq \frac{1}{N}$, there must exist an M such that $n > M$ implies $\frac{1}{N+1} \leq x_n \leq \frac{1}{N}$. Let $N' = \max(N, M)$, then when $n > N'$, we must have $f_n(x_n) = 0$ and hence $\lim_{n \to \infty} f_n(x_n) = 0 = f(x)$, which satisfies the requirement of the hypothesis. But Exercise 5 has proved that the convergence of $\{f_n\}$ is not uniform.

10. Letting (x) denote the fractional part of the real number x (see Exercise 16, Chap.4, for the definition), consider the function

$$f(x) = \sum_{n=1}^{\infty} \frac{(nx)}{n^2} \quad (x \text{ real}).$$

Find all discontinuities of f, and show that they form a countable dense set. Show that f is nevertheless Riemann-integrable on every bounded interval.

Proof: We have know that (x) is discontinuous where x is an integer. On the other hand, since $|\frac{(nx)}{n^2}| \le \frac{1}{n^2}$, and $\sum \frac{1}{n^2}$ converges, we have that $\sum \frac{(nx)}{n^2}$ converges uniformly, by Theorem 7.10. Let $f_N(x) = \sum_{n=1}^{N} \frac{(nx)}{n^2}$, then $f_N(x)$ is discontinuous where any (nx) is discontinuous $(1 \le n \le N)$. This means when $nx = m$, $m \in \mathbb{Z}$, $f_N(x)$ is discontinuous, which implies that when x is rational, $x = \frac{p}{q}$, $p, q \in \mathbb{Z}$ then $f_n(x)$ is discontinuous, for any $n \ge q$. Since f_N converges uniformly to f, if f_N is continuous for every N, f should also be continuous, by Theorem 7.12. Hence $f(x)$ is discontinuous at every rational point x, and clearly, $\mathbb{Q}$ is a countable dense subset of $\mathbb{R}$.
Suppose $[a, b]$ is any bounded interval, and suppose $g_n(x) = \frac{(nx)}{n^2}$, then we have $na \le nx \le nb$, if $x \in [a, b]$. Since the number of integers lying in $[na, nb]$ is finite, we know that g_n has only finitely many discontinuities in $[a, b]$. Since $|g_n(x)| = |\frac{(nx)}{n^2}| \le \frac{1}{n^2} \le 1$, we see that $g_n(x)$ is bounded on $[a, b]$, for every n, then according to Theorem 6.10, we have that $g_n \in \mathscr{R}$, for every n. Hence by the uniform converge of $\sum g_n(x)$ and the Corollary of Theorem 7.16, we have that $f \in \mathscr{R}$.

11. Suppose $\{f_n\}$, $\{g_n\}$ are defined on E, and

    (a) $\sum f_n$ has uniformly bounded partial sums;

    (b) $g_n \to 0$ uniformly on E;

    (c) $g_1(x) \ge g_2(x) \ge g_3(x) \ge \cdots$ for every $x \in E$.

    Prove that $\sum f_n g_n$ converges uniformly on E.

    Proof: Suppose $A_N(x) = \sum_{n=1}^{N} f_n(x)$, since $\sum f_n$ has uniformly bounded partial sums, we have $A_N(x) \le M$, for all $N \in \mathbb{N}$ and $x \in E$.
    Given any $\epsilon > 0$, there is an integer N' such that $g_n(x) \le (\epsilon/2M)$, for every $x \in E$, and $n \ge N'$, since $g_n(x)$ converges to 0 uniformly, and $g_n(x)$ is decreasing monotonically for every $x \in E$. For $N' \le p \le q$, we have $|\sum_{n=p}^{q} f_n(x)g_n(x)| = |\sum_{n=p}^{q} (A_n(x) - A_{n-1}(x))g_n(x)| = |\sum_{n=p}^{q} A_n(x)g_n(x) - \sum_{n=p-1}^{q-1} A_n(x)g_{n+1}(x)| = |\sum_{n=p}^{q-1} A_n(x)(g_n(x) - g_{n+1}(x)) + A_q(x)g_q(x) - A_{p-1}(x) g_p(x)| \le |\sum_{n=p}^{q-1} A_n(x)(g_n(x) - g_{n+1}(x))| + |A_q(x)g_q(x)| + |A_{p-1}(x)g_p(x)| \le M(|\sum_{n=p}^{q-1} (g_n(x) - g_{n+1}(x))| + |g_q(x)| + |g_p(x)|) = M|\sum_{n=p}^{q-1} (g_n(x) - g_{n+1}(x)) + g_q(x) + g_p(x)| = 2Mg_p(x) < \epsilon$, for every $x \in E$. Uniform convergence now follows from Cauchy's criterion.

12. Suppose g and $f_n (n = 1, 2, 3, ...)$ are defined on $(0, \infty)$, are Riemann-integrable on $[t, T]$ whenever $0 < t < T < \infty$, $|f_n| \le g$, $f_n \to f$ uniformly

on every compact subset of $(0, \infty)$, and

$$\int_0^\infty g(x)dx < \infty.$$

Prove that

$$\lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty f(x)dx.$$

(See Exercises 7 and 8 of Chap. 6 for the relevant definitions.)

Proof: Since $|f_n| \le g$ and $\int_0^\infty g(x)dx < \infty$, we have that $\int_0^\infty f_n(x)dx$ exists. What's more, since $f_n \to f$ uniformly, we must have $|f| \le g$, on every compact subset of $(0, \infty)$, and thus $\int_0^\infty f(x)dx$ exists.

Since $f_n \to f$ uniformly, we have that $f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$. Since $|f_n| \le g$, we have $-g \le f_n \le g$ and thus $0 \le f_n + g$, for every n. Let $g_n = \inf(f_i + g)$, where $i \ge n$. Then $0 \le g_1 \le g_2 \le \cdots$, $g_n \le (f_n + g)$, and $g_n \to (f + g)$ uniformly. Clearly, each $\int_0^\infty g_n dx = \int_0^\infty (f_n + g)dx$ exists. Therefore, we have

$$\int_0^\infty (f+g)dx \le \liminf_{n\to\infty} \int_0^\infty (f_n + g)dx,$$

or equivalently,

$$\int_0^\infty f dx \le \liminf_{n\to\infty} \int_0^\infty f_n dx.$$

Similarly, since $f_n \le g$, we have $g - f_n \ge 0$. Let $h_n = \inf(g - f_i)$, where $i \ge n$. Then $0 \le h_1 \le h_2 \le \cdots$, $h_n \le g - f_n$, and $h_n \to (g - f)$ uniformly. Clearly, each $\int_0^\infty h_n dx = \int_0^\infty (g - f_n)dx$ exists. Therefore, we have

$$\int_0^\infty (g-f)dx \le \liminf_{n\to\infty} \int_0^\infty (g - f_n)dx,$$

or equivalently,

$$-\int_0^\infty f dx \le \liminf_{n\to\infty} [-\int_0^\infty f_n dx],$$

which is equivalent to

$$\limsup_{n\to\infty} \int_0^\infty f_n dx \le \int_0^\infty f dx,$$

since

$$\limsup_{n\to\infty} \int_0^\infty f_n dx = -\liminf_{n\to\infty} [-\int_0^\infty f_n dx].$$

(See Exercise 5, Chap. 1).

Combining the above two results gives us

$$\int_0^\infty f dx \le \liminf_{n\to\infty} \int_0^\infty f_n dx \le \limsup_{n\to\infty} \int_0^\infty f_n dx \le \int_0^\infty f dx,$$

and hence

$$\int_0^\infty f dx = \liminf_{n\to\infty} \int_0^\infty f_n dx = \limsup_{n\to\infty} \int_0^\infty f_n dx = \lim_{n\to\infty} \int_0^\infty f_n dx,$$

which is the desired result.

13. Assume that $\{f_n\}$ is a sequence of monotonically increasing functions on $\mathbb{R}^1$ with $0 \le f_n(x) \le 1$ for all x and all n.

 (a) Prove that there is a function f and a sequence $\{n_k\}$ such that

$$f(x) = \lim_{k\to\infty} f_{n_k}(x)$$

 for every $x \in \mathbb{R}^1$. (The existence of such a pointwise convergent subsequence is usually called *Helly's selection theorem.*)
 Proof: (i) Some subsequence $\{f_{n_i}\}$ converges at all rational points r, say, to $f(r)$, by Theorem 7.23.
 (ii) Define $f(x)$, for any $x \in \mathbb{R}^1$, to be $\sup f(r)$, the sup being taken over all $r \le x$.
 (iii) Now, we will show that $f_{n_i}(x) \to f(x)$ at every x at which f is continuous. Since f is continuous at x, given any $\epsilon > 0$, there is a $\delta > 0$ such that $|y - x| < \delta$ implies $|f(y) - f(x)| < \epsilon/2$. Pick two rational numbers r_1 and r_2 from $N(x, \delta)$ such that $r_1 \le x \le r_2$. Since each f_{n_i} is monotonically increasing on $\mathbb{R}^1$, we have $f_{n_i}(r_1) \le f_{n_i}(x) \le f_{n_i}(r_2)$. And since $f_{n_i}(r) \to f(r)$, for every rational number r, there exists integers N_1 and N_2 such that $n_i > N_1$ implies $|f_{n_i}(r_1) - f(r_1)| < \epsilon/2$ and $n_i > N_2$ implies $|f_{n_i}(r_2) - f(r_2)| < \epsilon/2$. If we let $N = \max(N_1, N_2)$, then $n_i > N$ implies $|f_{n_i}(r_k) - f(r_k)| < \epsilon/2$, where $k = 1, 2$. This gives that $f(r_1) - \epsilon/2 < f_{n_i}(r_1) \le f_{n_i}(x) \le f_{n_i}(r_2) < f(r_2) + \epsilon/2$, when $n_i > N$. On the other hand, since $r_1, r_2 \in N(x, \delta)$, we have that $|f(r_1) - f(x)| < \epsilon/2$ and $|f(r_2) - f(x)| < \epsilon/2$, which gives that $f(r_1) > f(x) - \epsilon/2$ and $f(r_2) < f(x) + \epsilon/2$. Therefore, $f(x) - \epsilon < f(r_1) - \epsilon/2 < f_{n_i}(r_1) \le f_{n_i}(x) \le f_{n_i}(r_2) < f(r_2) + \epsilon/2 < f(x) + \epsilon$, or shortly, $f(x) - \epsilon < f_{n_i}(x) < f(x) + \epsilon$, when $n_i > N$, which gives $|f_{n_i}(x) - f(x)| < \epsilon$, when $n_i > N$. Hence, $f_{n_i}(x) \to f(x)$ at every x at which f is continuous.
 (iv) Now we will prove that $f(x)$ is monotonically increasing on $\mathbb{R}^1$. To see this, first suppose r_1 and r_2 to be two rational numbers and $r_1 \le r_2$. Since every f_{n_i} is monotonically increasing on $\mathbb{R}^1$, we have $f_{n_i}(r_1) \le f_{n_i}(r_2)$, for every n_i. Since $f_{n_i}(r) \to f(r)$ on every rational number r, we then have $f(r_1) \le f(r_2)$. Now suppose x_1 and x_2 be two real numbers such that $x_1 \le x_2$, then $f(x_1) = \sup f(r_1)$, $r_1 \le x_1$, and $f(x_2) = \sup f(r_2)$, $r_2 \le x_2$. Hence, $f(x_1) \le f(x_2)$ and $f(x)$ is monotonically increasing on $\mathbb{R}^1$. (Since if $x_1 = x_2$, clearly $f(x_1) = f(x_2)$, and if $x_1 < x_2$, there exist a rational number r such that $x_1 < r < x_2$, and thus $fx_1 \le f(r) \le f(x_2)$.) Then by the similar

argument as in Theorem 4.30, we can conclude that the set of points (denoted as E) where f is discontinuous is at most countable. Using Theorem 7.23 again on the sequence $\{f_{n_i}\}$ (note that $0 \le f_{n_i}(x) \le 1$ still holds, for all the n_i and x) gives us that there is a subsequence of $\{f_{n_i}\}$ which converges (to $f(x)$) for every $x \in E$, and therefore this subsequence of $\{f_{n_i}\}$ (and hence a subsequence of $\{f_n\}$) converges for every $x \in \mathbb{R}^1$, to a function $g(x)$ ($g(x) = f(x)$, where f is continuous at x, but we cannot conclude this for those discontinuities of f). This proves (a).

(b) If, moreover, f is continuous, prove that $f_{n_k} \to f$ uniformly on compact sets.
Proof:

14. Let f be a continuous real function on $\mathbb{R}^1$ with the following properties: $0 \le f(t) \le 1$, $f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & (0 \le t \le \frac{1}{3}) \\ 1 & (\frac{2}{3} \le t \le 1). \end{cases}$$

Put $\Phi(t) = (x(t), y(t))$, where

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t), \quad y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

Prove that Φ is *continuous* and that Φ maps $I = [0,1]$ *onto* the unit square $I^2 \subseteq \mathbb{R}^2$. If fact, show that Φ maps the Cantor set onto I^2.
Proof: First, we have $|2^{-n} f(3^{2n-1})| \le 2^{-n}$, and since $\sum 2^{-n}$ converges, $\sum 2^{-n} f(3^{2n-1}t)$ converges uniformly, for every t, by Theorem 7.10. Since f is continuous, each $2^{-n} f(3^{2n-1}t)$ is continuous, and hence both $x(t)$ and $y(t)$ is continuous. Therefore, $\Phi(t)$ is continuous, by Theorem 4.10. On the other hand, each $(x_0, y_0) \in I^2$ has the form

$$x_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1}, \quad y_0 = \sum_{n=1}^{\infty} 2^{-n} a_{2n}$$

where each a_i is 0 or 1. If

$$t_0 = \sum_{i=1}^{\infty} 3^{-i-1}(2a_i),$$

(by Exercise 19, Chap.3, t_0 is a point of the Cantor set) then we have $f(3^k t_0) = f(3^k \sum_{i=1}^{\infty} 3^{-i-1}(2a_i)) = f(\sum_{i=1}^{\infty} 3^{k-i-1}(2a_i)) = f(\sum_{i=1}^{k-1} 3^{k-i-1}(2a_i) + \sum_{i=k}^{\infty} 3^{k-i-1}(2a_i)) = f(2K + \sum_{i=k}^{\infty} 3^{k-i-1}(2a_i)) = f(\sum_{i=k}^{\infty} 3^{k-i-1}(2a_i))$
$= f(\sum_{i=1}^{\infty} 3^{-i}(2a_{i+k-1}))$.

If $a_k = 0$, then $0 \le \sum_{i=1}^{\infty} 3^{-i}(2a_{i+k-1}) = \sum_{i=2}^{\infty} 3^{-i}(2a_{i+k-1}) \le 2\sum_{i=2}^{\infty} 3^{-i} = 1/3$, and hence $f(\sum_{i=1}^{\infty} 3^{-i}(2a_{i+k-1})) = 0 = a_k$; if $a_k = 1$, then $2/3 \le \sum_{i=1}^{\infty} 3^{-i}(2a_{i+k-1}) \le 2/3 + \sum_{i=2}^{\infty} 3^{-i}(2a_{i+k-1}) \le 2/3 + 2\sum_{i=2}^{\infty} 3^{-i} = 2/3 + 1/3 = 1$, and hence $f(\sum_{i=1}^{\infty} 3^{-i}(2a_{i+k-1})) = 1 = a_k$. Now we have proved that $f(3^k t_0) = a_k$ and therefore, $f(3^{2n-1}t_0) = a_{2n-1}$, $f(3^{2n}t_0) = a_{2n}$, which gives that $x(t_0) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t_0) = \sum_{n=1}^{\infty} 2^{-n} a_{2n-1} = x_0$, and similarly, $y(t_0) = y_0$. This means, for each $(x_0, y_0) \in I^2$, we can find a $t_0 \in$ (Cantor set) E, such that $\Phi(t_0) = (x(t_0), y(t_0)) = (x_0, y_0)$. Hence, Φ maps E onto I^2. Since $E \subset I$, Φ also maps I onto I^2.

15. Suppose f is a real continuous function on $\mathbb{R}^1$, $f_n(t) = f(nt)$ for $n = 1, 2, 3, \ldots$, and $\{f_n\}$ is equicontinuous on $[0,1]$. What conclusions can you draw about f?
Solution: Since f_n is equicontinuous on $[0,1]$, then given any $\epsilon > 0$, there exists a $\delta > 0$, such that $|x - y| < \delta$ implies $|f_n(x) - f_n(y)| < \epsilon$, $x, y \in [0,1]$ and $n \in \mathbb{N}$. Or, equivalently, $|f(nx) - f(ny)| < \epsilon$, $x, y \in [0,1]$, $|x - y| < \delta$, and $n \in \mathbb{N}$. In particular, $|f(x) - f(y)| < \epsilon$, $x, y \in [0,1]$, and $|x - y| < \delta$. This means, f is uniformly continuous on $[0,1]$. Hence, in general, f is uniformly continuous on every interval $[0, n]$, but for each of these interval, the underlying $\delta' = n\delta$ is different for different n.

16. Suppose $\{f_n\}$ is an equicontinuous sequence of functions on a compact set K, and $\{f_n\}$ converges pointwise on K. Prove that $\{f_n\}$ converges uniformly on K.
Proof: Since $\{f_n\}$ is equicontinuous, given any $\epsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$, $x, y \in K$ implies $d(f_n(x), f_n(y)) < \epsilon$, for every n. Let $V(x, \delta)$ be the set of points $y \in K$ such that $d(y, x) < \delta$, then $\bigcup_{x \in K} V(x, \delta)$ forms an open cover of K. Since K is compact, there are finitely many points $x_1, x_2, \ldots, x_m \in K$ such that

$$K \subseteq V(x_1, \delta) \cup V(x_2, \delta) \cup \cdots \cup V(x_m, \delta). \quad (*)$$

Since f_n converges pointwise on K, there is an $N \in \mathbb{N}$ such that $n > N$ implies $d(f_i(x_s), f_j(x_s)) < \epsilon$, for every $i \ge N$, $j \ge N$, and $1 \le s \le m$. Given any $x \in K$, by (*), there is an x_s, $1 \le s \le m$, such that $x \in V(x_s, \delta)$. Therefore, we have $d(f_i(x), f_j(x)) \le d(f_i(x), f_i(x_s)) + d(f_i(x_s), f_j(x_s)) + d(f_j(x_s), f_j(x)) < 3\epsilon$, for $i \ge N$, $j \ge N$, and thus f_n converges uniformly on K, by Cauchy's criterion.

17. Define the notions of uniform convergence and equicontinuity for mappings into any metric space. Show that Theorems 7.9 and 7.12 are valid for mappings into any metric space, that Theorems 7.8 and 7.11 are valid

for mappings into any complete metric space, and that Theorems 7.10, 7.16, 7.17, 7.24, and 7.25 hold for vector-valued functions, that is, for mappings into any $\mathbb{R}^k$.

Solution: For mappings into any metric space, we have the following definitions for the notion of uniform convergence and quicontinuity, with minor modifications to Definition 7.7 and 7.22.

We say that a sequence of functions $\{f_n\}$, $n = 1, 2, 3, ...$, converges uniformly on E of to a function f if for every $\epsilon > 0$ there is an integer N such that $n \geq N$ implies $d_Y(f_n(x), f(x)) \leq \epsilon$ for all $x \in E$. Here, every f_n and f is a mapping from E to a metric space Y, and d_Y is the metric of Y.

A family $\mathscr{F}$ of functions f defined on a set E in a metric space X into a metric space Y is said to be equicontinuous on E if for every $\epsilon > 0$ there exists a $\delta > 0$ such that $d_Y(f(p), f(q)) < \epsilon$ whenever $d_X(p, q) < \delta$, $x \in E$, $y \in E$, and $f \in \mathscr{F}$. Here d_X and d_Y denote the metric of X and Y, respectively.

(a) The M_n in Theorem 7.9 should be rephrased as

$$M_n = \sup_{x \in E} d_Y(f_n(x), f(x))$$

in the current context, and clearly, this is still an immediate consequence of our modified definition of uniform convergence.

(b) Theorem 7.12 says that, if $\{f_n\}$ is a sequence of continuous functions on E, and if $f_n \to f$ uniformly on E, then f is continuous on E.
This statement needs to be proved differently in the new context as follows. Since $\{f_n\} \to f$ uniformly, given any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $d_Y(f_n(x), f(x)) < \epsilon$, for any $x \in E$. Now, fix any $x \in E$, and since f_N is continuous on E, there is a $\delta > 0$ such that $d_X(t, x) < \delta$, $t \in E$ implies $d_Y(f_N(t), f_N(x)) < \epsilon$. Hence we have $d_Y(f(t), f(x)) \leq d_Y(f(t), f_N(t)) + d_Y(f_N(t), f_N(x)) + d_Y(f_N(x), f(x)) < 3\epsilon$, for $t \in E$ and $d_X(t, x) < \delta$, which show that f is continuous on E.

(c) If we review the proof processes of Theorem 7.8 and 7.11, we can find that the required condition which may not hold in an arbitrary metric space is the equivalence of Cauchy sequences and convergent sequences. Since now we know that the given metric space is complete, this equivalence is guaranteed. Therefore, the proofs there remain true, and only the metrics need to be replaced.

(d) By reviewing the proof processes of Theorem 7.10, 7.16, 7.17, 7.24, and 7.25, it's clear that all these procedures hold in the context of $\mathbb{R}^k$. We only need to replace f by $\mathbf{f}$.

18. Let $\{f_n\}$ be a uniformly bounded sequence of functions which are Riemann-integrable on $[a, b]$, and put

$$F_n(x) = \int_a^x f_n(t)dt \quad (a \leq x \leq b).$$

Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a,b]$.
Proof: Theorem 6.20 has shown us that each F_n is continuous on $[a,b]$, and next we will show that $\{F_n\}$ is equicontinuous. Since $\{f_n\}$ is uniformly bounded, we have $|f_n(x)| \leq M$, for all n and $x \in [a,b]$. Then given any $\epsilon > 0$, choose a δ such that $0 < \delta < \epsilon/M$. When $|x-y| < \delta$, $x, y \in [a,b]$, we have

$$|F_n(x) - F_n(y)| = |\int_a^x f_n(t)dt - \int_a^y f_n(t)dt| = |\int_x^y f_n(t)dt|$$

$$\leq M|\int_x^y dt| = M|x-y| < M\delta < \epsilon,$$

for all n and $x, y \in [a,b], |x-y| < \delta$. Hence F_n is equicontinuous, by Definition 7.22. Furthermore, we have $|F_n(x)| = |\int_a^x f_n(t)dt| \leq M|\int_a^x dt| = M|x-a| \leq M(b-a)$, for every n and every $x \in [a,b]$. Thus $\{F_n\}$ is uniformly bounded (of course, pointwise bounded, too). Since $[a,b]$ is compact and $F_n \in \mathscr{C}([a,b])$, for every n, by Theorem 7.25, $\{F_n\}$ contains a uniformly convergent subsequence on $[a,b]$, which is the desired result.

19. Let K be a compact metric space, let S be a subset of $\mathscr{C}(K)$. Prove that S is compact (with respect to the metric defined in Section 7.14) if and only if S is uniformly closed, pointwise bounded, and equicontinuous. (If S is not equicontinuous, then S contains a sequence which has no equicontinuous subsequence, hence has no subsequence that converges uniformly on K.)
Proof: $\Rightarrow$: Suppose S is compact, then by Theorem 2.34, S is (uniformly) closed. Since every function in $\mathscr{C}(K)$ is bounded, and S is a subset of $\mathscr{C}(K)$, S is clearly pointwise bounded. Given any $\epsilon > 0$, for each $f \in S$, let $V(f, \epsilon)$ be the set of all functions $g \in S$ such that $d_{\mathscr{C}(K)}(f,g) = ||f-g|| < \epsilon$. Since S is compact, there are finitely many $f_i \in S$, $1 \leq i \leq m$, such that

$$S \subseteq V(f_1, \epsilon) \cup V(f_2, \epsilon) \cup \cdots \cup V(f_m, \epsilon).$$

Since each f_i, $1 \leq i \leq m$, is continuous, and K is compact, each f_i is uniformly continuous on K. Hence, there is a $\delta > 0$, such that $d(x,y) < \delta$, $x, y \in K$ implies $|f_i(x) - f_i(x)| < \epsilon$, for each $1 \leq i \leq m$. Here d is the metric of K. Now, for every $f \in S$, there is an f_s, $1 \leq s \leq m$, such that $f \in V(f_s, \epsilon)$, or, in other words, $||f - f_s|| < \epsilon$. We then have that

$$|f(x) - f(y)| \leq |f(x) - f_s(x)| + |f_s(x) - f_s(y)| + |f_s(y) - f(y)|$$

$$\leq ||f - f_s|| + |f_s(x) - f_s(y)| + ||f_s - f|| < 3\epsilon,$$

where $x, y \in K$ and $d(x,y) < \delta$. This gives that S is equicontinuous.
$\Leftarrow$: Suppose S is uniformly closed, pointwise bounded, and equicontinuous. Let E be any infinite subset of S, then E is pointwise bounded and equicontinuous, too. By Theorem 7.25(b), we have that E contains a uniformly convergent subsequence on K. Suppose $\{f_n\}$ is this subsequence,

and $\{f_n\}$ converges to f uniformly, then f is a limit point of E. Note that by Theorem 7.15, we know that $\mathscr{C}(K)$ is complete, so $f \in \mathscr{C}(K)$. What's more, since f is a limit point of E, f is also a limit point of S. Therefore, $f \in S$ since S is uniformly closed. Thus, by Exercise 2.26, we have that S is compact.

20. If f is continuous on $[0, 1]$ and if

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, ...),$$

prove that $f(x) = 0$ on $[0, 1]$.
Proof: If we can show that

$$\int_0^1 f^2(x) = 0,$$

then $f(x) = 0$ follows immediately. Since f is continuous on $[0, 1]$, by Theorem 7.26 (Weierstrass's Theorem), there exists a sequence of polynomials P_n such that $\lim_{n\to\infty} P_n(x) = f(x)$ uniformly on $[0, 1]$. Then by Theorem 7.16, we have that

$$\int_0^1 f^2(x)dx = \int_0^1 f(x)(\lim_{n\to\infty} P_n(x))dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)dx.$$

Since

$$\int_0^1 f(x)x^n dx = 0 \quad (n = 0, 1, 2, ...),$$

we thus have that

$$\int_0^1 f(x)P_n(x)dx = 0,$$

no matter which $P_n(x)$ is. Therefore,

$$\int_0^1 f^2(x)dx = 0$$

and hence $f(x) = 0$ on $[0, 1]$.

21. Let K be the unit circle in the complex plane (i.e., the set of all z with $|z| = 1$), and let $\mathscr{A}$ be the algebra of all functions of the form

$$f(e^{i\theta}) = \sum_{n=0}^{N} c_n e^{in\theta} \quad (\theta \text{ real}).$$

Then $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K, but nevertheless there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$.

Proof: Since $|z| = 1$, we can write $z = e^{i\theta}$ for some θ. The functions in $\mathscr{A}$ then can be rewritten as

$$f(z) = \sum_{n=0}^{N} c_n z^n.$$

Clearly, $\mathscr{A}$ separates points on K and $\mathscr{A}$ vanishes at no point of K. But there are continuous functions on K which are not in the uniform closure of $\mathscr{A}$. To see this, note that, for every $f \in \mathscr{A}$, we have

$$\int_0^{2\pi} f(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} (\sum_{n=0}^{N} c_n e^{in\theta})e^{i\theta}d\theta$$

$$= \int_0^{2\pi} (\sum_{n=0}^{N} c_n e^{i(n+1)\theta})d\theta = \sum_{n=0}^{N} \int_0^{2\pi} c_n e^{i(n+1)\theta}d\theta = 0.$$

And what's more, for every g in the closure of $\mathscr{A}$, we have $g = \lim_{n\to\infty} f_n$, and $f_n \to g$ uniformly, $f_n \in \mathscr{A}$. Therefore,

$$\int_0^{2\pi} ge^{i\theta}d\theta = \lim_{n\to\infty} \int_0^{2\pi} f_n e^{i\theta}d\theta = 0.$$

Pick $h(e^{i\theta}) = e^{-i\theta}$. Clearly, h is continuous on K. However,

$$\int_0^{2\pi} h(e^{i\theta})e^{i\theta}d\theta = \int_0^{2\pi} e^{-i\theta}e^{i\theta}d\theta = \int_0^{2\pi} d\theta = 2\pi \neq 0.$$

Thus, h is not in the closure of $\mathscr{A}$.

22. Assume $f \in \mathscr{R}(\alpha)$ on $[a, b]$, and prove that there are polynomials P_n such that

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 d\alpha = 0.$$

Proof: As in Exercise 6.11, for $u \in \mathscr{R}(\alpha)$, define

$$||u||_2 = \{\int_a^b |u|^2 d\alpha\}^{1/2}.$$

By Exercise 6.12, we known that, for $f \in \mathscr{R}(\alpha)$ and $\epsilon > 0$, there exists a continuous function g on $[a, b]$ such that $||f - g||_2 < \sqrt{\epsilon}/2$. What's more, by Weierstrass's Theorem, since g is continuous on $[a, b]$, there exists a sequence of polynomials P_n such that $\lim_{n\to\infty} P_n = g$ uniformly on $[a, b]$. This means, there is an integer N such that $n > N$ implies $|g - P_n| < \sqrt{\epsilon}/2\sqrt{\alpha(b) - \alpha(a)}$, i.e., $\int_a^b |g - P_n|^2 d\alpha < \epsilon/4$ and thus $||g - P_n||_2 < \sqrt{\epsilon}/2$. By Exercise 6.11, we have that

$$||f - P_n||_2 \leq ||f - g||_2 + ||g - P_n||_2 < \sqrt{\epsilon}, \quad \text{for } n > N.$$

Thus,

$$\int_a^b |f - P_n| d\alpha = (||f - P_n||_2)^2 < \epsilon, \quad \text{for } n > N.$$

Since ϵ is arbitrary, we have

$$\lim_{n\to\infty} \int_a^b |f - P_n|^2 d\alpha = 0.$$

23. Put $P_0 = 0$, and define, for $n = 0, 1, 2, ...,$

$$P_{n+1}(x) = P_n(x) + \frac{x^2 - P_n^2(x)}{2}.$$

Prove that

$$\lim_{n\to\infty} P_n(x) = |x|,$$

uniformly on $[-1, 1]$. (This makes it possible to prove the Stone-Weierstrass theorem without first proving Theorem 7.26.)

Proof: First, we will prove that $0 \le P_n(x) \le P_{n+1}(x) \le |x|$ if $|x| \le 1$ by induction.

(i)$n = 0$, then $P_1(x) = P_0(x) + \frac{x^2 - P_0^2(x)}{2} = \frac{x^2}{2} \ge 0 = P_0$, and clearly $P_1(x) \le |x|$ since $|x| \le 1$.

(ii) Suppose $n = k$, we have $0 \le P_k(x) \le P_{k+1}(x) \le |x|$, if $|x| \le 1$. When $n = k + 1$, we have that

$$\begin{aligned}|x| - P_{k+2}(x) &= |x| - P_{k+1}(x) - \frac{x^2 - P_{k+1}^2(x)}{2} \\ &= (|x| - P_{k+1}(x))(1 - \frac{|x| + P_{k+1}(x)}{2}).\end{aligned}$$

Since $P_{k+1}(x) \le |x|$, $|x| - P_{k+2}(x) \ge (|x| - P_{k+1}(x))(1 - |x|) \ge 0$, which gives $P_{k+2}(x) \le |x|$. What's more, we have $P_{k+2}(x) - P_{k+1}(x) = (|x| - P_{k+1}(x)) - (|x| - P_{k+2}(x)) = (|x| - P_{k+1}(x))(1 - (1 - \frac{|x|+P_{k+1}(x)}{2})) = (|x| - P_{k+1}(x))(\frac{|x|+P_{k+1}(x)}{2}) \ge 0$ and thus $P_{k+1}(x) \le P_{k+2}(x)$. Therefore, $0 \le P_{k+1}(x) \le P_{k+2}(x) \le |x|$ and we are done.

Next, since $|x| - P_n(x) = [|x| - P_{n-1}(x)][1 - \frac{|x|+P_{n-1}(x)}{2}]$ for $n \ge 1$, and $0 \le P_n(x) \le |x|$, we have that $|x| - P_n(x) \le [|x| - P_{n-1}(x)][1 - \frac{|x|}{2}]$. Apply this inequality n times, we get $|x| - P_n(x) \le (|x| - P_0)(1 - \frac{|x|}{2})^n = |x|(1 - \frac{|x|}{2})^n$. Assume $y = |x|$, and $f(y) = y(1 - \frac{y}{2})^n$. Then $f'(y) = (1 - \frac{y}{2})^n - \frac{ny}{2}(1 - \frac{y}{2})^{n-1}$. Let $f'(y) = 0$, we get $y_0 = \frac{2}{n+1}$. Since $f''(y) = -\frac{n}{2}(1 - \frac{y}{2})^{n-2}(2 - \frac{(n+1)}{2}y)$, we have $f''(y_0) = -\frac{n}{2}(1 - \frac{1}{n+1})^{n-2} < 0$. Therefore, y_0 is the point at which we get the maximum value of $f(y)$. Hence we have $f(y) \le f(y_0) = \frac{2}{n+1}(1 - \frac{1}{n+1})^n < \frac{2}{n+1}$, which gives $|x| - P_n(x) < \frac{2}{n+1}$. Then, given any $\epsilon > 0$, there is an integer N such that $n > N$ implies $|x| - P_n(x) < \frac{2}{n+1} < \epsilon$. Combined with the fact $0 \le P_n(x) \le P_{n+1}(x) \le |x|$, we have that $\lim_{n\to\infty} P_n(x) = |x|$ (by Theorem 3.14), and clearly, the convergence is uniform, if $x \in [-1, 1]$.

24. Let X be a metric space, with metric d. Fix a point $a \in X$. Assign to each $p \in X$ the function f_p defined by

$$f_p(x) = d(x,p) - d(x,a) \quad (x \in X).$$

Prove that $|f_p(x)| \le d(a,p)$ for all $x \in X$, and that therefore $f_p \in \mathscr{C}(X)$. Prove that

$$||f_p - f_q|| = d(p,q)$$

for all $p, q \in X$.
If $\Phi(p) = f_p$ it follows that Φ is an *isometry* (a distance-preserving mapping) of X onto $\Phi(X) \subseteq \mathscr{C}(X)$.
Let Y be the closure of $\Phi(X)$ in (X). Show that Y is complete.
Conclusion: X is isometric to a dense subset of a complete metric space Y.
Proof:
(i) Since $d(x,p) \le d(x,a) + d(p,a)$, we have $f_p(x) = d(x,p) - d(x,a) \le d(p,a) = d(a,p)$; and since $d(x,a) \le d(x,p) + d(p,a)$, we have $f_p(x) = d(x,p) - d(x,a) \ge -d(p,a) = -d(a,p)$. Therefore, $-d(a,p) \le f_p(x) \le d(a,p)$ and hence $|f_p(x)| \le d(a,p)$, for all $x \in X$. Since it's clear that $f_p(x)$ is continuous, therefore $f_p \in \mathscr{C}(X)$.
(ii) $||f_p - f_q|| = \sup\limits_{x \in X} |f_p(x) - f_q(x)| = \sup\limits_{x \in X} |d(x,p) - d(x,q)|$. Since $|d(x,p) - d(x,q)| \le d(p,q)$, for all $x \in X$, we have $\sup\limits_{x \in X} |d(x,p) - d(x,q)| \le d(p,q)$. On the other hand, $p \in X$ and $|d(p,p) - d(p,q)| = d(p,q)$, we thus have $\sup\limits_{x \in X} |d(x,p) - d(x,q)| = d(p,q)$, namely $||f_p - f_q|| = d(p,q)$, for all $p, q \in X$.
(iii) Let $\{f_n\}$ be any Cauchy sequence in Y, to show that Y is complete, we must show that $\{f_n\}$ converges to some $f \in Y$. Clearly, $\{f_n\}$ is a Cauchy sequence in $\mathscr{C}(X)$, and since $\mathscr{C}(X)$ is complete, $\{f_n\}$ must converge to some $f \in \mathscr{C}(X)$. Therefore, f is a limit point of $\mathscr{C}(X)$, and actually, a limit point of Y. Since Y is the closure of $\Phi(X)$ in $\mathscr{C}(X)$, Y is closed and thus $f \in Y$. Hence, Y is complete. (In fact, every closed subset E of a complete metric space X is complete. See the remark under Definition 3.12.)

25. Suppose ϕ is a continuous bounded real function in the strip defined by $0 \le x \le 1$, $-\infty < y < \infty$. Prove that the initial-value problem

$$y' = \phi(x,y), \quad y(0) = c$$

has a solution. (Note that the hypotheses of this existence theorem are less stringent than those of the corresponding uniqueness theorem; see Exercise 27, Chap.5.)
Proof: Fix n. For $i = 0, ..., n$, put $x_i = i/n$. Let f_n be a continuous function on $[0,1]$ such that $f_n(0) = c$, $f_n'(t) = \phi(x_i, f_n(x_i))$ if $x_i < t <$

x_{i+1}, and put $\Delta_n(t) = f_n'(t) - \phi(t, f_n(t))$, except at the points x_i, where $\Delta_n(t) = 0$. Then

$$f_n(x) = c + \int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt.$$

Choose $M < \infty$ so that $|\phi| \leq M$.
(a)$|f_n'| = |\phi(x_i, f_n(x_i))| \leq M$, and $|\Delta_n| \leq |f_n'| + |\phi| \leq 2M$. Clearly, $\Delta_n \in \mathscr{R}$ since ϕ is continuous on $[0,1]$, and $|f_n| \leq |c| + |\int_0^x [\phi(t, f_n(t)) + \Delta_n(t)]dt| = |c| + |\int_0^x f_n'(t)dt| \leq |c| + |\int_0^1 f_n'(t)dt| \leq |c| + M = M_1$ on $[0,1]$, for all n.
(b)We have $|f_n(x) - f_n(y)| = |\int_y^x f_n'(t)dt| \leq M|x-y|$, for all n and all $x, y \in [0,1]$. Then given any $\epsilon > 0$, we can pick $\delta = \epsilon/M > 0$, and when $|x-y| < \delta$ we get $|f(x) - f(y)| \leq M|x-y| < M\delta = \epsilon$, which is to say that $\{f_n\}$ is equicontinuous on $[0,1]$.
(c)By (a), (b) and Theorem 7.25(b), there is some subsequence $\{f_{n_k}\}$ of $\{f_n\}$ which converges to some f, uniformly on $[0,1]$.
(d)Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$, given any $\epsilon > 0$, we can pick a $\delta > 0$ such that when $|f_{n_k}(t) - f(t)| < \delta$, $|\phi(t, f_{n_k}(t)) - \phi(t, f(t))| < \epsilon$, for any $t \in [0,1]$. Since $f_{n_k} \to f$ uniformly, there is an $N > 0$ such that $n_k > N$ implies $|f_{n_k}(t) - f(t)| < \delta$, for all $t \in [0,1]$. Therefore, $\phi(t, f_{n_k}(t)) \to \phi(t, f(t))$ uniformly on $[0,1]$.
(e)Since ϕ is uniformly continuous on the rectangle $0 \leq x \leq 1$, $|y| \leq M_1$, for any given $\epsilon > 0$, there is a $r > 0$ such that $\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} < r$ implies $|\phi(x_1, y_1) - \phi(x_2, y_2)| < \epsilon$. Since f_n is uniformly continuous on $[0,1]$, there is a $\delta > 0$ such that $|x_i - t| < \delta$ implies $|f_n(x_i) - f_n(t)| < r/\sqrt{2}$. Let $\delta' = \min(\delta, r/\sqrt{2})$, then when $|x_i - t| < \delta'$, we have $\sqrt{(x_i - t)^2 + (f_n(x_i) - f_n(t))^2} < \sqrt{r^2} = r$ and therefore, $|\phi(x_i, f_n(x_i)) - \phi(t, f_n(t))| < \epsilon$, namely, $|\Delta_n(t)| < \epsilon$. Since $x_i = i/n$ and $t \in (x_i, x_{i+1})$, there is an integer N such that $n > N$ implies $|x_i - t| < 1/n < \delta'$, which means when $n > N$, $|\Delta_n(t)| < \epsilon$, for all $t \in [0,1]$. Hence, $\Delta_n(t) \to 0$ uniformly on $[0,1]$.
(f)Hence we have

$$\begin{aligned} f(x) = \lim_{n_k \to \infty} f_{n_k}(x) &= c + \lim_{n_k \to \infty} \int_0^x [\phi(t, f_{n_k}(t)) + \Delta_{n_k}(t)]dt \\ &= c + \int_0^x \phi(t, f(t))dt. \end{aligned}$$

This f is a solution solution of the given problem, since $f'(x) = \phi(x, f(x))$ and $f(0) = c$.

26. Prove an analogous existence theorem for the initial-value problem

$$\mathbf{y}' = \mathbf{\Phi}(x, \mathbf{y}), \quad \mathbf{y}(0) = \mathbf{c},$$

where now $\mathbf{c} \in R^k$, and $\mathbf{\Phi}$ is a continuous bounded mapping of the part of R^{k+1} defined by $0 \leq x \leq 1, \mathbf{y} \in R^k$. (Compare Exercise 28, Chap.5.)

Proof: Due to the similarity of the proof process as Exercise 7.25, here I just sketch the proof outlines.

Fix n. For $i = 0, ..., n$, put $x_i = i/n$. Let $\mathbf{f}_n$ be a continuous function on $[0, 1]$ such that $\mathbf{f}_n(0) = \mathbf{c}$, $\mathbf{f}'_n(t) = \mathbf{\Phi}(x_i, \mathbf{f}_n(x_i))$ if $x_i < t < x_{i+1}$, and put $\mathbf{\Delta}_n(t) = \mathbf{f}'_n(t) - \mathbf{\Phi}(t, \mathbf{f}_n(t))$, except at the points x_i, where $\mathbf{\Delta}_n(t) = 0$. Then

$$\mathbf{f}_n(x) = \mathbf{c} + \int_0^x [\mathbf{\Phi}(t, \mathbf{f}_n(t)) + \mathbf{\Delta}_n(t)]dt.$$

Choose $M < \infty$ so that $|\mathbf{\Phi}| \le M$.

(a)$|\mathbf{f}'_n| \le M$, $|\mathbf{\Delta}_n| \le 2M$, $\mathbf{\Delta}_n \in \mathscr{R}$, and $|\mathbf{f}_n| \le |\mathbf{c}| + M = M_1$, say, on $[0, 1]$, for all n.

(b)$\{\mathbf{f}_n\}$ is equicontinuous on $[0, 1]$, since $|\mathbf{f}'_n| \le M$.

(c)Some $\{\mathbf{f}_{n_k}\}$ converges to some $\mathbf{f}$, uniformly on $[0, 1]$, by using a vector-valued version of Theorem 7.25.

(d)Since $\mathbf{\Phi}$ is uniformly continuous on the rectangle $0 \le x \le 1$, $|\mathbf{y}| \le M_1$, $\mathbf{\Phi}(t, \mathbf{f}_{n_k}(t)) \to \mathbf{\Phi}(t, \mathbf{f}(t))$ uniformly on $[0, 1]$.

(e)$\mathbf{\Delta}_n(t) \to 0$ uniformly on $[0, 1]$, since $\mathbf{\Delta}_n(t) = \mathbf{\Phi}(x_i, \mathbf{f}_n(x_i)) - \mathbf{\Phi}(t, \mathbf{f}_n(t))$ in (x_i, x_{i+1}).

(f)Hence $\mathbf{f}(x) = \mathbf{c} + \int_0^x \mathbf{\Phi}(t, \mathbf{f}(t))dt$. This $\mathbf{f}$ is a solution of the given problem.

8 Some special functions

1. Define
$$f(x) = \begin{cases} e^{-1/x^2} & (x \neq 0), \\ 0 & (x = 0). \end{cases}$$
Prove that f has derivatives of all orders at $x = 0$, and that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, ...$
Proof: Let $y = 1/x$, then $f(x) = g(y) = e^{-y^2}$, when $x \neq 0$. Clearly, $g^{(n)}(y) = e^{-y^2}P_n(y)$, where $P_n(y)$ is some n-order polynomial of y, for $n = 1, 2, 3, ...$ Now we prove $f^{(n)}(0) = 0$ by induction.
(i)When $n = 1$, $f^{(1)}(0) = \lim\limits_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim\limits_{y\to\infty} ye^{-y^2} = 0$, by Theorem 8.6(f).
(ii)Suppose $f^{(k)}(0) = 0$. When $n = k+1$, $f^{(k+1)}(x) = \lim\limits_{x\to 0} \frac{f^{(k)}(x)-f^{(k)}(0)}{x-0} = \lim\limits_{y\to\infty} yg^{(k)}(y) = \lim\limits_{y\to\infty} ye^{-y^2}P_k(y) = 0$, according to Theorem 8.6(f).
Therefore, $f^{(n)}(0) = 0$, for $n = 1, 2, 3, ...$

2. Let a_{ij} be the number in the ith row and jth column of the array
$$\begin{array}{ccccc} -1 & 0 & 0 & 0 & \cdots \\ 1/2 & -1 & 0 & 0 & \cdots \\ 1/4 & 1/2 & -1 & 0 & \cdots \\ 1/8 & 1/4 & 1/2 & -1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

so that

$$a_{ij} = \begin{cases} 0 & (i < j), \\ -1 & (i = j), \\ 2^{j-i} & (i > j). \end{cases}$$

Prove that

$$\sum_i \sum_j a_{ij} = -2, \quad \sum_j \sum_i a_{ij} = 0.$$

Proof:

$$\sum_i \sum_j a_{ij} = \sum_i (-1 + \sum_{j<i} a_{ij}) = \sum_i (-1 + \sum_{j=1}^{i-1} a_{ij})$$

$$= \sum_i (-1 + \sum_{j=1}^{i-1} 2^{j-i}) = \sum_i (-1 + 2^{-i} \sum_{j=1}^{i-1} 2^j)$$

$$= \sum_i (-1 + 2^{-i} 2(2^{i-1} - 1)) = \sum_i (-1 + 1 - 2^{1-i}) = -2 \sum_{i=1}^{\infty} 2^{-i} = -2.$$

$$\sum_j \sum_i a_{ij} = \sum_j (-1 + \sum_{i>j} a_{ij}) = \sum_j (-1 + \sum_{i=j+1}^{\infty} a_{ij})$$

$$= \sum_j (-1 + \sum_{i=j+1}^{\infty} 2^{j-i}) = \sum_j (-1 + 2^j \sum_{i=j+1}^{\infty} 2^{-i})$$

$$= \sum_j (-1 + 2^j 2^{-j}) = \sum_j (-1 + 1) = \sum_j 0 = 0.$$

3. Prove that

$$\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$$

if $a_{ij} \geq 0$ for all i and j(the case $+\infty = +\infty$ may occur).
Proof: Suppose, first, $s = \sum_i \sum_j a_{ij}$ converges, that is, $s < +\infty$. Since $a_{ij} \geq 0$, $\sum_j |a_{ij}| = \sum_j a_{ij} = b_i$, and $\sum_i b_i$ converges. By Theorem 8.3, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.
Next, suppose $s = \sum_i \sum_j a_{ij} = +\infty$. Let $s_{mn} = \sum_{i=1}^m \sum_{j=1}^n a_{ij}$, given any $M > 0$, since $s = +\infty$ and $a_{ij} \geq 0$, there exists some m_1, n_1 such that $s_{m_1 n_1} > M$. Clearly $t_{n_1 m_1} = \sum_{j=1}^{n_1} \sum_{i=1}^{m_1} a_{ij} = s_{m_1 n_1} > M$, therefore, $t = \sum_j \sum_i a_{ij} = +\infty$.
Thus, $\sum_i \sum_j a_{ij} = \sum_j \sum_i a_{ij}$.

4. Prove the following limit relation:

(a) $\lim\limits_{x\to 0} \frac{b^x-1}{x} = \log b \quad (b>0)$.
Proof: By Theorem 5.13, we have

$$\lim_{x\to 0} \frac{b^x-1}{x} = \lim_{x\to 0} \frac{e^{x\log b}-1}{x} = \lim_{x\to 0} e^{x\log b} \log b = \log b.$$

(b) $\lim\limits_{x\to 0} \frac{\log(x+1)}{x} = 1$.
Proof: By Theorem 5.13, we have

$$\lim_{x\to 0} \frac{\log(x+1)}{x} = \lim_{x\to 0} \frac{1}{x+1} = 1.$$

(c) $\lim\limits_{x\to 0} (1+x)^{1/x} = e$.
Proof: Let $y = (1+x)^{1/x}$, then $\log y = \frac{\log(1+x)}{x}$, and

$$\lim_{x\to 0} \log y = \lim_{x\to 0} \frac{\log(1+x)}{x} = 1.$$

Since $\log(x)$ is continuous, this implies $\log(\lim_{x\to 0} y) = 1$. On the other hand, $\log e = 1$, and due to the monotonicity of $\log(x)$, we have $\lim_{x\to 0} y = e$, namely, $\lim_{x\to 0}(1+x)^{1/x} = e$.

(d) $\lim\limits_{n\to\infty} (1+\frac{x}{n})^n = e^x$.
Proof: If $x = 0$, then $\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{n\to\infty} 1 = 1 = e^0$. Suppose $x \neq 0$, let $y = n/x$, then $\lim_{n\to\infty}(1+\frac{x}{n})^n = \lim_{y\to\infty}[(1+\frac{1}{y})^y]^x$. Due to the continuity of x^α, this leads to $\lim_{y\to\infty}[(1+\frac{1}{y})^y]^x = [\lim_{y\to\infty}(1+\frac{1}{y})^y]^x = e^x$, by Theorem 3.31.

5. Find the following limits

(a) $\lim\limits_{x\to 0} \frac{e-(1+x)^{1/x}}{x}$.
Solution: Let $y = (1+x)^{1/x}$, then $\log y = 1/x \log(1+x)$, namely, $x\log y = \log(1+x)$. Differentiate both sides gives: $\log y + x(1/y)y' = 1/(1+x)$, i.e., $y' = (1/(1+x) - \log y)y/x = (1/(1+x) - 1/x\log(1+x))(1+x)^{1/x}/x = (\frac{1}{x(1+x)} - \frac{\log(1+x)}{x^2})(1+x)^{1/x} = (\frac{1}{x} - \frac{1}{1+x} - \frac{\log(1+x)}{x^2})(1+x)^{1/x}$, and therefore, $\lim_{x\to 0} y' = \lim_{x\to 0}(1/x - 1/(x+1) - 1/x)e = e\lim_{x\to 0}(-1/(x+1)) = -e$.
By Theorem 5.13, $\lim_{x\to 0} \frac{e-y}{x} = \lim_{x\to 0} -y' = e$.

(b) $\lim\limits_{n\to\infty} \frac{n}{\log n}[n^{1/n}-1]$.
Solution: Let $y = x^{1/x}$, then $\log y = \frac{1}{x}\log x$.
Therefore, $\lim_{x\to+\infty} \log y = 0$, by (45) on page 181, and hence $\lim_{x\to+\infty} y = 1$, due to the continuity and monotonicity of $\log(x)$. We then have

$$\lim_{n\to\infty} \frac{n}{\log n}[n^{1/n}-1] = \lim_{x\to+\infty} \frac{x}{\log x}[x^{1/x}-1] = \lim_{x\to+\infty} \frac{x^{1/x}-1}{\log x/x}$$

$$= \lim_{x\to+\infty} \frac{y'}{(1-\log x)/x^2},$$

by Theorem 5.13. With the similar process as in (a), we have $y' = \frac{x^{1/x}(1-\log x)}{x^2}$. Therefore,

$$\lim_{x\to+\infty} \frac{y'}{(1-\log x)/x^2} = \lim_{x\to+\infty} \frac{x^{1/x}(1-\log x)/x^2}{(1-\log x)/x^2} = \lim_{x\to+\infty} x^{1/x}$$
$$= \lim_{x\to+\infty} y = 1.$$

(c) $\lim\limits_{x\to 0} \frac{\tan x - x}{x(1-\cos x)}$.
Solution: Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ and $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$, we have $\cos x = 1 - x^2/2! + x^4/4! - \cdots$. Then

$$\lim_{x\to 0} \frac{\tan x - x}{x(1-\cos x)} = \lim_{x\to 0} \frac{\tan x - x}{x^3/2! - x^5/4! + \cdots} = \lim_{x\to 0} \frac{1/\cos^2 x - 1}{3x^2/2}$$
$$= \lim_{x\to 0} \frac{2\sin^2 x}{3x^2\cos^2 x} = \frac{2}{3}.$$

(d) $\lim\limits_{x\to 0} \frac{x-\sin x}{\tan x - x}$.
Solution: Similarly as in (c), we have $\sin x = x - x^3/3! + x^5/5! - \cdots$. Then

$$\lim_{x\to 0} \frac{x-\sin x}{\tan x - x} = \lim_{x\to 0} \frac{x^3/3! - x^5/5! + \cdots}{\tan x - x} = \lim_{x\to 0} \frac{3x^2/3!}{\sin^2 x/\cos^2 x}$$
$$= \lim_{x\to 0} \frac{x^2\cos^2 x}{2\sin^2 x} = \frac{1}{2}.$$

6. Suppose $f(x)f(y) = f(x+y)$ for all real x and y.

(a) Assuming that f is differentiable and not zero, prove that

$$f(x) = e^{cx}$$

where c is a constant.
Proof: Since f is differentiable, we have

$$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(x) - 1}{x},$$

since $f(0) = f(0+0) = f(0)f(0)$ and $f(0) \neq 0$(so $f(0) = 1$). Fix some integer m, and let $p_n = m/n$, then $p_n \to 0$ when $n \to \infty$, thus we have $\lim_{n\to\infty} \frac{f(m/n)-1}{m/n} = f'(0)$. Since $[f(m/n)]^n = f(n \cdot (m/n)) = f(m)$, we have $f(m/n) = f(m)^{1/n}$. Therefore, $f'(0) =$

$(1/m)\lim_{n\to\infty}\frac{f(m)^{1/n}-1}{1/n}=\log f(m)/m$, by Exercise 4(a). Note that the above process is immaterial with m, or in other words, for any integer m, we must have $\log f(m)/m=c$, where $c=f'(0)$. Hence, we have $f(m)=e^{cm}$. Then for any rational number $p=m/n$, $[f(p)]^n=f(np)=f(m)=e^{cm}$ and thus $f(p)=e^{cm/n}=e^{cp}$. Since f is differentiable, f is continuous, and since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $f(p)=e^{cp}$ for every $p\in\mathbb{Q}$, we have $f(x)=e^{cx}$ for every $x\in\mathbb{R}$, according to Exercise 4.4.

(b) Prove the same thing, assuming only that f is continuous.
Proof: $f(m)=f(m\cdot 1)=[f(1)]^m$, and hence $\log f(1)=\log f(m)/m$. Let $c=\log f(1)$, then $\log f(m)/m=c$ for every m. The following proof is the same as in (a). Note that only continuity of f is enough, and differentiable is not required.

7. If $0<x<\pi/2$, prove that

$$\frac{2}{\pi}<\frac{\sin x}{x}<1.$$

Proof: First, we will prove that $\sin x<x<\tan x$, for $0<x<\pi/2$. Let $f(x)=\tan x-x$, then $f'(x)=1/\cos^2 x-1=\sin^2 x/\cos^2 x>0$, which gives $f(x)>f(0)=0$, namely, $\tan x>x$. Next, let $g(x)=x-\sin x$, then $g'(x)=1-\cos x>0$, hence $g(x)>g(0)=0$, namely, $x>\sin x$. Therefore, $\sin x<x<\tan x$, for $0<x<\pi/2$.
Since $\frac{\sin x}{x}-1=\frac{\sin x-x}{x}<0$, we have $\frac{\sin x}{x}<1$; and let $h(x)=\frac{\sin x}{x}$, then $h'(x)=\frac{x\cos x-\sin x}{x^2}<0$, since $x<\tan x$ gives $x\cos x-\sin x<0$. Therefore, $h(x)>h(\pi/2)=2/\pi$, namely. $2/\pi<\sin x/x<1$.

8. For $n=0,1,2,..$, and x real, prove that

$$|\sin nx|\le n|\sin x|.$$

Note that this inequality may be false for other values of n. For instance,

$$|\sin\frac{1}{2}\pi|>\frac{1}{2}|\sin\pi|.$$

Proof: We prove this by induction.
(i)When $n=0$, $\sin 0x=0\cdot\sin x$ and the inequality holds.
(ii)Suppose the inequality holds when $n=k$, namely, $|\sin kx|\le k|\sin x|$. Let $n=k+1$, then $|\sin(k+1)x|=|\sin kx\cos x+\cos kx\sin x|\le|\sin kx\cos x|+|\cos kx\sin x|\le|\sin kx|+|\sin x|\le k|\sin x|+|\sin x|=(k+1)|\sin x|$. This proves the inequality.

9. (a) Put $s_N=1+\frac{1}{2}+\cdots+(1/N)$. Prove that

$$\lim_{N\to\infty}(s_N-\log N)$$

exists. (The limit, often denoted by γ, is called Euler's constant. Its numerical value is 0.5772.... It is not known whether γ is rational or not.)
Proof: Let $t_n = s_n - \log n$, it's sufficient to show that $\{t_n\}$ is a Cauchy sequence.
Suppose $n > m$, we have $|t_n - t_m| = |(s_n - \log n) - (s_m - \log m)| = |(s_n - s_m) + \log(m/n)| = |\sum_{k=m+1}^{n} 1/k + \log(m/n)|$.
It's easy to see that

$$\int_{m+1}^{n} \frac{1}{t} dt < \sum_{k=m+1}^{n} \frac{1}{k} < \int_{m}^{n} \frac{1}{t} dt,$$

and therefore

$$\log(\frac{n}{m+1}) < \sum_{k=m+1}^{n} \frac{1}{k} < \log(\frac{n}{m}),$$

which gives

$$\log(\frac{n}{m+1}) + \log(\frac{m}{n}) < \sum_{k=m+1}^{n} \frac{1}{k} + \log(\frac{m}{n}) < \log(\frac{n}{m}) \log(\frac{m}{n}),$$

namely,

$$\log(\frac{m}{m+1}) < t_n - t_m < 0.$$

Hence, $|t_n - t_m| < \log(1 + 1/m)$. Given any $\epsilon > 0$, there exists an integer $N > 0$ such that $m > N$ implies $\log(1 + 1/m) < \epsilon$ (or, $m > 1/(e^{\epsilon} - 1)$), namely, $|t_n - t_m| < \epsilon$, when $n > m > N$. Thus, $\{t_n\}$ is a Cauchy sequence, as we desire.

(b) Roughly how large must m be so that $N = 10^m$ satisfies $s_N > 100$?
Solution: Since $\{t_n\}$ converges to γ, there exists some $N > 0$, such that $n \geq N$ implies $|t_n - \gamma| < 0.1$, namely, $\gamma - 0.1 < t_n < \gamma + 0.1$, or, $\log n + \gamma - 0.1 < s_n < \log n + \gamma + 0.1$, which gives, $\log n < s_n < \log_n + 1$. Let $N = 10^m$, this gives $m \log 10 < s_N < m \log 10 + 1$. For $s_N > 100$, we must have $m \log 10 \geq 100$, which gives $m \geq 44$.

10. Prove that $\sum 1/p$ diverges; the sum extends over all primes. (This shows that the primes form a fairly substantial subset of the positive integers.)
Proof: Given N, let $p_1, ..., p_k$ be those primes that divide at least one integer $\leq N$. Then

$$\sum_{n=1}^{N} \frac{1}{n} \leq \prod_{j=1}^{k} \left(1 + \frac{1}{p_j} + \frac{1}{p_j^2} + \cdots\right)$$

$$= \prod_{j=1}^{k} \left(1 - \frac{1}{p_j}\right)^{-1} \leq \exp \sum_{j=1}^{k} \frac{2}{p_j}.$$

The last inequality holds because

$$(1-x)^{-1} \le e^{2x}$$

if $0 \le x \le \frac{1}{2}$. To see this, let $f(x) = e^{2x}(1-x)$, then $f'(x) = e^{2x}(1-2x) \ge 0$, for $0 \le x \le \frac{1}{2}$. Hence, $f(x) \ge f(0) = 1$, which gives $e^{2x} \ge (1-x)^{-1}$. Since $\sum \frac{1}{n}$ diverges, it's clear that $\sum \frac{1}{p}$ from above.

11. Suppose $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, and $f(x) \to 1$ as $x \to +\infty$. Prove that

$$\lim_{t \to 0} t \int_0^\infty e^{-tx} f(x) dx = 1 \quad (t > 0).$$

Proof: Fix some $t > 0$, e^{-tx} is strictly decreasing on $[0, +\infty)$. When $x = 0$, $e^{-tx} = 1$ and when $x \to +\infty$, $e^{-tx} \to 0$. Given any $\epsilon > 0$, since $f(x) \to 1$ as $x \to +\infty$, we can pick some $A' > 0$ such that $x > A'$ implies $|f(x) - 1| < \epsilon$, namely, $1 - \epsilon < f(x) < 1 + \epsilon$. We hence have

$$(1-\epsilon) t \int_{A'}^\infty e^{-tx} dx < t \int_{A'}^\infty e^{-tx} f(x) dx < (1+\epsilon) t \int_{A'}^\infty e^{-tx} dx.$$

Since

$$t \int_{A'}^\infty e^{-tx} dx = -\int_{A'}^\infty e^{-tx} d(-tx) = -e^{-tx}\Big|_{A'}^\infty = e^{-tA'}.$$

Let $t \to 0$, we have $e^{-tA'} \to 1$ and thus

$$1 - \epsilon < \lim_{t \to 0} t \int_{A'}^\infty e^{-tx} f(x) dx < 1 + \epsilon,$$

which gives

$$|\lim_{t \to 0} t \int_{A'}^\infty e^{-tx} f(x) dx - 1| < \epsilon.$$

Since $\epsilon > 0$ is arbitrary, this leads to

$$\lim_{t \to 0} t \int_{A'}^\infty e^{-tx} f(x) dx = 1.$$

On the other hand, since $f \in \mathscr{R}$ on $[0, A]$ for all $A < \infty$, $f \in \mathscr{R}$ on $[0, A']$, thus $|f| \in \mathscr{R}$ on $[0, A']$, by Theorem 6.13(b). Let $M = \int_0^{A'} |f(x)| dx$, we then have

$$|\int_0^{A'} e^{-tx} f(x) dx| \le \int_0^{A'} |e^{-tx}||f(x)| dx \le \int_0^{A'} |f(x)| dx = M.$$

Therefore,

$$0 \le |t \int_0^{A'} e^{-tx} f(x) dx| \le tM.$$

Let $t \to 0$, we have

$$\lim_{t\to 0} |t \int_0^{A'} e^{-tx} f(x)dx| = 0,$$

and thus

$$\lim_{t\to 0} t \int_0^{A'} e^{-tx} f(x)dx = 0.$$

We hence have

$$\lim_{t\to 0} t \int_0^{\infty} e^{-tx} f(x)dx = \lim_{t\to 0} t \int_0^{A'} e^{-tx} f(x)dx$$

$$+ \lim_{t\to 0} t \int_{A'}^{\infty} e^{-tx} f(x)dx = 0 + 1 = 1.$$

12. Suppose $0 < \delta < \pi$, $f(x) = 1$ if $|x| \le \delta$, $f(x) = 0$ if $\delta < |x| < \pi$, and $f(x + 2\pi) = f(x)$ for all x.

   (a) Compute the Fourier coefficients of f.
   Solution:

   $$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} dx = \frac{\delta}{\pi}.$$

   and

   $$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi} \int_{-\delta}^{\delta} e^{-inx}dx = \frac{\sin n\delta}{n\pi}, \quad (n \neq 0).$$

   (b) Conclude that

   $$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \frac{\pi - \delta}{2} \quad (0 < \delta < \pi).$$

   Proof: Let $y = \sum_{n=1}^{+\infty} c_n$. Since $c_{-n} = c_n$, we have $\sum_{n=-\infty}^{-1} c_n = \sum_{n=1}^{+\infty} c_{-n} = \sum_{n=1}^{+\infty} c_n = y$, and therefore, $2y + c_0 = \sum_{n=-\infty}^{+\infty} c_n = f(0)$ (since $f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx}$). This gives $2y + \frac{\delta}{\pi} = 1$, and thus $y = \frac{1-\delta/\pi}{2}$. So

   $$\sum_{n=1}^{\infty} \frac{\sin(n\delta)}{n} = \sum_{n=1}^{\infty} \pi c_n = \pi y = \frac{\pi - \delta}{2}.$$

   (c) Deduce from Parseval's theorem that

   $$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \frac{\pi - \delta}{2}.$$

Proof: Let $y = \sum_{n=1}^{+\infty} |c_n|^2$. Since $c_{-n} = c_n$, we have $\sum_{n=-\infty}^{-1} |c_n|^2 = \sum_{n=1}^{+\infty} |c_{-n}|^2 = \sum_{n=1}^{+\infty} |c_n|^2 = y$, and therefore,

$$2y + |c_0|^2 = \sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx,$$

by Parseval's theorem, which gives

$$2y + \frac{\delta^2}{\pi^2} = \frac{\delta}{\pi}, \quad i.e., \quad y = \frac{\delta/\pi - \delta^2/\pi^2}{2}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{\sin^2(n\delta)}{n^2\delta} = \sum_{n=1}^{\infty} \frac{\pi^2}{\delta}|c_n|^2 = \frac{\pi^2}{\delta} y = \frac{\pi - \delta}{2}.$$

(d) Let $\delta \to 0$ and prove that

$$\int_0^{\infty} (\frac{\sin x}{x})^2 dx = \frac{\pi}{2}.$$

Proof: Let $A > 0$ be any positive real number, and let $f(x) = (\frac{\sin x}{x})^2$. First we prove that $f \in \mathscr{R}$ on every $[0, A]$. Define

$$g(x) = \begin{cases} f(x) & (x > 0) \\ 1 & (x = 0) \end{cases}$$

Since

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} f(x) = 1 = g(0),$$

$g(x)$ is continuous on every $[0, A]$ and hence $g \in \mathscr{R}$ on $[0, A]$. Since

$$\int_0^A f(x)dx = \lim_{c \to 0} \int_c^A f(x)dx = \lim_{c \to 0} \int_c^A g(x)dx = \int_0^A g(x)dx,$$

$f \in \mathscr{R}$ on every $[0, A]$.
Let $P^* = \{x_0 = 0, x_1 = \delta, ..., x_m = A\}$(suppose $(m-1)\delta < A < m\delta$) be a partition on $[0, A]$ (see Exercise 6.7(a)). Suppose $P = \{y_0 = 0, y_1, ..., y_k = A\}$ is any other partition on $[0, A]$. Choose $\delta > 0$ small enough so that $x_{n_{i-1}} = n_{i-1}\delta \le y_i < x_{n_i} = n_i\delta$, for $1 \le i < k$. We then have

$$|\sum_{n_i}[f(n_i\delta)\Delta x_{n_i} - M_{1i}(y_i - x_{n_{i-1}}) - M_{2i}(x_{n_i} - y_i)]|$$

$$\le \sum_{n_i}[|f(n_i\delta)|\Delta x_{n_i} + M_{1i}(y_i - x_{n_{i-1}}) + M_{2i}(x_{n_i} - y_i)]$$

$$\le \sum_{n_i}[M\Delta x_{n_i} + M(y_i - x_{n_{i-1}}) + M(x_{n_i} - y_i)] = 2M\sum_{n_i}\Delta x_{n_i} = 2Mk\delta,$$

where $M = \sup_{x\in[0,A]} |f(x)|$, $M_{1i} = \sup_{x\in[y_{i-1},y_i]} |f(x)|$ and $M_{2i} = \sup_{x\in[y_i,y_{i+1}]} |f(x)|$. Therefore, we have

$$\sum_{n_i} f(n_i\delta)\Delta x_{n_i} \le \sum_{n_i}[M_{1i}(y_i - x_{n_{i-1}}) + M_{2i}(x_{n_i} - y_i)] + 2Mk\delta.$$

This gives

$$\sum_{n=1}^{m} f(x_n)\Delta x_n = \sum_{n=1}^{m} f(n\delta)\delta \le U(P, f) + 2Mk\delta,$$

and thus

$$\lim_{\delta\to 0} \sum_{n=1}^{m} f(x_n)\Delta x_n \le U(P, f).$$

Since P is arbitrary, we have

$$\lim_{\delta\to 0} \sum_{n=1}^{m} f(n\delta)\delta \le \bar{\int_0^A} f(x)dx, i.e., \lim_{\delta\to 0} \sum_{n=1}^{m} \frac{\sin^2(n\delta)}{n^2\delta} \le \bar{\int_0^A} f(x)dx.$$

On the other hand, we have

$$|\sum_{n_i}[f(n_i\delta)\Delta x_{n_i} - m_{1i}(y_i - x_{n_{i-1}}) - m_{2i}(x_{n_i} - y_i)]|$$

$$\le \sum_{n_i}[|f(n_i\delta)|\Delta x_{n_i} + m_{1i}(y_i - x_{n_{i-1}}) + m_{2i}(x_{n_i} - y_i)]$$

$$\le \sum_{n_i}[M\Delta x_{n_i} + M(y_i - x_{n_{i-1}}) + M(x_{n_i} - y_i)] = 2M\sum_{n_i} \Delta x_{n_i} = 2Mk\delta,$$

where $M = \sup_{x\in[0,A]} |f(x)|$, $m_{1i} = \inf_{x\in[y_{i-1},y_i]} |f(x)|$ and $m_{2i} = \inf_{x\in[y_i,y_{i+1}]} |f(x)|$. Therefore, we have

$$\sum_{n_i} f(n_i\delta)\Delta x_{n_i} \ge \sum_{n_i}[m_{1i}(y_i - x_{n_{i-1}}) + m_{2i}(x_{n_i} - y_i)] - 2Mk\delta.$$

This gives

$$\sum_{n=1}^{m} f(x_n)\Delta x_n = \sum_{n=1}^{m} f(n\delta)\delta \ge L(P, f) - 2Mk\delta,$$

and thus

$$\lim_{\delta\to 0} \sum_{n=1}^{m} f(x_n)\Delta x_n \ge L(P, f).$$

Since P is arbitrary, we have

$$\lim_{\delta\to 0}\sum_{n=1}^{m} f(n\delta)\delta \geq \underline{\int_0^A} f(x)dx, i.e., \lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} \geq \underline{\int_0^A} f(x)dx.$$

Hence,

$$\underline{\int_0^A} f(x)dx \leq \lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} \leq \overline{\int_0^A} f(x)dx.$$

Since $f \in \mathscr{R}$ on $[0, A]$,

$$\lim_{\delta\to 0}\sum_{n=1}^{m}\frac{\sin^2(n\delta)}{n^2\delta} = \int_0^A f(x)dx.$$

Thus,

$$\int_0^\infty (\frac{\sin x}{x})^2 dx = \int_0^\infty f(x)dx = \lim_{A\to\infty}\int_0^A f(x)dx$$

$$= \lim_{\delta\to 0}\sum_{n=1}^{\infty}\frac{\sin^2(n\delta)}{n^2\delta} = \lim_{\delta\to 0}\frac{\pi-\delta}{2} = \frac{\pi}{2}.$$

(e) Put $\delta = \pi/2$ in (c). What do you get?
Solution: We get

$$\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2} = \frac{\pi^2}{4}.$$

13. Put $f(x) = x$ if $0 \leq x < 2\pi$, and apply Parseval's theorem to conclude that

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Proof: By computing the Fourier coefficients of f, we get

$$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xdx = 0,$$

and

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{(-1)^{n+1}}{in} \quad (n \neq 0).$$

Thus $|c_n|^2 = \frac{1}{n^2}$. Let $y = \sum_{i=1}^{+\infty}|c_n|^2$, then $\sum_{n=-\infty}^{-1}|c_n|^2 = \sum_{n=1}^{+\infty}|c_{-n}|^2 = \sum_{n=1}^{+\infty}|c_n|^2 = y$. Hence

$$2y + |c_0|^2 = \sum_{n=-\infty}^{+\infty}|c_n|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2 dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx,$$

which gives

$$2y = \frac{\pi^2}{3}, \quad i.e., \quad y = \frac{\pi^2}{6}, \quad i.e., \quad \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

14. If $f(x) = (\pi - |x|)^2$ on $[-\pi, \pi]$, prove that

$$f(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx$$

and deduce that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}.$$

Proof: Compute the Fourier coefficients of f, we get

$$c_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 dx = \frac{\pi^2}{3},$$

and

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^2 e^{-inx}dx = \frac{2}{n^2} \quad (n \neq 0).$$

Hence,

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{inx} = \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{2}{n^2} e^{inx} + \sum_{n=-\infty}^{-1} \frac{2}{n^2} e^{inx}$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{+\infty} \frac{2}{n^2} e^{inx} + \sum_{n=1}^{+\infty} \frac{2}{n^2} e^{-inx} = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} (e^{inx} + e^{-inx})$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2} 2\cos nx = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} \cos nx.$$

Put $x = 0$, we get

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2} = \pi^2,$$

which gives

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Since $c_n = \frac{2}{n^2}$, we have $|c_n|^2 = \frac{4}{n^4}$, for $n \neq 0$. Let $y = \sum_{n=1}^{+\infty} |c_n|^2$, then since $c_n = c_{-n}$, we have

$$2y + |c_0|^2 = \sum_{n=-\infty}^{+\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (\pi - |x|)^4 dx,$$

which gives

$$2y + \frac{\pi^4}{9} = \frac{\pi^4}{5}, i.e., 2y = \frac{4\pi^4}{45}, i.e., y = \frac{2\pi^4}{45}.$$

Hence,

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{4}y = \frac{\pi^4}{90}.$$

15. With D_n as defined in (77), put

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x).$$

Prove that

$$K_N(x) = \frac{1}{N+1} \cdot \frac{1 - \cos(N+1)x}{1 - \cos x}$$

and that

(a) $K_N \geq 0$,
(b) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_N(x) dx = 1$,
(c) $K_N(x) \leq \frac{1}{N+1} \cdot \frac{2}{1-\cos\delta}$ if $0 < \delta \leq |x| \leq \pi$.

If $s_N = s_N(f; x)$ is the Nth partial sum of the Fourier series of f, consider the arithmetic means

$$\sigma_N = \frac{s_0 + s_1 + \cdots + s_N}{N+1}.$$

Prove that

$$\sigma_N(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_N(t) dt,$$

and hence prove the Fejer's theorem:
If f is continuous, with period 2π, then $\sigma_N(f; x) \to f(x)$ uniformly on $[-\pi, \pi]$.
Proof:
(i)Since

$$D_n(x) = \sum_{k=-n}^{n} e^{ikx} = \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})},$$

we have

$$\sum_{n=0}^{N} D_n(x) = \sum_{n=0}^{N} \frac{\sin(n+\frac{1}{2})x}{\sin(\frac{x}{2})} = \sum_{n=0}^{N} \frac{2\sin(\frac{x}{2})\sin(n+\frac{1}{2})x}{2\sin^2(\frac{x}{2})}$$

$$= \frac{1}{1-\cos x} \sum_{n=0}^{N} [\cos nx - \cos(n+1)x] = \frac{1-\cos(N+1)x}{1-\cos x}.$$

Hence,

$$K_N(x) = \frac{1}{N+1} \sum_{n=0}^{N} D_n(x) = \frac{1}{N+1} \cdot \frac{1-\cos(N+1)x}{1-\cos x}.$$

(ii)Next, we prove:

(a) The fact $K_N \geq 0$ is clear.

(b) Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-n}^{n} e^{ikx}dx = \sum_{k=-n}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikx}dx = 1,$$

we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}(\frac{1}{N+1}\sum_{n=0}^{N} D_n(x))dx = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} D_n(x)dx$$

$$= \frac{1}{N+1}\sum_{n=0}^{N} 1 = \frac{1}{N+1}\cdot(N+1) = 1.$$

(c) Let $g(x) = 1-\cos x$, then $g'(x) = \sin x$. If $0 < \delta \leq x \leq \pi$, then $\sin x \geq 0$ and thus $g(x) \geq g(\delta) = 1-\cos\delta$; and if $-\pi \leq x \leq -\delta$, then $\sin x \leq 0$ and thus $g(x) \geq g(-\delta) = 1-\cos\delta$. That is, $1-\cos x \geq 1-\cos\delta$, for $0 < \delta \leq |x| \leq \pi$. Therefore,

$$K_N(x) \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos x} \leq \frac{1}{N+1}\cdot\frac{2}{1-\cos\delta}, 0 < \delta \leq |x| \leq \pi.$$

(iii)Since

$$s_n(f;x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt,$$

we have

$$\sigma_N(f;x) = \frac{1}{N+1}\sum_{n=0}^{N} s_n(f;x) = \frac{1}{N+1}\sum_{n=0}^{N}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)dt$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)(\frac{1}{N+1}\sum_{n=0}^{N} D_n(t))dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)K_N(t)dt.$$

(iv)Since f is continuous on $[-\pi,\pi]$, f is uniformly continuous on $[-\pi,\pi]$. Given any $\epsilon > 0$, there exists some $0 < \delta < \pi$ such that $|x-y| < \delta$ implies $|f(x)-f(y)| < \epsilon/2$, for any $x, y \in [-\pi,\pi]$.

Due to (ii)(b), and $K_N(t) = K_N(-t)$, put $M = \sup|f(x)|$ for $x \in [-\pi, \pi]$, we have

$$|\sigma_N(f;x) - f(x)| = |\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t) - f(x))K_N(t)dt|$$

$$\leq 2M\cdot\frac{1}{2\pi}\int_{-\pi}^{-\delta}K_N(t)dt + \frac{1}{2\pi}\int_{-\delta}^{\delta}|f(x-t) - f(x)|K_N(t)dt + 2M\cdot\frac{1}{2\pi}\int_{\delta}^{\pi}K_N(t)dt$$

$$< \frac{2M}{\pi}\int_{\delta}^{\pi}K_N(t)dt + \frac{\epsilon}{2}(\frac{1}{2\pi}\int_{-\pi}^{\pi}K_N(t)dt) \leq \frac{2M}{\pi}(\pi-\delta)\frac{1}{N+1}\cdot\frac{2}{1-\cos\delta} + \frac{\epsilon}{2}$$

$$= \frac{4M(\pi-\delta)}{(N+1)\pi(1-\cos\delta)} + \frac{\epsilon}{2} < \epsilon, \qquad \text{for sufficiently large } N.$$

Therefore, $\sigma_N(f;x) \to f(x)$ uniformly on $[-\pi, \pi]$.

16. Prove a pointwise version of Fejer's theorem:
If $f \in \mathscr{R}$ and $f(x+)$, $f(x-)$ exist for some x, then

$$\lim_{n\to\infty}\sigma_N(f;x) = \frac{1}{2}[f(x+) + f(x-)].$$

Proof: Since $f(x+)$, $f(x-)$ exist for some x, Given any $\epsilon > 0$, there exist some $\delta_1 > 0$ and $\delta_2 > 0$ such that $x < t < x + \delta_1$ implies $|f(t) - f(x+)| < \epsilon/2$ and $x - \delta_2 < t < x$ implies $|f(t) - f(x-)| < \epsilon/2$. Let $\delta = \min(\delta_1, \delta_2)$, then $x < t < x + \delta$ implies $|f(t) - f(x+)| < \epsilon/2$ and $x - \delta < t < x$ implies $|f(t) - f(x-)| < \epsilon/2$. Since $f \in \mathscr{R}$, f is bounded, and thus $|f| \leq M$, for some $M > 0$. As in Exercise 15(iv), we have

$$|\sigma_N(f;x) - \frac{f(x+) + f(x-)}{2}| = |\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x-t) - \frac{f(x+) + f(x-)}{2})K_N(t)dt|$$

$$= |\frac{1}{2\pi}\int_{-\pi}^{-\delta}(f(x-t) - \frac{f(x+) + f(x-)}{2})K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(x-t) - \frac{f(x+) + f(x-)}{2})K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{\delta}^{\pi}(f(x-t) - \frac{f(x+) + f(x-)}{2})K_N(t)dt|$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{-\delta}|f(x-t) - \frac{f(x+) + f(x-)}{2}|K_N(t)dt$$

$$+|\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(x-t) - \frac{f(x+) + f(x-)}{2})K_N(t)dt|$$

$$+\frac{1}{2\pi}\int_{\delta}^{\pi}|f(x-t) - \frac{f(x+) + f(x-)}{2}|K_N(t)dt$$

$$\leq 2M \cdot \frac{1}{2\pi}\int_{-\pi}^{-\delta} K_N(t)dt + |\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(x-t) - \frac{f(x+)+f(x-)}{2})K_N(t)dt|$$

$$+2M\frac{1}{2\pi}\int_{\delta}^{\pi} K_N(t)dt \leq \frac{4M(\pi-\delta)}{(N+1)\pi(1-\cos\delta)}$$

$$+|\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(x-t) - \frac{f(x+)+f(x-)}{2})K_N(t)dt|,$$

and since

$$|\frac{1}{2\pi}\int_{-\delta}^{\delta}(f(x-t) - \frac{f(x+)+f(x-)}{2})K_N(t)dt| =$$

$$|\frac{1}{2\pi}\int_{-\delta}^{0}(f(x-t) - \frac{f(x+)+f(x-)}{2})K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{0}^{\delta}(f(x-t) - \frac{f(x+)+f(x-)}{2})K_N(t)dt|$$

$$|\frac{1}{2\pi}\int_{-\delta}^{0}[(f(x-t) - f(x+)) + \frac{f(x+)-f(x-)}{2}]K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{0}^{\delta}[(f(x-t) - f(x-)) + \frac{f(x-)-f(x+)}{2}]K_N(t)dt|$$

$$= |\frac{1}{2\pi}\int_{-\delta}^{0}(f(x-t) - f(x+))K_N(t)dt + \frac{1}{2\pi}\int_{-\delta}^{0}\frac{f(x+)-f(x-)}{2}K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{0}^{\delta}(f(x-t) - f(x-))K_N(t)dt + \frac{1}{2\pi}\int_{0}^{\delta}\frac{f(x-)-f(x+)}{2}K_N(t)dt|$$

$$= |\frac{1}{2\pi}\int_{-\delta}^{0}(f(x-t) - f(x+))K_N(t)dt + \frac{f(x+)-f(x-)}{2}\cdot\frac{1}{2\pi}\int_{0}^{\delta}K_N(t)dt$$

$$+\frac{1}{2\pi}\int_{0}^{\delta}(f(x-t) - f(x-))K_N(t)dt + \frac{f(x-)-f(x+)}{2}\cdot\frac{1}{2\pi}\int_{0}^{\delta}K_N(t)dt|$$

$$= |\frac{1}{2\pi}\int_{-\delta}^{0}(f(x-t) - f(x+))K_N(t)dt + \frac{1}{2\pi}\int_{0}^{\delta}(f(x-t) - f(x-))K_N(t)dt|$$

$$\leq \frac{1}{2\pi}\int_{-\delta}^{0}|f(x-t) - f(x+)|K_N(t)dt + \frac{1}{2\pi}\int_{0}^{\delta}|f(x-t) - f(x-)|K_N(t)dt|$$

$$< \frac{\epsilon}{2}\cdot(\frac{1}{2\pi}\int_{-\delta}^{0}K_N(t)dt + \frac{1}{2\pi}\int_{0}^{\delta}K_N(t)dt) = \frac{\epsilon}{2}\cdot\frac{1}{2\pi}\int_{-\delta}^{\delta}K_N(t)dt$$

$$\leq \frac{\epsilon}{2}\cdot\frac{1}{2\pi}\int_{-\pi}^{\pi}K_N(t)dt = \frac{\epsilon}{2},$$

we hence have

$$|\sigma_N(f;x) - \frac{f(x+)+f(x-)}{2}| < \frac{4M(\pi-\delta)}{(N+1)\pi(1-\cos\delta)} + \frac{\epsilon}{2} < \epsilon,$$

for sufficiently large N, which is the same to say:

$$\lim_{n\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+)+f(x-)].$$

17. Assume f is bounded and monotonic on $[-\pi,\pi)$, with Fourier coefficients c_n, as given by (62).

    (a) Use Exercise 17 of Chap. 6 to prove that $\{nc_n\}$ is a bounded sequence.
    Proof: We have

    $$nc_n = n\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = -\frac{1}{2i\pi}\int_{-\pi}^{\pi} f(x)(e^{-inx})'dx$$
    $$= -\frac{1}{2i\pi}(f(x)e^{-inx}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx}df),$$

    Since f is bounded, $|f| \le M$. Then

    $$|nc_n| = \frac{1}{2\pi}|f(x)e^{-inx}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} e^{-inx}df| \le$$
    $$\frac{1}{2\pi}(|f(\pi)e^{-in\pi}| + |f(-\pi)e^{in\pi}| + \int_{-\pi}^{\pi} |e^{-inx}|df)$$
    $$\le \frac{1}{2\pi}(2M + f(\pi) - f(-\pi)) \le \frac{1}{2\pi}\cdot 4M = \frac{2M}{\pi}.$$

    Therefore, $\{nc_n\}$ is bounded.

    (b) Combine (a) with Exercise 16 and with Exercise 14(e) of Chap. 3, to conclude that

    $$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2}[f(x+)+f(x-)]$$

    for every x.
    Proof: Let $a_0(f;x) = c_0, a_n(f;x) = c_ne^{inx} + c_{-n}e^{-inx}, n \ge 1$, and hence $a_n(f;x) = s_n(f;x) - s_{n-1}(f;x)$, for $n \ge 1$. Since $|na_n(f;x)| = |n(c_ne^{inx} + c_{-n}e^{-inx})| \le n(|c_ne^{inx}| + |c_{-n}e^{-inx}|) \le |nc_n| + |-nc_{-n}|$, $na_n(f;x)$ is bounded. Therefore, according to Exercise 14(e) of Chap. 3, since

    $$\lim_{N\to\infty} \sigma_N(f;x) = \frac{1}{2}[f(x+)+f(x-)],$$

    we have

    $$\lim_{N\to\infty} s_N(f;x) = \frac{1}{2}[f(x+)+f(x-)],$$

for every x.
Note that, since f is monotonic, $f(x+)$ and $f(x-)$ exist at every point of x of $(-\pi, \pi)$, by Theorem 4.29; and since f is monotonic, $f \in \mathscr{R}$, by Theorem 6.9. Therefore, the hypothesis of f in Exercise 16 holds.

(c) Assume only that $f \in \mathscr{R}$ on $[-\pi, \pi]$ and that f is monotonic in some segment $(\alpha, \beta) \subseteq [-\pi, \pi]$. Prove that the conclusion of (b) holds for every $x \in (\alpha, \beta)$. (This is an application of the localization theorem.)
Proof: Define a function g such that:
(i)g is bounded and monotonic on $[-\pi, \pi)$;
(ii)$g = f$ on (α, β).
By (b), we have

$$\lim_{N\to\infty} s_N(g; x) = \frac{1}{2}[g(x+) + g(x-)] = \frac{1}{2}[f(x+) + f(x-)],$$

for every $x \in (\alpha, \beta)$. By the localization theorem, we have

$$\lim_{N\to\infty} s_N(f; x) = \lim_{N\to\infty} s_N(g; x) = \frac{1}{2}[f(x+) + f(x-)],$$

for every $x \in (\alpha, \beta)$.

18. Define

$$f(x) = x^3 - \sin^2 x \tan x \quad g(x) = 2x^2 - \sin^2 x - x \tan x.$$

Find out, for each of these two functions, whether it is positive or negative for all $x \in (0, \pi/2)$, or whether it changes sign. Prove your answer.
Solution: I've no idea at the current time.

19. Suppose f is a continuous function on R^1, $f(x + 2\pi) = f(x)$, and α/π is irrational. Prove that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt$$

for every x.
Proof: We first prove this for $f(x) = e^{ikx}$.
(i)$k = 0$, then $f(x) = 1$, and we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(x + n\alpha) = 1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt.$$

(ii)$k \neq 0$, then we have

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ikt}dt = 0,$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha)=\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{ik(x+n\alpha)}$$

$$=\lim_{N\to\infty}\frac{1}{N}e^{ikx}\sum_{n=1}^{N} e^{ikn\alpha}=\lim_{N\to\infty}\frac{e^{ikx}}{N}\cdot\frac{e^{ik(N+1)\alpha}-e^{ik\alpha}}{e^{ik\alpha}-1}=0.$$

Note that this is due to the fact that α/π is irrational. To see this, let $\alpha=\beta\pi$, where β is some irrational number. Then $k\alpha=k\beta\pi\neq 2m\pi$, for every integer m. Hence $e^{ik\alpha}\neq 1$. On the other hand we have $k(N+1)\alpha=kN\alpha+k\alpha=kN\beta\pi+k\alpha\neq k\alpha+2m\pi$, for every integer m if $N\neq 0$, and therefore, $e^{ik(N+1)\alpha}\neq e^{ik\alpha}$ if $N\neq 0$. Since $|e^{i\theta}|=1$, for every real θ, we have the previous result and thus,

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x+n\alpha)=0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)dt.$$

Next, we will prove this for every f which satisfies the given hypothesis. We have

$$\sigma_N(f;x+m\alpha)=\frac{1}{N+1}\sum_{n=0}^{N} s_n=\frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} c_k e^{ik(x+m\alpha)},$$

then

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sigma_N(f;x+m\alpha)=\frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} c_k\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M} e^{ik(x+m\alpha)}$$

$$=\frac{1}{N+1}\sum_{n=0}^{N}\sum_{k=-n}^{n} c_k\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ikt}dt=\frac{1}{N+1}\sum_{n=0}^{N} c_0=c_0.$$

Since f is continuous, and $f(x+2\pi)=f(x)$, by Fejer's theorem, $\sigma_N(f;x)\to f(x)$ uniformly on $[-\pi,\pi]$, and thus uniformly on R^1. Thus given any $\epsilon>0$, there exists an integer $N_0>0$ such that $N>N_0$ implies

$$|\sigma_N(f;x+m\alpha)-f(x+m\alpha)|<\epsilon,$$

for any x and m. Hence,

$$|\frac{1}{M}\sum_{m=1}^{M}\sigma_N(f;x+m\alpha)-\frac{1}{M}\sum_{m=1}^{M} f(x+m\alpha)|$$

$$=|\frac{1}{M}\sum_{m=1}^{M}(\sigma_N(f;x+m\alpha)-f(x+m\alpha))|$$

$$\leq \frac{1}{M}\sum_{m=1}^{M}|\sigma_N(f;x+m\alpha)-f(x+m\alpha)| < \frac{1}{M}\sum_{m=1}^{M}\epsilon = \epsilon,$$

for $N > N_0$. Therefore, we have

$$|\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sigma_N(f;x+m\alpha) - \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}f(x+m\alpha)| < \epsilon,$$

for $N > N_0$, and thus

$$\lim_{N\to\infty}\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}\sigma_N(f;x+m\alpha) = \lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}f(x+m\alpha),$$

for every x, which gives

$$\lim_{M\to\infty}\frac{1}{M}\sum_{m=1}^{M}f(x+m\alpha) = \lim_{N\to\infty}c_0 = c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)dt.$$

20. The following simple computation yields a good approximation to Stirling's formula.
For $m = 1, 2, 3, ...$, define

$$f(x) = (m+1-x)\log m + (x-m)\log(m+1)$$

if $m \leq x \leq m+1$, and define

$$g(x) = \frac{x}{m} - 1 + \log m$$

if $m - \frac{1}{2} \leq x < m + \frac{1}{2}$. Draw the graphs of f and g. Note that $f(x) \leq \log x \leq g(x)$ if $x \geq 1$ and that

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

Integrate $\log x$ over $[1, n]$. Conclude that

$$\frac{7}{8} < \log(n!) - (n+\frac{1}{2})\log n + n < 1$$

for $n = 2, 3, 4, ...$ (*Note*: $\log\sqrt{2\pi} \sim 0.918$.) Thus

$$e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

Proof:
(i)First we prove $f(x) \leq \log x \leq g(x)$ if $x \geq 1$.

Since $(-\log x)'' = (-1/x)' = 1/x^2 > 0$ for $x > 0$, $-\log x$ is convex on $(0, +\infty)$. Hence,

$$-\log(\lambda m + (1-\lambda)(m+1)) \leq \lambda(-\log m) + (1-\lambda)(-\log(m+1))$$

for every $0 < \lambda < 1$. For $m \leq x \leq m+1$, put $\lambda = m+1-x$, we have $0 \leq \lambda \leq 1$, and therefore,

$$\log((m+1-x)m + (x-m)(m+1)) \geq (m+1-x)\log m + (x-m)\log(m+1),$$

namely,

$$\log x \geq (m+1-x)\log m + (x-m)\log(m+1) \quad i.e., \quad \log x \geq f(x).$$

On the other hand, put

$$h(x) = g(x) - \log x = \frac{x}{m} - 1 + \log m - \log x, \quad m - \frac{1}{2} \leq x < m + \frac{1}{2},$$

we have $h'(x) = 1/m - 1/x$. Let $h'(x) = 0$, we get $x = m$. Since $h''(x) = 1/x^2 > 0$ for any $m - \frac{1}{2} \leq x < m + \frac{1}{2}$, $h(x) \geq h(m) = 0$, which gives $g(x) \geq \log x$, for any $m - \frac{1}{2} \leq x < m + \frac{1}{2}$.
Note that the above statements hold for every $m = 1, 2, 3, ...$, and therefore, we have $f(x) \leq \log x \leq g(x)$ for all $x \geq 1$.
(ii)Next we will prove that

$$\int_1^n f(x)dx = \log(n!) - \frac{1}{2}\log n > -\frac{1}{8} + \int_1^n g(x)dx.$$

We have

$$\int_1^n f(x)dx = \sum_{m=1}^{n-1} \int_m^{m+1} f(x)dx$$

$$= \sum_{m=1}^{n-1} \int_m^{m+1} [(m+1-x)\log m + (x-m)\log(m+1)]dx$$

$$= \sum_{m=1}^{n-1} [\log(1 + \frac{1}{m}) \int_m^{m+1} xdx + (m+1)\log m - m\log(m+1)]$$

$$= \sum_{m=1}^{n-1} [\frac{2m+1}{2}\log(1 + \frac{1}{m}) + (m+1)\log m - m\log(m+1)]$$

$$= \sum_{m=1}^{n-1} [\frac{1}{2}\log(\frac{m+1}{m}) + \log m] = \frac{1}{2}\log n + \log((n-1)!) = \log(n!) - \frac{1}{2}\log n,$$

and

$$\int_{1/2}^{n+1/2} g(x)dx = \sum_{m=1}^{n} \int_{m-1/2}^{m+1/2} g(x)dx = \sum_{m=1}^{n} \int_{m-1/2}^{m+1/2} (\frac{x}{m} - 1 + \log m)dx$$

$$\sum_{m=1}^{n}[\log m-1+\frac{1}{m}\cdot\int_{m-1/2}^{m+1/2} xdx = \sum_{m=1}^{n}[\log m-1+\frac{1}{m}\cdot m] = \sum_{m=1}^{n}\log m = \log(n!).$$

Since

$$\int_{1/2}^{1} g(x)dx = \int_{1/2}^{1}(x-1)dx = -\frac{1}{8},$$

and

$$\int_{n}^{n+1/2} g(x)dx = \int_{n}^{n+1/2}(\frac{x}{n}-1+\log n)dx$$

$$= \frac{1}{2n}\cdot(n+\frac{1}{4})+\frac{1}{2}(\log n-1) = \frac{1}{8n}+\frac{1}{2}\log n,$$

we have

$$\int_{1}^{n} g(x)dx = \int_{1/2}^{n+1/2} g(x)dx - \int_{1/2}^{1} g(x)dx - \int_{n}^{n+1/2} g(x)dx$$

$$= \log(n!)+\frac{1}{8}-\frac{1}{8n}-\frac{1}{2}\log n = \int_{1}^{n} f(x)dx+\frac{1}{8}-\frac{1}{8n} < \int_{1}^{n} f(x)dx+\frac{1}{8}.$$

Therefore,

$$\int_{1}^{n} f(x)dx = \log(n!)-\frac{1}{2}\log n > -\frac{1}{8}+\int_{1}^{n} g(x)dx.$$

(iii) Since $f(x) \le \log x \le g(x)$ if $x \ge 1$, integrate over $[1,n]$ gives us

$$\int_{1}^{n} f(x)dx < \int_{1}^{n}\log xdx < \int_{1}^{n} g(x)dx,$$

namely,

$$\int_{1}^{n} f(x)dx < n\log n-(n-1) < \int_{1}^{n} g(x)dx, \quad i.e.,$$

$$\log(n!)-\frac{1}{2}\log n < n\log n-(n-1) < \log(n!)-\frac{1}{2}\log n+\frac{1}{8},$$

which gives

$$\frac{7}{8} < \log(n!)-(n+\frac{1}{2})\log n+n < 1.$$

Thus,

$$\frac{7}{8} < \log(\frac{n!e^n}{n^n\sqrt{n}}) < 1, \quad i.e., \quad e^{7/8} < \frac{n!}{(n/e)^n\sqrt{n}} < e.$$

Note that since

$$\frac{7}{8}-\log\sqrt{2\pi} < \log(n!)-(n+\frac{1}{2})\log n+n-\log\sqrt{2\pi} < 1-\log\sqrt{2\pi},$$

then

$$-0.043 < \log(\frac{n!e^n}{n^n\sqrt{2\pi n}}) < 0.082,$$

which gives

$$e^{-0.043} < \frac{n!e^n}{n^n\sqrt{2\pi n}} < e^{0.082}, \quad i.e., \quad 0.958 < \frac{n!}{(n/e)^n\sqrt{2\pi n}} < 1.085.$$

21. Let

$$L_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_n(t)|dt \quad (n = 1, 2, 3, ...).$$

Prove that there exists a constant $C > 0$ such that

$$L_n > C \log n \quad (n = 1, 2, 3, ...),$$

or, more precisely, that the sequence

$$\{L_n - \frac{4}{\pi^2}\log n\}$$

is bounded.
Proof: No idea at the current time...

22. If α is real and $-1 < x < 1$, prove Newton's binomial theorem

$$(1+x)^\alpha = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n.$$

Show also that

$$(1-x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)}x^n$$

if $-1 < x < 1$ and $\alpha > 0$.
Proof:
(i) First, we prove that

$$\lim_{n\to\infty} \left[\binom{n+m}{n}\right]^{1/n} = 1, \qquad m \in I^+.$$

Since

$$\binom{n+m}{n} = \frac{(n+m)!}{n!m!},$$

By Stirling's formula, we have

$$\lim_{n\to\infty} \left[\binom{n+m}{n}\right]^{1/n} = \lim_{n\to\infty} \left[\frac{(n+m)!}{n!m!}\right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{[(m+n)/e]^{m+n}\sqrt{2\pi(m+n)}}{m!(n/e)^n\sqrt{2\pi n}}\right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{(m+n)^m}{m!e^m} \cdot \left(1+\frac{m}{n}\right)^n \cdot \sqrt{\left(1+\frac{m}{n}\right)} \right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{(m+n)^m}{m!e^m} \cdot \left[\left(1+\frac{m}{n}\right)^{\frac{n}{m}}\right]^m \cdot \sqrt{\left(1+\frac{m}{n}\right)} \right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{(m+n)^m}{m!e^m} \cdot e^m \cdot 1 \right]^{1/n} = \lim_{n\to\infty} \left[\frac{(m+n)^m}{m!} \right]^{1/n}$$

$$= \lim_{n\to\infty} (m+n)^{m/n} = \lim_{n\to\infty} \left[n\left(1+\frac{m}{n}\right) \right]^{m/n} = 1.$$

(ii) Next, denote the right side by $f(x)$, we will prove that the series converges. Since $|x| < 1$, it's sufficient to show that

$$\limsup_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right|^{1/n} \leq 1.$$

(ii.a) If α is 0 or any positive integer, suppose $\alpha = N$, then $n > N$ implies $\alpha(\alpha-1)\cdots(\alpha-n+1) = 0$, and therefore,

$$\limsup_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right|^{1/n} = 0.$$

(ii.b) Suppose $\alpha > 0$ and α is not an integer, then there exists some integer $m \geq 0$ such that $m < \alpha < m+1$. Then we have $\alpha-(m+1)+1 = \alpha-m > 0$ and $\alpha - (m+2)+1 = \alpha-(m+1) < 0$. Therefore, when $n > m+1$, we can rewrite

$$\frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}$$

as

$$\alpha(\alpha-1)\cdots(\alpha-(m+1)+1) \cdot \frac{(\alpha-(m+2)+1)\cdots(\alpha-n+1)}{n!}.$$

Let

$$M = \left| \alpha(\alpha-1)\cdots(\alpha-(m+1)+1) \right|,$$

then

$$\left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right| = M \cdot \left| \frac{(\alpha-(m+2)+1)\cdots(\alpha-n+1)}{n!} \right|$$

$$= M \cdot \frac{((m+1)-\alpha)\cdots((n-1)-\alpha)}{n!}.$$

Since $m < \alpha < m+1$, we have

$$\frac{((m+1)-\alpha)\cdots((n-1)-\alpha)}{n!} \leq \frac{((m+1)-m)\cdots((n-1)-m)}{n!}$$

$$= \frac{(n-1-m)!}{n!} = \frac{(n-(m+1))!(m+1)!}{n!(m+1)!} = \frac{1}{\binom{n}{n-(m+1)}(m+1)!}.$$

Hence,

$$\limsup_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right|^{1/n}$$

$$= \limsup_{n\to\infty} \left[M \cdot \frac{((m+1)-\alpha)\cdots((n-1)-\alpha)}{n!} \right]^{1/n}$$

$$\leq \limsup_{n\to\infty} \left[M \cdot \frac{1}{\binom{n}{n-(m+1)}(m+1)!} \right]^{1/n} = 1,$$

due to the fact that

$$\limsup_{n\to\infty} \left[\binom{n}{n-(m+1)} \right]^{1/n} = \lim_{n\to\infty} \left[\binom{n}{n-(m+1)} \right]^{1/n}$$

$$\xrightarrow{n'=n-(m+1)}= \lim_{n'\to\infty} \left[\binom{n'+(m+1)}{n'} \right]^{(1/n')(n'/(n'+(m+1)))} = 1.$$

(ii.c) If $\alpha < 0$, we have $-\alpha > 0$. Suppose $m \leq -\alpha < m+1$, then

$$\left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right| = \frac{(-\alpha)((-\alpha)+1)\cdots((-\alpha)+(n-1))}{n!}$$

$$< \frac{(m+1)((m+1)+1)\cdots((m+1)+(n-1))}{n!} = \frac{(m+n)!}{n!m!} = \binom{n+m}{n}.$$

Therefore,

$$\limsup_{n\to\infty} \left| \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} \right|^{1/n} < \limsup_{n\to\infty} \left[\binom{n+m}{n} \right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\binom{n+m}{n} \right]^{1/n} = 1.$$

Combine (ii.a), (ii.b), (ii.c), we get the desired result that the series converges for any real α.
(iii) Next, we prove that

$$(1+x)f'(x) = \alpha f(x).$$

Since

$$f'(x) = \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n-1},$$

we have

$$(1+x)f'(x) = (1+x) \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n-1}$$

$$= \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n-1} + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n}$$

$$= \alpha + \sum_{n=2}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n-1}$$

$$+ \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n}$$

$$= \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-(n+1)+1)}{(n+1)!} (n+1) x^{n}$$

$$+ \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} n x^{n}$$

$$= \alpha + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} (\alpha-n+n) x^{n}$$

$$= \alpha(1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} x^{n}) = \alpha f(x).$$

(iv) Solve

$$(1+x)f'(x) = \alpha f(x)$$

gives us

$$(1+x)\frac{df(x)}{dx} = \alpha f(x), \quad i.e., \quad \frac{df(x)}{f(x)} = \alpha \frac{dx}{(1+x)},$$

and therefore,

$$\log f(x) = \alpha \log(1+x) + C, \quad i.e., \quad f(x) = C'(1+x)^{\alpha}.$$

Since $f(0) = 1$, we get $C' = 1$ and hence $f(x) = (1+x)^{\alpha}$.
(v) Substitute $-x$ as x, $-\alpha$ as α, we have

$$(1-x)^{-\alpha} = 1 + \sum_{n=1}^{\infty} \frac{(-\alpha)(-\alpha-1)\cdots(-\alpha-n+1)}{n!} (-x)^{n}$$

$$= 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} x^{n} = \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{n!\Gamma(\alpha)} x^{n},$$

since

$$\Gamma(n+\alpha) = (\alpha+n-1)\Gamma(\alpha+n-1) = \cdots = (\alpha+n-1)\cdots(\alpha+1)\alpha\Gamma(\alpha).$$

23. Let γ be a continuously differentiable *closed* curve in the complex plane, with parameter interval $[a,b]$, and assume that $\gamma(t) \neq 0$ for every $t \in [a,b]$. Define the *index* of γ to be

$$Ind(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt.$$

Prove that $Ind(\gamma)$ is always an integer.
Compute $Ind(\gamma)$ when $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$.
Explain why $Ind(\gamma)$ is often called the *winding number* of γ around 0.
Proof:
(i) Since γ is continuously differentiable, $\frac{\gamma'}{\gamma}$ is continuous on $[a,b]$. Define

$$\varphi(x) = \int_a^x \frac{\gamma'(t)}{\gamma(t)} dt, \quad x \in [a,b],$$

then $\varphi(a) = 0$ and by Theorem 6.20, $\varphi' = \frac{\gamma'}{\gamma}$.
Solve this equation gives us

$$\frac{d\varphi}{dx} = \frac{d\gamma}{\gamma dx}, \quad i.e., \quad d\varphi = \frac{d\gamma}{\gamma},$$

which gives

$$\varphi = \log\gamma + C, \quad i.e., \quad \gamma = C' \exp(\varphi),$$

where $C' \neq 0$ is some constant. Since $\gamma(a) = \gamma(b)$ (γ is closed), we must have $\exp\varphi(b) = \exp\varphi(a) = 1$(because $\varphi(a) = 0$). Note that

$$Ind(\gamma) = \frac{1}{2\pi i} \int_a^b \frac{\gamma'(t)}{\gamma(t)} dt = \frac{1}{2\pi i}(\varphi(b) - \varphi(a)) = \frac{\varphi(b)}{2\pi i},$$

and therefore, $\varphi(b) = 2\pi i Ind(\gamma)$. Combining with the fact $\exp\varphi(b) = 1$ gives that $Ind(\gamma)$ is an integer.
(ii) When $\gamma(t) = e^{int}$, $a = 0$, $b = 2\pi$, we have

$$Ind(\gamma) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{(e^{int})'}{e^{int}} dt = \frac{1}{2\pi i} in \int_0^{2\pi} dt = n.$$

(iii) Here, I will explain why $Ind(\gamma)$ is often called the *winding number* of γ around 0.
By (ii), if $Ind(\gamma) = n$, then we have $Ind(\gamma) = Ind(e^{int})$. By Theorem 6.17, e^{int}, $t \in [0, 2\pi]$ is rectifiable, and its length

$$\Lambda(e^{int}) = \int_0^{2\pi} |(e^{int})'| dt = \int_0^{2\pi} |ine^{int}| dt = 2\pi n.$$

Since the length of the unit circle on the complex plane is 2π, we know that the length of e^{int}, $t \in [0, 2\pi]$ is n times the length of the unit circle. What's more, since e^{int}, $t \in [0, 2\pi]$ has the same range as the unit circle, it seems as the curve e^{int}, $t \in [0, 2\pi]$ winds around 0 along the edge of the unit circle n times.

24. Let γ be as in Exercise 23, and assume in addition that the range of γ does not intersect the negative real axis. Prove that $Ind(\gamma) = 0$.
Proof: We first prove that for $0 \leq c < \infty$, $Ind(\gamma + c)$ is a continuous function of c.
Suppose $0 \leq x, y < \infty$, we have

$$|Ind(\gamma + x) - Ind(\gamma + y)| = |\frac{1}{2\pi i}\int_a^b [\frac{\gamma'(t)}{\gamma(t) + x} - \frac{\gamma'(t)}{\gamma(t) + y}]dt|$$

$$= \frac{1}{2\pi}|\int_a^b \frac{\gamma'(t)(y - x)}{(\gamma(t) + x)(\gamma(t) + y)}dt| \leq \frac{1}{2\pi}\int_a^b \frac{|\gamma'(t)||y - x|}{|\gamma(t) + x||\gamma(t) + y|}dt.$$

Since γ is continuously differentiable, $|\gamma'| \leq M$ for some $M > 0$, $|\gamma| \geq m$ for some $m > 0$, since $\gamma \neq 0$ and γ is continuous. (Or $m = \inf|\gamma(t)|$, for $t \in [a, b]$, and since γ is continuous (so $|\gamma|$ is continuous), we must have some $t_0 \in [a, b]$, for which $|\gamma(t_0)| = m$. Since $\gamma(t) \neq 0$ for every $t \in [a, b]$, $m > 0$.) On the other hand, since the range of γ does not intersect the negative real axis, we have $|\gamma + x| \geq |\gamma| \geq m$ and $|\gamma + y| \geq |\gamma| \geq m$. Hence,

$$\frac{1}{2\pi}\int_a^b \frac{|\gamma'(t)||y - x|}{|\gamma(t) + x||\gamma(t) + y|}dt \leq \frac{(b - a)M}{2\pi m^2}|y - x|.$$

Then given any $\epsilon > 0$, we can pick a δ such that $0 < \delta < \frac{2\pi m^2}{(b-a)M}\epsilon$. When $|y - x| < \delta$, we have

$$|Ind(\gamma + x) - Ind(\gamma + y)| \leq \frac{(b - a)M}{2\pi m^2}|y - x| < \frac{(b - a)M}{2\pi m^2}\delta < \epsilon.$$

This shows that for $0 \leq c < \infty$, $Ind(\gamma + c)$ is a continuous function of c. On the other hand, Exercise 23 tells us that $Ind(\gamma)$ is always an integer, so $Ind(\gamma + c)$ must be an integer, for every $0 \leq c < \infty$, since $\gamma + c$ satisfies the hypothesis of Exercise 23 if γ satisfies them. Therefore, $Ind(\gamma + c)$ must be a constant, for $0 \leq c < \infty$. To prove this, we only need to show that $Ind(\gamma + c)$ is constant on every interval $[0, A]$. Since $Ind(\gamma + c)$ is continuous, it is uniformly continuous on $[0, A]$. Then there exists a $\delta > 0$ such that $|y - x| < \delta$ implies $|Ind(\gamma + y) - Ind(\gamma + x)| < 1$. Since $Ind(\gamma + c)$ is integer-valued, this implies $Ind(\gamma + y) = Ind(\gamma + x)$ if $|y - x| < \delta$. Pick an r such that $0 < r < \delta$, then there is an integer N such that $Nr \leq A < (N + 1)r$. In each interval $[ir, (i + 1)r]$, where $0 \leq i \leq N$, $Ind(\gamma + c)$ is an constant, say $Ind(\gamma + ir)$. Hence we have $Ind(\gamma + ir) = Ind(\gamma + (i + 1)r)$, for every $0 \leq i \leq N$, which is equivalent to say that $Ind(\gamma + c)$ is constant on $[0, A]$, for every positive A and therefore, $Ind(\gamma + c)$ is constant on $[0, \infty)$. Since $Ind(\gamma + c) \to 0$ as $c \to \infty$, we have

$$Ind(\gamma) = Ind(\gamma + 0) = \lim_{c \to \infty} Ind(\gamma + c) = 0.$$

25. Suppose γ_1 and γ_2 are curves as in Exercise 23, and

$$|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)| \qquad (a \leq t \leq b).$$

Prove that $Ind(\gamma_1) = Ind(\gamma_2)$.
Proof: Put $\gamma = \gamma_2/\gamma_1$. Then $|1-\gamma| < 1$, hence $Ind(\gamma) = 0$, by Exercise 24. Also, since

$$\frac{\gamma'}{\gamma} = \frac{(\gamma_2/\gamma_1)'}{\gamma_2/\gamma_1} = \frac{(\gamma_2'\gamma_1 - \gamma_1'\gamma_2)/\gamma_1^2}{\gamma_2/\gamma_1} = \frac{\gamma_2'\gamma_1 - \gamma_1'\gamma_2}{\gamma_2\gamma_1} = \frac{\gamma_2'}{\gamma_2} - \frac{\gamma_1'}{\gamma_1},$$

we have

$$Ind(\gamma) = \frac{1}{2\pi i}\int_a^b \frac{\gamma'(t)}{\gamma(t)}dt = \frac{1}{2\pi i}\int_a^b [\frac{\gamma_2'(t)}{\gamma_2(t)} - \frac{\gamma_1'(t)}{\gamma_1(t)}]dt = Ind(\gamma_2) - Ind(\gamma_1),$$

which gives $Ind(\gamma_1) = Ind(\gamma_2)$.

26. Let γ be a *closed* curve in the complex plane (not necessarily differentiable) with parameter interval $[0, 2\pi]$, such that $\gamma(t) \neq 0$ for every $t \in [0, 2\pi]$. Choose $\delta > 0$ so that $|\gamma(t)| > \delta$ for all $t \in [0, 2\pi]$. If P_1 and P_2 are trigonometric polynomials such that $|P_j(t) - \gamma(t)| < \delta/4$ for all $t \in [0, 2\pi]$ (their existence is assured by Theorem 8.15), prove that

$$Ind(P_1) = Ind(P_2)$$

by applying Exercise 25.
Define this common value to be $Ind(\gamma)$.
Prove that the statements of Exercises 24 and 25 hold without any differentiability assumption.
Proof: Clearly, $P_j(t)$, $t \in [0, 2\pi]$ satisfy the hypothesis of Exercise 23, namely, they are continuously differentiable and closed nonzero curves. Since $|P_1(t) - \gamma(t)| < \delta/4$, we have $|P_1(t)| > |\gamma(t)| - \delta/4 > 3\delta/4$. Thus, $|P_1(t) - P_2(t)| = |(P_1(t) - \gamma(t)) - (P_2(t) - \gamma(t))| \leq |P_1(t) - \gamma(t)| + |P_2(t) - \gamma(t)| < \delta/2 < 3\delta/4 < |P_1(t)|$, and by Exercise 25, we have $Ind(P_1) = Ind(P_2)$.
If we define this common value to be $Ind(\gamma)$, the statements of Exercise 24 and 25 hold without any differentiability assumption.
First, consider the statement of Exercise 24. Since $|\gamma(t)| > \delta$ and $|P(t) - \gamma(t)| < \delta/4$, we have $|P(t)| > |\gamma(t)| - \delta/4 > 3\delta/4$. Thus if the range of γ does not intersect the negative real axis, so does P. Then we have $Ind(P) = 0$, since P satisfy the hypothesis of Exercise 23 and 24. Hence $Ind(\gamma) = 0$.
Next, consider the statement of Exercise 25. Let $g(t) = |\gamma_1(t)| - |\gamma_1(t) - \gamma_2(t)|$, for $t \in [a, b]$. Since $|\gamma_1(t) - \gamma_2(t)| < |\gamma_1(t)|$, for $t \in [a, b]$, we have $g(t) > 0$, for $t \in [a, b]$. Since γ_1 and γ_2 are continuous on $[a, b]$, g is continuous on $[a, b]$. Let $\mu = \inf g(t)$, $t \in [a, b]$, then there is some $t_0 \in [a, b]$ such that $g(t_0) = \mu$ and therefore, $\mu > 0$. We hence have $g(t) \geq \mu$, which gives $|\gamma_1(t)| \geq |\gamma_1(t) - \gamma_2(t)| + \mu$. Pick δ such that $0 < 3\delta/4 < \mu$, and $|\gamma_1| > \delta$, $|\gamma_2| > \delta$. Choose trigonometric polynomials P_1, P_2 such that $|P_i(t) - \gamma_i(t)| < \delta/4$ for all $t \in [0, 2\pi]$, $i = 1, 2$, thus $Ind(\gamma_i) = Ind(P_i)$, $i = 1, 2$. Then we have

$$|\gamma_1| - \delta/4 \geq |\gamma_1 - \gamma_2| + \mu - \delta/4 > |\gamma_1 - \gamma_2| + 3\delta/4 - \delta/4 = |\gamma_1 - \gamma_2| + \delta/2,$$

which gives

$$|P_1-P_2| = |(P_1-\gamma_1)+(\gamma_1-\gamma_2)+(\gamma_2-P_2)| \le |P_1-\gamma_1|+|\gamma_1-\gamma_2|+|\gamma_2-P_2|$$

$$< |\gamma_1 - \gamma_2| + \delta/2 < |\gamma_1| - \delta/4 < |P_1|.$$

By Exercise 25, we have $Ind(P_1) = Ind(P_2)$ and therefore $Ind(\gamma_1) = Ind(P_1) = Ind(P_2) = Ind(\gamma_2)$.

27. Let f be a continuous complex function defined in the complex plane. Suppose there is a positive integer n and a complex number $c \neq 0$ such that

$$\lim_{|z|\to\infty} z^{-n} f(z) = c.$$

Prove that $f(z) = 0$ for at least one complex number z.
Note that this is a generalization of Theorem 8.8.
Proof: Assume $f(z) \neq 0$ for all z, define $\gamma_r(t) = f(re^{it})$ for $0 \le r < \infty$, $0 \le t \le 2\pi$. We will prove the following statements about the curves γ_r.

(a) $Ind(\gamma_0) = 0$.
Since $\gamma_0(t) = f(0)$ for all $t \in [0, 2\pi]$, and $f(0) \neq 0$. We have

$$Ind(\gamma_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\gamma_0'(t)}{\gamma_0(t)} dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{0}{f(0)} dt = 0,$$

by Exercise 23.

(b) $Ind(\gamma_r) = n$ for all sufficiently large r.
Since

$$\lim_{|z|\to\infty} z^{-n} f(z) = c,$$

we have

$$\lim_{r\to\infty} r^{-n}e^{-int} f(re^{it}) = c, \quad i.e., \quad \lim_{r\to\infty} r^{-n}e^{-int}\gamma_r(t) = c.$$

Then when r is sufficiently large, $|r^{-n}e^{-int}\gamma_r(t) - c| < |c|$, for all $t \in [0, 2\pi]$. Let $g(t) = cr^n e^{int}$, $t \in [0, 2\pi]$, we hence have

$$|g(t) - \gamma_r(t)| = |\gamma_r(t) - g(t)| = |r^n e^{int}(r^{-n}e^{-int}\gamma_r(t) - c)|$$

$$= |r^n e^{int}| \cdot |r^{-n}e^{-int}\gamma_r(t) - c| < |c| \cdot |r^n e^{int}| = |cr^n e^{int}| = |g(t)|.$$

On the other hand, since $g(t)$ is continuously differentiable, we have

$$Ind(g(t)) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{g'(t)}{g(t)} dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(cr^n e^{int})'}{cr^n e^{int}} dt = n.$$

Therefore, by Exercise 25, we have $Ind(\gamma) = Ind(g) = n$, for sufficiently large r.

(c) $Ind(\gamma_r)$ is a continuous function of r, on $[0, \infty)$.
Fix any $x \in [0, \infty)$, for every $0 \le y < \infty$, we have $|\gamma_y(t) - \gamma_x(t)| = |f(ye^{it}) - f(xe^{it})|$. Since f is continuous, given any $\epsilon > 0$, we can pick a $\delta > 0$ such that $|ye^{it} - xe^{it}| = |y - x| < \delta$ implies $|f(ye^{it}) - f(xe^{it})| < \epsilon$ and thus $|\gamma_y(t) - \gamma_x(t)| < \epsilon$. Note that here different x may have different δ, since $[0, \infty)$ is closed, but not compact.
Let $\mu = \inf |f(z)|$. Since $||z^{-n}f(z)| - |c|| \le |z^{-n}f(z) - c|$ and $\lim\limits_{|z|\to\infty} z^{-n}f(z) = c$, we have $\lim\limits_{|z|\to\infty} |z^{-n}f(z)| = |c|$, which gives $|f(z)| \to \infty$ when $|z| \to \infty$, for otherwise, if $|f(z)| \le M$ for some $M > 0$, we must have $\lim\limits_{|z|\to\infty} |z^{-n}f(z)| = \lim\limits_{|z|\to\infty} |z|^{-n}|f(z)| = 0$, but $|c| \ne 0$ since $c \ne 0$. Now we have that there exists some $R > 0$ such that $|z| > R$ implies $|f(z)| > \mu$. Hence $\mu = \inf_{z\in\mathbb{C}} |f(z)| = \inf_{|z|\le R} |f(z)|$, and since the set $\{z|z \in \mathbb{C} \wedge |z| \le R\}$ is closed and bounded, thus compact, there must exist some z_0, $|z_0| \le R$, $|f(z_0)| = \mu$. Therefore, $\mu > 0$.
Now, fix any $x \in [0, \infty)$, we can find a $\delta > 0$ such that $|y - x| < \delta$, $y \in [0, \infty)$ implies $|\gamma_x(t) - \gamma_y(t)| = |\gamma_y(t) - \gamma_x(t)| < \mu \le |\gamma_x(t)|$, for any $t \in [0, 2\pi]$. By Exercise 25, we have $Ind(\gamma_y) = Ind(\gamma_x)$, if $|y - x| < \delta$. Hence $Ind(\gamma_r)$ is a continuous function of r, on $[0, \infty)$.

Note that (a), (b) and (c) are contradictory, since $n > 0$ but (c) and the fact that $Ind(\gamma_r)$ is integer-valued imply that $Ind(\gamma_r)$ must be constant on $[0, \infty)$.

28. Let $\bar{D}$ be the closed unit disc in the complex plane. (Thus $z \in \bar{D}$ if and only if $|z| \le 1$.) Let g be a continuous mapping of $\bar{D}$ into the unit circle T. (Thus, $|g(z)| = 1$ for every $z \in \bar{D}$.)
Prove that $g(z) = -z$ for at least one $z \in T$.
Proof: For $0 \le r \le 1$, $0 \le t \le 2\pi$, put $\gamma_r(t) = g(re^{it})$, and put $\psi(t) = e^{-it}\gamma_1(t)$. If $g(z) \ne -z$ for every $z \in T$, then $\psi(t) \ne e^{-it}(-e^{it}) = -1$ for every $t \in [0, 2\pi]$. Since $|\psi(t)| = |e^{-it}\gamma_1(t)| = |\gamma_1(t)| = |g(e^{it})| = 1$, $\psi(t) \ne -1$, hence $Ind(\psi) = 0$, by Exercise 24 and 26. It follows that $Ind(\gamma_1) = 1$. To see this, suppose $\delta > 0$ so that $|\gamma_1(t)| > \delta$ for all $t \in [0, 2\pi]$. Choose trigonometric polynomials P so that $|P(t) - \gamma_1(t)| < \delta/4$ for all $t \in [0, 2\pi]$. Then we have $|\psi(t)| = |e^{-it}\gamma_1(t)| = |\gamma_1(t)| > \delta$, and $|e^{-it}P(t) - \psi(t)| = |e^{-it}(P(t) - \gamma_1(t))| = |P(t) - \gamma_1(t)| < \delta/4$. Put $P^*(t) = e^{-it}P(t)$ which is also a trigonometric polynomial. Hence by Exercise 26, we must have $Ind(P^*(t)) = Ind(\psi(t)) = 0$. Therefore,

$$Ind(\gamma_1(t)) = Ind(P(t)) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{P'(t)}{P(t)}dt = \frac{1}{2\pi i}\int_0^{2\pi} \frac{(e^{it}P^*(t))'}{e^{it}P^*(t)}dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} [\frac{e^{it}P^{*\prime}(t) + ie^{it}P^*(t)}{e^{it}P^*(t)}]dt = \frac{1}{2\pi i}\int_0^{2\pi} [\frac{P^{*\prime}(t) + iP^*(t)}{P^*(t)}]dt$$

$$= \frac{1}{2\pi i}\int_0^{2\pi}[\frac{P^{*'}(t)}{P^*(t)} + i]dt = 1 + \frac{1}{2\pi i}\int_0^{2\pi} \frac{P^{*'}(t)}{P^*(t)}dt = 1 + Ind(P^*(t)) = 1.$$

But since $\gamma_0(t) = g(0) \neq 0$(note that $|g(z)| = 1$), $Ind(\gamma_0) = 0$, by Exercise 23.
Fix any $x \in [0,1]$, for every $y \in [0,1]$, we have $|\gamma_y(t) - \gamma_x(t)| = |g(ye^{it}) - g(xe^{it})|$. Since g is continuous, given any $\epsilon > 0$, there exists a $\delta > 0$ such that $|ye^{it} - xe^{it}| = |y - x| < \delta$ implies $|g(ye^{it}) - g(xe^{it})| < \epsilon$, namely, $|\gamma_y(t) - \gamma_x(t)| < \epsilon$ if $|y - x| < \delta$. Let $\epsilon = 1$, pick the required δ, we then get $|\gamma_x(t) - \gamma_y(t)| = |\gamma_y(t) - \gamma_x(t)| < 1 = |\gamma_x(t)| = |g(xe^{it})|$, and by Exercise 25, $Ind(\gamma_x) = Ind(\gamma_y)$ if $|y - x| < \delta$. Therefore, $Ind(\gamma_r)$ is a continuous function of r, on $[0,1]$. Then as the same reason of Exercise 27, $Ind(\gamma_0) = 0$ and $Ind(\gamma_1) = 1$ gives the contradictory.

29. Prove that every continuous mapping f of $\bar{D}$ into $\bar{D}$ has a fixed point in $\bar{D}$.
(This is the 2-dimensional case of Brouwer's fixed-point theorem.)
Proof: Assume $f(z) \neq z$ for every $z \in \bar{D}$. Associate to each $z \in \bar{D}$ the point $g(z) \in T$ which lies on the ray that starts at $f(z)$ and passes through z. Then g maps $\bar{D}$ into T, $g(z) = z$ if $z \in T$, and g is continuous, because

$$g(z) = z - s(z)[f(z) - z],$$

where $s(z)$ is the unique nonnegative root of a certain quadratic equation whose coefficients are continuous functions of f and z. By Exercise 28, $g(z) = -z$ for at least one $z \in T$. Suppose $z_0 \in T$ satisfies $g(z_0) = -z_0$, and on the other hand we must have $g(z_0) = z_0$ since $z_0 \in T$. This gives $-z_0 = z_0$, namely $z_0 = 0$, which contradicts the fact $z_0 \in T$.

30. Use Stirling's formula to prove that

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} = 1$$

for every real constant c.
Proof:

$$\lim_{x\to\infty} \frac{\Gamma(x+c)}{x^c\Gamma(x)} =$$

$$\lim_{x\to\infty} \frac{\frac{\Gamma(x+c)}{((x+c-1)/e)^{x+c-1}\sqrt{2\pi(x+c-1)}} \cdot ((x+c-1)/e)^{x+c-1}\sqrt{2\pi(x+c-1)}}{x^c \cdot \frac{\Gamma(x)}{((x-1)/e)^{x-1}\sqrt{2\pi(x-1)}} \cdot ((x-1)/e)^{x-1}\sqrt{2\pi(x-1)}}$$

$$= \lim_{x\to\infty} \frac{((x+c-1)/e)^{x+c-1}\sqrt{2\pi(x+c-1)}}{x^c \cdot ((x-1)/e)^{x-1}\sqrt{2\pi(x-1)}}$$

$$= \lim_{x\to\infty} \frac{(x+c-1)^{x+c-1}\sqrt{x+c-1}}{x^c \cdot e^c \cdot (x-1)^{x-1}\sqrt{x-1}}$$

$$= \lim_{x\to\infty} \frac{1}{e^c} \cdot (1 + \frac{c-1}{x})^c \cdot (1 + \frac{c}{x-1})^{x-1} \cdot \sqrt{1 + \frac{c}{x-1}}$$

$$= \frac{1}{e^c} \cdot 1 \cdot e^c \cdot 1 = 1.$$

31. In the proof of Theorem 7.26 it was shown that

$$\int_{-1}^{1} (1-x^2)^n dx \geq \frac{4}{3\sqrt{n}}$$

for $n = 1, 2, 3, ...$ Use Theorem 8.20 and Exercise 30 to show the more precise result

$$\lim_{n\to\infty} \sqrt{n} \int_{-1}^{1} (1-x^2)^n dx = \sqrt{\pi}.$$

Proof: We have

$$\sqrt{n} \int_{-1}^{1} (1-x^2)^n dx = 2\sqrt{n} \int_{0}^{1} (1-x^2)^n dx.$$

Let $t = x^2$, then $dt = 2xdx$, which gives $dx = \frac{1}{2\sqrt{t}}dt = \frac{1}{2}t^{-1/2}dt$, hence

$$2\sqrt{n} \int_0^1 (1-x^2)^n dx = 2\sqrt{n} \int_0^1 (1-t)^n \frac{1}{2} t^{-1/2} dt = \sqrt{n} \int_0^1 t^{-1/2} (1-t)^n dt$$

$$= \sqrt{n} \int_0^1 t^{1/2-1} (1-t)^{(n+1)-1} dt = \sqrt{n} \frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(1/2 + (n+1))}.$$

Then

$$\lim_{n\to\infty} \sqrt{n} \int_{-1}^{1} (1-x^2)^n dx = \lim_{n\to\infty} \sqrt{n} \frac{\Gamma(1/2)\Gamma(n+1)}{\Gamma(1/2 + (n+1))}$$

$$= \lim_{n\to\infty} \sqrt{n} \frac{(n+1)^{1/2}\Gamma(1/2)\Gamma(n+1)}{(n+1)^{1/2}\Gamma((n+1) + 1/2)}$$

$$= \lim_{n\to\infty} \sqrt{n} \frac{\Gamma(1/2)}{(n+1)^{1/2} \cdot \frac{\Gamma((n+1)+1/2)}{((n+1)^{1/2}\Gamma(n+1))}} = \Gamma(1/2) = \sqrt{\pi}.$$

9 Functions of several variables

1. If S is a nonempty subset of a vector space X, prove (as asserted in Sec. 9.1) that the span of S is a vector space.
Proof: Let E be the span of S. Suppose $\mathbf{y}_1$, $\mathbf{y}_2 \in E$. Then $\mathbf{y}_i = \sum_k c_{ik}\mathbf{x}_{ik}$, where $i = 1, 2$ and $\mathbf{x}_{ik} \in S$. We then have

$$\mathbf{y}_1 + \mathbf{y}_2 = \sum_k c_{1k}\mathbf{x}_{2k} + \sum_k c_{2k}\mathbf{x}_{2k} = \sum_j c_j\mathbf{x}_j,$$

where $\mathbf{x}_j \in S$; and

$$c\mathbf{y}_i = \sum_k (cc_{ik})\mathbf{x}_{ik} \qquad (i = 1, 2).$$

Therefore, both $\mathbf{y}_1 + \mathbf{y}_2 \in E$ and $c\mathbf{y}_i \in E (i = 1, 2)$. Hence E is a vector space.

2. Prove (as asserted in Sec. 9.6) that BA is linear if A and B are linear transformations. Prove also that A^{-1} is linear and invertible.
   Proof: We have

$$BA(\mathbf{x}_1 + \mathbf{x}_2) = B(A(\mathbf{x}_1 + \mathbf{x}_2)) = B(A\mathbf{x}_1 + A\mathbf{x}_2) = BA\mathbf{x}_1 + BA\mathbf{x}_2,$$

   and
$$BA(c\mathbf{x}) = B(A(c\mathbf{x})) = B(cA\mathbf{x}) = cBA\mathbf{x}.$$

   Hence BA is linear.
   Suppose $\mathbf{y}_1 = A\mathbf{x}_1$, $\mathbf{y}_2 = A\mathbf{x}_2$, and $\mathbf{y} = A\mathbf{x}$. Then

$$A^{-1}(\mathbf{y}_1 + \mathbf{y}_2) = A^{-1}(A\mathbf{x}_1 + A\mathbf{x}_2) = A^{-1}A(\mathbf{x}_1 + \mathbf{x}_2)$$

$$= \mathbf{x}_1 + \mathbf{x}_2 = A^{-1}\mathbf{y}_1 + A^{-1}\mathbf{y}_2,$$

   and
$$A^{-1}(c\mathbf{y}) = A^{-1}(cA\mathbf{x}) = A^{-1}A(c\mathbf{x}) = c\mathbf{x} = cA^{-1}\mathbf{y}.$$

   Hence A^{-1} is linear. Since A is invertible, A is one-to-one. Then A^{-1} is also one-to-one, and therefore A^{-1} is invertible, by Theorem 9.5.

3. Assume $A \in L(X, Y)$ and $A\mathbf{x} = \mathbf{0}$ only when $\mathbf{x} = \mathbf{0}$. Prove that A is then 1-1.
   Proof: Suppose that $A\mathbf{x} = A\mathbf{y}$, then $A(\mathbf{x}-\mathbf{y}) = \mathbf{0}$, which gives $\mathbf{x}-\mathbf{y} = \mathbf{0}$, namely $\mathbf{x} = \mathbf{y}$. Therefore, A is 1-1.

4. Prove (as asserted in Sec. 9.30) that null spaces and ranges of linear transformations are vector spaces.
   Proof:
   (i) Suppose $\mathbf{x} \in \mathscr{N}(A)$ and $\mathbf{y} \in \mathscr{N}(A)$, then $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{y} = \mathbf{0}$. Therefore $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \mathbf{0}$, $A(c\mathbf{x}) = cA\mathbf{x} = \mathbf{0}$, and thus $\mathbf{x} + \mathbf{y} \in \mathscr{N}(A)$, $c\mathbf{x} \in \mathscr{N}(A)$, which shows that $\mathscr{N}(A)$ is a vector space.
   (ii) Suppose $\mathbf{y}_1 \in \mathscr{R}(A)$ and $\mathbf{y}_2 \in \mathscr{R}(A)$, then there exist $\mathbf{x}_1 \in X$, $\mathbf{x}_2 \in X$ such that $A\mathbf{x}_1 = \mathbf{y}_1$ and $A\mathbf{x}_2 = \mathbf{y}_2$. Therefore, $\mathbf{y}_1 + \mathbf{y}_2 = A\mathbf{x}_1 + A\mathbf{x}_2 = A(\mathbf{x}_1 + \mathbf{x}_2) \in \mathscr{R}(A)$, and $c\mathbf{y}_i = cA\mathbf{x}_i = A(c\mathbf{x}_i) \in \mathscr{R}(A)$, which gives that $\mathscr{R}(A)$ is a vector space.

5. Prove that to every $A \in L(R^n, R^1)$ corresponds a unique $\mathbf{y} \in R^n$ such that $A\mathbf{x} = \mathbf{x} \cdot \mathbf{y}$. Prove also that $||A|| = |\mathbf{y}|$.
   Proof: Let $\mathbf{y} = (y_1, y_2, ..., y_n)^T$. Then we have $A\mathbf{e}_i = \mathbf{e}_i \cdot \mathbf{y}$, which gives $y_i = A\mathbf{e}_i$. Then for any $\mathbf{x} \in R^n$, suppose $\mathbf{x} = \sum_{i=1}^n x_i\mathbf{e}_i$, we have

$A\mathbf{x} = A\sum_{i=1}^{n} x_i\mathbf{e}_i = \sum_{i=1}^{n} x_i A\mathbf{e}_i = \sum_{i=1}^{n} x_i y_i = \mathbf{x}\cdot\mathbf{y}$. The existence and uniqueness of $\mathbf{y}$ then are proved.
Since $|A\mathbf{x}| = |\mathbf{x}\cdot\mathbf{y}| \le |\mathbf{x}|\cdot|\mathbf{y}|$, for all $\mathbf{x} \in R^n$, by Schwarz inequality, we thus have $||A|| \le |\mathbf{y}|$. On the other hand, since $\mathbf{y} \in R^n$, $A\mathbf{y} = \mathbf{y}\cdot\mathbf{y} = |\mathbf{y}|^2$, which gives $|A\mathbf{y}| = |\mathbf{y}|^2$. Since $|A\mathbf{y}| \le ||A||\cdot|\mathbf{y}|$, we have $|\mathbf{y}|^2 \le ||A||\cdot|\mathbf{y}|$, namely, $|\mathbf{y}| \le ||A||$. Therefore, $||A|| = |\mathbf{y}|$.

6. If $f(0,0) = 0$ and

$$f(x,y) = \frac{xy}{x^2+y^2} \qquad \text{if}(x,y) \ne (0,0),$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of R^2, although f is not continuous at $(0,0)$.
Proof: Suppose $\mathbf{x}_n = (\frac{1}{n}, \frac{1}{n})$, then $\mathbf{x}_n \to (0,0)$, as $n \to \infty$. But

$$\lim_{n\to\infty} f(\mathbf{x}_n) = \lim_{n\to\infty} f(\frac{1}{n}, \frac{1}{n}) = \lim_{n\to\infty} \frac{1/n^2}{2/n^2} = \frac{1}{2} \ne 0,$$

which implies f is not continuous at $(0,0)$.
On the other hand, $(D_1f)(x,y)$ and $(D_1f)(x,y)$ clearly exist at every point on the plane other than $(0,0)$. Since

$$(D_1f)(0,0) = \lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0} \frac{0-0}{x-0} = 0,$$

and

$$(D_2f)(0,0) = \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} = \frac{0-0}{y-0} = 0,$$

we conclude that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of R^2.

7. Suppose that f is a real-valued function defined in an open set $E \subseteq R^n$, and that the partial derivatives $D_1f, ..., D_nf$ are bounded in E. Prove that f is continuous in E.
Proof: Similarly as in the proof of Theorem 9.21, we fix $\mathbf{x} \in E$ and $\epsilon > 0$. Since D_if are bounded in E, $|D_if| \le M_i$ for some $M_i > 0$. Put $M = \max_{1\le i\le n} M_i$, we thus have $|D_if| \le M$, for $1 \le i \le n$. Suppose $\mathbf{h} = \sum h_j\mathbf{e}_j$, $|\mathbf{h}| < r = \epsilon/nM$, put $\mathbf{v}_0 = \mathbf{0}$, and $\mathbf{v}_k = h_1\mathbf{e}_1 + \cdots + h_k\mathbf{e}_k$, for $1 \le k \le n$. Then

$$f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) = \sum_{j=1}^{n} [f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})].$$

Since $\mathbf{v}_j = \mathbf{v}_{j-1} + h_j\mathbf{e}_j$, the mean value theorem (Theorem 5.10) shows that the jth summand is equal to

$$h_j(D_jf)(\mathbf{x}+\mathbf{v}_{j-1}+\theta_j h_j\mathbf{e}_j)$$

for some $\theta_j \in (0, 1)$. It follows that

$$|f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x})| = |\sum_{j=1}^{n} [f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})]|$$

$$\leq \sum_{j=1}^{n} |f(\mathbf{x}+\mathbf{v}_j) - f(\mathbf{x}+\mathbf{v}_{j-1})| = \sum_{j=1}^{n} |h_j (D_j f)(\mathbf{x}+\mathbf{v}_{j-1} + \theta_j h_j \mathbf{e}_j)|$$

$$\leq M \sum_{j=1}^{n} |h_j| \leq Mn|\mathbf{h}| < \epsilon,$$

which shows that f is continuous in E.

8. Suppose that f is a differentiable real function in an open set $E \subseteq R^n$, and that f has a local maximum at a point $\mathbf{x} \in E$. Prove that $f'(\mathbf{x}) = \mathbf{0}$.
Proof: Since
$$D_j f(\mathbf{x}) = \lim_{t \to 0} \frac{f(\mathbf{x}+t\mathbf{e}_j) - f(\mathbf{x})}{t},$$
and $f(\mathbf{x}+t\mathbf{e}_j) \leq f(\mathbf{x})$, when $|t| < r$ for some sufficiently small $r > 0$, due to the local maximum of $f(\mathbf{x})$. We thus have
$$\frac{f(\mathbf{x}+t\mathbf{e}_j) - f(\mathbf{x})}{t} \geq 0, \qquad t \in (0, r),$$
and
$$\frac{f(\mathbf{x}+t\mathbf{e}_j) - f(\mathbf{x})}{t} \leq 0, \qquad t \in (-r, 0),$$
and therefore $D_j f(\mathbf{x}) \geq 0$ and $D_j f(\mathbf{x}) \leq 0$, which give that $D_j f(\mathbf{x}) = 0$, for $1 \leq j \leq n$. It follows that $f'(\mathbf{x}) = \mathbf{0}$ according to Theorem 9.17.

9. If $\mathbf{f}$ is a differentiable mapping of a *connected* open set $E \subseteq R^n$ into R^m, and if $\mathbf{f}'(\mathbf{x}) = \mathbf{0}$ for every $\mathbf{x} \in E$, prove that $\mathbf{f}$ is constant in E.
Proof: Suppose that, on the contrary, $\mathbf{f}$ is not constant in E. Pick any $\mathbf{a} \in E$, and let $\mathbf{c} = \mathbf{f}(\mathbf{a})$. Let $A = \{\mathbf{p} | \mathbf{p} \in E \wedge \mathbf{f}(\mathbf{p}) = \mathbf{c}\}$ and $B = \{\mathbf{p} | \mathbf{p} \in E \wedge \mathbf{f}(\mathbf{p}) \neq \mathbf{c}\}$. Since $\mathbf{a} \in A$, $A \neq \emptyset$, and since $\mathbf{f}$ is not constant, $B \neq \emptyset$. Clearly, $E = A \cup B$ and $A \cap B = \emptyset$.
(i) First, we prove that for any $\mathbf{x} \in E$, there exists an $r > 0$ such that $|\mathbf{p} - \mathbf{x}| < r$ implies $\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{x})$. To see this, since E is open, given any $\mathbf{x} \in E$, there is an open ball $S \subseteq E$, with center at $\mathbf{x}$ and radius r. Since S is convex, by the corollary of Theorem 9.19, we known that $\mathbf{f}$ is constant in S, which is the desired result.
(ii) Next, we will prove that $\bar{A} \cap B = \emptyset$. Suppose $\mathbf{q} \in E$ is a limit point of A, then there is a $r_q > 0$ such that $|\mathbf{p} - \mathbf{q}| < r_q$ implies $\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q})$. Since $\mathbf{q}$ is a limit point of A, there is at least one $\mathbf{p} \in A$ which satisfies $|\mathbf{p} - \mathbf{q}| < r_q$. Therefore, we must have $\mathbf{f}(\mathbf{q}) = \mathbf{f}(\mathbf{p}) = \mathbf{c}$, and therefore $\mathbf{q} \in A$. It follow that $\bar{A} = A$ and hence $\bar{A} \cap B = A \cap B = \emptyset$.
(iii) Then, we will prove that $A \cap \bar{B} = \emptyset$. Suppose that, on the contrary,

$A \cap \bar{B} \neq \emptyset$. There exists at least one $\mathbf{q} \in A \cap \bar{B}$, that is, $\mathbf{f}(\mathbf{q}) = \mathbf{c}$. Since $A \cap B = \emptyset$, $\mathbf{q}$ is a limit point of B. By (i), we can find a $r_q > 0$ such that $|\mathbf{p} - \mathbf{q}| < r_q$ implies $\mathbf{f}(\mathbf{p}) = \mathbf{f}(\mathbf{q}) = \mathbf{c}$. Since $\mathbf{q}$ is a limit point of B, there exists at least one $\mathbf{p} \in B$ such that $|\mathbf{p} - \mathbf{q}| < r_q$. Therefore, $\mathbf{f}(\mathbf{p}) = \mathbf{c}$ and hence $\mathbf{p} \in A$, a contradictory since $A \cap B = \emptyset$. It follows that $A \cap \bar{B} = \emptyset$. By (ii) and (iii), we conclude that A and B are separated. Since $E = A \cup B$ and both A and B are nonempty, E is not connected, which is contradict to our assumption. Therefore, $\mathbf{f}$ must be constant in E.

10. If f is a real function defined in a convex open set $E \subseteq R^n$, such that $(D_1 f)(\mathbf{x}) = 0$ for every $\mathbf{x} \in E$, prove that $f(\mathbf{x})$ depends only on $x_2, ..., x_n$. Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.
Proof: Fix $x_2, ..., x_n$, we then get a subset S of E whose any two points only differs at x_1 in their coordinates. Let $\mathbf{a}$ and $\mathbf{b}$ be any two points in S, $\mathbf{a} = (a_1, x_2, ..., x_n)^T$ and $\mathbf{b} = (b_1, x_2, ..., x_n)^T$. Then $\lambda\mathbf{a} + (1-\lambda)\mathbf{b} = (\lambda a_1 + (1-\lambda)b_1, x_2, ..., x_n)^T$. Since E is convex, $\lambda\mathbf{a} + (1-\lambda)\mathbf{b} \in E$ and therefore, $\lambda\mathbf{a} + (1-\lambda)\mathbf{b} \in S$, for $0 < \lambda < 1$. Hence S is convex. Apply the corollary of Theorem 9.19, we conclude that f is constant in S, which is the desired result.
By Exercise 9, we know that the convexity of E can be replaced by a weaker condition, that is, every obtained S needs to be connected.

11. If f and g are differentiable real functions in R^n, prove that

$$\nabla(fg) = f\nabla g + g\nabla f$$

and that $\nabla(1/f) = -f^{-2}\nabla f$ wherever $f \neq 0$.
Proof: Since

$$(\nabla f)(\mathbf{x}) = \sum_{i=1}^{n} (D_i f)(\mathbf{x})\mathbf{e}_i,$$

we have

$$\begin{aligned}\nabla(fg) &= \sum_{i=1}^{n} D_i(fg)\mathbf{e}_i = \sum_{i=1}^{n} [(D_i f)g + f(D_i g)]\mathbf{e}_i \\ &= g\sum_{i=1}^{n}(D_i f)\mathbf{e}_i + f\sum_{i=1}^{n}(D_i g)\mathbf{e}_i = g\nabla f + f\nabla g,\end{aligned}$$

and

$$\begin{aligned}\nabla(1/f) &= \sum_{i=1}^{n}(D_i(1/f))\mathbf{e}_i = \sum_{i=1}^{n}\frac{0 - D_i f}{f^2}\mathbf{e}_i \\ &= -f^{-2}\sum_{i=1}^{n}(D_i f)\mathbf{e}_i = -f^{-2}\nabla f.\end{aligned}$$

12. Fix two real numbers a and b, $0 < a < b$. Define a mapping $\mathbf{f} = (f_1, f_2, f_3)$ of R^2 into R^3 by

$$f_1(s,t) = (b + a\cos s)\cos t$$
$$f_2(s,t) = (b + a\cos s)\sin t$$
$$f_3(s,t) = a\sin s.$$

Describe the range K of $\mathbf{f}$. (It is a certain compact subset of R^3.)

(a) Show that there are exactly 4 points $\mathbf{p} \in K$ such that

$$(\nabla f_1)(\mathbf{f}^{-1}(\mathbf{p})) = \mathbf{0}.$$

Find these points.
Solution: Since

$$\nabla f_1 = (-a\sin s\cos t, -(b + a\cos s)\sin t)^T,$$

$\nabla f_1 = \mathbf{0}$ gives $-a\sin s\cos t = 0$, $-(b + a\cos s)\sin t = 0$. It follows that

$$\cos t = 0, \qquad b + a\cos s = 0, \quad i.e., \quad \cos s = -b/a,$$

or

$$\sin t = 0, \qquad -a\sin s = 0, \quad i.e., \quad \sin s = 0.$$

Since $0 < a < b$, the first solution implies $\cos s = -b/a < -1$, which is impossible. The second solution gives four points in R^3, namely $\mathbf{p}_1 = (b + a, 0, 0)^T$, $\mathbf{p}_2 = (-b - a, 0, 0)^T$, $\mathbf{p}_3 = (b - a, 0, 0)^T$, and $\mathbf{p}_4 = (a - b, 0, 0)^T$.

(b) Determine the set of all $\mathbf{q} \in K$ such that

$$(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}.$$

Solution: Since $\nabla f_3 = (a\cos s, 0)$, $\nabla f_3 = \mathbf{0}$ implies $\cos s = 0$. Then the set of all $\mathbf{q} \in K$ satisfying $(\nabla f_3)(\mathbf{f}^{-1}(\mathbf{q})) = \mathbf{0}$ has elements $\mathbf{q}$ with form $\mathbf{q} = (b\cos t, b\sin t, a)$, or $\mathbf{q} = (b\cos t, b\sin t, -a)$.

(c) Show that one of the points $\mathbf{p}$ found in part (a) corresponds to a local maximum of f_1, one corresponds to a local minimum, and that the other two are neither (they are so-called "saddle points").
Which of the points $\mathbf{q}$ found in part (b) correspond to maxima or minima?
Solution: Clearly we have $-(a + b) \le f_1(s,t) \le (a + b)$, for all $(s,t) \in R^2$, therefore, $(b + a, 0, 0)^T$ corresponds to a local maximum of f_1, and $(-b - a, 0, 0)^T$ corresponds to a local minimum. The other two are neither.
Similarly, since $-a \le f_3(s,t) \le a$, for all $(s,t) \in R^2$, the points with form $\mathbf{q} = (b\cos t, b\sin t, a)$ correspond to maxima, and the points with form $\mathbf{q} = (b\cos t, b\sin t, -a)$ correspond to minima.

(d) Let λ be an irrational real number, and define $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$. Prove that $\mathbf{g}$ is a 1-1 mapping of R^1 onto a dense subset of K. Prove that

$$|\mathbf{g}'(t)|^2 = a^2 + \lambda^2(b + a\cos t)^2.$$

Proof: Since $\mathbf{g}(t) = \mathbf{f}(t, \lambda t)$, we have

$$g_1(t) = f_1(t, \lambda t) = (b + a\cos t)\cos(\lambda t)$$

$$g_2(t) = f_2(t, \lambda t) = (b + a\cos t)\sin(\lambda t)$$

$$g_3(t) = f_3(t, \lambda t) = a\sin t.$$

Suppose $t_1 \neq t_2$, but $\mathbf{g}(t_1) = \mathbf{g}(t_2)$, then $a\sin t_1 = a\sin t_2$, which gives $\sin t_1 = \sin t_2$. Therefore, $t_2 = t_1 + 2k\pi$, or $t_2 = (\pi - t_1) + 2k\pi$, where k is some integer. If $t_2 = t_1 + 2k\pi$, then $\cos t_2 = \cos t_1$, but $\lambda t_2 = \lambda(t_1 + 2k\pi) = \lambda t_1 + 2(k\lambda)\pi$, which means $\cos(\lambda t_2) \neq \cos(\lambda t_1)$ and $\sin(\lambda t_2) \neq \sin(\lambda t_1)$. Hence $\mathbf{g}(t_1) \neq \mathbf{g}(t_2)$, a contradiction. If $t_2 = (\pi - t_1) + 2k\pi$, we have $\lambda t_2 = \lambda(2k+1)\pi - \lambda t_1$, which also gives $\cos(\lambda t_2) \neq \cos(\lambda t_1)$ and $\sin(\lambda t_2) \neq \sin(\lambda t_1)$. Hence $\mathbf{g}(t_1) \neq \mathbf{g}(t_2)$, also a contradiction. Therefore, $\mathbf{g}$ is a 1-1 mapping.

Next, we will prove that the range of $\mathbf{g}$ is a dense subset of K. Let $E = \{(x, y) | \lambda(x + m \cdot 2\pi) = y + n \cdot 2\pi, m, n \in \mathbb{Z}\}$. If we can prove that E is dense in R^2, then we know that $\mathbf{f}(E)$ is dense in K, by Exercise 4 of Chap.4. Since for any $(x, y) \in E$, if we put $t = x + m \cdot 2\pi$, then $\lambda t = \lambda(x + m \cdot 2\pi) = y + n \cdot 2\pi$, which gives $\mathbf{g}(t) = \mathbf{f}(t, \lambda t) = \mathbf{f}(x + m \cdot 2\pi, y + n \cdot 2\pi) = \mathbf{f}(x, y)$. Suppose the range of $\mathbf{g}$ is G, it follows that $\mathbf{f}(E) \subseteq G$. Since $\mathbf{f}(E)$ is dense in K, G is dense in K and we are done.

Now, we begin to prove that E is dense in R^2. Given any $(x, y) \in R^2$ and any $\epsilon > 0$, we need to show that there is at least one $(s, t) \in E$ such that $(s, t) \in N_\epsilon(x, y)$. We put $s = x$, then it is sufficient to show that the set $F = \{\lambda(x + m \cdot 2\pi) + n \cdot 2\pi, m, n \in \mathbb{Z}\}$ is dense in R, since then we can find at least one $t \in F$ such that $|t - y| < \epsilon$. From Exercise 25(b), we have known that the set $S = \{m\lambda + n, m, n \in \mathbb{Z}\}$ is dense in R. Since $\lambda(x + m \cdot 2\pi) + n \cdot 2\pi = \lambda x + 2\pi(m\lambda + n)$, $|(\lambda(x + m \cdot 2\pi) + n \cdot 2\pi) - y| = |(\lambda x + 2\pi(m\lambda + n)) - y| = |2\pi(m\lambda + n) - (y - \lambda x)| = 2\pi|(m\lambda + n) - (y - \lambda x)/(2\pi)|$. Now pick m, n so that $|(m\lambda + n) - (y - \lambda x)/(2\pi)| < \epsilon/(2\pi)$. This can be done since S is dense in R. It follows that $|(\lambda(x + m \cdot 2\pi) + n \cdot 2\pi) - y| < \epsilon$, for the selected m, n. Therefore, F is dense in R.

Since

$$g_1'(t) = -a\sin t\cos(\lambda t) + (b + a\cos t)(-\lambda\sin(\lambda t))$$

$$= -b\lambda\sin(\lambda t) - a\sin t\cos(\lambda t) - \lambda a\cos t\sin(\lambda t)$$

$$g_2'(t) = -a\sin t\sin(\lambda t) + (b + a\cos t)(\lambda\cos(\lambda t))$$

$$= \lambda b\cos(\lambda t) - a\sin t\sin(\lambda t) + \lambda a\cos t\cos(\lambda t)$$

$$g_3'(t) = a\cos t,$$

we have

$$|\mathbf{g}'(t)|^2 = (g_1'(t))^2 + (g_2'(t))^2 + (g_3'(t))^2 = \lambda^2 b^2 + a^2\sin^2 t + \lambda^2 a^2\cos^2 t$$

$$+2ab\lambda\sin t\sin(\lambda t)\cos(\lambda t) + 2\lambda a^2\sin t\cos t\sin(\lambda t)\cos(\lambda t)$$

$$+2ab\lambda^2\cos t\sin^2(\lambda t) - 2ab\lambda\sin t\sin(\lambda t)\cos(\lambda t)$$

$$-2\lambda a^2\sin t\cos t\sin(\lambda t)\cos(\lambda t) + 2ab\lambda^2\cos t\cos^2(\lambda t) + a^2\cos^2 t$$

$$= \lambda^2 b^2 + a^2\sin^2 t + \lambda^2 a^2\cos^2 t + a^2\cos^2 t + 2ab\lambda^2\cos t$$

$$= a^2 + \lambda^2(b^2 + a^2\cos^2 t + 2ab\cos t) = a^2 + \lambda^2(b + a\cos t)^2.$$

(Note that we can also apply the chain rule to obtain $\mathbf{g}'(t)$.)

13. Suppose $\mathbf{f}$ is a differentiable mapping of R^1 into R^3 such that $|\mathbf{f}(t)| = 1$ for every t. Prove that $\mathbf{f}'(t)\cdot\mathbf{f}(t) = 0$.
Interpret this result geometrically.
Proof: Suppose $(f)(t) = (f_1(t), f_2(t), f_3(t))^T$, then $|\mathbf{f}(t)| = 1$ implies $f_1^2(t) + f_2^2(t) + f_3^2(t) = 1$, for every t. Let $g(t) = f_1^2(t) + f_2^2(t) + f_3^2(t)$, we then have $g(t) = 1$ for every t. Therefore, $g'(t) = 0$. On the other hand, we have $g'(t) = 2(f_1(t)f_1'(t) + f_2(t)f_2'(t) + f_3(t)f_3'(t)) = 2\mathbf{f}'(t)\cdot\mathbf{f}(t)$. Hence $\mathbf{f}'(t)\cdot\mathbf{f}(t) = 0$, for every t.
This means the direction of the tangent line of the curve at point $\mathbf{f}(t)$ is perpendicular to to direction of $\mathbf{f}(t)$.

14. Define $f(0,0) = 0$ and

$$f(x,y) = \frac{x^3}{x^2+y^2} \qquad \text{if}(x,y) \neq (0,0).$$

(a) Prove that $D_1 f$ and $D_2 f$ are bounded functions in R^2. (Hence f is continuous.)
Proof: At every point $(x,y) \neq (0,0)$, we have

$$|(D_1 f)(x,y)| = \frac{x^2(x^2+3y^2)}{(x^2+y^2)^2} \leq \frac{x^2(3x^2+3y^2)}{(x^2+y^2)^2} = 3\cdot\frac{x^2}{x^2+y^2} \leq 3,$$

and

$$|(D_2 f)(x,y)| = \frac{x^2\cdot(2|x||y|)}{(x^2+y^2)^2} \leq \frac{x^2(x^2+y^2)}{(x^2+y^2)^2} = \frac{x^2}{x^2+y^2} \leq 1.$$

Furthermore, we have

$$(D_1 f)(0,0) = \lim_{x\to 0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to 0}\frac{x-0}{x-0} = 1,$$

and

$$(D_2 f)(0,0) = \lim_{y \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y \to 0} \frac{0 - 0}{y - 0} = 0.$$

Hence, $D_1 f$ and $D_2 f$ are bounded.

(b) Let $\mathbf{u}$ be any unit vector in R^2. Show that the directional derivative $(D_{\mathbf{u}} f)(0,0)$ exists, and that its absolute value is at most 1.
Proof: By (a), we have $(D_{\mathbf{u}} f)(0,0) = (D_1 f)(0,0)u_1 + (D_2 f)(0,0)u_2 = u_1$. Hence $|(D_{\mathbf{u}} f)(0,0)| = |u_1| \leq |\mathbf{u}| \leq 1$.

(c) Let γ be a differentiable mapping of R^1 into R^2 (in other words, γ is a differentiable curve in R^2), with $\gamma(0) = (0,0)$ and $|\gamma'(0)| > 0$. Put $g(t) = f(\gamma(t))$ and prove that g is differentiable for every $t \in R^1$.
If $\gamma \in \mathscr{C}'$, prove that $g \in \mathscr{C}'$.
Proof:
(i) Since f is differentiable at every point other than $(0,0)$, by Theorem 9.21, it follows that g is differentiable for every $t \neq 0$ (note that in fact, we need γ to be a 1-1 mapping). So we need only to prove that g is differentiable for $t = 0$. Since γ is differentiable, we have $\gamma(t) - \gamma(0) = \gamma'(0)(t - 0) + r(t)$, where $\lim_{t \to 0} \frac{r(t)}{t} = 0$. Since $\gamma(0) = (0,0)$, $\gamma(t) = \gamma'(0)t + r(t)$, namely, $\gamma_1(t) = \gamma_1'(0)t + r_1(t)$ and $\gamma_2(t) = \gamma_2'(0)t + r_2(t)$, where $\gamma = (\gamma_1, \gamma_2)$. Then,

$$f(\gamma(t)) = \frac{\gamma_1^3(t)}{\gamma_1^2(t) + \gamma_2^2(t)} = \frac{(\gamma_1'(0)t + r_1(t))^3}{(\gamma_1'(0)t + r_1(t))^2 + (\gamma_2'(0)t + r_2(t))^2}$$

$$= \frac{(\gamma_1'(0))^3 t^3 + 3(\gamma_1'(0))^2 t^2 r_1(t) + 3\gamma_1'(0) t (r_1(t))^2 + (r_1(t))^3}{|\gamma'(0)|^2 t^2 + 2(\gamma_1'(0) r_1(t) + \gamma_2'(0) r_2(t))t + ((r_1(t))^2 + (r_2(t))^2)},$$

which gives

$$g'(0) = \lim_{t \to 0} \frac{g(t) - g(0)}{t - 0} = \lim_{t \to 0} \frac{f(\gamma(t)) - f(\gamma(0))}{t} = \lim_{t \to 0} \frac{f(\gamma(t))}{t}$$

$$= \lim_{t \to 0} \frac{(\gamma_1'(0))^3 t^3 + 3(\gamma_1'(0))^2 t^2 r_1(t) + 3\gamma_1'(0) t (r_1(t))^2 + (r_1(t))^3}{t[|\gamma'(0)|^2 t^2 + 2(\gamma_1'(0) r_1(t) + \gamma_2'(0) r_2(t))t + ((r_1(t))^2 + (r_2(t))^2)]}$$

$$= \lim_{t \to 0} \frac{(\gamma_1'(0))^3 + 3(\gamma_1'(0))^2 \frac{r_1(t)}{t} + 3\gamma_1'(0)(\frac{r_1(t)}{t})^2 + (\frac{r_1(t)}{t})^3}{|\gamma'(0)|^2 + 2(\gamma_1'(0)\frac{r_1(t)}{t} + \gamma_2'(0)\frac{r_2(t)}{t}) + ((\frac{r_1(t)}{t})^2 + (\frac{r_2(t)}{t})^2)}$$

$$= \frac{(\gamma_1'(0))^3}{|\gamma'(0)|^2}, \qquad \text{then the condition } |\gamma'(0)| > 0 \text{ implies } g'(0) \text{ exists .}$$

Hence g is differentiable for $t = 0$, and therefore differentiable for every $t \in R^1$.
(ii) Since f' is continuous at every point other than $(0,0)$, by Theorem 9.21, it follows that $g \in \mathscr{C}'$ for every $t \neq 0$, since $g'(t) = f'(\gamma(t))\gamma'(t)$ and f', γ, γ' are all continuous for $t \neq 0$. Hence we only need to

prove that g' is continuous at $t = 0$, namely, $\lim_{t\to 0} g'(t) = g'(0)$. Since $\gamma(t) = \gamma'(0)t + r(t) = t(\gamma'(0) + \frac{r(t)}{t})$,

$$g'(t) = f'(\gamma(t))\gamma'(t) = (D_1 f)(\gamma(t))\gamma_1'(t) + (D_2 f)(\gamma(t))\gamma_2'(t)$$

$$= \frac{(\gamma_1(t))^2((\gamma_1(t))^2 + 3(\gamma_2(t))^2)}{((\gamma_1(t))^2 + (\gamma_2(t))^2)^2}\gamma_1'(t) + \frac{-2(\gamma_1(t))^3\gamma_2(t)}{((\gamma_1(t))^2 + (\gamma_2(t))^2)^2}\gamma_2'(t),$$

we have

$$\lim_{t\to 0} g'(t) = \lim_{t\to 0}[\frac{(t(\gamma_1'(0) + \frac{r_1(t)}{t}))^2((t(\gamma_1'(0) + \frac{r_1(t)}{t}))^2 + 3(t(\gamma_2'(0) + \frac{r_2(t)}{t}))^2)}{((t(\gamma_1'(0) + \frac{r_1(t)}{t}))^2 + (t(\gamma_2'(0) + \frac{r_2(t)}{t}))^2)^2}\gamma_1'(t)$$

$$+\frac{-2(t(\gamma_1'(0) + \frac{r_1(t)}{t}))^3(t(\gamma_2'(0) + \frac{r_2(t)}{t}))}{(t(\gamma_1'(0) + \frac{r_1(t)}{t}))^2 + (t(\gamma_2'(0) + \frac{r_2(t)}{t}))^2)^2}\gamma_2'(t)]$$

$$= \lim_{t\to 0}[\frac{(\gamma_1'(0))^2((\gamma_1'(0))^2 + 3(\gamma_2'(0))^2)}{((\gamma_1'(0))^2 + (\gamma_2'(0))^2)^2}\gamma_1'(t) + \frac{-2(\gamma_1'(0))^3\gamma_2'(0)}{((\gamma_1'(0))^2 + (\gamma_2'(0))^2)^2}\gamma_2'(t)]$$

$$= \frac{(\gamma_1'(0))^3((\gamma_1'(0))^2 + 3(\gamma_2'(0))^2) - 2(\gamma_1'(0))^3(\gamma_2'(0))^2}{((\gamma_1'(0))^2 + (\gamma_2'(0))^2)^2} = \frac{(\gamma_1'(0))^3}{|\gamma'(0)|^2} = g'(0).$$

Therefore g' is continuous for $t = 0$, and thus $g \in \mathscr{C}'$.

(d) In spite of this, prove that f is not differentiable at $(0, 0)$.
Proof: We have

$$\lim_{h\to 0, k\to 0} \frac{|f(h,k) - f(0,0) - ((D_1 f)(0,0)h + (D_2 f)(0,0)k)|}{(h^2 + k^2)^{1/2}}$$

$$= \lim_{h\to 0, k\to 0} \frac{|h|k^2}{(h^2 + k^2)^{3/2}}.$$

If we put $\mathbf{x}_n = (\frac{1}{n}, \frac{1}{n})$, then $\mathbf{x}_n \to (0, 0)$ as $n \to \infty$. We thus get $\lim_{n\to\infty} \frac{\frac{1}{n}(\frac{1}{n})^2}{((\frac{1}{n})^2 + (\frac{1}{n})^2)^{3/2}} = \frac{1}{2\sqrt{2}} \neq 0$, which implies $\lim_{h\to 0, k\to 0} \frac{|h|k^2}{(h^2+k^2)^{3/2}} \neq 0$. Therefore, f is not differentiable at $(0, 0)$.

15. Define $f(0, 0) = 0$, and put

$$f(x, y) = x^2 + y^2 - 2x^2 y - \frac{4x^6 y^2}{(x^4 + y^2)^2}$$

if $(x, y) \neq (0, 0)$.

(a) Prove, for all $(x, y) \in R^2$, that

$$4x^4 y^2 \leq (x^4 + y^2)^2.$$

Conclude that f is continuous.
Proof: Since $(x^4+y^2)^2-4x^4y^2=(x^4-y^2)^2\geq 0$, it follows that $4x^4y^2\leq (x^4+y^2)^2$. Then

$$|f(x,y)|=|x^2+y^2-2x^2y-\frac{4x^6y^2}{(x^4+y^2)^2}|\leq x^2+y^2+2x^2|y|+\frac{4x^6y^2}{(x^4+y^2)^2}$$

$$=x^2+y^2+2x^2|y|+x^2\cdot\frac{4x^4y^2}{(x^4+y^2)^2}\leq x^2+y^2+2x^2|y|+x^2.$$

Since

$$x^2+y^2+2x^2|y|+x^2\to 0,\qquad \text{as } (x,y)\to(0,0),$$

we hence have $|f(x,y)|\to 0$, as $(x,y)\to(0,0)$. Therefore, $f(x,y)\to 0$, as $(x,y)\to(0,0)$, which means $f(x,y)$ is continuous at $(0,0)$. Since $f(x,y)$ is obviously continuous at every point other than $(0,0)$, we conclude that f is continuous.

(b) For $0\leq\theta\leq 2\pi$, $-\infty<t<\infty$, define

$$g_\theta(t)=f(t\cos\theta,t\sin\theta).$$

Show that $g_\theta(0)=0$, $g'_\theta(0)=0$, $g''_\theta(0)=2$. Each g_θ has therefore a strict local minimum at $t=0$.
In other words, the restriction of f to each line through $(0,0)$ has a strict local minimum at $(0,0)$.
Proof:

$$g_\theta(t)=f(t\cos\theta,t\sin\theta)=t^2-2t^3\cos^2\theta\sin\theta-\frac{4t^4\cos^6\theta\sin^2\theta}{(t^2\cos^4\theta+\sin^2\theta)^2},$$

hence $g_\theta(0)=0$.

$$g'_\theta(t)=2t-6t^2\cos^2\theta\sin\theta-\frac{16t^3\cos^6\theta\sin\theta(t^2\cos^4\theta(1+\sin\theta)+\sin^2\theta)}{(t^2\cos^4\theta+\sin^2\theta)^3},$$

hence $g'_\theta(0)=0$, and

$$g''_\theta(0)=\lim_{t\to 0}\frac{g'_\theta(t)-g'_\theta(0)}{t-0}=\lim_{t\to 0}\frac{g'_\theta(t)}{t}=\lim_{t\to 0}[2-6t\cos^2\theta\sin\theta$$

$$-\frac{16t^2\cos^6\theta\sin\theta(t^2\cos^4\theta(1+\sin\theta)+\sin^2\theta)}{(t^2\cos^4\theta+\sin^2\theta)^3}]=2.$$

Therefore, each g_θ has therefore a strict local minimum at $t=0$.

(c) Show that $(0,0)$ is nevertheless not a local minimum for f, since $f(x,x^2)=-x^4$.
Proof: Since $f(x,x^2)=-x^4$, given any $\epsilon>0$, pick the point $(x_0,y_0)=(1,1)$ if $\epsilon>\sqrt{2}$, and $(x_0,y_0)=(\epsilon/2,\epsilon^2/4)$ if $\epsilon\leq\sqrt{2}$. Then

$\sqrt{x_0^2 + y_0^2} = \sqrt{2} < \epsilon$ if $\epsilon > \sqrt{2}$, and $\sqrt{x_0^2 + y_0^2} = \sqrt{\epsilon^2/4 + \epsilon^4/16} < (\epsilon/2)(\sqrt{1+\epsilon^2/4}) \leq (\epsilon/2)(\sqrt{1+1/2}) < (\epsilon/2)\cdot 2 = \epsilon$ if $\epsilon \leq \sqrt{2}$. It follows that no matter which $\epsilon > 0$ chosen, we can pick a point (x_0, y_0) in $N_\epsilon(0,0)$ such that $f(x_0, y_0) < 0 = f(0,0)$. Therefore, $(0,0)$ is not a local minimum for f.

16. Show that the continuity of $\mathbf{f}'$ at the point $\mathbf{a}$ is needed in the inverse function theorem, even in the case $n = 1$: If

$$f(t) = t + 2t^2 \sin \frac{1}{t}$$

for $t \neq 0$, and $f(0) = 0$, then $f'(0) = 1$, f' is bounded in $(-1, 1)$, but f is not one-to-one in any neighborhood of 0.

Proof: we have

$$f'(0) = \lim_{t \to 0} \frac{f(t) - f(0)}{t - 0} = \lim_{t \to 0} \frac{f(t)}{t} = \lim_{t \to 0} [1 + 2t \sin \frac{1}{t}] = 1.$$

On the other hand, when $t \neq 0$,

$$f'(t) = 1 + 2[2t \sin \frac{1}{t} + t^2 \cos \frac{1}{t}(-\frac{1}{t^2})] = 1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t},$$

and hence

$$\lim_{t \to 0} f'(t) = \lim_{t \to 0} [1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t}] = \lim_{t \to 0} [1 - 2 \cos \frac{1}{t}] \neq 1.$$

It follows that $f'(t)$ is not continuous at 0.

Since $|f'(t)| = |1 + 4t \sin \frac{1}{t} - 2 \cos \frac{1}{t}| \leq 1 + 4|t| + 2 = 3 + 4|t| \leq 3 + 4 = 7$, when $0 < |t| < 1$. Thus f' is bounded in $(-1, 1)$.

Now we will show that f is not one-to-one in any neighborhood of 0. Suppose that, on the contrary, f is one-to-one in some neighborhood, say $N_\epsilon(0)$, of 0, where $\epsilon > 0$. Pick some r such that $0 < r < \epsilon$, then f is one-to-one on $[-r, r]$. Let $X = [-r, r]$, $f(X) = Y$, then f is a continuous 1-1 mapping from X onto Y. By Theorem 4.17, the inverse mapping f^{-1} defined on Y by $f^{-1}(f(x)) = x$, $x \in X$ is a continuous mapping of Y onto X. Let V be any open subset of X, since f^{-1} is continuous, $f(V)$ is also an open subset of Y, by Theorem 4.8. Then f is a continuous open mapping from X onto Y, by Exercise 15 of Chap.4, f is monotonic on X. Therefore, $f'(t) \geq 0$ for every $t \in [-r, r]$. But if we pick a sufficiently large n so that $0 < t_0 = \frac{1}{2n\pi} < r$, then $t_0 \in [-r, r]$ and

$$f'(t_0) = 1 + 4t_0 \sin \frac{1}{t_0} - 2 \cos \frac{1}{t_0} = 1 + 4\frac{1}{2n\pi} \sin(2n\pi) - 2\cos(2n\pi) = -1 < 0,$$

a contradiction. Therefore, f cannot be one-to-one in any neighborhood of 0.

Remark: In the above proof process, we use the fact that *every continuous open mapping is monotonic*. In fact, we can use the result that *every continuous injective mapping is monotonic*, which could achieve our conclusion more quickly.

17. Let $\mathbf{f} = (f_1, f_2)$ be the mapping of R^2 into R^2 given by

$$f_1(x,y) = e^x \cos y, \qquad f_2(x,y) = e^x \sin y.$$

(a) What is the range of f?
Solution: The range of $\mathbf{f}$ is R^2 except the point $(0,0)$.

(b) Show that the Jacobian of f is not zero at any point of R^2. Thus every point of R^2 has a neighborhood in which f is one-to-one. Nevertheless, f is not one-to-one in R^2.
Proof: The Jacobian of $\mathbf{f}$ is

$$\det \begin{bmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{bmatrix} = e^x > 0,$$

for every $(x,y) \in R^2$. Since $\mathbf{f} \in \mathscr{C}'$, by Theorem 9.21, every point of R^2 has a neighborhood in which $\mathbf{f}$ is one-to-one, by the inverse function theorem (Theorem 9.24). On the other hand, since $\mathbf{f}(x, y + 2n\pi) = \mathbf{f}(x,y)$, $\mathbf{f}$ is not one-to-one on R^2.

(c) Put $\mathbf{a} = (0, \pi/3)$, $\mathbf{b} = f(\mathbf{a})$, let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Find an explicit formula for $\mathbf{g}$, compute $\mathbf{f}'(\mathbf{a})$ and $\mathbf{g}'(\mathbf{b})$, and verify the formula (52).
Solution: Let $u = e^x \cos y$, $v = e^x \sin y$, then $u^2 + v^2 = e^{2x}$, which gives $x = \frac{1}{2}\log(u^2+v^2)$. Since $e^x = (u^2+v^2)^{1/2}$, we have $\cos y = u/e^x = \frac{u}{(u^2+v^2)^{1/2}}$, and $\sin y = v/e^x = \frac{v}{(u^2+v^2)^{1/2}}$, which gives $\tan y = v/u$. Since $\mathbf{b} = \mathbf{f}(\mathbf{a}) = \mathbf{f}(0, \pi/3) = (1/2, \sqrt{3}/2)$, and therefore, $y = \arctan(v/u)$. Let $\mathbf{g} = (g_1, g_2)$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$. Then

$$g_1(u,v) = \frac{1}{2}\log(u^2+v^2), \qquad g_2(u,v) = \arctan(v/u).$$

Clearly, $\mathbf{g}(\mathbf{b}) = \mathbf{g}(1/2, \sqrt{3}/2) = (0, \pi/3) = \mathbf{a}$.
What's more, since

$$\mathbf{g}'(u,v) = \begin{bmatrix} \frac{u}{u^2+v^2} & \frac{v}{u^2+v^2} \\ -\frac{v}{u^2+v^2} & \frac{u}{u^2+v^2} \end{bmatrix},$$

we have

$$\mathbf{f}'(\mathbf{a}) = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}, \qquad \mathbf{g}'(\mathbf{b}) = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix},$$

and hence $\mathbf{f}'(\mathbf{a}) \cdot \mathbf{g}'(\mathbf{b}) = I$, which gives

$$\mathbf{g}'(\mathbf{b}) = \{\mathbf{f}'(\mathbf{a})\}^{-1} = \{\mathbf{f}'(\mathbf{g}(\mathbf{b}))\}^{-1}.$$

Therefore (52) holds.

(d) What are the images under $\mathbf{f}$ of lines parallel to the coordinate axes?
Solution:
(i) If the lines are parallel to the x-axis, namely, $y = c$, where c is some constant, then $u = e^x \cos c$, $v = e^x \sin c$.
(a) If $c = n\pi + \frac{\pi}{2}$, then $u = 0$, $v = ke^x$, where $k = 1$ or $k = -1$.
(b) If $c = n\pi + \pi$, then $u = ke^x$, where $k = 1$ or $k = -1$, $v = 0$.
(c) Otherwise, $v/u = \tan c$, namely, $v = u \tan c$, $u > 0$ or $u < 0$.
For all the three cases the image under $\mathbf{f}$ is a radial with the unique end point $(0, 0)$ (but not including $(0, 0)$).
(ii) If the lines are parallel to the y-axis, namely, $x = c$, where c is some constant, then $u = e^c \cos y$, $v = e^c \sin y$. Let $k = e^c > 0$, then $u^2 + v^2 = k^2$. Therefore, the image under $\mathbf{f}$ is a circle of R^2 which centers at $(0, 0)$ and with radius k.

18. Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \qquad v = 2xy.$$

(a) The range of $\mathbf{f}$ is R^2.

(b) The Jacobian of $\mathbf{f}$ is

$$\det \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix} = 4(x^2 + y^2).$$

Thus every point of R^2 except $(0, 0)$ has a neighborhood in which $\mathbf{f}$ is one-to-one. Since $\mathbf{f}(x, x) = \mathbf{f}(-x, -x)$, $\mathbf{f}$ is not one-to-one on R^2.

(c) Let $\mathbf{a} = (a_1, a_2) \neq (0, 0)$, $\mathbf{b} = \mathbf{f}(\mathbf{a})$, and let $\mathbf{g}$ be the continuous inverse of $\mathbf{f}$, defined in a neighborhood of $\mathbf{b}$, such that $\mathbf{g}(\mathbf{b}) = \mathbf{a}$. Since $u^2 + v^2 = (x^2 - y^2)^2 + (2xy)^2 = (x^2 + y^2)^2$, we have $x^2 + y^2 = (u^2 + v^2)^{1/2}$. Thus $x^2 = \frac{1}{2}((u^2 + v^2)^{1/2} + u)$, and $y^2 = \frac{1}{2}((u^2 + v^2)^{1/2} - u)$. We then can obtain $\mathbf{g}$ according to the signs of a_1 and a_2.

19. Show that the system of equations

$$3x + y - z + u^2 = 0$$

$$x - y + 2z + u = 0$$

$$2x + 2y - 3z + 2u = 0$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.
Proof: We have

$$\mathbf{f}'(x, y, z, u) = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix},$$

and thus get

$$A_{x,y,u} = \begin{bmatrix} 3 & 1 & 2u \\ 1 & -1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, \quad A_{x,z,u} = \begin{bmatrix} 3 & -1 & 2u \\ 1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix},$$

$$A_{y,z,u} = \begin{bmatrix} 1 & -1 & 2u \\ -1 & 2 & 1 \\ 2 & -3 & 2 \end{bmatrix}, \quad A_{x,y,z} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & -1 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$$

It follows that

$$\det(A_{x,y,u}) = 8u - 12, \quad \det(A_{x,z,u}) = 21 - 14u,$$

$$\det(A_{y,z,u}) = 3 - 2u, \quad \det(A_{x,y,z}) = 0.$$

Since $\mathbf{f}(0,0,0,0) = 0$, and $\det(A_{x,y,u})$, $\det(A_{x,z,u})$, $\det(A_{y,z,u})$ all $\neq 0$ at $(0,0,0,0)$, the desired result then follows from the implicit function theorem (Theorem 9.28).

20. Take $n = m = 1$ in the implicit function theorem, and interpret the theorem (as well as its proof) graphically.
Solution: In the special case $n = m = 1$, the implicit function theorem can be stated as:
Let f be a $\mathscr{C}'$-mapping of an open set $E \subseteq R^2$ into R^1 (i.e., f is a continuously differentiable real function defined on a subset E of R^2), such that $f(a,b) = 0$ for some point $(a,b) \in E$. Assume that $(D_1 f)(a,b) \neq 0$, then there exist open sets $U \subseteq R^2$ and $W \subseteq R^1$, with $(a,b) \in U$ and $b \in W$, having the following property:
To every $y \in W$ corresponds a unique x such that $(x,y) \in U$ and $f(x,y) = 0$.
If this x is defined to be $g(y)$, then g is a $\mathscr{C}'$-mapping of W into R^1, $g(b) = a$, $f(g(y),y) = 0$, $(y \in W)$, and $g'(b) = -\frac{(D_2 f)(a,b)}{(D_1 f)(a,b)}$. The graphical interpretation is then easy to obtain.

21. Define f in R^2 by

$$f(x,y) = 2x^3 - 3x^2 + 2y^3 + 3y^2.$$

(a) Find the four points in R^2 at which the gradient of f is zero. Show that f has exactly one local maximum and one local minimum in R^2.
Solution: The gradient of f is $\nabla f = (6x^2 - 6x, 6y^2 + 6y)$. Let $\nabla f = \mathbf{0}$, then $6x^2 - 6x = 0$, which gives $x = 0$ or $x = 1$, and $6y^2 + 6y = 0$, which gives $y = 0$ or $y = -1$. So we get the four points at which the gradient of f is zero: $(0,0)$, $(0,-1)$, $(1,0)$ and $(1,-1)$. Since $f(0,0) = 0$, $f(0,-1) = 1$, $f(1,0) = -1$ and $f(1,-1) = 0$, we claim that $(0,-1)$ is the unique local maximum of f in R^2 and $(1,0)$ is the unique local minimum of f in R^2. (The uniqueness is clear since if any point $(x_0,y_0) \in R^2$ is a local maximum/minimum of f,

then $\nabla f(x_0, y_0) = \mathbf{0}$.) To see this, let's analyze each point one by one:
(i)The point (0,0): Let (h, k) be any point in a neighborhood of $(0, 0)$, then $f(h, k) = 2h^3 - 3h^2 + 2k^3 + 3k^2$. Let $k = h$, then $f(h, k) = f(h, h) = 4h^3$, which shows that $f(h, k) < 0$ if $h < 0$ and $f(h, k) > 0$ if $h > 0$. Therefore, $(0, 0)$ cannot be a local maximum/minimum of f.
(ii) The point (0,-1): Similarly, let $(h, -1 + k)$ be any point in a neighborhood of $(0, -1)$, then $f(h, -1+k) - 1 = 2h^3 - 3h^2 + 2(-1+k)^3 + 3(-1+k)^2 - 1 = 2h^3 - 3h^2 + 2k^3 - 3k^2 = 2(h^3+k^3) - 3(h^2+k^2)$. Since $h^3+k^3 \leq |h^3+k^3| \leq |h^3| + |k^3| = |h|h^2 + |k|k^2 \leq h^2+k^2$, if $|h|$, $|k|$ is sufficiently small (i.e.,$(h^2+k^2)^{1/2}$ is sufficiently small), we then have $f(h, -1+k) - 1 < 2(h^2+k^2) - 3(h^2+k^2) = -(h^2+k^2) < 0$, which gives $f(h, -1+k) < 1$ when $(h^2+k^2)^{1/2}$ is sufficiently small. This is the same to say that $(0, -1)$ is a local maximum of f.
(iii) The point (1,0): Similarly, let $(1+h, k)$ be any point in a neighborhood of $(1, 0)$, then $f(1+h, k) - (-1) = 2(1+h)^3 - 3(1+h)^2 + 2k^3 + 3k^2 + 1 = 2h^3 + 3h^2 + 2k^3 + 3k^2 = 2(h^3+k^3) + 3(h^2+k^2) > 2(-(h^2+k^2)) + 3(h^2+k^2) = h^2+k^2 > 0$, when $(h^2+k^2)^{1/2}$ is sufficiently small. This gives $f(1+h, k) > -1$ when $(h^2+k^2)^{1/2}$ is sufficiently small and therefore, $(1, 0)$ is a local minimum of f.
(iv) The point (1,-1): Similarly, let $(1+h, -1+k)$ be any point in a neighborhood of $(1, -1)$, then $f(1+h, -1+k) = 2(1+h)^3 - 3(1+h)^2 + 2(-1+k)^3 + 3(-1+k)^2 = 2h^3 + 3h^2 + 2k^3 - 3k^2$. Let $k = h$, then $f(1+h, -1+k) = f(1+h, -1+h) = 4h^3$. Therefore, $(1, -1)$ cannot be a local maximum/minimum of f, due to the same reason as (i).

(b) Let S be the set of all $(x, y) \in R^2$ at which $f(x, y) = 0$. Find those points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x (or for x in terms of y). Describe S as precisely as you can.
Solution: To satisfy the requirement in the hypothesis, we must have $6y^2 + 6y = 0$, which gives $y = 0$ or $y = -1$ and $f(x, y) = 0$. If $y = 0$, then $2x^3 - 3x^2 = 0$, which gives $x = 0$, or $x = 3/2$; If $y = -1$, then $2x^3 - 3x^2 + 1 = 0$, which gives $(x-1)^2(2x+1) = 0$, namely, $x = 1$ or $x = -1/2$. Thus the points of S that have no neighborhoods in which the equation $f(x, y) = 0$ can be solved for y in terms of x are $(0, 0)$, $(3/2, 0)$, $(1, -1)$ and $(-1/2, -1)$.

22. Give a similar discussion for

$$f(x, y) = 2x^3 + 6xy^2 - 3x^2 + 3y^2.$$

Solution: Since $\nabla f = (6x^2 + 6y^2 - 6x, 12xy + 6y)$, $\nabla f = \mathbf{0}$ gives $6x^2 + 6y^2 - 6x = 0$ and $12xy + 6y = 0$. $12xy + 6y = 0$ implies $y = 0$ or $x = -1/2$, and if $y = 0$, $6x^2 + 6y^2 - 6x = 0$ implies $x = 0$ or $x = 1$; if

$x = -1/2$, $6x^2 + 6y^2 - 6x = 0$ implies $6y^2 + 9/2 = 0$, which is impossible. Therefore, we get the two points at which $\nabla f = \mathbf{0}$, namely, $(0,0)$, $(1,0)$, and $f(0,0) = 0$, $f(1,0) = -1$.
(i) Since $f(h,h) = 8h^3$, $(0,0)$ is not a local maximum/minimum of f.
(ii) $f(1+h,k) - (-1) = 2(1+h)^3 + 6(1+h)k^2 - 3(1+h)^2 + 3k^2 + 1 = 2h^3 + 3h^2 + (9+6h)k^2 = (2h+3)(h^2+3k^2) > 0$, if $(h^2+k^2)^{1/2}$ is sufficiently small. Therefore, $(1,0)$ is a local minimum of f.
Suppose $f(x,y) = 0$, and $12xy + 6y = 0$. Then $y = 0$ or $x = -1/2$. If $y = 0$, $f(x,y) = 0$ implies $2x^3 - 3x^2 = 0$, which gives $x = 0$ or $x = 3/2$; If $x = -1/2$, $f(x,y) = 0$ implies $-1 = 0$, which is absurd. So the points of S that have no neighborhoods in which the equation $f(x,y) = 0$ can be solved for y in terms of x are $(0,0)$ and $(3/2,0)$.

23. Define f in R^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0,1,-1) = 0$, $(D_1 f)(0,1,-1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1,-1)$ in R^2, such that $g(1,-1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0.$$

Find $(D_1 g)(1,-1)$ and $(D_2 g)(1,-1)$.
Solution: Clearly, $f(0,1,-1) = 0$. Since $(D_1 f)(x, y_1, y_2) = 2xy_1 + e^x$, $(D_2 f)(x, y_1, y_2) = x^2$, and $(D_3 f)(x, y_1, y_2) = 1$, $(D_1 f)(0,1,-1) = 1 \neq 0$. Therefore, by the implicit function theorem, there exists a differentiable function g in some neighborhood of $(1,-1)$ in R^2, such that $g(1,-1) = 0$ and $f(g(y_1,y_2), y_1, y_2) = 0$. $g'(1,-1) = -(A_x)^{-1} A_y$, where $A_x = (D_1 f)(0,1,-1) = 1$, and $A_y = ((D_2 f)(0,1,-1), (D_3 f)(0,1,-1)) = (1,1)$. Hence, $g'(1,-1) = (-1,-1)$, which gives $(D_1 g)(1,-1) = -1$ and $(D_2 g)(1,-1) = -1$.

24. For $(x,y) \neq (0,0)$, define $\mathbf{f} = (f_1, f_2)$ by

$$f_1(x,y) = \frac{x^2 - y^2}{x^2 + y^2}, \qquad \frac{xy}{x^2 + y^2}.$$

Compute the rank of $\mathbf{f}'(x,y)$, and find the range of $\mathbf{f}$.
Solution: Since

$$\mathbf{f}'(x,y) = \begin{bmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4x^2y}{(x^2+y^2)^2} \\ \frac{y(y^2-x^2)}{(x^2+y^2)^2} & \frac{x(x^2-y^2)}{(x^2+y^2)^2} \end{bmatrix} = \frac{1}{(x^2+y^2)^2} \begin{bmatrix} 4xy^2 & -4x^2y \\ y(y^2-x^2) & x(x^2-y^2) \end{bmatrix},$$

then given any $(x,y) \in R^2$, we have $\mathbf{f}'(x,y)(x,y)^T = (0,0)^T$, which means $\mathscr{R}(\mathbf{f}'(x,y)) = \{(0,0)^T\}$. Therefore, the rank of $\mathbf{f}'(x,y)$ is 0.
Let $u = \frac{x^2-y^2}{x^2+y^2}$, $v = \frac{xy}{x^2+y^2}$, then $u^2 + 4v^2 = 1$, which means the range of $\mathbf{f}$ is an ellipse on R^2.

25. Suppose $A \in L(R^n, R^m)$, let r be the rank of A.

(a) Define S as in the proof of Theorem 9.32. Show that SA is a projection in R^n whose null space is $\mathscr{N}(A)$ and whose range is $\mathscr{R}(S)$.
Proof: Let $Y = \mathscr{R}(A)$, then $\dim Y = r$, which means Y has a base $\{\mathbf{y}_1, ..., \mathbf{y}_r\}$. Choose $\mathbf{z}_i \in R^n$ so that $A\mathbf{z}_i = \mathbf{y}_i (1 \le i \le r)$, and define a linear mapping S of Y into R^n by setting $S(c_i\mathbf{y}_i + \cdots + c_r\mathbf{y}_r) = c_1\mathbf{z}_1 + \cdots + c_r\mathbf{z}_r$ for all scalars $c_1, ..., c_r$.
Let $\mathbf{x} \in R^n$, and $\mathbf{y} = A\mathbf{x} \in Y$. Then $(SA)^2\mathbf{x} = SASA\mathbf{x} = SAS(\mathbf{y}) = S\mathbf{y} = SA\mathbf{x}$, by (68). which means $SA \in L(R^n)$ is a projection in R^n.
(i)Suppose $\mathbf{x} \in \mathscr{N}(A)$, then $A\mathbf{x} = \mathbf{0}$, which gives $SA\mathbf{x} = \mathbf{0}$, and hence $\mathbf{x} \in \mathscr{N}(SA)$.
On the other hand, suppose $\mathbf{x} \in \mathscr{N}(SA)$, then $SA\mathbf{x} = \mathbf{0}$, which gives $A(SA\mathbf{x}) = \mathbf{0}$. By (68), we have $A(SA\mathbf{x}) = AS(A\mathbf{x}) = A\mathbf{x}$, which shows that $A\mathbf{x} = \mathbf{0}$. Therefore, $\mathbf{x} \in \mathscr{N}(A)$.
Thus, $\mathscr{N}(SA) = \mathscr{N}(A)$.
(ii)Suppose $\mathbf{z} \in \mathscr{R}(S)$, then $\mathbf{z} = S\mathbf{y}$, for some $\mathbf{y} \in Y$. There is some $\mathbf{x} \in X$ such that $\mathbf{y} = A\mathbf{x}$, which shows $\mathbf{z} = SA\mathbf{x}$ and therefore, $\mathbf{z} \in \mathscr{R}(SA)$.
On the other hand, suppose $\mathbf{z} \in \mathscr{R}(SA)$, then $\mathbf{z} = SA\mathbf{x}$. Since $A\mathbf{x} \in Y$, we have $\mathbf{z} \in \mathscr{R}(S)$.
Hence, $\mathscr{R}(SA) = \mathscr{R}(S)$.

(b) Use (a) to show that

$$\dim \mathscr{N}(A) + \dim \mathscr{R}(A) = n.$$

Proof:
(i) Since for any $\mathbf{y} \in Y$, we can write $\mathbf{y} = \sum_{i=1}^{r} c_i\mathbf{y}_i$, for some c_i, $1 \le i \le r$. Then $S\mathbf{y} = \sum_{i=1}^{r} c_i\mathbf{z}_i$, that is, $\mathscr{R}(S)$ is spanned by $\mathbf{z}_i$, $1 \le i \le r$. Then $\dim\mathscr{R}(S) \le r$, by Theorem 9.2. Since $\mathscr{R}(SA) = \mathscr{R}(SA)$, by (a), $\dim\mathscr{R}(SA) = \dim\mathscr{R}(S) \le r$. Suppose $k = \dim\mathscr{R}(SA)$, $k \le r$, let $\{\mathbf{u}_1, ..., \mathbf{u}_k\}$ be a basis of $\mathscr{R}(SA)$, then R^n has a basis containing $\mathbf{u}_1, ..., \mathbf{u}_k$, by Theorem 9.3(c). Denote this basis by $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$, then for any $\mathbf{x} \in R^n$, $\mathbf{x}$ can be written as $\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{u}_i$. Since $\sum_{i=1}^{k} x_i\mathbf{u}_i \in \mathscr{R}(SA)$, we have $\sum_{i=k+1}^{n} x_i\mathbf{u}_i \in \mathscr{N}(SA)$, due to the fact that SA is a projection in R^n. In particular, $\mathbf{u}_i \in \mathscr{N}(SA)$, $k+1 \le i \le n$, and since $\mathbf{u}_i \in \mathscr{N}(SA)(k+1 \le i \le n)$ is independent, we must have $\dim\mathscr{N}(SA) \ge n-k \ge n-r$. Since $\mathscr{N}(SA) = \mathscr{N}(A)$, by (a), we hence have $\dim\mathscr{N}(A) \ge n-r$.
(ii) On the other hand, suppose $\dim\mathscr{N}(A) = s$, and let $\{\mathbf{v}_1, ..., \mathbf{v}_s\}$ be a basis of $\mathscr{N}(A)$, then by Theorem 9.3(c), R^n has a basis containing $\mathbf{v}_1, ..., \mathbf{v}_s$. Denote this basis by $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$, then for any $\mathbf{x} \in R^n$, $\mathbf{x}$ can be written as $\mathbf{x} = \sum_{i=1}^{n} x_i\mathbf{v}_i$. Since $\sum_{i=1}^{s} x_i\mathbf{v}_i \in \mathscr{N}(A)$, we then have $A\mathbf{x} = A(\sum_{i=1}^{n} x_i\mathbf{v}_i) = A(\sum_{i=1}^{s} x_i\mathbf{v}_i) + A(\sum_{i=s+1}^{n} x_i\mathbf{v}_i) = A(\sum_{i=s+1}^{n} x_i\mathbf{v}_i) = \sum_{i=s+1}^{n} x_iA\mathbf{v}_i \in \mathscr{R}(A)$. Hence $\{A\mathbf{v}_i\}(s+1 \le i \le n$ spans $\mathscr{R}(A)$, which gives $\dim\mathscr{R}(A) \le n-s$, namely, $r \le n-s$.

Therefore $\dim \mathscr{N}(A) = s \leq n - r$.
Combine (i) and (ii), we conclude that $\dim \mathscr{N}(A) = n - r = n - \dim \mathscr{R}(A)$. It follows that $\dim \mathscr{N}(A) + \dim \mathscr{R}(A) = n$.

26. Show that the existence (and even the continuity) of $D_{12}f$ does not imply the existence of $D_1 f$. For example, let $f(x,y) = g(x)$, where g is nowhere differentiable.
Proof: By Theorem 7.18, there exists a real continuous function on the real line which is nowhere differentiable. This gives the existence of g. Note that $D_2 f = 0$, which gives $D_{12} f = 0$, for any $(x,y) \in R^2$. Clearly, $D_{12} f$ is continuous. But $D_1 f$ does not exist at any x since g is nowhere differentiable.

27. Put $f(0,0) = 0$, and
$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$
if $(x,y) \neq (0,0)$. Prove that

(a) f, $D_1 f$, $D_2 f$ are continuous in R^2;
Proof:
(i) Since $|xy| \leq (x^2 + y^2)/2$,
$$|f(x,y)| = |\frac{xy(x^2 - y^2)}{x^2 + y^2}| = \frac{|xy| \cdot |x^2 - y^2|}{x^2 + y^2} \leq \frac{1}{2}|x^2 - y^2|,$$
and therefore,
$$\lim_{(x,y)\to(0,0)} |f(x,y)| \leq \lim_{(x,y)\to(0,0)} \frac{1}{2}|x^2 - y^2| = 0,$$
which gives that
$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0,0),$$
namely, f is continuous at $(0,0)$. Hence f is continuous in R^2.
(ii) For $(x,y) \neq (0,0)$, we have
$$(D_1 f)(x,y) = \frac{y(x^2 - y^2)}{x^2 + y^2}, \quad (D_2 f)(x,y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2},$$
and
$$(D_1 f)(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0} = \lim_{x\to 0} \frac{f(x,0)}{x} = 0,$$
$$(D_2 f)(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y - 0} = \lim_{y\to 0} \frac{f(0,y)}{y} = 0.$$

Since

$$|(D_1f)(x,y)| = |\frac{y(x^2-y^2)}{x^2+y^2}| = \frac{|y|\cdot|x^2-y^2|}{x^2+y^2} \leq |y|,$$

$$|(D_2f)(x,y)| = |\frac{x(x^4-4x^2y^2-y^4)}{(x^2+y^2)^2}| = \frac{|x|\cdot|x^4-4x^2y^2-y^4|}{(x^2+y^2)^2}$$

$$\leq |x|\left(\frac{|x^4-y^4|}{(x^2+y^2)^2} + \frac{4x^2y^2}{(x^2+y^2)^2}\right) = |x|\left(\frac{|x^2-y^2|}{x^2+y^2} + \frac{4x^2y^2}{(x^2+y^2)^2}\right)$$

$$\leq |x|(1+1) = 2|x|,$$

we have

$$\lim_{(x,y)\to(0,0)} |(D_1f)(x,y)| \leq \lim_{(x,y)\to(0,0)} |y| = 0,$$

$$\lim_{(x,y)\to(0,0)} |(D_2f)(x,y)| \leq \lim_{(x,y)\to(0,0)} 2|x| = 0,$$

which gives

$$\lim_{(x,y)\to(0,0)} (D_1f)(x,y) = 0 = (D_1f)(0,0),$$

$$\lim_{(x,y)\to(0,0)} (D_2f)(x,y) = 0 = (D_2f)(0,0).$$

Therefore, D_1f and D_2f are continuous at $(0,0)$ and thus are continuous in R^2.

(b) $D_{12}f$ and $D_{21}f$ exists at every point of R^2, and are continuous except at $(0,0)$;
Proof: We have

$$(D_{12}f)(0,0) = \lim_{(x,y)\to(0,0)} \frac{(D_2f)(x,0)-(D_2f)(0,0)}{x-0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(D_2f)(x,0)}{x} = \lim_{(x,y)\to(0,0)} \frac{x^4}{x^4} = 1,$$

and

$$(D_{21}f)(0,0) = \lim_{(x,y)\to(0,0)} \frac{(D_1f)(0,y)-(D_1f)(0,0)}{y-0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(D_1f)(0,y)}{y} = \lim_{(x,y)\to(0,0)} \frac{-y^2}{y^2} = -1.$$

For $(x,y) \neq (0,0)$, we have

$$(D_{12}f)(x,y) = \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3},$$

and

$$(D_{21}f)(x,y) = \frac{x^4-4x^2y^2-y^4}{(x^2+y^2)^3}.$$

Then $D_{12}f$ and $D_{21}f$ exist at every point of R^2. Note that we have

$$(D_{12}f)(\frac{1}{n}, \frac{1}{n}) = 0 \neq 1, \qquad (D_{21}f)(\frac{1}{n}, \frac{1}{n^2}) = 1 \neq -1,$$

and therefore, $D_{12}f$ and $D_{21}f$ are not continuous at $(0,0)$. It's clear that $D_{12}f$ and $D_{21}f$ are continuous at every point other than $(0,0)$.

(c) $(D_{12}f)(0,0) = 1$, and $(D_{21}f)(0,0) = -1$.
Proof: This has been showed in (b).

28. For $t \geq 0$, put

$$\varphi(x,t) = \begin{cases} x & (0 \leq x \leq \sqrt{t}) \\ -x + 2\sqrt{t} & (\sqrt{t} \leq x \leq 2\sqrt{t}) \\ 0 & (\text{otherwise}), \end{cases}$$

and put $\varphi(x,t) = -\varphi(x,|t|)$ if $t < 0$.
Show that φ is continuous on R^2, and

$$(D_2\phi)(x,0) = 0$$

for all x. Define

$$f(t) = \int_{-1}^{1} \varphi(x,t)dx.$$

Show that $f(t) = t$ if $|t| < \frac{1}{4}$. Hence

$$f'(0) \neq \int_{-1}^{1} (D_2\varphi)(x,0)dx.$$

Proof:
(i) First we show that φ is continuous on R^2. Note that we only need to prove for the case $t \geq 0$. If $t = 0$, we then have $\varphi(x,t) = 0$, for any x, and clearly it is continuous. Now we assume that $t > 0$. Fix some $(x_0, t_0) \in R^2$.
(a) If $0 < x_0 < \sqrt{t_0}$, then $\varphi(x_0,t_0) = x_0$. If we put $g(x,t) = \sqrt{t} - x$, then g is continuous on R^2 and $g(x_0,t_0) > 0$. Hence there is a neighborhood V_{r_1} of (x_0,t_0) such that $(x,t) \in V_{r_1}$ implies $g(x,t) > 0$. Similarly, if we put $f(x,t) = x$, then f is continuous on R^2 and $f(x_0,t_0) > 0$. Hence there is a neighborhood V_{r_2} of (x_0,t_0) such that $(x,t) \in V_{r_2}$ implies $f(x,t) > 0$. Let $r = \min(r_1, r_2)$, then for any (x,t) in V_r of (x_0,t_0), we have $f(x,t) > 0$ and $g(x,t) > 0$, namely, $0 < x < \sqrt{t}$. Therefore $\varphi(x,t) = x$ and given any $\epsilon > 0$, we can pick $\delta = \min(\epsilon, r)$, then for $(x,t) \in V_\delta$ of (x_0,t_0), we have $|\varphi(x,t) - \varphi(x_0,t_0)| = |x - x_0| < \delta \leq \epsilon$.
(b) If $\sqrt{t_0} < x_0 < 2\sqrt{t_0}$, then similarly as (a), we can pick an $r > 0$, such that $(x,t) \in V_r$ of (x_0,t_0) implies $\sqrt{t} < x < 2\sqrt{t}$. Then given any $\epsilon > 0$, we can pick $\delta = \min(\epsilon/2, \epsilon\sqrt{t_0}/4, r)$, then for any $(x,t) \in V_\delta$ of (x_0,t_0), we have $|\varphi(x,t) - \varphi(x_0,t_0)| = |(-x + 2\sqrt{t}) - (-x_0 + 2\sqrt{t_0})| =$

$|x_0 - x + 2(\sqrt{t} - \sqrt{t_0})| = |x_0 - x + 2\frac{t-t_0}{\sqrt{t}+\sqrt{t_0}}| \leq |x_0 - x| + 2\frac{|t-t_0|}{\sqrt{t}+\sqrt{t_0}} \leq |x_0 - x| + 2\frac{|t-t_0|}{\sqrt{t_0}} < \delta + 2\delta/\sqrt{t_0} \leq \epsilon/2 + \epsilon/2 = \epsilon$.
(c) If $x_0 < 0$ or $x_0 > 2\sqrt{t_0}$, then $\varphi(x_0, t_0) = 0$, and similarly as in (a) and (b), we can pick neighborhood V_r of (x_0, t_0) such that $(x, t) \in V_r$ implies $x < 0$ or $x > 2\sqrt{t}$, and therefore $\varphi(x, t) = 0$.
(d) If $x_0 = 0$, then $\varphi(x_0, t_0) = 0 < \sqrt{t_0}$, we then have $\lim_{(x,t)\to(0,t_0)} \varphi(x, t) = 0$. To see this, note that if $x < 0$, then $\varphi(x, t) = 0$ and thus $\lim_{x<0,(x,t)\to(0,t_0)} \varphi(x, t) = 0$; if $x > 0$, then $0 < x < \sqrt{t_0}$ implies that for any x we can pick a neighborhood of t_0 in which $0 < x < \sqrt{t}$ and therefore $\varphi(x, t) = x$, thus $\lim_{x>0,(x,t)\to(0,t_0)} \varphi(x, t) = 0$. Similarly, if $x_0 = \sqrt{t_0}$, then $\varphi(x_0, t_0) = \sqrt{t_0}$, and we have $\lim_{(x,t)\to(\sqrt{t_0},t_0)} \varphi(x, t) = \sqrt{t_0}$. If $x_0 = 2\sqrt{t_0}$, then $\varphi(x_0, t_0) = 0$, and we have $\lim_{(x,t)\to(2\sqrt{t_0},t_0)} \varphi(x, t) = 0$.
Combine the above cases, we conclude that φ is continuous when $t \geq 0$, and therefore is continuous on R^2.
(ii) We have

$$(D_2\varphi)(x, 0) = \lim_{t\to 0} \frac{\varphi(x, t) - \varphi(x, 0)}{t - 0} = \lim_{t\to 0} \frac{\varphi(x, t)}{t}.$$

If $x \leq 0$, then $\varphi(x, t) = 0$, for all t, and thus $(D_2\varphi)(x, 0) = 0$. If $x > 0$, then $x > 2\sqrt{|t|}$ when $|t|$ is sufficiently small, and therefore, $\varphi(x, t) = 0$, which gives $(D_2\varphi)(x, 0) = 0$. Hence $(D_2\varphi)(x, 0) = 0$, for all x.
(iii) If $|t| < 1/4$, then $\sqrt{|t|} < 1/2$ and therefore $2\sqrt{|t|} < 1$. Then

$$f(t) = \int_{-1}^{1} \varphi(x, t)dx = \int_{0}^{2\sqrt{|t|}} \varphi(x, t)dx$$

If $t \geq 0$, we have

$$f(t) = \int_{0}^{2\sqrt{t}} \varphi(x, t)dx = \int_{0}^{\sqrt{t}} xdx + \int_{\sqrt{t}}^{2\sqrt{t}} (-x + 2\sqrt{t})dx = t,$$

and if $t < 0$, we have

$$f(t) = \int_{0}^{2\sqrt{-t}} \varphi(x, t)dx = \int_{0}^{\sqrt{-t}} (-x)dx + \int_{\sqrt{-t}}^{2\sqrt{-t}} (x - 2\sqrt{-t})dx = t.$$

Thus, $f(t) = t$ if $|t| < 1/4$. Then $f'(0) = 1$. But $\int_{-1}^{1}(D_2\varphi)(x, 0)dx = 0$, which shows $f'(0) \neq \int_{-1}^{1}(D_2\varphi)(x, 0)dx$.

29. Let E be an open set in R^n. The classes $\mathscr{C}'(E)$ and $\mathscr{C}''(E)$ are defined in the text. By induction, $\mathscr{C}^{(k)}(E)$ can be defined as follows, for all positive integers k: To say that $f \in \mathscr{C}^{(k)}(E)$ means that the partial derivatives $D_1 f, ..., D_n f$ belong to $\mathscr{C}^{(k-1)}(E)$.
Assume $f \in \mathscr{C}^{(k)}(E)$, and show (by repeated application of Theorem 9.41) that the kth-order derivative

$$D_{i_1 i_2 \cdots i_k} f = D_{i_1} D_{i_2} ... D_{i_k} f$$

is unchanged if the subscripts i_1,...,i_k are permuted.
For instance, if $n \geq 3$, then

$$D_{1213}f = D_{3112}f$$

for every $f \in \mathscr{C}^{(4)}$.
Proof: It's sufficient to prove that for any permutation $i_1 i_2 \cdots i_k$ of $\{1, 2, ..., k\}$, $D_{i_1 i_2 \cdots i_k} = D_{12 \cdots k}$. Therefore, it's sufficient to prove that for any $p, q \in \{1, 2, ..., k\}$, $D_{i_1 \cdots i_p \cdots i_q \cdots i_k} = D_{i_1 \cdots i_q \cdots i_p \cdots i_k}$. So it's sufficient to prove that $D_{i_1 \cdots i_p i_q \cdots i_k} = D_{i_1 \cdots i_q i_p \cdots i_k}$ for any consecutive i_p, i_q. But this is implied by Theorem 9.41, since $f \in \mathscr{C}^{(k)}(E)$.

30. Let $f \in \mathscr{C}^{(m)}(E)$, where E is an open subset of R^n. Fix $\mathbf{a} \in E$, and suppose $\mathbf{x} \in R^n$ is so close to $\mathbf{0}$ that the points

$$\mathbf{p}(t) = \mathbf{a} + t\mathbf{x}$$

lie in E whenever $0 \leq t \leq 1$. Define

$$h(t) = f(\mathbf{p}(t))$$

for all $t \in R^1$ for which $\mathbf{p}(t) \in E$.

(a) For $1 \leq k \leq m$, show (by repeated application of the chain rule) that

$$\mathbf{h}^{(k)}(t) = \sum (D_{i_1 \cdots i_k} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_k}.$$

The sum extends over all ordered k-tuples $(i_1, ..., i_k)$ in which each i_j is one of the integers $1, ..., n$.
Proof: We prove this by induction on k.
(i) $k = 1$, then

$$h'(t) = f'(\mathbf{p}(t))\mathbf{p}'(t) = f'(\mathbf{p}(t))\mathbf{x} = \sum_{i=1}^{n} (D_i f)(\mathbf{p}(t)) x_i,$$

which is the desired result.
(ii) Suppose the result holds when $k = s$, that is

$$\mathbf{h}^{(s)}(t) = \sum (D_{i_1 \cdots i_s} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_s}.$$

The sum extends over all ordered s-tuples $(i_1, ..., i_s)$ in which each i_j is one of the integers $1, ..., n$. When $k = s + 1$, we have

$$h^{(s+1)}(t) = (h^{(s)}(t))' = [\sum (D_{i_1 \cdots i_s} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_s}]'$$

$$= \sum (D_{i_1 \cdots i_s} f)'(\mathbf{p}(t)) x_{i_1} \cdots x_{i_s} = \sum (\sum_{j=1}^{n} (D_{j i_1 \cdots i_s} f)(\mathbf{p}(t)) x_j) x_{i_1} \cdots x_{i_s}$$

$$= \sum (\sum_{j=1}^{n} (D_{ji_1 \cdots i_s} f)(\mathbf{p}(t)) x_j x_{i_1} \cdots x_{i_s}) = \sum (D_{i_1 \cdots i_{s+1}} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_{s+1}}.$$

The sum extends over all ordered $(s+1)$-tuples $(i_1, ..., i_{s+1})$ in which each i_j is one of the integers $1, ..., n$, which is the desired result.

(b) By Taylor's theorem (5.15),

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

for some $t \in (0,1)$. Use this to prove Taylor's theorem in n variables by showing that the formula

$$f(\mathbf{a} + \mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 \cdots i_k} f)(\mathbf{a}) x_{i_1} \cdots x_{i_k} + r(\mathbf{x})$$

represents $f(\mathbf{a}+\mathbf{x})$ as the sum of its so-called "Taylor polynomial of degree $m-1$," plus a remainder that satisfies

$$\lim_{\mathbf{x} \to \mathbf{0}} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

Each of the inner sums extends over all ordered k-tuples $(i_1, ..., i_k)$, as in part (a); as usual, the zero-order derivative of f is simply f, so that the constant term of the Taylor polynomial of f at $\mathbf{a}$ is $f(\mathbf{a})$.
Proof: By (a), $h(1) = f(\mathbf{p}(1)) = f(\mathbf{a}+\mathbf{x})$, and $h^{(k)}(0) = \sum (D_{i_1 \cdots i_k} f)(\mathbf{p}(0)) x_{i_1} \cdots x_{i_k} = \sum (D_{i_1 \cdots i_k} f)(\mathbf{a}) x_{i_1} \cdots x_{i_k}$, for $1 \le k \le m-1$, $h^{(0)}(0) = h(0) = f(\mathbf{p}(0)) = f(\mathbf{a})$.
$h^{(m)}(t) = \sum (D_{i_1 \cdots i_m} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_m}$, and let

$$r(\mathbf{x}) = \frac{h^{(m)}(t)}{m!} = \frac{\sum (D_{i_1 \cdots i_m} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_m}}{m!},$$

we then have

$$|\frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}}| = \frac{|r(\mathbf{x})|}{|\mathbf{x}|^{m-1}} = \frac{|\sum (D_{i_1 \cdots i_m} f)(\mathbf{p}(t)) x_{i_1} \cdots x_{i_m}|}{m! |\mathbf{x}|^{m-1}}$$

$$\le \frac{\sum |(D_{i_1 \cdots i_m} f)(\mathbf{p}(t))| |x_{i_1}| \cdots |x_{i_m}|}{m! |\mathbf{x}|^{m-1}} \le \frac{M |\mathbf{x}|^m}{m! |\mathbf{x}|^{m-1}} = \frac{M}{m!} |\mathbf{x}|.$$

Hence

$$\lim_{\mathbf{x} \to \mathbf{0}} |\frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}}| \le \lim_{\mathbf{x} \to \mathbf{0}} \frac{M}{m!} |\mathbf{x}| = 0,$$

and therefore,

$$\lim_{\mathbf{x} \to \mathbf{0}} |\frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}}| = 0, \quad i.e., \quad \lim_{\mathbf{x} \to \mathbf{0}} \frac{r(\mathbf{x})}{|\mathbf{x}|^{m-1}} = 0.$$

The formula

$$h(1) = \sum_{k=0}^{m-1} \frac{h^{(k)}(0)}{k!} + \frac{h^{(m)}(t)}{m!}$$

then gives us that

$$f(\mathbf{a}+\mathbf{x}) = \sum_{k=0}^{m-1} \frac{1}{k!} \sum (D_{i_1 \cdots i_k} f)(\mathbf{a}) x_{i_1} \cdots x_{i_k} + r(\mathbf{x}).$$

(c) Exercise 29 shows that repetition occurs in the Taylor polynomial as written in part (b). For instance, D_{113} occurs three times, as D_{113}, D_{131}, D_{311}. The sum of the corresponding three terms can be written in the form

$$3(D_1^2 D_3 f)(\mathbf{a}) x_1^2 x_3.$$

Prove (by calculating how often each derivative occurs) that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{(D_1^{s_1} \cdots D_n^{s_n} f)(\mathbf{a})}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n}.$$

Here the summation extends over all ordered n-tuples $(s_1, ..., s_n)$ such that each s_i is a nonnegative integer, and $s_1 + \cdots + s_n \leq m-1$.
Proof: Fix $s_1, ..., s_n$ and let $k = s_1 + \cdots + s_n$, then the coefficient of $x^{s_1} \cdots x^{s_n}$ in the Taylor polynomial in (b) will be

$$\frac{1}{k!} \cdot (D_1^{s_1} \cdots D_n^{s_n} f)(\mathbf{a}) \cdot \frac{k!}{s_1! \cdots s_n!} = \frac{(D_1^{s_1} \cdots D_n^{s_n} f)(\mathbf{a})}{s_1! \cdots s_n!},$$

which shows that the Taylor polynomial in (b) can be written in the form

$$\sum \frac{(D_1^{s_1} \cdots D_n^{s_n} f)(\mathbf{a})}{s_1! \cdots s_n!} x_1^{s_1} \cdots x_n^{s_n}.$$

31. Suppose $f \in \mathscr{C}^{(3)}$ in some neighborhood of a point $\mathbf{a} \in R^2$, the gradient of f is $\mathbf{0}$ at $\mathbf{a}$, but not all second-order derivatives of f are 0 at $\mathbf{a}$. Show how one can then determine from the Taylor polynomial of f at $\mathbf{a}$ (of degree 2) whether f has a local maximum, or a local minimum, or neither, at the point $\mathbf{a}$.
Extend this to R^n in place of R^2.
Proof: Since the gradient of f is $\mathbf{0}$ at $\mathbf{a}$, we have

$$f(\mathbf{a}+\mathbf{x}) = f(\mathbf{a}) + \frac{1}{2} \sum (D_{i_1 i_2} f)(\mathbf{a}) x_{i_1} x_{i_2} + r(\mathbf{x})$$

$$= f(\mathbf{a}) + \frac{1}{2}(x_1, x_2) \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + r(\mathbf{x}) = f(\mathbf{a}) + \frac{1}{2} \mathbf{x}^T H \mathbf{x} + r(\mathbf{x}),$$

then f has a local maximum at the point $\mathbf{a}$ if H is negatively determinant, and f has a local minimum at the point $\mathbf{a}$ if H is positively determinant.

If we can find $\mathbf{x}$ and $\mathbf{y}$ so that $\mathbf{x}^T H\mathbf{x} > 0$, but $\mathbf{y}^T H\mathbf{y} < 0$, then f has neither a local maximum or local minimum at the point $\mathbf{a}$.
The situation and conclusion in R^n is similar as in R^2, only now

$$H = \begin{bmatrix} D_{11}f & D_{12}f & \cdots & D_{1n}f \\ D_{21}f & D_{22}f & \cdots & D_{2n}f \\ \cdots & \cdots & \cdots & \cdots \\ D_{n1}f & D_{n2}f & \cdots & D_{nn}f \end{bmatrix}.$$

10 Integration of differential forms

1. Let H be a compact convex set in R^k, with nonempty interior. Let $f \in \mathscr{C}(H)$, put $f(\mathbf{x}) = 0$ in the complement of H, and define $\int_H f$ as in Definition 10.3.
Prove that $\int_H f$ is independent of the order in which the k integrations are carried out.
Proof: Since H is a compact set in R^k, H is closed and bounded, by Theorem 2.41. Therefore, H is contained by some I^k. Since $f(\mathbf{x}) = 0$ in the complement of H, we can define $\int_H f = \int_{I^k} f$. But note that f may be discontinuous on I^k.
Now, suppose $0 < \delta < 1$, given any $\mathbf{x} \in H^\circ$, we associate a set $B(\mathbf{x})$ of points which lie in $H' = H - H^\circ$ (namely, the limit points of H) to $\mathbf{x}$ such that $S(\mathbf{x}) = \{\mathbf{y}_i | 1 \leq i \leq k, \mathbf{y}_i \in H', y_{ij} = x_j, \text{for } j \neq i, \text{and } |y_{ii} - x_i| < \delta\}$. In the case that there are two or more $\mathbf{y}_i$ for fixed i, pick the one that gives the minimal $|y_{ii} - x_i|$(If again, two or more points satisfy this condition, pick any one of them). Then it's clear that given any $\mathbf{x} \in H^\circ$, $S(\mathbf{x})$ is uniquely defined, since H is convex. Next, define $d(\mathbf{x}) = \max_{\mathbf{y}_i \in S(\mathbf{x})} |\mathbf{y}_i - \mathbf{x}|$, for $\mathbf{x} \in H^\circ$, and define

$$g(\mathbf{x}) = \begin{cases} 1 & (\mathbf{x} \in H^\circ \text{ and } d(\mathbf{x}) \geq \delta) \\ d(\mathbf{x})/\delta & (\mathbf{x} \in H^\circ \text{ and } d(\mathbf{x}) < \delta) \\ 0 & (\mathbf{x} \notin H^\circ) \end{cases}$$

If we define $F(\mathbf{x}) = g(\mathbf{x}) f(\mathbf{x})$, $\mathbf{x} \in I^k$, then $F \in \mathscr{C}(I^k)$.
Put $\mathbf{y} = (x_1, ..., x_k)$, $\mathbf{x} = (\mathbf{y}, x_k)$. For each $\mathbf{y} \in I^{k-1}$, the set of all x_k such that $F(\mathbf{y}, x_k) \neq f(\mathbf{y}, x_k)$ is either empty or two segments either of whose length does not exceed δ. Since $0 \leq g \leq 1$, it follows that

$$|F_{k-1}(\mathbf{y}) - f_{k-1}(\mathbf{y})| = |\int_{a_k}^{b_k} (F(\mathbf{y}, x_k) - f(\mathbf{y}, x_k)) dx_k| \leq 2\delta ||f||, (\mathbf{y} \in I^{k-1}).$$

As $\delta \to 0$, we have that f_{k-1} is a uniform limit of a sequence of continuous functions. Thus $f_{k-1} \in \mathscr{C}(I^{k-1})$, and the further integrations present no problems.
This proves the existence of the integral $\int_H f$. Moreover, we have that

$$|\int_{I^k} F(\mathbf{x}) d\mathbf{x} - \int_{I^k} f(\mathbf{x}) d\mathbf{x}| \leq \delta ||f||,$$

and this is true, regardless of the order in which the k single integrations are carried out, by our definition of $S(\mathbf{x})$. Since $F \in \mathscr{C}(I^k)$, $\int F$ is unaffected by any change in this order, by Theorem 10.2. Hence the same is true of $\int f$.
This completes the proof.

2. For $i = 1, 2, 3, ...$, let $\varphi_i \in \mathscr{C}(R^1)$ have support in $(2^{-i}, 2^{1-i})$, such that $\int \varphi_i = 1$. Put

$$f(x,y) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)$$

Then f has compact support in R^2, f is continuous except at $(0,0)$, and

$$\int dy \int f(x,y)dx = 0 \quad \text{but} \quad \int dx \int f(x,y)dy = 1.$$

Observe that f is unbounded in every neighborhood of $(0,0)$.
Proof: Since $2^{1-i} \leq 1$, then the support of f must be bounded. By the definition of support, it is closed, and therefore, f has compact support in R^2, by Theorem 2.41.
Clearly, $(0,0)$ is not in the support of f and therefore, $f(0,0) = 0$. On the other hand, given any $x \in R^1$, there is at most one i such that $x \in (2^{-i}, 2^{1-i})$ and therefore, $\varphi_i(x) \neq 0$. Let $y = x$, then

$$f(x,x) = \sum_{i=1}^{\infty} [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(x) = \varphi_i^2(x),$$

for some i if we keep x fixed. Since $\int \varphi_i = 1$, we have that there is some $x_i \in (2^{-i}, 2^{1-i})$ such that $\varphi_i(x_i) > 1$, if i is sufficiently large. Suppose, on the contrary, this is not true, then $\int \varphi_i \leq 1 \cdot (2^{1-i} - 2^{-i}) = 2^{-i} < 1$, if i is sufficiently large, which is a contradictory since $\int \varphi_i = 1$. Note that $x_i \to 0$ if $i \to \infty$, and hence

$$\lim_{i \to \infty} f(x_i, x_i) = \varphi_i^2(x_i) \geq 1 \neq 0 = f(0,0).$$

Thus, f is not continuous at $(0,0)$. Fix any $(x_0, y_0) \in R^2$, $(x_0, y_0) \neq (0,0)$. Suppose $x_0 \in (2^{-j}, 2^{1-j})$ and $y_0 \in (2^{-k}, 2^{1-k})$, then $f(x_0, y_0) = [\varphi_k(x_0) - \varphi_{k+1}(x_0)]\varphi_k(y_0) = c\varphi_j(x_0)\varphi_k(y_0)$, where $c = 1$(if $j = k$), $c = -1$(if $j = k+1$) and $c = 0$(otherwise). Since $\varphi_i \in \mathscr{C}(R^1)$, $|\varphi_j| \leq M$, for some $M > 0$, and given any $\epsilon > 0$, we can get a $\delta > 0$ such that $|y - y_0| < \delta/2$ implies $y \in (2^{-k}, 2^{1-k})$ and $|\varphi_k(y) - \varphi_k(y_0)| < \epsilon/M$; $|x - x_0| < \delta/2$ implies $x \in (2^{-j}, 2^{1-j})$. Then $((x-x_0)^2 + (y-y_0)^2)^{1/2} < \delta$ and $|f(x,y) - f(x_0,y_0)| = |c\varphi_j(x)\varphi_k(y) - c\varphi_j(x_0)\varphi_k(y_0)| \leq |c|M|\varphi_k(y) - \varphi_k(y_0)| < \epsilon$, which means $f(x,y)$ is continuous at (x_0, y_0).
We have

$$\int dy \int f(x,y)dx = \int_{2^{-i}}^{2^{1-i}} dy \int [\varphi_i(x) - \varphi_{i+1}(x)]\varphi_i(y)dx, y \in (2^{-i}, 2^{1-i})$$

$$= \int_{2^{-i}}^{2^{1-i}} \varphi_i(y)dy \int [\varphi_i(x) - \varphi_{i+1}(x)]dx$$

$$= \int_{2^{-i}}^{2^{1-i}} \varphi_i(y)dy(\int \varphi_i(x)dx - \int \varphi_{i+1}(x)dx) = 0,$$

and

$$\int dx \int f(x,y)dy = \int_{1/2}^{1} dx \int \varphi_1(x)\varphi_1(y)dy$$

$$+ \int_{2^{-i}}^{2^{1-i}} dx \int \varphi_i(x)(\varphi_i(y) - \varphi_{i-1}(y))dy(x \in (2^{-i}, 2^{1-i}), i \geq 2)$$

$$= \int_{1/2}^{1} \varphi_1(x)dx \int \varphi_1(y)dy + \int_{2^{-i}}^{2^{1-i}} \varphi_i(x)dx \int (\varphi_i(y) - \varphi_{i-1}(y))dy$$

$$= \int_{1/2}^{1} \varphi_1(x)dx + \int_{2^{-i}}^{2^{1-i}} \varphi_i(x)dx(\int \varphi_i(y)dy - \int \varphi_{i-1}(y)dy) = 1.$$

Note that in the previous statement, actually given any $\epsilon > 0$, there is some $x_i \in (2^{-i}, 2^{1-i})$ such that $\varphi_i(x_i) > \epsilon$, if i is sufficiently large. Suppose that, on the contrary, this is not true, then $\int \varphi_i \leq \epsilon(2^{1-i} - 2^{-i}) = \epsilon 2^{-i} < 1$ if i is sufficiently large, which is contradict to the fact $\int \varphi_i = 1$. Therefore, we must have

$$\lim_{i \to \infty} f(x_i, x_i) = \varphi_i^2(x_i) \geq \epsilon.$$

Since $x_i \to 0$ as $i \to \infty$, this means in every neighborhood of $(0,0)$, we can pick some (x_i, x_i) such that $f(x_i, x_i) > \epsilon$. Since ϵ is arbitrary, this is equivalent to say that f is unbounded in every neighborhood of $(0,0)$.

3. (a) If $\mathbf{F}$ is as in Theorem 10.7, put $A = \mathbf{F}'(\mathbf{0})$, $\mathbf{F}_1(\mathbf{x}) = A^{-1}\mathbf{F}(\mathbf{x})$. Then $\mathbf{F}_1'(\mathbf{0}) = I$. Show that

$$\mathbf{F}_1(\mathbf{x}) = \mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x})$$

in some neighborhood of $\mathbf{0}$, for certain primitive mappings $\mathbf{G}_1$,...,$\mathbf{G}_n$. This gives another version of Theorem 10.7:

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}'(\mathbf{0})\mathbf{G}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1(\mathbf{x}).$$

Proof: The proof process is similar as that of Theorem 10.7. Let $\mathbf{H} = \mathbf{F}_1$, and put $\mathbf{H} = \mathbf{H}_1$. Assume $1 \leq m \leq n-1$, and make the following induction hypothesis (which evidently holds for $m = 1$):
V_m is a neighborhood of $\mathbf{0}$, $\mathbf{H}_m \in \mathscr{C}'(V_m)$, $\mathbf{H}_m(\mathbf{0}) = \mathbf{0}$,
$[\mathbf{H}_m'(\mathbf{0})](m;m) = 1$ and $P_{m-1}\mathbf{H}_m(\mathbf{x}) = P_{m-1}\mathbf{x}$, $(\mathbf{x} \in V_m)$.
We then have

$$\mathbf{H}_m(\mathbf{x}) = P_{m-1}\mathbf{x} + \sum_{i=m}^{n} \alpha_i(\mathbf{x})\mathbf{e}_i,$$

where $\alpha_m, ..., \alpha_n$ are real $\mathscr{C}'$-functions in V_m. Hence

$$\mathbf{H}'_m(\mathbf{0})\mathbf{e}_m = \sum_{i=m}^{n} (D_m \alpha_i)(\mathbf{0})\mathbf{e}_i.$$

Since $[\mathbf{H}'_m(\mathbf{0})](m; m) = 1$, the previous equation gives that $(D_m\alpha_m)(\mathbf{0}) = 1$.
Define

$$\mathbf{G}_m(\mathbf{x}) = \mathbf{x} + [\alpha_m(\mathbf{x}) - x_m]\mathbf{e}_m \qquad (\mathbf{x} \in V_m).$$

Then $\mathbf{G}_m \in \mathscr{C}'(V_m)$, $\mathbf{G}_m$ is primitive, and $\mathbf{G}'_m(\mathbf{0})$ is invertible, since $(D_m\alpha_m)(\mathbf{0}) = 1 \neq 0$.
The inverse function theorem shows therefore that there is an open set U_m, with $\mathbf{0} \in U_m \subseteq V_m$, such that $\mathbf{G}_m$ is a 1-1 mapping of U_m onto a neighborhood V_{m+1} of $\mathbf{0}$, in which $\mathbf{G}_m^{-1}$ is continuously differentiable. Define $\mathbf{H}_{m+1}$ by

$$\mathbf{H}_{m+1}(\mathbf{y}) = \mathbf{H}_m \circ \mathbf{G}_m^{-1}(\mathbf{y}) \qquad (\mathbf{y} \in V_{m+1}).$$

Then $\mathbf{H}_{m+1} \in \mathscr{C}'(V_{m+1})$, $\mathbf{H}_{m+1}(\mathbf{0}) = \mathbf{0}$, and $[\mathbf{H}'_{m+1}(\mathbf{0})](m+1; m+1) = 1$ (by the chain rule). Also, for $\mathbf{x} \in U_m$,

$$P_m\mathbf{H}_{m+1}(\mathbf{G}_m(\mathbf{x})) = P_m\mathbf{H}_m(\mathbf{x}) = P_m[P_{m-1}\mathbf{x} + \alpha_m(\mathbf{x})\mathbf{e}_m + \cdots]$$

$$= P_{m-1}\mathbf{x} + \alpha_m(\mathbf{x})\mathbf{e}_m = P_m\mathbf{G}_m(\mathbf{x})$$

so that

$$P_m\mathbf{H}_{m+1}(\mathbf{y}) = P_m\mathbf{y} \qquad (\mathbf{y} \in V_{m+1}).$$

Our induction hypothesis holds therefore with $m+1$ in place of m.
If we apply this with $m = 1, ..., n-1$, we successively obtain

$$\mathbf{F}_1 = \mathbf{H} = \mathbf{H}_1 = \mathbf{H}_2 \circ \mathbf{G}_1 = \cdots = \mathbf{H}_n \circ \mathbf{G}_{n-1} \circ \cdots \circ \mathbf{G}_1$$

in some neighborhood of $\mathbf{0}$. Since $\mathbf{H}_n(\mathbf{x}) = P_{n-1}\mathbf{x} + \alpha_n(\mathbf{x})\mathbf{e}_n$, $\mathbf{H}_n$ is primitive. This completes the proof.

(b) Prove that the mapping $(x, y) \to (y, x)$ of R^2 onto R^2 is not the composition of any two primitive mappings, in any neighborhood of the origin. (This shows that the flips B_i cannot be omitted from the statement of Theorem 10.7.)

Proof: We have $A = \mathbf{F}'(\mathbf{0}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and by (a), $\mathbf{F}(\mathbf{x}) = A\mathbf{G}_2 \circ \mathbf{G}_1(\mathbf{x})$, for certain primitive mappings $\mathbf{G}_1$ and $\mathbf{G}_2$. Therefore, $\mathbf{F}$ is not the composition of any two primitive mappings, in any neighborhood of the origin.

4. For $(x, y) \in R^2$, define

$$\mathbf{F}(x, y) = (e^x \cos y - 1, e^x \sin y).$$

Prove that $\mathbf{F} = \mathbf{G}_2 \circ \mathbf{G}_1$, where

$$\mathbf{G}_1(x, y) = (e^x \cos y - 1, y)$$

$$\mathbf{G}_2(u, v) = (u, (1 + u) \tan v)$$

are primitive in some neighborhood of $(0, 0)$.
Compute the Jacobians of $\mathbf{G}_1$, $\mathbf{G}_2$, $\mathbf{F}$ at (0,0). Define

$$\mathbf{H}_2(x, y) = (x, e^x \sin y)$$

and find

$$\mathbf{H}_1(u, v) = (h(u, v), v)$$

so that $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$ in some neighborhood of $(0, 0)$.
Solution: $\mathbf{G}_2 \circ \mathbf{G}_1(x, y) = \mathbf{G}_2(\mathbf{G}_1(x, y)) = \mathbf{G}_2(e^x \cos y - 1, y) = (e^x \cos y - 1, e^x \cos y \tan y) = (e^x \cos y - 1, e^x \sin y) = \mathbf{F}(x, y)$.
$\mathbf{J}_{\mathbf{G}_1}(0, 0) = 1$, $\mathbf{J}_{\mathbf{G}_2}(0, 0) = 1$, and $\mathbf{J}_{\mathbf{F}}(0, 0) = \mathbf{J}_{\mathbf{G}_2}(\mathbf{G}_1(0, 0)) \cdot \mathbf{J}_{\mathbf{G}_1}(0, 0) = \mathbf{J}_{\mathbf{G}_2}(0, 0) \cdot \mathbf{J}_{\mathbf{G}_1}(0, 0) = 1$.
Since $\mathbf{F} = \mathbf{H}_1 \circ \mathbf{H}_2$, we have $h(u, v) = e^x \cos y - 1$, $u = x$ and $v = e^x \sin y$. Hence $h(u, v) = \sqrt{e^{2u} - v^2} - 1$.

5. Formulate and prove an analogue of Theorem 10.8, in which K is a compact subset of an arbitrary metric space. (Replace the functions φ_i that occur in the proof of Theorem 10.8 by functions of the type constructed in Exercise 22 of Chap.4)
Proof: We need to prove the following version of Theorem 10.8:
Suppose K is a compact subset of an arbitrary metric space X, and $\{V_\alpha\}$ is an open cover of K. Then there exist functions ψ_1,...,$\psi_s \in \mathscr{C}(X)$ such that

   (a) $0 \leq \psi_i \leq 1$ for $1 \leq i \leq s$;
   (b) each ψ_i has its support in some V_α, and
   (c) $\psi_1(x) + \cdots + \psi_s(x) = 1$ for every $x \in K$.

Here is the proof:
Associate with each $x \in K$ an index $\alpha(x)$ so that $x \in V_{\alpha(x)}$. Then there are open balls $B(x)$ and $W(x)$, centered at x, with

$$\overline{B(x)} \subseteq W(x) \subseteq \overline{W(x)} \subseteq V_{\alpha(x)}.$$

Since K is compact, there are points $x_1, ..., x_s$ in K such that

$$K \subseteq B(x_1) \cup \cdots \cup B(x_s).$$

Define

$$\varphi_i(x) = \frac{\rho_{W^c(x_i)}(x)}{\rho_{W^c(x_i)}(x) + \rho_{\overline{B(x_i)}}(x)}, \quad (x \in X, 1 \leq i \leq s),$$

where $\rho_E(x) = \inf_{z\in E} d(x,z)$ and d is the metric of X. By Exercise 22 of Chap.4, we know that φ_i is a continuous function on X whose range lies in $[0,1]$, that $\varphi(x)=0$ on $W^c(x_i)$ and $\varphi(x)=1$ on $\overline{B(x)}$. Define $\psi_1=\varphi_1$ and

$$\psi_{i+1} = (1-\varphi_1)\cdots(1-\varphi_i)\varphi_{i+1}$$

for $i=1,...,s-1$.
Properties (a) and (b) then are clear. The relation

$$\psi_1+\cdots+\psi_i = 1-(1-\varphi_1)\cdots(1-\varphi_i)$$

is trivial for $i=1$. Suppose it holds for some $i<s$, addition of the above two equations yields the previous equation with $i+1$ in place of i. It follows that

$$\sum_{i=1}^{s}\psi_i(x) = 1-\prod_{i=1}^{s}[1-\varphi_i(x)] \quad (x\in X).$$

If $x\in K$, then $x\in B(x_i)$ for some i, hence $\varphi_i(x)=1$, and the product in the last equation is 0. This proves (c).

6. Strengthen the conclusion of Theorem 10.8 by showing that the functions ψ_i can be made differentiable, and even infinitely differentiable. (Use Exercise 1 of Chap.8 in the construction of the auxiliary functions φ_i.)
Proof: Let the radius of $\overline{B(\mathbf{x}_i)}$ and $\overline{W(\mathbf{x}_i)}$ be r_i and R_i, define φ_i to be

$$\varphi_i(\mathbf{x}) = \begin{cases} 1 & (|\mathbf{x}-\mathbf{x}_i| \le r_i \\ \exp(-\frac{|\mathbf{x}-r_i|^2}{|\mathbf{x}-R_i|^2}) & (r_i < |\mathbf{x}-\mathbf{x}_i| < R_i) \\ 0 & (|\mathbf{x}-\mathbf{x}_i| \ge R_i) \end{cases},$$

then the proof process is the same as that of Theorem 10.8.

7. (a) Show that the simplex Q^k is the smallest convex subset of R^k that contains $\mathbf{0},\mathbf{e}_1,...,\mathbf{e}_k$.
Proof: First we will prove the following statement by induction:
Suppose E is convex, $p_i\in E$, $\lambda_i>0$, and $\sum\lambda_i=1$, then $\sum\lambda_i x_i\in E$.
(i) The case $i=1$ is trivial;
(ii) Suppose the statement holds when $i=n$. Let $i=n+1$, then we have $\sum_{i=1}^{n+1}\lambda_i=1$, and

$$\sum_{i=1}^{n+1}\lambda_i x_i = \sum_{i=1}^{n}\lambda_i x_i + \lambda_{n+1}x_{n+1} = (1-\lambda_{n+1})\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i + \lambda_{n+1}x_{n+1}.$$

Since

$$\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}} = \frac{1}{1-\lambda_{n+1}}\sum_{i=1}^{n}\lambda_i = \frac{1}{1-\lambda_{n+1}}\cdot(1-\lambda_{n+1}) = 1,$$

$\sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i \in E$, by induction hypothesis. Therefore, $\sum_{i=1}^{n+1} \lambda_i x_i \in E$ due to the convexity of E.
Suppose now $E \subseteq R^k$, E is convex and E contains $\mathbf{e}_i$, $0 \le i \le k$, $\mathbf{e}_0 = \mathbf{0}$. For every $\mathbf{x} \in Q^k$, $\mathbf{x} = \sum_{i=0}^{k} \lambda_i \mathbf{e}_i$, $0 \le i \le k$, where $\lambda_i \le 0$, and $\sum_{i=0}^{k} \lambda_i = 1$. Therefore, we must have $\mathbf{x} \in E$, which means $Q^k \subseteq E$.

(b) Show that affine mapping take convex sets to convex sets.
Proof: Suppose E is convex, $\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{0}) + A\mathbf{x}$ is an affine mapping defined on E. For any $\mathbf{y}_1$, $\mathbf{y}_2 \in \mathbf{f}(E)$, there exists $\mathbf{x}_1$, $\mathbf{x}_2 \in E$ such that $\mathbf{f}(\mathbf{x}_1) = \mathbf{y}_1$ and $\mathbf{f}(\mathbf{x}_2) = \mathbf{y}_2$. Suppose $0 < \lambda < 1$, then $\lambda \mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 = \lambda(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_1) + (1-\lambda)(\mathbf{f}(\mathbf{0}) + A\mathbf{x}_2) = \mathbf{f}(\mathbf{0}) + A(\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2)$. Since E is convex, $\lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2 \in E$, and hence $\lambda\mathbf{y}_1 + (1-\lambda)\mathbf{y}_2 \in \mathbf{f}(E)$, which gives that $\mathbf{f}(E)$ is convex.

8. Let H be the parallelogram in R^2 whose vertices are $(1,1), (3,2), (4,5), (2,4)$. Find the affine map T which sends $(0,0)$ to $(1,1)$, $(1,0)$ to $(3,2)$, $(0,1)$ to $(2,4)$. Show that $\mathbf{J}_T = 5$. Use T to convert the integral

$$\alpha = \int_H e^{x-y} dxdy$$

to an integral over I^2 and thus compute α.
Solution:

$$T(x,y) = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Clearly, $\mathbf{J}_T = 5$, and therefore

$$\alpha = 5\int_{I^2} e^{t-2s} dtds = \frac{5}{2}(e-1)(1-e^{-2}).$$

9. Define $(x,y) = T(r,\theta)$ on the rectangle

$$0 \le r \le a, \qquad 0 \le \theta \le 2\pi$$

by the equations

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

Show that T maps this rectangle onto the closed disc D with center at $(0,0)$ and radius a, that T is one-to-one in the interior of the rectangle, and that $J_T(r,\theta) = r$. If $f \in \mathscr{C}(D)$, prove the formula for integration in polar coordinates:

$$\int_D f(x,y)dxdy = \int_0^a \int_0^{2\pi} f(T(r,\theta)) r dr d\theta.$$

Proof: Denote the rectangle $0 \le r \le a$, $0 \le \theta \le 2\pi$ by E. Since $x^2 + y^2 = r^2 \le a^2$, $T(E) \subseteq D$. Let (x_0, y_0) be any point of D, then

$x_0^2+y_0^2 \le a^2$. Pick r, $0 \le r \le a$, and $x_0^2+y_0^2 = r^2$, and let θ be $x_0 = r\cos\theta$, $y_0 = r\sin\theta$, $0 \le \theta \le 2\pi$. Then $(x_0,y_0) = T(r,\theta)$. Hence T maps E onto D.
In the interior of E, we have $0 < r < a$ and $0 < \theta < 2\pi$. Suppose $T(r_1,\theta_1) = (x,y)$ and $T(r_2,\theta_2) = (x,y)$, then $x^2+y^2 = r_1^2 = r_2^2$ and hence $r_1 = r_2 > 0$. Therefore $\cos\theta_1 = \cos\theta_2$ and $\sin\theta_1 = \sin\theta_2$, which gives $\theta_1 = \theta_2$, since $0 < \theta_1,\theta_2 < 2\pi$. Thus T is one-to-one in the interior of the rectangle E, and clearly $J_T(r,\theta) = r$.
Now, let D_0 be the interior of D, minus the interval from $(0,0)$ to $(a,0)$, and let E_0 be the interior of E. Then T maps E_0 one-to-one onto D_0, and $J_T(r,\theta) = r \neq 0$, for all $(r,\theta) \in E_0$. Theorem 10.9 thus applies to continuous functions whose support lies in D_0. To remove this restriction, let's proceed as in Example 10.4.
Define $\varphi(x,y)$ on R^2 as follows:
(i) $\varphi(x,y) = 0$, if $(x,y) \notin D_0$;
(ii) When $(x,y) \in D_0$, since T is one-to-one and onto from E_0 to D_0, there exist an unique $(r,\theta) \in E_0$ such that $T(r,\theta) = (x,y)$. Let $0 < \delta < a/2$.
(ii.a) $\varphi(x,y) = 1$, when $\delta \le r \le a-\delta$ and $\delta \le \theta \le 2\pi-\delta$;
(ii.b) $\varphi(x,y) = \frac{r}{\delta}$, when $0 < r < \delta$ and $\delta \le \theta \le 2\pi-\delta$;
(ii.c) $\varphi(x,y) = \frac{a-r}{\delta}$, when $a-\delta < r < a$ and $\delta \le \theta \le 2\pi-\delta$;
(ii.d) $\varphi(x,y) = \frac{\theta}{\delta}$, when $\delta \le r \le a-\delta$ and $0 < \theta < \delta$;
(ii.e) $\varphi(x,y) = \frac{2\pi-\theta}{\delta}$, when $\delta \le r \le a-\delta$ and $2\pi-\delta < \theta < 2\pi$;
(ii.f) $\varphi(x,y) = \frac{r\theta}{\delta^2}$, when $0 < r < \delta$ and $0 < \theta < \delta$;
(ii.g) $\varphi(x,y) = \frac{r(2\pi-\theta)}{\delta^2}$, when $0 < r < \delta$ and $2\pi-\delta < \theta < 2\pi$;
(ii.i) $\varphi(x,y) = \frac{(a-r)\theta}{\delta^2}$, when $a-\delta < r < a$ and $0 < \theta < \delta$;
(ii.j) $\varphi(x,y) = \frac{(a-r)(2\pi-\theta)}{\delta^2}$, when $a-\delta < r < a$ and $2\pi-\delta < \theta < 2\pi$.
Note that $\varphi(x,y)$ is continuous on R^2 since T is one-to-one and continuous on E_0. As in Example 10.4, define $F(x,y) = f(x,y)\varphi(x,y)$, $(x,y) \in D$, then $F \in \mathscr{C}(D)$. The area of R^2 where F is different from f is $2(\pi\delta^2 + \pi a^2 - \pi(a-\delta)^2 + [\pi a^2 - \pi\delta^2 - (\pi a^2 - \pi(a-\delta)^2)]\cdot\frac{\delta}{2\pi}) = 2(2\pi a\delta + \frac{\delta}{2}\cdot a(a-2\delta)) = \delta a(a-2\delta+4\pi)$. Hence

$$|\int_D F(x,y)dxdy - \int_D f(x,y)dxdy| \le \delta a(a-2\delta+4\pi)||f||.$$

As $\delta \to 0$, we get

$$\int_D f(x,y)dxdy = \lim_{\delta\to 0}\int_D F(x,y)dxdy.$$

Since the support of F lies in $D_0 = T(E_0)$, and T is a 1-1 $\mathscr{C}'$-mapping from E_0 to D_0, $J_T(r,\theta) = r \neq 0$ for all $(r,\theta) \in E_0$, Theorem 10.9 gives that

$$\int_D F(x,y)dxdy = \int_{D_0} F(x,y)dxdy = \int_{E_0} F(T(r,\theta))rdrd\theta,$$

and thus

$$\int_D f(x,y)dxdy = \lim_{\delta\to 0}\int_{E_0} F(T(r,\theta))rdrd\theta = \int_0^a \int_0^{2\pi} f(T(r,\theta))rdrd\theta.$$

10. Let $a \to \infty$ in Exercise 9 and prove that

$$\int_{R^2} f(x,y)dxdy = \int_0^\infty \int_0^{2\pi} f(T(r,\theta))rdrd\theta,$$

for continuous functions f that decrease sufficiently rapidly as $|x|+|y| \to \infty$. (Find a more precise formulation.) Apply this to

$$f(x,y) = \exp(-x^2 - y^2)$$

to derive formula (101) of Chap.8.
Proof: The above statement holds when

$$\lim_{r\to\infty} |f(T(r,\theta))|r^{2+\lambda} = 0, \qquad \lambda > 0$$

Let's see how to prove this. Since

$$\int_{R^2} f(x,y)dxdy = \lim_{a\to\infty}\int_D f(x,y)dxdy$$

$$= \lim_{a\to\infty}\int_0^a \int_0^{2\pi} f(T(r,\theta))rdrd\theta = \int_0^\infty \int_0^{2\pi} f(T(r,\theta))rdrd\theta,$$

if the last limit exists.
To prove the existence of the last limit, it is to say that given any $\epsilon > 0$, there exists an $A > 0$, such that $a' > a > A$ implies

$$|\int_a^{a'} \int_0^{2\pi} f(T(r,\theta))rdrd\theta| < \epsilon.$$

Pick $A_1 > 0$ such that $a > A_1$ implies $|f(T(r,\theta))|r^{2+\lambda} < 1$, namely, $|f(T(r,\theta))| < r^{-2-\lambda}$. Since

$$|\int_a^{a'} \int_0^{2\pi} f(T(r,\theta))rdrd\theta| \le \int_a^{a'} \int_0^{2\pi} |f(T(r,\theta))|rdrd\theta$$

$$< \int_a^{a'} \int_0^{2\pi} r^{-1-\lambda}drd\theta = \frac{2\pi}{\lambda}(a^{-\lambda} - a'^{-\lambda}),$$

for $a' > a > A_1$. Now, pick $A_2 > 0$ such that $a' > a > A_2$ implies $a'^{-\lambda} < a^{-\lambda} < A_2^{\lambda} < \frac{\lambda\epsilon}{4\pi}$. Let $A = \max(A_1, A_2)$, then when $a' > a > A$,

$$|\int_a^{a'} \int_0^{2\pi} f(T(r,\theta))rdrd\theta| < \frac{2\pi}{\lambda}(a^{-\lambda} - a'^{-\lambda})$$

$$< \frac{2\pi}{\lambda}(a^{-\lambda} + a'^{-\lambda}) < \frac{2\pi}{\lambda} \cdot \frac{\lambda\epsilon}{2\pi} = \epsilon.$$

Apply this result to $f(x, y) = \exp(-x^2 - y^2)$ gives that

$$\int_{R^2} \exp(-x^2 - y^2) dxdy = \int_0^\infty \int_0^{2\pi} \exp(-r^2) r dr d\theta = \pi.$$

Therefore,

$$\int_{-\infty}^{\infty} e^{-s^2} ds = \sqrt{\pi}.$$

11. Define $(u, v) = T(s, t)$ on the strip

$$0 < s < \infty, \qquad 0 < t < 1$$

by setting $u = s - st$, $v = st$. Show that T is a 1-1 mapping of the strip onto the positive quadrant Q in R^2. Show that $J_T(s, t) = s$.
For $x > 0$, $y > 0$, integrate

$$u^{s-1} e^{-u} v^{y-1} e^{-v}$$

over Q, use Theorem 10.9 to convert the integral to one over the strip, and derive formula (96) of Chap.8 in this way.
(For this application, Theorem 10.9 has to be extended so as to cover certain improper integrals. Provide this extension.)
Proof: For any $(u, v) \in Q$, we can pick (s, t) from the strip such that $s = u+v$ and $t = \frac{v}{u+v}$. Thus T is onto. Suppose (s_1, t_1), (s_2, t_2) are in the trip, and $T(s_1, t_1) = T(s_2, t_2)$, then $s_1 - s_1t_1 = s_2 - s_2t_2$ and $s_1t_1 = s_2t_2$. Summing up these two equations gives $s_1 = s_2 > 0$, and by the second equation we have $t_1 = t_2$. Hence T is one-to-one. Clearly, $J_T(s, t) = s$.
The extension of Theorem 10.9 here should be:
Suppose T is a 1-1 $\mathscr{C}'$-mapping of an open set $E \subseteq R^k$ into R^k such that $J_T(\mathbf{x}) \neq 0$ for all $\mathbf{x} \in E$. If f is a continuous function on R^k whose support lies in $T(E)$, then

$$\int_{R^k} f(\mathbf{y}) d\mathbf{y} = \int_{R^k} f(T(\mathbf{x})) |J_T(\mathbf{x})| d\mathbf{x}.$$

Now, we have

$$\int_Q u^{x-1} e^{-u} v^{y-1} e^{-v} dudv = \int_0^\infty \int_0^1 (s - st)^{x-1} e^{-(s-st)} (st)^{y-1} e^{-st} s ds dt$$

$$= \int_0^\infty s^{x+y-1} e^{-s} ds \int_0^1 t^{y-1} (1-t)^{x-1} dt = \Gamma(x+y) \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

namely,

$$\int_0^\infty u^{x-1} e^{-u} du \int_0^\infty v^{y-1} e^{-v} dv = \Gamma(x+y) \int_0^1 t^{x-1} (1-t)^{y-1} dt,$$

namely,

$$\Gamma(x)\Gamma(y) = \Gamma(x+y)\int_0^1 t^{x-1}(1-t)^{y-1}dt,$$

and therefore,

$$\int_0^1 t^{x-1}(1-t)^{y-1}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma x + y}.$$

12. Let I^2 be the set of all $\mathbf{u} = (u_1, ..., u_k) \in R^k$ with $0 \leq u_i \leq 1$ for all i; let Q^k be the set of all $\mathbf{x} = (x_1, ..., x_k) \in R^k$ with $x_i \geq 0$, $\sum x_i \leq 1$. (I^k is the unit cube; Q^k is the standard simplex in R^k.) Define $\mathbf{x} = T(\mathbf{u})$ by

$$x_1 = u_1, x_2 = (1-u_1)u_2, ..., x_k = (1-u_1)\cdots(1-u_{k-1})u_k.$$

Show that

$$\sum_{i=1}^k x_i = 1 - \prod_{i=1}^k (1-u_i).$$

Show that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S is defined in the interior of Q^k by $u_1 = x_1$ and

$$u_i = \frac{x_i}{1 - x_1 - \cdots - x_{i-1}}$$

for $i = 2, ..., k$. Show that

$$J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u^{k-1}),$$

and

$$J_S(\mathbf{x}) = [(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})]^{-1}.$$

Proof: We prove the first equation by induction.
(i) The equation holds when $k = 1$ trivially;
(ii) Suppose the equation holds when $k = n$, that is,

$$\sum_{i=1}^n x_i = 1 - \prod_{i=1}^n (1-u_i).$$

Let $k = n+1$, then

$$\sum_{i=1}^{n+1} x_i = \sum_{i=1}^n x_i + x_{n+1} = 1 - \prod_{i=1}^n (1-u_i)$$

$$+(1-u_1)\cdots(1-u_n)u_{n+1} = 1 - \prod_{i=1}^{n+1}(1-u_i).$$

The fact that T maps I^k onto Q^k, that T is 1-1 in the interior of I^k, and that its inverse S so defined can be checked easily.
Since $T'(\mathbf{u})$ is a lower-left triangular matrix,

$$J_T(\mathbf{u}) = (1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1}).$$

Note that $J_S(\mathbf{x}) = [J_T(\mathbf{u})]^{-1}$, and $1-u_1 = 1-x_1$, $1-u_i = \frac{1-x_1-\cdots-x_i}{1-x_1-\cdots-x_{i-1}}$.
Substitute these into $J_T(\mathbf{u})$, we can get

$$J_S(\mathbf{x}) = [(1-x_1)(1-x_1-x_2)\cdots(1-x_1-\cdots-x_{k-1})]^{-1}.$$

13. Let $r_1,...,r_k$ be nonnegative integers, and prove that

$$\int_{Q^k} x_1^{r_1}\cdots x_k^{r_k}d\mathbf{x} = \frac{r_1!\cdots r_k!}{(k+r_1+\cdots+r_k)!}.$$

Proof:

$$\int_{Q^k} x_1^{r_1}\cdots x_k^{r_k}d\mathbf{x} = \int_{I^k} u_1^{r_1}\cdots[(1-u_1)\cdots(1-u_{k-1})u_k]^{r_k}$$

$$(1-u_1)^{k-1}(1-u_2)^{k-2}\cdots(1-u_{k-1})d\mathbf{u}$$

$$= \int_{I^k} u_1^{r_1}\cdots u_k^{r_k}(1-u_1)^{r_2+\cdots+r_k+(k-1)}\cdots(1-u_{k-1})^{r_k+1}d\mathbf{u}$$

$$= (\int_0^1 u_1^{(r_1+1)-1}(1-u_1)^{r_2+\cdots+r_k+(k-1)}du_1)\cdots(\int_0^1 u_{k-1}^{(r_{k-1}+1)-1}$$

$$(1-u_{k-1})^{(r_k+2)-1}du_{k-1})\cdot(\int_0^1 u_k^{r_k}du_k) = \frac{\Gamma(r_1+1)\Gamma(r_2+\cdots+r_k+k)}{\Gamma(r_1+\cdots+r_k+(k+1))}$$

$$\cdot\frac{\Gamma(r_2+1)\Gamma(r_3+\cdots+r_k+(k-1))}{\Gamma(r_2+\cdots+r_k+k)}\cdots\frac{\Gamma(r_{k-1}+1)\Gamma(r_k+2)}{\Gamma(r_{k-1}+r_k+3)}\cdot\frac{1}{r_k+1}$$

$$= \frac{\Gamma(r_1+1)\cdots\Gamma(r_{k-1}+1)\Gamma(r_k+2)}{\Gamma(r_1+\cdots+r_k+(k+1))(r_k+1)} = \frac{r_1!\cdots r_{k-1}!(r_k+1)!}{(k+r_1+\cdots+r_k)!(r_k+1)}$$

$$= \frac{r_1!\cdots r_{k-1}!r_k!}{(k+r_1+\cdots+r_k)!}.$$

Note that the special case $r_1=\cdots=r_k=0$ shows that the volume of Q^k is $1/k!$.

14. Prove formula (46).
Proof: $s(j_1,...,j_k) = \prod_{p<q}\operatorname{sgn}(j_q-j_p) = (-1)^{N_{j_1}+\cdots+N_{j_k}}$, where N_{j_p} is the number of those j_q such that $q<p$ but $j_q>j_p$. Then it's clear that $\varepsilon(j_1,...,j_k) = s(j_1,...,j_k)$ due to the meaning of $\varepsilon(j_1,...,j_k)$.

15. If ω and λ are k- and m-forms, respectively, prove that

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

Proof: Because of (57), the result follows if it is proved for the special case

$$\omega = f dx_I, \quad \lambda = g dx_J,$$

where $f, g \in \mathscr{C}(E)$, dx_I is a basic k-form, and dx_J is a basic m-form. Then

$$\omega \wedge \lambda = fg dx_I \wedge dx_J, \qquad \lambda \wedge \omega = fg dx_J \wedge dx_I.$$

Since

$$dx_I \wedge dx_J = dx_{i_1} \wedge \cdots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \cdots \wedge dx_{j_m},$$

and

$$dx_J \wedge dx_I = dx_{j_1} \wedge \cdots \wedge dx_{j_m} \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k},$$

it's clear that

$$dx_I \wedge dx_J = (-1)^{km} dx_J \wedge dx_I$$

and therefore,

$$\omega \wedge \lambda = (-1)^{km} \lambda \wedge \omega.$$

16. If $k \le 2$ and $\sigma = [\mathbf{p}_0, \mathbf{p}_1, ..., \mathbf{p}_k]$ is an oriented affine k-simplex, prove that $\partial^2 \sigma = 0$, directly from the definition of the boundary operator ∂. Deduce from this that $\partial^2 \Psi = 0$ for every chain Ψ.
Proof: By 10.29,

$$\partial \sigma = \sum_{j=0}^{k} (-1)^j [\mathbf{p}_0, ..., \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, ..., \mathbf{p}_k].$$

Now, if $i < j$, let σ_{ij} be the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ and $\mathbf{p}_j$ from σ. We will show that each σ_{ij} occurs twice in $\partial^2 \sigma$, with opposite sign.
The $(k-1)$-simplex obtained by deleting $\mathbf{p}_i$ is:

$$\sigma_i = (-1)^i [\mathbf{p}_0, ..., \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, ..., \mathbf{p}_k],$$

and the $(k-1)$-simplex obtained by deleting $\mathbf{p}_j$ is:

$$\sigma_j = (-1)^j [\mathbf{p}_0, ..., \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, ..., \mathbf{p}_k].$$

Then the $(k-2)$-simplex obtained by deleting $\mathbf{p}_j$ from σ_i is:

$$\sigma_{ij} = (-1)^i (-1)^{j-1} [\mathbf{p}_0, ..., \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, ..., \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, ..., \mathbf{p}_k],$$

and the $(k-2)$-simplex obtained by deleting $\mathbf{p}_i$ from σ_j is:

$$\sigma_{ji} = (-1)^j (-1)^i [\mathbf{p}_0, ..., \mathbf{p}_{i-1}, \mathbf{p}_{i+1}, ..., \mathbf{p}_{j-1}, \mathbf{p}_{j+1}, ..., \mathbf{p}_k].$$

Clearly, σ_{ij} and σ_{ji} have opposite signs. Thus $\partial^2 \sigma = 0$.
Suppose $\Psi = \sum \Phi_i$, and $\Phi_i = T\sigma_i$, then $\partial \Psi = \sum \partial \Phi_i = \sum T(\partial \sigma_i)$, and therefore, $\partial^2 \Psi = \sum \partial^2 \Phi_i = \sum T(\partial^2 \sigma_i) = 0$.

17. Put $J^2 = \tau_1 + \tau_2$, where

$$\tau_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2], \qquad \tau_2 = -[\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1].$$

Explain why it is reasonable to call J^2 the positively oriented unit square in R^2. Show that ∂J^2 is the sum of 4 oriented affine 1-simplexes. Find these. What is $\partial(\tau_1 - \tau_2)$?
Proof: Clearly, both τ_1 and τ_2 have Jacobian 1¿0. Since

$$\partial\tau_1 = [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1],$$

$$\partial\tau_2 = -[\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2],$$

we have

$$\partial J^2 = \partial\tau_1 + \partial\tau_2 = [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2] + [\mathbf{e}_2, \mathbf{0}],$$

which is the same as ∂I^2. So it is reasonable to call J^2 the positively oriented unit square in R^2.
Since

$$\tau_1 - \tau_2 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1],$$

$$\partial(\tau_1 - \tau_2) = [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2]$$

$$= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{e}_1 + \mathbf{e}_2, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_1] + [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{e}_2 + \mathbf{e}_1, \mathbf{0}] + [\mathbf{0}, \mathbf{e}_2].$$

18. Consider the oriented affine 3-simplex

$$\sigma_1 = [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3]$$

in R^3. Show that σ_1 (regarded as a linear transformation) has determinant 1. Thus σ_1 is positively oriented.
Let $\sigma_2, ..., \sigma_6$ be five other oriented 3-simplexes, obtained as follows: There are five permutations (i_1, i_2, i_3) of $(1, 2, 3)$, distinct from $(1, 2, 3)$. Associate with each (i_1, i_2, i_3) the simplex

$$s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$$

where s is the sign that occurs in the definition of the determinant. (This is how τ_2 was obtained from τ_1 in Exercise 17.)
Show that $\sigma_2, ..., \sigma_6$ are positively oriented.
Put $J^3 = \sigma_1 + \cdots + \sigma_6$. Then J^3 may be called the positively oriented unit cube in R^3.
Show that ∂J^3 is the sum of 12 oriented affine 2-simplexes. (These 12 triangles cover the surface of the unit cube I^3.)
Show that $\mathbf{x} = (x_1, x_2, x_3)$ is in the range of σ_1 if and only if $0 \le x_3 \le x_2 \le x_1 \le 1$.
Show that the range of $\sigma_1, ..., \sigma_6$ have disjoint interiors, and that their

union covers I^3. (Compared with Exercise 13; note that 3!=6.)
Proof: Since $A\mathbf{e}_i = \mathbf{p}_i - \mathbf{p}_0$ for $1 \le i \le k$, we have

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

which gives $\det(A) = 1$. Thus σ_1 is positively oriented.
Let $A(i_1, i_2, i_3)$ be the corresponding linear transformation of $\sigma(i_1, i_2, i_3) = [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$. Define matrix $P(i_1, i_2, i_3)$ such that $P(k, i_k) = 1$ for $k = 1, 2, 3$ and $P(k, j) = 0$ otherwise. Then $P(i_1, i_2, i_3)A = A(i_1, i_2, i_3)$, and therefore, $\det[A(i_1, i_2, i_3)] = \det[A]\det[P(i_1, i_2, i_3)] = \det[P(i_1, i_2, i_3)]$. Note that $\det[P(i_1, i_2, i_3)] = s(i_1, i_2, i_3)a(1, i_1)a(2, i_2)a(3, i_3) = s(i_1, i_2, i_3)$, by Definition 9.33. Hence $\det[A(i_1, i_2, i_3)] = s(i_1, i_2, i_3)$. If we let $B(i_1, i_2, i_3)$ denote the corresponding linear transformation of $s(i_1, i_2, i_3)[\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}]$, then $\det[B(i_1, i_2, i_3)] = s(i_1, i_2, i_3)\det[A(i_1, i_2, i_3)] = (s(i_1, i_2, i_3))^2 = 1 > 0$. So $\sigma_2, ..., \sigma_6$ are positively oriented.
Note that $\partial\sigma(i_1, i_2, i_3) = s(i_1, i_2, i_3)\{[\mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] - [\mathbf{0}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] + [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2} + \mathbf{e}_{i_3}] - [\mathbf{0}, \mathbf{e}_{i_1}, \mathbf{e}_{i_1} + \mathbf{e}_{i_2}]\}$.
Thus, $\partial\sigma(1, 2, 3) = [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2]$,
$\partial\sigma(1, 3, 2) = -[\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3]$,
$\partial\sigma(2, 1, 3) = -[\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_1 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1]$,
$\partial\sigma(2, 3, 1) = [\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3]$,
$\partial\sigma(3, 2, 1) = -[\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_3 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2]$,
$\partial\sigma(3, 1, 2) = [\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1, \mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_3 + \mathbf{e}_1, \mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1]$, and thus

$$\begin{aligned} \partial J^3 = \sum_{i=1}^{6} \partial\sigma_i &= [\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3] - [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2] \\ &-[\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3, \mathbf{e}_1 + \mathbf{e}_3 + \mathbf{e}_2] + [\mathbf{0}, \mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_3] \\ &-[\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1, \mathbf{e}_2 + \mathbf{e}_1 + \mathbf{e}_3] + [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_1] \\ &[\mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3, \mathbf{e}_2 + \mathbf{e}_3 + \mathbf{e}_1] - [\mathbf{0}, \mathbf{e}_2, \mathbf{e}_2 + \mathbf{e}_3] \\ &-[\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2, \mathbf{e}_3 + \mathbf{e}_2 + \mathbf{e}_1] + [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_2] \\ &[\mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1, \mathbf{e}_3 + \mathbf{e}_1 + \mathbf{e}_2] - [\mathbf{0}, \mathbf{e}_3, \mathbf{e}_3 + \mathbf{e}_1]. \end{aligned}$$

$\Rightarrow$: If $0 \le x_3 \le x_2 \le x_1 \le 1$, let $u_3 = x_3$, $u_2 = x_2 - x_3$, $u_1 = x_1 - x_2$, then $u_i \ge 0$, $i = 1, 2, 3$, and $\sum u_i = x_1 \le 1$. Therefore, $\mathbf{u} = (u_1, u_2, u_3) \in Q^3$, and $\sigma_1(\mathbf{u}) = A\mathbf{u} = (u_1 + u_2 + u_3, u_2 + u_3, u_3) = (x_1, x_2, x_3) = \mathbf{x}$. Hence $\mathbf{x}$

is in the range of σ_1.
$\Leftarrow$: If $\mathbf{x}$ is in the range of σ_1, then there exists $\mathbf{u} \in Q^3$, such that $\sigma_1(\mathbf{u}) = A\mathbf{u} = \mathbf{x}$. That is $x_1 = u_1 + u_2 + u_3$, $x_2 = u_2 + u_3$, and $x_3 = u_3$, $u_i \leq 0$, $i = 1, 2, 3$, $\sum u_i \leq 1$. Therefore, $0 \leq x_3 \leq x_2 \leq x_1 \leq 1$.
Similarly as the above statements, we can prove that $\mathbf{x}$ is in the range of $\sigma(i_1, i_2, i_3)$ if and only if $0 \leq x_{i_3} \leq x_{i_2} \leq x_{i_1} \leq 1$. Then it's clear that the ranges of $\sigma_1, ..., \sigma_6$ have disjoint interiors. Since every σ_i has the same volume as Q^3 (since every σ_i has Jacobian 1), namely $1/3! = 1/6$, and thus 6 of them gives a total volume of 1, which is exactly the volume of I^3. Note that the range of every σ_i lies in I^3, and since they have disjoint interiors, we can conclude that their union covers I^3.

19. Let J^2 and J^3 be as in Exercise 17 and 18. Define

$$B_{01}(u,v) = (0,u,v), \qquad B_{11}(u,v) = (1,u,v),$$

$$B_{02}(u,v) = (u,0,v), \qquad B_{12}(u,v) = (u,1,v),$$

$$B_{03}(u,v) = (u,v,0), \qquad B_{13}(u,v) = (u,v,1).$$

These are affine, and map R^2 into R^3.
Put $\beta_{ri} = B_{ri}(J^2)$, for $r = 0, 1$, $i = 1, 2, 3$. Each β_{ri} is an affine-oriented 2-chain. (See Sec. 10.30.) Verify that

$$\partial J^3 = \sum_{i=1}^{3} (-1)^i (\beta_{0i} - \beta_{1i}),$$

in agreement with Exercise 18.

20. State conditions under which the formula

$$\int_\Phi f d\omega = \int_{\partial\Phi} fw - \int_\Phi (df) \wedge \omega$$

is valid, and show that it generalizes the formula for integration by parts.