Analysis

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Contents

1 Basis

1.1 Sequence Definitions

Sequences $\{x_k\}_{k=1,\ldots}$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 1 (Convergence: note $x_k \to x, \lim_{k\to\infty} x_k = x$)**.** *Given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x\| < \varepsilon \quad \forall k \geqslant N_\varepsilon$$

Definition 2 (Cauchy Sequence)**.** $\{x_k\}$ *is Cauchy if given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x_m\| < \varepsilon, \ \forall k, m \geqslant N_\varepsilon.$$

Note:

$$\{x_k\} \text{ converges } \Longleftrightarrow \{x_k\} \text{ is Cauchy}$$

Definition 3 (Subsequence)**.** *Infinite subset of* $\{x_k\}$*:* $\{x_k : k \in \mathcal{K}\}$ *or* $\{x_k\}_\mathcal{K}$*, where* $\mathcal{K}$ *is subset of* $\mathbb{Z}^+$*.*

Definition 4 (Limit point)**.** x *is a limit point of* $\{x_k\}$ *if* $\exists$ *a subsequence of* $\{x_k\}$ *that converges to* x*.*

Definition 5 (Bounded Sequence)**.**

$$\|x_k\| \leqslant b, \forall k$$

Results about Bounded sequences:
1. Every bounded has at least one limit point.
2. A bounded sequence converges iff it has a **unique limit point**.

1.2 Scalar Sequences

<u>**Scalar sequences**</u> $\{x_k\}, x_k \in \mathbb{R}$:

Proposition 1. *If* $\{x_k\}$ *is bounded above(below) and non-decreasing(non-increasing) it* ***converges****.*

Proposition 2. *The largest(smallest) limit point of* $\{x_k\}$ *is* $\lim_{k\to\infty} \sup x_k$ *(*$\lim_{k\to\infty} \inf x_k$*)*

Proposition 3. $\{x_k\}$ *converges* $\Longleftrightarrow -\infty < \lim_{k\to\infty} \inf x_k = \lim_{k\to\infty} \sup x_k < \infty$

1.3 Functions Basis

Definition 6 (Continunity)**.** *A real-valued function* f *is continuous at* x *if*
for every $\{x_k\}$ *converging to* x *satisfies that* $\lim_{k\to\infty} f(x_k) = f(x)$*.*
Equivalent: given $\varepsilon > 0, \exists \delta > 0$ *s.t.* $|f(x) - f(y)| < \varepsilon \quad \forall \|y - x\| < \delta$
f *is continuous if it is continuous at all points* x*.*

Definition 7 (Coercive)**.** *A real-valued function* $f : \& \to \mathbb{R}$ *is* <u>*coercive*</u> *if for* ***every*** $\{x_k\} \subset \&$ *s.t.* $\|x_k\| \to \infty, f(x_k) \to \infty$

Example 1 (Check coercive)**.**

1) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2$ - coercive
2) $x \in \mathbb{R}, f(x) = 1 - e^{-|x|}$ - not coercive
3) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (we need $f(x_k) \to \infty$ for all $\|x_k\| \to \infty$)

1.4 Sets

Definition 8 (Open Sets). *A set* $\& \subseteq \mathbb{R}^n$ *is open if*
$\forall x \in \&$ *we can draw a ball around* x *that is contained in* $\&$.
i.e. $\forall x \in \&, \exists \varepsilon > 0$ *s.t.* $\{y : \|y-x\| < \varepsilon\} \subseteq \&$

Definition 9 (Closed Sets). $\&$ *is closed if* $\&^c$ *is open*
Equivalent: if $\&$ *contains all limit points of all sequences in* $\&$

Example 2 (Closed and Open Sets).

1) $(1,2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open

2) $\mathbb{R}$ is both open and closed

3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open

4) $[1, \infty)$ is closed because its complement open

5) $(1,2]$ is neither open nor closed

Definition 10 (Bounded Set). *A is bounded if* $\exists M$ *s.t.* $\|x\| \leqslant M \quad \forall x \in \&$

Definition 11 (Compact Set). $\mathcal{L} \subseteq \mathbb{R}^n$ *is compact of it is closed and bounded.*

Example 3 (Compact Set). $[1,2] = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\}$; $\left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 4\right\}$

Definition 12 (Extreme of sets of scalars, $\sup A, \inf A$). *Let* $A \subset \mathbb{R}$.
- The infimum of A*, or inf* A *is largest* y *s.t.* $y \leqslant x, \forall x \in A$. *If no such* y *exists,* $\inf A = -\infty$
- Similar definition for supremum of A *(or wrote as* $\sup A$*).*

Proposition 4. *If* $\inf A (\sup A) = x^* \in A$, *then* $x^* = \min A (\max A)$

2 Functions

2.1 Extreme of Functions

Definition 13 (Extreme of Functions). *Let* $\& \subseteq \mathbb{R}^n, f : \& \to \mathbb{R}$

$$\inf_{x \in \&} f(x) = \inf\{f(x) : x \in \&\}$$

If $\exists x^* \in \&$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \&} f(x)$
x^* is called a **minimizer** of f, written as $x^* \in \arg\min_{x \in \&} f(x)$. If x^* is uniqne, we write $x^* = \arg\min_{x \in \&} f(x)$
Similarly, supremum and maximum of f.

2.1.1 Weierstrass' Theorem(Extreme value Theorem)

Theorem 1 (Weierstrass' Theorem(Extreme value Theorem)).
If f *is a* ***continuous*** *function on a* ***compact set***, $\& \subseteq \mathbb{R}^n$, *then* f *attains its min and max on* $\&$ *i.e.,*

$$\exists x_1 \in \& \text{ s.t. } f(x_1) = \inf_{x \in \&} f(x)$$
$$\exists x_2 \in \& \text{ s.t. } f(x_2) = \sup_{x \in \&} f(x)$$

Proof. (for existence of min; max is similar)
Let $\{\sigma_k\} \subseteq \&$ be s.t.

$$\inf_{x \in \&} f(x) \leq f(\sigma_k) \leq \inf_{x \in \&} f(x) + \frac{1}{k}$$

Then $\lim_{k\to\infty} f(\sigma_k) = \inf_{x\in\&} f(x)$
$\mathcal{L}$ is bounded $\Rightarrow \{\sigma_k\}$ has it least one limit point x,
$\mathcal{L}$ is closed $\Rightarrow x_1 \in \&$
f is continuous $\Rightarrow f(x_1) = \lim_{k\to\infty} f(\sigma_k) = \inf_{x\in\&} f(x)$ □

Corollary 1 (Corollary to WT)**.** *Let f be continuous on closed set $\&$ (not necessarily bounded). If f is coercive on $\&$ it attains its* min *on $\&$.*

Proof. Consider $\{\sigma_k\}$ as in proof of WT.
Since f is closed, $f(x) < \infty$, $\forall x \in \&$. And f is coercive on $\&$, which means $f(x) \to \infty$ if $\|x\| \to \infty$.
Hence, $\{\sigma_k\} \in \&$ is bounded. Rest of proof same as proof of WT. □

Example 4. $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) *Does f achieve its* min *and* max *on* $\mathcal{L}_1 = \left\{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leqslant 6\right\}$*?*
   - $\mathcal{L}_1$ is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over $\mathbb{R}^3$?
   - $f \to \infty$ whenever $\|x\| \to \infty \Rightarrow f$ is coercive.
   - $\mathbb{R}^3$ is closed.
   - $\Rightarrow f$ achieves its min. on $\mathbb{R}^3$ by corollary to WT.
   - max does not exist since $f \to \infty$ as $\|x\| \to \infty$.

3) Does f achieve its min and max over $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?
   - $\mathcal{L}_2$ is closed, but not bounded.
   - Since f is coercive, min achieved.
   - max does not exist since setting $x_1 = 0$ $x_2 = 3 - x_3$ and letting $x_3 \to \infty$ makes $f \to \infty$

3 Big $\mathcal{O}$ Notation

3.1 Definition

For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Example 5.

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$
$$n^3 + n + 2 = \Theta(n^3)$$
$$n^3 + n + 2 = o(n^4)$$

3.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \to a$ if $\limsup_{x \to a} \frac{|f(x)|}{g(x)} < \infty$.

Example 6. $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ *as* $\varepsilon \to 0$

4 Lipschitz Continuous

4.1 Definition

Definition 14. *Lipschitz continuous: if a function* $f : \mathbb{R}^n \to \mathbb{R}^m$ *satisfies*

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq \gamma \|\mathbf{x} - \mathbf{y}\|, \forall \mathbf{x}, \mathbf{y}$$

the function is called γ*-Lipschitz continuous;*

If f is γ-Lipschitz continuous, then it is also $(\gamma + 1)$-Lipschitz continuous
The minimal such γ is called a Lipschitz constant of function f
Remark: Here $\|\cdot\|$ can be any given norm of the space $\mathbb{R}^n$ and $\mathbb{R}^m$, such as Euclidean norm, ℓ_1-norm, etc.
When not specified, we assume it is Euclidean norm.

4.2 Example

Example 1: $f(x) = 2x$ is 2-Lipschitz continuous;
Example 2: What about $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).
Example 3: What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞.

4.3 Contraction Mapping

1. If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.
2. If $\gamma < 1$, then f is called a contraction mapping
Example 1: $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.
Example 2: $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

5 Fixed point theorem

1. Fixed point theorem: If f is a contraction mapping that maps $\mathbb{R}^n$ to itself, then the following two results hold:
1) There exists a unique fixed point x* satisfying

$$\mathbf{x}^* = f(\mathbf{x}^*)$$

2) In addition, the iterated function sequence

$$\mathbf{x}, f(\mathbf{x}), f(f(\mathbf{x})), \cdots,$$

converges to this unique fixed point $\mathbf{x}^*$ (independent of the initial point x).
2. Remark: This is a special case of "Banach fixed point theorem" (which applies to any complete metric space).