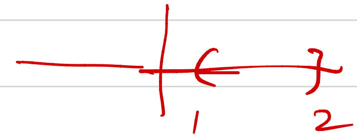
Closed and Open sets

A set $\mathcal{S} \subseteq \mathbb{R}^n$ is open if $\forall\, x \in \mathcal{S}$ we can draw a ball around x that is contained in $\mathcal{S}$,

i.e. $\forall\, x \in \mathcal{S}$, $\exists\, \varepsilon > 0$ s.t.

$$\{ y : \|y - x\| < \varepsilon \} \subseteq \mathcal{S}$$



$\mathcal{S}$ is closed if $\mathcal{S}^c$ is open

$\equiv$ if $\mathcal{S}$ contains all limit points of all sequences in $\mathcal{S}$.

Examples

$(1,2) = \{x \in \mathbb{R} : 1 < x < 2\}$ — open

$(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ — open

$[1, \infty)$ is closed because complement open

$(1, 2]$ is neither open nor closed

$\mathbb{R}$ is both open and closed

Compact Set $\mathcal{S} \subseteq \mathbb{R}^n$ is compact if it is closed and bounded. $\mathcal{S}$ is bounded if $\exists\, M$ s.t. $\|x\| \leq M \ \forall\, x \in \mathcal{S}$

Examples

$[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$

$\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

Extrema of sets of scalars Let $A \subset \mathbb{R}$.

- The infimum of A, or $\inf A$, is largest y s.t. $y \leq x$, $\forall\, x \in A$. If no such y exists, $\inf A = -\infty$
- Similar definition for supremum of A or $\sup A$.
- If $\inf A = x^*$ for some $x^* \in A$, then $x^* = \min A$ or minimum of A.
- similarly $\max A$ or maximum of A.