IE 516

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Contents

1 Lattice Programming

1.1 Lattice

Definition 1. *$(X, \geq)$ is a **lattice** if for any $x, y \in X$,*
$x \vee y = \inf\{z \in X | x \leq z, y \leq z\} \in X$
$x \wedge y = \sup\{z \in X | x \geq z, y \geq z\} \in X$

Definition 2. *$(X', \geq)$ is a **sublattice** of $(X, \geq)$: inherit $x \vee y$, $x \wedge y$ from X.*

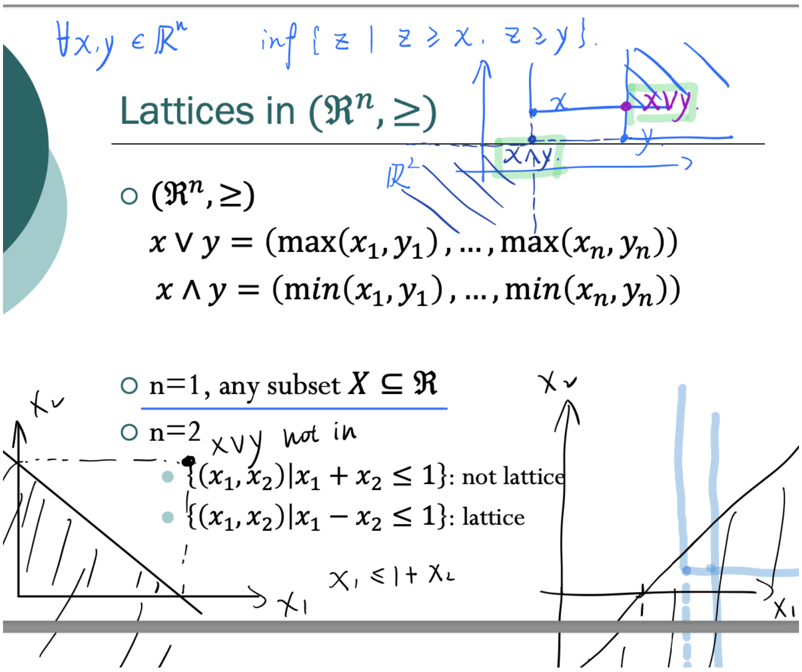


Figure 1:

Example 1. *Lattices:*

1. $\{0, 1\}^n$
2. Z^n
3. *a chain is a lattice (whose elements are ordered)*
4. *Intersection of two lattices*

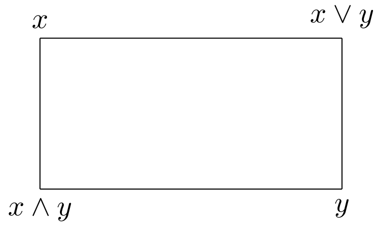
1.2 Supermodularity

1.2.1 Definition: Supermodular $g(x \vee y) + g(x \wedge y) \geq g(x) + g(y), \forall x, y \in X$

Definition 3. *A function $g : X \to \bar{\Re} (= \Re \cup \{+\infty\})$ is submodular if*

$$g(x \vee y) + g(x \wedge y) \leq g(x) + g(y), \forall x, y \in X$$

g is supermodular if $-g$ is submodular.



Claim 1. $\mathrm{dom}(g) = \{x \in X \mid g(x) < +\infty\}$ *is a lattice if* g *is submodular.*

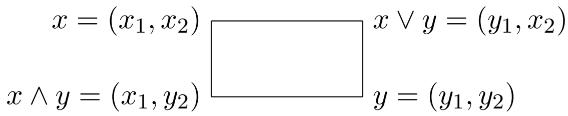
Proof. $\forall x, y \in \mathrm{dom}(g)$, prove $g(x \vee y) < +\infty$, $g(x \wedge y) < +\infty$. $\square$

1.2.2 Lemma: Supermodular $\Leftrightarrow \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0, \forall i \neq j$

Lemma 1. *Suppose* g *is twice partially differentiable in* $\Re^n$*. Then* g *is supermodular if and only if it has nonnegative cross partial derivatives, i.e.,*

$$\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0, \forall i \neq j$$

Proof.



$$x_1 \leq y_1;\ y_2 \leq x_2$$

$$\begin{aligned}
&g \text{ is supermodular} \\
\Leftrightarrow &g(x \vee y) - g(x) \geq g(y) - g(x \wedge y), \forall x, y \in X \\
&g(y_1, x_2) - g(x_1, x_2) \geq g(y_1, y_2) - g(x_1, y_2), \forall x, y \in X \\
&(\text{if } y_1 \to x_1,\ y_2 \text{ kept unchanged}) \\
&\frac{\partial g(x_1, x_2)}{\partial x_1} \geq \frac{\partial g(x_1, y_2)}{\partial x_1} \\
&(\text{if } y_2 \to x_2,\ y_2 \leq x_2) \\
&\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0,\ \forall i \neq j
\end{aligned}$$

$\square$

Note: Supermodularity $\approx$ Economic Complementarity
g is the profit function of selling products x_1 and x_2, $\frac{\partial}{\partial x_2}(\frac{\partial g(x_1, x_2)}{\partial x_1}) \geq 0$

Example 2 (Examples of Supermodular Functions)**.**

1. $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ $(\alpha_i \geq 0)$ *for* $x \geq 0$
2. $f(x, z) = \sum_{i=1}^n g_i(\alpha_i x_i - \beta_i z_i)$ *for any univariate concave function* $g_i : \Re \to \bar{\Re}$ $(\alpha_i \beta_i \geq 0)$
3. $f(x) = \sum_{i,j=1}^n a_{ij} x_i x_j = x^T A x$ *with* $a_{ij} = a_{ji}$ *is supermodular if and only if* $a_{ij} \geq 0$ $\forall i \neq j$

1.2.3 Lemma: Preservation of Supermodularity

Lemma 2 (Preservation of Supermodularity).

a) If f_i *is supermodular, then* $\lim_{i\to\infty} f_i(x), \sum_i \alpha_i f_i\ (\alpha_i \geq 0)$ *are supermodular*

b) If $f : \Re \to \Re$ *is convex and nondecreasing (nonincreasing) and* $g : \Re^n \to \Re$ *is increasing and supermodular (submodular), then* $f(g(x))$ *is supermodular*

c) Given $f : \Re^n \times \Re^m \to \Re$*, if* $f(\cdot, y)$ *is supermodular for all* y*, then* $E_\xi[f(x,\xi)]$ *is supermodular in* x

Lemma 3 (Supermodularity of composite functions).
If $X = \Pi_{i=1}^n X_i$ *and* $X_i \subseteq \Re, f_i(x_i) : X_i \to \Re$ *is increasing (decreasing) on* X_i *for* $i = 1, \ldots, n$*, and* $g(z_1, \ldots, z_n) : \Re^n \to \bar{\Re}$ *is supermodular in* $(z_1, \ldots, z_n)$*, then*

$$g(f_1(x_1), \ldots, f_n(x_n))$$

is supermodular on X

Lemma 4 (Topkis 1998). *If* X *is a lattice,* $f_i(x)$ *is increasing and supermodular (submodular) on* X *for* $i = 1, \ldots, k, Z_i$ *is a convex subset of* R^1 *containing the range of* $f_i(x)$ *on* X *or* $i = 1, \ldots, k$*, and* $g(z_1, \ldots, z_k, x)$ *is supermodular in* $(z_1, \ldots, z_k, x)$ *and is increasing (decreasing) and convex in* z_i *for fixed* z_{-i} *and* x*, then* $g(f_1(x), \ldots, f_k(x), x)$ *is supermodular on* X

1.3 Parametric Optimization Problems

Definition 4.

$$\begin{aligned} f(s) = &\max g(s,a) \\ &s.t.\ a \in A(s) \end{aligned}$$

S: subset of $\Re^m$
$A(s)$: finite dimensional
$C := \{(s,a) \mid s \in S, a \in A(s)\}$ (the graph of the constraint operator)
$A^*(s)$, the optimal solution set, is nonempty for every $s \in S$

Definition 5. *A set* $A(s)$ *is* ***ascending on*** S *if for* $s \leq s'$, $a \in A(s)$, $a' \in A(s')$*, we have* $a \wedge a' \in A(s)$ *and* $a \vee a' \in A(s')$*.*

Example 3. $A(s) = [s, +\infty)$ *is ascending on* S*.*

1.3.1 Theorem: Maximizer of supermodular func is ascending, the maximum value is also supermodular

Theorem 1 (Ascending Optimal Solutions and Preservation).
If

1. *S: sublattice of* $\Re^m$
2. $C := \{(s,a) \mid s \in S, a \in A(s)\}$ *is a sublattice*
3. g *is supermodular on* C

Then

1. *$A^*(s)$ is **ascending** on S. Under some conditions, the largest/smallest element of $A^*(s)$ exists, and is increasing in* s.

2. *$f(s)$ is supermodular.*

Proof. Take $s \leq s'$, $a^* \in A^*(s)$, $a'^* \in A^*(s')$, i.e.

$$g(s, a^*) = \max g(s, a) \text{ s.t. } a \in A(s)$$
$$g(s', a'^*) = \max g(s', a) \text{ s.t. } a \in A(s')$$

$$(s, a^*) \vee (s', a'^*) = (s', a^* \vee a'^*)$$
$$(s, a^*) \wedge (s', a'^*) = (s, a^* \wedge a'^*)$$

As we know C is a sublattice, we have

$$(s', a^* \vee a'^*) \in C \Rightarrow a^* \vee a'^* \in A(s')$$
$$(s, a^* \wedge a'^*) \in C \Rightarrow a^* \wedge a'^* \in A(s)$$

Hence,

$$g(s', a^* \vee a'^*) \leq g(s', a'^*); \; g(s, a^* \wedge a'^*) \leq g(s, a^*)$$

Since g is supermodular on C,

$$g(s', a^* \vee a'^*) + g(s, a^* \wedge a'^*) \geq g(s, a^*) + g(s', a'^*)$$
$$0 \geq g(s', a^* \vee a'^*) - g(s', a'^*) \geq g(s, a^*) - g(s, a^* \wedge a'^*) \leq 0$$

Hence,

$$g(s', a^* \vee a'^*) = g(s', a'^*); \; g(s, a^*) = g(s, a^* \wedge a'^*)$$

which means,

$$a^* \vee a'^* \in A^*(s'), \; a^* \wedge a'^* \in A^*(s)$$

Then, " $A^*(s)$ is ascending on S " is proved.
What's more, the largest elements of $A(s')$ and $A(s)$ are $a^* \vee a'^*$ and a^*, the smallest elements of $A(s')$ and $A(s)$ are a'^* and $a^* \wedge a'^*$, which are both increased as s increases to s'. ☐

Proof. $\forall s, s' \in S$, $a \in A^*(s), a' \in A^*(s')$.

$$\begin{aligned} f(s) + f(s') &= g(s, a) + g(s, a') \\ &\quad \text{(Since } g \text{ is supermodular on } C\text{)} \\ &\quad \leq g(s \wedge s', a \wedge a') + g(s \vee s', a \vee a') \\ &\quad \leq f(s \wedge s') + f(s \vee s') \end{aligned}$$

" $f(s)$ is supermodular " is proved. ☐

Example 4. *Pricing:* $p^*(c) = \text{argmax}_{p \geq c'}(p - c)D(p)$, $(c' > c)$

1. $C = \{(p, c) | c < c', p \geq c'\}$ is a sublattice of $\mathbb{R}^2$.
2. $g(p, c) = (p - c)D(p)$, $\frac{\partial^2 g(p,c)}{\partial p \partial c} = -D'(p) \geq 0 \Rightarrow g$ is supermodular on C.
Hence, $p^*(c)$ is increasing in c.

Example 5. *Newsvendor model:* $\min_{x \geq 0} f(x) = cx + h_+ E[(x - \xi)^+] + h_- E[(\xi - x)^+]$

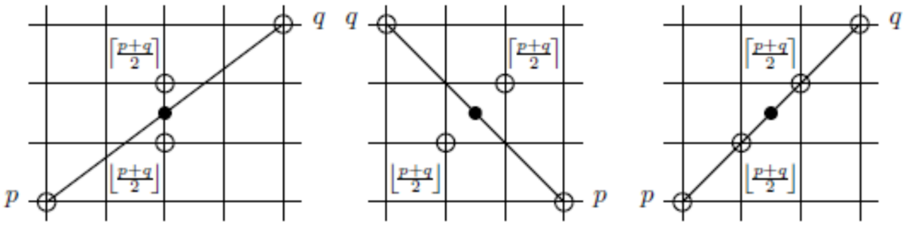


Figure 2: Discrete Midpoint Convexity

2 $L^\natural$-Convexity

2.1 Discrete Midpoint Convexity

Definition 6. *A function f is **discrete midpoint convexity** if*

$$f(\lceil \frac{p+q}{2} \rceil) + f(\lfloor \frac{p+q}{2} \rfloor) \leq f(p) + f(q)$$

2.2 $L^\natural$-Convexity on $\mathbb{Z}^n$

Definition 7. *A function $f : Z^n \to \bar{\Re}$ is called $L^\natural$ convex if f satisfies the discrete midpoint convexity.*

An equivalent definition: A function $f : Z^n \to \bar{\Re}$ is $L^\natural$-convex if and only if

$$g(x, \alpha) := f(x - \alpha e) = f([x_1 - \alpha, x_2 - \alpha, ..., x_n - \alpha]^T)$$

is submodular in (x, α) on Z^{n+1}(e : all-ones vector).

2.3 $L^\natural$-Convexity on $\mathcal{F}^n(\mathcal{F} = \mathbb{Z}$ or $\Re)$

Definition 8 (Murota 2003)**.**
A function $f : \mathcal{F}^{\mathrm{n}} \to \Re$ is $L^\natural$-convex if and only if $g(x, \xi) := f(x - \xi e)$ is submodular in $(x, \xi) \in \mathcal{F}^{\mathrm{n}} \times S$, where e is a vector with all components equal to 1 and S is the intersection of $\mathcal{F}$ with any unbounded interval in $\mathfrak{R}$. (f is required to be convex if $\mathcal{F} = \mathfrak{R}$)

Definition 9. *A set V is $L^\natural$-convex if and only if its indicator function $\delta_V(x)$ is $L^\natural$.*

$$\delta_V(x) = \begin{cases} +\infty & , x \notin V \\ 0 & , x \in V \end{cases}$$

$\Leftrightarrow$ *$g(x, \xi) = \delta_V(x - \xi e)$ is subnormal, i.e.*

$$g(x \vee y, \max\{\xi_x, \xi_y\}) + g(x \wedge y, \min\{\xi_x, \xi_y\}) \leq g(x, \xi_x) + g(y, \xi_y), \ \forall (x, \xi_x), (y, \xi_y)$$

If $x - \xi_x e, y - \xi_y e$ in V, $x \vee y - \max\{\xi_x, \xi_y\} e$ and $x \wedge y - \min\{\xi_x, \xi_y\}$ must in V.

Note: f is $L^\natural$-concave if $-f$ is $L^\natural$-convex.

2.4 Properties of $L^\natural$-Convexity

2.4.1 Proposition: $L^\natural$-convex $\Leftrightarrow a_{ij} \leq 0, \forall i \neq j, a_{ii} \geq 0, \sum_{j=1}^n a_{ij} \geq 0, \forall i$

Proposition 1. *A quadratic function $f(x) = \sum_{i,j=1}^n a_{ij}x_ix_j$ with $a_{ij} = a_{ji}$ is $\underline{L^\natural\text{-convex on } \mathcal{F} \text{ if and only if}}$ its Hessian is a diagonally dominated M-matrix*

$$a_{ij} \leq 0 \ \forall i \neq j, \quad a_{ii} \geq 0, \quad \sum_{j=1}^n a_{ij} \geq 0 \ \forall i$$

Proof.
$f(x)$ is $L^\natural$-convex $\Leftrightarrow g(x,\xi) = f(x - \xi e) = \sum_{i,j=1}^n a_{ij}(x_i - \xi)(x_j - \xi)$ is submodular in (x, ξ) i.e.

$$\frac{\partial^2 g}{\partial \xi \partial x_i} = \frac{\partial}{\partial \xi}(\sum_{j=1}^n a_{ij}(x_j - \xi) + \sum_{j=1}^n a_{ji}(x_j - \xi)) = -2\sum_{j=1}^n a_{ij} \leq 0$$
$$\frac{\partial^2 g}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_k}(\sum_{k=1}^n a_{ik}(x_k - \xi) + \sum_{k=1}^n a_{ki}(x_k - \xi)) = 2a_{ij} \leq 0$$

□

Proposition 2. *A twice continuous differentiable function $f : \Re^n \to \Re$ is $\underline{L^\natural\text{-convex if and only if}}$ its Hessian is a diagonally dominated M-matrix, that is*

$$a_{ij} \leq 0, \forall i \neq j, a_{ii} \geq 0, \sum_{j=1}^n a_{ij} \geq 0, \forall i$$

Proof.

$L^\natural$-convex $\Leftrightarrow g(x,\xi) = f(x - \xi e)$ is subnormal
(if twice differentiable)
$\Leftrightarrow \frac{\partial^2 g(x,\xi)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x-\xi e)}{\partial x_i \partial x_j} \leq 0, i \neq j,\ \frac{\partial^2 g(x,\xi)}{\partial x_i \partial \xi} = -\sum_{j=1}^n \frac{\partial^2 f(x-\xi e)}{\partial x_i \partial x_j} \leq 0,\ \forall (x,\xi) \in \Re^{n+1}$
$\Leftrightarrow \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, i \neq j;\ \sum_{j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \ \forall x \in \Re^n$ □

2.4.2 Corollary: $L^\natural$-convex $\longrightarrow$ convex + submodular

Corollary 1. *If a twice differentiable function f is $L^\natural$-convex, then the function is convex and submodular.*

Proof.

$\underline{a_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, i \neq j}$ means the cross partial derivatives are nonpositive, which equals to $\underline{f \text{ is submodular}}$.

$$\begin{aligned}
x^T \nabla^2 f(x) x &= \sum_{i,j=1}^{n} a_{ij} x_i x_j \\
&= \sum_{k}^{n} a_{kk} x_k^2 + \sum_{j=1}^{n} \sum_{i<j} a_{ij} 2 x_i x_j \\
&\geq \sum_{k}^{n} a_{kk} x_k^2 + \sum_{j=1}^{n} \sum_{i<j} a_{ij} (x_i^2 + x_j^2) \\
&\geq \sum_{k}^{n-1} a_{kk} x_k^2 + \sum_{j=1}^{n-1} \sum_{i<j} a_{ij} (x_i^2 + x_j^2) \\
&\dots \\
&\geq 0, \quad \forall x \in \Re^n
\end{aligned}$$

Then f is convex. □

Example 6.

- Given any univariate (discrete) convex function $g_i : \mathcal{F} \to \bar{\Re}$ and $h_{ij} : \mathcal{F} \to \Re$, the function $f : \mathcal{F}^n \to \bar{\Re}$ defined by

$$f(x) := \sum_{i} g_i(x_i) + \sum_{i \neq j} h_{ij}(x_i - x_j)$$

is $L^\natural$-convex.

Example 7.

- A set with a representation

$$\{x \in \mathcal{F}^n : l \leq x \leq u, x_i - x_j \leq v_{ij}, i \neq j\}$$

is $L^\natural$-convex, where $l, u \in \mathcal{F}^n, v_{ij} \in \mathcal{F}$.

2.4.3 Theorem: Minimizer of $L^\natural$-convex func is nondecreasing with bounded sensitivity, the minimum value is also $L^\natural$-convex

Theorem 2. *Assume $g : \mathcal{F}^n \times \mathcal{F}^m \to \overline{\Re}$ and set $C \subset \mathcal{F}^n \times \mathcal{F}^m$ are $L^\natural$-convex, define*

$$f(s) = \inf_{a:(s,a) \in C} g(s,a)$$

Then,

1. *The optimal solution set $A^*(s)$ is nondecreasing in s with bounded sensitivity i.e.,*

$$A^*(s + \omega e) \leq A^*(s) + \omega e, \ \forall \omega \in F_+$$

   (Zipkin 2008, Chen et al. 2018)

2. *f is $L^\natural - convex$. (Zipkin 2008)*

2.5 Relationship with Multimodularity

Definition 10. *A function $f(x_1, x_2, ..., x_n)$ is multimodular if $f(x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1})$ submodular in $(x_0, x_1, ..., x_n)$.*

Multimodularity and $L^\natural$-convexity are equivalent subject to an unimodular linear transformation.

3 Optimization with decisions truncated by random variables

$$\min_{u \in \mathcal{U}} E[f(u \wedge \xi)]$$

Question 1 (Supply uncertainty in SCM): u: ordering quantities; ξ: random capacities.
Question 2 (Demand uncertainty in RM): u: booking limits; ξ: random demands.
Difficulty: the object function is not convex (even if f is).

3.1 Unconstrained Problem

Consider

$$\tau^* = \min_{u \in \mathcal{F}^n} E[f(u \wedge \xi)]$$

$\mathcal{F}$ is either the real space or the set with all integers.
Random vector $\xi \in \mathcal{X} \subseteq \mathcal{F}^n$

3.1.1 Reformulation

Reformulation:

$$\begin{array}{lll} \min & E[f(v(\xi))] & \\ s.t. & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) & , \forall \xi \in \mathcal{X} \\ & v(\xi) = u \wedge \xi & , \forall \xi \in \mathcal{X} \end{array}.$$

Turn finding u^* into finding $v^*()$
$v()$ is not convex.

Theorem 3 (Equivalent Transformation, Chen, Gao and Pang 2018)**.** *Suppose that (Assumption I)*
(a) the function f is lower semi-continuous with $f(u) \to +\infty$ for $|u| \to +\infty$;
(b) the function f is componentwise (discrete) convex;
(c) the random vector ξ has independent components.
Then τ^* is also the optimal objective value of the following optimization problem:

$$\begin{array}{lll} \min & E[f(v(\xi))] & \\ s.t. & v(\xi) \leq \xi & , \forall \xi \in \mathcal{X} \\ & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) & , \forall \xi \in \mathcal{X} \end{array}.$$

3.1.2 $n = 1$

$\hat{u}$: minimizer of $f(u)$
Need to show

$$\min_u E[f(u \wedge \xi)] = \min_{v(\xi) \leq \xi} E[f(v(\xi))]$$

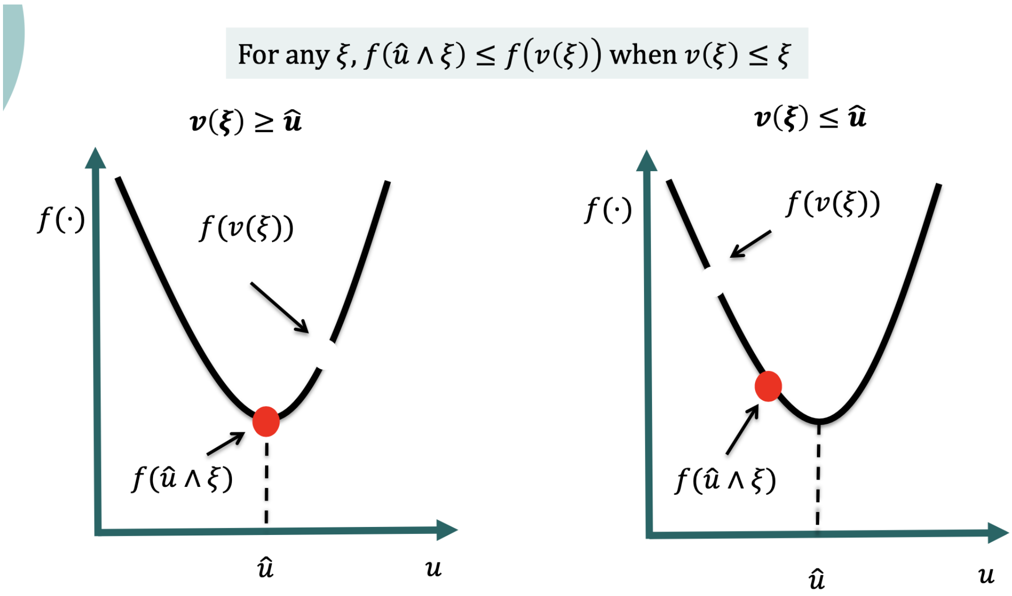


Figure 3: Easy to show $\forall \xi,\ f(\hat{u} \wedge \xi) \leq f(v(\xi))$ when $v(\xi) \leq \xi$

Easy to show $\forall \xi,\ f(\hat{u} \wedge \xi) \leq f(v(\xi))$ when $v(\xi) \leq \xi$. Then

$$\operatorname{argmin} E[f(u \wedge \xi)] = \hat{u} = \operatorname{argmin} f(u)$$

$$\begin{aligned}
E[f(\hat{u} \wedge \xi)] &\geq \min_u E[f(u \wedge \xi)] \\
&\geq \min_{v(\xi) \leq \xi} E[f(v(\xi))] \text{ (Consider } v^*(\xi) \geq u) \\
&\geq E[f(\hat{u} \wedge \xi)] \text{ (See the figure)} \\
\Rightarrow \quad & \min_u E[f(u \wedge \xi)] = \min_{v(\xi) \leq \xi} E[f(v(\xi))]
\end{aligned}$$

3.1.3 $n \geq 2$

$$\operatorname{argmin} E[f(u \wedge \xi)] \neq \hat{u}$$

Example 8.

$$f(u_1, u_2) = (u_1 + u_2 - 2)^2 + (u_1 - 1)^2 + (u_2 - 1)^2$$

ξ_1, ξ_2 *can take values* 0 *and* 2 *with equal probability.*
$\hat{u} = (1, 1)$
$\operatorname{argmin} E[f(u \wedge \xi)] = (1.2, 1.2)$

3.2 Transformation for Constrained Problem

$$\min_{u \in \mathcal{U}} E[f(u \wedge \xi)]$$

$$\downarrow$$

$$\begin{array}{lll} \min & E[f(v(\xi))] & \\ s.t. & v(\xi) \leq \xi & ,\forall \xi \in \mathcal{X} \\ & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) \in \mathcal{V} & ,\forall \xi \in \mathcal{X} \\ & \quad \mathcal{V} = \{u \wedge \xi \ |u \in \mathcal{U}, \xi \in \mathcal{X}\} & \end{array} .$$

Sufficient Conditions for the Transformation

(a) $\mathcal{U} = \{u|Au \leq b, u \geq l\}$, where $A \geq 0$

(b) $\mathcal{X}_j \subseteq [l_j, +\infty)$

(Example: some situations $l = (l_1, ..., l_n) = (0, ..., 0)$)

3.3 Generalization

$$\min_{u \in \mathcal{F}^n} l(u) + E[f(u \wedge \xi)]$$

- $l : \mathcal{F}^n \to \bar{\Re}$, $f : \mathcal{F}^n \to \bar{\Re}$
- $\xi \in \mathcal{X} \subseteq \mathcal{F}^n$
- ξ **dependent** (different from before !)

3.3.1 Positive Dependence

Let F_{ξ_i} be the joint CDF of $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n$ conditioned on ξ_i
$\{\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \mid \xi_i\}$ is stochastically increasing if $\int_S dF_{\xi_i}(w)$ is an increasing function of ξ_i for each increasing set S
$\{\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_n\}$ has positive dependence if $\{\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \mid \xi_i\}$ is stochastically increasing for all i

Proposition 3. *The collection of random variables generated by nonnegative linear combination of **independent log-concave** random variables has positive dependence.*

3.3.2 Transformation

Theorem 4 (Equivalent Transformation, Chen and Gao 2018)**.** *Suppose that (Assumption II)*

(1) the function f is lower semi-continuous with $f(u) \to +\infty$ for $|u| \to +\infty$;

(2) the function f is componentwise (discrete) convex and supermodular;

(3) the random vector ξ is positive dependent;

(4) $l(u)$ is componentwise increasing.

Then problem $\min_{u \in \mathcal{F}} l(u) + E[f(u \wedge \xi)]$ *has the **same optimal objective value** of*

$$\begin{array}{lll} \min & l(u) + E[f(v(\xi))] & \\ s.t. & v(\xi) \leq \xi & ,\forall \xi \in \mathcal{X} \\ & v(\xi) \leq u & ,\forall \xi \in \mathcal{X} \\ & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) & ,\forall \xi \in \mathcal{X} \\ & v_i(\xi_i) \text{ is increasing for all } i & \end{array} .$$

4 Single-Leg Capacity Allocation

(Seats reserved for future consumers)

4.1 Two-Class Model

Two periods: Period 1, random demand D_2 for price p_2; Period 2, random demand D_1 for price p_1.
$p_1 > p_2$
Provide y in period 1 and the remaining will be provided in period 2.

$$\begin{array}{rl} \max & p_1 E_{D_1,D_2}[D_1 \wedge (c-(c-y)\wedge D_2)] + p_2 E_{D_2}[(c-y)\wedge D_2] \\ s.t. & 0 \le y \le c, y \in \mathcal{F} \end{array} .$$

Where $\mathcal{F} = \mathbb{R}$ or $\mathbb{Z}$ and $a \wedge b = \min(a,b)$

4.1.1 Theorem: convex f, $\operatorname{argmin} E_D f(u\wedge D) = \operatorname{argmin} f(u)$

When D_2 is sufficiently high. Let $b = c-y$, and the question transferred to

$$\begin{array}{rl} \max & v(b) = p_1 E_{D_1}[D_1 \wedge (c-b)] + p_2 b \\ s.t. & 0 \le b \le c, b \in \mathcal{F} \end{array} .$$

$v(b)$ is a concave function.

Theorem 5. *Consider the following optimization problem*

$$\begin{array}{rl} \min & E_D f(u\wedge D) \\ s.t. & 0 \le u \le c, u \in \mathcal{F} \end{array} .$$

Assume D is a nonnegative random variable.
If f is ***convex*** *and $\mathcal{F} = \mathbb{R}$ or f is* ***discrete convex*** *and $\mathcal{F} = \mathbb{Z}$, then any optimal solution of*

$$\begin{array}{rl} \min & f(u) \\ s.t. & 0 \le u \le c, u \in \mathcal{F} \end{array} .$$

is also optimal for the former optimization problem.
(Actually, quasi-convexity suffices)

According to the $n = 1$ discussion of section 3, the theorem is easy to be proved. Then, the global-max in $v(b)$ is global-max for objective function.
Then we consider the equivalent minimum problem,

$$\begin{array}{rl} \max & \phi(y) = p_2 y - p_1 E_{D_1}[D_1 \wedge y] \\ s.t. & 0 \le y \le c, y \in \mathcal{F} \end{array} .$$

We need to find the optimal y^* minimize the $\phi(y)$. To simplify the analysis, we find the y° which is the optimal y regardless constraints. $y^* = \begin{cases} 0 & \text{if } y^\circ < 0 \\ y^\circ & \text{if } y^\circ \in [0,c] \\ c & \text{if } y^\circ > c \end{cases}$

4.1.2 Discrete, $\mathcal{F} = \mathbb{Z}$

$$\phi(y) - \phi(y-1) = p_2 - p_1 P(D_1 \geq y)$$

Then, the y° is

$$\begin{aligned}
\overline{y} &= \min\{y \in \mathbb{Z} : P(D_1 > y) < r\} \\
\underline{y} &= \max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\} \text{ \textbf{(Littlewood's rule)}} \\
y^\circ &= [\underline{y}, \overline{y}] \cap \mathbb{Z}
\end{aligned}$$

Where $r = \frac{p_2}{p_1}$, higher r causes lower y°.

Example 9. *Suppose that D_1 is a Poisson random variable with mean 80, the full fare is $p_1 = 100$ and the discounted fare is $p_2 = 60$*

$r = 60/100 = 0.6$, $y^* = \max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\} = 78$

4.1.3 Continuous, $\mathcal{F} = \mathbb{R}$

y° is the y s.t. $1 - F_1(y) = r$, where $F_1(\cdot)$ is the CDF of D_1.

$$y^\circ = F_1^{-1}(1-r)$$

Special Case: $D_1 \sim \mathcal{N}(\mu, \sigma^2)$

$$F_1(y) = \Phi(\frac{y-\mu}{\sigma})$$

$\Phi(\cdot)$ is the CDF of the standard normal $\mathcal{N}(0,1)$. Then,

$$y^\circ = \mu + \sigma\Phi^{-1}(1-r)$$

If $\frac{p_2}{p_1} = r < \frac{1}{2}$, y° increases as variance σ increases.

4.2 Multi-Class Model

- $p_1 > p_2 > \cdots > p_n$
- Lower class demand arrives earlier.
- Demand of different classes are independent.
- Control: demand to accept or reject.

4.2.1 Sequence of Events

At stage j with remaining capacity x,

1. Select booking limit b for class j, equivalently, protection level $y = x - b$ for classes l, $l < j$.
2. Demand D_j is realized.
3. Accept $b \wedge D_j$ of class j and collect revenue $p_j(b \wedge D_j)$.
4. Move on to stage $j-1$ with remaining capability $x - b \wedge D_j$.

4.2.2 Dynamic Programming

Set $f_j(x,b) = p_j b + V_{j-1}(x-b)$, $V_0(x) = 0, V_j(0) = 0$, $x = 0, 1, ..., c$ (discrete), $x \in [0, c]$ (continuous)

$$V_j(x) = \max_{b \in [0,x], b \in \mathcal{F}} \mathbb{E}[f_j(x, b \wedge D_j)] = \mathbb{E}[p_j(b \wedge D_j)] + \mathbb{E}[V_{j-1}(x - b \wedge D_j)]$$

Proposition 4. *(1). $\forall j$, f_j is $L^\natural-$concave, V_j is (discrete) convex; (2). The optimal solution of the dynamic programming b_j^* is the same as*

$$\max_{b \in [0,x], b \in \mathcal{F}} f_j(x,b) = p_j b + V_{j-1}(x-b)$$

Define y_{j-1}^* be the optimal solution of

$$\max_{y \geq 0, y \in \mathcal{F}} -p_j y + V_{j-1}(y)$$

Then

$$b_j^* = (x - y_{j-1}^*)^+$$

$$\begin{aligned} V_j(x) &= \mathbb{E}[f_j(x, (x - y_{j-1}^*)^+ \wedge D_j)] \\ &= \mathbb{E}[p_j(x - y_{j-1}^*)^+ \wedge D_j + V_{j-1}(x - (x - y_{j-1}^*)^+ \wedge D_j)] \\ &= \begin{cases} V_{j-1}(x) & \text{if } x \leq y_{j-1}^* \\ \mathbb{E}[p_j(x - y_{j-1}^*) \wedge D_j + V_{j-1}(x - (x - y_{j-1}^*) \wedge D_j)] & \text{if } x > y_{j-1}^* \end{cases} \end{aligned}$$

4.3 Discrete Case

Define

$$\Delta V_j(x) = V_j(x) - V_j(x-1)$$

Lemma 5. *If $x > y_{j-1}^*$, $\Delta V_j(x) = \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x - D_j)\}]$*

Proof.

$$\begin{aligned} \Delta V_j(x) &= p_j(\mathbb{E}[(x - y_{j-1}^*) \wedge D_j] - \mathbb{E}[(x - 1 - y_{j-1}^*) \wedge D_j]) \\ &\quad + \mathbb{E}[V_{j-1}(x - (x - y_{j-1}^*) \wedge D_j)] - \mathbb{E}[V_{j-1}(x - 1 - (x - 1 - y_{j-1}^*) \wedge D_j)] \\ &= \begin{cases} p_j & \text{if } x - y_{j-1}^* \leq D_j \\ \Delta V_{j-1}(x - D_j) & \text{if } x - y_{j-1}^* > D_j \end{cases} \\ &= \mathbb{E}[p_j \mathbb{I}(x - D_j \leq y_{j-1}^*) + \Delta V_{j-1}(x - D_j)\mathbb{I}(x - D_j > y_{j-1}^*)] \\ &\quad (\text{ Since } y_{j-1}^* \text{ maximizes } -p_j y + V_{j-1}(y), \\ &\quad \Delta V_{j-1}(y) > p_j \text{ if } y \leq y_{j-1}^* \text{ and } \Delta V_{j-1}(y) \leq p_j \text{ if } y > y_{j-1}^*) \\ &= \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x - D_j)\}] \end{aligned}$$

□

Proposition 5 (1.5 of GT 19)**.**

(i) $\Delta V_j(x+1) \leq \Delta V_j(x)$ *(proved by V_j is discrete concave)*

(ii) $\Delta V_{j+1}(x) \geq \Delta V_j(x)$

Proof.
If $x \leq y^*_{j-1}$,

$$\Delta V_j(x) = V_{j-1}(x) - V_{j-1}(x-1) = \Delta V_{j-1}(x)$$

If $x > y^*_{j-1}$ (i.e. $x - 1 \geq y^*_{j-1}$),

$$\begin{aligned}\Delta V_j(x) &= \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x - D_j)\}] \\ &\quad (\ V_{j-1}(x) \text{ is discrete concave}) \\ &\geq \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x)\}] \\ &\quad (\text{ Since } x > y^*_{j-1},\ V_{j-1}(x) < p_j) \\ &= \Delta V_{j-1}(x)\end{aligned}$$

□

Theorem 6 (part of 1.6 of GT 19)**.**
The optimal protection level at stage j is

$$y^*_{i-1} = \max\{y \in \mathbb{N}_+ : \Delta V_{j-1}(y) > p_j\}$$

Moreover, $y^*_{n-1} \geq y^*_{n-2} \geq \cdots \geq y^*_1 = y^*_0 = 0$

(Easy to prove: Since y^*_{j-1} maximizes $-p_j y + V_{j-1}(y)$, $\Delta V_{j-1}(y) > p_j$ if $y \leq y^*_{j-1}$ and $\Delta V_{j-1}(y) \leq p_j$ if $y > y^*_{j-1}$)
Note: Littlewood's rule is a special case for $n = 2$.

4.3.1 Discrete Case: Reformulation

$$\begin{aligned}V_j(x) &= \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j + V_{j-1}(x - (x - y^*_{j-1})^+ \wedge D_j)] \\ &= V_{j-1}(x) + \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j + (V_{j-1}(x - (x - y^*_{j-1})^+ \wedge D_j) - V_{j-1}(x))] \\ &= V_{j-1}(x) + \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j - \sum_{z=1}^{(x-y^*_{j-1})^+ \wedge D_j} \Delta V_{j-1}(x + 1 - z)] \\ &= V_{j-1}(x) + \mathbb{E}[\sum_{z=1}^{(x-y^*_{j-1})^+ \wedge D_j} (p_j - \Delta V_{j-1}(x + 1 - z))]\end{aligned}$$

$$\begin{aligned}V_j(x) &= V_{j-1}(x) + \mathbb{E}[\sum_{z=1}^{u^*}(p_j - \Delta V_{j-1}(x + 1 - z))] \\ u^* &= \min\{(x - y^*_{j-1})^+, D_j\} \\ y^*_{i-1} &= \max\{y \in \mathbb{N}_+ : p_j < \Delta V_{j-1}(y)\}\end{aligned}$$

- $y^*_1 \leq y^*_2 \leq \cdots \leq y^*_n$
- The "nested" booking limit $b^*_j = C - y^*_{j-1}$, $j = 2, ..., n$
  (nested booking limit is the total amount can be booked in $j, j+1, ..., n$)

$$b^*_j = y_j$$

- The marginal utility at j of choosing to reserve one more item in the next stage $j - 1$:

$$\pi_j(x) = \Delta V_{j-1}(x)$$

- The amount of selling at stage j

$$u^* = \begin{cases} 0 & \text{if } p_j < \pi_j(x) \\ \min\{\max\{z : p_j \geq \pi_j(x-z)\}, D_j\} & \text{if } p_j \geq \pi_j(x) \end{cases}$$

$p_j < \pi_j(x)$ means the marginal utility of reserving is larger than selling it now.

We can further compute, if $x > y^*_{j-1}$,

$$\Delta V_j(x) = p_j Pr(D_j \geq x - y^*_{j-1}) + \sum_{k=0}^{x-y^*_{j-1}-1} \Delta V_{j-1}(x-k) Pr(D_j = k)$$

If $x \leq y^*_{j-1}$, $\Delta V_j(x) = \Delta V_{j-1}(x)$.
Which will simplify the computation.

4.3.2 Discrete Case: Computation

The policy is implemented as follows:

1. At stage n, we start with $x_n = c$ units of inventory and we protect $y_{n-1}(x_n) = \min\left\{y^*_{n-1}, x_n\right\}$ units of capacity for fares $n-1, n-2, \ldots, 1$.
2. Therefore, we allow up to $\left[x_n - y^*_{n-1}\right]^+$units of capacity to be sold to fare class n.
3. We sell $\min\left\{\left[x_n - y^*_{n-1}\right]^+, D_n\right\}$ units of capacity to fare class n and we have a remaining capacity of $x_{n-1} = x_n - \min\left\{\left[x_n - y^*_{n-1}\right]^+, D_n\right\}$ at stage $n-1$.
4. We protect $y_{n-2}(x_{n-1}) = \min\left\{y^*_{n-2}, x_{n-1}\right\}$ units of capacity for fares $n-2, n-1, \ldots, 1$.
5. Therefore, we allow up to $\left[x_{n-1} - y^*_{n-2}\right]^+$units of capacity to be sold to fare class $n-1$.
6. We continue until we reach stage 1 with x_1 units of capacity, allowing $(x_1 - y_0)^+ = (x_1 - 0)^+ = x_1$ to be sold to fare class 1.

$$V_j(x) = \mathbb{E}[p_j \min\{(x - y^*_{j-1})^+, D_j\} + V_{j-1}(x - \min\{(x - y^*_{j-1})^+, D_j\})]$$

$y^*_0 = 0, V_0(x) = 0$, then we can compute $y^*_1, V_1(x)$,...
Backward: Use

$$\Delta V_j(x) = p_j Pr(D_j \geq x - y^*_{j-1}) + \sum_{k=0}^{x-y^*_{j-1}-1} \Delta V_{j-1}(x-k) Pr(D_j = k)$$
$$y^*_{j-1} = \max\{y \in \mathbb{N}_+ : p_j < \Delta V_{j-1}(y)\}$$

1. $V_1(x_1) = \mathbb{E}[p_1 \min\{x_1, D_1\}]$, then $\Delta V_1(x) = p_1 Pr(D_1 \geq x)$
2. $y^*_1 = \max\{y \in \mathbb{N}_+ : p_2 < \Delta V_1(y)\} = \max\{y : Pr(D_1 \geq y) > \frac{p_2}{p_1}\}$

$$\Delta V_2(x) = p_2 Pr(D_2 \geq x - y^*_1) + \sum_{k=0}^{x-y^*_1-1} p_1 Pr(D_1 \geq x-k) Pr(D_2 = k)$$

3. $y^*_2 = \max\{y \in \mathbb{N}_+ : p_3 < \Delta V_2(y)\} = \max\{y : Pr(\Delta V_1(y - D_2) > p_3)\}$

4. ...

The complexity is $O(nC^2)$

Example 10. *Suppose that there are five fare classes. The demand for all fare classes is a Poisson random variable. The fares and the expected demand for the five fare classes are given by* $(p_5, p_4, p_3, p_2, p_1) = (15, 35, 40, 60, 100)$ *and* $(\mathbb{E}D_5, \mathbb{E}D_4, \mathbb{E}D_3, \mathbb{E}D_2, \mathbb{E}D_1) = (120, 55, 50, 40, 15)$. *For this problem instance, the optimal protection levels are*

1. $V_1(x_1) = \mathbb{E}[100 \min\{x_1, D_1\}]$, then $\Delta V_1(x) = 100Pr(D_1 \geq x)$
2. $y_1^* = \max\{y : Pr(D_1 \geq y) > \frac{3}{5}\} = 14$

$$\Delta V_2(x) = 60Pr(D_2 \geq x - 14) + \sum_{k=0}^{x-15} 100Pr(D_1 \geq x - k)Pr(D_2 = k)$$

3. $y_2^* = \max\{y \in \mathbb{N}_+ : p_j <$

4.4 Continuous Case

Skip

4.5 Generalized Newsvendor Problem: High-before-low arrival pattern

Consider the problem of selecting c to maximize

$$\Pi_n(c) = V_n(c) - kc$$

Where $V_n(c)$ is the expected revenue to the multi-fare RM problem.
Assume high-before-low arrival pattern. Then

$$V_n(c) = \sum_{j=1}^{n} p_j \mathbb{E}[D_j \wedge (c - D_{1:j-1})^+]$$

and

$$\Delta V_n(c) = \sum_{j=1}^{n} (p_j - p_{j+1}) Pr(D_{1:j} > c)$$

Where $D_{1:j} = \sum_{l=1}^{j} D_l, p_{n+1} = 0$

4.6 Heuristics

When there are two classes, we find y^*: $\max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\}$
We try to use this form to simplify our computation,
EMSR (expected marginal seat revenue)

- **EMSR - a**

$$y_k^{j+1} = \max\{y : P(D_k \geq y) > \frac{p_{j+1}}{p_k}\}, k = j, j-1, ..., 1$$

$$y_j = \sum_{k=1}^{j} y_k^{j+1}$$

- **EMSR - b**

$$\bar{p}_j = \frac{\sum_{k=1}^{j} p_k \mathbb{E}[D_k]}{\sum_{k=1}^{j} \mathbb{E}[D_k]}$$
$$y_j = \max\{y : P(\sum_{k=1}^{j} D_k \geq y) > \frac{p_{j+1}}{\bar{p}_j}\}$$

4.7 Bounds on Optimal Expected Revenue

4.7.1 Upper Bound

$$\begin{aligned}
\overline{V}(c|D) &:= \max\{\sum_{j=1}^{n} p_j x_j | \sum_{j=1}^{n} x_j \leq c, 0 \leq x_j \leq D_j, j = 1, ..., n\} \\
V_n^U(c) &:= \mathbb{E}[\overline{V}(c|D)] \\
&= \sum_{j=1}^{n} (p_j - p_{j+1}) \sum_{k=1}^{c} Pr(D_{1:j} \geq k), \quad \text{(Set } p_{n+1} = 0\text{)}
\end{aligned}$$
$$\mathbb{E}[\overline{V}(c|D)] \leq \overline{V}(c|D) = \sum_{j=1}^{n} (p_j - p_{j+1}) \min\{\overline{D}_{1:j}, c\}$$

4.7.2 Lower Bound

Using zero protection level

$$\begin{aligned}
V_n^L(c) &= \sum_{j=1}^{n} p_j \mathbb{E}[\min\{D_k, (c - D_{j+1:n})^+\}] \\
&= \sum_{j=1}^{n} (p_j - p_{j-1}) \mathbb{E}[\min\{D_{j:n}, c\}], \quad \text{(Set } p_0 = 0\text{)}
\end{aligned}$$

4.8 Dynamical Models

- $p_1 \geq p_2 \geq \cdots \geq p_n$.
- T periods.
- At most one arrival each period.
- λ_{jt}: probability of an arrival of class j in period t.
- M_t: set of offered classes.

4.8.1 Discrete Time

$$\begin{aligned}
V_t(x) &= \sum_{j \in M_t} \lambda_{jt} \max\{p_j + V_{t-1}(x-1), V_{t-1}(x)\} + (1 - \sum_{j \in M_t} \lambda_{jt}) V_{t-1}(x) \\
&= V_{t-1}(x) + \sum_{j \in M_t} \lambda_{jt} [p_j - \Delta V_{t-1}(x)]^+ \\
&= V_{t-1}(x) + R_t(\Delta V_{t-1}(x))
\end{aligned}$$
Where we set $R_t(z) = \sum_{j \in M_t} \lambda_{jt} [p_j - z]^+$, $V_t(0) = 0$, $V_0(x) = 0, \forall x \geq 0$

4.8.2 Continuous Time: Poisson arrival

$$\frac{\partial V_t(x)}{\partial t} = R_t(\Delta V_t(x))$$

4.8.3 Optimal Policy: discrete time

Let

$$a(t,x) = \max\{j : p_j \geq \Delta V(t-1,x)\}$$

Optimal to accept all fares in the active set

$$A(t,x) = \{j \in M_t : j \leq a(t,x)\}$$

and reject the remaining fare classes

4.8.4 Structural Properties

Theorem 7 (1.18 of GT)**.**

- *$V_t(x)$ is increasing in t, x.*
- *$\Delta V_t(x)$ is increasing in t and decreasing in x.*
- *$a(t,x), A(t,x)$ is increasing in x.*

If $\lambda_{jt} \equiv \lambda_j > 0$, $M_t \equiv M = \{1,...,n\}$, then

- *$V_t(x)$ is strictly increasing and concave in t.*
- *$a(t,x), A(t,x)$ is decreasing in t.*

4.8.5 Discrete Case: Computation

$$V_t(x) = V_{t-1}(x) + \sum_{j \in M_t} \lambda_{jt}[p_j - \Delta V_{t-1}(x)]^+$$

$V_0(x) = 0$, then $V_1(x)$, then $\Delta V_1(x)$.
The complexity is $O(nCT)$ ($T \approx O(C)$)

5 Network Revenue Management with Independent Demands

5.1 Settings

- m resources with initial capacities $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{Z}_+^m$
- Time from $T, T-1, T-2, ...$ to 0.
- ODF kj: Itineraries $k = 1, ..., K$; Possible fares for itinerary k, p_{kj}, $j \in \{1, ..., n_k\}$. (Every itinerary may have n_k kinds of prices).

- Demand arrives as compound Poisson arrival process with rate λ_{tkj} at time t for ODF kj.
- Resources utilized by itinerary k: $A_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix}$, $a_{ik} \in \{0,1\}$ with $a_{ik} = 1$ if resource i is consumed by itinerary k.
- $V(t,x)$: the maximum total expected revenue that can be extracted when the remaining capacities are $x \in \mathbb{Z}_+^m$ and the remaining time is $t \in \mathbb{R}_+$.
- Decision: $u = \{u_{kj} : j = 1, ..., n_k, k = 1, ..., K\}$, $u_{kj} = \begin{cases} 1 & \text{accept a request for ODF } k_j \\ 0 & \text{others} \end{cases}$
- Feasible set of decisions: $u(x) = \{u_{kj} \in \{0,1\} : A_k u_{kj} \leq x, j = 1, ..., n_k, k = 1, ..., K\}$

5.2 HJB Equation

Assume now that the state is (t, x) and consider a time increment δt that is small enough so that we can approximate the probability of an arrival of a request for fare j of itinerary k by $\lambda_{tkj}\delta k$.

$$
\begin{aligned}
V(t,x) = &\sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\delta t \max_{u_{kj}\in\{0,1\}} \left[p_{kj}u_{kj} + V\left(t-\delta t, x - A_k u_{kj}\right)\right] \\
&+ \left\{1 - \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\delta t\right\} V(t-\delta t, x) + o(\delta t)
\end{aligned}
$$

where $o(\delta t)$ is a quantity that goes to zero faster than δt. Subtracting $V(t-\delta t, x)$ from both side of the equation, dividing by δt, and using the notation $\Delta_k V(t,x) = V(t,x) - V\left(t, x - A_k\right)$, we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$
\frac{\partial V(t,x)}{\partial t} = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - \Delta_k V(t,x)\right]^{+}
$$

with boundary conditions $V(t,0) = V(0,x) = 0$ for all $t \geq 0$ and all $x \geq 0$. Notice that term $[p_{kj} - \Delta V_k(t,x)]^{+}$is equivalent to the maximum of $p_{kj}u_{kj} + V(t, x - A_k u_{kj}) - V(t-\delta t, x)$ over $u_{kj} \in \{0,1\}$.

For any vector $z \geq 0$, Define

$$
R_t(u,z) := \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - z_k\right]u_{kj}
$$

and

$$
\mathcal{R}_t(z) := \max_{u} R_t(u,z) = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj} \max_{u_{jk}\in\{0,1\}} \left[p_{kj} - z_k\right]u_{kj} = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - z_k\right]^{+}
$$

Then

$$
\frac{\partial V(t,x)}{\partial t} = \mathcal{R}_t(\Delta V(t,x)), \quad \Delta V(t,x) = \begin{pmatrix} \Delta_1 V(t,x) \\ \Delta_2 V(t,x) \\ \vdots \\ \Delta_K V(t,x) \end{pmatrix}
$$

1. Let's aggregate ODF's into a single index.
2. $n = \sum_{k=1}^{K} n_k$
3. HJB equation:

$$\frac{\partial V(t,x)}{\partial t} = \mathcal{R}_t(\Delta V(t,x)) = \sum_{j\in M_t} \lambda_{tj}\left[p_j - \Delta_j V(t,x)\right]^+$$

   - $V(t,0) = V(0,x) = 0, \quad \forall t \geq 0, x \geq 0$
   - $M_t \subset \{1, ..., n\}$: offered set of fares at t
   - $\Delta_j V(t,x) = V(t,x) - V(t, x - A_j)$

4. Optimal Control:

$$u_j^*(t,x) = \begin{cases} 1 & \text{if } j \in M_t,\ A_j \leq x \text{ and } \underline{p_j \geq \Delta_j V(t,x)} \\ 0 & \text{others} \end{cases}$$

Compute exact $\Delta_j V(t,x)$ can be expensive, we can use heuristics to approx it by $\Delta_j \widetilde{V}(t,x)$

5.3 Upgrades

Let u_j be the set of products that can be used to fulfill a request for product j.
Customers are willing to take any products $k \in u_j$ at the price of product p_j.

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j\in M_t} \lambda_j \max_{k\in u_j} \left[p_j - \Delta_k V(t,x)\right]^+ = \sum_{j\in M_t} \lambda_{tj}\left[p_j - \hat{\Delta}_j V(t,x)\right]^+$$

where $\hat{\Delta}_j V(t,x) = \min_{k\in u_j} \Delta_k V(t,x)$ (Use the least valuable product to fulfill p_j's request.)

5.4 Upsells

Selling j instead of k to get higher revenue, but may be rejected by customers.

- γ_{jk}: revenue obtained from selling product j and fulfilling it with product $k \in u_j$.
- π_{jk}: probability a customer will accept the upgrade from product j to product k.

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j\in M_t} \lambda_j \max_{k\in u_j} \left[\pi_{jk}(r_{jk} - \hat{\Delta}_k V(t,x)) + (1-\pi_{jk})(p_j - \hat{\Delta}_j V(t,x))\right]$$

5.5 Linear programming-based upper bound

The discrete maximum problem is

$$V(t,x) = \max_{u\in U(x)} \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x - u_j A_j)]$$

Deterministic Linear Program
Let D_j be the aggregate demand for ODF j over $[0,T]$.

Then D_j is Poisson with parameter $\Lambda_j = \int_0^T \lambda_{sj} ds$.
Define

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j\in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \le c_i \quad \forall i \in M \\ & 0 \le y_j \le \Lambda_j \quad \forall j \in N. \end{aligned}$$

$$\begin{aligned} \bar{V}(T,c|D) := \max \quad & \textstyle\sum_{j\in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \le c_i \quad \forall i \in M \\ & 0 \le y_j \le D_j \quad \forall j \in N. \end{aligned}$$

Theorem 8 (2.2 of GT)**.**

$$V(T,C) \le \mathbb{E}[\bar{V}(T,c|D)] \le \bar{V}(T,c)$$

$\bar{V}(T,c)$ is the revenue of expected demand, $\mathbb{E}[\bar{V}(T,c|D)]$ is probability combination that is concave in D, so $\mathbb{E}[\bar{V}(T,c|D)] \le \bar{V}(T,c)$. And $V(T,C)$'s decision is feasible in $\mathbb{E}[\bar{V}(T,c|D)]$, so $V(T,C) \le \mathbb{E}[\bar{V}(T,c|D)]$.
<u>Dual formulation</u> of $\bar{V}(T,c)$

$$\begin{aligned} \bar{V}(T,c) := \min \quad & \textstyle\sum_{i\in M} c_i z_i + \sum_{j\in N} \Lambda_j \beta_j \\ \text{s.t.} \quad & \textstyle\sum_{i\in M} a_{ij} z_i + \beta_j \ge p_j \quad \forall j \in N \\ & z_i, \beta_j \ge 0 \quad \forall i \in M, \forall j \in N. \end{aligned}$$

We can simplify the formulation. Since $\beta_j \ge p_j - \sum_{i\in M} a_{ij} z_i$, $\beta_j \ge 0$ and dual is a minimization problem, we can rewrite $\beta_j = [p_j - \sum_{i\in M} a_{ij} z_i]^+$. Then,

$$\sum_{j\in N} \Lambda_j \beta_j = \sum_{j\in N} \Lambda_j [p_j - \sum_{i\in M} a_{ij} z_i]^+ = \int_0^T \mathcal{R}_t(A^T z) dt$$

so,

$$\bar{V}(T,c) = \min_{z\ge 0} \int_0^T \mathcal{R}_t(A^T z) dt + c^T z$$

The optimal solution z_i^* gives an estimation of the marginal value of the i^{th} resource. The approximation of $\Delta_j V(T,c)$ is $\sum_{i\in M} a_{ij} z_i^*$

<u>Bid-price Heuristic</u>
Accept ODF_j if and only if

$$p_j \ge \sum_{i\in M} a_{ij} z_i^* \text{ and } A_j \le x$$

<u>Probabilistic Admission Control (PAC) Heuristic</u>
Accept ODF_j with probability $\frac{y_j^*}{\Lambda_j}$ whenever $A_j \le x$.

Bid-price heuristic is not in general asymptotically optimal.
PAC heuristic is asymptotically optimal.

Theorem 9. *Let $\Pi^b(T,c)$ be the total expected revenue from PAC heuristic and $V^b(T,c)$ be the optimal total expected revenue corresponding to circumstance $b \ge 1$ with capacity bc and $b\lambda_{jt}$. Then*

$$\lim_{b\to\infty} \frac{\Pi^b(T,c)}{V^b(T,c)} = 1$$

5.6 Dynamic Programming Decomposition (DPD)

In this section, we describe two possible approaches for approximating the value functions $V(t,\cdot)$ for the discrete-time formulation

$$V(t,x) = V(t-1,x) + \mathcal{R}_t(\Delta V(t-1,x))$$

Consider the aggregated single index formulation

$$V(t,x) = \max_{u\in U(x)} \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x-u_jA_j)]$$

with $V(t,0) = V(0,x) = 0$ and $\sum_{j=1}^{n} \lambda_{tj} = 1, \lambda_{tj} \geq 0$. (scale can be standardized)

5.6.1 Deterministic Linear Program

The former DLP we use

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j\in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \leq c_i \quad \forall i \in M \\ & 0 \leq y_j \leq \Lambda_j \quad \forall j \in N. \end{aligned}$$

Its dual optimal value is $(z_1^*, z_2^*, ..., z_m^*)$. We choose an arbitrary resource i and relax the first set of constraints for all of the resources except for resource i by associating the dual multipliers $(z_1^*, z_2^*, ..., z_m^*)$ with them.

We relax the first constraints $\sum_{j\in N} a_{ij} y_j \leq c_i \quad \forall i \in M$, which won't change the objective value,

$$\begin{aligned} \max \sum_{j\in N} p_j y_j &= \sum_{j\in N} p_j y_j - \sum_{k\neq i}[\sum_{j\in N} a_{kj} y_j - c_k] z_k \\ &= \sum_{j\in N}[p_j - \sum_{k\neq i} a_{kj} z_k^*] y_j + \sum_{k\neq i} z_k^* c_k \end{aligned}$$

The new DLP is

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j\in N}[p_j - \sum_{k\neq i} a_{kj} z_k^*] y_j + \sum_{k\neq i} z_k^* c_k \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \leq c_i \quad \forall i \in M \\ & 0 \leq y_j \leq \Lambda_j \quad \forall j \in N. \end{aligned}$$

We can prove the optimal y^* and optimal objective values are the same.

Claim 2. *The optimal values y_j^* and optimal objective values of these two DLP are the same.*

(This claim can help prove the upperbound).

$$V(t,x) = \max_{u\in U(x)} \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x-u_jA_j)]$$

We consider the optimal total expected revenue in the single-resource revenue management problem for resource i, the corresponding price of ODF_j should be $p_j - \sum_{k\neq i} a_{kj} z_k^*$. Then the formulation is

$$v_i(t,x_i) = \max_{u\in U_i(x_i)} \sum_{j\in N} \lambda_{tj} \left\{ [p_j - \sum_{k\neq i} a_{kj} z_k^*] u_j + v_i(t-1, x_i - u_j a_{ij}) \right\}$$

We can prove that

- $v_i(T,c_i) \leq \bar{V}(T,c) - \sum_{k\neq i} z_k^* c_k$
- Theorem 2.11 of GT

$$V(t,x) \leq \min_{i\in M}\{v_i(t,x_i) + \sum_{k\neq i} z_k^* x_k\}$$

5.6.2 Lagrangian Relaxation

$$
\begin{array}{lll}
V(t,x) = \max & \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x - u_j A_j)] \\
\text{s.t.} & u_j A_j \le x \\
& u_j \in \{0,1\} \quad \forall \in N
\end{array}
$$

To demonstrate the Lagrangian relaxation strategy, we use decision variables $\{w_{ij} : i \in M, j \in N\}$ in the dynamic programming formulation of the network revenue management problem, where $w_{ij} = 1$ if we make ODF_j available for purchase on flight leg i, otherwise $w_{ij} = 0$.

$$
\begin{array}{lll}
V(t,x) = \max & \sum_{j\in N} \lambda_{tj} \left\{ p_j w_{\psi j} + V\left(t-1, x - \sum_{i\in M} w_{ij} a_{ij} e_i\right)\right\} \\
\text{s.t.} & a_{ij} w_{ij} \le x_i \\
& w_{ij} = w_{\psi j} \\
& w_{ij} \in \{0,1\}, w_{\psi j} \in \{0,1\} \quad \forall i \in M, j \in N
\end{array}
$$

We can relex the second set of constraints by adding Lagrange mutipliers $\{\alpha_{tij} : i \in M, j \in N\}$. Relaxed dynamic program:

$$
\begin{array}{lll}
V^{\alpha}(t,x) = \max & \sum_{j\in N} \lambda_{tj} \left\{ \sum_{i\in M} \alpha_{tij} w_{ij} + \left[p_j - \sum_{i\in M} \alpha_{tij}\right] w_{\psi j} + V^{\alpha}\left(t-1, x - \sum_{i\in M} w_{ij} a_{ij} e_i\right)\right\} \\
\text{s.t.} & a_{ij} w_{ij} \le x_i \\
& w_{ij} \in \{0,1\}, w_{\psi j} \in \{0,1\} \quad \forall i \in, j \in N
\end{array}
$$

Theorem 10 (2.13 of GT)**.** *Assume that the value functions $\{v_i^{\alpha}(t,\cdot) : t = 1,\dots,T\}$ are computed through the dynamic program*

$$
v_i^{\alpha}(t, x_i) = \max_{w_i \in \mathcal{U}_i(x_i)} \left\{ \sum_{j\in N} \lambda_{tj} \left\{ \alpha_{tij} w_{ij} + v_i^{\alpha}(t-1, x_i - w_{ij} a_{ij}) \right\} \right\}
$$

Then

$$
V^{\alpha}(t,x) = \sum_{i\in M} v_i^{\alpha}(t, x_i) + \sum_{\tau=1}^{t} \sum_{j\in N} \lambda_{\tau j} \left[p_j - \sum_{i\in M} \alpha_{\tau ij} \right]^{+}
$$

Theorem 11 (2.14 of GT)**.** *For any set of Lagrange multipliers α, we have*

$$
V(t,x) \le V^{\alpha}(t,x) \quad \forall x \in \mathbb{Z}_{+}^{m}, t = 1, ..., t
$$

The tightest possible upper bound, we can solve the problem

$$
\min_{\alpha \in \mathbb{R}^{Tmn}} V^{\alpha}(T, c)
$$

Lemma 6 (2.15 of GT)**.** *$V^{\alpha}(t,x)$ is a convex function of α for any $t = 1, ..., T$ and $x \in \mathbb{Z}_{+}^{m}$.*

Then compute $\min V^{\alpha}(T,c)$ can be easier.

6 Overbooking

7 Choice Modeling

Dependent Demand Models
With Choice Models, we can make the demand of offered products to the consumers dependent on the menu of the options.

7.1 Basic Models

7.1.1 Notations

- $N = \{1, \ldots, n\}$: the potential products,
- $S \subseteq N$: a subset of products that is offered (referred to as assortment).
- $\pi_j(S)$: probability that a consumer will purchase product j when S is offered.
- $\Pi(S) = \sum_{j \in S} \pi_j(S)$: probability of a sale when S is offered.
- $\pi_0(S) = 1 - \Pi(S)$: probability that the consumer selects outside alternative.

7.1.2 Assortment Optimization Problem (AOP)

Let

- p_j : The price of product $j \in N$.
- z_j : The unit cost of product $j \in N$.
- $p := (p_1, p_2, \ldots, p_n)$: The price vector
- $z := (z_1, z_2, \ldots, z_n)$: The unit cost vector.

If we offer S, the expected profit from offering subset S is given by:

$$R(S, z) := \sum_{j \in S} (p_j - z_j)\, \pi_j(S)$$

AOP aims at finding an assortment S which maximizes $R(S, z)$:

$$\mathcal{R}(z) := \max_{S \subseteq N} R(S, z)$$

7.1.3 Joint Assortment and Pricing Optimization Problem (JAPOP)

If the Price is also a Decision Variable, we should solve the following:

$$\mathcal{R}(z) := \max_{p \geq 0} \sum_{j \in N} (p_j - z_j)\, \pi_j(N, p)$$

For many choice models, if $p_j = \infty$ then, $\pi_j(N, p) = 0$. Thus optimizing over prices could implicitly select an assortment.

7.1.4 Maximum Utility Models (MUM)

Suppose a customer has a Full Ordering of the preferences among the products.

- Assigns cardinal utilities, say $\{u_i : i \in N\}$, to products and ranks them based on utilities.
- The available product with the highest utility is selected with probability 1.

7.1.5 Random Utility Models (RUM)

Add a random noise component to the utilities of the products:

$$U_i = u_i + \varepsilon_i, i \in N.$$

where the ε_i 's are mean zero, possibly dependent random variables.

7.2 Basic Attraction Model (BAM)

7.2.1 Settings

- In BAM each product has an attraction value $v_j > 0$, capturing attractiveness of product j. $v_0 > 0$ represents the attractiveness of the no-purchase alternative.
- The probability of purchasing each item is proportional to the attraction value:

$$\pi_j(S) = \frac{v_j}{v_0 + \sum_{i\in S} v_i} \quad \forall j \in S$$

Denote, for any $S \subseteq T$:

- $\pi_S(T) := \sum_{j\in S} \pi_j(T) = \frac{\sum_{j\in S} v_j}{v_0+\sum_{i\in T} v_i} = \frac{\sum_{j\in S} v_j}{\sum_{i\in T^+} v_i}$, the probability that a consumer selects a product in S when the set T is offered.
- $\pi_{S^+}(T) := \pi_S(T) + \pi_0(T) = \frac{v_0+\sum_{j\in S} v_j}{v_0+\sum_{i\in T} v_i} = \frac{\sum_{j\in S^+} v_j}{\sum_{i\in T^+} v_i}$, including the no-purchase alternative.

7.2.2 Luce Axioms for BAM

A discrete choice model satisfies the axioms iff it is of the BAM form:

- Axiom 1 :If $\pi_i(\{i\}) \in (0,1)$ for all $i \in T$, then for any $Q \subseteq S_+$, $S \subseteq T$:

$$\pi_Q(T) = \pi_Q(S)\pi_{S_+}(T)$$

$\frac{\sum_{i\in Q} v_i}{\sum_{i\in T^+} v_i} = \frac{\sum_{i\in Q} v_i}{\sum_{i\in S^+} v_i}\frac{\sum_{i\in S^+} v_i}{\sum_{i\in T^+} v_i}$

- Axiom 2: If $\pi_i(\{i\}) = 0$ for some $i \in T$, then for any $S \in T$ such that $i \in S$:

$$\pi_S(T) = \pi_{s\setminus\{i\}}(T\setminus\{i\})$$