Game Theory

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All models are wrong, but some are useful.

Contents

Chapter 1 Signalling Game

Based on

- "Kreps, D. M., & Sobel, J. (1994). Signalling. *Handbook of game theory with economic applications*, 2, 849-867."
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1.1 Canonical Game

Definition 1.1 (Canonical Game)

1. There are two players: $\mathbf{S}$ (sender) and $\mathbf{R}$ (receiver).
2. $\mathbf{S}$ holds more information than $\mathbf{R}$: the value of some random variable t with support $\mathcal{T}$. (We say that t is the **type** of $\mathbf{S}$)
3. Prior belief of $\mathbf{R}$ concerning t are given by a probability distribution ρ over $\mathcal{T}$ (common knowledge)
4. $\mathbf{S}$ sends a **signal** $s \in \mathcal{S}$ to $\mathbf{R}$ drawn from a signal set $\mathcal{S}$.
5. $\mathbf{R}$ receives this signal, and then takes an **action** $a \in \mathcal{A}$ drawn from a set $\mathcal{A}$ (which could depend on the signal s that is sent).
6. $\mathbf{S}$'s payoff is given by a function $u : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ and $\mathbf{R}$'s payoff is given by a function $v : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.

♣

1.2 Nash Equilibrium

Definition 1.2 (Strategy)

A **behavior strategy** for $\mathbf{S}$ is given by a function $\sigma : \mathcal{T} \times \mathcal{S} \to [0,1]$ such that $\sum_s \sigma(t,s)$ for each t.

A **behavior strategy** for $\mathbf{R}$ is given by a function $\alpha : \mathcal{S} \times \mathcal{A} \to [0,1]$ such that $\sum_a \alpha(s,a)$ for each t. ♣

Definition 1.3 (Nash Equilibrium)

Behavior strategies α and σ form a **Nash equilibrium** if and only if

1. For all $t \in \mathcal{T}$,

$$\sigma(t,s) > 0 \text{ implies } \textstyle\sum_a \alpha(s,a)u(t,s,a) = \max_{s' \in \mathcal{S}} \left(\sum_a \alpha(s',a)u(t,s',a)\right)$$

2. For each $s \in \mathcal{S}$ such that $\sum_t \sigma(t,s)\rho(t) > 0$,

$$\alpha(s,a) > 0 \text{ implies } \textstyle\sum_t \mu(t;s)v(t,s,a) = \max_{a'} \sum_t \mu(t;s)v(t,s,a')$$

where $\mu(t;s)$ is the $\mathbb{R}$'s posterior belief about t given s, $\mu(t;s) = \frac{\sigma(t,s)\rho(t)}{\sum_{t'}\sigma(t',s)\rho(t')}$ if $\sum_t \sigma(t,s)\rho(t) > 0$ and $\mu(t;s) = 0$ otherwise. ♣

Definition 1.4 (Separating & Pooling Equilibrium)

An equilibrium (σ, α) is called a **separating** equilibrium if each type t sends different signals; i.e., the set $\mathcal{S}$ can be partitioned into (disjoint) sets $\{\mathcal{S}_t; t \in \mathcal{S}\}$ such that $\sigma(t, \mathcal{S}_t) = 1$. An equilibrium (σ, α) is called a **pooling** equilibrium if there is a single signal s^* that is sent by all types; i.e., $\sigma(t, s^*) = 1$ for all $t \in \mathcal{T}$. ♣

1.3 Single-crossing

1.3.1 Situation over real line

Consider the situation that $\mathcal{T}, \mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$ and $\geq$ is the usual "greater than or equal to" relationship.

1. We let $\Delta\mathcal{A}$ denote the set of probability distributions on $\mathcal{A}$.
2. For each $s \in \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$, we let $\Delta\mathcal{A}(s, T')$ be the set of mixed strategies that are the best responses by $\mathbf{R}$ to $s \in \mathcal{S}$ for some probability distribution with support $\mathcal{T}'$.
3. For $\alpha \in \Delta\mathcal{A}$, we write $u(t, s, \alpha) \triangleq \sum_{a \in \mathcal{A}} u(t, s, a)\alpha(a)$.

Definition 1.5 (Single-crossing)

The data of the game are said to satisfy the **single-crossing property** if the following holds: If $t \in \mathcal{T}$, $(s, \alpha) \in \mathcal{S} \times \Delta\mathcal{A}$ and $(s', \alpha') \in \mathcal{S} \times \Delta\mathcal{A}$ are such that $\alpha \in \Delta\mathcal{A}(s, \mathcal{T})$, $\alpha' \in \Delta\mathcal{A}(s', \mathcal{T})$, $s > s'$ and $u(t, s, \alpha) \geq u(t, s', \alpha')$, then for all $t' \in T$ such that $t' > t$, $u(t', s, \alpha) \geq u(t', s', \alpha')$. ♣