Game Theory

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Mind offline, notes online.

Contents

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Based on

- "Kreps, D. M., & Sobel, J. (1994). Signalling. *Handbook of game theory with economic applications*, 2, 849-867."
- Mas-Colell, Whinston, and Green, Microeconomic Theory, Oxford University Press (1995).
- UIUC ECON 530 21Fall, Nolan H. Miller
- UC Berkeley ECON 201A 23Fall, 201B 24Spring
- UC Berkeley MATH 272 23Fall, Alexander Teytelboym
- Jehle, G., Reny, P.: Advanced Microeconomic Theory. Pearson, 3rd ed. (2011). Ch. 6.
- MIT 14.16 Strategy and Information, Mihai Manea

1.1 Basic Game Theory

1.1.1 Action and Domination Theorem

Let A be the finite set of possible actions and Ω be the finite set of possible states. A function can map the action and state to a value, $u(a, \omega)$. It can be represented by $\vec{u}(a) = \{u(a, \omega)\}_{\omega \in \Omega}$. It is common in game theory to assume the utility function is given or known.

A **mixed action** is a probability distribution over A, $\sigma \in \Delta(A)$.

A **belief** of the agent is a probability distribution over Ω, $\mu \in \Delta(\Omega)$.

Definition 1.1 (Optimal and Justifiable Mixed Action)

A mixed action $\sigma \in \Delta(A)$ is **optimal** given $\mu \in \Delta(\Omega)$ if

$$\mathbb{E}_\mu u(\sigma, \tilde{\omega}) \geq \mathbb{E}_\mu u(\sigma', \tilde{\omega}), \ \forall \sigma' \in \Delta(A)$$

A mixed action $\sigma \in \Delta(A)$ is **justifiable** if it is optimal for some belief $\mu \in \Delta(\Omega)$.

Definition 1.2 (Dominant and Dominated Action)

A mixed action $\sigma \in \Delta(A)$ is **dominant** if

$$u(\sigma, \omega) > u(\sigma', \omega), \ \forall \omega \in \Omega, \sigma' \in \Delta(A), \sigma \neq \sigma'$$

A mixed action $\sigma \in \Delta(A)$ is **dominated** if

$$u(\sigma, \omega) < u(\sigma', \omega), \ \forall \omega \in \Omega, \text{ and for some } \sigma' \in \Delta(A)$$

In this case we say σ' dominates σ.

Theorem 1.1 (Domination Theorem: Justifiable = Not Dominated)

A mixed action is justifiable if and only if it is not dominated.

Proof

$\Rightarrow$ is easily proved by the definition. We focus on proving $\Leftarrow$:

Let $\mathcal{U} = \{\vec{u}(\sigma) : \sigma \in \Delta(A)\}$ and σ^* be an undominated mixed action. Then, we have $\mathcal{U} \cap (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++}) = \emptyset$. Because $\mathcal{U}$ and $\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++}$ are disjoint, convex, and nonempty, we can use the Separating Hyperplane Theorem **??**: $\exists p \in \mathbb{R}^{\Omega}, p \neq 0$ such that $p \cdot a \leq p \cdot b, \forall a \in \mathcal{U}, b \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$.

Claim 1.1

$p \cdot \vec{u}(\sigma) \leq p \cdot \vec{u}(\sigma^*), \forall \sigma \in \Delta(A)$.

Proof

For any positive number m, $\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m}) \in \{\vec{u}(\sigma')\} + \mathbb{R}^{\Omega}_{++}$. So, for any $\sigma \in \Delta(A)$, $p \cdot \vec{u}(\sigma) \leq p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right)$. By taking limit, $p \cdot \vec{u}(\sigma^*) = \lim_{m \to \infty} p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right) \geq p \cdot \vec{u}(\sigma)$.

Claim 1.2

$p > 0$.

Proof

Prove by the contradiction. Suppose $p_\omega < 0$ for some $\omega \in \Omega$. Let $z = (\epsilon, ..., \epsilon) + M\mathbb{1}_\omega, M > 0, \epsilon > 0$. So, $\vec{u}(\sigma^*) + z \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$. We have $p \cdot \vec{u}(\sigma^*) \leq p \cdot (\vec{u}(\sigma^*) + z)$ by the result of SHT. There is a contradiction since $p_\omega < 0$. So, we have $p \geq 0$. Because $p \neq 0$, $p > 0$ is proved.

Finally, we normalize p to $\mu = \frac{1}{\sum_\omega p_\omega} p$. Then, σ^* is optimal for the belief μ, which means σ^* is justifiable.

1.1.2 Extensive Game

Definition 1.3 (History)

The sequences of actions are called **histories**. $h' = (\underbrace{a_1, ..., a_n}_{h\text{:prefix of } h'}, a_{n+1}, ...) \in H$. We call h' is the **continuation** of h. h is a **terminal** of H if there is no continuation of h in H. ($\emptyset \in H$.)

Definition 1.4 (Extensive form Perfect Information Game)

Am extensive form game with prefect information is defined as $G = \{N, A, H, Z, P, O, o, \succ_{n\in N}\}$, where N is the set of players, A is the set of actions, H is the set of all histories, Z is the set of all histories that are terminals, $P : H/Z \to N$ is a mapping from a non-terminal histories to a player (who moves after a non-terminal history), O is the set of outcomes, and o is a function from Z to O.

A PIG is finite horizon if there is a bound on the length of its histories.

We denote $A(h)$ as the actions available to player $P(h)$ after a history h.

Let $H_i = \{h \in H/Z : i = P(h)\}$ be the set of histories that player i moves after.

Definition 1.5 (Strategy)

A **strategy** is defined as a function $s_i : H_i \to A$ for which $s_i(h) \in A(h), \forall h \in H_i$. Let S_i be the set of all strategies available to the player i. A **strategy profile** is a collection of strategy $s = (s_i)_{i\in N}$.

Definition 1.6 (Subgame)

A **subgame** of a PIG $G = \{N, A, H, Z, P, O, o, \succ_{n\in N}\}$ is a game (a PIG) that starts after a given finite history $h \in H$. Formally, the subgame $G(h)$ associated with $h = (h_1, ..., h_n) \in H$ is $G(h) = \{N, A, H_h, Z, P_h, O, o_h, \succ_{n\in N}\}$, where

$$H_h = \{(a_1, a_2, ...) : (h_1, ..., h_n, a_1, a_2, ...) \in H\}$$

$$o_h(h') = o(hh'), P_h(h') = P(hh')$$

A strategy s of G defines a strategy s_h of $G(h)$ by $s_h(h') = s(hh')$.

Definition 1.7 (Subgame Perfect Equilibrium (SPNE))

A **subgame perfect equilibrium (SPNE)** of G is a strategy profile s^* such that for every subgame $G(h)$ it holds that $h' \mapsto s_i^*(hh')$ is an optimal strategy in $G(h)$, given beliefs that the rest of the players behave according to s_{-i}^* (or its restriction to $G(h)$).

Definition 1.8 (Profitable Deviation)

Let s be a strategy profile. We say that s_i' is a **profitable deviation** from s for player i at history h if s_i' is a strategy for G such that

$$o_h\left(s_i', s_{-i}\right) \succ_i o_h(s)$$

Note that a strategy profile has no profitable deviations iff it's a SPNE.

Theorem 1.2 (The one-deviation principle)

Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a finite horizon, extensive form game with perfect information. Let s be a strategy profile that is not a subgame perfect equilibrium. There exists some history h and a profitable deviation $\bar{s}_i$ for player $i = P(h)$ in $G(h)$ such that $\bar{s}_i(k) = s_i(k)$ for all $k \neq h$.

- Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a PIG.
- $A(\emptyset)$ is the set of allowed initial actions for player $i = P(\emptyset)$. For each $b \in A(\emptyset)$, let $s^{G(b)}$ be some strategy profile for the subgame $G(b)$.
- Given some $a \in A(\emptyset)$, we denote by s^a the strategy profile for G in which player $i = P(\emptyset)$ chooses the initial action a, and for each action $b \in A(\emptyset)$ the subgame $G(b)$ is played according to $s^{G(b)}$.
- So $s_i^a(\emptyset) = a$ and for every player $j, b \in A(\emptyset)$ and $bh \in H \backslash Z$, $s_j^a(bh) = s_j^{G(b)}(h)$.

Lemma 1.1 (Backward Induction)

Let $G = \left(N, A, H, Z, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a finite PIG. Assume that for each $b \in A(\emptyset)$ the subgame $G(b)$ has a subgame perfect equilibrium $s^{G(b)}$. Let $i = P(\emptyset)$ and let a be the $\succ_i$-maximizer over $A(\emptyset)$ of $o_a\left(s^{G(a)}\right)$. Then s^a is a subgame perfect equilibrium of G.

1.1.3 Strategic Form Game

Definition 1.9 (Normal Form Game)

A game in **normal form** is denoted by

$$G = \left(\underbrace{N}_{\text{players}}, \underbrace{\{S_i\}_{i \in N}}_{\text{Strategy Set}}, \underbrace{\{u_i(\cdot)\}_{i \in N}}_{\text{VNM utility}}\right)$$

$u_i : \prod_{i \in I} S_i \rightarrow \mathbb{R}$ is the utility function that maps all players' strategies to a player's utilities.

A finite game is a normal-form game in which the set of players N is a finite set, and the set of strategy profiles S is finite.

Definition 1.10 (Mixed/Pure Strategy)

A mixed strategy for player i is a probability distribution $\sigma_i \in \Delta(S_i)$.

Elements of S_i are called pure strategies.

Definition 1.11 (Dominant/Dominated Strategy)

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominant strategy** for i in G, if we have $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \neq \sigma_i, \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i$, $u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **weakly dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i$, $u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$ and there is a $\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$, $u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$

Lemma 1.2

1. A dominant strategy is always pure.
2. A strategy σ_i' dominates σ_i iff $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i})$, for all pure strategy profiles $s_{-i} \in S_{-i}$.

Definition 1.12 (Belief, Best Response)

A **belief** for player i is a probability distribution $\mu \in \Delta(S_{-i})$.

A strategy $\sigma_i \in \Delta(S_i)$ is the **best response** to beliefs μ if it solves the problem of $\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, s_{-i})$.

Denote the set of all best responses to μ by $\beta_i(\mu)$.

Lemma 1.3 (Mixed Strategy is BR iff its Pure Strategies are Indifferent)

A mixed strategy σ_i is in $\beta_i(\mu)$ iff every pure strategy in the support of σ_i is in $\beta_i(\mu)$. In particular, every strategy in the support of σ_i yields the same payoff to i.

Theorem 1.3 (Domination Theorem rephrased)

In a finite game, a strategy is dominated iff there is no belief to which it is a best response.

Definition 1.13 (Algorithm: Iterated Elimination of Dominated Strategies (IEDS))

Let $(N, (S_i), (u_i))$ be a finite game; $N = [n]$.

- We define (inductively) n sequences of sets of mixed strategies.
- Let $D_i^0 = \Delta(S_i)$.
- Given $D_1^{k-1}, \ldots, D_n^{k-1}$, let

$$D_i^k = \left\{\sigma_i : \nexists \bar{\sigma}_i : u_i(\sigma_i, \sigma_{-i}) < u_i(\bar{\sigma}_i, \sigma_{-i}) \, \forall \sigma_{-i} \in \times_{j \neq i} D_j^{k-1}\right\}.$$

- Note that $\{D_i^k\}$ is a decreasing sequence of sets.
- Let $D_i = \cap_{k=0}^{\infty} D_i^k$.
- The set $D = \times_{i=1}^n D_i$ be the set of strategies that survive the iterated elimination of dominated strategies.

A game is called **dominance-solvable** if D is a singleton.

Definition 1.14 (Rationalizable Strategies)

- $R_i^0 = \Delta(S_i)$.
- Given $R_1^{k-1}, \ldots, R_n^{k-1}$, Let

$$Z_i^k = \left\{s_i \in S_i : \sigma_i(s_i) > 0 \text{ for some } \sigma_i \in R_i^{k-1}\right\}$$
$$R_i^k = \left\{\sigma_i \in \Delta(S_i) : \exists \mu \in \Delta\left(\times_{j \neq i} Z_j^k\right) \text{ s.t. } \sigma_i \in \beta_i(\mu)\right\}$$

Note: $\{R_i^k\}_{k=0}^{\infty}$ is a decreasing sequence of sets.

Let $R_i = \cap_{k=0}^{\infty} R_i^k$.

The **rationalizable strategies** are the elements of $R = \times_{i=1}^n R_i$.

Lemma 1.4

In a finite game, R is always non-empty and contains a pure strategy profile.

Proposition 1.1

$\sigma_i \in \Delta(S_i)$ is **rationalizable** iff there are sets $Z_1, \ldots, Z_n, Z_j \subseteq S_j$ such that

1. $\sigma_i \in \beta_i(\mu_i)$ for some $\mu_i \in \Delta(\times_{h \neq i} Z_h)$.
2. for every $s_j \in Z_j$ there is $\mu_j \in \Delta(\times_{h \neq j} Z_h)$ such that $s_j \in \beta_j(\mu_j)$.

Corollary 1.1 (Rationalizable = IEDS)

Rationalizable strategies are exactly the strategies survive the iterated elimination of dominated strategies,

$$R = D$$

1.1.4 Nash Equilibrium and Existence

Definition 1.15 (Nash Equilibrium)

A strategy profile $\Sigma = (\sigma_1, ..., \sigma_I)$ is a **Nash** equilibrium of the game G if for every $i \in I$, we have: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Delta(S_i)$ (no profitable deviation). In other words,

1. σ_i is the best response to beliefs $\mu_i \in \Delta(S_{-i})$
2. $\mu_i = \sigma_{-i}$ (correct beliefs).

1. In rationalizable strategies, beliefs can be incorrect.
2. In a Nash equilibrium, beliefs are correct. Any strategy in a Nash equilibrium is rationalizable.

Definition 1.16 (Best Response Correspondence)

In a Nash equilibrium the player i's best response correspondence $\beta_i : \Delta(S_{-i}) \to 2^{\Delta(S_i)}$ is defined as $\beta_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})$. Let $\beta(\sigma) = \times_{i \in I} \beta_i(\sigma_{-i})$. Then σ is a Nash equilibrium iff $\beta(\sigma) = \sigma$. β is called the **best response correspondence** of the game.

Theorem 1.4 (Existence of Nash Equilibrium)

A Nash equilibrium exists in a finite game Γ, if for all $i \in I$,

(i). S_i is non-empty, convex, compact, subset of $\mathbb{R}^m$ (i.e., for some finite dimensions of real numbers).

(ii). $u_i(s_i, ..., s_I)$ is continuous in $(s_i, ..., s_I)$ and quasi-concave in any s_i.

Proof

We prove a lemma for the best response correspondence $\beta_i(s_{-i}) = \text{argmax}_{s_i \in S_i} u(s_i, s_{-i})$ firstly.

Lemma 1.5

Suppose $\{S_i\}_{i \in I}$ are non-empty. Suppose that S_i is compact and convex and u_i is continuous in $(s_i, ..., s_I)$ and quasi-concave in any s_i, then best response correspondence $\beta_i(s_{-i})$ is non-empty, convex-valued and uhc.

Proof

This lemma is proved by Berge's Maximum Theorem (Theroem **??**).

Consider the best response correspondence of the game β with $\beta(s_i, ..., s_I) = \{\beta_1(s_{-1}), ..., \beta_I(s_{-I})\}$.

As we proved β is non-empty, convex-valued and uhc from S to S where S is non-empty, compact, and convex. By the Kakutani's Fixed Point Theorem (Theorem **??**), we have β has a fixed point $s \in S$, which should be the Nash equilibrium.

1.1.5 Bayesian Game

Definition 1.17 (Bayesian Game)

A **Bayesian game** is defined by

$$\Gamma = (I, \Omega, \{A_i\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}, \{F_i\}_{i\in I})$$

where Ω is the state space, $u_i : A \times \Omega$ is i's payoff function, and $F_i \in \Delta(\Omega \times \Theta_i)$ is the (prior) distribution of the player i's type.

Definition 1.18 (Normal-form Bayesian game)

Assume a finite game. The **normal-form game** can be represented by

$$(I, (S_i, U_i)_{i\in I})$$

defined by letting S_i be the set of strategies based on types $s_i : \Theta_i \to A_i$ and

$$U_i(s) = \sum_{\omega\in\Omega} \sum_{(\theta_i)_{i\in I}\in\Theta} p(\omega, \theta_1, ..., \theta_I) u_i(s_1(\theta_1), ..., s_I(\theta_I), \omega)$$

for all $s \in S$.

A **Bayesian Nash equilibrium** (BNE) of a Bayesian game is a strategy profile $(s_1, ..., s_n)$ that is a Nash equilibrium of the derived normal-form game.

Definition 1.19 (Best Response, Interim Payoff)

s_i is a BR to s_{-i} iff for all θ_i, $s_i(\theta_i)$ maximizes the **interim payoff** of player i. The interim payoff is defined by the expected payoff given the type θ_i of player i by playing action a_i.

$$\mathbb{E}_{\omega\in\Omega, \tilde{\theta}_{-i}\in\Theta_{-i}}[u_i(a_i, s_{-i}(\tilde{\theta}_{-i}), \omega)|\theta_i]$$

1.1.6 Zero-sum Game

Let $G = (\{1, 2\}, (S_1, S_2), (u_1, u_2))$. Suppose there is a constant c so that $u_1(s) + u_2(s) = c, \forall s \in S$. Then G is equivalent to a zero-sum game.

Definition 1.20 (Saddle Point)

Let X, Y be sets and $f : X \times Y \to \mathbb{R}$ a real function. $(x^*, y^*) \in X \times Y$ is a **saddle point** of f if $x^* \in \text{argmax}_{x\in X} f(x, y^*)$ and $y^* \in \text{argmin}_{y\in Y} f(x^*, y)$. That is,

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \forall x \in X, y \in Y$$

Consider a zero-sum game of two players. The strategy of the player 1 is max-min strategy, which is given by

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2)$$

and the strategy of the player 2 is min-max strategy, which is given by

$$\min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

Proposition 1.2 (min-max≥max-min)

Min-max strategy is always better than max-min strategy. That is,

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

Proof

As $u(\sigma_1', \sigma_2) \leq \max_{\sigma_1} u(\sigma_1, \sigma_2)$ for all σ_1', we have

$$\min_{\sigma_2} u(\sigma_1', \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2), \forall \sigma_1'$$

$$\Rightarrow \max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

We shall prove that these are in fact equal in the zero-sum game.

Definition 1.21 (Value of a zero-sum game)

A **value** for a zero-sum game G is a number $v \in \mathbb{R}$ for which there exists a strategy profile $(\bar{\sigma}_1, \bar{\sigma}_2)$ such that

$$u(\bar{\sigma}_1, \sigma_2) \geq v \quad \text{for all } \sigma_2$$
$$u(\sigma_1, \bar{\sigma}_2) \leq v \quad \text{for all } \sigma_1$$

Note: $v = u(\bar{\sigma}_1, \bar{\sigma}_2)$. The value is unique (if its exists), and represents a guaranteed payoff for the players. (Uniqueness: Suppose there are two values, v and $v' > v$, achieved by profiles σ and σ'. Then when 2 plays σ_2 we have that $u(\sigma_1', \sigma_2) \geq v'$ because σ_1' guarantees v'. And when 1 plays σ_1' we have $v \geq u(\sigma_1', \sigma_2)$ because σ_2 guarantees v. So $v' > v$ leads to a contradiction.)

Proposition 1.3

The following statements are equivalent

1. $(\bar{\sigma}_1, \bar{\sigma}_2)$ is a Nash equilibrium.
2. $v = u(\bar{\sigma}_1, \bar{\sigma}_2)$.

Corollary 1.2

If (σ_1^*, σ_2^*) and $(\bar{\sigma}_1, \bar{\sigma}_2)$ are Nash equilibria of G, then so are the profiles $(\bar{\sigma}_1, \sigma_2^*)$ and $(\sigma_1^*, \bar{\sigma}_2)$.

Theorem 1.5 (Minimax Theorem)

Let G be a zero-sum game. There is a strategy profile (σ_1^*, σ_2^*) s.t.

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) = v = \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

where $v = u(\sigma_1^*, \sigma_2^*)$ is the value of the game and (σ_1^*, σ_2^*) is a Nash equilibrium.

Proof

Let $n_i = |S_i|$. Let $\vec{u}(\sigma_2) := \{u(s_1, \sigma_2) : s_1 \in S_1\}$. Let $\mathbb{C} = \{\vec{u}(\sigma_2) : \sigma_2 \in \Delta(S_2)\}$. We can find $\mathbb{C}$ is convex and compact.

Let $m(x) := \max\{x_i : i = 1, ..., n_1\}$. Then, the player 2's min max payoff is given by $v := \inf\{m(x) : x \in \mathbb{C}\}$. By compactness, exists strategy σ_2^* such that $\vec{u}(\sigma_2^*) = (v, v, ..., v)$. That is, $u(s_1, \sigma_2^*) = v, \forall s_1 \in S_1$.

Let $\mathbb{A} := \{z \in \mathbb{R}^{n_1} : z << (v, v, ..., v)\} = (v, v, ..., v) - \mathbb{R}^{n_1}_{++}$. As $\mathbb{C}$ and $\mathbb{A}$ are disjoint. By SHT, we can find a $p \neq 0$ s.t. $p \cdot \mathbb{A} \leq p \cdot \mathbb{C}$. $p > 0$ since $\mathbb{A}$ can have arbitrary small elements in any dimension. Then, we normalize p to be in $\Delta(S_1)$, denote it by σ_1^*.

By limitation, $v = \sigma_1^* \cdot (v, v, ..., v) = \lim_{\epsilon \to 0^+} \sigma_1^* \cdot (v - \epsilon, v - \epsilon, ..., v - \epsilon) \leq \sigma_1^* \cdot \vec{u}(\sigma_2), \forall \sigma_2 \in \Delta(S_2)$.

Hence, $u(s_1, \sigma_2^*) \leq m(\vec{u}(\sigma_2^*)) = v = u(\sigma_1^*, \sigma_2^*) \leq u(\sigma_1^*, \sigma_2)$

1.1.7 Correlated equilibrium

Suppose there is a mediator that give advices to each player based on a distribution $p \in \Delta(S)$. A player doesn't know other players' advices but knows the distribution p.

Definition 1.22 (Correlated Equilibrium)

A **correlated equilibrium** of G is any probability distribution $p \in \Delta(S)$ such that, for all i and $s_i, s_i' \in S_i$, the player i can't get a higher expected profit than by following the advice,

$$\sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i}) u_i(s_i', s_{-i})$$

where LHS is the expected profit of player i when he receives an advice s_i from the mediator.

Let G be a finite game,

Proposition 1.4

If we identify a Nash equilibrium σ of G with a probability distribution on $\Delta(S)$, then any Nash equilibrium of G is also a correlated equilibrium.

Proposition 1.5

The set of correlated equilibria is a non-empty, convex, compact subset of $\Delta(S)$.

For $p \in \Delta(S)$, the **marginal distribution** on S_i is given by

$$p_i(s_i) = \sum_{s_{-i} \in S_{-i}} p(s_i, s_{-i})$$

Proposition 1.6

A correlated equilibrium that is the independent mixture of its marginal distributions is a Nash equilibrium.

Proposition 1.7 (Correlated Equilibrium Strategy $\Rightarrow$ Rationalizable)

Let $G = (S_i, u_i)_{i=1}^n$ be a finite normal-form game, and ρ a correlated equilibrium of G. Suppose that the profile (s_i, s_{-i}) receives strictly positive probability in $\rho : \rho(s_i, s_{-i}) > 0$. s_i is rationalizable.

Proof

Let Z_i be the set of pure strategies of player i that receive strictly positive probability in ρ_i. For each $s_i \in Z_i$ we have that $\sum_{s_{-i} \in S_{-i}} \rho(s_i, s_{-i}) \geq \sum_{s_{-i} \in S_{-i}} \rho(s_i', s_{-i}), \forall s_i' \in S_i$, so for any $s_i' \in S_i$:

$$\begin{aligned}
\mathbb{E}_{\mu_i} u_i(s_i, s_{-i}) &= \sum_{s_{-i} \in S_{-i}} \frac{\rho(s_i, s_{-i})}{\sum_{\tilde{s}_{-i}} \in S_{-i} \rho(s_i, \tilde{s}_{-i})} u_i(s_i, s_{-i}) \\
&\geq \sum_{s_{-i} \in S_{-i}} \frac{\rho(s_i, s_{-i})}{\sum_{\tilde{s}_{-i}} \in S_{-i} \rho(s_i, \tilde{s}_{-i})} u_i(s_i', s_{-i}) = \mathbb{E}_{\mu_i} u_i(s_i', s_{-i})
\end{aligned}$$

where $\mu_i \in \Delta(S_{-i})$ are the beliefs over S_{-i} obtained by ρ conditioning on s_i. This means that s_i is a best response to beliefs μ_i. Since i and $s_i \in Z_i$ are arbitrary, we are done.

1.1.8 Quantal Response Equilibrium

Imagine choosing $a \in A$.

Definition 1.23 (Quantal Response (softmax))

A **quantal response** is a function $\gamma : \mathbb{R}^A \to \Delta(A)$ mapping from a vector of utility values v to a probability distribution over actions, which satisfies that

- $\gamma(v) \gg 0$ for all v (interior);
- γ is continuous,
- γ is monotonic $(v_h < v_j \to \gamma_h(v) < \gamma_j(v))$
- γ is responsive $\left(v_j < v_j' \to \gamma_j(v) < \gamma_j\left(v_j', v_{-j}\right)\right)$

Interpret $\gamma(v)$ as the probability of choosing each of $a \in A$ alternatives when v is the vector of utility values of the alternatives in A.

Interpretation: mistakes or random utility.

One common quantal response function is the logistic function:

$$\gamma_j(v) = \frac{e^{\lambda v_j}}{\sum_{h \in I} e^{\lambda v_h}}$$

for $\lambda > 0$, where λ captures how close the quantal response is to choosing according to the largest values of v.

Fix a normal-form game $G = \{N, \{S_i, i \in N\}, \{u_i, i \in N\}\}$. Let $\gamma_i : \mathbb{R}^{S_i} \to \Delta(S_i)$ be a quantal response for player i.

Definition 1.24 (Quantal Response Equilibrium)

A **quantal response equilibrium** of G is a strategy profile σ^* such that

$$\sigma_i^* = \gamma_i\left(\vec{u}_i(\sigma_{-i}^*)\right)$$

where $\vec{u}_i(\sigma_{-i}^*) = (u_i(s_i, \sigma_{-i}^*))_{s_i \in S_i}$.

Proposition 1.8

Every finite normal-form game, with any profile of quantal responses, has a quantal response equilibrium.

<u>Observation</u>: λ measures the distance to Nash.

Proposition 1.9

Let $\{\lambda^k\}$ be a sequence such that $\lim_{k\to\infty} \lambda^k = \infty$ and $\sigma^*(\lambda^k)$ be a QRE when the logistic quantal responses take parameter value λ^k. If $\{\sigma^*(\lambda^k)\}$ is a convergent sequence, it converges to a Nash equilibrium.

1.2 Knowledge and Common Knowledge

1.2.1 Knowledge and Information

1. Let Ω be a (finite) set of possible states of the world. Information is provided by a subset of Ω. The smaller a subset is, the more information it provides.
2. Subsets $E \subseteq \Omega$ are **events**.
3. **Definition 1.25 (Information Function)**

   We define the function $P : \Omega \to 2^{\Omega}$ an **information function**. $P(\omega)$ is the set of states that the agent considers possible when the actual state is ω.

   When the state is ω the decision-maker knows only that the state is in the set $P(\omega)$. Means that, if $\omega' \in P(\omega)$, then when the state is ω, information doesn't allow one to distinguish between ω and ω'.
4. **Definition 1.26 (Knowledge)**

   Knowledge is modeled through a function $K : 2^{\Omega} \to 2^{\Omega}$. $K(E)$ (which we write as KE) is the set of states at which the agent knows that the event E has occurred. That is, given an information function $P : \Omega \to 2^{\Omega}$, the **knowledge** $K : 2^{\Omega} \to 2^{\Omega}$ is defined as

   $$KE = \{\omega \in \Omega : P(\omega) \subseteq E\}$$

   We can say KE is the set of states that ``the agent knows E."
5. Given a state ω, $\{E \subseteq \Omega : KE \ni \omega\} = \{E \subseteq \Omega : P(\omega) \subseteq E\}$ is the set of all events that the agent "knows" (all events that the agent believes are possible). The most accurate information provided by it is $\cap\{E \subseteq \Omega : KE \ni \omega\}$. Hence,

   $$P(\omega) = \cap\{E \subseteq \Omega : KE \ni \omega\}$$

These two equations provide the back and fourth relationship between knowledge and information. However, we don't give any restrictions for the settings of the knowledge or the information function, so they can be any forms.

1.2.2 Partitional Information Function

Definition 1.27 (P1&P2 Conditions for Information Function)

Usually, we assume following two conditions of a information function:

P1. $\omega \in P(\omega)$ for every $\omega \in \Omega$. (Reality will not be excluded from agent's information structure.)

P2. If $\omega' \in P(\omega)$ then $P(\omega') = P(\omega)$.

Definition 1.28 (S5 Conditions for Knowledge)

There are 5 axioms of knowledge that can restrict the form of knowledge.

1. $K\Omega = \Omega$

   Alice knows that some state of the world has occurred.

2. $KA \cap KB = K(A \cap B)$

   Alice knows A and knows B iff she knows A and B.

3. $KA \subseteq A$ (Axiom of knowledge)

   If Alice knows A, then A has indeed occurred (some state in A is true).

4. $KKA = KA$ (Axiom of positive introspection)

   If Alice knows A then she knows that she knows A.

5. $(KA)^c = K\left((KA)^c\right)$ (Axiom of negative introspection)

   If Alice doesn't know A then she knows that she doesn't know A.

They are not independent.

Definition 1.29 (Partitional)

Information function P (and associated knowledge operator K) is **partitional** if $\{P(\omega) : \omega \in \Omega\}$ constitute a partition of Ω.

When P is partitional we abuse notation and denote the partition by P as well.

Theorem 1.6 (S5 ⇔ P1&P2 ⇔ Partitional)

For a P (and associated K), following conditions are equivalent:

- P/K is partitional;
- P satisfies P1 and P2.
- K satisfies S5.

Example 1.1 (Partitional)

$\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

1. Information Structure: $P(\omega_1) = P(\omega_2) = \{\omega_1, \omega_2\}$, $P(\omega_3) = \{\omega_3\}$, and $P(\omega_4) = \{\omega_4\}$.
2. Knowledge Function: $K(\{\omega_3, \omega_4\}) = \{\omega_3, \omega_4\}$, $K(\{\omega_1, \omega_3\}) = \{\omega_3\}$.

Example 1.2 (Non-Partitional)

$\Omega = \{\text{bark, don't bark}\}$. $P(\omega) = \begin{cases} \{\text{bark}\} & \text{if } \omega = \text{bark} \\ \{\text{bark, don't bark}\} & \text{if } \omega = \text{don't bark} \end{cases}$ Then, $K(\{\text{bark}\}) = \{\text{bark}\}$ and $K(\{\text{don't bark}\}) = \emptyset$. Axiom 5 is violated.

1.2.3 Self-evident and Algebra

Definition 1.30 (Self-evident)

An event $A \in 2^\Omega$ is **self-evident** if

$$KA = A$$

That is, the agent ``knows A'' if and only if A happens.

Proposition 1.10 (Self-evident $\Leftrightarrow$ Unions of Elements of Partition)

``A is self-evident'' if and only if it is the union (1 or more) of elements of the partition P.

Proof

A is self-evident iff $A = KA = \{\omega : P(\omega) \subseteq A\}$.

Definition 1.31 (Algebra)

A collection of events Σ is an **algebra** if it satisfies:

1. $\Omega \in \Sigma$;
2. If $A \in \Sigma$ then $A^c \in \Sigma$;
3. If $A, B \in \Sigma$ then $A \cup B \in \Sigma$.

Let Σ be **the collection of self-evident events**.

Corollary 1.3 (Set of Self-Evident Events is an Algebra)

Then Σ is an algebra (in fact, $\Sigma = \Sigma_P$, the algebra generated by the partition P).

Example 1.3

$\Omega = \{\omega_1, \omega_2, \omega_3\}$

1. $P_1 = \{\{\omega_1, \omega_2\}, \{\omega_3\}\}$, $\Sigma_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3\}, \Omega\}$.
2. $P_2 = \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}\}$, $\Sigma_2 = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\}$.
3. $P_3 = \{\{\omega_1\}, \{\omega_2, \omega_3\}\}$, $\Sigma_3 = \{\emptyset, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega\}$.

Corollary 1.4 ($KA \in \Sigma$)

For any A (despite whether it is in Σ) and K is partitional, we have

$$KA = \cup\{S \in \Sigma : S \subseteq A\} \in \Sigma$$

So KA is always self-evident. And, since any algebra is closed under unions, it follows that KA is the largest element of Σ that is contained in A.

Proof

We prove by two directions:

(1). $\underline{\omega \in KA \Rightarrow \omega \in \cup\{S \in \Sigma : S \subseteq A\}}$: Given $\omega \in KA$, we have $P(\omega) \subseteq A$. Since K is partitional, $P(\omega) \in \Sigma$. Therefore, $P(\omega) \in \{S \in \Sigma : S \subseteq A\}$. Hence, $\omega \in \cup\{S \in \Sigma : S \subseteq A\}$.

(2). $\underline{\omega \in \cup\{S \in \Sigma : S \subseteq A\} \Rightarrow \omega \in KA}$: Given $\omega \in \cup\{S \in \Sigma : S \subseteq A\}$, there exists $S \in \Sigma$ such that $S \subseteq A$ and $\omega \in S$. Since Σ is the collection of self-evident events, we have $KS = S$. Hence, $\omega \in KS \subseteq KA$.

Then, we can recover $P(\omega)$ from Σ by

$$P(\omega) = \cap\{S \in \Sigma : \omega \in S\} \in \Sigma$$

All in all, a knowledge space can be defined by P, K, or Σ.

1.2.4 Common Knowledge

Consider a finite set of N agents, each with a (partitional) knowledge function $K_i : 2^\Omega \to 2^\Omega, i \in N$.

Definition 1.32 (Refinement and Coarsening of Sub-algebras)

Let Σ, Π be two sub-algebras of some algebra. We say that Σ is a **refinement** of Σ and Π is a **coarsening** of Σ if $\Pi \subseteq \Sigma$.

Example 1.4

Consider two sub-algebras of $\{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\}$,

$$\Sigma = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_2, \omega_3\}, \Omega\} \text{ and } \Pi = \{\emptyset, \Omega\}$$

where $\Pi \subseteq \Sigma$. The Π provides a less inaccurate information structure than Σ.

Definition 1.33 (Meet and Join of Sub-algebras)

The **meet** of two algebras $\Sigma_1, \Sigma_2 \subseteq \Sigma$ is the finest sub-algebra of Σ that is a coarsening of each Σ_i.

Their **join** is the coarsest sub-algebra of Σ that is a refinement of each Σ_i.

Definition 1.34 (Common Knowledge)

An event A is said to be **common knowledge** at $\omega \in \Omega$ if for any sequence $i_1, i_2, ..., i_k \in N$ it holds that

$$\omega \in K_{i_1} K_{i_2} \cdots K_{i_k} A$$

Let $\Sigma_C = \cap_i \Sigma_i$ be the meet of the player's algebras.

Proposition 1.11

The following are equivalent:

1. $C \in \Sigma_C$.
2. $K_i C = C, \forall i \in N$.

Definition 1.35 (K_C)

Recall that $K_i A = \cup\{S \in \Sigma_i : S \subseteq A\}$. Analogously, we define a knowledge operator from Σ_C:

$$K_C A = \cup\{S \in \Sigma_C : S \subseteq A\}$$

Since $K_C A \in \Sigma_i$ for any i, we have $K_i K_C A = K_C A$. Hence, $K_C A$ is common knowledge at $\omega \in K_C A$. As Σ_C is also the set of $\{K_C A : A \in 2^\Omega\}$, we can conclude that

Proposition 1.12 ($E \in \Sigma_C \Leftrightarrow E$ is common knowledge at $\omega \in E$)

For any $E \in \Sigma_C$, E is common knowledge at $\omega \in E$. Conversely, if E is common knowledge at any $\omega \in E$, then $E \in \Sigma_C$ ($E = K_i E$ for any i).

Example 1.5

Consider the following game, which is a simplification of the popular board game "Clue." There are two decks of cards, with four cards each. Deck 1 has numbered cards, with numbers 1,2,3 and 4. Deck 2 has cards with the suites: $\heartsuit$, $\diamondsuit$, $\spadesuit$ and $\clubsuit$. One card from each deck is set aside; turned face down and not seen by any of the players. Of the remaining cards, Player 1 gets all the odd-numbered cards from Deck 1 while Player 2 gets the even cards. From the remainder of Deck 2, Player 1 gets the red cards ($\heartsuit$ and $\diamondsuit$) while 2 gets the black cards ($\spadesuit$ and $\clubsuit$). For example, if the cards set aside are 4 and $\diamondsuit$, then P1 gets 1, 3 and $\heartsuit$, while P2 gets 2, $\clubsuit$ and $\spadesuit$. Players don't see how many cards the other player received.

A state of the world is a pair of cards that was set aside (and the point of the game is to find out what these are).

Show that the event

$$E = \{(3, \clubsuit), (3, \spadesuit), (1, \clubsuit), (1, \spadesuit)\}$$

is common knowledge when the cards set aside are the number 3 and $\clubsuit$.

Answer The state is a pair (n, s), with n being a number and s a suit. At any state in E, n is odd and s is a black suit. So for any such state the players' information functions take values: $P_1(n, s) = \{(n, \clubsuit), (n, \spadesuit)\}$ and $P_2(n, s) = \{(1, s), (3, s)\}$.

Thus, for any $(n, s) \in E, P_i(n, s) \subseteq E$. This means that E is the union of elements of the partition P_i, and thus E is self-evident for each player. So $E \in \Sigma_1 \cap \Sigma_2$ and it is common knowledge at any $(n, s) \in E$.

1.2.5 Application: agreeing to disagree

Definition 1.36 (Belief Space)

Suppose

1. a finite set N of players,
2. a state-space Ω,
3. each agent endowed with a partitional knowledge K_i; resulting in an algebra Σ_i on Ω,
4. each agent endowed with a prior belief $\mu_i \in \Delta(\Omega)$

The tuple $(N, \Omega, \{\mu_i\}_{i \in N}, \{\Sigma_i\}_{i \in N})$ is a **belief space**.

Fix $\mu \in \Delta(\Omega)$, the expectation of a random variable $X : \Omega \to \mathbb{R}$ according to μ is denoted by $\mathbb{E}_\mu X$. Suppose that $\mu(B) \in (0, 1)$, a conditional probability can be written as $\mu(A|B) = \frac{\mu(A \cap B)}{\mu(B)}, \forall A \subseteq \Omega$. Conditional expectation $\mathbb{E}[X|B] = \mathbb{E}_{\mu(\cdot|B)} X$.

Given an algebra Σ_i (which represent a knowledge space), we define the expectation of X conditioned on player i's information at ω is

$$\mathbb{E}[X|\Sigma_i](\omega) = \mathbb{E}[X|P_i(\omega)] = \frac{\sum_{\omega' \in P_i(\omega)} \mu(\omega')X(\omega')}{\mu(P_i(\omega))}$$

(where we assume $\mu(P_i(\omega)) > 0$.)

An event that $\mathbb{E}[X|\Sigma_i] = q$ is the event that, given i's information (Σ_i), her conditional expectation for the random variable X is q: $\{\omega \in \Omega : \mathbb{E}[X|\Sigma_i](\omega) = q\}$.

Proposition 1.13

The **Law of Iterated (Total) Expectations**: let S be an element of an algebra Π, and let $X : \Omega \to \mathbb{R}$ be a r.v. Then,

$$\mathbb{E}[X|S] = \mathbb{E}[\mathbb{E}[X|\Pi]|S]$$

Consider a belief space $(N, \Omega, \{\mu_i\}_{i \in N}, \{\Sigma_i\}_{i \in N})$ and a **common prior** $\mu = \mu_i, \forall i \in N$.

Theorem 1.7 (Aumann's Agreement Theorem, 1976)

Let $X : \Omega \to \mathbb{R}$ be a random variable. Suppose that, at ω_0, it is common knowledge of posteriors that $\mathbb{E}[X|\Sigma_i] = q_i$ for $i = 1, ..., n$ and some $(q_1, q_2, ..., q_n) \in \mathbb{R}^n$. Then $q_1 = q_2 = \cdots = q_n$.

Proof

For $i \in N$, let A_i be the event that $\mathbb{E}[X|\Sigma_i] = q_i$. By the common knowledge hypothesis there is a $C \in \Sigma_C = \cap_i \Sigma_i$ such that $\omega_0 \in C \subset \cap_i A_i$. Hence, $\mathbb{E}[X|\Sigma_i](\omega) = q_i$ for all $\omega \in C$. Thus, for all i,

$$\mathbb{E}[X|C] = \mathbb{E}[\mathbb{E}[X|\Sigma_i]|C] = \mathbb{E}[q_i|C] = q_i, \forall i \in N$$

Corollary 1.5

If two players have common priors over a finite space, and it is common knowledge that their posteriors for some event S are q_1 and q_2, then $q_1 = q_2$.

1.3 Refinement

1.3.1 Trembling-hand Perfect Equilibrium

Definition 1.37 (ϵ-Constrained Equilibrium)

An **ϵ-constrained equilibrium** of a strategic form game is a mixed strategy profile σ^ϵ such that there exists $\bar{\epsilon} : \bigcup_{i \in I} S_i \to (0, \epsilon)$ such that for each player i,

$$\sigma_i^\epsilon \in \underset{\sigma_i \text{ s.t. } \sigma_i(s_i) \geq \bar{\epsilon}(s_i)}{\operatorname{argmax}} u_i(\sigma_i, \sigma_{-i}^\epsilon)$$

Definition 1.38 (Trembling-hand Perfect Equilibrium)

A strategy profile σ is a **trembling-hand perfect equilibrium** of a strategic form game if it is a limit of ϵ-constrained equilibria as $\epsilon \to 0$.

Note *Trembling-hand perfect equilibrium implies a SPE, while a SPE is not always a trembling-hand perfect equilibrium.*

This doesn't imply backward induction. (We do not rationalize the "trembling-hand".)

1.3.2 Forward Induction: Burning Money (Ben-Polath and Dekel, 1992)

Consider a game as following:

	A_2	B_2
A_1	9,6	0,4
B_1	4,0	6,9

Except the actions, the Player 1 can choose to Burn or Not Burn (i.e., lose 2.5 units of utility), and the Player 2 can see the Player 1's action.

There are four possible strategies of Player 1:

(S1). Burn and A_1

(S2). Burn and B_1

(S3). Not Burn and A_1

(S4). Not Burn and B_1

The potential payoffs of playing (S2) are 1.5 and 3.5, which is dominated by playing (S4). Then, if the Player 1 chooses Burn, he must play (S1). Thus, the Player 2 plays A_2, which gives $(6.5, 6)$. Therefore, the Player 1 chooses Burn (i.e., (S1)) can dominate (S4). Now, we only remain two possible strategies of Player 1, (S1) and (S3). (S1) is dominated by (S3), so (S3) is the optimal strategy of Player 1.

2 Dynamic Games

Terminology in Dynamic Games

- **Imperfect information**. Players don't observe everything that has happened in the past.
- **Incomplete information**. Players aren't sure about the precise game being played
    - Payoffs
    - Identity of other players
    - Possible actions
    - Map from actions to outcomes
    - What opponents know

In games of perfect and complete information, whenever a player moves they know everything that has happened in the past, and can compute exactly how the game will end given any profile of continuation strategies.

2.1 Extensive-Form Games with Perfect and Complete Information

Let us first review games with perfect and complete information, where players know everything that has happened in the past and can compute exactly how the game will end given any profile of continuation strategies.

For finite horizon game trees, we have discussed the extensive form game with perfect information (PIG), which is denoted by $G = \{N, A, H, Z, P, O, o, \succ_{n \in N}\}$, where:

- H represents the nodes (unique history of play), including an initial node
- $Z \subset H$ represents the terminal nodes
- P is a function mapping from H/Z (non-terminal history) to N (one of the players), where $N(h)$ indicates who moves at node h
- $A_i(h) \neq \emptyset$ represents the available actions for player i at node h where $N(h) = i$
- Action profiles can be identified with branches in the tree

- $u_i : Z \to \mathbb{R}$ represents the payoff functions

Now we shall allow two or more players to make simultaneous moves, and the P is a function from H/Z to $2^N/\{\emptyset\}$, the power set of the set of players, so that after history $h \in H$ the set of players who play simultaneously is $P(h)$.

Without losing generality, we can represent an extensive-form game as $\Gamma = (N, A, H, P, \{u_i\}_{i \in N})$.

For each player $i \in N$, let

$$H_i = \{h \in H/Z : i \in P(h)\}$$

be the set of histories at which i moves.

A (pure) **strategy** for player i is a function $s_i : H_i \to A$ for which

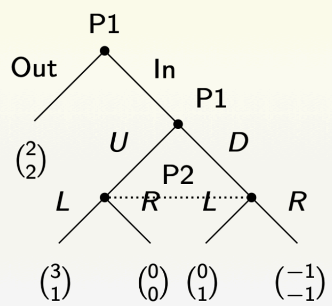
$$s_i(h) \in A_i(h), \; \forall h \in H_i$$

This represents a "complete plan of action" that specifies i's plan even at nodes that cannot be reached given the actions specified by s_i at upstream nodes. Given any profile of strategies s, there is a unique terminal node $z(s)$ that will be reached, and we denote $u_i(s) = u_i(z(s))$.

Let S_i be the set of all strategies available for player i. A strategy profile is a collection of all players' strategies.

Thus, we obtain a normal-form game $(N, (S_i, u_i)_{i \in N})$ defined from Γ.

2.1.1 Mixed and Behavioral Strategies



		P2: L	P2: R
P1	OU	$2, 2$	$2, 2$
	OD	$2, 2$	$2, 2$
	IU	$3, 1$	$0, 0$
	ID	$0, 1$	$-1, -1$

Figure 2.1: Example

In this example, a mixed strategy of the P1 can be represented by a probability distribution over $\{OU, OD, IU, ID\}$. To replicate mixed strategy, we can use **behavioral strategy** that an agent's (probabilistic) choice at each node is independent of his/her choices at other nodes.

Example 2.1

1. Mixed Strategy: OU with probability $\frac{1}{2}$ and ID with probability $\frac{1}{2}$.
2. Behavioral Strategy: play O and I with prob $\frac{1}{2}$ each at $h = \emptyset$; and then D with prob 1 at history $h = I$.

Definition 2.1 (Behavioral Strategy)

A **behavioral strategy** for player i is a function that maps histories $h \in H_i$ into probability distributions over $A_i(h)$,

$$\sigma_i(h) \in \Delta(A_i(h))$$

A profile of behavioral strategies, $\sigma = \{\sigma_i : i \in N\}$, includes a probability distribution over Z. So does a profile of mixed strategies. Then $u_i(\sigma)$ is the expected payoff.

Theorem 2.1 (Outcome Equivalent)

Behavioral and mixed strategies are ``outcome equivalent'' in games of **perfect recall**.

Without perfect recall, behavioral and mixed strategies are not equivalent.

2.1.2 Subgame Perfect Equilibrium

Definition 2.2

A strategy profile σ^* is a **Nash equilibrium** if, for all i,

1. σ_i^* is the best response to beliefs μ_i,
2. μ_i coincides with σ_{-i}^*.

When s is a pure-strategy Nash eq. $h(s)$ is the equilibrium path of s.

Definition 2.3

For games of perfect and complete information, a subgame of an extensive-form game $G = \{N, A, H, Z, P, O, o, \succ_{n \in N}\}$ is a game that starts after a given finite history $h \in H$. Formally, the subgame $\Gamma(h)$ associated with $h = (h_1, ..., h_n) \in H$ is $\Gamma(h) = (N, A, H_h, P_h, \{u_{i,h}\}_{i \in N})$, where $H_h = \{(a_1, a_2, ...) : (h_1, ..., h_n, a_1, a_2, ...) \in H\}$. $P_h(h') = P(hh')$ for all non-terminal $h' \in H_h$, and $u_{i,h} = u_i(hh')$ for any terminal $h' \in H$.

The game as a whole is a subgame. All other subgames are called proper subgames.

For a strategy profile σ: - $\sigma|h$ denotes the continuation strategy profile in the subgame beginning at h - $z(\sigma|h)$ denotes the terminal node reached by σ beginning from h - $u_i(\sigma|h)$ denotes the continuation payoffs

A behavioral strategy σ_i of i in Γ defines a behavioral strategy $\sigma_i(\cdot|h)$ of $\Gamma(h)$ by $\sigma_i(h'|h) = \sigma_i(hh')$.

Definition 2.4 (Subgame Perfect Equilibrium)

A **subgame perfect equilibrium**[SPNE/SPE] of Γ is a strategy profile σ^* such that for every subgame $\Gamma(h)$, the continuation strategy profile $\sigma^*|h$ is a Nash equilibrium of $\Gamma(h)$, i.e.,

$$u_i(\sigma^*|h) \geq u_i((\sigma_i', \sigma_{-i}^*)|h)$$

for every strategy σ_i' and every player i.

Over time $t = 1, ..., T$, the set of actions is A^T.

Lemma 2.1

σ^* is a SPNE iff its restriction $\sigma^*(\cdot|h)$ is SPNE of $\Gamma(h)$, for all subgames $\Gamma(h)$.

Theorem 2.2 (Zermelo's Theorem)

Any finite perfect (and complete) information game has at least one pure-strategy SPE which can be found by backward induction. Furthermore, any such game has a unique SPE if payoffs are generic (i.e., for any pair of terminal nodes $z \neq z'$ implies $u_i(z) \neq u_i(z')$ for all i).

2.1.3 One-Stage Deviation for Finite and Infinite Horizon Games

Definition 2.5 (Finite Horizon)

An extensive-form game has a **finite horizon** if there is a bound on the length of any history in Z.

In finite horizon games, SPNE can be found through backward induction. For infinite horizon games, we cannot use backward induction directly. Instead, we can think of backward induction as constructing a strategy that is unimprovable by one-stage deviations.

Definition 2.6 (One-Stage Deviation)

Let σ_i, σ_i' be two distinct strategies for player i. Let h be a node at which i moves. Let $d_h(\sigma_i, \sigma_i')$ be the strategy that coincides with σ_i at all nodes except h, where it is determined by σ_i', i.e.,

$$d_h(\sigma_i,\sigma_i')(h') = \begin{cases} \sigma_i(h') & \text{if } h' \neq h \\ \sigma_i'(h') & \text{if } h' = h \end{cases}$$

Such a strategy is called a one-stage deviation (at h).

Definition 2.7 (Unimprovable Strategy)

Fix a strategy profile σ. A strategy σ_i is **unimprovable by one-stage deviation** if for every σ_i' and every h at which i moves:

$$u_i(\sigma|h) \geq u_i((d_h(\sigma_i,\sigma_i'),\sigma_{-i})|h)$$

Proposition 2.1 (One-Stage Deviation Principle for Finite Games)

In any finite perfect and complete information game, any strategy profile that is unimprovable by one-stage deviations is a SPE (and the converse holds).

Proof

The converse is obvious. For the forward direction, suppose σ is unimprovable by one-stage deviations but not SPE. Then there exists a subgame $\Gamma(h)$ where some player i has a profitable deviation σ_i'. Consider the last node h' where σ_i and σ_i' differ. Then $d_{h'}(\sigma_i,\sigma_i')$ is a profitable one-stage deviation at h', contradicting unimprovability.

To extend this principle to infinite horizon games, we need an additional condition:

Definition 2.8 (Continuous At Infinity)

A perfect and complete information game is **continuous at infinity** if for every strategy profile σ, player i, history h, pair of strategies σ_i, σ_i', and $\epsilon > 0$, there exists an integer t such that the strategy $\tilde{\sigma}_i$ defined by:

$$\tilde{\sigma}_i(h') = \begin{cases} \sigma_i(h') & \text{if } h' \text{ is no more than } t \text{ moves past } h \\ \sigma_i'(h') & \text{otherwise} \end{cases}$$

earns a continuation payoff within ϵ of σ_i in the subgame beginning at h.

Remark

- A finite game is continuous at infinity
- A game with discounting and bounded payoffs is continuous at infinity

Proposition 2.2 (One-Stage Deviation Principle)

In any perfect and complete information game that is continuous at infinity, a strategy profile that is unimprovable by one-stage deviations is a subgame perfect Nash equilibrium (and the converse holds).

This principle provides the foundation for recursive dynamic programming approaches.

Definition 2.9 (Discounted Utility)

We focus on additively separable models with a sub-utility $u : A \to \mathbb{R}$ that is the same for each t. We model a **discounted utility** by assuming a discount factor $\delta \in [0, 1)$:

$$v(a_1, ...) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u(a_t)$$

Can think of δ as the probability that the game will go on for another period, and $1 - \delta$ the prob that the game will end.

"Ending" events are independent: A geometric distribution.

The probability that the game ends at time t is $(1 - \delta)\delta^{t-1}$ and $\sum(1 - \delta)\delta^t u(a_t)$ becomes expected utility.

Proposition 2.3

$v(a_1, ...) = (1 - \delta) \sum_{t=1}^{\infty} \delta^{t-1} u(a_t)$ is continuous at infinity.

2.2 Repeated Games

Let $G_0 = (N, \{A_i : i \in N\}, \{u_i : i \in N\})$ be a normal-form stage game, where N is the set of players, A_i is player i's finite action set, and u_i is player i's payoff function. Let $A = \Pi_{i \in N} A_i$ be the set of action profiles. We only consider games in which A is compact and each u_i is continuous.

Let T be the number of periods (or repetitions) of the repeated game. T can be either finite or infinite.

Definition 2.10 (Repeated Game)

A T-repeated, δ-discounted game of G_0 is an extensive form game $\Gamma = (N, A, H, P, \{v_i\}_{i \in N}, \delta)$, where:

1. $P(h) = N$ for all $h \in H/Z$ (simultaneous moves)
2. The set of histories H has terminal histories $Z = A^T$

3. For any sequence of action profiles $\{a(t)\}_{t=1}^{T}$, player i's payoff is:

$$v_i((a(t))_{t=1}^{T}) = \begin{cases} (1-\delta)\sum_{t=1}^{\infty}\delta^{t-1}u_i(a(t)), & \delta \in (0,1) \\ \sum_{t=1}^{T} u_i(a(t)), & \delta = 1 \end{cases}$$

For mixed strategies, let $\alpha_i \in \Delta(A_i)$ denote a mixed action for player i. We abuse notation and write $u_i(\alpha)$ for the expected payoff under mixed strategy profile α.

2.2.1 Nash Reversion Folk Theorem (Infinite Horizon)

Proposition 2.4 (Stage-Game NE as SPE)

If $\{\alpha(t)\}_{t=1}^{T}$ is a sequence of stage-game Nash equilibria, then playing $\alpha(t)$ at period t regardless of history is a subgame perfect equilibrium of the repeated game (whether finite or infinite).

Proposition 2.5 (SPE⇔No Profitable One-Stage Deviation (Infinite Horizon))

Let Γ be a discounted, infinitely repeated game. A strategy profile constitutes a subgame perfect equilibrium (SPE) of Γ if and only if no player can benefit from deviating in a single stage while conforming to the original strategy in all other stages.

Theorem 2.3 (Nash Reversion Folk Theorem (Friedman, 1971))

Let α^* be a Nash equilibrium of the stage game, and let $a \in A$ be a feasible action profile such that $u_i(a) > u_i(\alpha^*)$ for all $i \in N$. Then there exists $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium of the infinitely repeated game with discount factor δ where a is played in every period on the equilibrium path.

The proof is constructive: Consider the strategy profile where players play a on the equilibrium path and revert to playing the Nash equilibrium α^* forever if any player deviates. For sufficiently patient players (high δ), the long-term loss from punishment outweighs the short-term gain from deviation, making cooperation sustainable as a subgame perfect equilibrium.

2.2.2 General Folk Theorem

Definition 2.11 (Minmax Value)

Let M_{-j} be a profile of mixed strategies in the stage game for all players except j such that

$$M_{-j} \in \underset{\alpha_{-j}}{\text{argmin}} \max_{\alpha_j} u_j(\alpha_j, \alpha_{-j})$$

Player j's minmax value is:

$$\underline{v}_j = \max_{\alpha_j} u_j(\alpha_j, M_{-j})$$

A payoff profile $v = (v_1, \dots, v_n)$ is feasible if $v = u(a)$ for some $a \in A$.

Theorem 2.4 (General Folk Theorem (Fudenberg-Maskin, 1986))

Suppose a "full dimensionality condition" is satisfied. For any feasible payoff profile v such that $v_i > \underline{v}_i$ for all i, and any $\epsilon > 0$, there exists $\underline{\delta} < 1$ such that for all $\delta \in (\underline{\delta}, 1)$, there exists a subgame perfect equilibrium where the long-run payoff profile on the equilibrium path lies in the ϵ-neighborhood of v.

Remark If randomization is observable, we can implement v exactly in every period without the ϵ approximation. The proof is constructive: If a player deviates, that player is minmaxed for T periods before returning to cooperation. A key difficulty is that minmaxing player i may require player j to receive less than their minmax payoff, necessitating later compensation. The full dimensionality condition ensures such compensation is feasible.

This theorem shows the multiplicity of equilibria, making equilibrium selection an important open question studied through approaches like evolutionary arguments (Fudenberg-Maskin) and experiments (Dal Bo-Frechette).

2.2.3 Folk Theorem in Finite Horizon (Benoit-Krishna, 1985)

Remark The Nash reversion folk theorem implies that in the repeated prisoner's dilemma, (C, C) can be achieved. However, if the horizon is **finite**, backward induction implies there exists a unique SPE where only D can be played.

However, if the stage game has multiple Nash equilibria, then there exists a finite horizon such that any payoff profile that gives each player greater than the minmax payoff can be sustained as the time-average per-period payoff in a SPE!

Consider the following game:

	C_2	D_2	D_2'
C_1	1,1	-1,2	-5,-5
D_1	2,-1	0,0	-5,-5
D_1'	-5,-5	-5,-5	-2,-2

This game has two static pure Nash equilibria: (D_1, D_2) and (D_1', D_2').

For the 2-period case with $\delta = 1$, cooperation can be played in equilibrium through the following SPE:

- Play C_i at $t = 1$
- If (C_1, C_2) is played at $t = 1$, then play D_i
- Otherwise play D_i'

The key idea in finite horizon games is **using multiple Nash equilibria at a later stage as a reward/punishment device**. Note that in the repeated prisoner's dilemma, such multiplicity is absent.

Theorem 2.5 (Benoit-Krishna (1985))

If a stage game has multiple Nash equilibrium payoffs for each player, then there exists $\bar{T} < \infty$ such that for all horizon length $T > \bar{T}$, any payoff profile that gives each player greater than the minmax payoff can be sustained as the time-average per-period payoff in a SPE.

2.3 Repeated Games with Imperfect Monitoring

In repeated games with imperfect monitoring, players do not directly observe the outcome of the stage game, but rather receive signals about what occurred. This creates new challenges for sustaining cooperation.

There are two main types of imperfect monitoring:

- **Public monitoring**: All players observe the same signal about the stage game outcome
  - For example, in oligopoly settings firms may choose production quantities but only observe the resulting market price (as in the Green-Porter model)
- **Private monitoring**: Different players observe different private signals
  - For example, firms choosing prices but only observing their own sales volume (secret price cutting)

Private monitoring scenarios are particularly challenging to analyze since players have different information sets, requiring careful tracking of beliefs or restriction to "belief-free" equilibria.

Under public monitoring, the analysis becomes more tractable if we focus on **public strategies** - strategies that condition only on publicly observable information. We will focus primarily on games with public monitoring in what follows.

2.3.1 Public Monitoring Setup

Let us formalize the setup for games with public monitoring:

Stage Game:

- Players $i = 1, \ldots, I$ choose actions $a_i \in A_i$, where A_i is finite
- α_i denotes a mixed action
- $y \in Y$ is a public signal, where Y is finite
- Given action profile $a \in A$, signal y realizes with probability $\pi_y(a)$. We abuse notation and write $\pi_y(\alpha)$ for the distribution induced by mixed strategy profile α
- The expected payoff is $g_i(a) = \sum_{y \in Y} \pi_y(a) r_i(a_i, y)$ where $r_i(a_i, y)$ is player i's realized payoff
- Note that r_i depends on a_{-i} only through y (since players only observe the public signal)

Repeated Game:

- Public information at time t: $h^t = \{y^0, \ldots, y^{t-1}\}$
- Private information at time t: $z_i^t = \{a_i^0, \ldots, a_i^{t-1}\}$
- Player i's strategy σ_i maps (h^t, z_i^t) to mixed actions
- Payoffs: $u_i = (1-\delta) \sum_{t=0}^{\infty} \delta^t g_i(a)$

Examples:

- Observed actions: $Y = A$
- Green-Porter model: a_i is output, y is price, $\pi_y(a) = f(\sum_{i \in N} a_i)$
- Radner, Radner-Myerson-Maskin: a_i is effort, y is output

2.3.2 Public Strategies

Definition 2.12 (Public Strategy)

A strategy σ_i is a **public strategy** if for any h^t, $\sigma_i(h^t, z_i^t) = \sigma_i(h^t, \tilde{z}_i^t)$ holds for any two private histories $z_i^t, \tilde{z}_i^t$.

Key Observations:

1. Pure Nash equilibrium is equivalent to Nash equilibrium in public strategies, while this equivalence does not hold for mixed Nash equilibrium.

Reason: For mixed strategies, a player's belief about opponents' past actions depends on their own past actions (through public signals). If a player believes opponents' current actions depend on past play, they would want to condition their action on their own past actions. For pure strategies, beliefs are deterministic so best responses can be written as functions of public signals only.

2. If a player conditions only on public signals, opponents have no incentive to condition on private histories since those contain no information about the player's current or future actions.

2.3.3 Perfect Public Equilibrium

Definition 2.13 (Perfect Public Equilibrium)

A strategy profile σ is a **perfect public equilibrium (PPE)** if each σ_i is a public strategy and it induces a Nash equilibrium from each time t and public history h^t onwards.

We will focus on characterizing the set of PPE using recursive methods developed by Green and Porter (1984) and Abreu et al. (1986, 1990).

Definition 2.14 (Enforceability)

An action-payoff pair (α, v) is **enforceable** with respect to δ and $W \subseteq \mathbb{R}^I$ if there exists a continuation value function $w : Y \to W$ such that for each $i \in I$:

(i) $v_i = (1-\delta)g_i(\alpha) + \delta \sum_{y \in Y} \pi_y(\alpha) w_i(y)$

(ii) $\alpha_i \in \arg\max_{\alpha_i'} (1-\delta)g_i(\alpha_i', \alpha_{-i}) + \delta \sum_{y \in Y} \pi_y(\alpha_i', \alpha_{-i}) w_i(y)$

This means we can sustain players playing α by using payoff profiles in W as promised future continuation payoffs, such that α_i indeed solves player i's maximization problem.

Definition 2.15 (Generated Payoffs)

A payoff profile $v \in \mathbb{R}^I$ is **generated by** (δ, W) if there exists α such that (α, v) is enforceable with respect to (δ, W). Let $B(\delta, W)$ denote the set of all payoff profiles generated by (δ, W).

Let $E(\delta)$ denote the set of all PPE payoff profiles under discount factor δ.

Theorem 2.6 (APS)

$E(\delta) = B(\delta, E(\delta))$

where APS stands for Abreu-Pearce-Stacchetti: D. Abreu, D. Pearce, and E. Stacchetti. Optimal

cartel equilibria with imperfect monitoring. Journal of Economic Theory, 39(1):251–269, 1986.

Proof

For $E(\delta) \subseteq B(\delta, E(\delta))$: If $v \in E(\delta)$, the continuation payoff must lie in $E(\delta)$, satisfying (i). Since v is a PPE payoff, no player wants to deviate, satisfying (ii).

For $E(\delta) \supseteq B(\delta, E(\delta))$: Fix $v \in B(\delta, E(\delta))$. There exists α such that (α, v) is enforceable with respect to $(\delta, E(\delta))$, with continuation payoffs $w(y) \in E(\delta)$ for each y. Have players play α in the first period, then play a PPE with payoffs $w(y)$ if y realizes. Such PPE exists since $w(y) \in E(\delta)$. Therefore $v \in E(\delta)$.

Definition 2.16 (Self-Generation)

A set $W \subseteq \mathbb{R}^I$ is **self-generating** if $W \subseteq B(\delta, W)$.

This means for each $v \in W$, we can enforce v using only continuation payoffs in W. A simple example is a singleton set containing the Nash payoff profile.

Theorem 2.7

If W is self-generating and bounded, then $W \subseteq E(\delta)$.

Proof

Fix $v \in W$. Then $v \in B(\delta, W)$, so there exists α that players play at period 0 with continuation payoffs $w(y) \in W$ for each y. This defines period-0 actions, and actions at all periods can be defined recursively. Boundedness ensures this indeed induces payoffs v. Optimality follows from the one-shot deviation principle.

Key properties of $E(\delta)$:

- Compact
- May not be monotone or convex, but convexity generally implies monotonicity
- Higher δ means more patience so more can be sustained. With high δ, less variation in continuation payoffs is needed for given α
- $E(\delta)$ is guaranteed to be convex when players observe continuously-distributed public randomization device at period starts
- As $\delta \to 1$, randomization can be replicated by deterministic cycles, making $\lim_{\delta \to 1} E(\delta)$ convex

2.3.4 Using APS: Kandori (1992) Informativeness of Signals

A key insight from Kandori (1992) is that more informative signals cannot reduce the set of equilibrium payoffs. To formalize this, we use Blackwell's (1951) notion of informativeness.

Definition 2.17 (Signal Structure)

The public monitoring structure can be represented as a $|A| \times |Y|$ matrix Π, where each row sums to 1. Each entry $\pi_y(a)$ gives the probability of signal y given action profile a.

Definition 2.18 (Garbling)

A public monitoring structure Π' (for signal space Y') is a **garbling** of another structure Π (for signal space Y) if there exists a stochastic matrix Q such that $\Pi' = \Pi Q$.

Under a garbling, signal y' realizes with probability $q(y'|y)$ after first generating signal y according to Π. Therefore:

$$\pi'_{y'}(a) = \sum_{y \in Y} \pi_y(a) q(y'|y) \tag{2.1}$$

Note that garbling forms a partial order and does not require $|Y| < |Y'|$. When a public randomization device is available, any PPE under a garbled signal structure Π' can be replicated under the original structure Π. More generally:

Proposition 2.6

If Π' is a garbling of Π and $W \subseteq B(\delta, W', \Pi')$, then $W \subseteq B(\delta, \text{convhull}(W'), \Pi)$.

As a corollary, if $E_{\Pi'}(\delta)$ is convex, then $E_{\Pi'}(\delta) \subseteq E_{\Pi}(\delta)$. This formalizes the intuition that more informative signals allow for (weakly) more equilibrium outcomes.

2.4 Collusion in Dynamic Competition

Folk theorems applied to repeated games of (Cournot, Bertrand) competition yield powerful insights:

- Patient firms can sustain tacit collusion
- Collusion often involves simple strategies like maintaining constant prices (Bertrand) or quantities (Cournot)
- Price wars do not occur along the equilibrium path

2.4.1 Basic Model of Collusion

Consider two firms producing homogeneous goods with marginal cost c. The demand for firm i is given by:

$$D_i(p_i, p_j) = \begin{cases} D_i(p_i), & p_i < p_j, \\ \frac{1}{2}D_i(p_i), & p_i = p_j, \\ 0, & p_i > p_j \end{cases}$$

2.4.1.1 Equilibrium Analysis

For a finite horizon T, firm i's payoff is:

$$\sum_{t=1}^{T} \delta^{t-1} \Pi^i(p_{it}, p_{jt})$$

where $\delta = \frac{1}{1+r}$ is the discount factor.

Each firm observes the complete history up to period $t-1$ when choosing p_{it}.

In the single-period case ($T = 1$), we obtain the static Nash equilibrium: $p_1 = p_2 = c$.

For any finite horizon T, backward induction implies $p_{1t} = p_{2t} = c$ in all periods, since $p_{1T} = p_{2T} = c$ must hold in the final period.

With infinite horizon, while $p_{1t} = p_{2t} = c$ remains an equilibrium, collusion becomes possible. Firms can sustain the monopoly price $p^M := \text{argmax}_p (p-c)D(p)$ (yielding per-period profit $\Pi^M := \max_p (p-c)D(p)$) through trigger strategies that revert to competitive pricing upon deviation. This collusive equilibrium exists when:

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} \geq \Pi^M \Leftrightarrow \delta \geq \frac{1}{2}$$

(Folk Theorem)

2.4.1.2 Model Extensions

N **Firms** With N competitors, the collusion condition becomes:

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{N} \geq \Pi^M \Leftrightarrow \delta \geq 1 - \frac{1}{N}$$

Detection Lags If deviations are detected with a two-period lag, the condition becomes:

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} \geq (1+\delta)\Pi^M \Leftrightarrow \delta \geq \frac{1}{\sqrt{2}}$$

Barriers to Entry

Product Differentiation Greater product differentiation reduces the likelihood of collusion by weakening punishment threats.

Demand Fluctuations High demand increases firms' temptation to deviate, potentially leading to price wars during booms.

Secret Price Cuts with Demand Uncertainty When prices are unobservable and demand fluctuates, firms cannot distinguish between demand shocks and rivals' deviations. This ambiguity leads to partial punishments and price wars during recessions.

Cost Asymmetries

Multi-Market Contact Consider two markets: Market A operates every period while Market B operates every two periods.

In isolation, collusion requires $\delta \geq \frac{1}{2}$ for Market A and $\delta \geq \frac{1}{\sqrt{2}}$ for Market B.

With multi-market contact, the collusion condition becomes:

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} + \sum_{t=1}^{\infty} \delta^{2t-2} \frac{\Pi^M}{2} \geq 2\Pi^M$$

This allows collusion to be sustained for some $\delta \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$ in both markets.

Stochastic Market Continuation With market continuation probability x, the collusion condition becomes:

$$\sum_{t=1}^{\infty} (x\delta)^{t-1} \frac{\Pi^M}{2} \geq \Pi^M \Leftrightarrow \delta \geq \frac{1}{2x}$$

2.4.1.3 Empirical Framework

Consider market demand in period t and market s:

$$P_{ts} = f\left(q_{1ts} + q_{2ts}, Z_{ts}\right)$$

where q_{its} represents quantities and Z_{ts} denotes instrumental variables.

The cost structure is:

$$C_{its} = F_{its} + C^{VC}\left(q_{its}, w_{ts}\right)$$

where F_{its} represents fixed costs, $C^{VC}(\cdot)$ variable costs, and w_{ts} input prices.

Three market conduct scenarios:

1. Perfect competition: $P_{ts} = MC_{its}$

2. Perfect collusion: $(q_{1ts}, q_{2ts}) = \text{argmax}\, P_{ts}Q_{ts} - C_{1ts} - C_{2ts}$, where $Q_{ts} = q_{1ts} + q_{2ts}$

   First-order condition:

$$P_{ts} + Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}} - MC_{its} = 0$$

$$P_{ts} = MC_{its} - Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$$

3. Cournot competition: $P_{ts} = MC_{its} - \frac{1}{2}Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$

Market conduct can be empirically tested by estimating θ in:

$$P_{ts} = MC_{its} - \theta Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$$

Testing hypotheses:

1. Perfect competition: $H_0 : \theta = 0$
2. Perfect collusion: $H_0 : \theta = 1$
3. Cournot competition: $H_0 : \theta = \frac{1}{2}$

2.4.2 Rotemberg & Saloner (1986): Collusion in Booms and Busts

Analyze dynamic oligopolistic collusion under demand fluctuations. The central finding shows that collusion becomes more challenging during economic booms:

- Incentives to deviate strengthen during high-demand periods
- This constrains firms' ability to maintain high prices
- Leading to countercyclical pricing patterns

Model Framework:

- N identical firms producing a homogeneous product
- Dynamic setting with infinite-horizon Bertrand competition (not essential)
- Stochastic demand with i.i.d. shocks $\epsilon_t \in [a, b]$ distributed according to cdf F
- Market inverse demand function $P(Q_t, \epsilon_t)$ strictly increasing in demand shock ϵ
- Linear cost structure with constant marginal cost c
- Firms share common discount factor δ

Game Structure: The sequence of events in each period:

1. All firms observe demand shock ϵ_t (complete information)
2. Firms simultaneously choose prices p
3. Firms observe other firms' price choices (perfect monitoring)

Equilibrium Characterization: Under Bertrand competition, optimal punishment involves reverting to marginal cost pricing. Define $\Pi^M(\epsilon)$ as individual firm's share ($\frac{1}{N}$) of monopoly profits in state ϵ:

$$\Pi^M(\epsilon) = \frac{1}{N} \max_Q (P(Q,\epsilon) - c)Q$$

Deviation yields temporary profit of $N\Pi^M(\epsilon)$. Let W represent discounted future value under collusion. Deviation occurs when:

$$N\Pi^M(\epsilon) > \Pi^M(\epsilon) + \delta W \Leftrightarrow \Pi^M(\epsilon) > \frac{\delta W}{N-1}$$

Critical threshold ϵ^* is defined by $\Pi^M(\epsilon^*) = \frac{\delta W}{N-1}$. Equilibrium sustainable profits:

$$\Pi^S(\epsilon, \epsilon^*) = \begin{cases} \Pi^M(\epsilon) & \text{if } \epsilon \leq \epsilon^* \\ \frac{\delta W}{N-1} & \text{otherwise} \end{cases}$$

Equilibrium Value Function: The discounted value of future profits W can be computed as:

$$W(\epsilon^*) = \sum_{t=0}^{\infty} \delta^t \int_a^b \Pi^S(\epsilon, \epsilon^*) F(d\epsilon) = \frac{1}{1-\delta} \left[\int_a^{\epsilon^*} \Pi^M(\epsilon) F(d\epsilon) + (1 - F(\epsilon^*))\Pi^M(\epsilon^*) \right]$$

An equilibrium is characterized by the pair $(\epsilon^*, W(\epsilon^*))$. For such an equilibrium to exist, we need the function:

$$g(\epsilon) = \Pi^M(\epsilon) - \frac{\delta}{N-1} W(\epsilon)$$

to have a zero at some point.

Sufficient Conditions for Maximum Profit Equilibrium:

1. For $g(a) < 0$, the following inequality must hold:

$$N < \frac{1}{1-\delta}$$

2. For $g(b) > 0$, we require:

$$\frac{\delta}{1-\delta} \frac{1}{N-1} < \frac{\Pi^M(b)}{\int_a^b \Pi^M(\epsilon) F(d\epsilon)}$$

When these conditions are satisfied, the equilibrium demonstrates countercyclical pricing patterns:

- During economic downturns (low ϵ): Firms maintain monopoly-level pricing
- During economic booms (high ϵ): Firms set prices strictly below monopoly levels

Key Results: The equilibrium exhibits distinct countercyclical pricing dynamics - firms can sustain monopoly pricing during economic contractions, but face binding price constraints during expansions. These theoretical predictions find strong empirical support in analyses of supermarket pricing behavior.

2.4.3 Green & Porter (1984): Imperfect Monitoring

- Study collusion with imperfect monitoring of prices through secret price cuts

- Key insight: Price wars occur on equilibrium path and do not indicate cartel collapse
    - Examples: Joint Executive Committee, Marlboro Friday, Ready-to-Eat-Cereals
- Influential paper pioneering structural analysis and recursive methods for repeated games

Model Setup:

- N firms compete repeatedly by choosing quantities each period
- Constant marginal cost c of production
- Price in period t:

$$\theta_t p(\sum_{i=1}^{N} q_{it})$$

  where θ_t is i.i.d. across periods
- Imperfect monitoring: firms observe only market price and own quantity

Perfect Public Equilibrium (PPE):

- Firms accumulate private information over time
- Each firm's private information is payoff irrelevant for future periods
- Strategies condition only on publicly available information
- PPE satisfies one-shot deviation principle and is recursive

Collusive Equilibrium Structure:

- Collusive phase:
    - Game begins here
    - Each firm produces $\frac{1}{N}$ of monopoly quantity
    - Continues until price falls below trigger price $\bar{p}$
- Punishment phase:
    - Firms produce high quantity for T periods
    - Return to collusive phase after T periods

Key Idea:

- Due to demand uncertainty, equilibrium features alternating periods of collusion and price wars
- Green and Porter (1984) solve for this equilibrium using recursive dynamic programming methods

3 Signaling

3.1 Signaling Game

3.1.1 Canonical Game

Definition 3.1 (Canonical Game)

1. There are two players: **S** (sender) and **R** (receiver).
2. **S** holds more information than **R**: the value of some random variable t with support $\mathcal{T}$. (We say that t is the **type** of **S**)
3. Prior belief of **R** concerning t are given by a probability distribution ρ over $\mathcal{T}$ (common knowledge)
4. **S** sends a **signal** $s \in \mathcal{S}$ to **R** drawn from a signal set $\mathcal{S}$.
5. **R** receives this signal, and then takes an **action** $a \in \mathcal{A}$ drawn from a set $\mathcal{A}$ (which could depend on the signal s that is sent).
6. **S**'s payoff is given by a function $u : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ and **R**'s payoff is given by a function $v : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.

3.1.2 Nash Equilibrium

Definition 3.2 (Strategy)

A **behavior strategy** for **S** is given by a function $\sigma : \mathcal{T} \times \mathcal{S} \to [0, 1]$ such that $\sum_s \sigma(t, s)$ for each t.

A **behavior strategy** for **R** is given by a function $\alpha : \mathcal{S} \times \mathcal{A} \to [0, 1]$ such that $\sum_a \alpha(s, a)$ for each t.

Definition 3.3 (Nash Equilibrium)

Behavior strategies α and σ form a **Nash equilibrium** if and only if

1. For all $t \in \mathcal{T}$,

$$\sigma(t,s) > 0 \text{ implies } \textstyle\sum_a \alpha(s,a)u(t,s,a) = \max_{s' \in \mathcal{S}} \left(\sum_a \alpha(s',a)u(t,s',a)\right)$$

2. For each $s \in \mathcal{S}$ such that $\sum_t \sigma(t,s)\rho(t) > 0$,

$$\alpha(s,a) > 0 \text{ implies } \textstyle\sum_t \mu(t;s)v(t,s,a) = \max_{a'} \sum_t \mu(t;s)v(t,s,a')$$

   where $\mu(t;s)$ is the $\mathbb{R}$'s posterior belief about t given s, $\mu(t;s) = \frac{\sigma(t,s)\rho(t)}{\sum_{t'} \sigma(t',s)\rho(t')}$ if $\sum_t \sigma(t,s)\rho(t) > 0$ and $\mu(t;s) = 0$ otherwise.

Definition 3.4 (Separating & Pooling Equilibrium)

An equilibrium (σ, α) is called a **separating** equilibrium if each type t sends different signals; i.e., the set $\mathcal{S}$ can be partitioned into (disjoint) sets $\{\mathcal{S}_t; t \in \mathcal{S}\}$ such that $\sigma(t, \mathcal{S}_t) = 1$. An equilibrium (σ, α) is called a **pooling** equilibrium if there is a single signal s^* that is sent by all types; i.e., $\sigma(t, s^*) = 1$ for all $t \in \mathcal{T}$.

3.1.3 Single-crossing

3.1.3.1 Situation over real line

Consider the situation that $\mathcal{T}, \mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$ and $\geq$ is the usual "greater than or equal to" relationship.

1. We let $\Delta\mathcal{A}$ denote the set of probability distributions on $\mathcal{A}$.
2. For each $s \in \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$, we let $\Delta\mathcal{A}(s, T')$ be the set of mixed strategies that are the best responses by $\mathbf{R}$ to $s \in \mathcal{S}$ for some probability distribution with support $\mathcal{T}'$.
3. For $\alpha \in \Delta\mathcal{A}$, we write $u(t,s,\alpha) \triangleq \sum_{a \in \mathcal{A}} u(t,s,a)\alpha(a)$.

Definition 3.5 (Single-crossing)

The data of the game are said to satisfy the **single-crossing property** if the following holds: If $t \in \mathcal{T}$, $(s,\alpha) \in \mathcal{S} \times \Delta\mathcal{A}$ and $(s',\alpha') \in \mathcal{S} \times \Delta\mathcal{A}$ are such that $\alpha \in \Delta\mathcal{A}(s,\mathcal{T})$, $\alpha' \in \Delta\mathcal{A}(s',\mathcal{T})$, $s > s'$ and $u(t,s,\alpha) \geq u(t,s',\alpha')$, then for all $t' \in T$ such that $t' > t$, $u(t',s,\alpha) \geq u(t',s',\alpha')$.

3.2 Adverse Selection

Consider a labor market that has many identical firms. In competitive equilibrium, firms' profits are 0. Firms are price (wage) takers, risk-neutral, and CRS. There are continuum of workers with productivity levels $\theta \in [\underline{\theta}, \overline{\theta}]$ (Assume workers work if it is indifferent for them between employment and non-employment).

1. $\theta \sim F$, $F(\cdot)$ is a c.d.f. over $[\underline{\theta}, \overline{\theta}]$.
2. N is the total mass of workers.
3. Type θ worker has a reservation utility $r(\theta)$.

- Suppose the competitive equilibrium wages are $\theta = w^*(\theta)$.
- An allocation is denoted by $I : [\underline{\theta}, \overline{\theta}] \to \{0, 1\}$, where $I(\theta) = 0$ denotes θ is unemployed and $I(\theta) = 1$ denotes θ is employed.
- Aggregate welfare = sum of utilities of all participants

$$= N \int_{\underline{\theta}}^{\overline{\theta}} [I(\theta) \times \theta + [1 - I(\theta)] r(\theta)] \, dF(\theta)$$

Then we have the optimal allocation satisfies

$$I^*(\theta) \begin{cases} = 1, & \theta > r(\theta) \\ \in \{0, 1\} & \theta = r(\theta) \\ = 0, & \theta < r(\theta) \end{cases}$$

In the asymmetric information case,

Definition 3.6

w is CE wage if $w = \mathbb{E}[\theta | r(\theta) \leq w]$.

3.2.1 Adverse Selection

Assumption 3.1

(A1). r is strictly increasing in θ.

(A2). $F(\cdot)$ has a strictly positive density, $F(\theta) > 0, \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

(A3). $r(\theta) \leq \theta$ (outside option is worse than productivity, i.e., full employment is optimal).

Lemma 3.1

Under A1-A3, $\Phi(w) := \mathbb{E}[\theta | r(\theta) \leq w]$ is well-defined, continuous, and non-decreasing.

Hence, there exists underemployment, which makes 1^{st} welfare theorem fails. There may exist multiple CEs, where the one with the highest wage Pareto dominates others.

Example 3.1

Suppose $\theta \in [0,2]$, $F(\theta) = \frac{\theta}{2}$, $f(\theta) = \frac{1}{2}$, $r(\theta) = \alpha\theta, \alpha \in (0,1)$.

$$\mathbb{E}[\theta|r(\theta) \leq w] = \mathbb{E}\left[\theta|\theta \leq \frac{w}{\alpha}\right] = \begin{cases} 1, & w \geq 2\alpha \\ \frac{1}{F(\frac{w}{\alpha})}\int_0^{\frac{w}{\alpha}} \theta f(\theta)d\theta = \frac{w}{2\alpha}, & w \leq 2\alpha \end{cases}$$

CEs are given by $\mathbb{E}[\theta|r(\theta) \leq w] = w$. $w^* = 0$ is always CE and $w^* = 1$ is CE if $\alpha \leq \frac{1}{2}$.

3.2.2 Game Theoretical Approach

1. Suppose there are two firms setting wages simultaneously.
2. Workers observe the wages in stage 1 and make an employment decision.

Let W^* be the set of CE wages and $w^* := \max W^*$.

Lemma 3.2

$\forall w' \in (w^*, \overline{\theta}]$: $\mathbb{E}[\theta|r(\theta) \leq w'] < w'$.

Proof

Suppose by the contradiction that $\exists w' \in (w^*, \overline{\theta}]$ s.t. $\mathbb{E}[\theta|r(\theta) \leq w'] \geq w'$. Since $\mathbb{E}[\theta|r(\theta) \leq \overline{\theta}] < \overline{\theta}$, there must exist a $w'' \in [w', \overline{\theta})$ s.t. $\mathbb{E}[\theta|r(\theta) \leq w''] = w''$ by intermediate value theorem, which contradicts to the definition of w^*.

Proposition 3.1

(i). If $w^* > r(\underline{\theta})$ and $\exists \epsilon > 0$ s.t. $\mathbb{E}[\theta|r(\theta) \leq w'] > w', \forall w' \in (w^* - \epsilon, w^*)$. Then, there is a unique SPE where both firms set wage $= w^*$.

(ii). If $w^* = r(\underline{\theta})$ (complete market shutdown at w^*), there are multiple SPE that all give the same outcome as complete market shutdown where both firms set wage $= w^*$.

Proof

Lemma 3.3

In all SPE, firms make zero profits.

Proof

Suppose not, i.e., at least one firm makes strictly positive profits. Then, the total profits of firms 1&2,

$$\Pi = M(\bar{w})\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}] - \bar{w}\right] > 0$$

where $\bar{w}$ is the max wage set by the two firms and $M(\bar{w})$ is the mass of workers willing to work at $\bar{w}$. At least one firm, i, makes profit $\leq \frac{\Pi}{2}$. Then, i's profits from setting $\bar{w}+\delta$, with $\delta \to 0^+$, is higher:

$$\begin{aligned} &M(\bar{w}+\delta)\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}+\delta] - \bar{w} - +\delta\right] \\ \geq &M(\bar{w})\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}+\delta] - \bar{w} - +\delta\right] \to \Pi \text{ as } \delta \to 0 \end{aligned}$$

Hence, the i has incentive to deviate.

Lemma 3.4

In all SPE, firm i sets $w_i \leq w^*, i \in \{1, 2\}$.

Proof

Directly given by Lemma 3.2 and Lemma 3.3.

(i): In SPE, no firm i sets $w_i < w^*$: suppose $w_i < w^*$ and let $j \neq i$, take any w_j' s.t. $w_j' \in (w_i, w^*)$ and $w_j' > w^* - \epsilon$. Then, j gets profit: $M(w_j')\left[\mathbb{E}[\theta|r(\theta) \leq w_j'] - w_j'\right] > 0$ (by Case (i)'s conditions).

(ii): By Lemma 3.4, both firms set $w_i \leq w^* = r(\underline{\theta})$. Check that $\{(w_1, w_2) : w_1, w_2 \leq w^*\}$ is SPE wage profiles.

3.2.3 Planner Intervention

Planner can't observe the true type θ.

The planner's tools:

1. Take over the firms.
2. w_e, employment wage.
3. w_u, unemployment wage.

s.t. budget balanced.

Definition 3.7 (Constrained Efficient)

A CE w is **constrained efficient** if it cannot be Pareto improved upon by an intervention by the planner.

Proposition 3.2 ($w^* := \max W^*$ is constrained efficient)

Let W^* be the set of CE wages. $w^* := \max W^*$ is constrained efficient.

Proof

Note that both firms are making zero profits by the Lemma 3.3. Any CE wage $w \neq w^*$ can be Pareto improved by $\{w_e = w^*, w_u = 0\}$. Then, we prove w^* can't be Pareto improved.

1. Case 1: if w^* gives full-employment in CE, then w^* is Pareto efficient.
2. Case 1: suppose w^* doesn't give full-employment in CE.

   Consider taking an intervention $w_e \& w_u$. Then, $\{\theta : r(\theta) + w_u \leq w_e\} = [\underline{\theta}, \hat{\theta}]$ for some $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that

$$r(\hat{\theta}) + w_u = w_e \tag{3.1}$$

   The budget balanced gives

$$w_e F(\hat{\theta}) + w_u(1 - F(\hat{\theta})) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) \tag{3.2}$$

   Plug (3.1) into (3.2):

$$\begin{cases} w_u(\hat{\theta}) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) - r(\hat{\theta})F(\hat{\theta}) = F(\hat{\theta}) \left(\mathbb{E}[\theta|\theta \leq \hat{\theta}] - r(\hat{\theta})\right) \\ w_e(\hat{\theta}) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) + r(\hat{\theta})(1 - F(\hat{\theta})) \end{cases}$$

   Let θ^* be s.t. $r(\theta^*) = w^*$. Because w^* is a CE price, $\mathbb{E}[\theta|\theta \leq \theta^*] = r(\theta^*) = w^*$. So, CE with w^* can be implemented by $w_u(\theta^*) = 0$ and $w_e(\theta^*) = w^*$.

   (a). If $\hat{\theta} < \theta^*$. $\underline{\theta}$ is worse off under the intervention since $w_e(\hat{\theta}) < w^*$.

   (b). If $\hat{\theta} > \theta^*$. $\overline{\theta}$ is worse off under the intervention since $w_u(\hat{\theta}) = F(\hat{\theta}) \left(\mathbb{E}[\theta|\theta \leq \hat{\theta}] - r(\hat{\theta})\right) < 0$ by the Lemma 3.2

3.2.4 Signaling

Suppose the worker $\theta \in [\underline{\theta}, \overline{\theta}]$ can properly and costlessly reveal his type to the firms. Then,

Lemma 3.5

All workers revel their types.

Spence's Job Market Signaling Model One worker has productivity $\theta \in \{\theta_L, \theta_H\}$ with $P(\theta_H) = \lambda$. The worker signal through his education with cost $e > 0$. The education doesn't change his productivity. The payoff of the worker is the wage minus the cost:

$$u(w, e|\theta) = w - c(e, \theta)$$

where $c(0,\theta) = 0, c_e(e,\theta) := \frac{\partial c(e,\theta)}{\partial e} > 0, c_\theta(e,\theta) := \frac{\partial c(e,\theta)}{\partial \theta} < 0$, and $c_{e\theta}(e,\theta) := \frac{\partial^2 c(e,\theta)}{\partial e \partial \theta} < 0$ (Single-Crossing Property, the difference $c(e,\theta_L) - c(e,\theta_H)$ is increasing in e (i.e., $c_e(e,\theta_L) - c_e(e,\theta_H) > 0$), which means if $c(e,\theta_L)$ and $c(e,\theta_H)$ intersect as functions of e, they only intersect at one time.)

1. Stage 0: Nature chooses the $\theta \in \{\theta_L, \theta_H\}$ with $P(\theta_H) = \lambda$.
2. Stage 1: The worker learns θ and chooses $e(\theta) \geq 0$.
3. Stage 2: Firms observe $e(\theta)$. Then, they simultaneously make wage offers w_1 and w_2.
4. Stage 3: The worker observes w_1, w_2 and makes employment decision.

Let $r(\theta_L)$ and $r(\theta_H) = 0$. Let $\mu(e) \in [0,1]$ be the probability that in the beginning of stage 2, firms think that the worker is θ_H type with probability $\mu(e)$ when observing e. The corresponding expected productivity (the highest wage) that the firm can pay is

$$w(e) = \mu(e)\theta_H + (1-\mu(e))\theta_L$$

In stage 2, both firm will set $w(e)$ (complete competition).

Definition 3.8 (Perfect Bayesian Equilibrium)

A PBE is a strategy profile ($e^*(\theta_L)$, $e^*(\theta_H)$, $w_1^* : \mathbb{R}_+ \to \mathbb{R}$, $w_2^* : \mathbb{R}_+ \to \mathbb{R}$), and a belief $\mu^* : \mathbb{R} \to [0,1]$ such that

1. $\forall \theta \in \{\theta_L, \theta_H\}$, the worker strategy optimal given firm strategies.
2. The belief $\mu^*(e)$ is derived from $\lambda, e^*(\theta_L), e^*(\theta_H)$ via Bayes' rule whenever possibly (on the equilibrium path). Outside the equilibrium path the belief $\mu^*(e)$ is arbitrarily.
3. Firms offer wages that form a NE of the stage 2 game, where their belief $\mu^*(e)$ about their workers' type. (sequential rationality).

We simplify the game by backward induction:

1. Stage 3: The worker chooses the highest wage off if it is ≥ 0.
2. Stage 2: After observing $e(\theta)$, firms chooses the wage as the expected productivity in NE,

$$w^*(e) = \mu^*(e)\theta_H + (1-\mu^*(e))\theta_L$$

   because it is a Bertrend competition.

Separating Equilibrium In separating equilibrium, $e^*(\theta_L) \neq e^*(\theta_H)$.

Lemma 3.6

In any separating PBE, $w^*(e^*(\theta)) = \theta, \forall \theta \in \{\theta_L, \theta_H\}$.

Proof

By Bayes' rule, after firm observe $e^*(\theta_L)$, $\mu^*(e^*(\theta_L)) = 0$. Then, $w^*(e^*(\theta_L)) = \theta_L$. ($e^*(\theta_H)$ is similar.)

Lemma 3.7

In separating PBE, low type always chooses zero education, $\theta^*(\theta_L) = 0$.

Proof

If not, the low type worker always has profitable deviation, $\theta^*(\theta_L) = 0$.

Lemma 3.8

Define $\underline{e}$ and $\overline{e}$ such that

1. $\theta_L = \theta_H - c(\underline{e}, \theta_L)$ (the lowest effort can prevent the low type from mimicking high type) and
2. $\theta_L = \theta_H - c(\overline{e}, \theta_H)$ (the highest effort can prevent the high type from pooling with low type).

Then, in all separating PBEs, $e \in [\underline{e}, \overline{e}]$.

Conversely, $\forall \hat{e} \in [\underline{e}, \overline{e}]$, there is a separating PBE where $e^*(\theta_H) = \hat{e}$.

These different PBEs are Pareto ranked. High type prefers the PBE with a lower e (the best is the one with $e^*(\theta_H) = \underline{e}$.)

Pooling PBE $e^*(\theta) = e^*, \theta \in \{\theta_L, \theta_H\}$, $\mu^*(e^*) = \lambda$, and $w^*(e^*) = \mathbb{E}[\theta]$.

Lemma 3.9

Define e' by $\theta_L = \mathbb{E}[\theta] - c(e', \theta_L)$ (the highest effort can prevent the low type from choosing $e = 0$ and get $w = \theta_L$.)

Then, for all pooling PBE, $e^*(\theta_L) = e^*(\theta_H) = e^* \in [0, e']$. Conversely, for all $\hat{e} \in [0, e']$, there is a pooling PBE with $e^* = \hat{e}$.

3.2.5 Cho-Kreps Intuitive Criterion

Definition 3.9 (Equilibrium Dominated Message)

A message is **equilibrium dominated** for a type if the type must do strictly worse by sending the message than it does in equilibrium (i.e., payoff in eq. is strictly better than maximum payoff from deviating).

Definition 3.10 (Cho-Kreps Intuitive Criterion)

If an information set is off the eq. path and a message is eq. dominated for a type, then beliefs should assign zero probability to the message coming from that type (if possible).

Fix a PBE $e^*(\theta), \theta \in \{\theta_L, \theta_H\}, \mu^*(\cdot)$ (We know $w_1^*(e) = w_2^*(e) = \mu(e)\theta_H + (1-\mu(e))\theta_L$). Let $u^*(\theta), \theta \in \{\theta_L, \theta_H\}$ be the PBE utility of the type θ worker.

The criterion requires the (off-path) belief $\mu^*(e) := P(\tilde{\theta} = \theta_H | e) = 1 - P(\tilde{\theta} = \theta_L | e)$ satisfies

$$P(\tilde{\theta} = \theta | e) = 0, \forall e, \theta$$

such that

1. $u^*(\theta) > \max_{w \in [\underline{\theta}, \overline{\theta}]} [w - c(e, \theta)]$
2. $\exists \theta'$ s.t. $u^*(\theta') \leq \max_{w \in [\underline{\theta}, \overline{\theta}]} [w - c(e, \theta')]$ (make sure the sum of beliefs given e is nonzero.)

In this application, the only PBE that survives Intuitive Criterion is the best separating PBE, $e^*(\theta_H) = \underline{e}$ (the lowest effort).

3.2.6 Grossman-Perry-Farrell Equilibrium

For the equilibrium refinement, we can also introduce the Grossman-Perry-Farrell equilibrium based on the perfect sequential equilibrium (grossman1986sequential) and the neologism-proof equilibrium (farrell1993meaning). We formally define the Grossman-Perry-Farrell equilibrium by ruling out the self-signaling sets in Perfect Bayesian Equilibrium (bertomeu2018verifiable,glode2018voluntary). A binary example is given as follows:

Definition 3.11 (Grossman-Perry-Farrell Equilibrium (GPFE))

A pure-strategy perfect Bayesian equilibrium $(p_L^*, p_H^*, b^*(\cdot))$ is a ``Grossman-Perry-Farrell equilibrium'' (GPFE) if there does not exist a self-signaling set, which is defined by a set $\chi \subseteq \{L, H\}$ such that there exists a price p' such that

$$\chi = \{j \in \{L, H\} : U(p', \mu_\chi) > U(p_j^*, b^*(p_j^*))\},$$

where $\mu_\chi = \frac{q_L \rho_e \mathbf{1}_{L \in \chi} + q_H (1-\rho_e) \mathbf{1}_{H \in \chi}}{\rho_e \mathbf{1}_{L \in \chi} + (1-\rho_e) \mathbf{1}_{H \in \chi}}$ is the average quality of types in χ based on the relative prior probabilities.

Note *The GPFE is a strong refinement of PBE that may lead to no equilibrium exists.*

4 Screening

4.1 Screening Model

Workers can undertake a contractible/observable task level $t \geq 0$. The utility of a worker is defined by $u(w,t,\theta) := w - c(t,\theta)$, where $c(\cdot,\cdot)$ satisfies the same assumption as in signaling model 3.2.4. The Game follows

1. Stage 1: Two firms simultaneously determine sets of contracts, (w,t).
2. Stage 2: The worker observes all offer contracts and makes employment decision. (If indifference, choose lower task contract, favor employment over unemployment. If contracts of firms are indifferent, choose each with probability 1/2.)

The null contract is $(w,t) = (0,0)$. Assume WLOG at stage 1, each firm appears a non-empty set of contracts.

Perfect Information

Proposition 4.1 (Perfect Information)

If firms can observe the worker types, then in SPE firms make zero profit and type θ_i worker signs $(w_i^*, t_i^*) = (\theta_i, 0)$.

Proof

Claim 4.1

Firms make zero profits from this contract.

Proof

Suppose not,

- $w_i^* > \theta_i \Rightarrow$ negative profits, firms benefit from offering null contract.
- $w_i^* < \theta_i \Rightarrow$ Let Π be the total profits of the firms. Then one of the firms makes profit $\leq \frac{\Pi}{2}$. Then, this firm can benefit from offering $(w_i^* + \Delta, t_i^*)$, where $\Delta \to 0^+$.

Then, we prove the firms must choose $(w_i^*, t_i^*) = (\theta_i, 0)$. Suppose by the way of contradiction that $t_i^* > 0$. Then, one firm can profitably deviate by offering $(w_i^*, 0)$.

Asymmetric Information

Lemma 4.1

In any SPE, firms obtain zero profits,

Proof

Firms must make profits ≥ 0. Suppose by the way of contradiction that the total profit $\Pi > 0$. Let (w_L, t_L) be the contract signed by θ_L and (w_H, t_H) be the contract signed by θ_H. One firm can profitably deviate by offering $(w_L + \Delta, t_L)$ and $(w_H + \Delta, t_H)$, where $\Delta \in (0, \Pi)$.

Lemma 4.2

There is **no** pooling SPE.

Proof

Suppose for a contradiction, $\exists$ an SPE where both worker types sign $(w_p = \mathbb{E}[\theta], t_p)$. Suppose one firm offers (w_p, t_p), then another firm can only employ high type workers by offering $(\tilde{w}, \tilde{t})$, where $\tilde{w} - c(\tilde{t}, \theta_H) > \mathbb{E}[\theta] - c(t_p, \theta_H)$, $\tilde{w} - c(\tilde{t}, \theta_L) < \mathbb{E}[\theta] - c(t_p, \theta_L)$, and $\tilde{w} < \theta_H$. (The existence is given by $\frac{\partial^2 c(t,\theta)}{\partial t \partial \theta} < 0$.)

Lemma 4.3

Let (w_L, t_L) be the contract signed by θ_L and (w_H, t_H) be the contract signed by θ_H in separating SPE. Then, $w_L = \theta_L$ and $w_H = \theta_H$.

Proof

Suppose $w_i > \theta_i, i \in \{L, H\}$, firms benefit from not offering this contract. So, $w_L \leq \theta_L$ and $w_H \leq \theta_H$.

1. $\underline{w_L = \theta_L}$: Suppose $w_L < \theta_L$. Either firm can profitably deviate by setting (w_L', t_L) such that $w_L < w_L' < \theta_L$. This offer can win all low-type workers and get a positive profit from hiring them. If $w_L' - c(t_L, \theta_H) \geq w_H - c(t_H, \theta_H)$, the offer can also hire high-type workers, which also give positive profit for the firm. Hence, there is a contradiction.
2. $\underline{w_H = \theta_H}$: Suppose $w_H < \theta_H$, firms get positive profits, which contradicts to the Lemma 4.1.

Lemma 4.4

θ_L signs the contract $(\theta_L, 0)$ in SPE.

Proof

Suppose $t_L > 0$. One firm can profitably deviate by offering $(\theta_L - \Delta, 0)$.

Proposition 4.2

In any (pure strategy) SPE, θ_L signs $(w_L, t_L) = (\theta_L, 0)$ and θ_H signs $(w_H, t_H) = (\theta_H, t_H)$, where t_H solves

$$\theta_H - c(t_H, \theta_L) = \theta_L$$

If $\lambda := P(\theta_H)$ is high, the pure SPE may not exist (exist $(\tilde{w}, \tilde{t})$ can attract both types and make positive profit).

Cross subsidizing deviation by a firm (prices one product above its market value to fund another product), $(\tilde{w}, \tilde{t})$ (signed by low type) and $(\tilde{\tilde{w}}, \tilde{\tilde{t}})$ (signed by high type), is a profitable deviation if λ is large enough.

5 Bargaining

Bargaining refers to ”a process to determine the terms of trade that is not adequately captured by off-the-shelf oligopoly models.” (Loertscher and Marx, 2021).

5.1 Axiomatic Complete Information Bargaining

The axiomatic approach abstracts away from the specifics of the bargaining process, focusing instead on identifying “reasonable” or “natural” properties that outcomes should satisfy.

5.1.1 Bilateral Negotiations

Let X denote the *set of possible agreements* and D the *disagreement* outcome.

Example 5.1

Suppose $X = \{(x_1, x_2) : x_1 + x_2 = 1, x_i \geq 0\}$ and $D = (0, 0)$.

Each player i has preferences represented by a utility function u_i defined over $X \cup \{D\}$. The set of possible payoffs, denoted by U, is defined as:

$$U = \{(v_1, v_2) : u_1(x) = v_1, u_2(x) = v_2 \text{ for some } x \in X\},$$
$$d = (u_1(D), u_2(D)).$$

Definition 5.1 (Bargaining Problem)

A **bargaining problem** is a pair (U, d) where $U \subseteq \mathbb{R}^2$ and $d \in U$. Typically, we assume:

1. U is a convex and compact set.
2. There exists some $v \in U$ such that $v > d$ (i.e., $v_i > d_i$ for all i).

We denote the set of all possible bargaining problems by B. A bargaining solution is a function $f : B \to U$.

Definition 5.2 (Axioms)

We study bargaining solutions $f(\cdot)$ that satisfy the following axioms:

1. **Axiom 1 (Pareto Efficiency)**: A bargaining solution $f(U, d)$ is *Pareto efficient* if there does not exist a $(v_1, v_2) \in U$ such that $v \geq f(U, d)$ and $v_i > f_i(U, d)$ for some i.

   Note *Intuition: An inefficient outcome is unlikely, as it leaves room for renegotiation.*

2. **Axiom 2 (Symmetry)**: A bargaining solution f is *symmetric* if for any symmetric bargaining problem (U, d) (i.e., $(u_1, u_2) \in U$ if and only if $(u_2, u_1) \in U$, and $d_1 = d_2$), we have $f_1(U, d) = f_2(U, d)$.

   Note *Intuition: If the players are indistinguishable, the agreement should not favor one over the other. (This axiom can be relaxed to account for bargaining power.)*

3. **Axiom 3 (Invariance to Linear Transformations)**: A bargaining solution f is *invariant* if for any bargaining problem (U, d) and all $\alpha_i \in (0, \infty)$, $\beta_i \in \mathbb{R}$ $(i = 1, 2)$, if we consider the transformed bargaining problem (U', d') with

$$U' = \{(\alpha_1 u_1 + \beta_1, \alpha_2 u_2 + \beta_2) : (u_1, u_2) \in U\},$$

$$d' = (\alpha_1 d_1 + \beta_1, \alpha_2 d_2 + \beta_2),$$

   then $f_i(U', d') = \alpha_i f_i(U, d) + \beta_i$ for $i = 1, 2$.

   Note *Intuition: Utility functions are merely one cardinal representation of ordinal preferences. They lack intrinsic cardinal meaning, so monotonic (especially linear) transformations should not affect the outcome.*

4. **Axiom 4 (Independence of Irrelevant Alternatives)**: A bargaining solution f is *independent* if for any two bargaining problems (U, d) and (U', d) with $U' \subseteq U$ and $f(U, d) \in U'$, we have $f(U', d) = f(U, d)$.

   Note *Intuition: Removing options that were not chosen should not change the outcome. This axiom arguably reflects behavioral assumptions and may require additional justification.*

5.1.2 Nash Bargaining Solution

Definition 5.3 (Nash Bargaining Solution)

The Nash (1950) bargaining solution f_N is defined as:

$$f_N(U, d) = \underset{u \in U, u \geq d}{\text{argmax}} \, (u_1 - d_1)(u_2 - d_2)$$

Given the assumptions on (U, d), the solution to this optimization problem exists (if U is compact and the objective function is continuous) and is unique (if the objective function is strictly quasi-concave).

Theorem 5.1 (Nash, 1950)

f_N is the unique bargaining solution that satisfies the four axioms.

The Nash bargaining solution can be generalized to account for unequal bargaining weights:

$$f_\beta(U,d) = \underset{u \in U, u \geq d}{\operatorname{argmax}} \, (u_1 - d_1)^\beta (u_2 - d_2)^{1-\beta}$$

This solution is straightforward to compute and can be microfounded using an alternating offers bargaining game, as in Rubinstein (1982). See, for example, Binmore et al. (1986). Essentially, it selects a point on the efficient frontier (above d).

The Nash bargaining solution assumes no breakdown in negotiations.

Note ***Weakness:*** *In Nash bargaining, the disagreement point plays a disproportionately significant role. If the negotiation is over a set of options that dominates d, it is unclear why the disagreement point should matter (e.g., Binmore et al. (1986)). This characteristic of Nash bargaining often has a substantial impact in empirical applications.*

5.1.3 Rubinstein's Alternating-Offers Model

Rubinstein (1982) provides a non-cooperative foundation for bargaining by modeling the actual process of negotiation. Two players negotiate over how to divide a ”pie” of size 1 through alternating offers:

- At $t = 0$, player A makes an offer $(x_A, 1 - x_A)$ where x_A is A's share
- If B accepts, game ends with that division
- If B rejects, at $t = 1$, B makes a counteroffer $(1 - x_B, x_B)$
- If A accepts, game ends; if not, roles switch and game continues

Players discount future payoffs with factors $\delta_A, \delta_B \in (0, 1)$. Zero payoff in periods without agreement.

Theorem 5.2 (Rubinstein, 1982)

The game has a unique subgame perfect Nash equilibrium characterized by:

1. Immediate agreement in period 0

2. Equilibrium shares when player i proposes:

$$x_A = \frac{1 - \delta_B}{1 - \delta_A \delta_B}$$
$$x_B = \frac{1 - \delta_A}{1 - \delta_A \delta_B}$$

Proof

[Sketch] Key steps:

1. In any subgame where A offers, B accepts if getting $\geq \delta_B x_B$ (discounted continuation value)
2. When B offers, A accepts if getting $\geq \delta_A x_A$
3. This yields system of equations:

$$x_A = 1 - \delta_B x_B$$
$$x_B = 1 - \delta_A x_A$$

4. Solving gives unique stationary equilibrium shares
5. One-shot deviation principle proves uniqueness

Remark[Connection to Nash Bargaining] As $\delta_A, \delta_B \to 1$, equilibrium shares approach the symmetric Nash bargaining solution. This provides a non-cooperative foundation for the axiomatic approach.

Remark[First-Mover Advantage] Player A's equilibrium share x_A exceeds B's discounted share $\delta_B x_B$, reflecting advantage of making first offer. This advantage diminishes as players become more patient.

5.1.4 Multiple Simultaneous Negotiations (Nash-in-Nash)

1. Consider a scenario with a finite set of agents $\{1, ..., N\}$.
2. Let $\mathcal{G}$ represent the set of feasible pairs, ij, that can potentially form agreements to collaborate.
3. Denote p_{ij} as the transfer from agent j to agent i if they reach an agreement.
4. The value to agent i when a set $A \subseteq \mathcal{G}$ of agreements is realized is $\pi_i(A)$. The net payoff to i is $\pi_i(A) + p_i$, where p_i is the total payment received by i.

Note *Agreements may generate externalities, but payments themselves do not.*

5. For $B \subset A \subset \mathcal{G}$, define $\Delta\pi_i(A, B) = \pi_i(A) - \pi_i(A \setminus B)$ as the marginal value of adding the set B of agreements, given that $A \setminus B$ is already realized.

Assumption 5.1 (Gains from Trade)

For all $ij \in \mathcal{G}$, $\Delta\pi_i(\mathcal{G}, ij) + \Delta\pi_j(\mathcal{G}, ij) > 0$.

Definition 5.4 (Nash-in-Nash Bargaining Solution)

In the Nash-in-Nash solution, under the gains-from-trade assumption, all agreements are reached. The transfer from j to i is given by:

$$p_{ij}^N = \underset{p}{\operatorname{argmax}} \left(\Delta\pi_i(\mathcal{G}, ij) + p\right)^{b_i} \left(\Delta\pi_j(\mathcal{G}, ij) - p\right)^{b_j} = \frac{b_i \Delta\pi_j(\mathcal{G}, ij) - b_j \Delta\pi_i(\mathcal{G}, ij)}{b_i + b_j}.$$

In other words, the outcome between each pair is the bilateral Nash bargaining solution, given the "equilibrium" conjecture that all other agreements are reached. This represents a Nash equilibrium in Nash bargaining.

Remark **Weakness:**

1. Inherits the limitations of the Nash bargaining solution.
2. Assumes binary agreements.
3. Considers externalities only over agreements, not lump-sum payments.
4. Relies on "passive beliefs": if the negotiation between i and j breaks down, i negotiates with k as if the agreement between i and j were still in place.
5. Assumes complete information.

5.2 Incomplete Information Bargaining

5.2.1 Bilateral Trade with Incomplete Information

Consider a bilateral trade setting with:

- A buyer B with value $v \in [\underline{v}, \bar{v}]$ for a good
- A seller S with value (or production cost) $c \in [\underline{c}, \bar{c}]$
- Types are independently distributed

By the revelation principle, we can focus on incentive compatible direct mechanisms, which consist of:

Definition 5.5 (Direct Mechanism)

A direct mechanism in bilateral trade is characterized by:

1. An allocation rule $q : [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}] \to [0, 1]$ determining the probability of trade
2. Payment rules $p_j : [\underline{v}, \bar{v}] \times [\underline{c}, \bar{c}] \to \mathbb{R}$ determining the payment made by agent $j \in \{B, S\}$

Definition 5.6 (Mechanism Properties)

A mechanism (q, p) is:

1. **Bayesian incentive compatible (BIC)** if truthful reporting is optimal for each agent at the interim stage (knowing only their own type).
2. **Interim IR** if each agent receives a non-negative expected payoff at the interim stage.
3. **Dominant strategy incentive compatible (DSIC)** if truthful reporting is optimal for each agent, conditional on each profile of the others' types.
4. **Ex-post IR** if each agent receives a non-negative payoff ex-post.
5. **No-deficit in expectation** if $E[p_B(v, c) + p_S(v, c)] \geq 0$
6. **Budget balanced in expectation** if $E[p_B(v, c) + p_S(v, c)] = 0$
7. **Efficient** if trade occurs whenever $v > c$, and never if $v < c$

Remark BIC, Interim IR, and no-deficit seem reasonable conditions which should be satisfied, at a minimum.

Theorem 5.3 (Myerson-Satterthwaite Impossibility Theorem)

In the bilateral trade setting, if $\underline{v} < \bar{c}$, then there exists no mechanism that is simultaneously efficient, Bayesian incentive compatible (BIC), interim individually rational (IR), and does not run a deficit in expectation.

In other words, whenever agents have the option to walk away and play a Bayes-Nash equilibrium, trade cannot be efficient.

Proof

Recall the Revenue Equivalence theorem, which states that we can write the payments made by agent i as a function of the allocation rule, plus the surplus that the mechanism leaves to i when they have the least-favorable type ($\underline{v}$ for buyer, $\bar{c}$ for seller).

Consider the following mechanism that implements the efficient allocation rule:

- Trade occurs if and only if $v \geq c$. The buyer pays $p_B = \max\{c, \underline{v}\}$, and the seller receives $-p_S = \min\{v, \bar{c}\}$. Otherwise, no trade occurs and no payments are made.

This mechanism is dominant strategy incentive compatible (DSIC), and therefore a fortiori BIC. Moreover, the payoff of a type $\underline{v}$ buyer and type $\bar{c}$ seller are both zero. Furthermore, $p_B(v, c) + p_S(v, c) \leq 0$ for all c, with strict inequality for almost all types.

By revenue equivalence, any other efficient, IR, and BIC mechanism must have weakly lower revenue (strictly lower if either agent's IR constraint does not bind).

5.2.2 k-Double Auction

The k-double auction, proposed by Chatterjee and Samuelson (1983), provides a non-cooperative foundation for bargaining in bilateral trade:

- Buyer submits a sealed bid b
- Seller submits a sealed offer s
- Trade occurs if and only if $b \geq s$
- If trade occurs, payment from buyer to seller is $kb + (1-k)s$

Assume that agents play a Bayes-Nash equilibrium and can walk away before the game begins. This corresponds exactly to the Bayesian incentive compatibility (BIC) and Interim individual rationality (IR) constraints for the direct mechanism. Note that the outcome of the k-double auction is budget balanced by construction.

Example 5.2 (Special Equilibrium, posted price)

Here is a special equilibrium in the k-double auction: Fix a price $p \in (\underline{c}, \bar{v})$ and consider the following strategies:

- Buyer: Bid p when $v \geq p$, and bid $\underline{c}$ otherwise
- Seller: Ask p when $c \leq p$, and ask $\bar{v}$ otherwise

Since $kp + (1-k)p = p$, this is equivalent to a posted price mechanism (on the equilibrium path).

Note

1. *Fixing one agent's strategy, the specified strategy is optimal for the other agent ex-post (even if they knew the other's type). Therefore this mechanism can be implemented as a dominant strategy incentive compatible (DSIC) direct mechanism.*
2. *Any $p \in (\underline{c}, \bar{v})$ will work as this fixed price.*

Theorem 5.4 (DSIC Characterization)

If $\underline{c} \leq \bar{v}$, then any budget-balanced, ex-post individually rational, dominant strategy incentive compatible (DSIC) mechanism that is not Pareto dominated must be a randomization over posted price mechanisms.

Proof

[Sketch] Consider any mechanism satisfying the conditions. For any realization of types (v, c):

- If trade occurs, the payment must be independent of v (by buyer's DSIC) and independent of c (by seller's DSIC)
- Therefore the payment must be a fixed price p whenever trade occurs
- By ex-post IR, trade can only occur when $v \geq p \geq c$
- By Pareto efficiency, trade must occur whenever $v \geq p \geq c$

This describes exactly a posted price mechanism. Any mechanism that is not a randomization over such mechanisms would be Pareto dominated by one that is.

This result shows that the posted price equilibrium we found in the k-double auction is essentially the only way to implement trade with dominant strategies. The impossibility of efficient trade extends beyond Bayesian implementation to dominant strategy implementation.

Remark What can happen under Bayesian implementation? The same impossibility result extends:

- Even if we relax from dominant strategy to Bayesian incentive compatibility (BIC)
- And from ex-post IR to interim IR
- We still **cannot** do better than randomizing over posted prices

Theorem 5.5 (BIC Characterization)

If $\underline{c} \leq \bar{v}$, then any budget-balanced, interim individually rational, Bayesian incentive compatible mechanism that is not Pareto dominated must be interim equivalent to a randomization over posted price mechanisms.

Proof

[Sketch] The proof follows similar logic to the DSIC case:

- By BIC, the interim allocation and payment rules must be monotone
- Budget balance and interim IR together constrain the possible transfers
- Any mechanism violating these properties would be Pareto dominated
- The only mechanisms satisfying all constraints are (interim) equivalent to randomizations over posted prices

This stronger impossibility result shows that even with the weaker implementation concept of BIC:

- We cannot achieve better outcomes than simple posted prices
- The bilateral trade problem remains fundamentally constrained
- The Myerson-Satterthwaite impossibility extends beyond just efficient trade

Remark[Summary of Bilateral Trade] Our analysis of bilateral trade can be summarized as follows:

- We started by specifying a specific (indirect) game form, the k-double auction
- For any k, the set of equilibria of this game was exactly the set of all "reasonable" outcomes (satisfying Bayesian incentive compatibility, interim individual rationality, budget balance, and Pareto-undominated)
- These outcomes were all induced by (distributions over) posted prices
- Which posted prices should we expect to see? This depends on the distribution of bargaining power!

5.2.3 Bargaining Power in k-Double Auction

Remark[Bargaining Power in k-Double Auction] Regarding bargaining power in the k-double auction:

- In the k-double auction, price is determined by $kb + (1-k)s$
- One might reasonably think that k measures bargaining power
- However, we found that regardless of k, we get the same set of equilibria
- (Note: This fact would be missed if focusing only on restricted classes of equilibria, like those with smooth bidding strategies)
- One might be tempted to conclude that "in this model, bargaining power doesn't matter for outcomes"
- But this conclusion is nonsense!

Remark[Rethinking Bargaining Power] We need to step back and ask: What is bargaining power?

- Bargaining power is whatever determines which outcome is selected from among the set of reasonable outcomes
- Therefore, k is not the right notion of bargaining power in the k-double auction
- What is? It's the price p!
- Alternatively, we can say the mechanism maximizes a weighted sum of buyer and seller surplus, where the weights determine which p to choose
- This is essentially equivalent to choosing p, but provides a way to connect predictions across different bargaining scenarios

5.3 Pay Transparency and Reputation (Valenzuela-Stookey, 2023)

Valenzuela-Stookey (2023) provides a model of pay transparency:

- The model can be viewed as bargaining between many sellers and one buyer

- It illustrates why passive beliefs, as used in Nash-in-Nash, might be a problematic assumption
- Particularly with incomplete information, passive beliefs can hide interesting phenomena
- The model demonstrates how transparency affects bargaining power and reputation effects

5.3.1 Model Setup

- N sellers, each offering one item at zero marginal cost
- One buyer with additive demand (can buy from multiple sellers)
- Buyer's value per item is either high (h) or low (l)
- Buyer's type is initially unknown to sellers

5.3.2 Timing

The game proceeds in two periods:

1. Period 1:
    - Sellers simultaneously make take-it-or-leave-it price offers
    - Buyer observes all offers and decides which to accept
2. Information Revelation (Transparency):
    - Some subset of sellers observes outcomes of other transactions
3. Period 2:
    - Sellers who observed buyer paying above l in period 1 infer type h
    - These sellers demand price h
    - All other sellers demand price l

5.3.3 Equilibrium Analysis

- The buyer has incentives to maintain a reputation for being a low type
- This means the buyer's actions in period 1 are not independent across sellers
- For a fixed set of period-1 offers:
    - If the buyer accepts a price above l from seller k
    - They become more willing to accept a price above l from seller i
- Sellers anticipate this in equilibrium
- When pricing in period 1, sellers know:
    - If the buyer rejects their offer
    - They will also reject all other offers priced above l

- Beliefs are very much not passive!

5.3.4 Implications

These reputation effects have important implications for prices and transparency:

- Transparency increases buyer's incentives to maintain reputation
- This lowers equilibrium prices in period 1
- But increases price volatility in period 2

5.4 Strategic Delay in Bargaining with Two-Sided Uncertainty (Cramton, 1992)

A seller with valuation S and a buyer with valuation B are bargaining over the price of an object. The valuation is symmetric and private, that is, B and S are i.i.d. drawn from a distribution F with density f over $[0, 1]$.

An outcome of the game is the time and the price, $\langle t, p \rangle$. The discount rate is r, that is, the payoff to S is $e^{-rt}(p-S)$ and the payoff to B is $e^{-rt}(B-p)$. The discount rate r and the valuation distribution F are common knowledge.

As in Admati and Perry (1987), the players alternate making offers with a minimum time of $t^0 = -\frac{1}{r}\log\delta$ between offers. Initially, both traders have the option of making the first offer or terminating negotiations (at time $t \geq -t^0$). If the traders happen to make initial offers at the same time, then a fair coin is flipped to determine which offer stands as the initial offer. After an offer is made, the other trader has three possible responses: (1) a counter-offer, (2) acceptance, or (3) termination.

Suppose that trader $T \in \{S, B\}$ makes the first offer p1 after a delay of Δ_1, and that in round i the offer p_i is made after a delay Δ_i beyond the minimum time t_0 between offers. The history after n rounds is $h^n = \{T, (\Delta_i, p_i)_{i=1,\dots,n}\}$.

The pure strategy of the seller and the buyer are denoted by π_S and π_B. The profile of strategies is $\pi = \{\pi_S, \pi_B : \forall(S, B)\}$, which result in an outcome $\{t(S, B), p(S, B)\}$ that depends on the traders' valuations (S, B). "No trade" is represented by $t = \infty$. Since all actions are publicly observed, S's belief about B's valuation is independent of S after any history h^n. The belief after h^n can be denoted by $\mu = \{F_B(\cdot \mid h^n), F_S(\cdot \mid h^n)\}$.