Statistics: Math

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All models are wrong, but some are useful.

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Chapter 1 Review of statistics

1.1 Random Vectors

Suppose a random vector $\vec{Z} \in \mathbb{R}^{m\times 1}$ has mean $\vec{\mu} = \mathbb{E}(\mathbf{Z}) = [\mathbb{E}(Z_1), \mathbb{E}(Z_2), \cdots, \mathbb{E}(Z_m)]^T$ and variance-covariance matrix: $\Sigma_{m\times m} = Cov(\mathbf{Z}) = \mathbb{E}((\mathbf{Z}-\mu)(\mathbf{Z}-\mu)^T) = \begin{bmatrix} Var(Z_1) & \cdots & Cov(Z_1, Z_m) \\ \cdots & \cdots & \cdots \\ Cov(Z_m, Z_1) & \cdots & Var(Z_m) \end{bmatrix}$.

1.2 Affine Transformation

(1) The affine transformation $\mathbf{W} = \mathbf{a}_{n\times 1} + \mathbf{B}_{n\times m}\mathbf{Z}_{m\times 1}$ should have following properties:

$$\mathbb{E}(\mathbf{W}) = \mathbf{a} + \mathbf{B}\mu,\ Cov(\mathbf{W}) = \mathbf{B}\Sigma\mathbf{B}^T$$

(2) The affine transformation $\mathbf{W} = \mathbf{v}^T\mathbf{Z} = v_1Z_1 + ... + v_mZ_m$ should have following properties:

$$\mathbb{E}(\mathbf{W}) = \mathbf{v}^T\mu = \sum_{i=1}^{m} v_i\mu_i$$

$$Var(\mathbf{W}) = \mathbf{v}^T\Sigma\mathbf{v} = \sum_{i=1}^{m} v_i^2 Var(Z_i) + 2\sum_{i<j} v_iv_jCov(Z_i, Z_j)$$

$$\text{i.e. } \mathbb{E}(\mathbf{AZ}) = \mathbf{A}\mathbb{E}(Z);\ Var(\mathbf{AZ}) = \mathbf{A}Var(\mathbf{Z})\mathbf{A}^T$$

(3)

$$\begin{aligned} Cov(\mathbf{AX}, \mathbf{BY}) &= \mathbb{E}[(\mathbf{AX} - \mathbf{A}\mathbb{E}(X))(\mathbf{BY} - \mathbf{B}\mathbb{E}(Y))^T] \\ &= \mathbf{A}\mathbb{E}[(\mathbf{X} - \mathbb{E}(X))(\mathbf{Y} - \mathbb{E}(Y))^T]\mathbf{B}^T \\ &= \mathbf{A}Cov(\mathbf{X}, \mathbf{Y})\mathbf{B}^T \end{aligned}$$