



Analysis and Something Else

Author: Wenxiao Yang

Institute: Haas School of Business, University of California Berkeley

Date: 2023

All models are wrong, but some are useful.

Contents

Chapter 1 Logic	1
1.1 Main Methods of Proof (@ Lec 01 of ECON 204)	1
1.1.1 Proof by Induction	1
1.1.2 Proof by Deduction	1
1.1.3 Proof by Contradiction	1
1.1.4 Proof by Contraposition	1
Chapter 2 Analysis Basis	2
2.1 Real Number \mathbb{R} (@ Lec 02 of ECON 204)	2
2.1.1 Order Axiom	2
2.1.2 Completeness Axiom	2
2.1.3 Supremum sup \mathbb{X} , Infimum inf \mathbb{X} for $\mathbb{X} \subseteq \mathbb{R}$	3
2.1.4 Theorem: sup / inf in set is max / min	3
2.1.5 Transformation of sup / inf	3
2.1.6 The Supremum Property: Nonempty set in \mathbb{R} is bounded above/below $\Rightarrow \exists \text{sup} / \text{inf}$.	3
2.1.7 Archimedean Property	4
2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)	4
2.2.1 Metric Space (\mathbb{X}, d) and Metric $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$	4
2.2.2 Theorem: Subset forms a metric space	4
2.2.3 Norm $\ \cdot\ $ and Normed Vector Space $(V, \ \cdot\)$	4
2.2.4 Subset may not form a normed vector space	5
2.2.5 Theorem: metric can be defined by norm	5
2.2.6 Cauchy-Schwarz Inequality	5
2.2.7 Lipschitz-equivalent Norm	6
2.2.8 Ball, Radius, Diameter, and Distance	6
2.3 Set Theory	7
2.3.1 Well Defined Set	7
2.3.2 Numerically Equivalent: \exists bijection (@ Lec 01 of ECON 204)	7
2.3.3 Finite (@ Lec 01 of ECON 204)	7

2.3.4	Countable Set: numerically equivalent to \mathbb{N} (@ Lec 01 of ECON 204)	7
2.3.5	Power Set (@ Lec 02 of ECON 204)	7
2.3.6	Theorem (Cantor): The power set of \mathbb{N} is uncountable (@ Lec 02 of ECON 204)	8
2.3.7	Cardinalities of Sets (@ Lec 02 of ECON 204)	8
2.3.8	Pigeonhole Principle: $ A > B \Rightarrow$ no injective function $\sigma : A \rightarrow B$	8
2.3.9	B^A : Sets of Function	8
2.3.10	Bounded Set (@ Lec 03 of ECON 204)	9
2.3.11	Open, Closed Set (@ Lec 04 of ECON 204)	9
2.3.12	★ Theorem: Union of open sets is open, Intersection of finite open sets is open	10
2.3.13	★ Corollary: Intersection of closed sets is closed, Union of finite closed sets is closed	10
2.3.14	Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)	10
2.3.15	Dense Set: $\bar{A} = X$	11
2.3.16	Compact Set	12
2.3.17	Sublevel Set	12
2.3.18	Set Operations	12
2.3.19	Connected Sets (@ Lec 07 of ECON 204)	13
2.3.20	Theorem: Connected Set over $\mathbb{E}^1 \Leftrightarrow$ Interval (@ Lec 07 of ECON 204)	13
2.3.21	Theorem: Continuous Image of Connected Set is Connected	14
2.3.22	Intermediate Value Theorem proved by connectedness	14
2.4	Sequences	14
2.4.1	Convergence of Sequences (@ Lec 03 of ECON 204)	14
2.4.2	Limit Point (@ Lec 03 of ECON 204)	14
2.4.3	Cluster Point (@ Lec 03 of ECON 204)	15
2.4.4	Proposition: Sequences in \mathbb{R}^n , Bounded above and non-decreasing sequence \Rightarrow Converge	15
2.4.5	Theorem: Increasing/Decreasing Sequences in \mathbb{R}^n , Limit is sup/inf (@ Lec 03 of ECON 204)	16
2.4.6	Definition: \limsup and \liminf (@ Lec 03 of ECON 204)	16
2.4.7	Exists Limit $\Leftrightarrow \lim = \limsup = \liminf$ (@ Lec 03 of ECON 204)	16
2.4.8	Rising Sun Lemma: Sequence in \mathbb{R}^n contains monotone subsequence (@ Lec 03 of ECON 204)	16
2.4.9	Bolzano-Weierstrass: Bounded Sequence in \mathbb{R}^n contains a convergent subsequence (@ Lec 03 of ECON 204)	17

2.5	Complete Metric Spaces (@ Lec 05 of ECON 204)	17
2.5.1	Cauchy Sequence	17
2.5.2	Theorem: Convergent \Rightarrow Cauchy	17
2.5.3	Complete Metric Space and Banach Space	17
2.5.4	\star Theorem: Subset of Complete Metric Space "is Complete" \Leftrightarrow "is Closed"	18
2.6	Compactness over Metric Spaces (@ Lec 06 of ECON 204)	18
2.6.1	Open Cover	18
2.6.2	Definition of Compact Set	18
2.6.3	Theorem: Closed Subset of Compact Set is Compact	19
2.6.4	Theorem: Compact Set over Metric Space is Closed	19
2.6.5	Definition of Sequentially Compact	20
2.6.6	Theorem: Compact \Leftrightarrow Sequentially Compact	20
2.6.7	Definition of Totally Bounded: $\forall \varepsilon > 0 : \exists x_1, \dots, x_n \in A$ s.t. $A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$	20
2.6.8	\star Theorem: Compact \Leftrightarrow Complete/Closed and Totally Bounded	21
2.6.9	\star Heine-Borel Theorem: With Euclidean Metric, Compact \Leftrightarrow Closed and Bounded	21
2.6.10	\star Theorem: Continuous Images of Compact Set is Compact	21
Chapter 3 Functions		22
3.1	Definitions of Function	22
3.1.1	Image, Preimage, Fiber	22
3.1.2	Function Image's Properties	22
3.1.3	Composition of functions	23
3.1.4	Function Composition is Associative	23
3.1.5	Homeomorphism (@ Lec 04 of ECON 204)	23
3.2	Injection, Surjection, Bijection	23
3.2.1	Definitions: Injective, surjective, bijective	23
3.2.2	Lemma 1.1.7: injective/surjective/bijective is preserved in composition	23
3.2.3	Theorem: A function is bijective $\Leftrightarrow \exists$ inverse	24
3.3	Function Continuity	24
3.3.1	Continuous Function in \mathbb{R} with Euclidean norm	24
3.3.2	Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)	24
3.3.3	Theorem: Continuous \Leftrightarrow Preimage of open set is open (@ Lec 04 of ECON 204)	25
3.3.4	Limit characterization of continuity: continuous at $x_0 \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$	26

3.3.5	Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)	26
3.3.6	Uniform Continuity (@ Lec 04 of ECON 204)	26
3.3.7	Theorem: Continuous function with Compact domain is Uniformly Continuous (@ Lec 06 of ECON 204)	26
3.3.8	Lipschitz Continuous (@ Lec 04 of ECON 204)	27
3.3.9	Contraction (from the view of Lipschitz constant)	28
3.4	Extreme Value, Intermediate Value of Functions	28
3.4.1	Intermediate Value Theorem (@ Lec 02 of ECON 204)	28
3.4.2	Coercive Function	28
3.4.3	Extreme of Functions	29
3.4.4	Weierstrass' Theorem (Extreme value Theorem) (@ Lec 06 of ECON 204)	29
3.4.5	Corollary: Continuous and Coercive function attains its Minimum on Closed Set	30
3.5	Monotonic Function (@ Lec 05 of ECON 204)	30
3.5.1	Theorem: Monotonically Increasing \Rightarrow One-sided Limits exist	30
3.5.2	Theorem: Monotonically Increasing \Rightarrow Set of Points are discontinuous is finite (possibly empty) or countable	31
3.6	Differentiable Function (@ Lec 11 of ECON 204)	31
3.6.1	Definition of Differentiable in \mathbb{R}	31
3.6.2	General Definition of Differentiable	32
3.6.3	" O " and " o " Notations	32
3.6.4	Jacobian and Error Term	32
3.6.5	Theorem: Differentiable \Rightarrow Partial Derivatives (and Jacobian) exist	33
3.6.6	Theorem: Partial Derivatives exist and Continuous \Rightarrow Differentiable	34
3.6.7	Directional Derivatives	34
3.6.8	Chain Rule: $dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$ and $DF_{x_0} = Dg(f(x_0))Df(x_0)$	34
3.6.9	Mean Value Theorem: $\exists z \in l(x, y), \ f(y) - f(x)\ \leq \ df_z\ \ y - x\ $	34
3.6.10	Taylor's Theorem	35
3.6.11	C^k Function and Continuously Differentiable	36
3.6.12	Taylor's Theorem with C^k	36
Chapter 4	Differential Equations (@ Lec 14/15 of ECON 204)	39
4.1	Initial Value Problem	39
4.1.1	Initial Value Problem	39

4.1.2 Theorem: Initial Value Problem, Continuous → Existence; Lipschitz → Uniqueness	39
4.2 Autonomous Differential Equation	40
4.2.1 Autonomous Differential Equation	40
4.2.2 Complex Exponentials	41
4.2.3 Solutions of Simple Case: $y'(t) = ay(t)$	41
4.2.4 Solutions of Special Case: Linear Differential Equations with Constant Coefficients	41
4.2.5 The Form of Real Solutions	44
4.3 Higher Order Differential Equations	45
4.3.1 Second Order Linear Differential Equations	45
4.4 Inhomogeneous Linear Differential Equations with Nonconstant Coefficients	45
4.4.1 Theorem: General Solution of Inhomo = General Solution of Homo + Particular Solution of Inhomo	46
4.4.2 Matrix Exponentials Notations	46
4.4.3 Theorem: Particular Solution of Inhomogeneous Linear Differential Equations with Constant Coefficients (Matrix)	47
Chapter 5 Big \mathcal{O} and Small o Notation	48
5.1 Definition	48
5.1.1 Extension	49

Chapter 1 Logic

1.1 Main Methods of Proof (@ Lec 01 of ECON 204)

1.1.1 Proof by Induction

1.1.2 Proof by Deduction

1.1.3 Proof by Contradiction

1.1.4 Proof by Contraposition

- $\neg P$ ("not P ") means " P is false".
- $P \wedge Q$ (" P and Q ") means " P is true and Q is true."
- $P \vee Q$ (" P or Q ") means " P is true or Q is true (or possibly both)."
- $\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.
- $P \Rightarrow Q$ (" P implies Q ") means "whenever P is satisfied, Q is also satisfied."

Statement: Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Definition 1.1 (Contrapositive)

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.



Theorem 1.1 (Prove Contrapositive Instead)

$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.



Proposition 1.1

$[(P \wedge Q) \Rightarrow R] \Leftrightarrow [P \Rightarrow (Q \Rightarrow R)]$: Given P , if Q then R .

$\Leftrightarrow [(P \Rightarrow R) \vee (Q \Rightarrow R)]$: Either $P \Rightarrow R$ holds or $Q \Rightarrow R$ holds



Chapter 2 Analysis Basis

2.1 Real Number \mathbb{R} (@ Lec 02 of ECON 204)

\mathbb{R} is a field with the usual operations $+$, \cdot , additive identity 0, and multiplicative identity 1.

2.1.1 Order Axiom

Proposition 2.1 (Order Axiom)

There is a complete ordering \leq , i.e. \leq is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that $\forall \alpha, \beta \in \mathbb{R}$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.

The order is compatible with $+$ and \cdot , i.e. $\forall \alpha, \beta, \gamma \in \mathbb{R}$

1. $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.
2. $\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$.



2.1.2 Completeness Axiom

Proposition 2.2 (Completeness Axiom)

Suppose $L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H$ satisfy $\forall l \in L, h \in H, l \leq h$. Then,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall l \in L, h \in H, l \leq \alpha \leq h$$

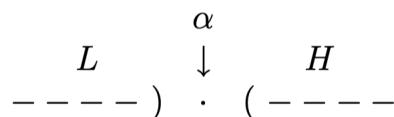


Figure 2.1: Completeness Axiom

Claim 2.1

The Completeness Axiom differentiates \mathbb{R} from \mathbb{Q} :

\mathbb{Q} satisfies all the axioms for \mathbb{R} except the Completeness Axiom.



2.1.3 Supremum $\sup \mathbb{X}$, Infimum $\inf \mathbb{X}$ for $\mathbb{X} \subseteq \mathbb{R}$

Definition 2.1 (Supremum and Infimum)

- (1). Suppose \mathbb{X} is bounded above. The **supremum** of \mathbb{X} , written $\sup \mathbb{X}$, is the least upper bound for \mathbb{X} , i.e. $\sup \mathbb{X}$ satisfies
 - (a). $\sup \mathbb{X} \geq x, \forall x \in \mathbb{X}$ ($\sup \mathbb{X}$ is an upper bound).
 - (b). $\forall y < \sup \mathbb{X}, \exists x \in \mathbb{X}$ s.t. $x > y$ (there is no smaller upper bound).
- (2). Suppose \mathbb{X} is bounded below. The **infimum** of \mathbb{X} , written $\inf \mathbb{X}$, is the greatest lower bound for \mathbb{X} , i.e. $\inf \mathbb{X}$ satisfies
 - (a). $\inf \mathbb{X} \leq x, \forall x \in \mathbb{X}$ ($\inf \mathbb{X}$ is a lower bound).
 - (b). $\forall y > \inf \mathbb{X}, \exists x \in \mathbb{X}$ s.t. $x < y$ (there is no greater lower bound).
- (3). If \mathbb{X} is not bounded above, write $\sup \mathbb{X} = \infty$. If \mathbb{X} is not bounded below, write $\inf \mathbb{X} = -\infty$. By convention, $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.



2.1.4 Theorem: sup / inf in set is max / min

Proposition 2.3 (sup / inf in set is max / min)

If $\inf A = x^* \in A$ ($\sup A = x^* \in A$), then $x^* = \min A$ ($x^* = \max A$).



2.1.5 Transformation of sup / inf

Theorem 2.1

Let A and B be nonempty sets of real numbers,

1. both of them bounded above, $\sup\{a + b : a \in A, b \in B\} = \sup A + \sup B$.
2. bounded above and below respectively, $\sup\{a - b : a \in A, b \in B\} = \sup A - \inf B$
3. $\inf A = -\sup(-A)$



2.1.6 The Supremum Property: Nonempty set in \mathbb{R} is bounded above/below $\Rightarrow \exists \sup / \inf$

Proposition 2.4 (The Supremum Property)

Every nonempty set of real numbers that is bounded above has a supremum, which is a real number.

Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.



Theorem 2.2

The Supremum Property (Prop 2.4) and the Completeness Axiom (Prop 2.2) are equivalent.



2.1.7 Archimedean Property

Theorem 2.3 (Archimedean Property)

$\forall x \in \mathbb{R}, y \in \mathbb{R}^+, \exists n \in \mathbb{N} \text{ s.t. } ny > x.$



2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)

2.2.1 Metric Space (\mathbb{X}, d) and Metric $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$

Definition 2.2 (Metric Space)

A **metric space** is a pair (\mathbb{X}, d) , where \mathbb{X} is a set and $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ a function satisfying

1. Non-negative: $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in \mathbb{X}.$
2. Symmetric: $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}.$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathbb{X}.$

A function $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ satisfying 1-3 is called a **metric** on \mathbb{X} .



A metric gives a notion of distance between elements of \mathbb{X} .

2.2.2 Theorem: Subset forms a metric space

Theorem 2.4 (Subset forms a metric space)

Given a metric space (X, d) , (Y, d) for some $Y \subseteq X$ is still a metric space.



2.2.3 Norm $\|\cdot\|$ and Normed Vector Space $(V, \|\cdot\|)$

Definition 2.3 (Norm)

Let V be a vector space over \mathbb{R} . A **norm** on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_+$ satisfying

1. Non-negative: $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0, \forall x \in V.$
2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V.$
3. $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, x \in V.$



A norm gives a notion of length of a vector in V .

Definition 2.4 (Normed Vector Space)

A **normed vector space** is a vector space over \mathbb{R} equipped with a norm, $(V, \|\cdot\|)$.



Example 2.1 Normed Vector Space

- \mathbf{E}^n : n -dimensional Euclidean space.

$$V = \mathbb{R}^n, \|x\|_2 = |x| = \sqrt{\sum_{i=1}^n (x_i)^2}$$

- $V = \mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i|$ (the "taxi cab" norm or L^1 norm)
- $V = \mathbb{R}^n, \|x\|_\infty = \max \{|x_1|, \dots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)
- $C([0, 1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
- $C([0, 1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$
- $C([0, 1]), \|f\|_1 = \int_0^1 |f(t)| dt$

where $C([0, 1])$ is the space of all continuous real-valued functions on $[0, 1]$.

2.2.4 Subset may not form a normed vector space

Given a normed vector space $(V, \|\cdot\|)$, is $(W, \|\cdot\|)$ for some $W \subseteq V$, W may not be a vector space (e.g. it may not contain the zero vector)!

2.2.5 Theorem: metric can be defined by norm

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 2.5 (Metric can be defined by norm)

Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ be defined by $d(v, w) = \|v - w\|$. Then (V, d) is a metric space.



2.2.6 Cauchy-Schwarz Inequality

Theorem 2.6 (Cauchy-Schwarz Inequality)

If $v, w \in \mathbb{R}^n$, then

$$\left(\sum_{i=1}^n v_i w_i \right)^2 \leq \left(\sum_{i=1}^n v_i^2 \right) \left(\sum_{i=1}^n w_i^2 \right)$$

$$\|v \cdot w\|^2 \leq \|v\|^2 \|w\|^2$$

$$\|v \cdot w\| \leq \|v\| \|w\|$$



2.2.7 Lipschitz-equivalent Norm

Definition 2.5 (Lipschitz-equivalent)

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the same vector space V are said to be **Lipschitz-equivalent** (or **equivalent**) if $\exists m, M > 0$ s.t. $\forall x \in V$,

$$m\|x\| \leq \|x\|^* \leq M\|x\|$$

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V, x \neq 0$,

$$m \leq \frac{\|x\|^*}{\|x\|} \leq M$$



Theorem 2.7

All norms on \mathbb{R}^n are equivalent.



However, infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0, 1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \in [0, \frac{1}{n}] \\ 0, & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then $\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} \rightarrow 0$, which means there is no lower bound $m > 0$.

2.2.8 Ball, Radius, Diameter, and Distance

In a metric space (X, d) , define

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\ &= \text{open ball with center } x \text{ and radius } \varepsilon \\ B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\ &= \text{closed ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

We can use the metric d to define a generalization of "radius". In a metric space (X, d) , define the *diameter* of a subset $S \subseteq X$ by

$$\text{diam}(S) = \sup \{d(s, s') : s, s' \in S\}$$

Similarly, we can define the *distance from a point to a set*, and *distance between sets*, as follows:

$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf \{d(a, b) : a \in A, b \in B\} \end{aligned}$$

2.3 Set Theory

2.3.1 Well Defined Set

Definition 2.6

A set S is **well defined** if an object a is either $a \in S$ or $a \notin S$.



2.3.2 Numerically Equivalent: \exists bijection (@ Lec 01 of ECON 204)

Definition 2.7

Two sets A, B are **numerically equivalent** (or have the same cardinality) if there is a bijection $f : A \rightarrow B$, that is, 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$).



2.3.3 Finite (@ Lec 01 of ECON 204)

Definition 2.8 (Finite Set)

A set is either **finite** or **infinite**. A set is **finite** if it is numerically equivalent to $\{1, \dots, n\}$ for some n . A set that is not finite is infinite.



2.3.4 Countable Set: numerically equivalent to \mathbb{N} (@ Lec 01 of ECON 204)

We give a more precise definition to classify infinite set:

Definition 2.9 (Countable Set)

An infinite set is **countable** if it is numerically equivalent to \mathbb{N} . An infinite set that is not countable is called **uncountable**.



Theorem 2.8 (Countable \mathbb{Q})

The set of rational numbers \mathbb{Q} is countable.



2.3.5 Power Set (@ Lec 02 of ECON 204)

Definition 2.10 (Power Set: the set of all subsets)

*For any set A , we denote by $\mathcal{P}(A)$ the collection of all subsets of A . $\mathcal{P}(A)$ is the **power set** of A .*



We may also use the notation 2^A (in Berkeley ECON 204).

2.3.6 Theorem (Cantor): The power set of \mathbb{N} is uncountable (@ Lec 02 of ECON 204)

Theorem 2.9 (Cantor)

$\mathcal{P}(\mathbb{N})$ (or denoted by $2^{\mathbb{N}}$), the set of all subsets of \mathbb{N} , is uncountable.



2.3.7 Cardinalities of Sets (@ Lec 02 of ECON 204)

Definition 2.11 (Cardinality)

If A is a set, $|A| = \text{cardinality of } A = \# \text{ of elements}$



$n \in \mathbb{N}, |\{1, \dots, n\}| = n; |\emptyset| = 0 (\emptyset = \text{empty set}).$

Proposition 2.5 (Facts about cardinality)

1. If A is numerically equivalent to $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, then $|A| = n$.
2. A and B are numerically equivalent if and only if $|A| = |B|$.
3. If $|A| = n$ (finite) and A is a proper subset of B (that is, $A \subset B$ and $A \neq B$) then $|A| < |B|$.
4. If A is countable and B is uncountable, then $n < |A| < |B|, \forall n \in \mathbb{N}$.
5. If $A \subseteq B$, then $|A| \leq |B|$. (if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable.)
6. If there is an injection $\sigma : A \rightarrow B$, we can write $|A| \leq |B|$;
7. If there is a surjection $\sigma : A \rightarrow B$, we can write $|A| \geq |B|$;
8. If there is a bijection $\sigma : A \rightarrow B$, we can write $|A| = |B|$.



2.3.8 Pigeonhole Principle: $|A| > |B| \Rightarrow \text{no injective function } \sigma : A \rightarrow B$

Theorem 2.10 (Pigeonhole Principle)

If A and B are sets and $|A| > |B|$, then there is no injective function $\sigma : A \rightarrow B$.



2.3.9 B^A : Sets of Function

If A, B are sets, then $B^A = \{\sigma : A \rightarrow B | \sigma \text{ a function}\}$.

Example 2.2 $n \in \mathbb{Z}$, we define a function $f : B^{\{1, \dots, n\}} \rightarrow B^n (= B \times B \times B \times \dots \times B)$ by the equation $f(\sigma) = \{\sigma(1), \dots, \sigma(n)\}$, where $\sigma : \{1, \dots, n\} \rightarrow B$. The f is a bijection.

Proof 2.1

1. *Injective*:

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), \dots, \sigma_1(n)\} = \{\sigma_2(1), \dots, \sigma_2(n)\}$$

Since $\sigma : \{1, \dots, n\} \rightarrow B$, it is sufficient to prove $\sigma_1 = \sigma_2$.

2. *Surjective*:

$$\forall \{b_1, \dots, b_n\}, \text{ we have } \sigma^*(x) = b_x, x = 1, \dots, n. \text{ s.t. } f(\sigma^*) = \{b_1, \dots, b_n\}$$

Example 2.3

$$C(\mathbb{R}, \mathbb{R}) = \{\text{continuous functions } \sigma : \mathbb{R} \rightarrow \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

2.3.10 Bounded Set (@ Lec 03 of ECON 204)**Definition 2.12 (Bounded Set)**

In a metric space (X, d) , a subset $S \subseteq X$ is **bounded** if $\exists x \in X, \beta \in \mathbb{R}$ such that $\forall s \in S, d(s, x) \leq \beta$. 

Equivalent: $\exists \beta > 0$ and $s \in S$ such that $S \subseteq B_{\beta}(s)$.

2.3.11 Open, Closed Set (@ Lec 04 of ECON 204)**Definition 2.13 (Open Sets)**

Let (X, d) be a metric space. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is **open** if

$\forall x \in \mathbb{X}$ we can draw a ball around x that is contained in \mathbb{X} .

i.e. $\forall x \in \mathbb{X}, \exists \varepsilon > 0$ s.t. $B_{\varepsilon}(x) = \{y : d(y, x) < \varepsilon\} \subseteq \mathbb{X}$ 

Example 2.4 In a metric space (X, d) , if a is an isolated point of X , then the singleton set $\{a\}$ is open: Since a is not a limit point of X , we can find a neighborhood $B_{\varepsilon}(a)$ such that $B_{\varepsilon}(a) \cap X = \{a\}$. So $B_{\varepsilon}(a) \subseteq \{a\}$ and x is an interior point. Hence, every point is an interior point and the set is open.

Definition 2.14 (Closed Sets)

\mathbb{X} is **closed** if \mathbb{X}^c is open. 

Theorem 2.11 (Equivalent definition: Closed Sets)

Equivalent: if A in a metric space (X, d) contains all limit points of all sequences in A , A is closed.

$$\{x_n\} \subset A, x_n \rightarrow x \in X \Rightarrow x \in A$$

**Example 2.5 (Closed and Open Sets on \mathbb{E}_1 i.e., \mathbb{R} with the usual Euclidean metric)**

- 1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open
- 2) \mathbb{R} is both open and closed
- 3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open
- 4) $[1, \infty)$ is closed because its complement open
- 5) $(1, 2]$ is neither open nor closed

Example 2.6 (Closed and Open Sets on other metric space) In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon]$.

Theorem 2.12 (Empty Set and Full Set are both open and closed)

In any metric space (X, d) both \emptyset and X are open, and both \emptyset and X are closed.



Theorem 2.13 (Every neighborhood is an open set)

Every neighborhood $B_\varepsilon(x)$ is an open set.



Theorem 2.14 (Finite sets are closed in any metric space)

Finite sets are closed in any metric space (X, d) .



Proposition 2.6

Any set in a metric space (X, d) can be written as the intersection of open sets.



2.3.12 ★ Theorem: Union of open sets is open, Intersection of finite open sets is open

Theorem 2.15 (Union of open sets is open, Intersection of finite open sets is open)

In any metric space (X, d) ,

1. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.
2. The intersection of a finite collection of open sets is open.



2.3.13 ★ Corollary: Intersection of closed sets is closed, Union of finite closed sets is closed

Corollary 2.1 (Intersection of closed sets is closed, Union of finite closed sets is closed)

In any metric space (X, d) ,

1. The intersection of an arbitrary collection of closed sets is closed.
2. The union of a finite collection of closed sets is closed.



2.3.14 Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)

Given a set $S \subseteq X$, the **point** of X can be classified into three types relative to S :

- **Interior (points)**, denoted $\text{int}(S)$: $\vec{x} \in S$ for which there exists some $B(\vec{x}, r) \subseteq S$, is the largest open set contained in S (the union of all open sets contained in S).

$$x \in \text{int } A \Leftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$$

- **Exterior (points)**, denoted $\text{ext}(S)$: $\vec{x} \notin S$ for which there exists some $B(\vec{x}, r)$ containing no points of S , is the largest open set contained in $X \setminus S$.

$$x \in \text{ext } A \Leftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A^C$$

- **Boundary (points)** denoted $\partial(S)$ or $bd(S)$: all other points (for which any $B(\vec{x}, r)$ will contain some points of S and some points outside S).

$$x \in \partial A \Leftrightarrow \forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^C \neq \emptyset$$

- **Closure of S** , denoted \bar{S} or $\text{cl}(S) = \text{int}(S) \cup \partial(S)$, is the smallest closed set containing S (the intersection of all closed sets containing S).

$$x \in \text{cl } A \Leftrightarrow \forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset$$

- Moreover, boundary satisfies $\partial(S) = \overline{(X \setminus S)} \cap \bar{S}$.

Proposition 2.7

- (1) $\text{int } A$, $\text{ext } A$ and ∂A partition X .
- (2) $\text{int } A$ and ∂A partition $\text{cl } A$.
- (3) A set S is **open** iff $S = \text{int}(S)$ - i.e., S does not contain any of its boundary points.
- (4) A set S is **closed** iff $S = \bar{S} = \text{int}(S) \cup \partial(S)$ - i.e., S contains all of its boundary points.
- (5) A set S is **open** and **closed** iff $\partial(S) = \emptyset$.
- (6) $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$.



2.3.15 Dense Set: $\bar{A} = X$

Definition 2.15 (Dense Set)

Let (X, d) be a metric space. A set $A \subseteq X$ is **dense** in X if $\bar{A} = X$.



Example 2.7 \mathbb{Q} is dense in \mathbb{R} : for all $x \in \mathbb{Q}$, let $y_n = \frac{\lfloor 10^n x \rfloor}{10^n}$ with $n \in \mathbb{N}$. Then, $\{y_n\} \subseteq \mathbb{Q}$, and $y_n \rightarrow x$ since $|y_n - x| < 10^{-n}$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n > N$, $|y_n - x| < \varepsilon$ (or $y_n \in B_\varepsilon(x)$). This implies $B_\varepsilon(x) \cap \mathbb{Q} \neq \emptyset$, so $\bar{\mathbb{Q}} = \mathbb{R}$.

Corollary 2.2 (\mathbb{Q}^n is dense in \mathbb{R}^n)

\mathbb{Q}^n is dense in \mathbb{R}^n for $n \in \mathbb{N}$.



Theorem 2.16

If A is dense in X , for all $x \in X$: $\forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset$.

**Proof 2.2**

Since A is dense in X , then for any $x \in X = \bar{A}$, $\forall \varepsilon > 0, B_\varepsilon(x) \cap A \neq \emptyset$ by the definition of closure.

Example 2.8 If \mathcal{A} is a collection of open subsets of \mathbb{R}^n , pairwise disjoint, that is, $A_\lambda \cap A_{\lambda'} = \emptyset$ if $\lambda \neq \lambda'$, then \mathcal{A} is at most countable.

Let Λ be the set of indices of \mathcal{A} , i.e., $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$. Choose an arbitrary $\lambda \in \Lambda$, and choose an arbitrary point $x \in A_\lambda$. Since A_λ is open, $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq A_\lambda$. Because \mathbb{Q}^n is dense in \mathbb{R}^n , $\exists q_\lambda \in \mathbb{Q}^n$ s.t. $q_\lambda \in B_\varepsilon(x) \subseteq A_\lambda$. Therefore, $\forall \lambda \in \Lambda, \exists q_\lambda \in \mathbb{Q}^n$ s.t. $q_\lambda \in B_\varepsilon(x) \subseteq A_\lambda$. Therefore, $\forall \lambda \in \Lambda, \exists q_\lambda \in \mathbb{Q}^n$ s.t. $q_\lambda \in A_\lambda$. Note that \mathcal{A} is a collection of pairwise disjoint open sets, then $\forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda', q_\lambda \neq q'_{\lambda'}$. Therefore, we can consider a function $f : \mathcal{A} \rightarrow \mathbb{Q}^n$, with $f(A_\lambda) = q_\lambda, \forall \lambda \in \Lambda$. Since $\forall \lambda, \lambda' \in \Lambda, \lambda \neq \lambda', q_\lambda \neq q'_{\lambda'}$, then f is an injection. This indicates the cardinality of \mathbb{Q}^n is weakly larger than \mathcal{A} , then \mathcal{A} is at most countable.

2.3.16 Compact Set**Definition 2.16 (Compact Set)**

$\mathbb{X} \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.



Example 2.9 Compact Set $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}; \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

2.3.17 Sublevel Set**Definition 2.17 (Sublevel Set)**

The sublevel set of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (for some level $c \in \mathbb{R}$) is the set

$$\overline{L_c} = \{x \in \mathbb{R}^n : f(x) \leq c\}$$

**2.3.18 Set Operations****Definition 2.18**

A binary operation on a set A is a function $* : A \times A \rightarrow A$.

The operation is associative if $a * (b * c) = (a * b) * c, \forall a, b, c \in A$.

The operation is commutative if $a * b = b * a, \forall a, b \in A$.

**Example 2.10**

$+$, \circ are both *associative* and *commutative* operations on $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$; $-$ is neither *associative* nor *commutative* operation on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not \mathbb{N} .

Definition 2.19

A subset $H \subset S$ is closed under $*$ if $a * b \in H$ for all $a, b \in H$.

**Definition 2.20**

* has identity element $e \in S$ if $a * e = e * a = a$ for all $s \in S$.

**2.3.19 Connected Sets (@ Lec 07 of ECON 204)****Definition 2.21 (Separated/Connected Set)**

Two sets A, B in a metric space are **separated** if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

where \bar{A} is the closure of A .

A set in a metric space is **connected** if it cannot be written as the union of two nonempty separated sets.



Equivalent Definition:

Definition 2.22 (Connected Set)

A set Y in a metric space X is **connected** if there do not exist *open sets* A and B such that $A \cap B = \emptyset$,

$Y \subseteq A \cup B$ and $A \cap Y \neq \emptyset$ and $B \cap Y \neq \emptyset$.



Note that separation does not require that $\bar{A} \cap \bar{B} = \emptyset$. For example, $[0, 1) \cup (1, 2]$ is not connected, since $[0, 1)$ and $(1, 2]$ are separated.

2.3.20 Theorem: Connected Set over $\mathbb{E}^1 \Leftrightarrow$ Interval (@ Lec 07 of ECON 204)**Theorem 2.17 (Connected Set over $\mathbb{E}^1 \Leftrightarrow$ Interval)**

A set $S \subseteq \mathbb{E}^1$ of real numbers is **connected** if and only if it is an **interval**, i.e. if $x, y \in S$ and $z \in (x, y)$,

then $z \in S$.



Notice that this result is only valid in \mathbb{R} . For example, connected sets in \mathbb{R}^n need not be intervals, or even convex.

2.3.21 Theorem: Continuous Image of Connected Set is Connected

Theorem 2.18 (Continuous Image of Connected Set is Connected)

Let X be a metric space and $f : X \rightarrow Y$ be continuous. If C is a connected subset of X , then $f(C)$ is connected.



2.3.22 Intermediate Value Theorem proved by connectedness

Corollary 2.3 (Intermediate Value Theorem)

If $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and $f(a) < d < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = d$.



Proof 2.3

Since $[a, b]$ is an interval, it is connected. So $f([a, b])$ is connected, hence $f([a, b])$ is an interval.

$f(a) \in f([a, b])$, and $f(b) \in f([a, b])$, and $d \in [f(a), f(b)]$; since $f([a, b])$ is an interval, $d \in f([a, b])$, i.e. there exists $c \in [a, b]$ such that $f(c) = d$. Since $f(a) < d < f(b)$, $c \neq a, c \neq b$, so $c \in (a, b)$.

2.4 Sequences

Sequences $\{x_k\}_{k=1}^{\infty}$, or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 2.23 (Subsequence)

Suppose $\{x_n\}$ is a sequence and $n_1 < n_2 < \dots$, then $\{x_{n_k}\}$ is called a **subsequence**.



2.4.1 Convergence of Sequences (@ Lec 03 of ECON 204)

Definition 2.24 (Convergence: note $x_k \rightarrow x, \lim_{k \rightarrow \infty} x_k = x$)

Let (X, d) be a metric space. A sequence $\{x_k\}$ converges to x (written $x_k \rightarrow x$ or $\lim_{k \rightarrow \infty} x_k = x$) if

$\forall \varepsilon > 0, \exists N_{\varepsilon}$ s.t. $d(x_k, x) < \varepsilon, \forall k \geq N_{\varepsilon}$



2.4.2 Limit Point (@ Lec 03 of ECON 204)

Definition 2.25 (Limit Point)

x is a **limit point** of $\{x_k\}$ if \exists a subsequence of $\{x_k\}$ that converges to x .



Definition 2.26 (Limit Point, alternative definition)

Let (X, d) be a metric space. A point $x \in X$ is a **limit point** of a set $A \subseteq X$ if

$$\forall \varepsilon > 0, B_{\varepsilon}(x) \cap A \not\subseteq \{x\}$$



Theorem 2.19 (Set of limit points of any set $A \subseteq \mathbb{R}^n$ is closed.)

The set of limit points of any set $A \subseteq \mathbb{R}^n$ is closed.

**Proof 2.4**

Suppose \mathcal{S} is the set of limit points of A . Let $D = \mathbb{R}^n \setminus \mathcal{S}$. Take an arbitrary $x \in D$. Then x is not the limit point of A , so $\exists \varepsilon_x > 0$ s.t. $B_{\varepsilon_x}(x) \cap A \subseteq \{x\}$. For any $y \in B_{\varepsilon_x}(x)$, since $B_{\varepsilon_x}(x)$ is open, $\exists \varepsilon_y > 0$ s.t. $B_{\varepsilon_y}(y) \subseteq B_{\varepsilon_x}(x)$. Let $\varepsilon_m = \min\{d(x, y), \varepsilon_y\}$ such that $B_{\varepsilon_m}(y) \subseteq B_{\varepsilon_x}(x)$ and $y \notin B_{\varepsilon_m}(y)$. So, $B_{\varepsilon_m}(y) \cap A = \emptyset$. So, y is not a limit point of A , that is $y \in \mathbb{R}^n \setminus \mathcal{S} = D$. Therefore, $\forall y \in B_{\varepsilon_x}(x) \Rightarrow y \in D$. We can conclude $B_{\varepsilon_x}(x) \subseteq D, \forall x \in D$. So, D is open.

Theorem 2.20 (Subsequence has the same limit as the original sequence)

All subsequences of a convergent sequence converge to the same limit as the original sequence.

**Theorem 2.21 (Uniqueness of Limits)**

In a metric space (X, d) , if $x_k \rightarrow x$ and $x_k \rightarrow x'$, then $x = x'$.

**2.4.3 Cluster Point (@ Lec 03 of ECON 204)****Definition 2.27 (Cluster Point)**

An element c is a **cluster point** of a sequence $\{x_n\}$ in a metric space (X, d) if $\forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N}, \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$



Example 2.11 $x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ even} \\ \frac{1}{n}, & \text{if } n \text{ odd} \end{cases}$ has cluster points $\{0, 1\}$.

Theorem 2.22 (Cluster Point \Leftrightarrow exists subsequence converges to it)

Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X . Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k \rightarrow \infty} x_{n_k} = c$.

**2.4.4 Proposition: Sequences in \mathbb{R}^n , Bounded above and non-decreasing sequence \Rightarrow Converge****Proposition 2.8**

If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it converges.



2.4.5 Theorem: Increasing/Decreasing Sequences in \mathbb{R}^n , Limit is sup/inf (@ Lec 03 of ECON 204)

Theorem 2.23

Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ ($\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$). In particular, the limit exists.



2.4.6 Definition: \limsup and \liminf (@ Lec 03 of ECON 204)

Consider a sequence $\{x_n\}$ of real numbers. Let

$$\alpha_n = \sup\{x_k : k \geq n\}$$

and

$$\beta_n = \inf\{x_k : k \geq n\}$$

Either $\alpha_n = +\infty$ for all n , or $\alpha_n \in \mathbb{R}$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots$. Either $\beta_n = -\infty$ for all n , or $\beta_n \in \mathbb{R}$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \dots$

Definition 2.28 (Lim Sups and Lim Insets)

$$\limsup_{n \rightarrow \infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

$$\liminf_{n \rightarrow \infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$



2.4.7 Exists Limit $\Leftrightarrow \lim = \limsup = \liminf$ (@ Lec 03 of ECON 204)

Theorem 2.24

Let $\{x_n\}$ be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\}$$

$$\Leftrightarrow \lim \sup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \gamma$$



2.4.8 Rising Sun Lemma: Sequence in \mathbb{R}^n contains monotone subsequence (@ Lec 03 of ECON 204)

Theorem 2.25 (Rising Sun Lemma)

Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.



2.4.9 Bolzano-Weierstrass: Bounded Sequence in \mathbb{R}^n contains a convergent subsequence (@ Lec 03 of ECON 204)

Theorem 2.26 (Bolzano-Weierstrass)

Every bounded sequence of real numbers contains a convergent subsequence.



2.5 Complete Metric Spaces (@ Lec 05 of ECON 204)

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

2.5.1 Cauchy Sequence

Definition 2.29 (Cauchy Sequence)

A sequence $\{x_k\}$ in a metric space (X, d) is **Cauchy** if $\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t.

$$d(x_n, x_m) < \varepsilon, \forall n, m > N(\varepsilon)$$



Note: Cauchy property depends only on the sequence and the metric d , not on the ambient metric space.

Note: In $\mathbb{E}^1 = (\mathbb{R}, \|\cdot\|)$, $\{x_k\}$ converges $\iff \{x_k\}$ is Cauchy.

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to. Any sequence that does converge must be Cauchy, however, by the argument above.

2.5.2 Theorem: Convergent \Rightarrow Cauchy

Theorem 2.27 (Convergent \Rightarrow Cauchy)

Every convergent sequence in a metric space is Cauchy.



2.5.3 Complete Metric Space and Banach Space

Definition 2.30 (Complete Metric Spaces)

A metric space (X, d) is **complete** if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$.



Example 2.12

1. Consider the earlier example of $X = (0, 1]$ with d the usual Euclidean metric. Since $x_n = \frac{1}{n}$ is Cauchy but does not converge in that metric space, $((0, 1], d)$ is not complete.
2. \mathbb{Q} is not complete in the Euclidean metric: $x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} \rightarrow \sqrt{2}$.

Definition 2.31 (Banach Space)

A **Banach space** is a *normed space* that is complete in the metric generated by its norm.

**Theorem 2.28 (\mathbb{E}^1 is complete)**

\mathbb{R} is complete with the usual metric (so \mathbb{E}^1 is a Banach space).

**Theorem 2.29 (\mathbb{E}^n is complete)**

\mathbb{E}^n is complete for every $n \in \mathbb{N}$.

**Theorem 2.30 ($(C(X), \|\cdot\|_\infty)$ is Complete)**

Given $X \subseteq \mathbb{R}^n$, let $C(X)$ be the set of bounded continuous functions from X to \mathbb{R} with

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Then $C(X)$ is a Banach space.

**2.5.4 ★ Theorem: Subset of Complete Metric Space "is Complete" \Leftrightarrow "is Closed"****Theorem 2.31 (Subset of Complete Metric Space "is Complete" \Leftrightarrow "is Closed")**

Suppose (X, d) is a complete metric space and $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if

Y is a closed subset of X . (where $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$; $d|_Y : \mathbb{Y} \times \mathbb{Y} \rightarrow \mathbb{R}_+$)

**2.6 Compactness over Metric Spaces (@ Lec 06 of ECON 204)****2.6.1 Open Cover****Definition 2.32 (Open Cover)**

A collection of sets $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of $A \subseteq X$ if

1. U_λ is open for all $\lambda \in \Lambda$ and
2. $A \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$

**2.6.2 Definition of Compact Set****Definition 2.33 (Compact Set)**

A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A . In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A , there exist $n \in N$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that

$$A \subseteq \bigcup_{i=1}^n U_{\lambda_i}$$

**Example 2.13**

1. $(0, 1]$ is not a compact set over \mathbb{E}^1 : Consider the open cover $\{(\frac{1}{m}, 2) : m \in \mathbb{N}\}$.
2. $[0, \infty)$ is not a compact set over \mathbb{E}^1 : Consider the open cover $\{(-1, m) : m \in \mathbb{N}\}$.

2.6.3 Theorem: Closed Subset of Compact Set is Compact

Theorem 2.32 (Closed Subset of Compact Set is Compact)

Every closed subset A of a compact metric space (X, d) is compact.



Proof 2.5

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A .

Let $U'_\lambda = U_\lambda \cup (X \setminus A)$. Since A is closed, $X \setminus A$ is open; Since U_λ and $X \setminus A$ are both open, U'_λ is open.

By the definition of open set, if $a \in A$, $a \in \cup_{\lambda \in \Lambda} U_\lambda$. If $a \in X \setminus A$, $a \in U'_\lambda$ for sure. So, $X \subseteq \cup_{\lambda \in \Lambda} U'_\lambda$.

Hence, we can conclude $\{U_\lambda \cup (X \setminus A) : \lambda \in \Lambda\}$ is an open cover of X .

Since X is compact, there exists $n \in N$ and $\lambda_1, \dots, \lambda_n \in \Lambda$ such that $X \subseteq \cup_{i=1}^n U'_{\lambda_i}$.

Then,

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \text{ for some } i \end{aligned}$$

Hence, we can conclude $A \subseteq \cup_{i=1}^n U_{\lambda_i}$. Thus, A is compact.

2.6.4 Theorem: Compact Set over Metric Space is Closed

Theorem 2.33 (Compact Set over Metric Space is Closed)

If A is a compact subset of the metric space (X, d) , then A is closed.



Proof 2.6

Suppose $X \setminus A$ is not open, so we can find a point $x \in X \setminus A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbb{N}$, let

$$U_n = X \setminus B_{1/n}[x]$$

Each U_n is open, and

$$\cup_{n \in \mathbb{N}} U_n = X \setminus \{x\} \supseteq A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbb{N}\}$ is an open cover for A . Since A is compact, there is a finite

subcover $\{U_{n_1}, \dots, U_{n_k}\}$. Let $n = \max\{n_1, \dots, n_k\}$. Then

$$\begin{aligned} U_n &= X \setminus B_{1/n}[x] \\ &\supseteq X \setminus B_{1/n_j}[x] \quad (j = 1, \dots, k) \\ U_n &\supseteq \bigcup_{j=1}^k U_{n_j} \\ &\supseteq A \end{aligned}$$

But $A \cap B_{1/n}[x] \neq \emptyset$, so $A \not\subseteq X \setminus B_{1/n}[x] = U_n$. This is a contradiction, which proves that A is closed.

2.6.5 Definition of Sequentially Compact

Definition 2.34 (Sequentially Compact)

A set A in a metric space (X, d) is **sequentially compact** if every sequence of elements of A contains a convergent subsequence whose limit lies in A .



2.6.6 Theorem: Compact \Leftrightarrow Sequentially Compact

Theorem 2.34 (Compact \Leftrightarrow Sequentially Compact)

A set A in a metric space (X, d) is **compact** if and only if it is **sequentially compact**.



2.6.7 Definition of Totally Bounded: $\forall \varepsilon > 0 : \exists x_1, \dots, x_n \in A$ s.t. $A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i)$.

Definition 2.35 (Totally Bounded)

A set A in a metric space (X, d) is **totally bounded** if, for every $\varepsilon > 0$,

$$\exists x_1, \dots, x_n \in A \text{ s.t. } A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i).$$



Claim 2.2

"Totally bounded" is stronger than "bounded".



Example 2.14

1. $X = [0, 1]$ is totally bounded with Euclidean metric: Given $\varepsilon > 0$, let $x_i = \frac{i}{n}$, $i = 1, \dots, n - 1$.
2. $X = [0, 1]$ is not totally bounded with discrete metric $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$: Given $\varepsilon = \frac{1}{2}$, $B_\varepsilon(x_i) = \{x_i\}$. Hence, $\bigcup_i^n B_\varepsilon(x_i) = \{x_1, \dots, x_n\} \not\supseteq [0, 1]$.

Note: X is bounded but not totally bounded.

2.6.8 ★ Theroem: Compact \Leftrightarrow Complete/Closed and Totally Bounded

Theorem 2.35 (Subset is Compact \Leftrightarrow Complete and Totally Bounded)

Let A be a subset of a metric space (X, d) . Then A is **compact** if and only if A is **complete** and **totally bounded**.



Proof 2.7

Compact \Rightarrow totally bounded (by definition). Compact \Leftrightarrow sequentially compact \Rightarrow every Cauchy sequence converges to a limit in the subset (i.e., Complete).

Conversely, suppose complete and totally bounded. Because totally bounded, we can extract a Cauchy subsequence. Because complete, subsequence converges to a limit in the subset, which shows sequentially compact and hence compact.

Corollary 2.4 (Subset is Compact \Leftrightarrow Closed and Totally Bounded)

Let A be a subset of a metric space (X, d) . Then A is **compact** if and only if A is **closed** and **totally bounded**.



Proof 2.8

Directly by theorem 2.31 and theorem 2.35.

2.6.9 ★ Heine-Borel Theorem: With Euclidean Metric, Compact \Leftrightarrow Closed and Bounded

Claim 2.3 (With Euclidean Metric, Compact \Leftrightarrow Closed and Bounded)

If $A \subseteq \mathbb{E}^n$, then A is **totally bounded** if and only if A is **bounded**.



Theorem 2.36 (Heine-Borel Theorem: With Euclidean Metric, Compact \Leftrightarrow Closed and Bounded)

If $A \subseteq \mathbb{E}^n$, then A is **compact** if and only if A is **closed** and **bounded**.



2.6.10 ★ Theorem: Continuous Images of Compact Set is Compact

Theorem 2.37 (Continuous Images of Compact Set is Compact)

Let (X, d) and (Y, ρ) be metric spaces. If $f : X \rightarrow Y$ is continuous and C is a compact subset of (X, d) , then $f(C)$ is compact in (Y, ρ) .



Chapter 3 Functions

3.1 Definitions of Function

Definition 3.1 (Function)

Function is a rule $\sigma : A \rightarrow B$ that assigns an element B to *every* element of A . $\forall a \in A, \sigma(a) \in B$.



3.1.1 Image, Preimage, Fiber

Definition 3.2

1. A is the domain of σ , B is the range of σ .
2. We call $\sigma(a) = \text{value of } \sigma \text{ at } a$ as the image of a .
3. A set $C \subset B$, we call $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$ as the preimage of C .
4. An element $b \in B$, we call $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$ as the fiber of b .



3.1.2 Function Image's Properties

Theorem 3.1

Let X, Y be sets, $f : X \rightarrow Y$ be a function, $A, B \subseteq X$ be arbitrary subsets of X , and $C, D \subseteq Y$ be arbitrary subsets of Y . Then

1. $f(A \cap B) \subseteq f(A) \cap f(B)$. (*Strict subset example:* $y = x^2$. $A = (-2, 0)$ and $B = (0, 2)$.)
2. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
3. $f(A \cup B) = f(A) \cup f(B)$
4. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
5. $A \subseteq f^{-1}(f(A))$ (*Strict subset example:* $y = x^2$, $A = [0, 2]$. Then $f(A) = [0, 4]$, $f^{-1}(f(A)) = [-2, 2]$.)
6. $f(f^{-1}(C)) \subseteq C$ (*Strict subset example:* $X = [0, 1], Y = [0, 2], f(x) = x$. Let $C = [0.5, 1.5]$. Then $f^{-1}(C) = [0.5, 1]$, $f(f^{-1}(C)) = [0.5, 1]$.)



3.1.3 Composition of functions

Definition 3.3 (Function Composition)

The function composition \circ is an operation that takes two functions $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$, , and produces a function $\tau \circ \sigma : A \rightarrow C$ that fulfills $\forall a \in A, (\tau \circ \sigma)(a) = \tau(\sigma(a))$.



3.1.4 Function Composition is Associative

Proposition 3.1 (Associativity of Functions)

Suppose $\sigma : A \rightarrow B, \tau : B \rightarrow C, \rho : C \rightarrow D$ are functions and \circ is the function composition, then

$$\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma.$$



3.1.5 Homeomorphism (@ Lec 04 of ECON 204)

Definition 3.4 (Homeomorphism)

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is called a **homeomorphism** if it is one-to-one, onto, continuous, and its inverse function is continuous.



3.2 Injection, Surjection, Bijection

3.2.1 Definitions: Injective, surjective, bijective

A function $\sigma : A \rightarrow B$ is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

3.2.2 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

Lemma 3.1 (Lemma 1.1.7)

Suppose $\sigma : A \rightarrow B, \tau : B \rightarrow C$ are functions,

If σ, τ are injective, then $\tau \circ \sigma$ is injective.

If σ, τ are surjective, then $\tau \circ \sigma$ is surjective.

If σ, τ are bijective, then $\tau \circ \sigma$ is bijective.



3.2.3 Theorem: A function is bijective $\Leftrightarrow \exists$ inverse

Theorem 3.2 (Bijective $\Leftrightarrow \exists$ Inverse)

A function $\sigma : A \rightarrow B$ is a bijection if and only if \exists a function $\tau : B \rightarrow A$ such that

$$\sigma \circ \tau = id_B = \text{identity on } B(id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$

Such τ is unique, called inverse of σ , $\tau = \sigma^{-1}$.



3.3 Function Continuity

3.3.1 Continuous Function in \mathbb{R} with Euclidean norm

Definition 3.5 (Continuity at Point)

A real-valued function f is continuous at x if

"For every $\{x_k\}$ converging to x satisfies that $\lim_{k \rightarrow \infty} f(x_k) = f(x)$ ".

Equivalent definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \|y - x\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$



Continuity at x_0 requires:

1. $f(x_0)$ is defined; and
2. either
 - x_0 is an isolated point of X , i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{x\}$; or
 - $\lim_{x \rightarrow x_0} f(x)$ exists and equals $f(x_0)$

Definition 3.6 (Continuous Function)

A real-valued function f is continuous if it is continuous at all points in its domain.



3.3.2 Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)

Definition 3.7 (Continuity in Metric Spaces)

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is **continuous** at a point $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon.$$



3.3.3 Theorem: Continuous \Leftrightarrow Preimage of open set is open (@ Lec 04 of ECON 204)

Theorem 3.3 (Continuous \Leftrightarrow Preimage of open set is open)

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \rightarrow Y$. Then f is continuous if and only if

$$f^{-1}(A) = \{x \in X : f(x) \in A\} \text{ is open in } X \quad \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$



Proposition 3.2

Let $A \subseteq \mathbb{R}$ be an open set and $f : A \rightarrow \mathbb{R}$. The two following statements are equivalent:

- (a). f is continuous.
- (b). for all $c \in \mathbb{R}$, the sets $E[f < c] = \{x \in A \mid f(x) < c\}$ and $E[f > c] = \{x \in A \mid f(x) > c\}$ are open.



Proof 3.1

1. " $(a) \Rightarrow (b)$ ": Suppose f is continuous. Consider any $c \in \mathbb{R}$. Since $(-\infty, c)$ and (c, ∞) are open and f is continuous, $E[f < c] = f^{-1}((-\infty, c))$ and $E[f > c] = f^{-1}((c, \infty))$ are open.

2. " $(b) \Rightarrow (a)$ ": Suppose $E[f < c] = \{x \in A \mid f(x) < c\}$ and $E[f > c] = \{x \in A \mid f(x) > c\}$ are open for all $c \in \mathbb{R}$. Let $C \subseteq \mathbb{R}$ be an arbitrary open set over \mathbb{R} . There are following cases about the C :

- (1). $C = (-\infty, k)$, where $k \in \mathbb{R}$. In this case $f^{-1}(C) = E[f < k]$, which is open by the given condition.
- (2). $C = (k, \infty)$, where $k \in \mathbb{R}$. In this case $f^{-1}(C) = E[f > k]$, which is open by the given condition.
- (3). $C = \emptyset$. In this case $f^{-1}(C) = \{x \in A \mid f(x) \in \emptyset\} = E[f > k_1] \cap E[f < k_2]$, $\forall k_1 \geq k_2$, which is a finite intersection of two open sets. So, $f^{-1}(C)$ is open.
- (4). $C = (k_1, k_2)$, where $k_1, k_2 \in \mathbb{R}$ s.t. $k_1 < k_2$. In this case $f^{-1}(C) = \{x \in A \mid f(x) \in (k_1, k_2)\} = E[f > k_1] \cap E[f < k_2]$, which is a finite intersection of two open sets. So, $f^{-1}(C)$ is open.
- (5). $C = (-\infty, k_2) \cup (k_1, \infty)$. In this case $f^{-1}(C) = \{x \in A \mid f(x) \in (-\infty, k_2) \cup (k_1, \infty)\} = E[f > k_1] \cup E[f < k_2]$, $\forall k_1, k_2 \in \mathbb{R}$ s.t. $k_1 > k_2$. $f^{-1}(C)$ is a union of two open set, which is open.
- (6). $C = \mathbb{R}$. In this case $f^{-1}(C) = \{x \in A \mid f(x) \in \mathbb{R}\} = \{x \in A \mid f(x) \in (-\infty, k_2) \cup (k_1, \infty)\} = E[f > k_1] \cup E[f < k_2]$, $\forall k_1, k_2 \in \mathbb{R}$ s.t. $k_1 < k_2$. $f^{-1}(C)$ is a union of two open set, which is open.

Hence, $f^{-1}(C) = \{x \in A : f(x) \in C\}$ is open $\forall C \subseteq \mathbb{R}$ s.t. C is open in \mathbb{R}

3.3.4 Limit characterization of continuity: continuous at $x_0 \Leftrightarrow \lim_{x \rightarrow x_0} f(x) = f(x_0)$

Theorem 3.4 (Limit Characterization of Continuity)

If x_0 is a limit point of X , then f is **continuous at x_0** if and only if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$



3.3.5 Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)

Theorem 3.5 (Continuity is preserved in composition)

Let (X, d_X) , (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $g \circ f : X \rightarrow Z$ is continuous.



Proof 3.2

Proved by previous theorem 3.3.

3.3.6 Uniform Continuity (@ Lec 04 of ECON 204)

Definition 3.8 (Uniformly Continuous)

Suppose $f : (X, d) \rightarrow (Y, \rho)$. f is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$



Claim 3.1

Uniformly Continuous implies (is stronger than) Continuous.

$(f \text{ is continuous if } \forall x_0 \in X, \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon)$



Given $\varepsilon > 0$, "uniformly continuous" requires $\delta(\varepsilon)$ that works for all $x_0 \in X$.

3.3.7 Theorem: Continuous function with Compact domain is Uniformly Continuous (@ Lec 06 of ECON 204)

Theorem 3.6 (Continuous function with Compact domain is Uniformly Continuous)

Let (X, d) and (Y, ρ) be metric spaces, C a **compact** subset of X , and $f : C \rightarrow Y$ a **continuous** function.

Then f is **uniformly continuous** on C .



Proof 3.3

Fix $\varepsilon > 0$. Since f is continuous, given $c \in C$, we can find $\delta(c) > 0$ such that

$$\forall x \in C, d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

Let $U_c = B_{\delta(c)}(c)$, then $\{U_c : c \in C\}$ is an open cover of C . Since C is compact, there is a finite subcover $\{U_{c_1}, \dots, U_{c_n}\} = \{B_{\delta(c_1)}(c_1), \dots, B_{\delta(c_n)}(c_n)\}$. Let $\delta = \min\{\delta(c_1), \dots, \delta(c_n)\}$.

Given $x, y \in C$ with $d(x, y) < \delta$, note that $x \in U_{c_i}$ for some $i \in \{1, \dots, n\}$, so $d(x, c_i) < \delta(c_i)$.

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) = 2\delta(c_i) \end{aligned}$$

So,

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

That is, $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$, which proves that f is uniformly continuous.

3.3.8 Lipschitz Continuous (@ Lec 04 of ECON 204)

Definition 3.9 (Lipschitz (Continuous) in Normed Vector Space)

Let X, Y be normed vector spaces, $\mathbb{E} \subseteq X$.

(1). A function $f : X \rightarrow Y$ is **Lipschitz** on \mathbb{E} satisfies

$$\exists \gamma > 0, \|f(\mathbf{x}) - f(\mathbf{y})\|_Y \leq \gamma \|\mathbf{x} - \mathbf{y}\|_X, \forall \mathbf{x}, \mathbf{y} \in \mathbb{E}$$

or we call γ -**Lipschitz continuous**;

(2). f is **locally Lipschitz** on \mathbb{E} if

$$\forall x_0 \in \mathbb{E}, \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap \mathbb{E}$$



Claim 3.2

- 1). If f is γ -Lipschitz continuous, then it is also $(\gamma + 1)$ -Lipschitz continuous. The minimal such γ is called a Lipschitz constant of function f
- 2). The definition can be extended to general metric spaces by replacing $\|\cdot\|_X$ and $\|\cdot\|_Y$ to general metrics.



Example 3.1 (Lipschitz)

1. $f(x) = 2x$ is 2-Lipschitz continuous;
2. What about $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).
3. What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞ .

Example 3.2 (Locally Lipschitz) Every C^1 function is locally Lipschitz. (Recall that a function $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous.)

Claim 3.3 (Lipschitz continuity vs. continuity vs. uniform continuity)

Lipschitz continuity is stronger than either continuity or uniform continuity:

- *locally Lipschitz \Rightarrow continuous*
- *Lipschitz \Rightarrow uniformly continuous*

**3.3.9 Contraction (from the view of Lipschitz constant)**

- a). If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.
- b). If the Lipschitz constant $\gamma < 1$, then f is called a contraction mapping.

Example 3.3

1. $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.
2. $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

3.4 Extreme Value, Intermediate Value of Functions**3.4.1 Intermediate Value Theorem (@ Lec 02 of ECON 204)****Theorem 3.7 (Intermediate Value Theorem)**

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.

**3.4.2 Coercive Function****Definition 3.10 (Coercive)**

A real-valued function $f : \mathbb{X} \rightarrow \mathbb{R}$ is coercive if for every $\{x_k\} \subset \mathbb{X}$ s.t. $\|x_k\| \rightarrow \infty$, $f(x_k) \rightarrow \infty$

**Example 3.4 Check coercive**

- 1) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2$ - coercive
- 2) $x \in \mathbb{R}$, $f(x) = 1 - e^{-|x|}$ - not coercive
- 3) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (we need $f(x_k) \rightarrow \infty$ for all $\|x_k\| \rightarrow \infty$)

3.4.3 Extreme of Functions

Definition 3.11 (Extreme of Functions)

Let $\mathbb{X} \subseteq \mathbb{R}^n$, $f : \mathbb{X} \rightarrow \mathbb{R}$

$$\inf_{x \in \mathbb{X}} f(x) = \inf\{f(x) : x \in \mathbb{X}\}$$



If $\exists x^* \in \mathbb{X}$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \mathbb{X}} f(x)$

x^* is called a **minimizer** of f , written as $x^* \in \arg \min_{x \in \mathbb{X}} f(x)$. If x^* is unique, we write $x^* = \arg \min_{x \in \mathbb{X}} f(x)$

Similarly, supremum and maximum of f .

3.4.4 Weierstrass' Theorem (Extreme value Theorem) (@ Lec 06 of ECON 204)

Theorem 3.8 (Weierstrass' Theorem (Extreme value Theorem))

Given a **compact set** C in a metric space (X, d) and a **continuous function** $f : C \rightarrow \mathbb{R}$, then f is bounded on C and attains its minimum and maximum on C i.e.,

$$\exists x_1 \in C \text{ s.t. } f(x_1) = \inf f(C)$$

$$\exists x_2 \in C \text{ s.t. } f(x_2) = \sup f(C)$$



Proof 3.4

Since C is compact and f is continuous, $f(C) \subset \mathbb{R}$ is compact by theorem 2.37. Hence, $f(C)$ is closed and bounded.

1. *Proof of maximum:* Let $M = \sup f(C) < \infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So $y_m \rightarrow M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e., there exists $c \in C$ such that $f(c) = M = \sup f(C)$.

2. *Proof of minimum:* Let $M = \inf f(C) > -\infty$. Then $\forall m > 0$ there exists $y_m \in f(C)$ such that

$$M \geq y_m \geq M + \frac{1}{m}$$

So $y_m \rightarrow M$ and $\{y_m\} \subseteq f(C)$. Since $f(C)$ is closed, $M \in f(C)$, i.e., there exists $c \in C$ such that $f(c) = M = \inf f(C)$.

3.4.5 Corollary: Continuous and Coercive function attains its Minimum on Closed Set

Corollary 3.1 (Continuous and Coercive function attains its Minimum on Closed Set)

Let f be continuous and coercive on closed set \mathbb{X} (not necessarily bounded), it attains its min on \mathbb{X} .



Proof 3.5

Consider $\{\sigma_k\}$ as in proof of WT.

Since f is closed, $f(x) < \infty, \forall x \in \mathbb{X}$. And f is coercive on \mathbb{X} , which means $f(x) \rightarrow \infty$ if $\|x\| \rightarrow \infty$.

Hence, $\{\sigma_k\} \in \mathbb{X}$ is bounded. Rest of proof same as proof of WT.

Example 3.5 $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) Does f achieve its min and max on $\mathbb{X}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leq 6\}$?

- \mathbb{X}_1 is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over \mathbb{R}^3 ?

- $f \rightarrow \infty$ whenever $\|x\| \rightarrow \infty \Rightarrow f$ is coercive.

- \mathbb{R}^3 is closed.

$\Rightarrow f$ achieves its min. on \mathbb{R}^3 by corollary to WT.

- max does not exist since $f \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

3) Does f achieve its min and max over $\mathbb{X}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?

- \mathbb{X}_2 is closed, but not bounded.

- Since f is coercive, min achieved.

- max does not exist since setting $x_1 = 0, x_2 = 3 - x_3$ and letting $x_3 \rightarrow \infty$ makes $f \rightarrow \infty$

3.5 Monotonic Function (@ Lec 05 of ECON 204)

Definition 3.12

A function $f : \mathbb{R} \rightarrow \mathbb{R}$ is **monotonically increasing** if $y > x \Rightarrow f(y) \geq f(x)$.



3.5.1 Theorem: Monotonically Increasing \Rightarrow One-sided Limits exist

Theorem 3.9

Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \rightarrow \mathbb{R}$ be monotonically increasing. Then the one-sided limits:

$f(t^+) = \lim_{u \rightarrow t^+} f(u)$ and $f(t^-) = \lim_{u \rightarrow t^-} f(u)$ exist and are real numbers for all $t \in (a, b)$.



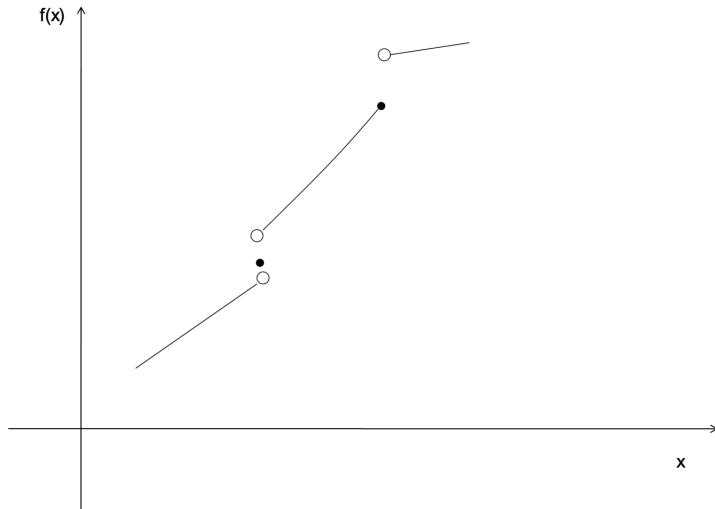


Figure 3.1: A monotonic function has only simple jump discontinuities.

3.5.2 Theorem: Monotonically Increasing \Rightarrow Set of Points are discontinuous is finite (possibly empty) or countable

Theorem 3.10

Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \rightarrow \mathbb{R}$ be monotonically increasing. Then $D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$ is finite (possibly empty) or countable.



3.6 Differentiable Function (@ Lec 11 of ECON 204)

3.6.1 Definition of Differentiable in \mathbb{R}

Definition 3.13

Let $f : I \rightarrow \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval. f is **differentiable** at $x \in I$ if

$$\lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = a$$

for some $a \in \mathbb{R}$.



This is equivalent to $\exists a \in \mathbb{R}$ such that:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x + h) - (f(x) + ah)}{h} = 0 \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x + h) - (f(x) + ah)}{h} \right| < \varepsilon \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \frac{|f(x + h) - (f(x) + ah)|}{|h|} < \varepsilon \\ \Leftrightarrow & \lim_{h \rightarrow 0} \frac{|f(x + h) - (f(x) + ah)|}{|h|} = 0 \end{aligned}$$

Recall that the limit considers h near zero, but not $h = 0$.

Proposition 3.3

f is **differentiable** $\Rightarrow f$ is **continuous**.



3.6.2 General Definition of Differentiable

Definition 3.14

If $X \subseteq \mathbb{R}^n$ is open, $f : X \rightarrow \mathbb{R}^m$ is **differentiable** at $x \in X$ if $\exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t.

$$\lim_{h \rightarrow 0, h \in \mathbb{R}^n} \frac{|f(x + h) - (f(x) + T_x(h))|}{|h|} = 0$$

f is differentiable if it is differentiable at all $x \in X$.



Note that T_x is uniquely determined by Equation (1). h is a small, nonzero element of \mathbb{R}^n ; $h \rightarrow 0$ from any direction, from above, below, along a spiral, etc. The definition requires that one linear operator T_x works no matter how h approaches zero. In this case, $f(x) + T_x(h)$ is the best linear approximation to $f(x + h)$ for small h .

3.6.3 "O" and "o" Notations

Definition 3.15 ("O" and "o")

1. $y = O(|h|^n)$ as $h \rightarrow 0$ – read “ y is big-Oh of $|h|^n$ ” – means $\exists K, \delta > 0$ s.t. $|h| < \delta \Rightarrow |y| \leq K|h|^n$
2. $y = o(|h|^n)$ as $h \rightarrow 0$ – read “ y is little-Oh of $|h|^n$ ” – means $\lim_{h \rightarrow 0} \frac{|y|}{|h|^n} = 0$



Note Note that the statement $y = O(|h|^{n+1})$ as $h \rightarrow 0$ implies $y = o(|h|^n)$ as $h \rightarrow 0$.

In both cases, $y \rightarrow 0$ as $h \rightarrow 0$, but the *rate* is different.

Definition 3.16 (New Definition)

Using this notation, note that f is **differentiable** at $x \Leftrightarrow$

$$\exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m) \text{ such that } f(x + h) = f(x) + T_x(h) + o(|h|) \text{ as } h \rightarrow 0$$



3.6.4 Jacobian and Error Term

Definition 3.17 (Jacobian, Error Term)

- df_x is the linear transformation T_x
- $Df(x)$ is the matrix of df_x with respect to the standard basis. This is called the **Jacobian** or Jacobian matrix of f at x
- $E_f(h) = f(x + h) - (f(x) + df_x(h))$ is the **error term**

Using this notation, f is differentiable at $x \Leftrightarrow E_f(h) = o(h)$ as $h \rightarrow 0$



Now compute $Df(x) = (a_{ij})$. Let $\{e_1, \dots, e_n\}$ be the standard basis of \mathbb{R}^n . Look in direction e_j (note that $|\gamma e_j| = |\gamma|$).

$$o(\gamma) = f(x + \gamma e_j) - (f(x) + T_x(\gamma e_j))$$

$$\begin{aligned} &= f(x + \gamma e_j) - \left(f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\ &= f(x + \gamma e_j) - \left(f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right) \end{aligned}$$

For $i = 1, \dots, m$, let f^i denote the i^{th} component of the function f :

$$f^i(x + \gamma e_j) - (f^i(x) + \gamma a_{ij}) = o(\gamma)$$

$$\text{so, } a_{ij} = \frac{\partial f^i}{\partial x_j}(x)$$

3.6.5 Theorem: Differentiable \Rightarrow Partial Derivatives (and Jacobian) exist

Theorem 3.11 (Differentiable \Rightarrow Partial Derivatives (and Jacobian) exist)

Suppose $X \subseteq \mathbb{R}^n$ is open and $f : X \rightarrow \mathbb{R}^m$ is differentiable at $x \in X$. Then $\frac{\partial f^i}{\partial x_j}$ exists at x for

$1 \leq i \leq m, 1 \leq j \leq n$, and

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

i.e. the Jacobian at x is the matrix of partial derivatives at x .



Note If f is differentiable at x , then all first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ exist at x . However, the converse is false: existence of all the first-order partial derivatives does not imply that f is differentiable.

Example 3.6 $f(X) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & X \neq (0, 0) \\ 0, & X = (0, 0) \end{cases}$

3.6.6 Theorem: Partial Derivatives exist and Continuous \Rightarrow Differentiable

The missing piece is continuity of the partial derivatives:

Theorem 3.12 (Partial Derivatives exist and Continuous \Rightarrow Differentiable)

If all the first-order partial derivatives $\frac{\partial f^i}{\partial x_j}$ ($1 \leq i \leq m, 1 \leq j \leq n$) exist and are continuous at x , then f is differentiable at x .



3.6.7 Directional Derivatives

Definition 3.18 (Directional Derivative)

Suppose $X \subseteq \mathbb{R}^n$ open, $f : X \rightarrow \mathbb{R}^m$ is differentiable at x , and $|u| = 1$.

$$\begin{aligned} f(x + \gamma u) - (f(x) + T_x(\gamma u)) &= o(\gamma) \text{ as } \gamma \rightarrow 0 \\ \Rightarrow f(x + \gamma u) - (f(x) + \gamma T_x(u)) &= o(\gamma) \text{ as } \gamma \rightarrow 0 \\ \Rightarrow \lim_{\gamma \rightarrow 0} \frac{f(x + \gamma u) - f(x)}{\gamma} &= T_x(u) = Df(x)u \end{aligned}$$

i.e. the **directional derivative** in the direction u (with $|u| = 1$) is

$$Df(x)u \in \mathbb{R}^m$$



3.6.8 Chain Rule: $dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$ and $DF_{x_0} = Dg(f(x_0))Df(x_0)$

Theorem 3.13 (Chain Rule)

Let $X \subseteq \mathbb{R}^n$, $Y \subseteq \mathbb{R}^m$ be open, $f : X \rightarrow Y$, $g : Y \rightarrow \mathbb{R}^p$. Let $x_0 \in X$ and $F = g \circ f$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$, then $F = g \circ f$ is differentiable at x_0 and

$$dF_{x_0} = dg_{f(x_0)} \circ df_{x_0} \text{ (composition of linear transformations)}$$

$$DF_{x_0} = Dg(f(x_0))Df(x_0) \text{ (matrix multiplication)}$$



3.6.9 Mean Value Theorem: $\exists z \in l(x, y), \|f(y) - f(x)\| \leq \|df_z\| \|y - x\|$

Theorem 3.14 (Mean Value Theorem, Univariate Case: $\mathbb{R} \rightarrow \mathbb{R}$)

Let $a, b \in \mathbb{R}$. Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b) . Then there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

that is

$$f(b) - f(a) = f'(c)(b - a)$$



Note The Mean Value Theorem is useful for estimating bounds on functions and error terms in approximation of functions.

Definition 3.19 (Line Segment)

$l(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$ is the **line segment** from x to y .



Theorem 3.15 (Mean Value Theorem, $\mathbb{R}^n \rightarrow \mathbb{R}$)

Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is differentiable on an open set $X \subseteq \mathbb{R}^n$, $x, y \in X$ and $l(x, y) \subseteq X$. Then there exists $z \in l(x, y)$ such that

$$f(y) - f(x) = Df(z)(y - x)$$



Theorem 3.16 (Mean Value Theorem, General Case: $\mathbb{R}^n \rightarrow \mathbb{R}^m$)

Suppose $X \subseteq \mathbb{R}^n$ is open and $f : X \rightarrow \mathbb{R}^m$ is differentiable. If $x, y \in X$ and $l(x, y) \subseteq X$, then there exists $z \in l(x, y)$ such that

$$\begin{aligned} \|f(y) - f(x)\| &\leq \|df_z(y - x)\| \\ &\leq \|df_z\| \|y - x\| \end{aligned}$$



3.6.10 Taylor's Theorem

Theorem 3.17 (Taylor's Theorem)

Let $f : I \rightarrow \mathbb{R}$ be n -times differentiable, where $I \subseteq \mathbb{R}$ is an open interval. If $x, x + h \in I$, then

$$f(x + h) = f(x) + \sum_{k=1}^{n-1} \frac{f^{(k)}(x)h^k}{k!} + E_n$$

where $f^{(k)}$ is the k^{th} derivative of f and

$$E_n = \frac{f^{(n)}(x + \lambda h)h^n}{n!} \text{ for some } \lambda \in (0, 1)$$



Theorem 3.18 (Taylor's Theorem in \mathbb{R}^1)

Let $f : I \rightarrow \mathbb{R}$ be n times differentiable, where $I \subseteq \mathbb{R}$ is an open interval and $x \in I$. Then,

$$f(x + h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + o(h^n) \text{ as } h \rightarrow 0$$

If f is $(n + 1)$ times continuously differentiable (i.e. all derivatives up to order $n + 1$ exist and are continuous), then

$$f(x + h) = f(x) + \sum_{k=1}^n \frac{f^{(k)}(x)h^k}{k!} + o(h^{n+1}) \text{ as } h \rightarrow 0$$



3.6.11 C^k Function and Continuously Differentiable

Definition 3.20 (C^k Function)

f is C^k if all partial derivatives of order less than or equal to k exist and are continuous in X .



Definition 3.21 (Continuously Differentiable)

Let $X \subseteq \mathbb{R}^n$ be open. A function $f : X \rightarrow \mathbb{R}^m$ is **continuously differentiable** on X if

1. f is differentiable on X and
2. df_x is a continuous function of x from X to $L(\mathbb{R}^n, \mathbb{R}^m)$, with operator norm $\|df_x\|$.



Theorem 3.19 (Continuously Differentiable $\Leftrightarrow C^1$)

Suppose $X \subseteq \mathbb{R}^n$ is open and function $f : X \rightarrow \mathbb{R}^m$ is **continuously differentiable** if and only if f is C^1 .



3.6.12 Taylor's Theorem with C^k

Linear Terms

Theorem 3.20

Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}^m$ is differentiable, then

$$f(x + h) = f(x) + Df(x)h + o(h) \text{ as } h \rightarrow 0$$



The previous theorem is essentially a restatement of the definition of differentiability.

Theorem 3.21

Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}^m$ is C^2 , then

$$f(x + h) = f(x) + Df(x)h + O(|h|^2) \text{ as } h \rightarrow 0$$



Quadratic Terms

We treat each component of the function separately, so consider $f : X \rightarrow \mathbb{R}$, $X \subseteq \mathbb{R}^n$ an open set. Let

$$D^2 f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2}(x) & \frac{\partial^2 f}{\partial x_2 \partial x_1}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_1}(x) \\ \frac{\partial^2 f}{\partial x_1 \partial x_2}(x) & \frac{\partial^2 f}{\partial x_2^2}(x) & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_1 \partial x_n}(x) & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2}(x) \end{pmatrix}$$

$$f \in C^2 \Rightarrow \frac{\partial^2 f}{\partial x_i \partial x_j}(x) = \frac{\partial^2 f}{\partial x_j \partial x_i}(x)$$

$\Rightarrow D^2 f(x)$ is symmetric

$\Rightarrow D^2 f(x)$ has an orthonormal basis of eigenvectors and thus can be diagonalized

Theorem 3.22 (Stronger Version of Taylor's Theorem)

Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}$ is C^2 , then

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2}h^T(D^2 f(x))h + O(|h|^2) \text{ as } h \rightarrow 0$$

If $f \in C^3$,

$$f(x + h) = f(x) + Df(x)h + \frac{1}{2}h^T(D^2 f(x))h + O(|h|^3) \text{ as } h \rightarrow 0$$



Definition 3.22

We say f has a **saddle** at x if $Df(x) = 0$ but f has neither a local maximum nor a local minimum at x .



Corollary 3.2

Suppose $X \subseteq \mathbb{R}^n$ is open and $x \in X$. If $f : X \rightarrow \mathbb{R}$ is C^2 , then there is an orthonormal basis $\{v_1, \dots, v_n\}$ and corresponding eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$ of $D^2 f(x)$ such that

$$\begin{aligned} f(x + h) &= f(x + \gamma_1 v_1 + \cdots + \gamma_n v_n) \\ &= f(x) + \sum_{i=1}^n (Df(x)v_i) \gamma_i + \frac{1}{2} \sum_{i=1}^n \lambda_i \gamma_i^2 + o(|\gamma|^2) \end{aligned}$$

where $\gamma_i = h \cdot v_i$.

1. If $f \in C^3$, we may strengthen $o(|\gamma|^2)$ to $O(|\gamma|^3)$.

2. If f has a local maximum or local minimum at x , then

$$Df(x) = 0$$

3. If $Df(x) = 0$, then

$\lambda_1, \dots, \lambda_n > 0 \Rightarrow f$ has a local minimum at x

$\lambda_1, \dots, \lambda_n < 0 \Rightarrow f$ has a local maximum at x

$\lambda_i < 0$ for some i , $\lambda_j > 0$ for some $j \Rightarrow f$ has a saddle at x

$\lambda_1, \dots, \lambda_n \geq 0, \lambda_i > 0$ for some $i \Rightarrow f$ has a local minimum or a saddle at x

$\lambda_1, \dots, \lambda_n \leq 0, \lambda_i < 0$ for some $i \Rightarrow f$ has a local maximum or a saddle at x

$\lambda_1 = \dots = \lambda_n = 0$ gives no information.



Chapter 4 Differential Equations (@ Lec 14/15 of ECON 204)

4.1 Initial Value Problem

4.1.1 Initial Value Problem

Definition 4.1 (Differential Equation)

A **differential equation** is an equation of the form $y'(t) = F(y(t), t)$.



where $F : U \rightarrow \mathbb{R}^n$ and U is an open subset of $\mathbb{R}^n \times \mathbb{R}$.

Definition 4.2 (Initial Value Problem)

1. An initial value problem is a differential equation combined with an initial condition

$$y(t_0) = y_0$$

where $(y_0, t_0) \in U$

2. A solution of the initial value problem is a differentiable function $y : (a, b) \rightarrow \mathbb{R}^n$ such that $t_0 \in (a, b)$, $y(t_0) = y_0$ and, for all $t \in (a, b)$, $y'(t) = F(y(t), t)$.
3. The general solution of the differential equation is the family of all solutions for all initial values $(y_0, t_0) \in U$.



4.1.2 Theorem: Initial Value Problem, Continuous \rightarrow Existence; Lipschitz \rightarrow Uniqueness

Theorem 4.1 (Initial Value Problem, Continuous \rightarrow Existence; Lipschitz \rightarrow Uniqueness)

Consider the initial value problem

$$y'(t) = F(y(t), t), \quad y(t_0) = y_0$$

U is an open subset of $\mathbb{R}^n \times \mathbb{R}$ containing (y_0, t_0) .

1. Suppose $F : U \rightarrow \mathbb{R}^n$ is continuous. Then the initial value problem has a solution.

2. If, in addition, F is Lipschitz in y on U

(i.e. there is a constant K such that for all $(y, t), (\hat{y}, t) \in U$, $|F(y, t) - F(\hat{y}, t)| \leq K|y - \hat{y}|$),
then there is an interval (a, b) containing t_0 such that the solution is unique on (a, b) .



Example 4.1 Consider the initial value problem

$$y'(t) = 1 + y^2(t), \quad y(0) = 0$$

Here, we have $F(y, t) = 1 + y^2$ which is Lipschitz in y over $U = V \times \mathbb{R}$, provided that V is bounded, but not

over all of $\mathbb{R} \times \mathbb{R}$. The theorem tells us that the initial value problem has a unique solution over some interval of t , (a, b) , with $0 \in (a, b)$.

We can find $y(t) = \tan t$ is a unique solution in $(a, b) \subset (-\frac{\pi}{2}, \frac{\pi}{2})$ with $a < 0, b > 0$.

Example 4.2 Consider the initial value problem

$$y'(t) = 2\sqrt{|y|}, y(0) = 0$$

As $F(y) = 2\sqrt{|y|}$ is not locally Lipschitz in y at $y = 0$, we can find for all $\alpha \geq 0$,

$$y_\alpha(t) = \begin{cases} 0, & t \leq \alpha \\ (t - \alpha)^2, & t \geq \alpha \end{cases}$$

is a solution of the initial value problem, which is not unique.

4.2 Autonomous Differential Equation

4.2.1 Autonomous Differential Equation

Definition 4.3 (Autonomous Differential Equation)

An **autonomous differential equation** is a differential equation of the form $y'(t) = F(y(t))$ where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ depends on t only through the value of $y(t)$.

A **stationary point** of an autonomous differential equation is a point $y_s \in \mathbb{R}^n$ such that $F(y_s) = 0$.



We study the qualitative properties of autonomous differential equations by looking for stationary points. The constant function

$$y(t) = y_s$$

is a solution (and the unique solution when F is Lipschitz) of the initial value problem

$$y' = F(y), y(t_0) = y_s$$

If F is C^2 , then Taylor's Theorem implies that near a stationary point y_s ,

$$\begin{aligned} F(y_s + h) &= F(y_s) + DF(y_s)h + O(|h|^2) \\ &= DF(y_s)h + O(|h|^2) \end{aligned}$$

Thus, when we are sufficiently close to the stationary point, the solutions of the autonomous differential equation are closely approximated by the solutions of the linear differential equation

$$y' = (y - y_s)' = DF(y_s)(y - y_s)$$

Thus, we study solutions of linear differential equations, using linear algebra. For this we first recall the formulation of complex exponentials, which are central in the general solution of linear differential equations.

4.2.2 Complex Exponentials

By the Taylor series, $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. If $x = a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$, we have

$$\begin{aligned} e^x &= e^{a+ib} = e^a e^{ib} \\ &= e^a \left(\sum_{k=0}^{\infty} (-1)^k \frac{b^{2k}}{(2k)!} + i \sum_{k=0}^{\infty} (-1)^k \frac{b^{2k+1}}{(2k+1)!} \right) \\ &= e^a (\cos b + i \sin b) \end{aligned}$$

Hence,

1. If $a < 0$, then $e^{tx} \rightarrow 0$ as $t \rightarrow \infty$.
2. If $a > 0$, then $|e^{tx}| \rightarrow \infty$ as $t \rightarrow \infty$.
3. If $a = 0$, then $|e^{tx}| \rightarrow 1$ for all $t \in \mathbb{R}$.

4.2.3 Solutions of Simple Case: $y'(t) = ay(t)$

Proposition 4.1 (Simple Case)

Given $y'(t) = ay(t)$, we have

$$y(t) = ce^{at}, c \in \mathbb{R}^*$$



Proof 4.1

$$\begin{aligned} y'(t) &= \frac{dy}{dt} = ay \\ \frac{dy}{y} &= adt \\ \int \frac{1}{y} dy &= \int adt \\ \ln y &= at + c', c' \in \mathbb{R} \\ y(t) &= e^{c'} e^{at} = ce^{at}, c \in \mathbb{R}^* \end{aligned}$$

Proposition 4.2 (Simple Case with Initial Value)

Given initial value (t_0, y_0) , $y'(t) = ay(t)$, we have

$$y(t) = y_0 e^{a(t-t_0)}$$



4.2.4 Solutions of Special Case: Linear Differential Equations with Constant Coefficients

Let $M \in \mathbb{R}^{n \times n}$. The linear differential equation

$$y' = (y - y_s)' = M(y - y_s)$$

has a complete solution in closed form.

The matrix representation

$$M = DF(y_s)$$

need not be symmetric, hence may not be diagonalizable. If M is diagonalizable over \mathbb{C} , the complete solution takes the following simple form:

Theorem 4.2 (Solutions of Linear Differential Equations with Constant Coefficients)

Consider the linear differential equation

$$y' = (y - y_s)' = M(y - y_s)$$

where M is a real $n \times n$ matrix. Suppose that M can be diagonalized over the complex field \mathbb{C} . Let U be the standard basis of \mathbb{R}^n and $V = \{v_1, \dots, v_n\}$ be a basis of (complex) eigenvectors corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{C}$ of M . Then the solution of the initial value problem is given by

$$y(t) = y_s + P^{-1} \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n(t-t_0)} \end{bmatrix} P(y(t_0) - y_s) \quad (4.1)$$

where $P = \text{Mtx}_{V,U}(id)$, and the general complex solution is obtained by allowing $y(t_0)$ to vary over \mathbb{C}^n ; it has n complex degrees of freedom. The general real solution is obtained by allowing $y(t_0)$ to vary over \mathbb{R}^n ; it has n real degrees of freedom. Every real solution is a linear combination of the real and imaginary parts of a complex solution. In particular,

1. If the real part of each eigenvalue is less than zero, all solutions converge to y_s .
2. If the real part of each eigenvalue is greater than zero, all solutions diverge from y_s and tend to infinity.
3. If the real parts of some eigenvalues are less than zero and the real parts of other eigenvalues are greater than zero, solutions follow roughly hyperbolic paths.
4. If the real parts of all eigenvalues are zero, all solutions follow closed cycles around y_s .



Note If one or more of the eigenvalues are complex, each of the three matrices in Equation 4.1 will contain complex entries, but the product of the three matrices is real. Thus, if the initial condition y_0 is real, Equation 4.1 gives us a real solution; indeed, it gives us the unique solution of the initial value problem.



Note Given a fixed time t_0 , the general real solution is obtained by varying the initial values of $y(t_0)$ over \mathbb{R}^n , which provides n real degrees of freedom. You might think that varying t_0 provides one additional degree of freedom, but it doesn't. Given any solution satisfying the initial condition $y(t_0) = y_0$, the solution is defined on

some interval $(t_0 - \delta, t_0 + \delta)$; given $t_1 \in (t_0 - \delta, t_0 + \delta)$, let $y_1 = y(t_1)$; then the solution with initial condition $y_1 = y(t_1)$ is the same as the solution with initial condition $y_0 = y(t_0)$. The same holds true for the general complex solution.

Proof 4.2

Let $P = \text{Mtx}_{V,U}(id)$. Rewrite the differential equation in terms of a new variable

$$z = Py$$

Let $z_s = Py_s$, then we have

$$\begin{aligned} z - z_s &= P(y - y_s) \\ (z - z_s)' &= z' = Py' \\ &= PM(y - y_s) \\ &= PMP^{-1}(z - z_s) \\ &= B(z - z_s) \end{aligned}$$

where $B = \text{diag}(\lambda_1, \dots, \lambda_n)$.

Thus, the i^{th} component of $(z(t) - z_s)$ satisfies

$$(z(t) - z_s)_i' = \lambda_i(z(t) - z_s)_i$$

by the proposition 4.2, we have

$$(z(t) - z_s)_i = e^{\lambda_i(t-t_0)}(z(t_0) - z_s)_i$$

so,

$$z(t) - z_s = \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n(t-t_0)} \end{bmatrix} (z(t_0) - z_s)$$

$$\begin{aligned}
y(t) - y_s &= P^{-1} \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n(t-t_0)} \end{bmatrix} (z(t_0) - z_s) \\
&= P^{-1} \begin{bmatrix} e^{\lambda_1(t-t_0)} & 0 & 0 & 0 \\ 0 & e^{\lambda_2(t-t_0)} & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & e^{\lambda_n(t-t_0)} \end{bmatrix} P(y(t_0) - y_s)
\end{aligned}$$

4.2.5 The Form of Real Solutions

We can determine the form of the real solutions once we know the eigenvalues. In an important special case, we can solve for the solution of the initial value problem without calculating the diagonalization, as in Equation 4.1.

Theorem 4.3

Consider the differential equation

$$y' = (y - y_s)' = M(y - y_s)$$

Suppose that the matrix M can be diagonalized over \mathbb{C} . Let the eigenvalues of M with the correct multiplicity be

$$a_1 + ib_1, a_1 - ib_1, \dots, a_m + ib_m, a_m - ib_m, a_{m+1}, \dots, a_{n-m}$$

Then for each fixed $i = 1, \dots, n$, every real solution is of the form

$$(y(t) - y_s)_i = \sum_{j=1}^m e^{a_j(t-t_0)} (C_{ij} \cos b_j(t-t_0) + D_{ij} \sin b_j(t-t_0)) + \sum_{j=m+1}^{n-m} C_{ij} e^{a_j(t-t_0)}$$

The n^2 parameters

$$\{C_{ij} : i = 1, \dots, n; j = 1, \dots, n-m\} \cup \{D_{ij} : i = 1, \dots, n; j = 1, \dots, m\}$$

have n real degrees of freedom. The parameters are uniquely determined from the n real initial conditions of an initial value problem.



4.3 Higher Order Differential Equations

When we consider a differential equation of order n is an equation of the form

$$y^{(n)}(t) = F(y(t), y'(t), \dots, y^{n-1}(t), t)$$

We can always rewrite

$$\bar{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \\ \vdots \\ y^{n-1}(t) \end{bmatrix}$$

4.3.1 Second Order Linear Differential Equations

Consider the second order differential equation $y'' = cy + by'$ with $b, c \in \mathbb{R}$.

Rewrite $\bar{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$, then

$$\bar{y}'(t) = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix} \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix} \bar{y}$$

By the Theorem 4.2, we can now solve it. Specifically, the eigenvalues of $\begin{bmatrix} 0 & 1 \\ c & b \end{bmatrix}$ are $\frac{b \pm \sqrt{b^2 + 4c}}{2}$.

Example 4.3 Consider the second order linear differential equation $y'' = 2y + y'$, we can solve the eigenvalues $\lambda_1 = 2, \lambda_2 = -1$ and eigenvectors $v_1 = (1, 2), v_2 = (1, -1)$. The general solution of $\bar{y}(t) = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ can be solved by the Theorem 4.2.

4.4 Inhomogeneous Linear Differential Equations with Nonconstant Coefficients

Consider the inhomogeneous linear differential equation

$$y' = M(t)y + H(t) \quad (1)$$

where M is continuous function from t to the set of $n \times n$ matrices; and H is continuous function from t to \mathbb{R}^n .

4.4.1 Theorem: General Solution of Inhomo = General Solution of Homo + Particular Solution of Inhomo

There is a close relationship between solutions of the *inhomogeneous* linear differential equation 1 and the associated *homogeneous* linear differential equation

$$y' = M(t)y \quad (2)$$

Theorem 4.4 (General Solution of Inhomo=General Solution of Homo+Particular Solution of Inhomo)

The general solution of the inhomogeneous linear differential equation 1 is

$$y_h + y_p$$

where y_h is the general solution of the homogeneous linear differential equation 2 and y_p is any particular solution of the inhomogeneous linear differential equation 1.



Hence, to find general solution of inhomogeneous equation:

1. Find general solution of homogeneous equation;
2. Find a particular solution of inhomogeneous equation;
3. Add these to get general solution of inhomogeneous equation

4.4.2 Matrix Exponentials Notations

In analogy with how we define e^x for $x \in \mathbb{R}$, for an $n \times n$ matrix M we define

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M + \frac{M^2}{2} + \dots$$

and

$$e^{tM} = \sum_{k=0}^{\infty} \frac{t^k M^k}{k!}$$

The matrix exponential satisfies many properties analogous to the exponential function in \mathbb{R} and \mathbb{C} . Here are a few of the most important properties, which can be established fairly directly from the definitions above.

- If D is a diagonal matrix with diagonal d_1, \dots, d_n ,

$$e^D = \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n} \end{pmatrix}$$

- $e^{A+B} = e^A e^B$ if $AB = BA$ (this is not necessarily valid for matrices A and B that do not commute)
- $e^{P^{-1}AP} = P^{-1}e^A P$

Then, if M is diagonalizable, so $M = P^{-1}DP$ for a diagonal matrix D , then

$$e^M = P^{-1}e^D P = P^{-1} \begin{pmatrix} e^{d_1} & 0 & \cdots & 0 \\ 0 & e^{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{d_n} \end{pmatrix} P$$

- $g(t) = e^{tM}$ is differentiable (in fact C^∞) and $g'(t) = M e^{tM}$

4.4.3 Theorem: Particular Solution of Inhomogeneous Linear Differential Equations with Constant Coefficients (Matrix)

Theorem 4.5 (Particular Solution with Constant Coefficients (Matrix))

Consider the inhomogeneous linear differential equation 1, and suppose that $M(t)$ is a constant matrix M , independent of t . A particular solution of the inhomogeneous linear differential equation 1, satisfying the initial condition $y_p(t_0) = y_0$, is given by

$$y_p(t_0) = e^{(t-t_0)M} y_0 + \int_{t_0}^t e^{(t-s)M} H(s) ds \quad (3)$$



Chapter 5 Big \mathcal{O} and Small o Notation

5.1 Definition

Definition 5.1 ("O" and "o")

1. $y = O(|h|^n)$ as $h \rightarrow 0$ – read “ y is big-Oh of $|h|^n$ ” – means $\exists K, \delta > 0$ s.t. $|h| < \delta \Rightarrow |y| \leq K|h|^n$
2. $y = o(|h|^n)$ as $h \rightarrow 0$ – read “ y is little-Oh of $|h|^n$ ” – means $\lim_{h \rightarrow 0} \frac{|y|}{|h|^n} = 0$



Complexity:

Definition 5.2

A sequence $f(n)$ is $O(1)$ if $\lim_{n \rightarrow \infty} f(n) < \infty$.



Definition 5.3

A sequence $f(n)$ is $O(g(n))$ if $\frac{f(n)}{g(n)}$ is $O(1)$.



Definition 5.4

A sequence $f(n)$ is $o(1)$ if $\lim_{n \rightarrow \infty} \sup f(n) = 0$.



Definition 5.5

A sequence $f(n)$ is $o(g(n))$ if $\lim_{n \rightarrow \infty} \sup \frac{f(n)}{g(n)} = 0$.



Definition 5.6

A sequence $f(n)$ is asymptotic to $g(n)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$. (This is denoted by $f(n) \sim g(n)$ as $a \rightarrow \infty$)



For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$.

Example 5.1

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$

$$n^3 + n + 2 = \Theta(n^3)$$

$$n^3 + n + 2 = o(n^4)$$

5.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$ if $\limsup_{x \rightarrow a} \frac{|f(x)|}{g(x)} < \infty$.

Example 5.2 $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \rightarrow 0$