Analysis

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2022

目录

1 Sequence

Sequences $\{x_k\}_{k=1},\ldots$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 1 (Convergence: note $x_k \to x, \lim_{k\to\infty} x_k = x$)**.** *Given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x\| < \varepsilon \quad \forall k \geqslant N_\varepsilon$$

Definition 2 (Cauchy Sequence)**.** $\{x_k\}$ *is Cauchy if given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x_m\| < \varepsilon,\ \forall k, m \geqslant N_\varepsilon.$$

Note:

$$\{x_k\} \text{ converges } \Longleftrightarrow \{x_k\} \text{ is Cauchy}$$

Definition 3 (Subsequence)**.** *Infinite subset of* $\{x_k\}$*:* $\{x_k : k \in \mathcal{K}\}$ *or* $\{x_k\}_{\mathcal{K}}$*, where* $\mathcal{K}$ *is subset of* $\mathbb{Z}^+$*.*

Definition 4 (Limit point)**.** x *is a limit point of* $\{x_k\}$ *if* $\exists$ *a subsequence of* $\{x_k\}$ *that converges to* x*.*

Definition 5 (Bounded Sequence)**.**

$$\|x_k\| \leqslant b, \forall k$$

Results about Bounded sequences:
1. Every bounded has at least one limit point.
2. A bounded sequence converges iff it has a **unique limit point**.
Scalar sequences $\{x_k\}, x_k \in \mathbb{R}$
(below) (increasing) above and non-decreasing
- If $\{x_k\}$ is bounded above and non-decreasing it Converges
- The largest limit point of $\{x_k\}$ is $\lim_{k\to\infty}\sup x_k$
- $\{x_k\}$ lonvergos $\Longleftrightarrow -\infty < \lim_{k\to\infty}\inf_k = \lim_{k\to\infty} x_k < \infty$
D)efinition (contimuity) A real-valued function f is Continnous at x if for every $\{x_k\}$ converging to x $\lim_{k\to\infty} f(x_k) = f(x)$
Equivalenty, given $\varepsilon > 0, \exists\delta > 0$ s.t.
$|f(x) - f(y)| < \varepsilon \quad \forall\|y - x\| < \delta$
f is contimuons if it is contimuons at all points x
Definition (coercive) A real-valued function $f : \mathcal{A} \to \mathbb{R}$ is coercive if for every $\{x_k\} \subset$ & s.t. $\|x_k\| \to \infty, f(x_k) \to \infty$
Examples 1) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 -$ coercive
2) $x \in \mathbb{R}, f(x) = 1 - e^{-|x|} -$ not coercive
3) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 - 2x_1x_2$
$= (x_1 - x_2)^2 -$ not coercive Closed and open sets
A set & $\subseteq \mathbb{R}^n$ is open if $\forall x \in$ & we can dran a ball around x that is contanicd in &,

le. $\forall x \in \&, \exists \varepsilon > 0$ s.t.

$\{y : \|y - x\| < \varepsilon\} \leq -8$

$+\frac{1}{1} z$

& is closed if &c is open

$\equiv$ if & contans all limit points of all sequences in &

Examples $(1,2) = \{x \in \mathbb{R} : 1 < x < 2\}-$ open

$\mathbb{R}$ is both

$(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}-$ open

open and closed $[1, \infty)$ is closed because complement open

$(1,2]$ is neither open nor closed

Compact Set $\mathcal{L} \subseteq \mathbb{R}^n$ is compact of it is closed and bermded. A is bounded if $\exists M$ s.t. $\|x\| \leqslant M \quad \forall x \in -\&$

Examples

$[1,2] = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\}$

$\{x \in \mathbb{R}^2$ of scalars

$x_1^2 + x_2^2 \leqslant 4\}$

Extrema of sets of scalars Let A $C\mathbb{R}$.

- The infinum of A, or inf A is largest y s.t. $y \leqslant x, \forall x \in A$. If no such y exists, inf $A = -\infty$
- Similar definition for supremum of f or sup A.
- If mif $A = x^*$ for some $x^* \in A$, then

$x^* = \min A$ or minimem of A.

- similarly max A or maximum of A. Examples 1) $A = [1,2]$, inf $A = \min A = 1$

anp $A = \max A = 2$

2) $A = (1,2]$, inf $A = 1$, not achieved

3) $A = (1, \infty)$, sup $A = mxA = 2$

sup $A =$ no maximum

Extrema I Functiono Let $\subseteq \mathbb{R}^n, f : \& \rightarrow \mathbb{R}$

$$\operatorname{inif}_{x \in \&} f(x) = \operatorname{mif}\{f(x) : x \in \&\}.$$

If $\exists x^* \in \&$ s.t. $\inf f(x) = f(x^*)$ Then

f achieves (attains) its minimum and

$$f(x^*) = \min_{x \in \mathcal{L}} f(x)$$

x^* is called a minimizer of f, written as

$$x^* \in \arg\min_{x \in \mathcal{L}} f(x) \text{ If } x^* \text{ is unigne we write } x^* = \arg\min_{x \in s} f(x).$$

Similarly, supremum and maximum of f.

$$f(x) = x, \quad x \in (-1, 2)$$

$$\text{Example 1} \quad \begin{aligned} &\sup f(x) = 2, \text{ neither achleved} \\ &\inf f(x) = -1 \end{aligned}$$

Exanple 2 $f(x) = x^2, x \in \mathbb{R}$

$$\begin{aligned} &f(x) = x^2, x \in \mathbb{R} \\ &\inf f(x) = \min f(x) = 0, x^* = 0 \\ &\text{sip } f(x) = \infty, \text{ mox doen wot exist.} \end{aligned}$$