Lattice Programming, $L^\natural$-Convexity, and little Revenue Management

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Contents

1 Lattice Programming

1.1 Lattice

Definition 1. *$(X, \geq)$ is a **lattice** if for any $x, y \in X$,*
$x \vee y = \inf\{z \in X | x \leq z, y \leq z\} \in X$
$x \wedge y = \sup\{z \in X | x \geq z, y \geq z\} \in X$

Definition 2. *$(X', \geq)$ is a **sublattice** of $(X, \geq)$: inherit $x \vee y$, $x \wedge y$ from X.*

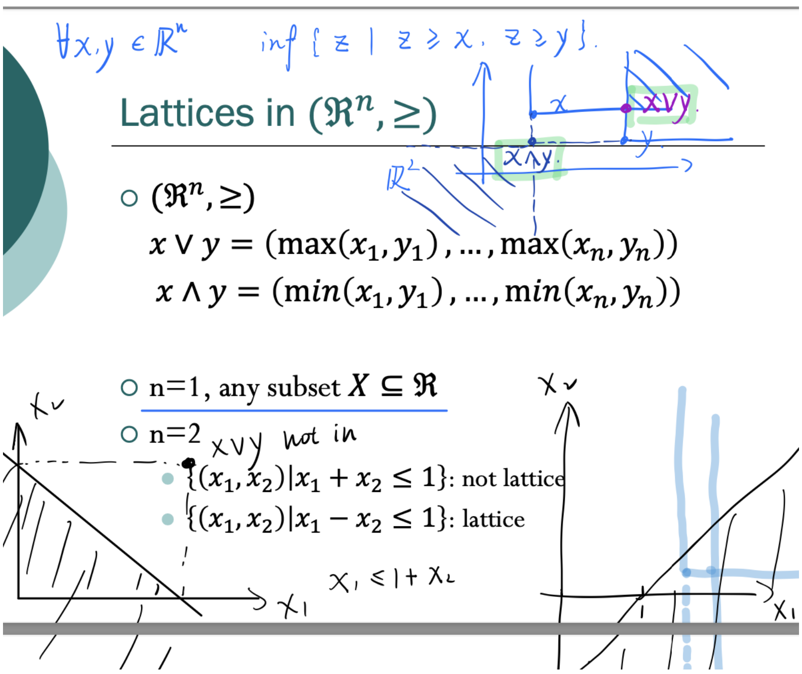


Figure 1:

Example 1. *Lattices:*

1. $\{0, 1\}^n$
2. Z^n
3. *a chain is a lattice (whose elements are ordered)*
4. *Intersection of two lattices*

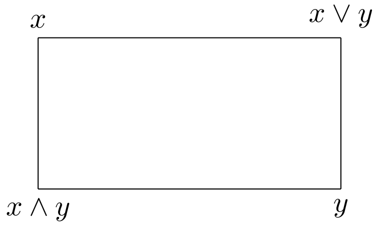
1.2 Supermodularity

1.2.1 Definition: Supermodular $g(x \vee y) + g(x \wedge y) \geq g(x) + g(y), \forall x, y \in X$

Definition 3. *A function $g : X \to \bar{\Re}(= \Re \cup \{+\infty\})$ is submodular if*

$$g(x \vee y) + g(x \wedge y) \leq g(x) + g(y), \forall x, y \in X$$

g is supermodular if $-g$ is submodular.



Claim 1. $\mathrm{dom}(g) = \{x \in X \mid g(x) < +\infty\}$ *is a lattice if g is submodular.*

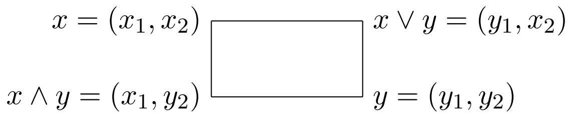
Proof. $\forall x, y \in \mathrm{dom}(g)$, prove $g(x \vee y) < +\infty$, $g(x \wedge y) < +\infty$. $\square$

1.2.2 Lemma: Supermodular $\Leftrightarrow \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0, \forall i \neq j$

Lemma 1. *Suppose g is twice partially differentiable in $\Re^n$. Then g is supermodular if and only if it has nonnegative cross partial derivatives, i.e.,*

$$\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0, \forall i \neq j$$

Proof.



$$x_1 \leq y_1;\ y_2 \leq x_2$$

$$\begin{aligned}
& g \text{ is supermodular} \\
\Leftrightarrow & g(x \vee y) - g(x) \geq g(y) - g(x \wedge y), \forall x, y \in X \\
& g(y_1, x_2) - g(x_1, x_2) \geq g(y_1, y_2) - g(x_1, y_2), \forall x, y \in X \\
& (\text{if } y_1 \to x_1,\ y_2 \text{ kept unchanged}) \\
& \frac{\partial g(x_1, x_2)}{\partial x_1} \geq \frac{\partial g(x_1, y_2)}{\partial x_1} \\
& (\text{if } y_2 \to x_2,\ y_2 \leq x_2) \\
& \frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0,\ \forall i \neq j
\end{aligned}$$

$\square$

Note: Supermodularity $\approx$ Economic Complementarity
g is the profit function of selling products x_1 and x_2, $\frac{\partial}{\partial x_2}(\frac{\partial g(x_1, x_2)}{\partial x_1}) \geq 0$

Example 2 (Examples of Supermodular Functions)**.**

1. $f(x) = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ $(\alpha_i \geq 0)$ *for* $x \geq 0$
2. $f(x, z) = \sum_{i=1}^n g_i (\alpha_i x_i - \beta_i z_i)$ *for any univariate concave function* $g_i : \Re \to \bar{\Re}$ $(\alpha_i \beta_i \geq 0)$
3. $f(x) = \sum_{i,j=1}^n a_{ij} x_i x_j = x^T A x$ *with* $a_{ij} = a_{ji}$ *is supermodular if and only if* $a_{ij} \geq 0$ $\forall i \neq j$

1.2.3 Lemma: Preservation of Supermodularity

Lemma 2 (Preservation of Supermodularity)**.**

a) If f_i is supermodular, then $\lim_{i\to\infty} f_i(x), \sum_i \alpha_i f_i\,(\alpha_i \geq 0)$ are supermodular

b) If $f : \Re \to \Re$ is convex and nondecreasing (nonincreasing) and $g : \Re^n \to \Re$ is increasing and supermodular (submodular), then $f(g(x))$ is supermodular

c) Given $f : \Re^n \times \Re^m \to \Re$, if $f(\cdot, y)$ is supermodular for all y, then $E_\xi[f(x, \xi)]$ is supermodular in x

Lemma 3 (Supermodularity of composite functions)**.**
If $X = \Pi_{i=1}^n X_i$ and $X_i \subseteq \Re, f_i\,(x_i) : X_i \to \Re$ is increasing (decreasing) on X_i for $i = 1, \ldots, n$, and $g\,(z_1, \ldots, z_n) : \Re^n \to \bar{\Re}$ is supermodular in $(z_1, \ldots, z_n)$, then

$$g\left(f_1\left(x_1\right), \ldots, f_n\left(x_n\right)\right)$$

is supermodular on X

Lemma 4 (Topkis 1998)**.** *If X is a lattice, $f_i(x)$ is increasing and supermodular (submodular) on X for $i = 1, \ldots, k, Z_i$ is a convex subset of R^1 containing the range of $f_i(x)$ on X or $i = 1, \ldots, k$, and $g\,(z_1, \ldots, z_k, x)$ is supermodular in $(z_1, \ldots, z_k, x)$ and is increasing (decreasing) and convex in z_i for fixed z_{-i} and x, then $g\,(f_1(x), \ldots, f_k(x), x)$ is supermodular on X*

1.3 Parametric Optimization Problems

Definition 4.

$$\begin{aligned} f(s) = &\max g(s, a) \\ &s.t.\ a \in A(s) \end{aligned}$$

S: subset of $\Re^m$
$A(s)$: finite dimensional
$C := \{(s, a) \mid s \in S, a \in A(s)\}$ (the graph of the constraint operator)
$A^*(s)$, the optimal solution set, is nonempty for every $s \in S$

Definition 5. *A set $A(s)$ is* ***ascending on*** *S if for $s \leq s'$, $a \in A(s)$, $a' \in A(s')$, we have $a \wedge a' \in A(s)$ and $a \vee a' \in A(s')$.*

Example 3. *$A(s) = [s, +\infty)$ is ascending on S.*

1.3.1 Theorem: Maximizer of supermodular func is ascending, the maximum value is also supermodular

Theorem 1 (Ascending Optimal Solutions and Preservation)**.**
If

1. *S: sublattice of $\Re^m$*
2. *$C := \{(s, a) \mid s \in S, a \in A(s)\}$ is a sublattice*
3. *g is supermodular on C*

Then

1. *$A^*(s)$ is **ascending** on S. Under some conditions, the largest/smallest element of $A^*(s)$ exists, and is increasing in* s.

2. *$f(s)$ is supermodular.*

Proof. Take $s \leq s'$, $a^* \in A^*(s)$, $a'^* \in A^*(s')$, i.e.

$$g(s,a^*) = \max g(s,a) \text{ s.t. } a \in A(s)$$
$$g(s',a'^*) = \max g(s',a) \text{ s.t. } a \in A(s')$$

$$(s,a^*) \vee (s',a'^*) = (s', a^* \vee a'^*)$$
$$(s,a^*) \wedge (s',a'^*) = (s, a^* \wedge a'^*)$$

As we know C is a sublattice, we have

$$(s', a^* \vee a'^*) \in C \Rightarrow a^* \vee a'^* \in A(s')$$
$$(s, a^* \wedge a'^*) \in C \Rightarrow a^* \wedge a'^* \in A(s)$$

Hence,

$$g(s', a^* \vee a'^*) \leq g(s', a'^*);\; g(s, a^* \wedge a'^*) \leq g(s, a^*)$$

Since g is supermodular on C,

$$g(s', a^* \vee a'^*) + g(s, a^* \wedge a'^*) \geq g(s,a^*) + g(s',a'^*)$$
$$0 \geq g(s', a^* \vee a'^*) - g(s',a'^*) \geq g(s,a^*) - g(s, a^* \wedge a'^*) \leq 0$$

Hence,

$$g(s', a^* \vee a'^*) = g(s', a'^*);\; g(s, a^*) = g(s, a^* \wedge a'^*)$$

which means,

$$a^* \vee a'^* \in A^*(s'),\; a^* \wedge a'^* \in A^*(s)$$

Then, " $A^*(s)$ is ascending on S " is proved.
What's more, the largest elements of $A(s')$ and $A(s)$ are $a^* \vee a'^*$ and a^*, the smallest elements of $A(s')$ and $A(s)$ are a'^* and $a^* \wedge a'^*$, which are both increased as s increases to s'. □

Proof. $\forall s, s' \in S$, $a \in A^*(s), a' \in A^*(s')$.

$$\begin{aligned} f(s) + f(s') &= g(s,a) + g(s,a') \\ &\quad \text{(Since } g \text{ is supermodular on } C\text{)} \\ &\leq g(s \wedge s', a \wedge a') + g(s \vee s', a \vee a') \\ &\leq f(s \wedge s') + f(s \vee s') \end{aligned}$$

" $f(s)$ is supermodular " is proved. □

Example 4. *Pricing:* $p^*(c) = \text{argmax}_{p \geq c'}(p-c)D(p)$, $(c' > c)$

1. $C = \{(p,c) | c < c', p \geq c'\}$ is a sublattice of $\mathbb{R}^2$.
2. $g(p,c) = (p-c)D(p)$, $\frac{\partial^2 g(p,c)}{\partial p \partial c} = -D'(p) \geq 0 \Rightarrow g$ is supermodular on C.
Hence, $p^*(c)$ is increasing in c.

Example 5. *Newsvendor model:* $\min_{x \geq 0} f(x) = cx + h_+ E[(x-\xi)^+] + h_- E[(\xi - x)^+]$

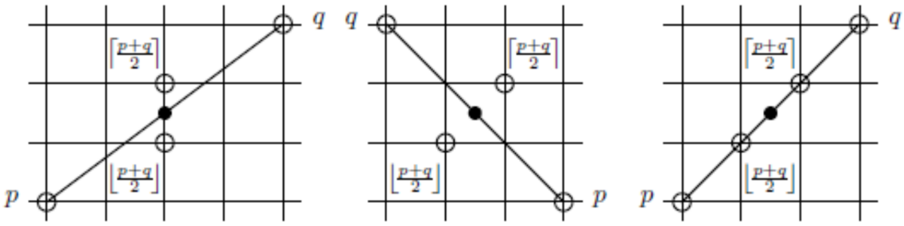


Figure 2: Discrete Midpoint Convexity

2 $L^\natural$-Convexity

2.1 Discrete Midpoint Convexity

Definition 6. *A function f is **discrete midpoint convexity** if*

$$f(\lceil \frac{p+q}{2} \rceil) + f(\lfloor \frac{p+q}{2} \rfloor) \leq f(p) + f(q)$$

2.2 $L^\natural$-Convexity on $\mathbb{Z}^n$

Definition 7. *A function $f : Z^n \to \bar{\Re}$ is called $L^\natural$ convex if f satisfies the discrete midpoint convexity.*

An equivalent definition: A function $f : Z^n \to \bar{\Re}$ is $L^\natural$-convex if and only if

$$g(x, \alpha) := f(x - \alpha e) = f([x_1 - \alpha, x_2 - \alpha, ..., x_n - \alpha]^T)$$

is submodular in (x, α) on Z^{n+1}(e : all-ones vector).

2.3 $L^\natural$-Convexity on $\mathcal{F}^n(\mathcal{F} = \mathbb{Z}$ or $\Re)$

Definition 8 (Murota 2003)**.**
A function $f : \mathcal{F}^{\mathrm{n}} \to \Re$ is $L^\natural$-convex if and only if $g(x, \xi) := f(x - \xi e)$ is submodular in $(x, \xi) \in \mathcal{F}^{\mathrm{n}} \times S$, where e is a vector with all components equal to 1 and S is the intersection of $\mathcal{F}$ with any unbounded interval in $\Re$. (f is required to be convex if $\mathcal{F} = \Re$)

Definition 9. *A set V is $L^\natural$-convex if and only if its indicator function $\delta_V(x)$ is $L^\natural$.*

$$\delta_V(x) = \begin{cases} +\infty & , x \notin V \\ 0 & , x \in V \end{cases}$$

$\Leftrightarrow$ *$g(x, \xi) = \delta_V(x - \xi e)$ is subnormal, i.e.*

$$g(x \vee y, \max\{\xi_x, \xi_y\}) + g(x \wedge y, \min\{\xi_x, \xi_y\}) \leq g(x, \xi_x) + g(y, \xi_y), \ \forall (x, \xi_x), (y, \xi_y)$$

If $x - \xi_x e, y - \xi_y e$ in V, $x \vee y - \max\{\xi_x, \xi_y\}e$ and $x \wedge y - \min\{\xi_x, \xi_y\}$ must in V.

Note: f is $L^\natural$-concave if $-f$ is $L^\natural$-convex.

2.4 Properties of $L^\natural$-Convexity

2.4.1 Proposition: $L^\natural$-convex $\Leftrightarrow a_{ij} \leq 0, \forall i \neq j, a_{ii} \geq 0, \sum_{j=1}^n a_{ij} \geq 0, \forall i$

Proposition 1. *A quadratic function $f(x) = \sum_{i,j=1}^n a_{ij} x_i x_j$ with $a_{ij} = a_{ji}$ is $\underline{L^\natural\text{-convex on } \mathcal{F} \text{ if and only if}}$ its Hessian is a diagonally dominated M-matrix*

$$a_{ij} \leq 0 \ \forall i \neq j, \quad a_{ii} \geq 0, \quad \sum_{j=1}^n a_{ij} \geq 0 \ \forall i$$

Proof.
$f(x)$ is $L^\natural$-convex $\Leftrightarrow g(x,\xi) = f(x - \xi e) = \sum_{i,j=1}^n a_{ij}(x_i - \xi)(x_j - \xi)$ is submodular in (x,ξ) i.e.

$$\frac{\partial^2 g}{\partial \xi \partial x_i} = \frac{\partial}{\partial \xi}(\sum_{j=1}^n a_{ij}(x_j - \xi) + \sum_{j=1}^n a_{ji}(x_j - \xi)) = -2\sum_{j=1}^n a_{ij} \leq 0$$
$$\frac{\partial^2 g}{\partial x_j \partial x_i} = \frac{\partial}{\partial x_k}(\sum_{k=1}^n a_{ik}(x_k - \xi) + \sum_{k=1}^n a_{ki}(x_k - \xi)) = 2a_{ij} \leq 0$$

□

Proposition 2. *A twice continuous differentiable function $f : \Re^n \to \Re$ is $\underline{L^\natural\text{-convex if and only if}}$ its Hessian is a diagonally dominated M-matrix, that is*

$$a_{ij} \leq 0, \forall i \neq j, a_{ii} \geq 0, \sum_{j=1}^n a_{ij} \geq 0, \forall i$$

Proof.

$L^\natural$-convex $\Leftrightarrow g(x,\xi) = f(x - \xi e)$ is subnormal
(if twice differentiable)
$\Leftrightarrow \frac{\partial^2 g(x,\xi)}{\partial x_i \partial x_j} = \frac{\partial^2 f(x-\xi e)}{\partial x_i \partial x_j} \leq 0, i \neq j,\ \frac{\partial^2 g(x,\xi)}{\partial x_i \partial \xi} = -\sum_{j=1}^n \frac{\partial^2 f(x-\xi e)}{\partial x_i \partial x_j} \leq 0,\ \forall (x,\xi) \in \Re^{n+1}$
$\Leftrightarrow \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, i \neq j;\ \sum_{j=1}^n \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \ \forall x \in \Re^n$ □

2.4.2 Corollary: $L^\natural$-convex $\longrightarrow$ convex + submodular

Corollary 1. *If a twice differentiable function f is $L^\natural$-convex, then the function is convex and submodular.*

Proof.

$\underline{a_{ij} = \frac{\partial^2 f(x)}{\partial x_i \partial x_j} \leq 0, i \neq j}$ means the cross partial derivatives are nonpositive, which equals to $\underline{f \text{ is submodular}}$.

$$
\begin{aligned}
x^T \nabla^2 f(x) x &= \sum_{i,j=1}^{n} a_{ij} x_i x_j \\
&= \sum_{k}^{n} a_{kk} x_k^2 + \sum_{j=1}^{n} \sum_{i<j} a_{ij} 2 x_i x_j \\
&\geq \sum_{k}^{n} a_{kk} x_k^2 + \sum_{j=1}^{n} \sum_{i<j} a_{ij} (x_i^2 + x_j^2) \\
&\geq \sum_{k}^{n-1} a_{kk} x_k^2 + \sum_{j=1}^{n-1} \sum_{i<j} a_{ij} (x_i^2 + x_j^2) \\
& \dots \\
&\geq 0, \quad \forall x \in \Re^n
\end{aligned}
$$

Then f is convex. □

Example 6.

- Given any univariate (discrete) convex function $g_i : \mathcal{F} \to \bar{\Re}$ and $h_{ij} : \mathcal{F} \to \Re$, the function $f : \mathcal{F}^n \to \bar{\Re}$ defined by

$$f(x) := \sum_{i} g_i(x_i) + \sum_{i \neq j} h_{ij}(x_i - x_j)$$

is $L^\natural$-convex.

Example 7.

- A set with a representation

$$\{x \in \mathcal{F}^n : l \leq x \leq u, x_i - x_j \leq v_{ij}, i \neq j\}$$

is $L^\natural$-convex, where $l, u \in \mathcal{F}^n, v_{ij} \in \mathcal{F}$.

2.4.3 Theorem: Minimizer of $L^\natural$-convex func is nondecreasing with bounded sensitivity, the minimum value is also $L^\natural$-convex

Theorem 2. *Assume $g : \mathcal{F}^n \times \mathcal{F}^m \to \overline{\Re}$ and set $C \subset \mathcal{F}^n \times \mathcal{F}^m$ are $L^\natural$-convex, define*

$$f(s) = \inf_{a:(s,a) \in C} g(s, a)$$

Then,

1. *The optimal solution set $A^*(s)$ is nondecreasing in s with bounded sensitivity i.e.,*

$$A^*(s + \omega e) \leq A^*(s) + \omega e, \ \forall \omega \in F_+$$

(Zipkin 2008, Chen et al. 2018)

2. *f is $L^\natural - convex$. (Zipkin 2008)*

2.5 Relationship with Multimodularity

Definition 10. *A function $f(x_1, x_2, ..., x_n)$ is multimodular if $f(x_1 - x_0, x_2 - x_1, ..., x_n - x_{n-1})$ submodular in $(x_0, x_1, ..., x_n)$.*

Multimodularity and $L^\natural$-convexity are equivalent subject to an unimodular linear transformation.

3 Optimization with decisions truncated by random variables

$$\min_{u \in \mathcal{U}} E[f(u \wedge \xi)]$$

Question 1 (Supply uncertainty in SCM): u: ordering quantities; ξ: random capacities.
Question 2 (Demand uncertainty in RM): u: booking limits; ξ: random demands.
Difficulty: the object function is not convex (even if f is).

3.1 Unconstrained Problem

Consider

$$\tau^* = \min_{u \in \mathcal{F}^n} E[f(u \wedge \xi)]$$

$\mathcal{F}$ is either the real space or the set with all integers.
Random vector $\xi \in \mathcal{X} \subseteq \mathcal{F}^n$

3.1.1 Reformulation

Reformulation:

$$\begin{array}{lll} \min & E[f(v(\xi))] & \\ s.t. & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) & , \forall \xi \in \mathcal{X} \\ & v(\xi) = u \wedge \xi & , \forall \xi \in \mathcal{X} \end{array}.$$

Turn finding u^* into finding $v^*()$
$v()$ is not convex.

Theorem 3 (Equivalent Transformation, Chen, Gao and Pang 2018)**.** *Suppose that (Assumption I)*
(a) the function f is lower semi-continuous with $f(u) \to +\infty$ for $|u| \to +\infty$;
(b) the function f is componentwise (discrete) convex;
(c) the random vector ξ has independent components.
Then τ^* is also the optimal objective value of the following optimization problem:

$$\begin{array}{lll} \min & E[f(v(\xi))] & \\ s.t. & v(\xi) \leq \xi & , \forall \xi \in \mathcal{X} \\ & v(\xi) = (v_1(\xi_1), ..., v_n(\xi_n)) & , \forall \xi \in \mathcal{X} \end{array}.$$

3.1.2 $n = 1$

$\hat{u}$: minimizer of $f(u)$
Need to show

$$\min_u E[f(u \wedge \xi)] = \min_{v(\xi) \leq \xi} E[f(v(\xi))]$$

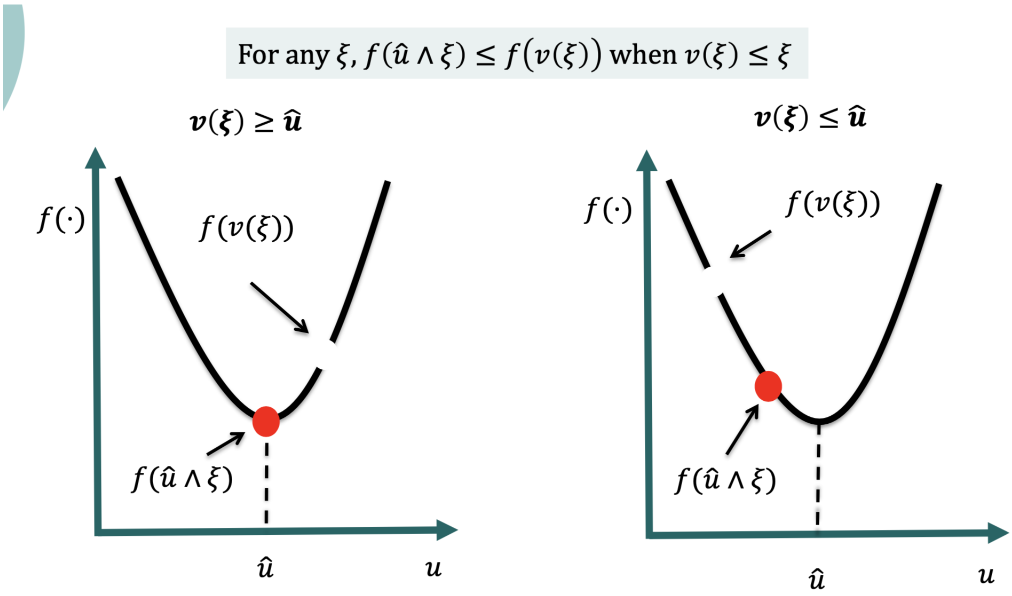


Figure 3: Easy to show $\forall\xi,\ f(\hat{u} \wedge \xi) \leq f(v(\xi))$ when $v(\xi) \leq \xi$

Easy to show $\forall\xi,\ f(\hat{u} \wedge \xi) \leq f(v(\xi))$ when $v(\xi) \leq \xi$. Then

$$\operatorname{argmin} E[f(u \wedge \xi)] = \hat{u} = \operatorname{argmin} f(u)$$

$$\begin{aligned} E[f(\hat{u} \wedge \xi)] &\geq \min_u E[f(u \wedge \xi)] \\ &\geq \min_{v(\xi) \leq \xi} E[f(v(\xi))] \text{ (Consider } v^*(\xi) \geq u) \\ &\geq E[f(\hat{u} \wedge \xi)] \text{ (See the figure)} \\ \Rightarrow \quad & \min_u E[f(u \wedge \xi)] = \min_{v(\xi) \leq \xi} E[f(v(\xi))] \end{aligned}$$

3.1.3 $n \geq 2$

$$\operatorname{argmin} E[f(u \wedge \xi)] \neq \hat{u}$$

Example 8.

$$f(u_1, u_2) = (u_1 + u_2 - 2)^2 + (u_1 - 1)^2 + (u_2 - 1)^2$$

ξ_1, ξ_2 *can take values* 0 *and* 2 *with equal probability.*
$\hat{u} = (1, 1)$
$\operatorname{argmin} E[f(u \wedge \xi)] = (1.2, 1.2)$

3.2 Transformation for Constrained Problem

$$\min_{u \in \mathcal{U}} E[f(u \wedge \xi)]$$

$$\downarrow$$

$$
\begin{array}{lll}
\min & E[f(v(\xi))] & \\
s.t. & v(\xi)\leq \xi & ,\forall \xi \in \mathcal{X} \\
& v(\xi)= (v_1(\xi_1),...,v_n(\xi_n)) \in \mathcal{V} & ,\forall \xi \in \mathcal{X} \\
& \mathcal{V}= \{u \wedge \xi \mid u \in \mathcal{U}, \xi \in \mathcal{X}\} &
\end{array}
.
$$

Sufficient Conditions for the Transformation

(a) $\mathcal{U} = \{u|Au \leq b, u \geq l\}$, where $A \geq 0$

(b) $\mathcal{X}_j \subseteq [l_j, +\infty)$

(Example: some situations $l = (l_1, ..., l_n) = (0, ..., 0)$)

3.3 Generalization

$$\min_{u\in\mathcal{F}^n} l(u) + E[f(u \wedge \xi)]$$

- $l : \mathcal{F}^n \to \bar{\Re}$, $f : \mathcal{F}^n \to \bar{\Re}$
- $\xi \in \mathcal{X} \subseteq \mathcal{F}^n$
- ξ **dependent** (different from before !)

3.3.1 Positive Dependence

Let F_{ξ_i} be the joint CDF of $\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n$ conditioned on ξ_i
$\{\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \mid \xi_i\}$ is stochastically increasing if $\int_S dF_{\xi_i}(w)$ is an increasing function of ξ_i for each increasing set S
$\{\xi_1, \ldots, \xi_{i-1}, \xi_i, \xi_{i+1}, \ldots, \xi_n\}$ has positive dependence if $\{\xi_1, \ldots, \xi_{i-1}, \xi_{i+1}, \ldots, \xi_n \mid \xi_i\}$ is stochastically increasing for all i

Proposition 3. *The collection of random variables generated by nonnegative linear combination of* ***independent log-concave*** *random variables has positive dependence.*

3.3.2 Transformation

Theorem 4 (Equivalent Transformation, Chen and Gao 2018)**.** *Suppose that (Assumption II)*

(1) the function f is lower semi-continuous with $f(u) \to +\infty$ for $|u| \to +\infty$;

(2) the function f is componentwise (discrete) convex and supermodular;

(3) the random vector ξ is positive dependent;

(4) $l(u)$ is componentwise increasing.

Then problem $\min_{u\in\mathcal{F}} l(u) + E[f(u \wedge \xi)]$ *has the* ***same optimal objective value*** *of*

$$
\begin{array}{lll}
\min & l(u) + E[f(v(\xi))] & \\
s.t. & v(\xi)\leq \xi & ,\forall \xi \in \mathcal{X} \\
& v(\xi)\leq u & ,\forall \xi \in \mathcal{X} \\
& v(\xi)= (v_1(\xi_1),...,v_n(\xi_n)) & ,\forall \xi \in \mathcal{X} \\
& v_i(\xi_i) \text{ is increasing for all } i &
\end{array}
.
$$

4 Single-Leg Capacity Allocation

(Seats reserved for future consumers)

4.1 Two-Class Model

Two periods: Period 1, random demand D_2 for price p_2; Period 2, random demand D_1 for price p_1. $p_1 > p_2$

Provide y in period 1 and the remaining will be provided in period 2.

$$\begin{aligned} \max \quad & p_1 E_{D_1,D_2}[D_1 \wedge (c - (c-y) \wedge D_2)] + p_2 E_{D_2}[(c-y) \wedge D_2] \\ s.t. \quad & 0 \le y \le c, y \in \mathcal{F} \end{aligned} .$$

Where $\mathcal{F} = \mathbb{R}$ or $\mathbb{Z}$ and $a \wedge b = \min(a, b)$

4.1.1 Theorem: convex f, $\operatorname{argmin} E_D f(u \wedge D) = \operatorname{argmin} f(u)$

When D_2 is sufficiently high. Let $b = c - y$, and the question transferred to

$$\begin{aligned} \max \quad & v(b) = p_1 E_{D_1}[D_1 \wedge (c-b)] + p_2 b \\ s.t. \quad & 0 \le b \le c, b \in \mathcal{F} \end{aligned} .$$

$v(b)$ is a concave function.

Theorem 5. *Consider the following optimization problem*

$$\begin{aligned} \min \quad & E_D f(u \wedge D) \\ s.t. \quad & 0 \le u \le c, u \in \mathcal{F} \end{aligned} .$$

Assume D is a nonnegative random variable.
*If f is **convex** and $\mathcal{F} = \mathbb{R}$ or f is **discrete convex** and $\mathcal{F} = \mathbb{Z}$, then any optimal solution of*

$$\begin{aligned} \min \quad & f(u) \\ s.t. \quad & 0 \le u \le c, u \in \mathcal{F} \end{aligned} .$$

is also optimal for the former optimization problem.
(Actually, quasi-convexity suffices)

According to the $n = 1$ discussion of section 3, the theorem is easy to be proved. Then, the global-max in $v(b)$ is global-max for objective function.
Then we consider the equivalent minimum problem,

$$\begin{aligned} \max \quad & \phi(y) = p_2 y - p_1 E_{D_1}[D_1 \wedge y] \\ s.t. \quad & 0 \le y \le c, y \in \mathcal{F} \end{aligned} .$$

We need to find the optimal y^* minimize the $\phi(y)$. To simplify the analysis, we find the y° which is the optimal y regardless constraints. $y^* = \begin{cases} 0 & \text{if } y^\circ < 0 \\ y^\circ & \text{if } y^\circ \in [0, c] \\ c & \text{if } y^\circ > c \end{cases}$

4.1.2 Discrete, $\mathcal{F} = \mathbb{Z}$

$$\phi(y) - \phi(y-1) = p_2 - p_1 P(D_1 \geq y)$$

Then, the y° is

$$\begin{aligned} \overline{y} &= \min\{y \in \mathbb{Z} : P(D_1 > y) < r\} \\ \underline{y} &= \max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\} \ \textbf{(Littlewood's rule)} \\ y^\circ &= [\underline{y}, \overline{y}] \cap \mathbb{Z} \end{aligned}$$

Where $r = \frac{p_2}{p_1}$, higher r causes lower y°.

Example 9. *Suppose that D_1 is a Poisson random variable with mean 80, the full fare is $p_1 = 100$ and the discounted fare is $p_2 = 60$*

$r = 60/100 = 0.6$, $y^* = \max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\} = 78$

4.1.3 Continuous, $\mathcal{F} = \mathbb{R}$

y° is the y s.t. $1 - F_1(y) = r$, where $F_1(\cdot)$ is the CDF of D_1.

$$y^\circ = F_1^{-1}(1 - r)$$

Special Case: $D_1 \sim \mathcal{N}(\mu, \sigma^2)$

$$F_1(y) = \Phi(\frac{y - \mu}{\sigma})$$

$\Phi(\cdot)$ is the CDF of the standard normal $\mathcal{N}(0, 1)$. Then,

$$y^\circ = \mu + \sigma \Phi^{-1}(1 - r)$$

If $\frac{p_2}{p_1} = r < \frac{1}{2}$, y° increases as variance σ increases.

4.2 Multi-Class Model

- $p_1 > p_2 > \cdots > p_n$
- Lower class demand arrives earlier.
- Demand of different classes are independent.
- Control: demand to accept or reject.

4.2.1 Sequence of Events

At stage j with remaining capacity x,

1. Select booking limit b for class j, equivalently, protection level $y = x - b$ for classes l, $l < j$.
2. Demand D_j is realized.
3. Accept $b \wedge D_j$ of class j and collect revenue $p_j(b \wedge D_j)$.
4. Move on to stage $j - 1$ with remaining capability $x - b \wedge D_j$.

4.2.2 Dynamic Programming

Set $f_j(x,b) = p_j b + V_{j-1}(x-b)$, $V_0(x) = 0, V_j(0) = 0$, $x = 0, 1, ..., c$ (discrete), $x \in [0, c]$ (continuous)

$$V_j(x) = \max_{b\in[0,x], b\in\mathcal{F}} \mathbb{E}[f_j(x, b\wedge D_j)] = \mathbb{E}[p_j(b\wedge D_j)] + \mathbb{E}[V_{j-1}(x - b\wedge D_j)]$$

Proposition 4. *(1). $\forall j$, f_j is $L^\natural-$concave, V_j is (discrete) convex; (2). The optimal solution of the dynamic programming b_j^* is the same as*

$$\max_{b\in[0,x], b\in\mathcal{F}} f_j(x,b) = p_j b + V_{j-1}(x-b)$$

Define y_{j-1}^* be the optimal solution of

$$\max_{y\geq 0, y\in\mathcal{F}} -p_j y + V_{j-1}(y)$$

Then

$$b_j^* = (x - y_{j-1}^*)^+$$

$$\begin{aligned}
V_j(x) &= \mathbb{E}[f_j(x, (x-y_{j-1}^*)^+ \wedge D_j)] \\
&= \mathbb{E}[p_j(x-y_{j-1}^*)^+ \wedge D_j + V_{j-1}(x - (x-y_{j-1}^*)^+ \wedge D_j)] \\
&= \begin{cases} V_{j-1}(x) & \text{if } x \leq y_{j-1}^* \\ \mathbb{E}[p_j(x-y_{j-1}^*) \wedge D_j + V_{j-1}(x - (x-y_{j-1}^*) \wedge D_j)] & \text{if } x > y_{j-1}^* \end{cases}
\end{aligned}$$

4.3 Discrete Case

Define

$$\Delta V_j(x) = V_j(x) - V_j(x-1)$$

Lemma 5. *If $x > y_{j-1}^*$, $\Delta V_j(x) = \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x - D_j)\}]$*

Proof.

$$\begin{aligned}
\Delta V_j(x) &= p_j(\mathbb{E}[(x-y_{j-1}^*)\wedge D_j] - \mathbb{E}[(x-1-y_{j-1}^*)\wedge D_j]) \\
&\quad + \mathbb{E}[V_{j-1}(x - (x-y_{j-1}^*)\wedge D_j)] - \mathbb{E}[V_{j-1}(x-1-(x-1-y_{j-1}^*)\wedge D_j)] \\
&= \begin{cases} p_j & \text{if } x - y_{j-1}^* \leq D_j \\ \Delta V_{j-1}(x - D_j) & \text{if } x - y_{j-1}^* > D_j \end{cases} \\
&= \mathbb{E}[p_j \mathbb{I}(x - D_j \leq y_{j-1}^*) + \Delta V_{j-1}(x-D_j)\mathbb{I}(x - D_j > y_{j-1}^*)] \\
&\quad (\text{ Since } y_{j-1}^* \text{ maximizes } -p_j y + V_{j-1}(y), \\
&\quad \Delta V_{j-1}(y) > p_j \text{ if } y \leq y_{j-1}^* \text{ and } \Delta V_{j-1}(y) \leq p_j \text{ if } y > y_{j-1}^*) \\
&= \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x-D_j)\}]
\end{aligned}$$

□

Proposition 5 (1.5 of GT 19)**.**

(i) $\Delta V_j(x+1) \leq \Delta V_j(x)$ *(proved by V_j is discrete concave)*

(ii) $\Delta V_{j+1}(x) \geq \Delta V_j(x)$

Proof.
If $x \leq y^*_{j-1}$,

$$\Delta V_j(x) = V_{j-1}(x) - V_{j-1}(x-1) = \Delta V_{j-1}(x)$$

If $x > y^*_{j-1}$ (i.e. $x - 1 \geq y^*_{j-1}$),

$$\begin{aligned}\Delta V_j(x) &= \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x - D_j)\}] \\ &\text{(} V_{j-1}(x) \text{ is discrete concave)} \\ &\geq \mathbb{E}[\min\{p_j, \Delta V_{j-1}(x)\}] \\ &\text{(Since } x > y^*_{j-1}, V_{j-1}(x) < p_j) \\ &= \Delta V_{j-1}(x)\end{aligned}$$

□

Theorem 6 (part of 1.6 of GT 19)**.**
The optimal protection level at stage j is

$$y^*_{i-1} = \max\{y \in \mathbb{N}_+ : \Delta V_{j-1}(y) > p_j\}$$

Moreover, $y^*_{n-1} \geq y^*_{n-2} \geq \cdots \geq y^*_1 = y^*_0 = 0$

(Easy to prove: Since y^*_{j-1} maximizes $-p_j y + V_{j-1}(y)$, $\Delta V_{j-1}(y) > p_j$ if $y \leq y^*_{j-1}$ and $\Delta V_{j-1}(y) \leq p_j$ if $y > y^*_{j-1}$)
Note: Littlewood's rule is a special case for $n = 2$.

4.3.1 Discrete Case: Reformulation

$$\begin{aligned}V_j(x) &= \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j + V_{j-1}(x - (x - y^*_{j-1})^+ \wedge D_j)] \\ &= V_{j-1}(x) + \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j + (V_{j-1}(x - (x - y^*_{j-1})^+ \wedge D_j) - V_{j-1}(x))] \\ &= V_{j-1}(x) + \mathbb{E}[p_j(x - y^*_{j-1})^+ \wedge D_j - \sum_{z=1}^{(x-y^*_{j-1})^+ \wedge D_j} \Delta V_{j-1}(x + 1 - z)] \\ &= V_{j-1}(x) + \mathbb{E}[\sum_{z=1}^{(x-y^*_{j-1})^+ \wedge D_j} (p_j - \Delta V_{j-1}(x + 1 - z))]\end{aligned}$$

$$\begin{aligned}V_j(x) &= V_{j-1}(x) + \mathbb{E}[\sum_{z=1}^{u^*}(p_j - \Delta V_{j-1}(x + 1 - z))] \\ u^* &= \min\{(x - y^*_{j-1})^+, D_j\} \\ y^*_{i-1} &= \max\{y \in \mathbb{N}_+ : p_j < \Delta V_{j-1}(y)\}\end{aligned}$$

- $y^*_1 \leq y^*_2 \leq \cdots \leq y^*_n$
- The "nested" booking limit $b^*_j = C - y^*_{j-1}$, $j = 2, ..., n$
  (nested booking limit is the total amount can be booked in $j, j+1, ..., n$)

$$b^*_j = y_j$$

- The marginal utility at j of choosing to reserve one more item in the next stage $j - 1$:

$$\pi_j(x) = \Delta V_{j-1}(x)$$

- The amount of selling at stage j

$$u^* = \begin{cases} 0 & \text{if } p_j < \pi_j(x) \\ \min\{\max\{z : p_j \geq \pi_j(x-z)\}, D_j\} & \text{if } p_j \geq \pi_j(x) \end{cases}$$

$p_j < \pi_j(x)$ means the marginal utility of reserving is larger than selling it now.

We can further compute, if $x > y_{j-1}^*$,

$$\Delta V_j(x) = p_j Pr(D_j \geq x - y_{j-1}^*) + \sum_{k=0}^{x-y_{j-1}^*-1} \Delta V_{j-1}(x-k) Pr(D_j = k)$$

If $x \leq y_{j-1}^*$, $\Delta V_j(x) = \Delta V_{j-1}(x)$.
Which will simplify the computation.

4.3.2 Discrete Case: Computation

The policy is implemented as follows:

1. At stage n, we start with $x_n = c$ units of inventory and we protect $y_{n-1}(x_n) = \min\left\{y_{n-1}^*, x_n\right\}$ units of capacity for fares $n-1, n-2, \ldots, 1$.
2. Therefore, we allow up to $\left[x_n - y_{n-1}^*\right]^+$units of capacity to be sold to fare class n.
3. We sell $\min\left\{\left[x_n - y_{n-1}^*\right]^+, D_n\right\}$ units of capacity to fare class n and we have a remaining capacity of $x_{n-1} = x_n - \min\left\{\left[x_n - y_{n-1}^*\right]^+, D_n\right\}$ at stage $n-1$.
4. We protect $y_{n-2}(x_{n-1}) = \min\left\{y_{n-2}^*, x_{n-1}\right\}$ units of capacity for fares $n-2, n-1, \ldots, 1$.
5. Therefore, we allow up to $\left[x_{n-1} - y_{n-2}^*\right]^+$units of capacity to be sold to fare class $n-1$.
6. We continue until we reach stage 1 with x_1 units of capacity, allowing $(x_1 - y_0)^+ = (x_1 - 0)^+ = x_1$ to be sold to fare class 1.

$$V_j(x) = \mathbb{E}[p_j \min\{(x - y_{j-1}^*)^+, D_j\} + V_{j-1}(x - \min\{(x - y_{j-1}^*)^+, D_j\})]$$

$y_0^* = 0, V_0(x) = 0$, then we can compute $y_1^*, V_1(x)$,...
Backward: Use

$$\Delta V_j(x) = p_j Pr(D_j \geq x - y_{j-1}^*) + \sum_{k=0}^{x-y_{j-1}^*-1} \Delta V_{j-1}(x-k) Pr(D_j = k)$$
$$y_{j-1}^* = \max\{y \in \mathbb{N}_+ : p_j < \Delta V_{j-1}(y)\}$$

1. $V_1(x_1) = \mathbb{E}[p_1 \min\{x_1, D_1\}]$, then $\Delta V_1(x) = p_1 Pr(D_1 \geq x)$
2. $y_1^* = \max\{y \in \mathbb{N}_+ : p_2 < \Delta V_1(y)\} = \max\{y : Pr(D_1 \geq y) > \frac{p_2}{p_1}\}$

$$\Delta V_2(x) = p_2 Pr(D_2 \geq x - y_1^*) + \sum_{k=0}^{x-y_1^*-1} p_1 Pr(D_1 \geq x-k) Pr(D_2 = k)$$

3. $y_2^* = \max\{y \in \mathbb{N}_+ : p_3 < \Delta V_2(y)\} = \max\{y : Pr(\Delta V_1(y - D_2) > p_3)\}$

4. ...

The complexity is $O(nC^2)$

Example 10. *Suppose that there are five fare classes. The demand for all fare classes is a Poisson random variable. The fares and the expected demand for the five fare classes are given by* $(p_5, p_4, p_3, p_2, p_1) = (15, 35, 40, 60, 100)$ *and* $(\mathbb{E}D_5, \mathbb{E}D_4, \mathbb{E}D_3, \mathbb{E}D_2, \mathbb{E}D_1) = (120, 55, 50, 40, 15)$*. For this problem instance, the optimal protection levels are*

1. $V_1(x_1) = \mathbb{E}[100 \min\{x_1, D_1\}]$, then $\Delta V_1(x) = 100Pr(D_1 \geq x)$
2. $y_1^* = \max\{y : Pr(D_1 \geq y) > \frac{3}{5}\} = 14$

$$\Delta V_2(x) = 60Pr(D_2 \geq x - 14) + \sum_{k=0}^{x-15} 100Pr(D_1 \geq x - k)Pr(D_2 = k)$$

3. $y_2^* = \max\{y \in \mathbb{N}_+ : p_j <$

4.4 Continuous Case

Skip

4.5 Generalized Newsvendor Problem: High-before-low arrival pattern

Consider the problem of selecting c to maximize

$$\Pi_n(c) = V_n(c) - kc$$

Where $V_n(c)$ is the expected revenue to the multi-fare RM problem.
Assume high-before-low arrival pattern. Then

$$V_n(c) = \sum_{j=1}^{n} p_j \mathbb{E}[D_j \wedge (c - D_{1:j-1})^+]$$

and

$$\Delta V_n(c) = \sum_{j=1}^{n} (p_j - p_{j+1})Pr(D_{1:j} > c)$$

Where $D_{1:j} = \sum_{l=1}^{j} D_l, p_{n+1} = 0$

4.6 Heuristics

When there are two classes, we find y^*: $\max\{y \in \mathbb{Z} : P(D_1 \geq y) > r\}$
We try to use this form to simplify our computation,
EMSR (expected marginal seat revenue)

- **EMSR - a**

$$y_k^{j+1} = \max\{y : P(D_k \geq y) > \frac{p_{j+1}}{p_k}\}, k = j, j-1, ..., 1$$

$$y_j = \sum_{k=1}^{j} y_k^{j+1}$$

- **EMSR - b**

$$\bar{p}_j = \frac{\sum_{k=1}^{j} p_k \mathbb{E}[D_k]}{\sum_{k=1}^{j} \mathbb{E}[D_k]}$$

$$y_j = \max\{y : P(\sum_{k=1}^{j} D_k \geq y) > \frac{p_{j+1}}{\bar{p}_j}\}$$

4.7 Bounds on Optimal Expected Revenue

4.7.1 Upper Bound

$$\begin{aligned}
\overline{V}(c|D) &:= \max\{\sum_{j=1}^{n} p_j x_j | \sum_{j=1}^{n} x_j \leq c, 0 \leq x_j \leq D_j, j = 1, ..., n\} \\
V_n^U(c) &:= \mathbb{E}[\overline{V}(c|D)] \\
&= \sum_{j=1}^{n} (p_j - p_{j+1}) \sum_{k=1}^{c} Pr(D_{1:j} \geq k), \quad \text{(Set } p_{n+1} = 0\text{)}
\end{aligned}$$

$$\mathbb{E}[\overline{V}(c|D)] \leq \overline{V}(c|D) = \sum_{j=1}^{n} (p_j - p_{j+1}) \min\{\overline{D}_{1:j}, c\}$$

4.7.2 Lower Bound

Using zero protection level

$$\begin{aligned}
V_n^L(c) &= \sum_{j=1}^{n} p_j \mathbb{E}[\min\{D_k, (c - D_{j+1:n})^+\}] \\
&= \sum_{j=1}^{n} (p_j - p_{j-1}) \mathbb{E}[\min\{D_{j:n}, c\}], \quad \text{(Set } p_0 = 0\text{)}
\end{aligned}$$

4.8 Dynamical Models

- $p_1 \geq p_2 \geq \cdots \geq p_n$.
- T periods.
- At most one arrival each period.
- λ_{jt}: probability of an arrival of class j in period t.
- M_t: set of offered classes.

4.8.1 Discrete Time

$$\begin{aligned}
V_t(x) &= \sum_{j \in M_t} \lambda_{jt} \max\{p_j + V_{t-1}(x-1), V_{t-1}(x)\} + (1 - \sum_{j \in M_t} \lambda_{jt}) V_{t-1}(x) \\
&= V_{t-1}(x) + \sum_{j \in M_t} \lambda_{jt} [p_j - \Delta V_{t-1}(x)]^+ \\
&= V_{t-1}(x) + R_t(\Delta V_{t-1}(x))
\end{aligned}$$

Where we set $R_t(z) = \sum_{j \in M_t} \lambda_{jt} [p_j - z]^+$, $V_t(0) = 0$, $V_0(x) = 0, \forall x \geq 0$

4.8.2 Continuous Time: Poisson arrival

$$\frac{\partial V_t(x)}{\partial t} = R_t(\Delta V_t(x))$$

4.8.3 Optimal Policy: discrete time

Let

$$a(t,x) = \max\{j : p_j \geq \Delta V(t-1,x)\}$$

Optimal to accept all fares in the active set

$$A(t,x) = \{j \in M_t : j \leq a(t,x)\}$$

and reject the remaining fare classes

4.8.4 Structural Properties

Theorem 7 (1.18 of GT)**.**

- *$V_t(x)$ is increasing in t, x.*
- *$\Delta V_t(x)$ is increasing in t and decreasing in x.*
- *$a(t,x), A(t,x)$ is increasing in x.*

If $\lambda_{jt} \equiv \lambda_j > 0$, $M_t \equiv M = \{1,...,n\}$, then

- *$V_t(x)$ is strictly increasing and concave in t.*
- *$a(t,x), A(t,x)$ is decreasing in t.*

4.8.5 Discrete Case: Computation

$$V_t(x) = V_{t-1}(x) + \sum_{j \in M_t} \lambda_{jt}[p_j - \Delta V_{t-1}(x)]^+$$

$V_0(x) = 0$, then $V_1(x)$, then $\Delta V_1(x)$.
The complexity is $O(nCT)$ ($T \approx O(C)$)

5 Network Revenue Management with Independent Demands

5.1 Settings

- m resources with initial capacities $c = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{pmatrix} \in \mathbb{Z}_+^m$
- Time from $T, T-1, T-2, ...$ to 0.
- ODF kj: Itineraries $k = 1, ..., K$; Possible fares for itinerary k, p_{kj}, $j \in \{1, ..., n_k\}$. (Every itinerary may have n_k kinds of prices).

- Demand arrives as compound Poisson arrival process with rate λ_{tkj} at time t for ODF kj.

- Resources utilized by itinerary k: $A_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{mk} \end{pmatrix}$, $a_{ik} \in \{0,1\}$ with $a_{ik} = 1$ if resource i is consumed by itinerary k.

- $V(t,x)$: the maximum total expected revenue that can be extracted when the remaining capacities are $x \in \mathbb{Z}_+^m$ and the remaining time is $t \in \mathbb{R}_+$.

- Decision: $u = \{u_{kj} : j = 1, ..., n_k, k = 1, ..., K\}$, $u_{kj} = \begin{cases} 1 & \text{accept a request for ODF } k_j \\ 0 & \text{others} \end{cases}$

- Feasible set of decisions: $u(x) = \{u_{kj} \in \{0,1\} : A_k u_{kj} \leq x, j = 1, ..., n_k, k = 1, ..., K\}$

5.2 HJB Equation

Assume now that the state is (t,x) and consider a time increment δt that is small enough so that we can approximate the probability of an arrival of a request for fare j of itinerary k by $\lambda_{tkj}\delta k$.

$$\begin{aligned} V(t,x) = &\sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\delta t \max_{u_{kj}\in\{0,1\}} \left[p_{kj}u_{kj} + V\left(t-\delta t, x - A_k u_{kj}\right)\right] \\ &+ \left\{1 - \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\delta t\right\} V(t-\delta t, x) + o(\delta t) \end{aligned}$$

where $o(\delta t)$ is a quantity that goes to zero faster than δt. Subtracting $V(t-\delta t, x)$ from both side of the equation, dividing by δt, and using the notation $\Delta_k V(t,x) = V(t,x) - V\left(t, x - A_k\right)$, we obtain the Hamilton-Jacobi-Bellman (HJB) equation

$$\frac{\partial V(t,x)}{\partial t} = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - \Delta_k V(t,x)\right]^+$$

with boundary conditions $V(t,0) = V(0,x) = 0$ for all $t \geq 0$ and all $x \geq 0$. Notice that term $[p_{kj} - \Delta V_k(t,x)]^+$is equivalent to the maximum of $p_{kj}u_{kj} + V(t, x- \ A_k u_{kj}) - V(t-\delta t, x)$ over $u_{kj} \in \{0,1\}$.

For any vector $z \geq 0$, Define

$$R_t(u,z) := \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - z_k\right] u_{kj}$$

and

$$\mathcal{R}_t(z) := \max_u R_t(u,z) = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj} \max_{u_{jk}\in\{0,1\}} \left[p_{kj} - z_k\right] u_{kj} = \sum_{k=1}^{K}\sum_{j=1}^{n_k} \lambda_{tkj}\left[p_{kj} - z_k\right]^+$$

Then

$$\frac{\partial V(t,x)}{\partial t} = \mathcal{R}_t(\Delta V(t,x)), \quad \Delta V(t,x) = \begin{pmatrix} \Delta_1 V(t,x) \\ \Delta_2 V(t,x) \\ \vdots \\ \Delta_K V(t,x) \end{pmatrix}$$

1. Let's aggregate ODF's into a single index.
2. $n = \sum_{k=1}^{K} n_k$
3. HJB equation:

$$\frac{\partial V(t,x)}{\partial t} = \mathcal{R}_t(\Delta V(t,x)) = \sum_{j \in M_t} \lambda_{tj} \left[p_j - \Delta_j V(t,x)\right]^+$$

   - $V(t,0) = V(0,x) = 0, \quad \forall t \geq 0, x \geq 0$
   - $M_t \subset \{1, ..., n\}$: offered set of fares at t
   - $\Delta_j V(t,x) = V(t,x) - V(t, x - A_j)$
4. Optimal Control:

$$u_j^*(t,x) = \begin{cases} 1 & \text{if } j \in M_t,\ A_j \leq x \text{ and } \underline{p_j \geq \Delta_j V(t,x)} \\ 0 & \text{others} \end{cases}$$

Compute exact $\Delta_j V(t,x)$ can be expensive, we can use heuristics to approx it by $\Delta_j \widetilde{V}(t,x)$

5.3 Upgrades

Let u_j be the set of products that can be used to fulfill a request for product j.
Customers are willing to take any products $k \in u_j$ at the price of product p_j.

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j \in M_t} \lambda_j \max_{k \in u_j} \left[p_j - \Delta_k V(t,x)\right]^+ = \sum_{j \in M_t} \lambda_{tj} \left[p_j - \hat{\Delta}_j V(t,x)\right]^+$$

where $\hat{\Delta}_j V(t,x) = \min_{k \in u_j} \Delta_k V(t,x)$ (Use the least valuable product to fulfill p_j's request.)

5.4 Upsells

Selling j instead of k to get higher revenue, but may be rejected by customers.

- γ_{jk}: revenue obtained from selling product j and fulfilling it with product $k \in u_j$.
- π_{jk}: probability a customer will accept the upgrade from product j to product k.

$$\frac{\partial V(t,x)}{\partial t} = \sum_{j \in M_t} \lambda_j \max_{k \in u_j} \left[\pi_{jk}(r_{jk} - \hat{\Delta}_k V(t,x)) + (1 - \pi_{jk})(p_j - \hat{\Delta}_j V(t,x))\right]$$

5.5 Linear programming-based upper bound

The discrete maximum problem is

$$V(t,x) = \max_{u \in U(x)} \sum_{j=1}^{n} \lambda_{tj} [p_j u_j + V(t-1, x - u_j A_j)]$$

Deterministic Linear Program
Let D_j be the aggregate demand for ODF j over $[0, T]$.

Then D_j is Poisson with parameter $\Lambda_j = \int_0^T \lambda_{sj} ds$.
Define

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j\in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \le c_i \quad \forall i \in M \\ & 0 \le y_j \le \Lambda_j \quad \forall j \in N. \end{aligned}$$

$$\begin{aligned} \bar{V}(T,c|D) := \max \quad & \textstyle\sum_{j\in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j\in N} a_{ij} y_j \le c_i \quad \forall i \in M \\ & 0 \le y_j \le D_j \quad \forall j \in N. \end{aligned}$$

Theorem 8 (2.2 of GT)**.**

$$V(T,C) \le \mathbb{E}[\bar{V}(T,c|D)] \le \bar{V}(T,c)$$

$\bar{V}(T,c)$ is the revenue of expected demand, $\mathbb{E}[\bar{V}(T,c|D)]$ is probability combination that is concave in D, so $\mathbb{E}[\bar{V}(T,c|D)] \le \bar{V}(T,c)$. And $V(T,C)$'s decision is feasible in $\mathbb{E}[\bar{V}(T,c|D)]$, so $V(T,C) \le \mathbb{E}[\bar{V}(T,c|D)]$.
Dual formulation of $\bar{V}(T,c)$

$$\begin{aligned} \bar{V}(T,c) := \min \quad & \textstyle\sum_{i\in M} c_i z_i + \sum_{j\in N} \Lambda_j \beta_j \\ \text{s.t.} \quad & \textstyle\sum_{i\in M} a_{ij} z_i + \beta_j \ge p_j \quad \forall j \in N \\ & z_i, \beta_j \ge 0 \quad \forall i \in M, \forall j \in N. \end{aligned}$$

We can simplify the formulation. Since $\beta_j \ge p_j - \sum_{i\in M} a_{ij} z_i$, $\beta_j \ge 0$ and dual is a minimization problem, we can rewrite $\beta_j = [p_j - \sum_{i\in M} a_{ij} z_i]^+$. Then,

$$\sum_{j\in N} \Lambda_j \beta_j = \sum_{j\in N} \Lambda_j [p_j - \sum_{i\in M} a_{ij} z_i]^+ = \int_0^T \mathcal{R}_t(A^T z) dt$$

so,

$$\bar{V}(T,c) = \min_{z\ge 0} \int_0^T \mathcal{R}_t(A^T z) dt + c^T z$$

The optimal solution z_i^* gives an estimation of the marginal value of the i^{th} resource. The approximation of $\Delta_j V(T,c)$ is $\sum_{i\in M} a_{ij} z_i^*$

Bid-price Heuristic
Accept ODF_j if and only if

$$p_j \ge \sum_{i\in M} a_{ij} z_i^* \text{ and } A_j \le x$$

Probabilistic Admission Control (PAC) Heuristic
Accept ODF_j with probability $\frac{y_j^*}{\Lambda_j}$ whenever $A_j \le x$.

Bid-price heuristic is not in general asymptotically optimal.
PAC heuristic is asymptotically optimal.

Theorem 9. *Let $\Pi^b(T,c)$ be the total expected revenue from PAC heuristic and $V^b(T,c)$ be the optimal total expected revenue corresponding to circumstance $b \ge 1$ with capacity bc and $b\lambda_{jt}$. Then*

$$\lim_{b\to\infty} \frac{\Pi^b(T,c)}{V^b(T,c)} = 1$$

5.6 Dynamic Programming Decomposition (DPD)

In this section, we describe two possible approaches for approximating the value functions $V(t,\cdot)$ for the discrete-time formulation

$$V(t,x) = V(t-1,x) + \mathcal{R}_t(\Delta V(t-1,x))$$

Consider the aggregated single index formulation

$$V(t,x) = \max_{u \in U(x)} \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x - u_j A_j)]$$

with $V(t,0) = V(0,x) = 0$ and $\sum_{j=1}^{n} \lambda_{tj} = 1, \lambda_{tj} \geq 0$. (scale can be standardized)

5.6.1 Deterministic Linear Program

The former DLP we use

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j \in N} p_j y_j \\ \text{s.t.} \quad & \textstyle\sum_{j \in N} a_{ij} y_j \leq c_i \quad \forall i \in M \\ & 0 \leq y_j \leq \Lambda_j \quad \forall j \in N. \end{aligned}$$

Its dual optimal value is $(z_1^*, z_2^*, ..., z_m^*)$. We choose an arbitrary resource i and relax the first set of constraints for all of the resources except for resource i by associating the dual multipliers $(z_1^*, z_2^*, ..., z_m^*)$ with them.

We relax the first constraints $\sum_{j \in N} a_{ij} y_j \leq c_i \quad \forall i \in M$, which won't change the objective value,

$$\begin{aligned} \max \sum_{j \in N} p_j y_j &= \sum_{j \in N} p_j y_j - \sum_{k \neq i} [\sum_{j \in N} a_{kj} y_j - c_k] z_k \\ &= \sum_{j \in N} [p_j - \sum_{k \neq i} a_{kj} z_k^*] y_j + \sum_{k \neq i} z_k^* c_k \end{aligned}$$

The new DLP is

$$\begin{aligned} \bar{V}(T,c) := \max \quad & \textstyle\sum_{j \in N} [p_j - \sum_{k \neq i} a_{kj} z_k^*] y_j + \sum_{k \neq i} z_k^* c_k \\ \text{s.t.} \quad & \textstyle\sum_{j \in N} a_{ij} y_j \leq c_i \quad \forall i \in M \\ & 0 \leq y_j \leq \Lambda_j \quad \forall j \in N. \end{aligned}$$

We can prove the optimal y^* and optimal objective values are the same.

Claim 2. *The optimal values y_j^* and optimal objective values of these two DLP are the same.*

(This claim can help prove the upperbound).

$$V(t,x) = \max_{u \in U(x)} \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x - u_j A_j)]$$

We consider the optimal total expected revenue in the single-resource revenue management problem for resource i, the corresponding price of ODF_j should be $p_j - \sum_{k \neq i} a_{kj} z_k^*$. Then the formulation is

$$v_i(t, x_i) = \max_{u \in U_i(x_i)} \sum_{j \in N} \lambda_{tj} \left\{ [p_j - \sum_{k \neq i} a_{kj} z_k^*] u_j + v_i(t-1, x_i - u_j a_{ij}) \right\}$$

We can prove that

- $v_i(T, c_i) \leq \bar{V}(T,c) - \sum_{k \neq i} z_k^* c_k$
- Theorem 2.11 of GT

$$V(t,x) \leq \min_{i \in M} \{ v_i(t, x_i) + \sum_{k \neq i} z_k^* x_k \}$$

5.6.2 Lagrangian Relaxation

$$\begin{array}{lll} V(t,x) = \max & \sum_{j=1}^{n} \lambda_{tj}[p_j u_j + V(t-1, x - u_j A_j)] \\ \text{s.t.} & u_j A_j \leq x \\ & u_j \in \{0,1\} \quad \forall \in N \end{array}$$

To demonstrate the Lagrangian relaxation strategy, we use decision variables $\{w_{ij} : i \in M, j \in N\}$ in the dynamic programming formulation of the network revenue management problem, where $w_{ij} = 1$ if we make ODF_j available for purchase on flight leg i, otherwise $w_{ij} = 0$.

$$\begin{array}{lll} V(t,x) = \max & \sum_{j\in N} \lambda_{tj} \left\{ p_j w_{\psi j} + V\left(t-1, x - \sum_{i\in M} w_{ij} a_{ij} e_i\right)\right\} \\ \text{s.t.} & a_{ij} w_{ij} \leq x_i \\ & w_{ij} = w_{\psi j} \\ & w_{ij} \in \{0,1\}, w_{\psi j} \in \{0,1\} \quad \forall i \in M, j \in N \end{array}$$

We can relex the second set of constraints by adding Lagrange mutipliers $\{\alpha_{tij} : i \in M, j \in N\}$. Relaxed dynamic program:

$$\begin{array}{lll} V^{\alpha}(t,x) = \max & \sum_{j\in N} \lambda_{tj} \left\{ \sum_{i\in M} \alpha_{tij} w_{ij} + \left[p_j - \sum_{i\in M} \alpha_{tij}\right] w_{\psi j} \; + V^{\alpha}\left(t-1, x - \sum_{i\in M} w_{ij} a_{ij} e_i\right)\right\} \\ \text{s.t.} & a_{ij} w_{ij} \leq x_i \\ & w_{ij} \in \{0,1\}, w_{\psi j} \in \{0,1\} \quad \forall i \in, j \in N \end{array}$$

Theorem 10 (2.13 of GT). *Assume that the value functions $\{v_i^{\alpha}(t,\cdot) : t = 1,\ldots,T\}$ are computed through the dynamic program*

$$v_i^{\alpha}(t, x_i) = \max_{w_i \in \mathcal{U}_i(x_i)} \left\{ \sum_{j\in N} \lambda_{tj} \left\{ \alpha_{tij} w_{ij} + v_i^{\alpha}(t-1, x_i - w_{ij} a_{ij}) \right\} \right\}$$

Then

$$V^{\alpha}(t,x) = \sum_{i\in M} v_i^{\alpha}(t, x_i) + \sum_{\tau=1}^{t} \sum_{j\in N} \lambda_{\tau j} \left[p_j - \sum_{i\in M} \alpha_{\tau ij} \right]^{+}$$

Theorem 11 (2.14 of GT). *For any set of Lagrange multipliers α, we have*

$$V(t,x) \leq V^{\alpha}(t,x) \quad \forall x \in \mathbb{Z}_{+}^{m}, t = 1, \ldots, t$$

The tightest possible upper bound, we can solve the problem

$$\min_{\alpha \in \mathbb{R}^{Tmn}} V^{\alpha}(T,c)$$

Lemma 6 (2.15 of GT). *$V^{\alpha}(t,x)$ is a convex function of α for any $t = 1, \ldots, T$ and $x \in \mathbb{Z}_{+}^{m}$.*

Then compute $\min V^{\alpha}(T,c)$ can be easier.