Analysis

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目录

1 Basis

1.1 Sequence Definitions

Sequences $\{x_k\}_{k=1}, \ldots$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 1 (Convergence: note $x_k \to x, \lim_{k\to\infty} x_k = x$)**.** *Given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x\| < \varepsilon \quad \forall k \geqslant N_\varepsilon$$

Definition 2 (Cauchy Sequence)**.** $\{x_k\}$ *is Cauchy if given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x_m\| < \varepsilon, \ \forall k, m \geqslant N_\varepsilon.$$

Note:

$$\{x_k\} \text{ converges } \Longleftrightarrow \{x_k\} \text{ is Cauchy}$$

Definition 3 (Subsequence)**.** *Infinite subset of* $\{x_k\}$*:* $\{x_k : k \in \mathcal{K}\}$ *or* $\{x_k\}_\mathcal{K}$*, where* $\mathcal{K}$ *is subset of* $\mathbb{Z}^+$*.*

Definition 4 (Limit point)**.** *x is a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to x.*

Definition 5 (Bounded Sequence)**.**

$$\|x_k\| \leqslant b, \forall k$$

Results about Bounded sequences:

1. Every bounded has at least one limit point.
2. A bounded sequence converges iff it has a **unique limit point**.

1.2 Scalar Sequences

<u>**Scalar sequences**</u> $\{x_k\}, x_k \in \mathbb{R}$:

Proposition 1. *If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it **converges**.*

Proposition 2. *The largest(smallest) limit point of $\{x_k\}$ is $\lim_{k\to\infty} \sup x_k$ ($\lim_{k\to\infty} \inf x_k$)*

Proposition 3. $\{x_k\}$ *converges* $\Longleftrightarrow -\infty < \lim_{k\to\infty} \inf x_k = \lim_{k\to\infty} \sup x_k < \infty$

1.3 Functions Basis

Definition 6 (Continunity)**.** *A real-valued function f is continuous at x if*
for every $\{x_k\}$ converging to x satisfies that $\lim_{k\to\infty} f(x_k) = f(x)$.
Equivalent: given $\varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon \quad \forall \|y - x\| < \delta$
f is continuous if it is continuous at all points x.

Definition 7 (Coercive)**.** *A real-valued function $f : \& \to \mathbb{R}$ is <u>coercive</u> if for **every** $\{x_k\} \subset \&$ s.t.* $\|x_k\| \to \infty, f(x_k) \to \infty$

Example 1 (Check coercive).

1) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2$ - coercive
2) $x \in \mathbb{R}, f(x) = 1 - e^{-|x|}$ - not coercive
3) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (需要所以 $\|x_k\| \to \infty$ 都满足)

1.4 Sets

Definition 8 (Open Sets). *A set $\& \subseteq \mathbb{R}^n$ is open if*
$\forall x \in \&$ we can draw a ball around x that is contained in $\&$.
i.e. $\forall x \in \&, \exists \varepsilon > 0$ s.t. $\{y : \|y - x\| < \varepsilon\} \leq -\sigma$

Definition 9 (Closed Sets). *$\&$ is closed if $\&^c$ is open*
Equivalent: if $\&$ contains all limit points of all sequences in $\&$

Example 2 (Closed and Open Sets).

1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open

2) $\mathbb{R}$ is both open and closed

3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open

4) $[1, \infty)$ is closed because its complement open

5) $(1, 2]$ is neither open nor closed

Definition 10 (Bounded Set). *A is bounded if $\exists M$ s.t. $\|x\| \leqslant M \quad \forall x \in \&$*

Definition 11 (Compact Set). *$\mathcal{L} \subseteq \mathbb{R}^n$ is compact of it is closed and bounded.*

Example 3 (Compact Set). $[1, 2] = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\}$; $\left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 4\right\}$

Definition 12 (Extreme of sets of scalars, $\sup A, \inf A$). *Let $A \subset \mathbb{R}$.*
- The infimum of A, or inf A is largest y s.t. $y \leqslant x, \forall x \in A$. If no such y exists, $\inf A = -\infty$
- Similar definition for supremum of A (or wrote as $\sup A$).

Proposition 4. *If $\inf A (\sup A) = x^* \in A$, then $x^* = \min A (\max A)$*

2 Functions

2.1 Extreme of Functions

Definition 13 (Extreme of Functions). *Let $\& \subseteq \mathbb{R}^n, f : \& \to \mathbb{R}$*

$$\inf_{x \in \&} f(x) = \inf\{f(x) : x \in \&\}$$

If $\exists x^* \in \&$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x\in\&} f(x)$ x^* is called a **minimizer** of f, written as $x^* \in \arg\min_{x\in\&} f(x)$. If x^* is uniqne, we write $x^* = \arg\min_{x\in\&} f(x)$
Similarly, supremum and maximum of f.

2.1.1 Weierstrass' Theorem(Extreme value Theorem)

Theorem 1 (Weierstrass' Theorem(Extreme value Theorem)).
*If f is a **continuous** function on a **compact set**, $\& \subseteq \mathbb{R}^n$, then f attains its min and max on $\&$ i.e.,*

$$\exists x_1 \in \& \ s.t. \ f(x_1) = \inf_{x\in\&} f(x)$$

$$\exists x_2 \in \& \ s.t. \ f(x_2) = \sup_{x\in\&} f(x)$$

证明. (for existence of min; max is similar)
Let $\{\sigma_k\} \subseteq \&$ be s.t.

$$\inf_{x\in\&} f(x) \le f(\sigma_k) \le \inf_{x\in\&} f(x) + \frac{1}{k}$$

Then $\lim_{k\to\infty} f(\sigma_k) = \inf_{x\in\&} f(x)$
$\mathcal{L}$ is bounded $\Rightarrow \{\sigma_k\}$ has it least one limit point x,
$\mathcal{L}$ is closed $\Rightarrow x_1 \in \&$
f is continuous $\Rightarrow f(x_1) = \lim_{k\to\infty} f(\sigma_k) = \inf_{x\in\&} f(x)$ □

Corollary 1 (Corollary to WT). *Let f be continuous on closed set $\&$ (not necessarily bounded). If f is coercive on $\&$ it attains its* min *on $\&$.*

证明. Consider $\{\sigma_k\}$ as in proof of WT.
Since f is closed, $f(x) < \infty$, $\forall x \in \&$. And f is coercive on $\&$, which means $f(x) \to \infty$ if $\|x\| \to \infty$.
Hence, $\{\sigma_k\} \in \&$ is bounded. Rest of proof same as proof of WT. □

Example 4. $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) *Does f achieve its* min *and* max *on $\mathcal{L}_1 = \left\{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leqslant 6\right\}$?*

   - $\mathcal{L}_1$ is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over $\mathbb{R}^3$?

   - $f \to \infty$ whenever $\|x\| \to \infty \Rightarrow f$ is coercive.

   - $\mathbb{R}^3$ is closed.

   $\Rightarrow f$ achieves its min. on $\mathbb{R}^3$ by corollary to WT.

   - max does not exist since $f \to \infty$ as $\|x\| \to \infty$.

3) Does f achieve its min and max over $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?

- $\mathcal{L}_2$ is closed, but not bounded.

- Since f is coercive, min achieved.

- max does not exist since setting $x_1 = 0$ $x_2 = 3 - x_3$ and letting $x_3 \to \infty$ makes $f \to \infty$