Microeconomic Theory

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All models are wrong, but some are useful.

0.1 Equilibrium in Auctions with Entry

- Levin, D., & Smith, J. L. (1994). Equilibrium in auctions with entry. The American Economic Review, 585-599.

A single item offered to a group of N potential bidders. There are two stage: each potential entrant decides whether to enter by a fixed cost c, then an auction is conducted among n participants (the number of bidders who enter).

Assumption

1. *The seller and all potential bidders are risk-neutral.*
2. *We assume the seller's valuation is* 0 *and the possible value's range being* $[\underline{v}, \bar{v}]$*, where* $\underline{v} < 0 < \bar{v}$*. This case is equivalent to the case that the seller's valuation is* $-\underline{v}$ *and the possible value's range being* $[0, \bar{v} - \underline{v}]$.
3. *The domain of possible values for the item* (V) *and the domain of estimates* (x) *are compact:* $V \in [0, \bar{v} - \underline{v}]$ *and* $x \in [0, \bar{x}]$.
4. *Information is symmetric and bidders randomly draw values from the same distribution.*
5. *The auction mechanism* (m) *and the number of potential bidders* (N) *are common knowledge, and the number of actual bidders is revealed prior to stage 2.*
6. *A unique symmetric Nash equilibrium exists and individual behavior conforms to the symmetric Nash equilibrium.*

Given the number of bidders n, cost c, and mechanism m, the *ex-ante* expected gain from entering and bidding according to the symmetric Nash equilibrium of each potential entrant is denoted by $\mathbb{E}[\pi \mid n, m]$.

If $\mathbb{E}[\pi \mid n, m]$ is decreasing in n, $\exists$ an unqiue integer n^* such that $\mathbb{E}[\pi \mid n^*, m] \geq 0 > \mathbb{E}[\pi \mid n^* + 1, m]$. We focus on the case that $n^* \in (0, N)$.

A symmetric entry equilibrium must yield the same probability of entry for all potential bidders, which is denoted by $q^* \in (0, 1)$ and each potential entrant must be indifferent between entering or not:

$$\sum_{n=1}^{N} \left[\binom{N-1}{n-1} (q^*)^{n-1} (1-q^*)^{N-n} \mathbb{E}[\pi \mid n, m] \right] = 0 \tag{1}$$

where $\binom{N-1}{n-1} (q^*)^{n-1} (1-q^*)^{N-n}$ is the probability that exactly $n-1$ rivals also enter. The number of bidders has mean $\bar{n} := q^* N$ and variance $\bar{n}(1-q^*)$.

We focus on the mechanism that a bidder wins and pays for the item only if his bid is the highest.

Note *Different to the original paper, we focus on the "free entry" case (i.e., the mechanism can be denoted* *by the reserve prices $R = \{R_1, ..., R_N\}$, where R_n means the reserve price enforced by the seller if n bidders enter).*

We let T represent the event that trade occurs (i.e., the highest bid is greater than the reserve price) and let $T_n(R_n) := \Pr[V_{(n)} \geq R_n]$ represent the probability of trade given n bidders enter and the seller's mechanism R.

Using symmetry, a bidder's *ex-ante* expected profit, conditional on entering an auction with n bidders, can be written as $\frac{V_n - W_n}{n} - c$, where $V_n := \mathbb{E}[V_{(n)} \mid V_{(n)} \geq R_n]$ is the expected value of the item to the highest bidder and W_n is the expected payment this bidder makes to the seller, both conditional on trade occurring under the given mechanism.

We use Ω to denote $\{R, c, N\}$ and $B_i(q, \Omega)$ to denote the i^{th} bidder's expected profit from entering when all $N - 1$ rivals are using arbitrary entry probability q:

$$\begin{aligned} B_i(q, \Omega) &= \sum_{n=1}^{N} \left[\binom{N-1}{n-1} (q)^{n-1} (1-q)^{N-n} T_n(R_n) \frac{V_n - W_n}{n} \right] - c \\ &= \frac{1}{Nq} \sum_{n=1}^{N} \left[\binom{N}{n} (q)^{n} (1-q)^{N-n} T_n(R_n) \left(V_n - W_n\right) \right] - c \end{aligned}$$

If the i^{th} bidder also elects to enter with probability q, the expected profit of all N parties must be:

$$\begin{aligned} B(q, \Omega) &= NqB_i(q, \Omega) \\ &= \sum_{n=1}^{N} p_n T_n(R_n)(V_n - W_n) - \bar{n}c \end{aligned}$$

where $p_n := \binom{N}{n} (q)^n (1-q)^{N-n}$ is the binomial probability that exactly n bidders enter in total.

The seller's expected revenue is

$$\Pi(q, \Omega) = \sum_{n=1}^{N} p_n T_n(R_n) W_n$$

and the total social welfare is

$$S(q, \Omega) = B(q, \Omega) + \Pi(q, \Omega) = \sum_{n=1}^{N} p_n T_n(R_n) V_n - \bar{n}c$$

To make it is indifferent between entering and not, $q^* \in (0, 1)$, we must have $B_i(q^*, \Omega) = 0$, i.e.,

$$\frac{1}{Nq^*} \sum_{n=1}^{N} \left[\binom{N}{n} (q^*)^{n} (1-q^*)^{N-n} T_n(R_n) \left(V_n - W_n\right) \right] - c = 0 \tag{2}$$

Then, we can define the symmetric entry equilibrium given Ω, $q^* = q(\Omega)$.

Assumption *In this paper, we focus on the mechanism that the expected profit of an entrant is negatively correlated with the number of bidders, which is equivalent to $\frac{\partial q^*}{\partial c} < 0$ and $\frac{\partial q^*}{\partial (V_i - W_i)} < 0$ (Lemma 1 in original* *paper).*

Hence, we can find that the total social welfare is equivalent to the expected revenue of the seller.

$$S(q^*, \Omega) = \sum_{n=1}^{N} p_n T_n(R_n) V_n - \bar{n}c$$
$$= \Pi(q^*, \Omega) = \sum_{n=1}^{N} p_n T_n(R_n) W_n$$

Proposition 0.1 (Revenue Equivalence Holds for Symmetric Entry)

Any two mechanisms that are revenue-equivalent with fixed n and R remain revenue-equivalent with induced entry.

Proposition 0.2 (Optimal Reservation Price is the Seller's Value)

Any mechanism that maximizes the seller's expected revenue also induces socially optimal entry. Such a mechanism has reservation price $R = 0$.

Proof 0.1

We want to maximize $\sum_{n=1}^{N} p_n T_n(R_n) V_n - \bar{n}c$, where $T_n(R_n)V_n$ can be written as

$$\begin{aligned} T_n(R_n)V_n &= \Pr[V_{(n)} \geq R_n]\mathbb{E}[V_{(n)} \mid V_{(n)} \geq R_n] \\ &= \Pr[V_{(n)} \geq R_n] \int_{R_n}^{\bar{v}} x \frac{f_{V_{(n)}}(x)}{\Pr[V_{(n)} \geq R_n]} dx \\ &= \int_{R_n}^{\bar{v}} x f_{V_{(n)}}(x) dx \end{aligned}$$

which is maximized at $R_n = 0$.

Then, the seller's expected revenue and the total social welfare can be written as

$$S(q^*, \Omega) = \underbrace{\sum_{n=1}^{N} \binom{N}{n} (q^*)^n (1-q^*)^{N-n}}_{:=p_n} \underbrace{\int_0^{\bar{v}} x f_{V_{(n)}}(x) dx}_{:=\tilde{V}_n} - \bar{n}c$$

Similarly, we can write $T_n(R_n)W_n$ as $\tilde{W}_n$ when $R_n = 0$. Hence, the (2) can be written as

$$\frac{1}{Nq^*} \sum_{i=1}^{n} p_n \left(\tilde{V}_n - \tilde{W}_n \right) = c \tag{3}$$

Lemma 0.1

Given $R_n = 0$, in independent private value auctions, $\frac{\partial S}{\partial q}(q^*, \Omega) = 0$.

Proof 0.2

Given $S(q,\Omega) = \sum_{n=1}^{N} p_n \tilde{V}_n - \bar{n}c$,

$$\begin{aligned}
\frac{\partial S(q,\Omega)}{\partial q} &= \sum_{n=1}^{N} \frac{\partial p_n}{\partial q} T_n(R_n) V_n - Nc \\
&= \sum_{n=1}^{N} \begin{pmatrix} N \\ n \end{pmatrix} \left[n q^{n-1}(1-q)^{N-n} - (N-n) q^n (1-q)^{N-n-1} \right] \tilde{V}_n - Nc \\
&= \frac{\sum_{n=1}^{N} p_n \tilde{V}_n (n - qN)}{q(1-q)} - Nc
\end{aligned} \tag{4}$$

Substituting (3), *we get*

$$\begin{aligned}
\frac{\partial S(q^*,\Omega)}{\partial q} &= \frac{\sum_{n=1}^{N} p_n \tilde{V}_n (n - q^* N)}{q^*(1-q^*)} - \frac{1}{q^*} \sum_{i=1}^{n} p_n \left(\tilde{V}_n - \tilde{W}_n \right) \\
&= \frac{1}{q^*} \sum_{n=1}^{N} p_n \left[\frac{\tilde{V}_n (n - q^* N)}{1 - q^*} - \left(\tilde{V}_n - \tilde{W}_n \right) \right]
\end{aligned}$$

In independent private value auctions, the expected payment on average is given by the second-highest value. Hence,

$$\tilde{V}_n - \tilde{W}_n = n \left(\tilde{V}_n - \tilde{V}_{n-1} \right)$$

We don't prove this in detail here (Milgrom and Weber, 1982 theorem 0). The intuition is: $\tilde{V}_n - \tilde{V}_{n-1}$ *is the difference between the highest value among first* n *values and the highest value among first* $n-1$ *values. There is only* $\frac{1}{n}$ *probability that the* n^{th} *value is the highest value among the first* n *values. However,* $\tilde{V}_n - \tilde{W}_n$ *always gives the difference between the highest value and the second-highest value among first* n *values.*

Then,

$$\begin{aligned}
\frac{\partial S(q^*,\Omega)}{\partial q} &= \frac{1}{q^*} \sum_{n=1}^{N} p_n \left[\frac{\tilde{V}_n (n - q^* N)}{1 - q^*} - n \left(\tilde{V}_n - \tilde{V}_{n-1} \right) \right] \\
&= \frac{1}{q^*} \sum_{n=1}^{N} p_n \left[n \tilde{V}_{n-1} - \frac{\tilde{V}_n q^* (N - n)}{1 - q^*} \right] \\
&= \frac{1}{q^*} \left[\sum_{n=1}^{N} p_n n \tilde{V}_{n-1} - \sum_{n=1}^{N-1} p_{n+1} (n+1) \tilde{V}_n \right] \\
&= \frac{1}{q^*} p_1 \tilde{V}_0 = 0
\end{aligned}$$

Proposition 0.3 (Effect of Market Thickness)

In independent private value auctions, the level of social welfare generated by optimal auctions decreases monotonically as N *increases beyond* n^*.

Proof 0.3

Given $N > n^*$, the symmetric equilibrium gives $q^s = q^s(N) < 1$. Suppose N drops by 1 to $N-1$ while each remaining member continuous to use $q^S(N)$. The impact on social welfare is

$$\begin{aligned}\Delta S &= \sum_{n=1}^{N} p_n \tilde{V}_n - \sum_{n=1}^{N-1} \phi_n \tilde{V}_n - q^s c \\ &= \sum_{n=1}^{N} p_n \tilde{V}_n \frac{n - q^s N}{(1-q^s)N} - q^s c\end{aligned}$$

where $\phi_n = \begin{pmatrix} N-1 \\ n \end{pmatrix} (q^s)^n (1-q^s)^{N-n-1} = \frac{N-n}{N(1-q^s)} p_n$. By (4) and Lemma 0.1, we have

$$\Delta S = \frac{q^s}{N} \frac{\partial S}{\partial q} = 0$$

Hence, dropping one potential entrant while holding entry probabilities constant leaves social welfare unchanged. By relaxing the restriction on entry probability, the q^s will increase and then increase the level of social welfare.

Corollary 0.1

The expected revenue of any seller who use his optimal mechanism increases monotonically as N decreases beyond n^*. ♡