



# Analysis and Something Else

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*All models are wrong, but some are useful.*

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# Chapter 1 Logic

## 1.1 Main Methods of Proof (@ Lec 01 of ECON 204)

### 1.1.1 Proof by Induction

### 1.1.2 Proof by Deduction

### 1.1.3 Proof by Contradiction

### 1.1.4 Proof by Contraposition

- $\neg P$  ("not  $P$ ") means " $P$  is false".
- $P \wedge Q$  (" $P$  and  $Q$ ") means " $P$  is true and  $Q$  is true."
- $P \vee Q$  (" $P$  or  $Q$ ") means " $P$  is true or  $Q$  is true (or possibly both)."
- $\neg P \wedge Q$  means  $(\neg P) \wedge Q$ ;  $\neg P \vee Q$  means  $(\neg P) \vee Q$ .
- $P \Rightarrow Q$  (" $P$  implies  $Q$ ") means "whenever  $P$  is satisfied,  $Q$  is also satisfied."

**Statement:** Formally,  $P \Rightarrow Q$  is equivalent to  $\neg P \vee Q$ .

#### Definition 1.1 (Contrapositive)

The *contrapositive* of the statement  $P \Rightarrow Q$  is the statement  $\neg Q \Rightarrow \neg P$ .



#### Theorem 1.1 (Prove Contrapositive Instead)

$P \Rightarrow Q$  is true if and only if  $\neg Q \Rightarrow \neg P$  is true.



#### Proposition 1.1

$[(P \wedge Q) \Rightarrow R] \Leftrightarrow [P \Rightarrow (Q \Rightarrow R)]$ : Given  $P$ , if  $Q$  then  $R$ .

$\Leftrightarrow [(P \Rightarrow R) \vee (Q \Rightarrow R)]$ : Either  $P \Rightarrow R$  holds or  $Q \Rightarrow R$  holds



## Chapter 2 Analysis Basis

### 2.1 Real Number $\mathbb{R}$ (@ Lec 02 of ECON 204)

$\mathbb{R}$  is a field with the usual operations  $+$ ,  $\cdot$ , additive identity 0, and multiplicative identity 1.

#### 2.1.1 Order Axiom

##### Proposition 2.1 (Order Axiom)

There is a complete ordering  $\leq$ , i.e.  $\leq$  is reflexive, transitive, antisymmetric ( $\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$ ) with the property that  $\forall \alpha, \beta \in \mathbb{R}$  either  $\alpha \leq \beta$  or  $\beta \leq \alpha$ .

The order is compatible with  $+$  and  $\cdot$ , i.e.  $\forall \alpha, \beta, \gamma \in \mathbb{R}$

1.  $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$ .
2.  $\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$ .



#### 2.1.2 Completeness Axiom

##### Proposition 2.2 (Completeness Axiom)

Suppose  $L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H$  satisfy  $\forall l \in L, h \in H, l \leq h$ . Then,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall l \in L, h \in H, l \leq \alpha \leq h$$

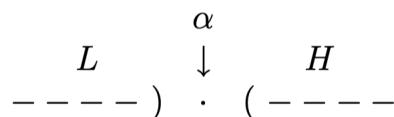


Figure 2.1: Completeness Axiom

##### Claim 2.1

The Completeness Axiom differentiates  $\mathbb{R}$  from  $\mathbb{Q}$ :

$\mathbb{Q}$  satisfies all the axioms for  $\mathbb{R}$  except the Completeness Axiom.



### 2.1.3 Supremum $\sup \mathbb{X}$ , Infimum $\inf \mathbb{X}$ for $\mathbb{X} \subseteq \mathbb{R}$

#### Definition 2.1 (Supremum and Infimum)

- (1). Suppose  $\mathbb{X}$  is bounded above. The **supremum** of  $\mathbb{X}$ , written  $\sup \mathbb{X}$ , is the least upper bound for  $\mathbb{X}$ , i.e.  $\sup \mathbb{X}$  satisfies
  - (a).  $\sup \mathbb{X} \geq x, \forall x \in \mathbb{X}$  ( $\sup \mathbb{X}$  is an upper bound).
  - (b).  $\forall y < \sup \mathbb{X}, \exists x \in \mathbb{X}$  s.t.  $x > y$  (there is no smaller upper bound).
- (2). Suppose  $\mathbb{X}$  is bounded below. The **infimum** of  $\mathbb{X}$ , written  $\inf \mathbb{X}$ , is the greatest lower bound for  $\mathbb{X}$ , i.e.  $\inf \mathbb{X}$  satisfies
  - (a).  $\inf \mathbb{X} \leq x, \forall x \in \mathbb{X}$  ( $\inf \mathbb{X}$  is a lower bound).
  - (b).  $\forall y > \inf \mathbb{X}, \exists x \in \mathbb{X}$  s.t.  $x < y$  (there is no greater lower bound).
- (3). If  $\mathbb{X}$  is not bounded above, write  $\sup \mathbb{X} = \infty$ . If  $\mathbb{X}$  is not bounded below, write  $\inf \mathbb{X} = -\infty$ . By convention,  $\sup \emptyset = -\infty$ ,  $\inf \emptyset = +\infty$ .



### 2.1.4 Theorem: sup / inf in set is max / min

#### Proposition 2.3 (sup / inf in set is max / min)

If  $\inf A = x^* \in A$  ( $\sup A = x^* \in A$ ), then  $x^* = \min A$  ( $x^* = \max A$ ).



### 2.1.5 Transformation of sup / inf

#### Theorem 2.1

Let  $A$  and  $B$  be nonempty sets of real numbers,

1. both of them bounded above,  $\sup\{a + b : a \in A, b \in B\} = \sup A + \sup B$ .
2. bounded above and below respectively,  $\sup\{a - b : a \in A, b \in B\} = \sup A - \inf B$
3.  $\inf A = -\sup(-A)$



### 2.1.6 The Supremum Property: Nonempty set in $\mathbb{R}$ is bounded above/below $\Rightarrow \exists \sup / \inf$

#### Proposition 2.4 (The Supremum Property)

Every nonempty set of real numbers that is bounded above has a supremum, which is a real number.

Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.



#### Theorem 2.2

The Supremum Property (Prop 2.4) and the Completeness Axiom (Prop 2.2) are equivalent.



### 2.1.7 Archimedean Property

**Theorem 2.3 (Archimedean Property)**

$\forall x \in \mathbb{R}, y \in \mathbb{R}^+, \exists n \in \mathbb{N}$  s.t.  $ny > x$ .



## 2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)

### 2.2.1 Metric Space $(\mathbb{X}, d)$ and Metric $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$

**Definition 2.2 (Metric Space)**

A **metric space** is a pair  $(\mathbb{X}, d)$ , where  $\mathbb{X}$  is a set and  $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$  a function satisfying

1. Non-negative:  $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in \mathbb{X}$ .
2. Symmetric:  $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$ .
3. Triangle inequality:  $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathbb{X}$ .

A function  $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$  satisfying 1-3 is called a **metric** on  $\mathbb{X}$ .



A metric gives a notion of distance between elements of  $\mathbb{X}$ .

### 2.2.2 Theorem: Subset forms a metric space

**Theorem 2.4 (Subset forms a metric space)**

Given a metric space  $(X, d)$ ,  $(Y, d)$  for some  $Y \subseteq X$  is still a metric space.



### 2.2.3 Norm $\|\cdot\|$ and Normed Vector Space $(V, \|\cdot\|)$

**Definition 2.3 (Norm)**

Let  $V$  be a vector space over  $\mathbb{R}$ . A **norm** on  $V$  is a function  $\|\cdot\| : V \rightarrow \mathbb{R}_+$  satisfying

1. Non-negative:  $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0, \forall x \in V$ .
2. Triangle inequality:  $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$ .
3.  $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, x \in V$ .



A norm gives a notion of length of a vector in  $V$ .

**Definition 2.4 (Normed Vector Space)**

A **normed vector space** is a vector space over  $\mathbb{R}$  equipped with a norm,  $(V, \|\cdot\|)$ .



### Example 2.1 Normed Vector Space

- $\mathbf{E}^n$  :  $n$ -dimensional Euclidean space.

$$V = \mathbb{R}^n, \|x\|_2 = |x| = \sqrt{\sum_{i=1}^n (x_i)^2}$$

- $V = \mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i|$  (the "taxi cab" norm or  $L^1$  norm)
- $V = \mathbb{R}^n, \|x\|_\infty = \max \{|x_1|, \dots, |x_n|\}$  (the maximum norm, or sup norm, or  $L^\infty$  norm)
- $C([0, 1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
- $C([0, 1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$
- $C([0, 1]), \|f\|_1 = \int_0^1 |f(t)| dt$

where  $C([0, 1])$  is the space of all continuous real-valued functions on  $[0, 1]$ .

#### 2.2.4 Subset may not form a normed vector space

Given a normed vector space  $(V, \|\cdot\|)$ , is  $(W, \|\cdot\|)$  for some  $W \subseteq V$ ,  $W$  may not be a vector space (e.g. it may not contain the zero vector)!

#### 2.2.5 Theorem: metric can be defined by norm

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

**Theorem 2.5 (Metric can be defined by norm)**

Let  $(V, \|\cdot\|)$  be a normed vector space. Let  $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$  be defined by  $d(v, w) = \|v - w\|$ . Then  $(V, d)$  is a metric space.



#### 2.2.6 Cauchy-Schwarz Inequality

**Theorem 2.6 (Cauchy-Schwarz Inequality)**

If  $v, w \in \mathbb{R}^n$ , then

$$\left( \sum_{i=1}^n v_i w_i \right)^2 \leq \left( \sum_{i=1}^n v_i^2 \right) \left( \sum_{i=1}^n w_i^2 \right)$$

$$\|v \cdot w\|^2 \leq \|v\|^2 \|w\|^2$$

$$\|v \cdot w\| \leq \|v\| \|w\|$$



### 2.2.7 Lipschitz-equivalent Norm

#### Definition 2.5 (Lipschitz-equivalent)

Two norms  $\|\cdot\|$  and  $\|\cdot\|^*$  on the same vector space  $V$  are said to be **Lipschitz-equivalent** (or **equivalent**) if  $\exists m, M > 0$  s.t.  $\forall x \in V$ ,

$$m\|x\| \leq \|x\|^* \leq M\|x\|$$

Equivalently,  $\exists m, M > 0$  s.t.  $\forall x \in V, x \neq 0$ ,

$$m \leq \frac{\|x\|^*}{\|x\|} \leq M$$



#### Theorem 2.7

All norms on  $\mathbb{R}^n$  are equivalent.



However, infinite-dimensional spaces support norms that are not equivalent. For example, on  $C([0, 1])$ , let  $f_n$  be the function

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \in [0, \frac{1}{n}] \\ 0, & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then  $\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} \rightarrow 0$ , which means there is no lower bound  $m > 0$ .

### 2.2.8 Ball, Radius, Diameter, and Distance

In a metric space  $(X, d)$ , define

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\ &= \text{open ball with center } x \text{ and radius } \varepsilon \\ B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\ &= \text{closed ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

We can use the metric  $d$  to define a generalization of "radius". In a metric space  $(X, d)$ , define the *diameter* of a subset  $S \subseteq X$  by

$$\text{diam}(S) = \sup \{d(s, s') : s, s' \in S\}$$

Similarly, we can define the *distance from a point to a set*, and *distance between sets*, as follows:

$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf \{d(a, b) : a \in A, b \in B\} \end{aligned}$$

## 2.3 Set Theory

### 2.3.1 Well Defined Set

#### Definition 2.6

A set  $S$  is **well defined** if an object  $a$  is either  $a \in S$  or  $a \notin S$ .



### 2.3.2 Numerically Equivalent (@ Lec 01 of ECON 204)

#### Definition 2.7

Two sets  $A, B$  are **numerically equivalent** (or have the same cardinality) if there is a bijection  $f : A \rightarrow B$ , that is, 1-1 ( $a \neq a' \Rightarrow f(a) \neq f(a')$ ), and onto ( $\forall b \in B, \exists a \in A$  s.t.  $f(a) = b$ ).



### 2.3.3 Finite, Countable Set (@ Lec 01 of ECON 204)

#### Definition 2.8 (Finite Set)

A set is either **finite** or **infinite**. A set is **finite** if it is numerically equivalent to  $\{1, \dots, n\}$  for some  $n$ . A set that is not finite is infinite.



We give a more precise definition to classify infinite set:

#### Definition 2.9 (Countable Set)

An infinite set is **countable** if it is numerically equivalent to  $\mathbb{N}$ . An infinite set that is not countable is called **uncountable**.



#### Theorem 2.8 (Countable $\mathbb{Q}$ )

*The set of rational numbers  $\mathbb{Q}$  is countable.*



### 2.3.4 Power Set (@ Lec 02 of ECON 204)

#### Definition 2.10 (Power Set: the set of all subsets)

For any set  $A$ , we denote by  $\mathcal{P}(A)$  the collection of all subsets of  $A$ .  $\mathcal{P}(A)$  is the **power set** of  $A$ .



We may also use the notation  $2^A$  (in Berkeley ECON 204).

### 2.3.5 Theorem (Cantor): The power set of $\mathbb{N}$ is uncountable (@ Lec 02 of ECON 204)

#### Theorem 2.9 (Cantor)

$\mathcal{P}(\mathbb{N})$  (or denoted by  $2^{\mathbb{N}}$ ), the set of all subsets of  $\mathbb{N}$ , is uncountable.



### 2.3.6 Cardinalities of Sets (@ Lec 02 of ECON 204)

#### Definition 2.11 (Cardinality)

If  $A$  is a set,  $|A| = \text{cardinality of } A = \# \text{ of elements}$



$n \in \mathbb{N}, |\{1, \dots, n\}| = n; |\emptyset| = 0 (\emptyset = \text{empty set}).$

#### Proposition 2.5 (Facts about cardinality)

1. If  $A$  is numerically equivalent to  $\{1, \dots, n\}$  for some  $n \in \mathbb{N}$ , then  $|A| = n$ .
2.  $A$  and  $B$  are numerically equivalent if and only if  $|A| = |B|$ .
3. If  $|A| = n$  (finite) and  $A$  is a proper subset of  $B$  (that is,  $A \subset B$  and  $A \neq B$ ) then  $|A| < |B|$ .
4. If  $A$  is countable and  $B$  is uncountable, then  $n < |A| < |B|, \forall n \in \mathbb{N}$ .
5. If  $A \subseteq B$ , then  $|A| \leq |B|$ . (if  $B$  is countable and  $A \subseteq B$ , then  $A$  is at most countable, that is,  $A$  is either empty, finite, or countable.)
6. If there is an injection  $\sigma : A \rightarrow B$ , we can write  $|A| \leq |B|$ ;
7. If there is a surjection  $\sigma : A \rightarrow B$ , we can write  $|A| \geq |B|$ ;
8. If there is a bijection  $\sigma : A \rightarrow B$ , we can write  $|A| = |B|$ .



### 2.3.7 Pigeonhole Principle: $|A| > |B| \Rightarrow \text{no injective function } \sigma : A \rightarrow B$

#### Theorem 2.10 (Pigeonhole Principle)

If  $A$  and  $B$  are sets and  $|A| > |B|$ , then there is no injective function  $\sigma : A \rightarrow B$ .



### 2.3.8 $B^A$ : Sets of Function

If  $A, B$  are sets, then  $B^A = \{\sigma : A \rightarrow B | \sigma \text{ a function}\}$ .

**Example 2.2**  $n \in \mathbb{Z}$ , we define a function  $f : B^{\{1, \dots, n\}} \rightarrow B^n (= B \times B \times B \times \dots \times B)$  by the equation  $f(\sigma) = \{\sigma(1), \dots, \sigma(n)\}$ , where  $\sigma : \{1, \dots, n\} \rightarrow B$ . The  $f$  is a bijection.

#### Proof 2.1

1. *Injective:*

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), \dots, \sigma_1(n)\} = \{\sigma_2(1), \dots, \sigma_2(n)\}$$

Since  $\sigma : \{1, \dots, n\} \rightarrow B$ , it is sufficient to prove  $\sigma_1 = \sigma_2$ .

2. *Surjective:*

$$\forall \{b_1, \dots, b_n\}, \text{ we have } \sigma^*(x) = b_x, x = 1, \dots, n. \text{ s.t. } f(\sigma^*) = \{b_1, \dots, b_n\}$$

**Example 2.3**

$$C(\mathbb{R}, \mathbb{R}) = \{\text{continuous functions } \sigma : \mathbb{R} \rightarrow \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

**2.3.9 Bounded Set (@ Lec 03 of ECON 204)****Definition 2.12 (Bounded Set)**

In a metric space  $(X, d)$ , a subset  $S \subseteq X$  is **bounded** if  $\exists x \in X, \beta \in \mathbb{R}$  such that  $\forall s \in S, d(s, x) \leq \beta$ .



Equivalent:  $\exists \beta > 0$  and  $s \in S$  such that  $S \subseteq B_{\beta}(s)$ .

**2.3.10 Open, Closed Set (@ Lec 04 of ECON 204)****Definition 2.13 (Open Sets)**

Let  $(X, d)$  be a metric space. A set  $\mathbb{X} \subseteq \mathbb{R}^n$  is **open** if

$\forall x \in \mathbb{X}$  we can draw a ball around  $x$  that is contained in  $\mathbb{X}$ .

i.e.  $\forall x \in \mathbb{X}, \exists \varepsilon > 0$  s.t.  $B_{\varepsilon}(x) = \{y : d(y, x) < \varepsilon\} \subseteq \mathbb{X}$



**Example 2.4** In a metric space  $(X, d)$ , if  $a$  is an isolated point of  $X$ , then the singleton set  $\{a\}$  is open: Since  $a$  is not a limit point of  $X$ , we can find a neighborhood  $B_{\varepsilon}(a)$  such that  $B_{\varepsilon}(a) \cap X = \{a\}$ . So  $B_{\varepsilon}(a) \subseteq \{a\}$  and  $x$  is an interior point. Hence, every point is an interior point and the set is open.

**Definition 2.14 (Closed Sets)**

$\mathbb{X}$  is **closed** if  $\mathbb{X}^c$  is open.

**Theorem 2.11 (Equivalent definition: Closed Sets)**

Equivalent: if  $A$  in a metric space  $(X, d)$  contains all limit points of all sequences in  $A$ ,  $A$  is closed.

$$\{x_n\} \subset A, x_n \rightarrow x \in X \Rightarrow x \in A$$

**Example 2.5 (Closed and Open Sets on  $\mathbb{E}_1$  i.e.,  $\mathbb{R}$  with the usual Euclidean metric)**

- 1)  $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$  - open
- 2)  $\mathbb{R}$  is both open and closed
- 3)  $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$  - open
- 4)  $[1, \infty)$  is closed because its complement open
- 5)  $(1, 2]$  is neither open nor closed

**Example 2.6 (Closed and Open Sets on other metric space)** In the metric space  $[0, 1]$ ,  $[0, 1]$  is open. With  $[0, 1]$  as the underlying metric space,  $B_{\varepsilon}(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon]$ .

**Theorem 2.12 (Empty Set and Full Set are both open and closed)**

In any metric space  $(X, d)$  both  $\emptyset$  and  $X$  are open, and both  $\emptyset$  and  $X$  are closed.

**Theorem 2.13 (Every neighborhood is an open set)**

Every neighborhood  $B_\varepsilon(x)$  is an open set.

**Theorem 2.14 (Finite sets are closed in any metric space)**

Finite sets are closed in any metric space  $(X, d)$ .

**Proposition 2.6**

Any set in a metric space  $(X, d)$  can be written as the intersection of open sets.

**2.3.11 Theorem: Union of open sets is open, Intersection of finite open sets is open****Theorem 2.15 (Union of open sets is open, Intersection of finite open sets is open)**

In any metric space  $(X, d)$ ,

1. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.
2. The intersection of a finite collection of open sets is open.

**2.3.12 Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)**

Given a set  $S \subseteq X$ , the **point** of  $X$  can be classified into three types relative to  $S$ :

- **Interior (points)**, denoted  $\text{int}(S)$ :  $\vec{x} \in S$  for which there exists some  $B(\vec{x}, r) \subseteq S$ , is the largest open set contained in  $S$  (the union of all open sets contained in  $S$ ).

$$x \in \text{int } A \Leftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A$$

- **Exterior (points)**, denoted  $\text{ext}(S)$ :  $\vec{x} \notin S$  for which there exists some  $B(\vec{x}, r)$  containing no points of  $S$ , is the largest open set contained in  $X \setminus S$ .

$$x \in \text{ext } A \Leftrightarrow \exists \varepsilon > 0 : B_\varepsilon(x) \subseteq A^C$$

- **Boundary (points)** denoted  $\partial(S)$  or  $\text{bd}(S)$ : all other points (for which any  $B(\vec{x}, r)$  will contain some points of  $S$  and some points outside  $S$ ).

$$x \in \partial A \Leftrightarrow \forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset \text{ and } B_\varepsilon(x) \cap A^C \neq \emptyset$$

- **Closure of  $S$** , denoted  $\bar{S}$  or  $\text{cl}(S) = \text{int}(S) \cup \partial(S)$ , is the smallest closed set containing  $S$  (the intersection of all closed sets containing  $S$ ).

$$x \in \text{cl } A \Leftrightarrow \forall \varepsilon > 0 : B_\varepsilon(x) \cap A \neq \emptyset$$

- Moreover, boundary satisfies  $\partial(S) = \overline{(X \setminus S)} \cap \bar{S}$ .

### Proposition 2.7

- (1)  $\text{int}A$ ,  $\text{ext}A$  and  $\partial A$  partition  $X$ .
- (2)  $\text{int}A$  and  $\partial A$  partition  $\text{cl}A$ .
- (3) A set  $S$  is **open** iff  $S = \text{int}(S)$  - i.e.,  $S$  does not contain any of its boundary points.
- (4) A set  $S$  is **closed** iff  $S = \bar{S} = \text{int}(S) \cup \partial(S)$  - i.e.,  $S$  contains all of its boundary points.
- (5) A set  $S$  is **open** and **closed** iff  $\partial(S) = \emptyset$ .
- (6)  $\overline{A \cap B} \subseteq \bar{A} \cap \bar{B}$ .



### 2.3.13 Dense Set: $\bar{A} = X$

#### Definition 2.15 (Dense Set)

Let  $(X, d)$  be a metric space. A set  $A \subseteq X$  is **dense** in  $X$  if  $\bar{A} = X$ .



**Example 2.7**  $\mathbb{Q}$  is dense in  $\mathbb{R}$ : for all  $x \in \mathbb{Q}$ , let  $y_n = \frac{\lfloor 10^n x \rfloor}{10^n}$  with  $n \in \mathbb{N}$ . Then,  $\{y_n\} \subseteq \mathbb{Q}$ , and  $y_n \rightarrow x$  since  $|y_n - x| < 10^{-n}$ . Then  $\exists N \in \mathbb{N}$  s.t.  $\forall n > N$ ,  $|y_n - x| < \varepsilon$  (or  $y_n \in B_\varepsilon(x)$ ). This implies  $B_\varepsilon(x) \cap \mathbb{Q} \neq \emptyset$ , so  $\bar{\mathbb{Q}} = \mathbb{R}$ .

#### Corollary 2.1 ( $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$ )

$\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$  for  $n \in \mathbb{N}$ .



#### Theorem 2.16

If  $A$  is dense in  $X$ , for all  $x \in X$ :  $\forall \varepsilon > 0$ ,  $B_\varepsilon(x) \cap A \neq \emptyset$ .



#### Proof 2.2

Since  $A$  is dense in  $X$ , then for any  $x \in X = \bar{A}$ ,  $\forall \varepsilon > 0$ ,  $B_\varepsilon(x) \cap A \neq \emptyset$  by the definition of closure.

**Example 2.8** If  $\mathcal{A}$  is a collection of open subsets of  $\mathbb{R}^n$ , pairwise disjoint, that is,  $A_\lambda \cap A_{\lambda'} = \emptyset$  if  $\lambda \neq \lambda'$ , then  $\mathcal{A}$  is at most countable.

Let  $\Lambda$  be the set of indices of  $\mathcal{A}$ , i.e.,  $\mathcal{A} = \{A_\lambda\}_{\lambda \in \Lambda}$ . Choose an arbitrary  $\lambda \in \Lambda$ , and choose an arbitrary point  $x \in A_\lambda$ . Since  $A_\lambda$  is open,  $\exists \varepsilon > 0$  s.t.  $B_\varepsilon(x) \subseteq A_\lambda$ . Because  $\mathbb{Q}^n$  is dense in  $\mathbb{R}^n$ ,  $\exists q_\lambda \in \mathbb{Q}^n$  s.t.  $q_\lambda \in B_\varepsilon(x) \subseteq A_\lambda$ . Therefore,  $\forall \lambda \in \Lambda$ ,  $\exists q_\lambda \in \mathbb{Q}^n$  s.t.  $q_\lambda \in B_\varepsilon(x) \subseteq A_\lambda$ . Therefore,  $\forall \lambda \in \Lambda$ ,  $\exists q_\lambda \in \mathbb{Q}^n$  s.t.  $q_\lambda \in A_\lambda$ . Note that  $\mathcal{A}$  is a collection of pairwise disjoint open sets, then  $\forall \lambda, \lambda' \in \Lambda$ ,  $\lambda \neq \lambda'$ ,  $q_\lambda \neq q'_{\lambda'}$ . Therefore, we can consider a function  $f : \mathcal{A} \rightarrow \mathbb{Q}^n$ , with  $f(A_\lambda) = q_\lambda$ ,  $\forall \lambda \in \Lambda$ . Since  $\forall \lambda, \lambda' \in \Lambda$ ,  $\lambda \neq \lambda'$ ,  $q_\lambda \neq q'_{\lambda'}$ , then  $f$  is an injection. This indicates the cardinality of  $\mathbb{Q}^n$  is weakly larger than  $\mathcal{A}$ , then  $\mathcal{A}$  is at most countable.

### 2.3.14 Compact Set

#### Definition 2.16 (Compact Set)

$\mathbb{X} \subseteq \mathbb{R}^n$  is compact if it is closed and bounded.



**Example 2.9 Compact Set**  $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}; \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

### 2.3.15 Sublevel Set

#### Definition 2.17 (Sublevel Set)

The sublevel set of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  (for some level  $c \in \mathbb{R}$ ) is the set

$$\overline{L_c} = \{x \in \mathbb{R}^n : f(x) \leq c\}$$



### 2.3.16 Set Operations

#### Definition 2.18

A binary operation on a set  $A$  is a function  $* : A \times A \rightarrow A$ .

The operation is *associative* if  $a * (b * c) = (a * b) * c, \forall a, b, c \in A$ .

The operation is *commutative* if  $a * b = b * a, \forall a, b \in A$ .



### Example 2.10

$+, \circ$  are both *associative* and *commutative* operations on  $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$ ;  $-$  is neither *associative* nor *commutative* operation on  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ , but not  $\mathbb{N}$ .

#### Definition 2.19

A subset  $H \subset S$  is closed under  $*$  if  $a * b \in H$  for all  $a, b \in H$ .



#### Definition 2.20

$*$  has identity element  $e \in S$  if  $a * e = e * a = a$  for all  $a \in S$ .



### 2.3.17 Connected Sets (@ Lec 07 of ECON 204)

#### Definition 2.21 (Separated/Connected Set)

Two sets  $A, B$  in a metric space are **separated** if

$$\bar{A} \cap B = A \cap \bar{B} = \emptyset$$

where  $\bar{A}$  is the closure of  $A$ .

A set in a metric space is **connected** if it cannot be written as the union of two nonempty separated sets.



**Equivalent Definition:** A set  $Y$  in a metric space  $X$  is connected if there do not exist *open sets*  $A$  and  $B$  such that  $A \cap B = \emptyset$ ,  $Y \subseteq A \cup B$  and  $A \cap Y \neq \emptyset$  and  $B \cap Y \neq \emptyset$ .

**Note** that separation does not require that  $\bar{A} \cap \bar{B} = \emptyset$ . For example,  $[0, 1) \cup (1, 2]$  is not connected, since  $[0, 1)$  and  $(1, 2]$  are separated.

### 2.3.18 Theorem: Connected Set over $\mathbb{E}^1 \Leftrightarrow$ Interval (@ Lec 07 of ECON 204)

#### Theorem 2.17 (Connected Set over $\mathbb{E}^1 \Leftrightarrow$ Interval)

A set  $S \subseteq \mathbb{E}^1$  of real numbers is **connected** if and only if it is an **interval**, i.e. if  $x, y \in S$  and  $z \in (x, y)$ , then  $z \in S$ .



Notice that this result is only valid in  $\mathbb{R}$ . For example, connected sets in  $\mathbb{R}^n$  need not be intervals, or even convex.

### 2.3.19 Theorem: Continuous Image of Connected Set is Connected

#### Theorem 2.18 (Continuous Image of Connected Set is Connected)

Let  $X$  be a metric space and  $f : X \rightarrow Y$  be continuous. If  $C$  is a connected subset of  $X$ , then  $f(C)$  is connected.



### 2.3.20 Intermediate Value Theorem proved by connectedness

#### Corollary 2.2 (Intermediate Value Theorem)

If  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, and  $f(a) < d < f(b)$ , then there exists  $c \in (a, b)$  such that  $f(c) = d$ .



#### Proof 2.3

Since  $[a, b]$  is an interval, it is connected. So  $f([a, b])$  is connected, hence  $f([a, b])$  is an interval.  $f(a) \in f([a, b])$ , and  $f(b) \in f([a, b])$ , and  $d \in [f(a), f(b)]$ ; since  $f([a, b])$  is an interval,  $d \in f([a, b])$ , i.e. there exists  $c \in [a, b]$  such that  $f(c) = d$ . Since  $f(a) < d < f(b)$ ,  $c \neq a$ ,  $c \neq b$ , so  $c \in (a, b)$ .

## 2.4 Sequences

Sequences  $\{x_k\}_{k=1}^{\infty}$ , ... or  $\{x_k\}$ ,  $x_k \in \mathbb{R}^n$

#### Definition 2.22 (Subsequence)

Suppose  $\{x_n\}$  is a sequence and  $n_1 < n_2 < \dots$ , then  $\{x_{n_k}\}$  is called a **subsequence**.



### 2.4.1 Convergence of Sequences (@ Lec 03 of ECON 204)

**Definition 2.23 (Convergence: note  $x_k \rightarrow x, \lim_{k \rightarrow \infty} x_k = x$ )**

Let  $(X, d)$  be a metric space. A sequence  $\{x_k\}$  converges to  $x$  (written  $x_k \rightarrow x$  or  $\lim_{k \rightarrow \infty} x_k = x$ ) if  $\forall \varepsilon > 0, \exists N_\varepsilon$  s.t.  $d(x_k, x) < \varepsilon, \forall k \geq N_\varepsilon$



### 2.4.2 Limit Point (@ Lec 03 of ECON 204)

**Definition 2.24 (Limit Point)**

$x$  is a **limit point** of  $\{x_k\}$  if  $\exists$  a subsequence of  $\{x_k\}$  that converges to  $x$ .



**Definition 2.25 (Limit Point, alternative definition)**

Let  $(X, d)$  be a metric space. A point  $x \in X$  is a **limit point** of a set  $A \subseteq X$  if

$$\forall \varepsilon > 0, B_\varepsilon(x) \cap A \not\subseteq \{x\}$$



**Theorem 2.19 (Set of limit points of any set  $A \subseteq \mathbb{R}^n$  is closed.)**

The set of limit points of any set  $A \subseteq \mathbb{R}^n$  is closed.



**Proof 2.4**

Suppose  $\mathcal{S}$  is the set of limit points of  $A$ . Let  $D = \mathbb{R}^n \setminus \mathcal{S}$ . Take an arbitrary  $x \in D$ . Then  $x$  is not the limit point of  $A$ , so  $\exists \varepsilon_x > 0$  s.t.  $B_{\varepsilon_x}(x) \cap A \subseteq \{x\}$ . For any  $y \in B_{\varepsilon_x}(x)$ , since  $B_{\varepsilon_x}(x)$  is open,  $\exists \varepsilon_y > 0$  s.t.  $B_{\varepsilon_y}(y) \subseteq B_{\varepsilon_x}(x)$ . Let  $\varepsilon_m = \min\{d(x, y), \varepsilon_y\}$  such that  $B_{\varepsilon_m}(y) \subseteq B_{\varepsilon_x}(x)$  and  $x \notin B_{\varepsilon_m}(y)$ . So,  $B_{\varepsilon_m}(y) \cap A = \emptyset$ . So,  $y$  is not a limit point of  $A$ , that is  $y \in \mathbb{R}^n \setminus \mathcal{S} = D$ . Therefore,  $\forall y \in B_{\varepsilon_x}(x) \Rightarrow y \in D$ . We can conclude  $B_{\varepsilon_x}(x) \subseteq D, \forall x \in D$ . So,  $D$  is open.

**Theorem 2.20 (Subsequence has the same limit as the original sequence)**

All subsequences of a convergent sequence converge to the same limit as the original sequence.



**Theorem 2.21 (Uniqueness of Limits)**

In a metric space  $(X, d)$ , if  $x_k \rightarrow x$  and  $x_k \rightarrow x'$ , then  $x = x'$ .



### 2.4.3 Cluster Point (@ Lec 03 of ECON 204)

**Definition 2.26 (Cluster Point)**

An element  $c$  is a **cluster point** of a sequence  $\{x_n\}$  in a metric space  $(X, d)$  if  $\forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\}$  is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N}, \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$



**Example 2.11**  $x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ even} \\ \frac{1}{n}, & \text{if } n \text{ odd} \end{cases}$  has cluster points  $\{0, 1\}$ .

**Theorem 2.22 (Cluster Point  $\Leftrightarrow$  exists subsequence converges to it)**

Let  $(X, d)$  be a metric space,  $c \in X$ , and  $\{x_n\}$  a sequence in  $X$ . Then  $c$  is a cluster point of  $\{x_n\}$  if and only if there is a subsequence  $\{x_{n_k}\}$  such that  $\lim_{k \rightarrow \infty} x_{n_k} = c$ .



#### 2.4.4 Proposition: Sequences in $\mathbb{R}^n$ , Bounded above and non-decreasing sequence $\Rightarrow$ Converge

**Proposition 2.8**

If  $\{x_k\}$  is bounded above(below) and non-decreasing(non-increasing) it converges.



#### 2.4.5 Theorem: Increasing/Decreasing Sequences in $\mathbb{R}^n$ , Limit is sup/inf (@ Lec 03 of ECON 204)

**Theorem 2.23**

Let  $\{x_n\}$  be an increasing (decreasing) sequence of real numbers. Then  $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$  ( $\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$ ). In particular, the limit exists.



#### 2.4.6 Definition: $\limsup$ and $\liminf$ (@ Lec 03 of ECON 204)

Consider a sequence  $\{x_n\}$  of real numbers. Let

$$\alpha_n = \sup\{x_k : k \geq n\}$$

and

$$\beta_n = \inf\{x_k : k \geq n\}$$

Either  $\alpha_n = +\infty$  for all  $n$ , or  $\alpha_n \in \mathbb{R}$  and  $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots$ . Either  $\beta_n = -\infty$  for all  $n$ , or  $\beta_n \in \mathbb{R}$  and  $\beta_1 \leq \beta_2 \leq \beta_3 \leq \dots$

**Definition 2.27 (Lim Sups and Lim Insets)**

$$\limsup_{n \rightarrow \infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

$$\liminf_{n \rightarrow \infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$



### 2.4.7 Exists Limit $\Leftrightarrow \lim = \limsup = \liminf$ (@ Lec 03 of ECON 204)

#### Theorem 2.24

Let  $\{x_n\}$  be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} x_n = \gamma \in \mathbb{R} \cup \{-\infty, \infty\}$$

$$\Leftrightarrow \limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \gamma$$



### 2.4.8 Rising Sun Lemma: Sequence in $\mathbb{R}^n$ contains monotone subsequence (@ Lec 03 of ECON 204)

#### Theorem 2.25 (Rising Sun Lemma)

Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.



### 2.4.9 Bolzano-Weierstrass: Bounded Sequence in $\mathbb{R}^n$ contains a convergent subsequence (@ Lec 03 of ECON 204)

#### Theorem 2.26 (Bolzano-Weierstrass)

Every bounded sequence of real numbers contains a convergent subsequence.



## 2.5 Complete Metric Spaces (@ Lec 05 of ECON 204)

Roughly, a metric space is complete if “every sequence that ought to converge to a limit has a limit to converge to.”

### 2.5.1 Cauchy Sequence

#### Definition 2.28 (Cauchy Sequence)

A sequence  $\{x_k\}$  in a metric space  $(X, d)$  is **Cauchy** if  $\forall \varepsilon > 0, \exists N(\varepsilon)$  s.t.

$$d(x_n, x_m) < \varepsilon, \forall n, m > N(\varepsilon)$$



**Note:** Cauchy property depends only on the sequence and the metric  $d$ , not on the ambient metric space.

**Note:** In  $\mathbb{E}^1 = (\mathbb{R}, \|\cdot\|)$ ,  $\{x_k\}$  converges  $\Leftrightarrow \{x_k\}$  is Cauchy.

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to. Any sequence that does converge must be Cauchy, however, by the argument above.

### 2.5.2 Theorem: Convergent $\Rightarrow$ Cauchy

**Theorem 2.27 (Convergent  $\Rightarrow$  Cauchy)**

Every convergent sequence in a metric space is Cauchy.



### 2.5.3 Complete Metric Space and Banach Space

**Definition 2.29 (Complete Metric Spaces)**

A metric space  $(X, d)$  is **complete** if every Cauchy sequence  $\{x_n\} \subseteq X$  converges to a limit  $x \in X$ .



**Example 2.12**

1. Consider the earlier example of  $X = (0, 1]$  with  $d$  the usual Euclidean metric. Since  $x_n = \frac{1}{n}$  is Cauchy but does not converge in that metric space,  $((0, 1], d)$  is not complete.
2.  $\mathbb{Q}$  is not complete in the Euclidean metric:  $x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} \rightarrow \sqrt{2}$ .

**Definition 2.30 (Banach Space)**

A **Banach space** is a normed space that is complete in the metric generated by its norm.



**Theorem 2.28 ( $\mathbb{E}^1$  is complete)**

$\mathbb{R}$  is complete with the usual metric (so  $\mathbb{E}^1$  is a Banach space).



**Theorem 2.29 ( $\mathbb{E}^n$  is complete)**

$\mathbb{E}^n$  is complete for every  $n \in \mathbb{N}$ .



**Theorem 2.30 ( $(C(X), \|\cdot\|_\infty)$  is Complete)**

Given  $X \subseteq \mathbb{R}^n$ , let  $C(X)$  be the set of bounded continuous functions from  $X$  to  $\mathbb{R}$  with

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Then  $C(X)$  is a Banach space.



### 2.5.4 Theorem: Subset of Complete Metric Space is Complete $\Leftrightarrow$ Subset is Closed

**Theorem 2.31 (Subset of Complete Metric Space is Complete  $\Leftrightarrow$  Subset is Closed)**

Suppose  $(X, d)$  is a complete metric space and  $Y \subseteq X$ . Then  $(Y, d|_Y)$  is complete if and only if  $Y$  is a closed subset of  $X$ . (where  $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ ;  $d|_Y : \mathbb{Y} \times \mathbb{Y} \rightarrow \mathbb{R}_+$ )



## 2.6 Compactness over Metric Spaces (@ Lec 06 of ECON 204)

### 2.6.1 Open Cover

#### Definition 2.31 (Open Cover)

A collection of sets  $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$  in a metric space  $(X, d)$  is an **open cover** of  $A \subseteq X$  if

1.  $U_\lambda$  is open for all  $\lambda \in \Lambda$  and
2.  $A \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda$



### 2.6.2 Definition of Compact Set

#### Definition 2.32 (Compact Set)

A set  $A$  in a metric space is **compact** if every open cover of  $A$  contains a finite subcover of  $A$ . In other words, if  $\{U_\lambda : \lambda \in \Lambda\}$  is an open cover of  $A$ , there exist  $n \in N$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that

$$A \subseteq \bigcup_{i=1}^n U_{\lambda_i}$$



#### Example 2.13

1.  $(0, 1]$  is not a compact set over  $\mathbb{E}^1$ : Consider the open cover  $\{(\frac{1}{m}, 2) : m \in \mathbb{N}\}$ .
2.  $[0, \infty)$  is not a compact set over  $\mathbb{E}^1$ : Consider the open cover  $\{(-1, m) : m \in \mathbb{N}\}$ .

### 2.6.3 Theorem: Closed Subset of Compact Set is Compact

#### Theorem 2.32 (Closed Subset of Compact Set is Compact)

*Every closed subset  $A$  of a compact metric space  $(X, d)$  is compact.*



#### Proof 2.5

Let  $\{U_\lambda : \lambda \in \Lambda\}$  be an open cover of  $A$ .

Let  $U'_\lambda = U_\lambda \cup (X \setminus A)$ . Since  $A$  is closed,  $X \setminus A$  is open; Since  $U_\lambda$  and  $X \setminus A$  are both open,  $U'_\lambda$  is open.

By the definition of open set, if  $a \in A$ ,  $a \in \bigcup_{\lambda \in \Lambda} U'_\lambda$ . If  $a \in X \setminus A$ ,  $a \in U'_\lambda$  for sure. So,  $X \subseteq \bigcup_{\lambda \in \Lambda} U'_\lambda$ .

Hence, we can conclude  $\{U_\lambda \cup (X \setminus A) : \lambda \in \Lambda\}$  is an open cover of  $X$ .

Since  $X$  is compact, there exists  $n \in N$  and  $\lambda_1, \dots, \lambda_n \in \Lambda$  such that  $X \subseteq \bigcup_{i=1}^n U'_{\lambda_i}$ .

Then,

$$a \in A \Rightarrow a \in X$$

$$\Rightarrow a \in U'_{\lambda_i} \text{ for some } i$$

$$\Rightarrow a \in U_{\lambda_i} \text{ for some } i$$

Hence, we can conclude  $A \subseteq \bigcup_{i=1}^n U_{\lambda_i}$ . Thus,  $A$  is compact.

### 2.6.4 Theorem: Compact Set over Metric Space is Closed

**Theorem 2.33 (Compact Set over Metric Space is Closed)**

If  $A$  is a compact subset of the metric space  $(X, d)$ , then  $A$  is closed.



**Proof 2.6**

Suppose  $X \setminus A$  is not open, so we can find a point  $x \in X \setminus A$  such that, for every  $\varepsilon > 0$ ,  $A \cap B_\varepsilon(x) \neq \emptyset$ , and hence  $A \cap B_\varepsilon[x] \neq \emptyset$ . For  $n \in \mathbb{N}$ , let

$$U_n = X \setminus B_{1/n}[x]$$

Each  $U_n$  is open, and

$$\cup_{n \in \mathbb{N}} U_n = X \setminus \{x\} \supseteq A$$

since  $x \notin A$ . Therefore,  $\{U_n : n \in \mathbb{N}\}$  is an open cover for  $A$ . Since  $A$  is compact, there is a finite subcover  $\{U_{n_1}, \dots, U_{n_k}\}$ . Let  $n = \max\{n_1, \dots, n_k\}$ . Then

$$\begin{aligned} U_n &= X \setminus B_{1/n}[x] \\ &\supseteq X \setminus B_{1/n_j}[x] \quad (j = 1, \dots, k) \\ U_n &\supseteq \cup_{j=1}^k U_{n_j} \\ &\supseteq A \end{aligned}$$

But  $A \cap B_{1/n}[x] \neq \emptyset$ , so  $A \not\subseteq X \setminus B_{1/n}[x] = U_n$ . This is a contradiction, which proves that  $A$  is closed.



### 2.6.5 Definition of Sequentially Compact

**Definition 2.33 (Sequentially Compact)**

A set  $A$  in a metric space  $(X, d)$  is **sequentially compact** if every sequence of elements of  $A$  contains a convergent subsequence whose limit lies in  $A$ .



### 2.6.6 Theorem: Compact $\Leftrightarrow$ Sequentially Compact

**Theorem 2.34 (Compact  $\Leftrightarrow$  Sequentially Compact)**

A set  $A$  in a metric space  $(X, d)$  is **compact** if and only if it is **sequentially compact**.



### 2.6.7 Definition of Totally Bounded

#### Definition 2.34 (Totally Bounded)

A set  $A$  in a metric space  $(X, d)$  is **totally bounded** if, for every  $\varepsilon > 0$ ,

$$\exists x_1, \dots, x_n \in A \text{ s.t. } A \subseteq \bigcup_{i=1}^n B_\varepsilon(x_i).$$



#### Claim 2.2

"Totally bounded" is stronger than "bounded".



#### Example 2.14

1.  $X = [0, 1]$  is totally bounded with Euclidean metric: Given  $\varepsilon > 0$ , let  $x_i = \frac{i}{n}, i = 1, \dots, n - 1$ .

2.  $X = [0, 1]$  is not totally bounded with discrete metric  $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$ : Given  $\varepsilon = \frac{1}{2}$ ,

$B_\varepsilon(x_i) = \{x_i\}$ . Hence,  $\bigcup_i^n B_\varepsilon(x_i) = \{x_1, \dots, x_n\} \not\supseteq [0, 1]$ .

**Note:**  $X$  is bounded but not totally bounded.

### 2.6.8 Theroem: Compact $\Leftrightarrow$ Complete/Closed and Totally Bounded

#### Theorem 2.35 (Subset is Compact $\Leftrightarrow$ Complete and Totally Bounded)

Let  $A$  be a subset of a metric space  $(X, d)$ . Then  $A$  is **compact** if and only if  $A$  is **complete** and **totally bounded**.



#### Proof 2.7

$\text{Compact} \Rightarrow \text{totally bounded}$  (by definition).  $\text{Compact} \Leftrightarrow \text{sequentially compact} \Rightarrow \text{every Cauchy sequence converges to a limit in the subset (i.e., Complete)}$ .

Conversely, suppose complete and totally bounded. Because totally bounded, we can extract a Cauchy subsequence. Because complete, subsequence converges to a limit in the subset, which shows sequentially compact and hence compact.

#### Corollary 2.3 (Subset is Compact $\Leftrightarrow$ Closed and Totally Bounded)

Let  $A$  be a subset of a metric space  $(X, d)$ . Then  $A$  is **compact** if and only if  $A$  is **closed** and **totally bounded**.



#### Proof 2.8

Directly by theorem 2.31 and theorem 2.35.

### 2.6.9 Heine-Borel Theorem: With Euclidean Metric, Compact $\Leftrightarrow$ Closed and Bounded

**Claim 2.3 (With Euclidean Metric, Compact  $\Leftrightarrow$  Closed and Bounded)**

If  $A \subseteq \mathbb{E}^n$ , then  $A$  is **totally bounded** if and only if  $A$  is **bounded**.



**Theorem 2.36 (Heine-Borel Theorem: With Euclidean Metric, Compact  $\Leftrightarrow$  Closed and Bounded)**

If  $A \subseteq \mathbb{E}^n$ , then  $A$  is **compact** if and only if  $A$  is **closed and bounded**.



### 2.6.10 Theorem: Continuous Images of Compact Set is Compact

**Theorem 2.37 (Continuous Images of Compact Set is Compact)**

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. If  $f : X \rightarrow Y$  is continuous and  $C$  is a compact subset of  $(X, d)$ , then  $f(C)$  is compact in  $(Y, \rho)$ .



# Chapter 3 Functions

## 3.1 Definitions of Function

### Definition 3.1 (Function)

*Function* is a rule  $\sigma : A \rightarrow B$  that assigns an element  $B$  to *every* element of  $A$ .  $\forall a \in A, \sigma(a) \in B$ .



### 3.1.1 Image, Preimage, Fiber

#### Definition 3.2

1.  $A$  is the domain of  $\sigma$ ,  $B$  is the range of  $\sigma$ .
2. We call  $\sigma(a) = \text{value of } \sigma \text{ at } a$  as the image of  $a$ .
3. A set  $C \subset B$ , we call  $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$  as the preimage of  $C$ .
4. An element  $b \in B$ , we call  $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$  as the fiber of  $b$ .



### 3.1.2 Function Image's Properties

#### Theorem 3.1

Let  $X, Y$  be sets,  $f : X \rightarrow Y$  be a function,  $A, B \subseteq X$  be arbitrary subsets of  $X$ , and  $C, D \subseteq Y$  be arbitrary subsets of  $Y$ . Then

1.  $f(A \cap B) \subseteq f(A) \cap f(B)$ . (*Strict subset example:*  $y = x^2$ .  $A = (-2, 0)$  and  $B = (0, 2)$ .)
2.  $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
3.  $f(A \cup B) = f(A) \cup f(B)$
4.  $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
5.  $A \subseteq f^{-1}(f(A))$  (*Strict subset example:*  $y = x^2$ ,  $A = [0, 2]$ . Then  $f(A) = [0, 4]$ ,  $f^{-1}(f(A)) = [-2, 2]$ .)
6.  $f(f^{-1}(C)) \subseteq C$  (*Strict subset example:*  $X = [0, 1], Y = [0, 2], f(x) = x$ . Let  $C = [0.5, 1.5]$ . Then  $f^{-1}(C) = [0.5, 1]$ ,  $f(f^{-1}(C)) = [0.5, 1]$ .)



### 3.1.3 Composition of functions

#### Definition 3.3 (Function Composition)

The function composition  $\circ$  is an operation that takes two functions  $\sigma : A \rightarrow B$  and  $\tau : B \rightarrow C$ , , and produces a function  $\tau \circ \sigma : A \rightarrow C$  that fulfills  $\forall a \in A, (\tau \circ \sigma)(a) = \tau(\sigma(a))$ .



### 3.1.4 Function Composition is Associative

#### Proposition 3.1 (Associativity of Functions)

Suppose  $\sigma : A \rightarrow B, \tau : B \rightarrow C, \rho : C \rightarrow D$  are functions and  $\circ$  is the function composition, then

$$\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma.$$



### 3.1.5 Homeomorphism (@ Lec 04 of ECON 204)

#### Definition 3.4 (Homeomorphism)

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A function  $f : X \rightarrow Y$  is called a **homeomorphism** if it is one-to-one, onto, continuous, and its inverse function is continuous.



## 3.2 Injection, Surjection, Bijection

### 3.2.1 Definitions: Injective, surjective, bijective

A function  $\sigma : A \rightarrow B$  is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

### 3.2.2 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

#### Lemma 3.1 (Lemma 1.1.7)

Suppose  $\sigma : A \rightarrow B, \tau : B \rightarrow C$  are functions,

If  $\sigma, \tau$  are injective, then  $\tau \circ \sigma$  is injective.

If  $\sigma, \tau$  are surjective, then  $\tau \circ \sigma$  is surjective.

If  $\sigma, \tau$  are bijective, then  $\tau \circ \sigma$  is bijective.



### 3.2.3 Theorem: A function is bijective $\Leftrightarrow \exists$ inverse

#### Theorem 3.2 (Bijective $\Leftrightarrow \exists$ Inverse)

A function  $\sigma : A \rightarrow B$  is a bijection if and only if  $\exists$  a function  $\tau : B \rightarrow A$  such that

$$\sigma \circ \tau = id_B = \text{identity on } B(id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$

Such  $\tau$  is unique, called inverse of  $\sigma$ ,  $\tau = \sigma^{-1}$ .



## 3.3 Function Continuity

### 3.3.1 Continuous Function in $\mathbb{R}$ with Euclidean norm

#### Definition 3.5 (Continuity at Point)

A real-valued function  $f$  is continuous at  $x$  if

"For every  $\{x_k\}$  converging to  $x$  satisfies that  $\lim_{k \rightarrow \infty} f(x_k) = f(x)$ ".

Equivalent definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \|y - x\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$



Continuity at  $x_0$  requires:

1.  $f(x_0)$  is defined; and
2. either
  - $x_0$  is an isolated point of  $X$ , i.e.  $\exists \varepsilon > 0$  s.t.  $B_\varepsilon(x) = \{x\}$ ; or
  - $\lim_{x \rightarrow x_0} f(x)$  exists and equals  $f(x_0)$

#### Definition 3.6 (Continuous Function)

A real-valued function  $f$  is continuous if it is continuous at all points in its domain.



### 3.3.2 Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)

#### Definition 3.7 (Continuity in Metric Spaces)

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces. A function  $f : X \rightarrow Y$  is **continuous** at a point  $x_0 \in X$  if

$$\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon.$$



### 3.3.3 Theorem: Continuous $\Leftrightarrow$ Preimage of open set is open (@ Lec 04 of ECON 204)

**Theorem 3.3 (Continuous  $\Leftrightarrow$  Preimage of open set is open)**

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces, and  $f : X \rightarrow Y$ . Then  $f$  is continuous if and only if

$$f^{-1}(A) = \{x \in X : f(x) \in A\} \text{ is open in } X \quad \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$



**Proposition 3.2**

Let  $A \subseteq \mathbb{R}$  be an open set and  $f : A \rightarrow \mathbb{R}$ . The two following statements are equivalent:

(a).  $f$  is continuous.

(b). for all  $c \in \mathbb{R}$ , the sets  $E[f < c] = \{x \in A \mid f(x) < c\}$  and  $E[f > c] = \{x \in A \mid f(x) > c\}$  are open.



**Proof 3.1**

1. " $(a) \Rightarrow (b)$ ": Suppose  $f$  is continuous. Consider any  $c \in \mathbb{R}$ . Since  $(-\infty, c)$  and  $(c, \infty)$  are open and  $f$  is continuous,  $E[f < c] = f^{-1}((-\infty, c))$  and  $E[f > c] = f^{-1}((c, \infty))$  are open.

2. " $(b) \Rightarrow (a)$ ": Suppose  $E[f < c] = \{x \in A \mid f(x) < c\}$  and  $E[f > c] = \{x \in A \mid f(x) > c\}$  are open for all  $c \in \mathbb{R}$ . Let  $C \subseteq \mathbb{R}$  be an arbitrary open set over  $\mathbb{R}$ . There are following cases about the  $C$ :

(1).  $C = (-\infty, k)$ , where  $k \in \mathbb{R}$ . In this case  $f^{-1}(C) = E[f < k]$ , which is open by the given condition.

(2).  $C = (k, \infty)$ , where  $k \in \mathbb{R}$ . In this case  $f^{-1}(C) = E[f > k]$ , which is open by the given condition.

(3).  $C = \emptyset$ . In this case  $f^{-1}(C) = \{x \in A \mid f(x) \in \emptyset\} = E[f > k_1] \cap E[f < k_2], \forall k_1 \geq k_2$ , which is a finite intersection of two open sets. So,  $f^{-1}(C)$  is open.

(4).  $C = (k_1, k_2)$ , where  $k_1, k_2 \in \mathbb{R}$  s.t.  $k_1 < k_2$ . In this case  $f^{-1}(C) = \{x \in A \mid f(x) \in (k_1, k_2)\} = E[f > k_1] \cap E[f < k_2]$ , which is a finite intersection of two open sets. So,  $f^{-1}(C)$  is open.

(5).  $C = (-\infty, k_2) \cup (k_1, \infty)$ . In this case  $f^{-1}(C) = \{x \in A \mid f(x) \in (-\infty, k_2) \cup (k_1, \infty)\} = E[f > k_1] \cup E[f < k_2], \forall k_1, k_2 \in \mathbb{R}$  s.t.  $k_1 > k_2$ .  $f^{-1}(C)$  is a union of two open set, which is open.

(6).  $C = \mathbb{R}$ . In this case  $f^{-1}(C) = \{x \in A \mid f(x) \in \mathbb{R}\} = \{x \in A \mid f(x) \in (-\infty, k_2) \cup (k_1, \infty)\} = E[f > k_1] \cup E[f < k_2], \forall k_1, k_2 \in \mathbb{R}$  s.t.  $k_1 < k_2$ .  $f^{-1}(C)$  is a union of two open set, which is open.

Hence,  $f^{-1}(C) = \{x \in A : f(x) \in C\}$  is open  $\forall C \subseteq \mathbb{R}$  s.t.  $C$  is open in  $\mathbb{R}$

### 3.3.4 Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)

#### Theorem 3.4 (Continuity is preserved in composition)

Let  $(X, d_X)$ ,  $(Y, d_Y)$  and  $(Z, d_Z)$  be metric spaces. If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then  $g \circ f : X \rightarrow Z$  is continuous.



#### Proof 3.2

Proved by previous theorem 3.3.

### 3.3.5 Uniform Continuity (@ Lec 04 of ECON 204)

#### Definition 3.8 (Uniformly Continuous)

Suppose  $f : (X, d) \rightarrow (Y, \rho)$ .  $f$  is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$



#### Claim 3.1

**Uniformly Continuous** implies (is stronger than) **Continuous**.

( $f$  is continuous if  $\forall x_0 \in X, \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0$  s.t.  $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$ )



Given  $\varepsilon > 0$ , "uniformly continuous" requires  $\delta(\varepsilon)$  that works for all  $x_0 \in X$ .

### 3.3.6 Theorem: Continuous function with Compact domain is Uniformly Continuous (@ Lec 06 of ECON 204)

#### Theorem 3.5 (Continuous function with Compact domain is Uniformly Continuous)

Let  $(X, d)$  and  $(Y, \rho)$  be metric spaces,  $C$  a **compact** subset of  $X$ , and  $f : C \rightarrow Y$  a **continuous** function.

Then  $f$  is **uniformly continuous** on  $C$ .



#### Proof 3.3

Fix  $\varepsilon > 0$ . Since  $f$  is continuous, given  $c \in C$ , we can find  $\delta(c) > 0$  such that

$$\forall x \in C, d(x, c) < 2\delta(c) \Rightarrow \rho(f(x), f(c)) < \frac{\varepsilon}{2}$$

Let  $U_c = B_{\delta(c)}(c)$ , then  $\{U_c : c \in C\}$  is an open cover of  $C$ . Since  $C$  is compact, there is a finite subcover  $\{U_{c_1}, \dots, U_{c_n}\} = \{B_{\delta(c_1)}(c_1), \dots, B_{\delta(c_n)}(c_n)\}$ . Let  $\delta = \min\{\delta(c_1), \dots, \delta(c_n)\}$ .

Given  $x, y \in C$  with  $d(x, y) < \delta$ , note that  $x \in U_{c_i}$  for some  $i \in \{1, \dots, n\}$ , so  $d(x, c_i) < \delta(c_i)$ .

$$\begin{aligned} d(y, c_i) &\leq d(y, x) + d(x, c_i) \\ &< \delta + \delta(c_i) \\ &\leq \delta(c_i) + \delta(c_i) = 2\delta(c_i) \end{aligned}$$

So,

$$\begin{aligned} \rho(f(x), f(y)) &\leq \rho(f(x), f(c_i)) + \rho(f(c_i), f(y)) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

That is,  $d(x, y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$ , which proves that  $f$  is uniformly continuous.

### 3.3.7 Lipschitz Continuous (@ Lec 04 of ECON 204)

#### Definition 3.9 (Lipschitz (Continuous) in Normed Vector Space)

Let  $X, Y$  be normed vector spaces,  $\mathbb{E} \subseteq X$ .

(1). A function  $f : X \rightarrow Y$  is **Lipschitz** on  $\mathbb{E}$  satisfies

$$\exists \gamma > 0, \|f(\mathbf{x}) - f(\mathbf{y})\|_Y \leq \gamma \|\mathbf{x} - \mathbf{y}\|_X, \forall \mathbf{x}, \mathbf{y} \in \mathbb{E}$$

or we call  $\gamma$ -**Lipschitz continuous**;

(2).  $f$  is **locally Lipschitz** on  $\mathbb{E}$  if

$$\forall x_0 \in \mathbb{E}, \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap \mathbb{E}$$



#### Claim 3.2

- 1). If  $f$  is  $\gamma$ -Lipschitz continuous, then it is also  $(\gamma + 1)$ -Lipschitz continuous. The minimal such  $\gamma$  is called a Lipschitz constant of function  $f$
- 2). The definition can be extended to general metric spaces by replacing  $\|\cdot\|_X$  and  $\|\cdot\|_Y$  to general metrics.



#### Example 3.1 (Lipschitz)

1.  $f(x) = 2x$  is 2-Lipschitz continuous;
2. What about  $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$ , where  $\mathbf{A}$  is a matrix? Spectral norm  $\|\mathbf{A}\|_2$  (for Euclidean norm).
3. What about  $f(x) = x^2$ ? Not Lipschitz continuous, or the Lipschitz constant is  $\infty$ .

**Example 3.2 (Locally Lipschitz)** Every  $C^1$  function is locally Lipschitz. (Recall that a function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$  is said to be  $C^1$  if all its first partial derivatives exist and are continuous.)

**Claim 3.3 (Lipschitz continuity vs. continuity vs. uniform continuity)**

*Lipschitz continuity is stronger than either continuity or uniform continuity:*

- *locally Lipschitz  $\Rightarrow$  continuous*
- *Lipschitz  $\Rightarrow$  uniformly continuous*

**3.3.8 Contraction (from the view of Lipschitz constant)**

- a). If the Lipschitz constant  $\gamma \leq 1$ , then  $f$  is called a non-expansive mapping.
- b). If the Lipschitz constant  $\gamma < 1$ , then  $f$  is called a contraction mapping.

**Example 3.3**

1.  $f(x) = 2x$  is not a contraction mapping;  $f(x) = 0.5x$  is.
2.  $f(x) = Ax$  is a contraction mapping (with respect to Euclidean norm) iff  $\|A\|_2 < 1$ .

**3.4 Extreme Value, Intermediate Value of Functions****3.4.1 Intermediate Value Theorem (@ Lec 02 of ECON 204)****Theorem 3.6 (Intermediate Value Theorem)**

*Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous, and  $f(a) < d < f(b)$ . Then there exists  $c \in (a, b)$  such that  $f(c) = d$ .*

**3.4.2 Coercive Function****Definition 3.10 (Coercive)**

*A real-valued function  $f : \mathbb{X} \rightarrow \mathbb{R}$  is coercive if for every  $\{x_k\} \subset \mathbb{X}$  s.t.  $\|x_k\| \rightarrow \infty$ ,  $f(x_k) \rightarrow \infty$*

**Example 3.4 Check coercive**

- 1)  $x \in \mathbb{R}^2$ ,  $f(x) = x_1^2 + x_2^2$  - coercive
- 2)  $x \in \mathbb{R}$ ,  $f(x) = 1 - e^{-|x|}$  - not coercive
- 3)  $x \in \mathbb{R}^2$ ,  $f(x) = x_1^2 + x_2^2 - 2x_1x_2$  - not coercive (we need  $f(x_k) \rightarrow \infty$  for all  $\|x_k\| \rightarrow \infty$ )

### 3.4.3 Extreme of Functions

#### Definition 3.11 (Extreme of Functions)

Let  $\mathbb{X} \subseteq \mathbb{R}^n$ ,  $f : \mathbb{X} \rightarrow \mathbb{R}$

$$\inf_{x \in \mathbb{X}} f(x) = \inf\{f(x) : x \in \mathbb{X}\}$$



If  $\exists x^* \in \mathbb{X}$  s.t.  $\inf f(x) = f(x^*)$ . Then,  $f$  achieves (attains) its minimum and  $f(x^*) = \min_{x \in \mathbb{X}} f(x)$

$x^*$  is called a **minimizer** of  $f$ , written as  $x^* \in \arg \min_{x \in \mathbb{X}} f(x)$ . If  $x^*$  is unique, we write  $x^* = \arg \min_{x \in \mathbb{X}} f(x)$

Similarly, supremum and maximum of  $f$ .

### 3.4.4 Weierstrass' Theorem (Extreme value Theorem) (@ Lec 06 of ECON 204)

#### Theorem 3.7 (Weierstrass' Theorem (Extreme value Theorem))

Given a **compact** set  $C$  in a metric space  $(X, d)$  and a **continuous** function  $f : C \rightarrow \mathbb{R}$ , then  $f$  is bounded on  $C$  and attains its minimum and maximum on  $C$  i.e.,

$$\exists x_1 \in C \text{ s.t. } f(x_1) = \inf f(C)$$

$$\exists x_2 \in C \text{ s.t. } f(x_2) = \sup f(C)$$



#### Proof 3.4

Since  $C$  is compact and  $f$  is continuous,  $f(C) \subset \mathbb{R}$  is compact by theorem 2.37. Hence,  $f(C)$  is closed and bounded.

1. *Proof of maximum:* Let  $M = \sup f(C) < \infty$ . Then  $\forall m > 0$  there exists  $y_m \in f(C)$  such that

$$M - \frac{1}{m} \leq y_m \leq M$$

So  $y_m \rightarrow M$  and  $\{y_m\} \subseteq f(C)$ . Since  $f(C)$  is closed,  $M \in f(C)$ , i.e., there exists  $c \in C$  such that  $f(c) = M = \sup f(C)$ .

2. *Proof of minimum:* Let  $M = \inf f(C) > -\infty$ . Then  $\forall m > 0$  there exists  $y_m \in f(C)$  such that

$$M \geq y_m \geq M + \frac{1}{m}$$

So  $y_m \rightarrow M$  and  $\{y_m\} \subseteq f(C)$ . Since  $f(C)$  is closed,  $M \in f(C)$ , i.e., there exists  $c \in C$  such that  $f(c) = M = \inf f(C)$ .

### 3.4.5 Corollary: Continuous and Coercive function attains its Minimum on Closed Set

**Corollary 3.1 (Continuous and Coercive function attains its Minimum on Closed Set)**

Let  $f$  be continuous and coercive on closed set  $\mathbb{X}$  (not necessarily bounded), it attains its min on  $\mathbb{X}$ .



**Proof 3.5**

Consider  $\{\sigma_k\}$  as in proof of WT.

Since  $f$  is closed,  $f(x) < \infty, \forall x \in \mathbb{X}$ . And  $f$  is coercive on  $\mathbb{X}$ , which means  $f(x) \rightarrow \infty$  if  $\|x\| \rightarrow \infty$ .

Hence,  $\{\sigma_k\} \in \mathbb{X}$  is bounded. Rest of proof same as proof of WT.

**Example 3.5**  $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) Does  $f$  achieve its min and max on  $\mathbb{X}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leq 6\}$ ?

-  $\mathbb{X}_1$  is compact and  $f$  is continuous. Both min and max are achieved (WT).

2) Does  $f$  achieve its min and max over  $\mathbb{R}^3$ ?

-  $f \rightarrow \infty$  whenever  $\|x\| \rightarrow \infty \Rightarrow f$  is coercive.

-  $\mathbb{R}^3$  is closed.

$\Rightarrow f$  achieves its min. on  $\mathbb{R}^3$  by corollary to WT.

- max does not exist since  $f \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ .

3) Does  $f$  achieve its min and max over  $\mathbb{X}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$ ?

-  $\mathbb{X}_2$  is closed, but not bounded.

- Since  $f$  is coercive, min achieved.

- max does not exist since setting  $x_1 = 0, x_2 = 3 - x_3$  and letting  $x_3 \rightarrow \infty$  makes  $f \rightarrow \infty$

## 3.5 Monotonic Function (@ Lec 05 of ECON 204)

**Definition 3.12**

A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **monotonically increasing** if  $y > x \Rightarrow f(y) \geq f(x)$ .



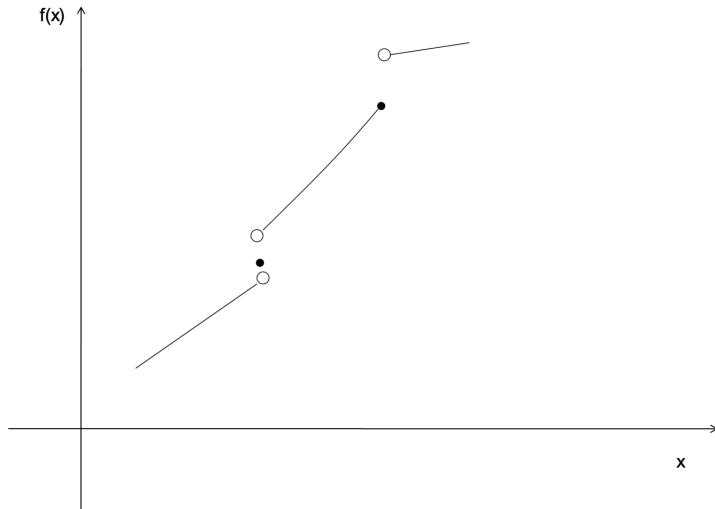
### 3.5.1 Theorem: Monotonically Increasing $\Rightarrow$ One-sided Limits exist

**Theorem 3.8**

Let  $a, b \in \mathbb{R}$  with  $a < b$ , and let  $f : (a, b) \rightarrow \mathbb{R}$  be monotonically increasing. Then the one-sided limits:

$f(t^+) = \lim_{u \rightarrow t^+} f(u)$  and  $f(t^-) = \lim_{u \rightarrow t^-} f(u)$  exist and are real numbers for all  $t \in (a, b)$ .





**Figure 3.1:** A monotonic function has only simple jump discontinuities.

### 3.5.2 Theorem: Monotonically Increasing $\Rightarrow$ Set of Points are discontinuous is finite (possibly empty) or countable

#### Theorem 3.9

Let  $a, b \in \mathbb{R}$  with  $a < b$ , and let  $f : (a, b) \rightarrow \mathbb{R}$  be monotonically increasing. Then  $D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$  is finite (possibly empty) or countable.



## 3.6 Contraction Mapping Theorem (@ Lec 05 of ECON 204)

### 3.6.1 Contractions

#### Definition 3.13

Let  $(X, d)$  be a nonempty complete metric space. An operator is a function  $T : X \rightarrow X$ . An operator  $T$  is a **contraction of modulus  $\beta$**  if  $\beta < 1$  and

$$d(T(x), T(y)) \leq \beta d(x, y), \forall x, y \in X$$



A contraction shrinks distances by a *uniform* factor  $\beta < 1$ .

### 3.6.2 Theorem: Contraction $\Rightarrow$ Uniformly Continuous

#### Theorem 3.10 (Contraction $\Rightarrow$ Uniformly Continuous)

*Every contraction is uniformly continuous.*



**Proof 3.6**

Let  $\delta = \frac{\varepsilon}{\beta}$ .

**3.6.3 Theorem: Blackwell's Sufficient Conditions for Contraction**

Let  $X$  be a set, and let  $B(X)$  be the set of all bounded functions from  $X$  to  $\mathbb{R}$ . Then  $(B(X), \|\cdot\|_\infty)$  is a normed vector space.

(Notice that below we use shorthand notation that identifies a constant function with its constant value in  $\mathbb{R}$ , that is, we write interchangeably  $a \in \mathbb{R}$  and  $a : X \rightarrow \mathbb{R}$  to denote the function such that  $a(x) = a, \forall x \in X$ .)

**Theorem 3.11 (Blackwell's Sufficient Conditions)**

Consider  $B(X)$  with the sup norm  $\|\cdot\|_\infty$ . Let  $T : B(X) \rightarrow B(X)$  be an operator satisfying

1. (monotonicity)  $f(x) \leq g(x), \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x), \forall x \in X$
2. (discounting)  $\exists \beta \in (0, 1)$  such that for every  $a \geq 0$  and  $x \in X$ ,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then  $T$  is a contraction with modulus  $\beta$ .

**Proof 3.7**

Fix  $f, g \in B(X)$ . By the definition of the sup norm,

$$f(x) \leq g(x) + \|f - g\|_\infty \quad \forall x \in X$$

Then

$$(Tf)(x) \leq (T(g + \|f - g\|_\infty))(x) \leq (Tg)(x) + \beta \|f - g\|_\infty \quad \forall x \in X$$

where the first inequality above follows from monotonicity, and the second from discounting. Thus

$$(Tf)(x) - (Tg)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X$$

Reversing the roles of  $f$  and  $g$  above gives

$$(Tg)(x) - (Tf)(x) \leq \beta \|f - g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta \|f - g\|_\infty$$

Thus  $T$  is a contraction with modulus  $\beta$

## 3.7 Fixed Point Theorem (@ Lec 05 of ECON 204)

### 3.7.1 Fixed Point

#### Definition 3.14 (Fixed Point)

A **fixed point** of an operator  $T$  is element  $x^* \in X$  such that  $T(x^*) = x^*$ .



### 3.7.2 Contraction Mapping Theorem: Existence and Uniqueness of Fixed Point

#### Theorem 3.12 (Contraction Mapping Theorem)

Let  $(X, d)$  be a nonempty complete metric space and  $T : X \rightarrow X$  a contraction with modulus  $\beta < 1$ .

Then

1.  $T$  has a unique fixed point  $x^*$ .
2. For every  $x_0 \in X$ , the sequence defined by

$$x_1 = T(x_0)$$

$$x_2 = T(x_1) = T(T(x_0)) = T^2(x_0)$$

$\vdots$

$$x_{n+1} = T(x_n) = T^{n+1}(x_0)$$

converges to  $x^*$ .



Note that the theorem asserts both the **existence** and **uniqueness** of the fixed point, as well as giving an **algorithm** to find the fixed point of a contraction.

#### Proof 3.8

Define the sequence  $\{x_n\}$  as above. Then,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$

Then for any  $n > m$ ,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_1, x_0) \sum_{i=m}^{n-1} \beta^i \\ &< d(x_1, x_0) \sum_{i=m}^{\infty} \beta^i \\ &= \frac{\beta^m}{1-\beta} d(x_1, x_0) \rightarrow 0 \text{ as } m \rightarrow \infty \end{aligned}$$

Fixed  $\varepsilon > 0$ , we can choose  $N(\varepsilon)$  such that  $\forall n, m > N(\varepsilon)$ ,

$$d(x_n, x_m) < \frac{\beta^m}{1 - \beta} d(x_1, x_0) < \varepsilon$$

Therefore,  $\{x_n\}$  is Cauchy. Since  $(X, d)$  is complete,  $x_n \rightarrow x^*$  for some  $x^* \in X$ .

Next we show that  $x^*$  is a fixed point of  $T$ .

$$\begin{aligned} T(x^*) &= T\left(\lim_{n \rightarrow \infty} x_n\right) \\ &= \lim_{n \rightarrow \infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n \rightarrow \infty} x_{n+1} \\ &= x^* \end{aligned}$$

so  $x^*$  is a fixed point of  $T$ .

Finally, we show that there is at most one fixed point. Suppose  $x^*$  and  $y^*$  are both fixed points of  $T$ , so

$T(x^*) = x^*$  and  $T(y^*) = y^*$ . Then

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \beta d(x^*, y^*) \\ \Rightarrow (1 - \beta)d(x^*, y^*) &\leq 0 \\ \Rightarrow d(x^*, y^*) &\leq 0 \end{aligned}$$

So  $d(x^*, y^*) = 0$ , which implies  $x^* = y^*$ .

### 3.7.3 Theorem: Fixed Point's Continuous Dependence on Parameters

#### Theorem 3.13 (Continuous Dependence on Parameters)

Let  $(X, d)$  and  $(\Omega, \rho)$  be two metric spaces and  $T : X \times \Omega \rightarrow X$ . For each parameter  $\omega \in \Omega$  let  $T_\omega : X \rightarrow X$  be defined by  $T_\omega(x) = T(x, \omega)$ .

Suppose (1).  $(X, d)$  is complete, (2).  $T$  is continuous in  $\omega$  (that is  $T(x, \cdot) : \Omega \rightarrow X$  is continuous for each  $x \in X$ ), and (3).  $\exists \beta < 1$  such that  $T_\omega$  is a contraction of modulus  $\beta \forall \omega \in \Omega$ .

Then the fixed point function (about parameter  $\omega$ )  $x^* : \Omega \rightarrow X$  defined by  $x^*(\omega) = T_\omega(x^*(\omega))$  is continuous.



## 3.8 Correspondence (@ Lec 07 of ECON 204)

### 3.8.1 Definition of Correspondence

#### Definition 3.15 (Correspondence)

A **correspondence**  $\Psi : X \rightarrow 2^Y$  from  $X$  to  $Y$  is a function from  $X$  to  $2^Y$ , that is,  $\Psi(x) \subseteq Y$  for every  $x \in X$ . ( $2^Y$  is the set of all subsets of  $Y$ )



**Example 3.6** Let  $u : \mathbb{R}_+^n \rightarrow \mathbb{R}$  be a continuous utility function,  $y > 0$  and  $p \in \mathbb{R}_{++}^n$ , that is,  $p_i > 0$  for each  $i$ .

Define  $\Psi : \mathbb{R}_{++}^n \times \mathbb{R}_{++} \rightarrow 2^{\mathbb{R}_+^n}$  by

$$\Psi(p, y) = \operatorname{argmax} u(x)$$

$$\text{s.t. } x \geq 0$$

$$p \cdot x \leq y$$

$\Psi$  is the demand correspondence associated with the utility function  $u$ ; typically  $\Psi(p, y)$  is multi-valued.

### 3.8.2 Continuity of Correspondences

#### Definition 3.16 (Upper/Lower Hemicontinuous)

Let  $X \subseteq \mathbb{E}^n$ ,  $Y \subseteq \mathbb{E}^m$ , and  $\Psi : X \rightarrow 2^Y$ .

1.  $\Psi$  is **upper hemicontinuous** (uhc) at  $x_0 \in X$  if, for every open set  $V \supseteq \Psi(x_0)$ , there is an open set  $U$  with  $x_0 \in U$  (which requires elements around  $x_0$  all satisfy) such that

$$\Psi(x) \subseteq V \text{ for every } x \in U \cap X$$

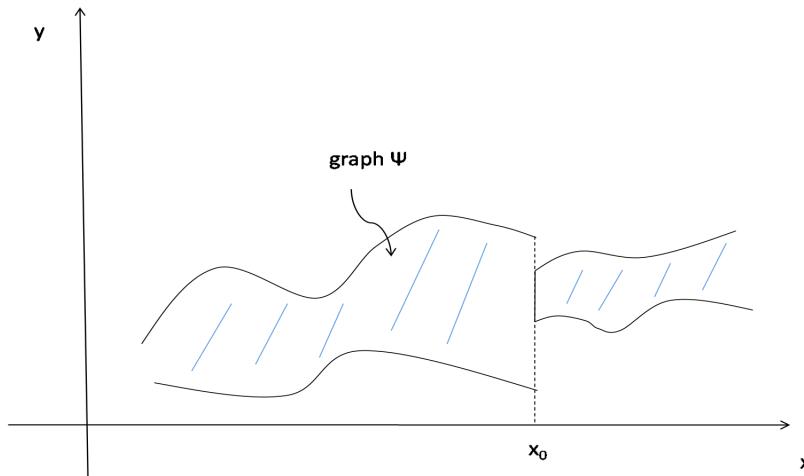
Upper hemicontinuity reflects the requirement that  $\Psi$  doesn't "jump down/implode in the limit" at  $x_0$ . (A set to "jump down" at the limit  $x_0$ : It should mean the set suddenly gets smaller – it "implodes in the limit" – that is, there is a sequence  $x_n \rightarrow x_0$  and points  $y_n \in \Psi(x_n)$  that are far from every point of  $\Psi(x_0)$  as  $n \rightarrow \infty$ .)

2.  $\Psi$  is **lower hemicontinuous** (lhc) at  $x_0 \in X$  if, for every open set  $V$  such that  $\Psi(x_0) \cap V \neq \emptyset$ , there is an open set  $U$  with  $x_0 \in U$  such that

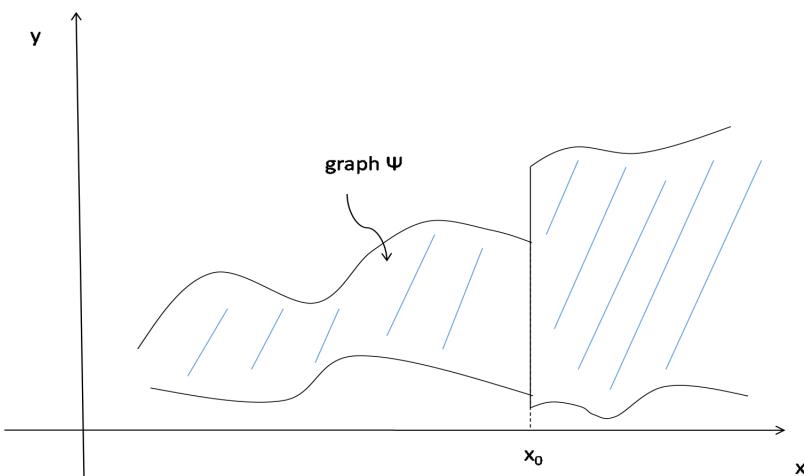
$$\Psi(x) \cap V \neq \emptyset \text{ for every } x \in U \cap X$$

Lower hemicontinuity reflects the requirement that  $\Psi$  doesn't "jump up/explode in the limit" at  $x_0$ . (A set to "jump up" at the limit  $x_0$ : It should mean that the set suddenly gets bigger – it "explodes in the limit" – that is, there is a sequence  $x_n \rightarrow x_0$  and a point  $y_0 \in \Psi(x_0)$  that is far from every point of  $\Psi(x_n)$  as  $n \rightarrow \infty$ .)

3.  $\Psi$  is **continuous** at  $x_0 \in X$  if it is both **uhc** and **lhc** at  $x_0$ .
4.  $\Psi$  is upper hemicontinuous (respectively lower hemicontinuous, continuous) if it is uhc (respectively lhc, continuous) at every  $x \in X$ .



**Figure 3.2:** The correspondence  $\Psi$  “implodes in the limit” at  $x_0$ .  $\Psi$  is not upper hemicontinuous at  $x_0$ .



**Figure 3.3:** The correspondence  $\Psi$  “explodes in the limit” at  $x_0$ .  $\Psi$  is not lower hemicontinuous at  $x_0$ .

### 3.8.3 Graph of Correspondence

An alternative notion of continuity looks instead at properties of the graph of the correspondence.

**Definition 3.17 (Graph of Correspondence)**

The **graph** of a correspondence  $\Psi : X \rightarrow 2^Y$  is the set

$$\text{graph}\Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$



By the definition of continuous function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ , each convergent sequence  $\{(x_n, y_n)\}$  in graph  $f$  converges to a point  $(x, y)$  in graph  $f$ , that is, graph  $f$  is closed.

**Definition 3.18 (Closed Graph)**

Let  $X \subseteq \mathbb{E}^n$ ,  $Y \subseteq \mathbb{E}^m$ . A correspondence  $\Psi : X \rightarrow 2^Y$  has closed graph if its graph is a closed subset of  $X \times Y$ , that is, if for any sequences  $\{x_n\} \subseteq X$  and  $\{y_n\} \subseteq Y$  such that  $x_n \rightarrow x \in X$ ,  $y_n \rightarrow y \in Y$  and  $y_n \in \Psi(x_n)$  for each  $n$ , then  $y \in \Psi(x)$ .



**Example 3.7** Consider the correspondence  $\Psi(x) = \begin{cases} \{\frac{1}{x}\}, & \text{if } x \in (0, 1] \\ \{0\}, & \text{if } x = 0 \end{cases}$  ("implode in the limit")

Let  $V = (-0.1, 0.1)$ . Then  $\Psi(0) = \{0\} \subseteq V$ , but no matter how close  $x$  is to 0,  $\Psi(x) = \{\frac{1}{x}\} \not\subseteq V$ , so  $\Psi$  is not uhc at 0. However, note that  $\Psi$  has closed graph.

**3.8.4 Theorem:**  $\Psi(x) = \{f(x)\}$  is uhc  $\Leftrightarrow f$  is continuous**Theorem 3.14 ( $\Psi(x) = \{f(x)\}$  is uhc  $\Leftrightarrow f$  is continuous)**

Let  $X \subseteq \mathbb{E}^n$ ,  $Y \subseteq \mathbb{E}^m$  and  $f : X \rightarrow Y$ . Let  $\Psi : X \rightarrow 2^Y$  be defined by  $\Psi(x) = \{f(x)\}$  for all  $x \in X$ .

Then  $\Psi$  is **uhc** if and only if  $f$  is **continuous**.

**3.8.5 Closed-valued and Compact-valued Correspondences****Definition 3.19 (Closed-valued and Compact-valued Correspondences)**

A correspondence  $\Psi : X \rightarrow 2^Y$  is called **closed-valued** if  $\Psi(x)$  is a closed subset of  $Y$  for all  $x$ ;  $\Psi$  is called **compact-valued** if  $\Psi(x)$  is compact for all  $x$ .



For closed-valued correspondences these concepts can be more tightly connected. A closed-valued and upper hemicontinuous correspondence must have closed graph. For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

**Theorem 3.15 (Properties)**

Let  $X \subseteq \mathbb{E}^n$ ,  $Y \subseteq \mathbb{E}^m$ , and  $\Psi : X \rightarrow 2^Y$ .

- (i).  $\Psi$  is **closed-valued** and **uhc**  $\Rightarrow \Psi$  has **closed graph**.
- (ii).  $\Psi$  is **closed-valued** and **uhc**  $\Leftrightarrow \Psi$  has **closed graph**. (If  $Y$  is **compact**)

(iii). If  $\Psi$  has **closed graph** and there is an **open set**  $W$  with  $x_0 \in W$  and a **compact set**  $Z$  such that

$x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$ , then  $\Psi$  is **uhc** at  $x_0$ .



## 3.9 Differentiable Function (@ Lec 11 of ECON 204)

### 3.9.1 Definition of Differentiable in $\mathbb{R}$

#### Definition 3.20

Let  $f : I \rightarrow \mathbb{R}$ , where  $I \subseteq \mathbb{R}$  is an open interval.  $f$  is **differentiable** at  $x \in I$  if

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = a$$

for some  $a \in \mathbb{R}$ .



This is equivalent to  $\exists a \in \mathbb{R}$  such that:

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{f(x+h) - (f(x) + ah)}{h} = 0 \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \left| \frac{f(x+h) - (f(x) + ah)}{h} \right| < \varepsilon \\ \Leftrightarrow & \forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } 0 < |h| < \delta \Rightarrow \frac{|f(x+h) - (f(x) + ah)|}{|h|} < \varepsilon \\ \Leftrightarrow & \lim_{h \rightarrow 0} \frac{|f(x+h) - (f(x) + ah)|}{|h|} = 0 \end{aligned}$$

Recall that the limit considers  $h$  near zero, but not  $h = 0$ .

### 3.9.2 General Definition of Differentiable

#### Definition 3.21

If  $X \subseteq \mathbb{R}^n$  is **open**,  $f : X \rightarrow \mathbb{R}^m$  is **differentiable** at  $x \in X$  if  $\exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m)$  s.t.

$$\lim_{h \rightarrow 0, h \in \mathbb{R}^n} \frac{|f(x+h) - (f(x) + T_x(h))|}{|h|} = 0$$

$f$  is differentiable if it is differentiable at all  $x \in X$ .



Note that  $T_x$  is uniquely determined by Equation (1).  $h$  is a small, nonzero element of  $\mathbb{R}^n$ ;  $h \rightarrow 0$  from any direction, from above, below, along a spiral, etc. The definition requires that one linear operator  $T_x$  works no matter how  $h$  approaches zero. In this case,  $f(x) + T_x(h)$  is the best linear approximation to  $f(x+h)$  for small  $h$ .

### 3.9.3 "O" and "o" Notations

#### Definition 3.22 ("O" and "o")

1.  $y = O(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is big-Oh of  $|h|^n$ ” – means  $\exists K, \delta > 0$  s.t.  $|h| < \delta \Rightarrow |y| \leq K|h|^n$
2.  $y = o(|h|^n)$  as  $h \rightarrow 0$  – read “ $y$  is little-Oh of  $|h|^n$ ” – means  $\lim_{h \rightarrow 0} \frac{|y|}{|h|^n} = 0$

 **Note** Note that the statement  $y = O(|h|^{n+1})$  as  $h \rightarrow 0$  implies  $y = o(|h|^n)$  as  $h \rightarrow 0$ .

In both cases,  $y \rightarrow 0$  as  $h \rightarrow 0$ , but the *rate* is different.

#### Definition 3.23 (New Definition)

Using this notation, note that  $f$  is **differentiable** at  $x \Leftrightarrow$

$$\exists T_x \in L(\mathbb{R}^n, \mathbb{R}^m) \text{ such that } f(x + h) = f(x) + T_x(h) + o(|h|) \text{ as } h \rightarrow 0$$

### 3.9.4 Jacobian and Error Term

#### Definition 3.24 (Jacobian, Error Term)

- $df_x$  is the linear transformation  $T_x$
- $Df(x)$  is the matrix of  $df_x$  with respect to the standard basis. This is called the **Jacobian** or Jacobian matrix of  $f$  at  $x$
- $E_f(h) = f(x + h) - (f(x) + df_x(h))$  is the **error term**

Using this notation,  $f$  is differentiable at  $x \Leftrightarrow E_f(h) = o(h)$  as  $h \rightarrow 0$

Now compute  $Df(x) = (a_{ij})$ . Let  $\{e_1, \dots, e_n\}$  be the standard basis of  $\mathbb{R}^n$ . Look in direction  $e_j$  (note that  $|\gamma e_j| = |\gamma|$ ).

$$o(\gamma) = f(x + \gamma e_j) - (f(x) + T_x(\gamma e_j))$$

$$\begin{aligned} &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right) \\ &= f(x + \gamma e_j) - \left( f(x) + \begin{pmatrix} \gamma a_{1j} \\ \vdots \\ \gamma a_{mj} \end{pmatrix} \right) \end{aligned}$$

For  $i = 1, \dots, m$ , let  $f^i$  denote the  $i^{\text{th}}$  component of the function  $f$ :

$$f^i(x + \gamma e_j) - (f^i(x) + \gamma a_{ij}) = o(\gamma)$$

$$\text{so, } a_{ij} = \frac{\partial f^i}{\partial x_j}(x)$$

### 3.9.5 Theorem: Differentiable $\Rightarrow$ Partial Derivatives (and Jacobian) exist

#### Theorem 3.16 (Differentiable $\Rightarrow$ Partial Derivatives (and Jacobian) exist)

Suppose  $X \subseteq \mathbb{R}^n$  is open and  $f : X \rightarrow \mathbb{R}^m$  is differentiable at  $x \in X$ . Then  $\frac{\partial f^i}{\partial x_j}$  exists at  $x$  for  $1 \leq i \leq m, 1 \leq j \leq n$ , and

$$Df(x) = \begin{pmatrix} \frac{\partial f^1}{\partial x_1}(x) & \cdots & \frac{\partial f^1}{\partial x_n}(x) \\ \vdots & \ddots & \vdots \\ \frac{\partial f^m}{\partial x_1}(x) & \cdots & \frac{\partial f^m}{\partial x_n}(x) \end{pmatrix}$$

i.e. the Jacobian at  $x$  is the matrix of partial derivatives at  $x$ .



**Note** If  $f$  is differentiable at  $x$ , then all first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  exist at  $x$ . However, the converse is false: existence of all the first-order partial derivatives does not imply that  $f$  is differentiable.

**Example 3.8**  $f(X) = \begin{cases} \frac{x_1 x_2}{x_1^2 + x_2^2}, & X \neq (0, 0) \\ 0, & X = (0, 0) \end{cases}$

### 3.9.6 Theorem: Partial Derivatives exist and Continuous $\Rightarrow$ Differentiable

The missing piece is continuity of the partial derivatives:

#### Theorem 3.17 (Partial Derivatives exist and Continuous $\Rightarrow$ Differentiable)

If all the first-order partial derivatives  $\frac{\partial f^i}{\partial x_j}$  ( $1 \leq i \leq m, 1 \leq j \leq n$ ) exist and are continuous at  $x$ , then  $f$  is differentiable at  $x$ .



### 3.9.7 Directional Derivatives

Suppose  $X \subseteq \mathbb{R}^n$  open,  $f : X \rightarrow \mathbb{R}^m$  is differentiable at  $x$ , and  $|u| = 1$ .

$$f(x + \gamma u) - (f(x) + T_x(\gamma u)) = o(\gamma) \text{ as } \gamma \rightarrow 0$$

$$\Rightarrow f(x + \gamma u) - (f(x) + \gamma T_x(u)) = o(\gamma) \text{ as } \gamma \rightarrow 0$$

$$\Rightarrow \lim_{\gamma \rightarrow 0} \frac{f(x + \gamma u) - f(x)}{\gamma} = T_x(u) = Df(x)u$$

i.e. the **directional derivative** in the direction  $u$  (with  $|u| = 1$ ) is

$$Df(x)u \in \mathbb{R}^m$$

**3.9.8 Chain Rule:**  $dF_{x_0} = dg_{f(x_0)} \circ df_{x_0}$  and  $DF_{x_0} = Dg(f(x_0))Df(x_0)$ **Theorem 3.18 (Chain Rule)**

Let  $X \subseteq \mathbb{R}^n$ ,  $Y \subseteq \mathbb{R}^m$  be open,  $f : X \rightarrow Y$ ,  $g : Y \rightarrow \mathbb{R}^p$ . Let  $x_0 \in X$  and  $F = g \circ f$ . If  $f$  is differentiable at  $x_0$  and  $g$  is differentiable at  $f(x_0)$ , then  $F = g \circ f$  is differentiable at  $x_0$  and

$$dF_{x_0} = dg_{f(x_0)} \circ df_{x_0} \text{ (composition of linear transformations)}$$

$$DF_{x_0} = Dg(f(x_0))Df(x_0) \text{ (matrix multiplication)}$$

**3.9.9 Mean Value Theorem:**  $\exists c \in (a, b), f(b) - f(a) = f'(c)(b - a)$ **Theorem 3.19 (Mean Value Theorem, Univariate Case:  $\mathbb{R} \rightarrow \mathbb{R}$ )**

Let  $a, b \in \mathbb{R}$ . Suppose  $f : [a, b] \rightarrow \mathbb{R}$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Then there exists  $c \in (a, b)$  such that

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

that is

$$f(b) - f(a) = f'(c)(b - a)$$



**Note** The Mean Value Theorem is useful for estimating bounds on functions and error terms in approximation of functions.

**Definition 3.25 (Line Segment)**

$l(x, y) = \{\alpha x + (1 - \alpha)y : \alpha \in [0, 1]\}$  is the **line segment** from  $x$  to  $y$ .

**Theorem 3.20 (Mean Value Theorem,  $\mathbb{R}^n \rightarrow \mathbb{R}$ )**

Suppose  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable on an open set  $X \subseteq \mathbb{R}^n$ ,  $x, y \in X$  and  $l(x, y) \subseteq X$ . Then there exists  $z \in l(x, y)$  such that

$$f(y) - f(x) = Df(z)(y - x)$$

**Theorem 3.21 (Mean Value Theorem, General Case:  $\mathbb{R}^n \rightarrow \mathbb{R}^m$ )**

Suppose  $X \subseteq \mathbb{R}^n$  is open and  $f : X \rightarrow \mathbb{R}^m$  is differentiable. If  $x, y \in X$  and  $l(x, y) \subseteq X$ , then there exists  $z \in l(x, y)$  such that

$$\|f(y) - f(x)\| \leq \|df_z(y - x)\|$$

$$\leq \|df_z\| \|y - x\|$$



# Chapter 4 Big $\mathcal{O}$ and Small $o$ Notation

## 4.1 Definition

### Complexity:

#### Definition 4.1

A sequence  $f(n)$  is  $O(1)$  if  $\lim_{n \rightarrow \infty} f(n) < \infty$ .



#### Definition 4.2

A sequence  $f(n)$  is  $O(g(n))$  if  $\frac{f(n)}{g(n)}$  is  $O(1)$ .



#### Definition 4.3

A sequence  $f(n)$  is  $o(1)$  if  $\lim_{n \rightarrow \infty} \sup f(n) = 0$ .



#### Definition 4.4

A sequence  $f(n)$  is  $o(g(n))$  if  $\lim_{n \rightarrow \infty} \sup \frac{f(n)}{g(n)} = 0$ .



#### Definition 4.5

A sequence  $f(n)$  is asymptotic to  $g(n)$  if  $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$ . (This is denoted by  $f(n) \sim g(n)$  as  $a \rightarrow \infty$ )



For two scalar functions  $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$ , where  $x \in \mathbb{R}$ , we write:

1.  $f(x) = \mathcal{O}(g(x))$  if  $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} < \infty$ ; we say  $f$  is dominated by  $g$  asymptotically.
2.  $f(x) = \Omega(g(x))$  if  $\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0$ .
3.  $f(x) = \Theta(g(x))$  if  $f(x) = \mathcal{O}(g(x))$  and  $f(x) = \Omega(g(x))$  both hold.
4.  $f(x) = o(g(x))$  if  $\liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$ .

### Example 4.1

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$

$$n^3 + n + 2 = \Theta(n^3)$$

$$n^3 + n + 2 = o(n^4)$$

### 4.1.1 Extension

$f(x) = \mathcal{O}(g(x))$  as  $x \rightarrow a$  if  $\limsup_{x \rightarrow a} \frac{|f(x)|}{g(x)} < \infty$ .

**Example 4.2**  $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$  as  $\varepsilon \rightarrow 0$