# MATH 417 Lec06-15

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# 1 Integers

## 1.1 Proposition 1.4.1: Properties of integers $\mathbb{Z}$

**Proposition 1** (Proposition 1.4.1.). *The following hold in the integers $\mathbb{Z}$:*

(i) *Addition* and *multiplication* are *commutative* and *associative* operations in $\mathbb{Z}$.

(ii) $0 \in \mathbb{Z}$ is an identity element for addition; that is, $\forall a \in \mathbb{Z}, 0 + a = a$.

(iii) Every $a \in \mathbb{Z}$ has an additive inverse, denoted $-a$ and given by $-a = (-1)a$, satisfying $a + (-a) = 0$.

(iv) $1 \in \mathbb{Z}$ is an identity element for multiplication; that is, for all $a \in \mathbb{Z}$, $1a = a$.

(v) The *distributive* law holds: $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$.

(vi) Both $\mathbb{N} = \{x \in Z | x \geq 0\}$ and $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$ are *closed* under *addition* and *multiplication*. That is, if $x$ and $y$ are in one of these sets, then $x + y$ and $xy$ are also in that set.

(vii) For any two nonzero integers $a, b \in \mathbb{Z}, |ab| \geq \max\{|a|, |b|\}$. Strict inequality holds if $|a| > 1$ and $|b| > 1$.

From this we get cancellation.

$$ab = ac \Rightarrow b = c \ or \ a = 0$$

## 1.2 Divide

Suppose $a, b \in \mathbb{Z}, b \neq 0$, <u>b divides a</u> if $\exists m \in \mathbb{Z}$, so that $a = bm, b|a$. Otherwise, write $b \nmid a$.

### 1.2.1 Proposition 1.4.2: properties of integer division

**Proposition 2** (Proposition 1.4.2). $\forall a, b \in \mathbb{Z}$

(i) if $a \neq 0$, then $a|0$

(ii) if $a|1$, then $a = \pm 1$

(iii) if $a|b$ & $b|a$, then $a = \pm b$

(iv) if $a|b$ & $b|c$, then $a|c$

(v) if $a|b$ & $a|c$, then $a|(mc + nb)\forall m, n \in \mathbb{Z}$

### 1.2.2 Prime, Greatest common divisor $gcd(a, b)$

$p > 1, p \in \mathbb{Z}$ is called *prime* if the only divisors are $\pm 1, \pm p$.

Given $a, b \in \mathbb{Z}, a, b \neq 0$, the *greatest common divisor* of $a$ and $b$ is $c \in \mathbb{Z}, c > 0$ s.t.

(1) $c|a$ and $c|b$

(2) if $d|a, d|b$, then $d|c$

The $c$ is unique, we write it $gcd(a, b)$.

### 1.2.3  Proposition 1.4.7: Euclidean Algorithm

**Proposition 3** (Proposition 1.4.7(Euclidean Algorithm))**.** *Given $a, b \in \mathbb{Z}, b \neq 0$, then $\exists q, r \in \mathbb{Z}$ s.t.* $a = qb + r, 0 \leq r \leq |b|$.

**Example 1** (Exercise 1.4.3)**.** *For the pair $(a, b) = (130, 95)$, find $gcd(a, b)$ using the Euclidean Algorithm and express it in the form $gcd(a, b) = m_0 a + n_0 b$ for $m_0, n_0 \in Z$.*

$$130 = 95 + 35; \quad 95 = 2 \times 35 + 25$$
$$35 = 25 + 10; \quad 25 = 2 \times 10 + 5$$
$$10 = 2 \times 5 + 0$$

$$5 = 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35$$
$$= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130$$

### 1.2.4  Proposition 1.4.8:$gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$

**Proposition 4** (Proposition 1.4.8)**.** $\forall a, b \in \mathbb{Z}$, *not both 0, $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$.*

i.e. $\exists m_0, n_0 \in \mathbb{Z}$ s.t. $gcd(a, b) = m_0 a + n_0 b$.

证明.
Let $c$ be the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$. $c = m_0 a + n_0 b > 0$.
Let $d = ma + nb \in M$, $d = qc + r$ where $0 \leq r < c$ (by Euclidean Algorithm).
$d - qc = r = (m - qm_0)a + (n - qn_0)b \in M$. So $r = 0$. $\Rightarrow d = qc$. So $c|d$.
$a = 1a + 0b \in M \Rightarrow c|a$, $b = 0a + 1b \in M \Rightarrow c|b$.
If $t|a, t|b$ then $t|m_0 a + n_0 b \Rightarrow t|c$.
$\Rightarrow c = gcd(a, b)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

### 1.2.5  Proposition 1.4.10: $b, c$ are *relatively prime*, $b|ac \Rightarrow b|a$

**Proposition 5** (Proposition 1.4.10)**.** *Suppose $a, b, c \in \mathbb{Z}$. If $b, c$ are relatively prime i.e. $gcd(b, c) = 1$ and $b|ac$, then $b|a$.*

证明.
$gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$ s.t. $1 = mb + nc \Rightarrow a = amb + anc$
$b|nac, b|amb \Rightarrow b|a$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

### 1.2.6  Corollary 1.4.11: $p|ab \Rightarrow p|a$ **or** $p|b$

**Corollary 1** (Corollary 1.4.11)**.** $a, b, p \in \mathbb{Z}, p > 1$ *prime. If $p|ab$, then $p|a$ or $p|b$.*

证明.

If $p|b$, done. Otherwise, $gcd(p, b) = 1$. By Prop 1.4.10, $p|a$. $\qquad\square$

### 1.2.7  Theorem 1.4.3: Any integer $a > 1$ has a unique prime factorization

**Lemma 1** (Lemma 1.4.4)**.** *[Existence of Theorem 1.4.3] Any integer $a > 1$ has a prime factorization*

Proved by induction.

**Theorem 1** (Theorem 1.4.3)**.**

Any integer $a > 1$ has a unique prime factorization: $a = p_1^{k_1} \cdot p_2^{k_2} \cdot ... p_n^{k_n}$ where $p_i > 1$ is prime, $k_i \in \mathbb{Z}_+, \forall i = 1, ..., n, p_i \neq p_j, \forall i \neq j$.

证明.

Existence(Lemma 1.4.4)

Uniqueness: $a = p_1^{n_1} \cdot p_2^{n_2} \cdot ... p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot ... q_j^{r_j}$.

$p_1 > p_2 > ... > p_k, q_1 > q_2 > ... > q_j, n_i, r_i \geq 1$

$p_1 | a \Rightarrow \exists\ q_i\ s.t.\ p_1 | q_i$. Similarly, $\exists\ q_i\ s.t.\ q_1 | p_{i'}$

$q_1 \leq p_i' \leq p_1 \leq q_i \Rightarrow q_1 = p_i' = p_1 = q_i$

We can also know $n_1 = r_1$, otherwise we would have two prime factorizations of the quotient where the largest primes are different by dividing $p_1^{\min\{n_1, r_1\}}$.

Then we can get $b = p_2^{n_2} \cdot ... p_k^{n_k} = q_2^{r_2} \cdot ... q_j^{r_j}$. Then prove it by induction. $\qquad\square$

## 2  Modular arithmetic

### 2.1  <u>congruence modulo</u> $n$:$a \equiv b$ ***mod*** $n$**, *if*** $n|(a - b)$

Given $n \in \mathbb{Z}_+$, define a relation on $\mathbb{Z}$: <u>**congruence modulo** $n$</u>

$$a \equiv b\ mod\ n,\ if\ n|(a - b)$$

read"$a$ is congruent to $b$ mod $n$".

### 2.1.1 Proposition 1.5.1: Congruence modulo $n$ is an <u>equivalence relation</u>

**Proposition 6** (Proposition 1.5.1). *Congruence modulo n is an equivalence relation $\forall n \in \mathbb{Z}_+$*

证明.

1. <u>Reflexive</u>: $\forall a \in \mathbb{Z}, n|0 = (a-a)$, so $a \equiv a \bmod n$.

2. <u>Symmetric</u>: $\forall a, b \in \mathbb{Z}, a \equiv b \bmod n$, so $n|(a-b)$ i.e. $a - b = kn$ for some $k \in \mathbb{Z} \Rightarrow b - a = -kn \Rightarrow n|(b-a) \Rightarrow b \equiv a \bmod n$.

2. <u>Transitive</u>: $\forall a, b, c \in \mathbb{Z}, a \equiv b \bmod n, b \equiv c \bmod n$. So, $n|(a-b), n|(b-c)n|(a-b) + (b-c) = (a-c) \Rightarrow a \equiv c \bmod n$. $\qquad\square$

## 2.2 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

将给定 $n$，相同余数的数分为一组

Fix $n \in \mathbb{Z}_+$, we call $[a]_n = [a]$ the **<u>congruence class</u>** of a modulo n.

$$[a] = \{b \in \mathbb{Z} | b \equiv a \bmod n\} = \{a + kn | k \in \mathbb{Z}\}$$

### 2.2.1 Set of congruence classes of mod $n$: $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], ..., [n-1]\}$

The set of *congruence classes* of mod $n$ is denoted $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

**Proposition 7** (Proposition 1.5.2.). *For any $n \geq 1$ there are exactly n congruences classes modulo n, which we may write as*

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

证明.

For any $a \in \mathbb{Z}$. By Euclidean algorithm, $a = qn + r$, $q, r \in \mathbb{Z}$, $0 \leq r < n \Rightarrow a \in [r]$. So, $\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$.

When $0 \leq a < b \leq n - 1$, $n \nmid (b-a)$, so $[a] \neq [b]$ the $n$ congruence classes listed are all distinct. Hence, there are exactly $n$ congruence classes. $\qquad\square$

### 2.2.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

**Lemma 2** (Lemma 1.5.3). *$\forall a, b, c, d, n \in \mathbb{Z}, n > 1$, $a \equiv c \bmod n$, $b \equiv d \bmod n$. Then $a + b \equiv c + d \bmod n$, $ab \equiv cd \bmod n$*

证明.

$n|(a-c), n|(b-d) \Rightarrow n|(a-c) + (b-d) = (a+b) - (c-d)$

$a = c + kn, k \in \mathbb{Z}; b = d + jn, j \in \mathbb{Z}$. $ab = cd + cjn + dkn + kjn^2 \Rightarrow n|(ab-cd)$. $\qquad\square$

Fix $n \in \mathbb{Z}$, we define addition+ and multiplication· on $\mathbb{Z}_n$:

$$[a] + [b] = [a+b] = \{a + b + (k+j)n | k, j \in \mathbb{Z}\}$$

$$[a] \cdot [b] = [ab] = \{ab + (aj + bk + kjn)n | k, j \in \mathbb{Z}\}$$

This is well defined, follows Lemma 1.5.3.

**Proposition 8** (Proposition 1.5.5.)**.** *Let $a, b, c, d, n \in \mathbb{Z}, n \geq 1$, then*

*(i) Addition and multiplication are commutative and associative operations in $\mathbb{Z}_n$.*

*(ii) $[a] + [0] = [a]$.*

*(iii) $[-a] + [a] = [0]$.*

*(iv) $[1][a] = [a]$.*

*(v) $[a]([b] + [c]) = [a][b] + [a][c]$.*

证明.

$\square$

### 2.2.3   Units(i.e. invertible) in Congruence Classes

将与 $n$ 互质的数分为一组

Say $[a] \in \mathbb{Z}_n$ is a **unit** or is **invertible** if $\exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [1]$.

### 2.2.4   Proposition 1.5.6: Set of units in congruence classes: $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \text{ \textit{is a unit}}\} = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$

The set of **invertible** elements in $\mathbb{Z}_n$ will be denoted $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \text{ is a unit}\}$.

**Proposition 9** (Proposition 1.5.6.)**.** *For all $n \geq 1$, we have $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$.*

证明.

By Proposition 1.4.8, we know there exists $b, c$ s.t. $ab + cn = 1$. So, $ab \equiv 1 \ mod \ n$, $[1] = [ab] = [a][b]$. So, $\{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\} \subset \mathbb{Z}_n^\times$

$[a]$ *is a unit* $\Rightarrow \exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow gcd(a, n) = 1$. So, $\mathbb{Z}_n^\times \subset \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$. $\square$

**Note 1.** *Inverse of $[a]$ is unique, i.e. $[b] = [a]^{-1}$ is unique.*

$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$

### 2.2.5   Corollary 1.5.7: if $p$ is prime, $\varphi(p) = \mathbb{Z}_p^\times = \{[1], [2], ..., [p-1]\}$

**Corollary 2** (Corollary 1.5.7)**.** *If $p \geq 2$ is prime, $\mathbb{Z}_p^\times = \{[1], [2], ..., [p-1]\}$.*

## 2.3   <u>Euler phi-function:</u> $\varphi(n) = |\mathbb{Z}_n^\times|$

<u>Euler phi-function:</u> $\varphi(n) = |\mathbb{Z}_n^\times|$.

$p$ prime, $\varphi(p) = p - 1$.

**2.3.1** $m|n$, $\pi_{m,n}([a]_n) = [a]_m$

**Example 2** (Exercise 1.5.4). *If $m|n$, we can define $\pi_{m,n} : \mathbb{Z}_n \to \mathbb{Z}_m$ by $\pi_{m,n}([a]_n) = [a]_m$. Prove it is well-defined.*

证明.

We write $[a]_n = [c]_n$, verify that $[a]_m = [c]_m$.

Since $m|n$, there exists $k \in \mathbb{Z}$ s.t. $n = km$.

$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z}$ s.t. $c = a + jn$.

$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m$ $\qquad\qquad\square$

## 2.4 Theorem 1.5.8(Chinese Remainder Theorem): $n = mk, gcd(m, k) = 1$, $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

**Theorem 2** (Theorem 1.5.8(Chinese Remainder Theorem)). *If $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F : \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_k$ which is given by $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$, then $F$ is a bijection.*

证明.

(1)Injective: $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$ i.e. $a \equiv b \bmod m, a \equiv b \bmod n$. $\exists i, j \in \mathbb{Z}$ s.t. $b = a + im = a + jk \Rightarrow k|im$. Since $gcd(m, k) = 1, k|i \Rightarrow n = mk|im$. Then $[b]_n = [a]_n + [im]_n = [a]_n$.

(2)Surjective: prove $\forall u, v \in \mathbb{Z}, \exists a\mathbb{Z}$ s.t. $[a]_m = [u]_m, [a]_k = [v]_k$.

Since $gcd(m, k) = 1, \exists s, t \in \mathbb{Z}$ so that $1 = sm + tk$.

Let $a = (1 - tk)u + (1 - sm)v$, $[a]_m = [(u - v)sm + v]_m = [v]_m$, $[a]_k = [(v - u)tk + u]_k = [u]_k$. $\quad\square$

**Note 2.** $F([a]_n[b]_n) = F([ab]_n) = ([ab]_m, [ab]_k) = ([a]_m[b]_m, [a]_k[b]_k)$

Since $F$ is a bijection, $[ab]_n = [1]_n$ iff $([a]_m[b]_m, [a]_k[b]_k) = ([1]_m, [1]_k)$.

**2.4.1 Proposition 1.5.9+Corollary 1.5.10:** $m, n, k > 0, n = mk, gcd(m, k) = 1$, **then** $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, **then** $\varphi(n) = \varphi(m)\varphi(k)$

**Proposition 10** (Proposition 1.5.9+Corollary 1.5.10). *If $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$.*

**2.5 prime factorization:** $n = p_1^{r_1}...p_k^{r_k}$, **then** $\varphi(n) = (p_1 - 1)p_1^{r_1 - 1}...(p_k - 1)p_k^{r_k - 1}$

**Proposition 11.** *If $n \in \mathbb{Z}$ is positive integre with prime factorization $n = p_1^{r_1}...p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1 - 1}...(p_k - 1)p_k^{r_k - 1}$*

证明.

$\mathbb{Z}_{p^r} = \{[0], [1], ..., [p^r - 1]\}$, the number of multiples of $p$ is $\frac{p^r}{p} = p^{r-1}$. Then $\varphi(p^r) = |\mathbb{Z}_{p^r}^\times| = $

$p^r - p^{r-1} = (p-1)p^{r-1}$. So,

$$\varphi(n) = \varphi(p_1^{r_1})...\varphi(p_k^{r_k}) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$$

$\square$

# 3 Complex numbers

$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$, $\mathbb{R} = \{a + 0i | a \in \mathbb{R}\} \subset \mathbb{C}$

Addition & multiplication

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = ac + bci + adi + bdi^2$$
$$= (ac - bd) + (bc + ad)i$$

**Complex conjugation**: $z = a + bi, \bar{z} = a - bi, \overline{zw} = \bar{z}\bar{w}$

**Absolute value**: $|z| = \sqrt{a^2 + b^2}, |z|^2 = z\bar{z}$

**Additive inverse**: $-z = -a - bi$

**Multiplicative inverse**: $z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

$$z \in \mathbb{C}, \overline{z + \bar{z}} = \bar{z} + \bar{\bar{z}} = z + \bar{z}$$
$$Real\ part:\ Re(z) = \frac{z + \bar{z}}{2}$$
$$Imaginary\ part:\ Im(z) = \frac{z - \bar{z}}{2i}$$

## 3.1 Geometric Meaning of Addition and Multiplication

Addition: parallelogram law

Multiplication:

$$z = a + bi \neq 0$$
$$= r\cos\theta + r\sin\theta i$$
$$= r(\cos\theta + i\sin\theta)$$
$$|z|^2 = a^2 + b^2 = r^2$$

$$z = r(\cos\theta + i\sin\theta)$$
$$w = s(\cos\phi + i\sin\phi)$$
$$zw = rs[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)]$$
$$= rs[cos(\theta + \phi) + i\sin(\theta + \phi)]$$
$$= |z||w|[cos(\theta + \phi) + i\sin(\theta + \phi)]$$



We will write,

$$\cos\theta + i\sin\theta = e^{i\theta}$$
$$e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}$$
$$z = |z|e^{i\theta}$$
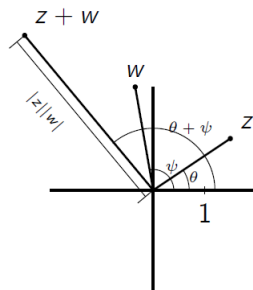
## 3.2 Theorem 2.1.1: $f(x) = a_0 + a_1 x + ... + a_n x^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then $f$ has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$

**Theorem 3** (Theorem 2.1.1). *Supose a nonconstant polynomial $f(x) = a_0 + a_1 x + ... + a_n x^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then $f$ has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$.*

### 3.2.1 Corollary 2.1.2: $f(x) = a_n \prod_{i=1}^{n}(x - k_i) = a_n(x - k_1)(x - k_2)...(x - k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$

**Corollary 3** (Corollary 2.1.2). *Every nonconstant polynomial with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$ can be factored as $f(x) = a_n \prod_{i=1}^{n}(x - k_i) = a_n(x - k_1)(x - k_2)...(x - k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$.*

### 3.2.2 Corollary 2.1.3: $a_i \in \mathbb{R}$, $f$ can be expresses as a product of linear and quadratic polynomials

**Corollary 4** (Corollary 2.1.3). *If $f(x) = a_0 + a_1 x + ... + a_n x^n$ is a nonconstant polynomial $a_0, a_1, ..., a_n \in \mathbb{R}, a_n \neq 0$. Then $f$ can be expresses as a product of linear and quadratic polynomials.*

这里 $a_0, a_1, ..., a_n$ 是实数!

证明.

(1)Obviously, the corollary holds at $n = 1$ and $n = 2$.

(2)Suppose the corollary holds for all situations that $n < k$.

When $n = k$, $f(x) = a_0 + a_1 x + ... + a_k x^k, a_k \neq 0$.

By F.T.A., $f$ has a root $\alpha$ in $\mathbb{C}$.

If $\alpha \in \mathbb{R}$, long division $f(x) = q(x)(x - \alpha)$. $q$ has real coefficients, *degree of $q = k - 1$*. Since the corollary holds at $n = k - 1$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

If $\alpha \notin \mathbb{R}$

$$0 = f(\alpha) = a_0 + a_1 \alpha + ... + a_k \alpha^k$$
$$0 = \overline{f(\alpha)} = a_0 + a_1 \bar{\alpha} + ... + a_n \bar{\alpha}^n = f(\bar{\alpha})$$

Since $\bar{\alpha} \neq \alpha$, $(x - \alpha)(x - \bar{\alpha})|f$.

$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2$ is a polynomial with coefficients in $\mathbb{R}$. So $f(x) = q(x)(x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2)$, $q$ has real coefficients with degree $k - 2$. The corollary also holds at $n = k - 2$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

$\square$

# 4   Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity $\&$ inverse(M,A))

**Definition**: A <u>field</u> is a nonempty set $\mathbb{F}$ with two operations:

1. <u>addition</u>, written $a + b, \forall a, b \in \mathbb{F}$;

2. <u>multiplication</u>, written $a \cdot b = ab, \forall a, b \in \mathbb{F}$.

such that:

(i) *addition* and *multiplication* are <u>associative</u> and <u>commutative</u>

(ii) *multiplication* <u>distributes</u> over *addition*: $a(b + c) = ab + ac, \forall a, b, c \in \mathbb{F}$

(iii) $\exists$ an <u>additive identity</u> $0 \in \mathbb{F}$ s.t. $0 + a = a, \forall a \in \mathbb{F}$.

(iv)$\forall a \in \mathbb{F}$, $\exists$ an <u>additive inverse</u> $-a$ s.t. $a + (-a) = 0, \forall a \in \mathbb{F}$.

(v) $\exists$ a <u>multiplicative identity</u>: $1 \in \mathbb{F}$ s.t. $1a = a, \forall a \in \mathbb{F}, 1 \neq 0$.

(vi) $\forall a \in \mathbb{F}, a \neq 0$, $a$ has a <u>multiplicative inverse</u> $a^{-1} = \frac{1}{a} \in \mathbb{F} : a \cdot \frac{1}{a} = 1$.

**Proposition 12** (Proposition 2.2.2). $\mathbb{F}$ *a field*, $a, b \in \mathbb{F}$, *then*

(i) If $a + b = b$ then $a = 0$

(ii) If $ab = b$ and $b \neq 0$, then $a = 1$

(iii) $0a = 0$

(iv) If $a + b = 0$, then $b = -a$

(v) If $a \neq 0$ and $ab = 1$, then $b = a^{-1}$

**Example 3.** *$\mathbb{Z}_4$ is not a field. Because $[2]_4$ doesn't have <u>multiplicative inverse</u> in $\mathbb{Z}_4$.*

## 4.1 Subfield $(\mathbb{K}, +, \cdot)$: $\mathbb{K} \subseteq \mathbb{F}$, closed under $+, \cdot$ and inverse

<u>**Definition**</u>:Suppose $\mathbb{F}$ is a field and $\mathbb{K} \subseteq \mathbb{F}$ s.t.

$$0, 1 \in \mathbb{K}$$
$$\forall a, b \in \mathbb{K}, a + b, ab, -a, a^{-1}(if \ a \neq 0) \in \mathbb{K}$$

We call $\mathbb{K}$ a <u>subfield</u> of $\mathbb{F}$.

**Example 4.** $\mathbb{Q} \subseteq \mathbb{R}, \mathbb{R} \subseteq \mathbb{C}, \mathbb{Q} \subseteq \mathbb{C}$

**Example 5.** $\mathbb{K} \subseteq \mathbb{Z}_p$ *a subfield* $\Rightarrow \mathbb{K} = \mathbb{Z}_p$. *Prove by induction.*

### 4.1.1 Proposition 2.2.3: Subfield 继承 operations 自成一 field

**Proposition 13** (Proposition 2.2.3)**.** *Suppose $\mathbb{K} \subset \mathbb{F}$ is a subfield of a field $\mathbb{F}$ Then the operations of $\mathbb{F}$ make $\mathbb{K}$ into a field.*

$\Rightarrow$We can prove a set is a field by proving it is a subfield of a known field.

# 5 Polynomials

Let $\mathbb{F}$ be any field. A polynomial over $\mathbb{F}$ in variable $x$ is a formal sum:

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

where $n \geq 0$ is an integer, $a_1, a_1, ..., a_n \in \mathbb{F}$.

Polynomial is a squence $\{a_k\}_{k=0}^{\infty}$ with $a_m = 0, \forall m > n$.

## 5.1 $\mathbb{F}[x]$: Polynomial ring 在一个 field 上形成的所有多项式 (方程) 的集合

Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients in the field $\mathbb{F}$.

$$\mathbb{F}[x] = \{\sum_{i=0}^{n} a_i x^i | n \geq 0, n \in \mathbb{Z}, a_0, ..., a_n \in \mathbb{F}\}$$

We call the $\mathbb{F}[x]$ *polynomial ring* over the field $\mathbb{F}$.

$$f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{n} a_j x^j \in \mathbb{F}[x]$$

$$f + g = \sum_{i=0}^{n} (a_i + b_i) x^i \in \mathbb{F}[x]$$

$$fg(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{n} a_j x^j) = \sum_{i=0}^{2n} (\sum_{j=0}^{i} a_j b_{i-j}) x^i$$

### 5.1.1 Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

**Proposition 14** (Proposition 2.3.2). *Suppose $\mathbb{F}$ is any field. Then,*

(i) Addition and multiplication are commutative & associative operations on $\mathbb{F}[x]$

(ii) Multiplication distributes over addition

(iii) $0 \in \mathbb{F}$, is additive identity in $F[x] : \forall f \in \mathbb{F}[x], f + 0 = 0$

(iv) $\forall f \in \mathbb{F}[x], f = (-1)f$ is the additive inverse: $f + (-1)f = 0$.

(v) $1 \in \mathbb{F}$, is the multiplicative identity in $\mathbb{F}[x] : 1f = f, \forall f \in \mathbb{F}[x]$

## 5.2 Degree of a Polynomial: $deg(f)$

$f = \sum_{i=0}^{n} a_i x^i$, $deg(f) =$ degree of $f$ is,

$$deg(f) = \begin{cases} 0 & \text{if } f \text{ is constant, } f \neq 0 \\ n & \text{if } a_n \neq 0 \text{ in above } (a_n = \text{leading coefficent}) \\ -\infty & \text{if } f = 0 \end{cases}$$

Define $-\infty + a = a + (-\infty) = -\infty \ \forall a \in \mathbb{Z} \cup \{-\infty\}$

### 5.2.1 Lemma 2.3.3: $deg(fg) = deg(f) + deg(g), \ deg(f + g) \leq \max\{deg(f), deg(g)\}$

**Lemma 3** (Lemma 2.3.3). *For any field $\mathbb{F}$ and $f$ , $g \in \mathbb{F}[x]$,*

$$deg(fg) = deg(f) + deg(g)$$

$$deg(f + g) \leq \max\{deg(f), deg(g)\}$$

## 5.3 Corollary 2.3.5: Unit(invertible) in $\mathbb{F}[x]$: constant$\neq 0$ iff $deg(f) = 0$

**Corollary 5** (Corollary 2.3.5). *For any field $\mathbb{F}$ and $f \in \mathbb{F}[x]$, Then $f$ is a <u>unit</u>(i.e. invertible) in $\mathbb{F}[x]$ iff $deg(f) = 0$.*

证明.

Obviously, $deg(f) = 0 \Rightarrow f$ is a unit.

Suppose $f$ is a unit, i.e. $\exists g \in \mathbb{F}[x]$ s.t. $fg = 1$.

$0 = deg(fg) = deg(f) + deg(g) \Rightarrow deg(f), deg(g) \geq 0 \Rightarrow deg(f) = 0, deg(g) = 0.$ □

## 5.4 Irreducible Polynomials: "无法分解为两个 $degree \geq 1$ 的多项式积"的多项式: 至少一个是 constant (i.e. $degree = 0$)

A nonconstant polynomial $f$ is irreducible if $f = uv$, $u, v \in \mathbb{F}[x]$, then either $u$ or $v$ is a unit(i.e., constant$\neq 0$)

## 5.5 Theorem 2.3.6: nonconstant polynomials 可以被唯一地分解

**Theorem 4** (Theorem 2.3.6). *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is any nonconstant. Then $f = ap_1p_2 \ldots p_k$ where $a \in \mathbb{F}$, $p_1, \ldots p_k \in \mathbb{F}[x]$ are irreducible monic polynomials (monic = i.e. leading coeff. 1). If $f = bq_1q_2 \ldots q_r$ with $b \in \mathbb{F}$ and $q_1, q_2, \ldots, q_r \in \mathbb{F}[x]$ monic irreducible, then $a = b, k = r$, and after reindexing $p_i = q_i$, $\forall i$*

**Lemma 4** (Lemma 2.3.7). *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is nonconstant monic polynomial. Then $f = p_1p_2 \ldots p_k$ where each $p_i$ is monic irreducible.*

证明.

Prove it by induction. When $deg(f) = 1$, $f = uv$, $u, v \in \mathbb{F}[x]$, $deg(f) = deg(u) + deg(v) \Rightarrow$ one of these is 0.

Suppose the lemma holds for all degree$< n$. When $deg(f) = n$,

Either $f$ is irreducible, done.

Suppose $f = uv$ with/ $deg(u), deg(v) \geq 1$

$\Rightarrow deg(u), deg(v) < n \Rightarrow u = p_1p_2 \ldots p_k, v = q_1q_2 \ldots q_j$ So, $f = p_1p_2 \ldots p_kq_1q_2 \ldots q_j$. $\qquad \square$

**Example 6.** $x^2 - 1 \in \mathbb{Q}[x]$ *reducible*

$x - 1, x + 1 \in \mathbb{Q}[x]$ *irreducible*

$x^2 + 1 \in \mathbb{Q}[x]$ *irreducible*

$x^2 + 1 \in \mathbb{C}[x]$ *reducible*

$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$ *reducible*

## 5.6 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$, $f$ divides $g$, $f|g$ means $\exists u \in \mathbb{F}[x]$ s.t. $g = fu$.

**Proposition 15** (Proposition 2.3.8). *$f, h, g \in \mathbb{F}[x]$, then*

(i) If $f \neq 0, f|0$

(ii) If $f|1$, $f$ is nonzero constant

(iii) If $f|g$ and $g|f$, then $f = cg$ for some $c \in \mathbb{F}$

(iv) If $f|g$ and $g|h$, then $f|h$

(v) If $f|g$ and $f|h$, then $f|(ug + vh)$ for all $u, v \in \mathbb{F}[x]$.

### 5.6.1 Greatest common divisor of $f$ and $g$: is not unique, we denote monic Greatest common divisor as $gcd(f, g)$

If $f, g \in \mathbb{F}[x]$ are nonzero polynomials, a <u>greatest common divisor</u> of $f$ and $g$ is a polynomial $h \in \mathbb{F}[x]$ such that

(i) $h|f$ and $h|g$, and

(ii) if $k \in \mathbb{F}[x]$ and $k|f$ and $k|g$, then $k|h$.

the $gcd$ is not unique, but the monic $gcd$ is unique. We call it **the monic greatest common divisor**, denote it $gcd(f, g)$.

**Example 7.**

$$x^2 - 1, x^2 - 2x + 1 \in \mathbb{Q}[x]$$
$$(x - 1)(x + 1), (x - 1)^2 \in \mathbb{Q}[x]$$
$$x - 1 = gcd(x^2 - 1, x^2 - 2x + 1)$$

### 5.6.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

**Proposition 16** (Proposition 2.3.9). *Given $f, g \in \mathbb{F}[x]$, $g \neq 0$, then $\exists q, r \in \mathbb{F}[x]$ s.t. $deg(r) < deg(g)$ and $f = qg + r$*

**Example 8.**

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$

### 5.6.3 Proposition 2.3.10: $gcd(f, g)$ 是 degree 最小的 f,g 的线性组合

**Proposition 17** (Proposition 2.3.10). *Any 2 nonzero polynomials $f, g \in \mathbb{F}[x]$ have a gcd in $\mathbb{F}[x]$. In fact among all polynomials in the set $M = \{uf + vg | u, v \in \mathbb{F}[x]\}$ any nonconstant of minimal degree are gcds.*

证明.

$h \in M$, $deg(h) = d$ minimal. Let $k|f$ and $k|g \Rightarrow k|uf + vg$, $\forall u, v \Rightarrow k|h$.

Suppose $h' \in M$ is any nonzero element. $deg(h') \geq deg(h) \Rightarrow \exists q, r \in \mathbb{F}, deg(r) < deg(h)$ $h' = qh + r$.
$r = h' - qh \in M$. Since $deg(h) = d$ is nonconstant minimal degree, $r = 0 \Rightarrow h' = qh$. So $\exists q_1, q_2 \in \mathbb{F}[x]$, $1f + 0g = q_1 h, 0f + 1g = q_2 h \Rightarrow h|g, h|f$. $\qquad \square$

**Example 9.**

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$

$$f = 3g + x^2 - 3x + 2$$

$$g = (x+1)(x^2 - 3x + 2) + x - 1$$

$$x^2 - 3x + 2 = (x-2)(x-1)$$

$$\Rightarrow gcd(f,g) = x - 1$$

$$x - 1 = g - (x+1)(x^2 - 3x + 2) = g - (x+1)(f - 3g) = (3x+4)g - (x+1)f$$

**Example 10.** *Find a greatest common divisor of $f = x^3 - x^2 - x + 1$ and $g = x^2 - 3x + 2$ in $\mathbb{Q}[x]$, and express it in form $uf + vg, \; u, v \in \mathbb{Q}[x]$.*

$$f = (x+2)g + 3x - 3$$

$$g = \frac{1}{3}(x-2)(3x-3)$$

$$gcd(f,g) = 3x - 3$$

$$3x - 3 = f - (x+2)g$$

### 5.6.4 Proposition 2.3.12: $gcd(f,g) = 1, f|gh \Rightarrow f|h$

**Proposition 18** (Proposition 2.3.12). *If $f, g, h \in \mathbb{F}[x]$, $gcd(f,g) = 1$, and $f|gh$, then $f|h$.*

### 5.6.5 Corollary 2.3.13: irreducible $f$, $f|gh \Rightarrow f|g$ or $f|h$

**Corollary 6** (Corollary 2.3.13). *If $f \in \mathbb{F}[x]$ is irreducible, and $f|gh$, then $f|g$ or $f|h$.*

Since $f$ is irreducible, we have two possible situations:

1. $gcd(f,g) = f$, i.e. $f|g$ done.

2. $gcd(f,g) = 1$, then according to Prop 2.3.12, we can know $f|h$.

## 5.7 Roots

<u>Root:</u>$\alpha \in \mathbb{F}$ is a root of $f$ if $f(\alpha) = 0$.

### 5.7.1 Corollary 2.3.16(of Euclidean Algorithm): $f$ 可被分为 $(x - \alpha)q + f(\alpha)$i.e. if $\alpha$ is a root, then $(x - \alpha)|f$

**Corollary 7** (Corollary 2.3.16(of Euclidean Algorithm)). *$\forall f \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$, there exists a polynomial $q \in \mathbb{F}[x]$ s.t. $f = (x - \alpha)q + f(\alpha)$. In particular , if $\alpha$ is a root, then $(x - \alpha)|f$.*

## 5.8 Multiplicity

If $\alpha$ is a root of $f$ , say its *multiplicity* is $m$, if $x - \alpha$ appears $m$ times in irreducible factorization.

### 5.8.1 *Sum of multiplicity* $\leq deg(f)$

**Proposition 19** (Proposition 2.3.17). *Given a nonconstant polynomial $f \in \mathbb{F}[x]$, the number of roots of $f$, counted with multiplicity, is at most $deg(f)$.*

## 5.9 Roots in a filed may not in its subfield

Note if $\mathbb{F} \subset \mathbb{K}$, then $\mathbb{F}[x] \subset \mathbb{K}$. $f \in \mathbb{F}[x]$ may have no roots in $\mathbb{F}$, but could have roots in $\mathbb{K}$

**Example 11.** $x^n - 1 \in \mathbb{Q}[x]$ *has a root in $\mathbb{Q}$: 1; has 2 roots if $n$ even: $\pm 1$*
$\underline{roots\ in\ \mathbb{C}}: \zeta_n = e^{\frac{2\pi i}{n}}$, *then $\zeta_n^n = e^{2\pi i} = 1$; $(\zeta_n^k)^n = e^{2\pi k i} = 1$ So, the roots: $\{e^{\frac{2\pi k i}{n}} | k = 0, ..., n-1\}$*
*The roots of $x^n - d$: $\{e^{\frac{2\pi k i}{n}} \sqrt{d} | k = 0, ..., n-1\}$*

# 6 Linear Algebra

## 6.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$)

A $\underline{\text{vector space}}$ over a field $\mathbb{F}$ is a set $V$ w/ an operation $\underline{\text{addition}} + : V \times V \to V$ and an operation $\underline{\text{scalar multiplication}}\ \mathbb{F} \times V \to V$

(1) Addition is associative & commutative

(2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$

(3) $1v = v \forall v \in V$ (where $1 \in \mathbb{F}$ is multi. id. in $\mathbb{F}$ )

(4) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ \alpha(\beta v) = (\alpha\beta)v$

(5) $\forall v \in V,\ (-1)v = -v$ we have $v + (-v) = 0$

(6) $\forall \alpha \in \mathbb{F},\ v, u \in V,\ \alpha(v + u) = \alpha v + \alpha u$

(7) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ (\alpha + \beta)v = \alpha v + \beta v$

### 6.1.1 A field is a vector space over its subfield

**Example 12.** $\mathbb{K} \subset \mathbb{F}$ *is a subfield of a field $\mathbb{F}$. Then $\mathbb{F}$ is a vector space over $\mathbb{K}$. (Since $\mathbb{F} \subset \mathbb{F}[x]$, then $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$.)*

### 6.1.2 Vector subspace

Suppose that $V$ is a vector space over $\mathbb{F}$. A $\underline{\text{vector subspace}}$ or just $\underline{\text{subspace}}$ is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W$, $av \in W$, $\forall v, w \in W$, $a \in \mathbb{F}$.

**Example 13.** $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$, *then $\mathbb{L}$ is a subspace of $\mathbb{F}$ over $\mathbb{K}$.*

## 6.2 Linear independent, Linear combination

## 6.3 span V, basis, dimension, Proposition 2.4.10

A set of elements $v_1, ..., v_n \in V$ is said to **span** $V$ if every vector $v \in V$ can be expressed as a linear combination of $v_1, ..., v_n$. If $v_1, ..., v_n$ spans and is linearly independent, then we call the set a ***basis*** for $V$.

**Proposition 20** (Proposition 2.4.10.)**.** *Suppose $V$ is a vector space over a field $\mathbb{F}$ having a basis $\{v_1, ..., v_n\}$ with $n \geq 1$.*

(i) For all $v \in V$ , $v = a_1 v_1 + ... + a_n v_n$ for exactly one $(a_1, ..., a_n) \in \mathbb{F}^n$.
(ii) If $w_1, ..., w_n$ span $V$ , then they are linearly independent.
(iii)If $w_1, ..., w_n$ are linearly independent, then they span $V$.
If a vector space $V$ over $\mathbb{F}$ has a basis with $n$ vectors, then $V$ is said to be n-dimensional (over $\mathbb{F}$) or is said to have **dimension** $n$.

### 6.3.1 Standard basis vectors

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1) \in \mathbb{F}^n$$

are a basis for $\mathbb{F}^n$ called the **standard basis vectors**.

## 6.4 Linear transformation

Given two vector spaces $V$ and $W$ over $\mathbb{F}$ a **linear transformation** is a function $T : V \to W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$ , we have

$$T(av) = aT(v) \text{ and } T(v + w) = T(v) + T(w)$$

**Proposition 21** (Proposition 2.4.15.)**.** *If $V$ and $W$ are vector spaces and $v_1, ..., v_n$ is a basis for $V$ then any function from $\{v_1, ..., v_n\} \to W$ extends uniquely to a linear transformation $V \to W$.*

Any $v \in V$, $\exists (a_1, ..., a_n)$ s.t. $v = a_1 v_1 + ... + a_n v_n$. Then $T(v) = T(a_1 v_1 + ... + a_n v_n) = a_1 T(v_1) + ... + a_n T(v_n)$

### 6.4.1 Corollary 2.4.16: 一个线性变换对应一个矩阵 *bijection* $\mathcal{L}(V, M) \to M_{m \times n}(\mathbb{F})$

**Corollary 8** (Corollary 2.4.16.)**.** *If $v_1, ..., v_n$ is a basis for a vector space $V$ and $w_1, ..., w_n$ is a basis for a vector space $W$ (both over $\mathbb{F}$), then any linear transformation $T : V \to W$ determines (and is*

*determined by) the $m \times n$ matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & ... & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}^T = A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from $V$ to $W$; $M_{m \times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in $\mathbb{F}$. $T \to A(T)$ defines a *bijection* $\mathcal{L}(V, M) \to M_{m \times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** $T$.

### 6.4.2 Proposition 2.4.19: 线性变换矩阵相乘仍为线性变换矩阵

**Proposition 22** (Proposition 2.4.19). *Suppose that $V$ , $W$, and $U$ are vector spaces over $\mathbb{F}$, with fixed chosen bases. If $T : V \to W$ and $S : W \to U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \to U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

## 6.5 GL(V): invertible(bijective) linear transformations $V \to V$

Given a vector space $V$ over $F$, we let $GL(V) \subset \mathcal{L}(V, V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \text{ is a bijection}\} = \mathcal{L}(V, V) \cap Sym(V)$$

# 7 Euclidean geometry basics

## 7.1 Euclidean distance, inner product

**Euclidean distance** on $\mathbb{R}^n$:

$$|x - y| = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

**Euclidean inner product**:

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n = x^T y$$

## 7.2 Isometry of $\mathbb{R}^n$: a bijection $\mathbb{R}^n \to \mathbb{R}^n$ preserves distance

An **isometry** of $\mathbb{R}^n$ is a bijection $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x - y|, \ \forall x, y \in \mathbb{R}^n$$

### 7.2.1  $Isom(\mathbb{R}^n)$: set of all isometries of $\mathbb{R}^n$

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of $\mathbb{R}^n$,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n | |\Phi(x) - \Phi(y)| = |x - y|, \ \forall x, y \in \mathbb{R}^n\}$$

### 7.2.2  $Isom(\mathbb{R}^n)$ is closed under $\circ$ and inverse

**Proposition 23.** $\Phi, \Psi \in Isom(\mathbb{R}^n)$, then $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$

证明.
Since $\Phi, \Psi$ are bijections, so is $\Phi \circ \Psi$. Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x - y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x - y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

$\square$

### 7.3  $A \in GL(n, \mathbb{R})$, $T_A(v) = Av$: $A^t A = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a *invertible linear transofrmations* $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t(Aw) = v^t A^t A w$$

$$A^t A = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)} \ T_A \in Isom(\mathbb{R}^n)$$

### 7.4  Linear isometries i.e. orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\}$

We define the all isometries in *invertible linear transofrmations* $\mathbb{R}^n \to \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\} \subset GL(n, \mathbb{R})$$

#### 7.4.1  Special orthogonal group $SO(n) = \{A \in O(n) | det(A) = 1\}$: orthogonal group with $det(A) = 1$

$O(n)$ are the matrices representing linear isometries of $\mathbb{R}^n$. $1 = det(I) = det(A^t A) = det(A^t)det(A) = det(A)^2 \Rightarrow det(A) = 1$ or $det(A) = -1$. We use **special orthogonal group** represents $A$ with $det(A) = 1$,

$$SO(n) = \{A \in O(n) | det(A) = 1\}$$

### 7.5  translation: $\tau_v(x) = x + v$

Define a *translation* by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \to \mathbb{R}^n, \ \tau_v(x) = x + v$$

### 7.5.1 translation is an isometry

**Note 3** (Exercise 2.5.3). $\forall v \in \mathbb{R}^n, \tau_v$ *is an isometry.*

证明. $|\tau_v(x) - \tau_v(y)| = |(x+v) - (y+v)| = |x-y|$ □

## 7.6 The composition of a translation and an orthogonal transformation is an isometry $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since *the composition of isometries is an isometry,* $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries**.

### 7.6.1 Theorem 2.5.3: All isometries can be represented by a composition of *a translation* and *an orthogonal transformation,* $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

**Theorem 5** (Theorem 2.5.3). $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

# 8 Group $(G, *)$: a set with an operation(associative, identity, inverse)

A *group* is a nonempty set $G$ with an operation $* : G \times G \to G$ s.t. the following holds
(1) $*$ is associative
(2) $\exists e \in G$ an identity element $e * g = g * e = g \ \forall g \in G$
(3) $\forall g \in G, \exists g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$

## 8.1 $(Sym(X), \circ)$ symmetric/permutation group of $X$

**Example 14.** *If $X$ is any nonempty set, permutation group of $X : \{\sigma : X \to X | \sigma$ is a bijection\}, then*

1. $\circ$ is associative;
2. $id : X \to X, \ id(x) = x \ \forall x \in X$ is the idenity;
3. $\sigma \in Sym(X), \sigma^{-1} \in Sym(X)$ is the inverse function.
$(Sym(X), \circ)$ **is a group called the symmetric group of** $X$

## 8.2 Order of a group $(G, *)$

$|G| =$Order of a group $(G, *)$

## 8.3 Abelian group (group that $*$ satisfies commutative)

$(\mathbb{Z}, +)$ is a group and $+$ is commutative, we call this kind of groups(statify commutative) *abelian group*.

**Example 15.** *If $\mathbb{F}$ is a field, then $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \cdot)$ are abelian group.*

**Example 16.** *If $V$ is a vector space over $\mathbb{F}$, then $(V, +)$ abelian group.*

As we know a $V$ is a vector space over $\mathbb{F}$ means $V$ is a field whose subfields include $\mathbb{F}$.

## 8.4 $H$: Subgroup of $(G, *)$ ($H \neq \emptyset \subset G$ closed under $*$; inverse $g^{-1} \in H$), written as $H < G$

$H \neq \emptyset \subset G$ is a *subgroup* of $(G, *)$ if,
1. $\forall\ g, h \in H, g * h \in H$.
2. $\forall\ g \in H, g^{-1} \in H$.
write $H < G$ if $H$ is a subgroup of $(G, *)$.

### 8.4.1 Proposition 2.6.8: $H < G$, $(H, *)$ is a group: A group's operation with its any subgroup is also a group

**Proposition 24** (Proposition 2.6.8). *If $(G, *)$ is a group, $H \subset G$ is a subgroup, then $(H, *)$ is a group.*

**Example 17.** *$(G, *)$ is a group, then $e < G$, $G < G$.*

**Example 18.** *$\mathbb{K} \subset \mathbb{F}$ is a subfield, then $\mathbb{K} < \mathbb{F}$, $\mathbb{K}^\times < \mathbb{F}^\times$.*

**Example 19.** *$W \subset V$ is a vector subspace, $W < V$.*

**Example 20.** *$1 \in S^1 \subset \mathbb{C}^\times$, $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. $S^1$ is a subgroup.*

证明.
$S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. For any $e^{i\theta}, e^{i\psi} \in S^1$, $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1, e^{-i\theta} \in S^1$. $\qquad\qquad \square$

**Example 21.** *$Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$*

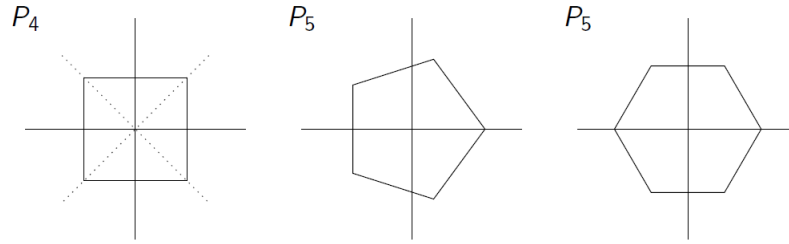**Example 22.** *If $\mathbb{F}$ is a field, $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \to \mathbb{F} \in Sym(\mathbb{F}) | \sigma(a+b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$*

**Example 23.** *Dihedral Groups:*

保留多边形
Let $P_n \subset \mathbb{R}^2$ be a regular $n - gon$

$D_n < Isom(\mathbb{R}^2)$, $D_n = \{\Phi \in Isom(\mathbb{R}^2) | \Phi(P_n) = P_n\}$

$P_4$        $P_5$        $P_5$

# 9   Ring $(R, +, \cdot)$: $+$ is associative, commutative, identity, inverse $\in R$; $\cdot$ is associative, distributes over $+$

**Definition 1.** *A ring is a nonempty set with two operations, called addition and multiplication, $(R, +, \cdot)$ such that*

(1): $(R, +)$ is an ablian group: i.e. $+$ is associatve and commucative. $0, -a \in R$

(2): $\cdot$ is associative.

(3): $\cdot$ distributes over $+$: $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

## 9.1   Commutative ring: ring's $\cdot$ is commutative

If "$\cdot$" is commutative, we call $(R, +, \cdot)$ a commutative ring.

## 9.2   Ring with 1: exists multiplication identity $1 \in R$

If there exists an element $1 \in R \backslash \{0\}$ such that $a1 = 1a = a$, $\forall a \in R$,then we say that $R$ is a ring with 1.

## 9.3   Field $\mathbb{F}$ is a commutative ring with 1; $\mathbb{F}[x]$ is also a commutative ring with 1

Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))
Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

## 9.4   $S \subset R$: Subring (closed under $+$ and $\cdot$; addictive inverse $-a \in S$)

### 9.4.1   Proposition 2.6.27: $(S, +, \cdot)$ is a ring

**Proposition 25** (Proposition 2.6.27). *If $S \subset R$ is a subring, then $+, \cdot$ make $S$ into a ring.*

# 10 Group theory

## 10.1 Properties of Group Operation

### 10.1.1 Proposition 3.1.1: $g * h = h$ or $h * g = h$, then $g = e$; $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

**Proposition 26** (Proposition 3.1.1). *Let $(G, *)$ be a group with identity $e \in G$, then*

(1) if $g, h \in G$ and either $g * h = h$ or $h * g = h$, then $g = e$
(2) if $g, h \in G$ and $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

### 10.1.2 Corollary 3.1.: $e^{-1} = e$, $(g^{-1})^{-1} = g$, $(g * h)^{-1} = h^{-1} * g^{-1}$

**Corollary 9** (Corollary 3.1.2). $e^{-1} = e$, $(g^{-1})^{-1} = g$, $(g * h)^{-1} = h^{-1} * g^{-1}$

### 10.1.3 Proposition 3.1.3: $g * h = k * h$ or $h * g = h * k$, then $g = k$

**Proposition 27** (Proposition 3.1.3). *If $g * h = k * h$ or $h * g = h * k$, then $g = k$.*

### 10.1.4 Proposition 3.1.4: $g * x = h$ and $x * g = h$ have unique solutions $x \in G$.

**Proposition 28** (Proposition 3.1.4). *$g * x = h$ and $x * g = h$ have unique solutions $x \in G$.*

## 10.2 Power of an Element

We define $g^n$ recursively for $n \geq 0$ by setting $g^0 = e$ and for $n \geq 1$, we set $g^n = g^{n-1} * g$. For $n \leq 0$, we define $g^n = (g^{-1})^{-n}$.

### 10.2.1 Proposition 3.1.5: $g^n * g^m = g^{n+m}$, $(g^n)^m = g^{nm}$

**Proposition 29** (Proposition 3.1.5). *(1) $g^n * g^m = g^{n+m}$; (2) $(g^n)^m = g^{nm}$*

## 10.3 $(G \times H, \circledast)$: __Direct Product__ of $G$ and $H$

$(G, *)$ a group $(H, \star)$ a group. Define an operation on $G \times H$, $\circledast$:

$$(h, k) \circledast (h', k') = (h * h', k * k')$$

### 10.3.1 Proposition 3.1.7: $(G \times H, \circledast)$ is a group

**Proposition 30** (Proposition 3.1.7). *$(G \times H, \circledast)$ is a group. The identity is $(e_G, e_H)$, inverse is $(g^{-1}, h^{-1})$*

usually written as

$$(h, k)(h', k') = (hh', kk')$$

## 10.4   Subgroups and cyclic groups

### 10.4.1   Proposition 3.2.2: Intersection of a Collection of Subgroups is a group

**Proposition 31** (Proposition 3.2.2)**.** *Let $G$ be a group and suppose $\mathcal{H}$ is any collection of subgroups of $G$. Then $K = \cap_{H \in \mathcal{H}} H < G$ is a subgroup of $G$.*

### 10.4.2   Subgroup Generated by $A$: $\langle A \rangle = \cap_{H < G; A \subset H} H$

We define **Subgroup Generated by $A$:**

$$\langle A \rangle = \cap_{H \in \mathcal{H}(A)} H$$

where $\mathcal{H}(A)$ is the set of all subgroups of $G$ containing the set $A$:

$$\mathcal{H}(A) = \{H < G | A \subset H \ \text{and} \ H \ \text{is a subgroup of} \ G\}$$

### 10.4.3   <u>Cyclic Subgroup</u> generated by $a$: $\langle a \rangle = \cap_{H < G; a \in H} H$ ($G$ is <u>cyclic</u> if exists $g$, $\langle g \rangle = G$)

If $A = \{a\}$, then $\langle a \rangle\, (= \langle \{a\} \rangle) = $ the <u>cyclic subgroup</u> generated by $a$

Say $G$ is <u>cyclic</u> if $\exists g \in G$, s.t. $G = \langle g \rangle$; $g$ is called a <u>generator</u> for $G$ in this case.

### 10.4.4   Proposition 3.2.3: $\langle g \rangle = \{g^n | n \in \mathbb{Z}\}$

**Proposition 32** (Proposition 3.2.3)**.** *Let $G$ be a group, $g \in G$. Then*

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}$$

### 10.4.5   Corollary 3.2.4: $G$ is a cyclic group $\Rightarrow$ $G$ is abelian

**Corollary 10** (Corollary 3.2.4)**.** *If $G$ is a cyclic group (i.e. exits $g \in G$ s.t. $\langle g \rangle = G$), then $G$ is abelian (i.e. commutative).*

### 10.4.6   Equivalent properties of order of $g$: $|g| = |\langle g \rangle| < \infty$

**Proposition 33** (Proposition 3.2.6)**.** *Let $G$ be a group for $g \in G$, the following are equivalent:*

(i) $|g| < \infty$
(ii) $\exists n \neq m$ in $\mathbb{Z}$ so that $g^n = g^m$
(iii) $\exists n \in \mathbb{Z}$, $n \neq 0$ so that $g^n = e$
(iv) $\exists n \in \mathbb{Z}_+$ so that $g^n = e$
If $|g| < \infty$, then $|g| = $ smallest $n \in \mathbb{Z}_+$ so that $g^n = e$, and $\langle g \rangle = \{e, g, g^2, \ldots, g^{n-1}\} = \{g^n \mid n = 0, \ldots, n-1\}$

**10.4.7** $(\mathbb{Z}, +)$ **Theorem 3.2.9:** $H < \mathbb{Z}$ **is a subgroup** $\Rightarrow H = \{0\}$ **or** $H = \langle d \rangle$**;** $\langle a \rangle < \langle b \rangle$ **if and only if** $b|a$

**Theorem 6** (Theorem 3.2.9). *If $H < \mathbb{Z}$ is a subgroup, then either $H = \{0\}$ , or else $H = \langle d \rangle$ , where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently, $a \to \langle a \rangle$ defines a **bijection** from $N = \{0, 1, 2, ...\}$ to the set of subgroups of $\mathbb{Z}$. Furthermore, for $a, b \in \mathbb{Z}_+$, we have $\langle a \rangle < \langle b \rangle$ if and only if $b|a$.*

**10.4.8** $(\mathbb{Z}_n, +)$ **Theorem 3.2.10:** $H < \mathbb{Z}_n$ **is a subgroup** $\Rightarrow H = \langle [d] \rangle$**;** $\langle [d] \rangle < \langle [d'] \rangle$ **if and only if** $d'|d$

**Theorem 7** (Theorem 3.2.10). *For any $n \geq 2$, if $H < \mathbb{Z}_n$ is a subgroup, then there is a positive divisor $d$ of $n$ so that*

$$H = \langle [d] \rangle$$

*Furthermore, this defines a bijection between divisors of $H$ and subgroups of $\mathbb{Z}_n$. Furthermore, if $d, d' > 0$ are two divisors of $n$, then $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$.*

If $H = \langle [d] \rangle$ is a subgroup of $H$, then $[n] \in H$, so $d|n$. And $|H| = |\langle [d] \rangle| = \frac{n}{d}$, so $|H||d$

### 10.4.9 Subgroup Lattice

The set of all subgroups of a group of G, together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup $\{e\}$ at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.

## 参考文献

[1] Christopher J Leininger Introduction to Abstract Algebra (Draft) 2017.
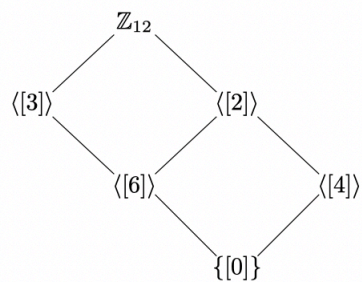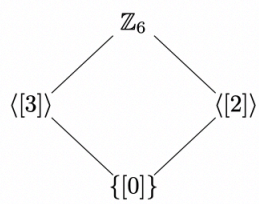
Writing down the subgroup lattice is as easy as writing down the divisibility lattice in which $n$ is placed at the bottom, 1 at the top, and all intermediate divisors in between, connected by edges when there is divisibility. The congruence class of the divisor generates the corresponding subgroup in the subgroup lattice.
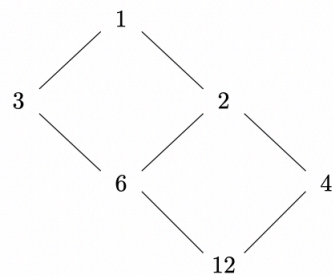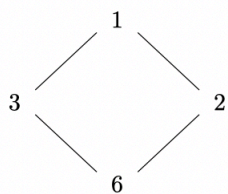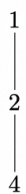


图 1: