

Analysis

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2022

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1 Basis

1.1 Sequence Definitions

Sequences $\{x_k\}_{k=1}, \dots$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 1 (Convergence: note $x_k \rightarrow x, \lim_{k \rightarrow \infty} x_k = x$). Given $\varepsilon > 0$, $\exists N_\varepsilon$ s.t.

$$\|x_k - x\| < \varepsilon \quad \forall k \geq N_\varepsilon$$

Definition 2 (Cauchy Sequence). $\{x_k\}$ is Cauchy if given $\varepsilon > 0$, $\exists N_\varepsilon$ s.t.

$$\|x_k - x_m\| < \varepsilon, \quad \forall k, m \geq N_\varepsilon.$$

Note:

$$\{x_k\} \text{ converges} \iff \{x_k\} \text{ is Cauchy}$$

Definition 3 (Subsequence). Infinite subset of $\{x_k\}$: $\{x_k : k \in \mathcal{K}\}$ or $\{x_k\}_{\mathcal{K}}$, where \mathcal{K} is subset of \mathbb{Z}^+ .

Definition 4 (Limit point). x is a limit point of $\{x_k\}$ if \exists a subsequence of $\{x_k\}$ that converges to x .

Definition 5 (Bounded Sequence).

$$\|x_k\| \leq b, \forall k$$

Results about Bounded sequences:

1. Every bounded has at least one limit point.
2. A bounded sequence converges iff it has a **unique limit point**.

1.2 Scalar Sequences

Scalar sequences $\{x_k\}, x_k \in \mathbb{R}$:

Proposition 1. If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it **converges**.

Proposition 2. The largest(smallest) limit point of $\{x_k\}$ is $\lim_{k \rightarrow \infty} \sup x_k$ ($\lim_{k \rightarrow \infty} \inf x_k$)

Proposition 3. $\{x_k\}$ converges $\iff -\infty < \lim_{k \rightarrow \infty} \inf x_k = \lim_{k \rightarrow \infty} \sup x_k < \infty$

1.3 Functions Basis

Definition 6 (Continuity). A real-valued function f is continuous at x if for every $\{x_k\}$ converging to x satisfies that $\lim_{k \rightarrow \infty} f(x_k) = f(x)$.

Equivalent: given $\varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon \quad \forall \|y - x\| < \delta$

f is continuous if it is continuous at all points x .

Definition 7 (Coercive). A real-valued function $f : \& \rightarrow \mathbb{R}$ is coercive if for **every** $\{x_k\} \subset \&$ s.t. $\|x_k\| \rightarrow \infty, f(x_k) \rightarrow \infty$

Example 1 (Check coercive).

- 1) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2$ - coercive
- 2) $x \in \mathbb{R}, f(x) = 1 - e^{-|x|}$ - not coercive
- 3) $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (需要所以 $\|x_k\| \rightarrow \infty$ 都满足)

1.4 Sets

Definition 8 (Open Sets). *A set $\& \subseteq \mathbb{R}^n$ is open if*

$\forall x \in \&$ we can draw a ball around x that is contained in $\&$.

i.e. $\forall x \in \&, \exists \varepsilon > 0$ s.t. $\{y : \|y - x\| < \varepsilon\} \subseteq \&$

Definition 9 (Closed Sets). *$\&$ is closed if $\&^c$ is open*

Equivalent: if $\&$ contains all limit points of all sequences in $\&$

Example 2 (Closed and Open Sets).

- 1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open
- 2) \mathbb{R} is both open and closed
- 3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open
- 4) $[1, \infty)$ is closed because its complement open
- 5) $(1, 2]$ is neither open nor closed

Definition 10 (Bounded Set). *A is bounded if $\exists M$ s.t. $\|x\| \leq M \quad \forall x \in \&$*

Definition 11 (Compact Set). *$\mathcal{L} \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.*

Example 3 (Compact Set). $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}; \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

Definition 12 (Extreme of sets of scalars, $\sup A, \inf A$). *Let $A \subset \mathbb{R}$.*

- The infimum of A , or $\inf A$ is largest y s.t. $y \leq x, \forall x \in A$. If no such y exists, $\inf A = -\infty$

- Similar definition for supremum of A (or wrote as $\sup A$).

Proposition 4. *If $\inf A(\sup A) = x^* \in A$, then $x^* = \min A(\max A)$*

2 Functions

2.1 Extreme of Functions

Definition 13 (Extreme of Functions). *Let $\& \subseteq \mathbb{R}^n, f : \& \rightarrow \mathbb{R}$*

$$\inf_{x \in \&} f(x) = \inf\{f(x) : x \in \&\}$$

If $\exists x^* \in \mathcal{X}$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \mathcal{X}} f(x)$. x^* is called a **minimizer** of f , written as $x^* \in \arg \min_{x \in \mathcal{X}} f(x)$. If x^* is unique, we write $x^* = \arg \min_{x \in \mathcal{X}} f(x)$

Similarly, supremum and maximum of f .

2.1.1 Weierstrass' Theorem(Extreme value Theorem)

Theorem 1 (Weierstrass' Theorem(Extreme value Theorem)).

If f is a **continuous** function on a **compact set**, $\mathcal{X} \subseteq \mathbb{R}^n$, then f attains its min and max on \mathcal{X} i.e.,

$$\exists x_1 \in \mathcal{X} \text{ s.t. } f(x_1) = \inf_{x \in \mathcal{X}} f(x)$$

$$\exists x_2 \in \mathcal{X} \text{ s.t. } f(x_2) = \sup_{x \in \mathcal{X}} f(x)$$

证明. (for existence of min; max is similar)

Let $\{\sigma_k\} \subseteq \mathcal{X}$ be s.t.

$$\inf_{x \in \mathcal{X}} f(x) \leq f(\sigma_k) \leq \inf_{x \in \mathcal{X}} f(x) + \frac{1}{k}$$

Then $\lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \mathcal{X}} f(x)$

\mathcal{X} is bounded $\Rightarrow \{\sigma_k\}$ has at least one limit point x ,

\mathcal{X} is closed $\Rightarrow x_1 \in \mathcal{X}$

f is continuous $\Rightarrow f(x_1) = \lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \mathcal{X}} f(x)$ □

Corollary 1 (Corollary to WT). Let f be continuous on closed set \mathcal{X} (not necessarily bounded). If f is coercive on \mathcal{X} it attains its min on \mathcal{X} .

证明. Consider $\{\sigma_k\}$ as in proof of WT.

Since f is closed, $f(x) < \infty, \forall x \in \mathcal{X}$. And f is coercive on \mathcal{X} , which means $f(x) \rightarrow \infty$ if $\|x\| \rightarrow \infty$.

Hence, $\{\sigma_k\} \in \mathcal{X}$ is bounded. Rest of proof same as proof of WT. □

Example 4. $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) Does f achieve its min and max on $\mathcal{L}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leq 6\}$?

- \mathcal{L}_1 is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over \mathbb{R}^3 ?

- $f \rightarrow \infty$ whenever $\|x\| \rightarrow \infty \Rightarrow f$ is coercive.

- \mathbb{R}^3 is closed.

$\Rightarrow f$ achieves its min. on \mathbb{R}^3 by corollary to WT.

- max does not exist since $f \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

- 3) Does f achieve its min and max over $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?
- \mathcal{L}_2 is closed, but not bounded.
 - Since f is coercive, min achieved.
 - max does not exist since setting $x_1 = 0$ $x_2 = 3 - x_3$ and letting $x_3 \rightarrow \infty$ makes $f \rightarrow \infty$

3 Big \mathcal{O} Notation

3.1 Definition

For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$.

Example 5.

$$\begin{aligned} n^3 + n + 2 &= \Omega(1), n^3 + n + 2 = \Omega(n^2) \\ n^3 + n + 2 &= \Theta(n^3) \\ n^3 + n + 2 &= o(n^4) \end{aligned}$$

3.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$ if $\limsup_{x \rightarrow a} \frac{|f(x)|}{g(x)} < \infty$.

Example 6. $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \rightarrow 0$

4 Lipschitz Continuous

4.1 Definition

Definition 14. *Lipschitz continuous: if a function $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ satisfies*

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq \gamma \|\mathbf{x} - \mathbf{y}\|, \forall \mathbf{x}, \mathbf{y}$$

the function is called γ -Lipschitz continuous;

If f is γ -Lipschitz continuous, then it is also $(\gamma + 1)$ -Lipschitz continuous

The minimal such γ is called a Lipschitz constant of function f

Remark: Here $\|\cdot\|$ can be any given norm of the space \mathbb{R}^n and \mathbb{R}^m , such as Euclidean norm, ℓ_1 -norm, etc.

When not specified, we assume it is Euclidean norm.

4.2 Example

Example 1: $f(x) = 2x$ is 2-Lipschitz continuous;

Example 2: What about $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where \mathbf{A} is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).

Example 3: What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞ .

4.3 Contraction Mapping

1. If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.

2. If $\gamma < 1$, then f is called a contraction mapping

Example 1: $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.

Example 2: $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

5 Fixed point theorem

1. Fixed point theorem: If f is a contraction mapping that maps \mathbb{R}^n to itself, then the following two results hold:

1) There exists a unique fixed point \mathbf{x}^* satisfying

$$\mathbf{x}^* = f(\mathbf{x}^*)$$

2) In addition, the iterated function sequence

$$\mathbf{x}, f(\mathbf{x}), f(f(\mathbf{x})), \dots,$$

converges to this unique fixed point \mathbf{x}^* (independent of the initial point x).

2. Remark: This is a special case of "Banach fixed point theorem" (which applies to any complete metric space).