Linear Algebra

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目录

1 Vector Space

1.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$)

A vector space over a field $\mathbb{F}$ is a set V w/ an operation addition $+ : V \times V \to V$ and an operation scalar multiplication $\mathbb{F} \times V \to V$
(1) Addition is associative & commutative
(2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$
(3) $1v = v \forall v \in V$(where $1 \in \mathbb{F}$ is multi. id. in $\mathbb{F}$)
(4) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ \alpha(\beta v) = (\alpha\beta)v$
(5) $\forall v \in V,\ (-1)v = -v$ we have $v + (-v) = 0$
(6) $\forall \alpha \in \mathbb{F},\ v, u \in V,\ \alpha(v + u) = \alpha v + \alpha u$
(7) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ (\alpha + \beta)v = \alpha v + \beta v$

1.2 A field is a vector space over its subfield

Example 1. *$\mathbb{K} \subset \mathbb{F}$ is a subfield of a field $\mathbb{F}$. Then $\mathbb{F}$ is a vector space over $\mathbb{K}$. (Since $\mathbb{F} \subset \mathbb{F}[x]$, then $\mathbb{F}[x]$ is a vector space over $\mathbb{F}$.)*

1.3 Vector subspace

Suppose that V is a vector space over $\mathbb{F}$. A vector subspace or just subspace is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W,\ av \in W,\ \forall v, w \in W,\ a \in \mathbb{F}$.

Example 2. *$\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$, then $\mathbb{L}$ is a subspace of $\mathbb{F}$ over $\mathbb{K}$.*

1.4 Linear independent, Linear combination

1.5 span V, basis, dimension

A set of elements $v_1, ..., v_n \in V$ is said to **span** V if every vector $v \in V$ can be expressed as a linear combination of $v_1, ..., v_n$. If $v_1, ..., v_n$ spans and is linearly independent, then we call the set a ***basis*** for V.

Proposition 1 (Proposition 2.4.10.). *Suppose V is a vector space over a field $\mathbb{F}$ having a basis $\{v_1, ..., v_n\}$ with $n \geq 1$.*

(i) For all $v \in V$, $v = a_1v_1 + ... + a_nv_n$ for exactly one $(a_1, ..., a_n) \in \mathbb{F}^n$.
(ii) If $w_1, ..., w_n$ span V , then they are linearly independent.
(iii)If $w_1, ..., w_n$ are linearly independent, then they span V.
If a vector space V over $\mathbb{F}$ has a basis with n vectors, then V is said to be n-dimensional (over $\mathbb{F}$) or is said to have **dimension** n.

1.6 Standard basis vectors

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1) \in \mathbb{F}^n$$

are a basis for $\mathbb{F}^n$ called the **standard basis vectors**.

1.7 Linear transformation

Given two vector spaces V and W over $\mathbb{F}$ a **linear transformation** is a function $T : V \to W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$, we have

$$T(av) = aT(v) \ and \ T(v + w) = T(v) + T(w)$$

Proposition 2 (Proposition 2.4.15.)**.** *If V and W are vector spaces and $v_1, ..., v_n$ is a basis for V then any function from $\{v_1, ..., v_n\} \to W$ extends uniquely to a linear transformation $V \to W$.*

Any $v \in V$, $\exists(a_1, ..., a_n)$ s.t. $v = a_1v_1 + ... + a_nv_n$. Then $T(v) = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$

1.8 一个线性变换对应一个矩阵，线性变换矩阵相乘仍为线性变换矩阵

Corollary 1 (Corollary 2.4.16.)**.** *If $v_1, ..., v_n$ is a basis for a vector space V and $w_1, ..., w_n$ is a basis for a vector space W (both over $\mathbb{F}$), then any linear transformation $T : V \to W$ determines (and is determined by) the $m \times n$ matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & ... & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}^T = A \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from V to W; $M_{m\times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in $\mathbb{F}$. $T \to A(T)$ defines a *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** T.

Proposition 3 (Proposition 2.4.19)**.** *Suppose that V , W, and U are vector spaces over $\mathbb{F}$, with fixed chosen bases. If $T : V \to W$ and $S : W \to U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \to U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

1.9 GL(V): invertible linear transformations $V \to V$

Given a vector space V over F, we let $GL(V) \subset \mathcal{L}(V,V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V,V) | T \ \textit{is a bijection}\} = \mathcal{L}(V,V) \cap Sym(V)$$

2 Euclidean geometry basics

2.1 Norm

2.1.1 Vector's Norm

Vector $x \in \mathbb{R}^n$-n-dim Euclidean space

$$x = (x_1, \ldots, x_n) \equiv \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^\top = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Norm of x, $\|x\|$ satisfies properties:

(a) $\|x\| \geqslant 0$

(b) $\|x\| = 0 \Leftrightarrow x = 0$

(c) $\|cx\| = |c|\|x\|$, for $c \in \mathbb{R}$

(d) $\|x+y\| \leqslant \|x\| + \|y\| \longleftarrow$ Triangle Ineq.

Enclidean Norm (default $\rho = 2$): $\|x\| = \sqrt{x^\top x} = \sqrt{\sum_{i=1}^n {x_i}^2}$

Other norms:

1. l_1-norm : $\|x\|_1 = \sum_{i=1}^n |x_i|$
2. l_ρ-norm : $\|x\|_\rho = \sqrt[\rho]{\sum_{i=1}^n |x_i|^\rho}$
3. Supremum norm or l_∞-norm : $\|x\|_\infty = \max_i |x_i|$

2.1.2 Matrix's Norm

$A \in \mathbb{R}^{n \times m}$ is a matrix

$\|Ax\| \leqslant \|A\|\|x\|, \|AB\| \leqslant \|A\|\|B\|$

Default is $\rho = 1$: $\|A\| = \max_{\|x\|=1} \|Ax\|$. 即找到最大的绝对值和的“列”。

$\|A\|_F = \sqrt{\sum_{i,j} a_{ij}^2}$ (Frobenius norm)

$\|A\|_1 = \max_j \sum_{i=1}^n |A_{ij}|$ [Π]

$\|A\|_\infty = \max_2 \sum_{j=1}^n |A_{ij}|$ [□]

$\|A\|_2 = \max_k \sigma_k, \sigma_k$ is the singurai veruee of A

$\|A\| = \max\left(\frac{\|A\times\|}{\|\times\|}\right) \Rightarrow \|A\| \geqslant \frac{\|A\times\|}{\|\times\|}$

$\|A\times\| \leqslant \|A\|\|x\|$

2.2 Euclidean distance, inner product

Euclidean distance on $\mathbb{R}^n$:

$$|x-y| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}$$

Euclidean inner product:

$$x \cdot y = x_1 y_1 + \cdots + x_n y_n = x^T y$$

Two important results for Euchidean norm:
1) Pythagorean Theorem: If $x^\top y = 0$,

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2$$

2) Cauchy - Schwarz Inequality:

$$\left|x^\top y\right| \leqslant \|x\|\|y\|$$
$$" = " \text{ iff } x = \alpha y \text{ for some } \alpha \in \mathbb{R}$$

2.3 Isometry

An **isometry** of $\mathbb{R}^n$ is a bijection $\varPhi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance, which means,

$$|\varPhi(x) - \varPhi(y)| = |x-y|,\ \forall x, y \in \mathbb{R}^n$$

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of $\mathbb{R}^n$,

$$Isom(\mathbb{R}^n) = \{\varPhi : \mathbb{R}^n \to \mathbb{R}^n \,|\, |\varPhi(x) - \varPhi(y)| = |x-y|,\ \forall x, y \in \mathbb{R}^n\}$$

Proposition 4. *$\varPhi, \varPsi \in Isom(\mathbb{R}^n)$, then $\varPhi \circ \varPsi, \varPhi^{-1} \in Isom(\mathbb{R}^n)$*

证明.
Since $\varPhi, \varPsi$ are bijections, so is $\varPhi \circ \varPsi$. Moreover,

$$|\varPhi \circ \varPsi(x) - \varPhi \circ \varPsi(y)| = |\varPhi(\varPsi(x)) - \varPhi(\varPsi(y))| = |\varPsi(x) - \varPsi(y)| = |x-y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x-y| = |id(x) - id(y)| = |\varPhi \circ \varPhi^{-1}(x) - \varPhi \circ \varPhi^{-1}(y)| = |\varPhi^{-1}(x) - \varPhi^{-1}(y)|$$

□

2.4 Linear isometries i.e. orthogonal group

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a *invertible linear transofrmations* $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t(Aw) = v^t A^t Aw$$

$$A^t A = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$$

We define the all isometries in *invertible linear transofrmations* $\mathbb{R}^n \to \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\} \subset GL(n, \mathbb{R})$$

2.5 Special orthogonal group

$O(n)$ are the matrices representing linear isometries of $\mathbb{R}^n$. $1 = det(I) = det(A^t A) = det(A^t)det(A) = det(A)^2 \Rightarrow det(A) = 1$ or $det(A) = -1$. We use **special orthogonal group** represents A with $det(A) = 1$,

$$SO(n) = \{A \in O(n) | det(A) = 1\}$$

2.6 translation

Define a *translation* by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \to \mathbb{R}^n, \ \tau_v(x) = x + v$$

Note 1 (Exercise 2.5.3)**.** *$\forall v \in \mathbb{R}^n, \tau_v$ is an isometry.*

证明. $|\tau_v(x) - \tau_v(y)| = |(x + v) - (y + v)| = |x - y|$ □

2.7 All isometries can be represented by a composition of *a translation* and *an orthogonal transformation*

Since *the composition of isometries is an isometry,* $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries**.

Theorem 1. $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

参考文献

[1] MATH 417: Christopher J Leininger Introduction to Abstract Algebra (Draft) 2017.

[2] MATH 484

[3] ECE 490