

# Analysis

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# 1 Basis

## 1.1 Sequence Definitions

Sequences  $\{x_k\}_{k=1}, \dots$  or  $\{x_k\}, x_k \in \mathbb{R}^n$

**Definition 1** (Convergence: note  $x_k \rightarrow x, \lim_{k \rightarrow \infty} x_k = x$ ). Given  $\varepsilon > 0$ ,  $\exists N_\varepsilon$  s.t.

$$\|x_k - x\| < \varepsilon \quad \forall k \geq N_\varepsilon$$

**Definition 2** (Cauchy Sequence).  $\{x_k\}$  is Cauchy if given  $\varepsilon > 0$ ,  $\exists N_\varepsilon$  s.t.

$$\|x_k - x_m\| < \varepsilon, \quad \forall k, m \geq N_\varepsilon.$$

**Note:**

$$\{x_k\} \text{ converges} \iff \{x_k\} \text{ is Cauchy}$$

**Definition 3** (Subsequence). Infinite subset of  $\{x_k\}$ :  $\{x_k : k \in \mathcal{K}\}$  or  $\{x_k\}_{\mathcal{K}}$ , where  $\mathcal{K}$  is subset of  $\mathbb{Z}^+$ .

**Definition 4** (Limit point).  $x$  is a limit point of  $\{x_k\}$  if  $\exists$  a subsequence of  $\{x_k\}$  that converges to  $x$ .

**Definition 5** (Bounded Sequence).

$$\|x_k\| \leq b, \forall k$$

Results about Bounded sequences:

1. Every bounded has at least one limit point.
2. A bounded sequence converges iff it has a **unique limit point**.

## 1.2 Scalar Sequences

Scalar sequences  $\{x_k\}, x_k \in \mathbb{R}$ :

**Proposition 1.** If  $\{x_k\}$  is bounded above(below) and non-decreasing(non-increasing) it **converges**.

**Proposition 2.** The largest(smallest) limit point of  $\{x_k\}$  is  $\lim_{k \rightarrow \infty} \sup x_k$  ( $\lim_{k \rightarrow \infty} \inf x_k$ )

**Proposition 3.**  $\{x_k\}$  converges  $\iff -\infty < \lim_{k \rightarrow \infty} \inf x_k = \lim_{k \rightarrow \infty} \sup x_k < \infty$

## 1.3 Functions Basis

**Definition 6** (Continuity). A real-valued function  $f$  is continuous at  $x$  if for every  $\{x_k\}$  converging to  $x$  satisfies that  $\lim_{k \rightarrow \infty} f(x_k) = f(x)$ .

Equivalent: given  $\varepsilon > 0, \exists \delta > 0$  s.t.  $|f(x) - f(y)| < \varepsilon \quad \forall \|y - x\| < \delta$

$f$  is continuous if it is continuous at all points  $x$ .

**Definition 7** (Coercive). A real-valued function  $f : \& \rightarrow \mathbb{R}$  is coercive if for **every**  $\{x_k\} \subset \&$  s.t.  $\|x_k\| \rightarrow \infty, f(x_k) \rightarrow \infty$

**Example 1** (Check coercive).

- 1)  $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2$  - coercive
- 2)  $x \in \mathbb{R}, f(x) = 1 - e^{-|x|}$  - not coercive
- 3)  $x \in \mathbb{R}^2, f(x) = x_1^2 + x_2^2 - 2x_1x_2$  - not coercive (需要所以  $\|x_k\| \rightarrow \infty$  都满足)

## 1.4 Sets

**Definition 8** (Open Sets). *A set  $\& \subseteq \mathbb{R}^n$  is open if*

*$\forall x \in \&$  we can draw a ball around  $x$  that is contained in  $\&$ .*

*i.e.  $\forall x \in \&, \exists \varepsilon > 0$  s.t.  $\{y : \|y - x\| < \varepsilon\} \subseteq \&$*

**Definition 9** (Closed Sets).  *$\&$  is closed if  $\&^c$  is open*

*Equivalent: if  $\&$  contains all limit points of all sequences in  $\&$*

**Example 2** (Closed and Open Sets).

- 1)  $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$  - open
- 2)  $\mathbb{R}$  is both open and closed
- 3)  $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$  - open
- 4)  $[1, \infty)$  is closed because its complement open
- 5)  $(1, 2]$  is neither open nor closed

**Definition 10** (Bounded Set). *A is bounded if  $\exists M$  s.t.  $\|x\| \leq M \quad \forall x \in \&$*

**Definition 11** (Compact Set).  *$\mathcal{L} \subseteq \mathbb{R}^n$  is compact if it is closed and bounded.*

**Example 3** (Compact Set).  $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}; \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

**Definition 12** (Extreme of sets of scalars,  $\sup A, \inf A$ ). *Let  $A \subset \mathbb{R}$ .*

- *The infimum of  $A$ , or  $\inf A$  is largest  $y$  s.t.  $y \leq x, \forall x \in A$ . If no such  $y$  exists,  $\inf A = -\infty$*
- *Similar definition for supremum of  $A$  (or wrote as  $\sup A$ ).*

**Proposition 4.** *If  $\inf A(\sup A) = x^* \in A$ , then  $x^* = \min A(\max A)$*

## 2 Functions

### 2.1 Extreme of Functions

**Definition 13** (Extreme of Functions). *Let  $\& \subseteq \mathbb{R}^n, f : \& \rightarrow \mathbb{R}$*

$$\inf_{x \in \&} f(x) = \inf\{f(x) : x \in \&\}$$

If  $\exists x^* \in \&$  s.t.  $\inf f(x) = f(x^*)$ . Then,  $f$  achieves (attains) its minimum and  $f(x^*) = \min_{x \in \&} f(x)$ .  $x^*$  is called a **minimizer** of  $f$ , written as  $x^* \in \arg \min_{x \in \&} f(x)$ . If  $x^*$  is unique, we write  $x^* = \arg \min_{x \in \&} f(x)$

Similarly, supremum and maximum of  $f$ .

### 2.1.1 Weierstrass' Theorem(Extreme value Theorem)

**Theorem 1** (Weierstrass' Theorem(Extreme value Theorem)).

If  $f$  is a **continuous** function on a **compact set**,  $\& \subseteq \mathbb{R}^n$ , then  $f$  attains its min and max on  $\&$  i.e.,

$$\exists x_1 \in \& \text{ s.t. } f(x_1) = \inf_{x \in \&} f(x)$$

$$\exists x_2 \in \& \text{ s.t. } f(x_2) = \sup_{x \in \&} f(x)$$

证明. (for existence of min; max is similar)

Let  $\{\sigma_k\} \subseteq \&$  be s.t.

$$\inf_{x \in \&} f(x) \leq f(\sigma_k) \leq \inf_{x \in \&} f(x) + \frac{1}{k}$$

Then  $\lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \&} f(x)$

$\mathcal{L}$  is bounded  $\Rightarrow \{\sigma_k\}$  has at least one limit point  $x$ ,

$\mathcal{L}$  is closed  $\Rightarrow x_1 \in \&$

$f$  is continuous  $\Rightarrow f(x_1) = \lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \&} f(x)$  □

**Corollary 1** (Corollary to WT). Let  $f$  be continuous on closed set  $\&$  (not necessarily bounded). If  $f$  is coercive on  $\&$  it attains its min on  $\&$ .

证明. Consider  $\{\sigma_k\}$  as in proof of WT.

Since  $f$  is closed,  $f(x) < \infty, \forall x \in \&$ . And  $f$  is coercive on  $\&$ , which means  $f(x) \rightarrow \infty$  if  $\|x\| \rightarrow \infty$ .

Hence,  $\{\sigma_k\} \in \&$  is bounded. Rest of proof same as proof of WT. □

**Example 4.**  $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) Does  $f$  achieve its min and max on  $\mathcal{L}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leq 6\}$ ?

-  $\mathcal{L}_1$  is compact and  $f$  is continuous. Both min and max are achieved (WT).

2) Does  $f$  achieve its min and max over  $\mathbb{R}^3$ ?

-  $f \rightarrow \infty$  whenever  $\|x\| \rightarrow \infty \Rightarrow f$  is coercive.

-  $\mathbb{R}^3$  is closed.

$\Rightarrow f$  achieves its min. on  $\mathbb{R}^3$  by corollary to WT.

- max does not exist since  $f \rightarrow \infty$  as  $\|x\| \rightarrow \infty$ .

- 3) Does  $f$  achieve its min and max over  $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$ ?
- $\mathcal{L}_2$  is closed, but not bounded.
  - Since  $f$  is coercive, min achieved.
  - max does not exist since setting  $x_1 = 0$   $x_2 = 3 - x_3$  and letting  $x_3 \rightarrow \infty$  makes  $f \rightarrow \infty$

### 3 Big $\mathcal{O}$ Notation

#### 3.1 Definition

For two scalar functions  $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$ , where  $x \in \mathbb{R}$ , we write:

1.  $f(x) = \mathcal{O}(g(x))$  if  $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} < \infty$ ; we say  $f$  is dominated by  $g$  asymptotically.
2.  $f(x) = \Omega(g(x))$  if  $\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0$ .
3.  $f(x) = \Theta(g(x))$  if  $f(x) = \mathcal{O}(g(x))$  and  $f(x) = \Omega(g(x))$  both hold.
4.  $f(x) = o(g(x))$  if  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$ .

**Example 5.**

$$\begin{aligned} n^3 + n + 2 &= \Omega(1), n^3 + n + 2 = \Omega(n^2) \\ n^3 + n + 2 &= \Theta(n^3) \\ n^3 + n + 2 &= o(n^4) \end{aligned}$$

##### 3.1.1 Extension

$f(x) = \mathcal{O}(g(x))$  as  $x \rightarrow a$  if  $\limsup_{x \rightarrow a} \frac{|f(x)|}{g(x)} < \infty$ .

**Example 6.**  $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$  as  $\varepsilon \rightarrow 0$