

# Sequential Assortment Optimization

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## 1 Rational Inattention Foundation for Multinomial Logit Model

### 1.1 References:

Matějka, F., & McKay, A. (2015). Rational inattention to discrete choices: A new foundation for the multinomial logit model. *American Economic Review*, 105(1), 272-98.

(Too complicated, the proof can be simplified to the format below. However, the proof cost function over purchase probability is omitted in this paper and I can't reproduce it.)

Ben Hebert Stanford Graduate School of Business: Rational Inattention Theory and Evidence (presentation at Israel Institute for Advanced Studies <https://www.youtube.com/watch?v=l4gx3KxHzo>).

Gabaix, X. (2019). Behavioral inattention. In *Handbook of Behavioral Economics: Applications and Foundations 1* (Vol. 2, pp. 261-343). North-Holland.

(Don't have full proof, change the cost function in the first paper to Kullback–Leibler distance (which is also deduced from information theory, I will use this setting))

*I also adjust some settings to make it close to assortment optimization settings (and can be easily understood).*

### 1.2 Model

A consumer makes an action on action set  $A = \{1, \dots, N\}$  (Buy which product)

$V \subset \mathbb{R}^N$  is the set of states of nature,  $\mathbf{v}$  is a state of nature in  $V$  (The reality: products' values to the consumer.)

The consumer makes an action  $i \in A$  at state  $\mathbf{v}$  and get utility  $v_i$ .

**If consumer knows  $\mathbf{v}$** , this problem is simply:

$$i^* \in \operatorname{argmax}_i v_i$$

However, we need to discuss the situation that the consumer doesn't know  $\mathbf{v}$ . She has to infer the  $\mathbf{v}$  from some information.

**When consumer doesn't know  $\mathbf{v}$** , she has belief  $b \in \Delta(V)$ . ( $\Delta(V)$  is the set of all possibility distributions on  $V$ ) Then,

$$i^*(b) \in \operatorname{argmax}_i \mathbb{E}_b[v_i]$$

(Choose the product with the highest expected value under belief  $b$ ) Set the map from beliefs to utility (not actual utility, the utility under belief) is

$$\pi(b) = \max_i \mathbb{E}_b[v_i]$$

Now, suppose the consumer can choose to pay attention to change the belief  $b_0$  (original belief) by getting additional information (signal)  $s \in S$  ( $S$  is the set of all possible signals that can be gained by paying more attention by the consumer.) Then, the consumer gets a posterior belief  $b_s \in \Delta(V)$  after observing  $s$ .

When the consumer pay attention to get more information in different state  $\mathbf{v}$ , the signal  $s$  is generated with different probability. The consumer can also get different information about  $\mathbf{v}$  under different signal  $s$ . (When the state is good, the signal is more likely to be good. Inversely, when the signal is good, the state is more likely to be good.)

The consumer uses an **information strategy** to deal with the information (represented by the joint distribution of  $s, \mathbf{v}$ ,  $f(s, \mathbf{v})$ ), which represents how the consumer transform information between signal  $s$  and  $\mathbf{v}$ .

To fulfill the consistency requirement, the original belief without any information

$$b_{0,\mathbf{v}} = f_V(\mathbf{v}) = \int_s f(s, \mathbf{v}) ds$$

is the marginal distribution over  $\mathbf{v}$  (i.e.,  $f_V(\mathbf{v})$ ). (It is a restriction to  $f(s, \mathbf{v})$ )

The updated posterior belief

$$b_{s,\mathbf{v}} = f(\mathbf{v}|s) = \frac{f(s, \mathbf{v})}{f_S(s)}$$

The optimization problem for the consumer is

$$\max_f \int_s \pi(b_s) f_S(s) ds - \hat{c}(f)$$

$\hat{c}(f)$  is the cost of paying attention, which will be discussed later.

For strategy  $f$ , the set of signals that lead to purchasing product  $i$  is  $S_i = \{s : i^*(b_s) = i\}$

Probability of selecting  $i$  condition on state  $\mathbf{v}$

$$P_i(\mathbf{v}) = \int_{s \in S_i} f(s|\mathbf{v}) ds$$

The unconditional probability of selecting  $i$

$$P_i^0 = \int_{\mathbf{v}} P_i(\mathbf{v}) f_V(\mathbf{v}) d\mathbf{v} = \int_{\mathbf{v}} \int_{s \in S_i} f(s|\mathbf{v}) ds f_V(\mathbf{v}) d\mathbf{v} = \int_{\mathbf{v}} \int_{s \in S_i} f(s, \mathbf{v}) ds d\mathbf{v}$$

Then, we can infer that

$$\begin{aligned} \int_s \pi(b_s) f_S(s) ds &= \sum_{i=1}^N \int_{s \in S_i} \mathbb{E}_{b_s}[v_i] f_S(s) ds \\ &= \sum_{i=1}^N \int_{s \in S_i} \int_{\mathbf{v}} f(\mathbf{v}|s) v_i d\mathbf{v} f_S(s) ds \\ &= \sum_{i=1}^N \int_{s \in S_i} \int_{\mathbf{v}} f(s, \mathbf{v}) v_i d\mathbf{v} ds \\ &= \sum_{i=1}^N \int_{s \in S_i} \int_{\mathbf{v}} f(s|\mathbf{v}) f_V(\mathbf{v}) v_i d\mathbf{v} ds \\ &= \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) f_V(\mathbf{v}) v_i d\mathbf{v} \end{aligned}$$

Now the optimization problem can be transformed to selecting optimal  $\{P_i(\mathbf{v})\}_{\mathbf{v} \in V}$ .

Now let's consider the cost function of attention. We use the entropy based cost function to measure the effect of the **information strategy**  $\{f(s, \mathbf{v})\}_{s \in S, \mathbf{v} \in V}$  (Now replaced by  $\{P_i(\mathbf{v})\}_{\mathbf{v} \in V}$ ).

Actually  $P_i(\mathbf{v})$  is a strategy production (consumer infer expected weights of  $\mathbf{v}$  from  $s$ , which will generate final outcome  $P_i(\mathbf{v})$ ), on the contrary, we can infer weight  $\mathbf{v}$  from  $\{P_i(\mathbf{v})\}_{\mathbf{v} \in V}$

If  $P_i(\mathbf{v}) = P_i^0, \forall \mathbf{v} \in V$ , signal (information)  $s$  has no effect on final outcome (i.e. the consumer don't pay any attention, she makes decisions based on prior beliefs). If  $P_i(\mathbf{v})$  differs from  $P_i^0$ , signal  $s$  will have stronger effect on final outcome (i.e. the consumer pay some attention to process information to adjust decisions).

Hence, We measure the cost from paying attention by how the  $P_i(\mathbf{v})$  deviates from  $P_i^0$ . If the  $P_i(\mathbf{v})$  deviation is larger, the **information strategy** is more effective, which may cost more attention (i.e. higher cost). We use an entropy based Kullback-Leibler distance,

$$\hat{c}(f) \equiv \lambda \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) \log \left( \frac{P_i(\mathbf{v})}{P_i^0} \right) f_V(\mathbf{v}) d\mathbf{v}$$

$\lambda$  is the unit cost of paying attention.

Then, the optimization problem is

$$\begin{aligned} \max_{\{P_i(\mathbf{v})\}_{\mathbf{v} \in V, i=1, \dots, N}} \quad & \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) f_V(\mathbf{v}) v_i d\mathbf{v} - \lambda \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) \log \left( \frac{P_i(\mathbf{v})}{P_i^0} \right) f_V(\mathbf{v}) d\mathbf{v} \\ \text{s.t.} \quad & \sum_{i=1}^N P_i(\mathbf{v}) = 1, \forall \mathbf{v} \in V \end{aligned}$$

Lagrangian:

$$L = \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) f_V(\mathbf{v}) v_i d\mathbf{v} - \lambda \sum_{i=1}^N \int_{\mathbf{v}} P_i(\mathbf{v}) \log \left( \frac{P_i(\mathbf{v})}{P_i^0} \right) f_V(\mathbf{v}) d\mathbf{v} + \int_{\mathbf{v}} \mu(\mathbf{v}) \left( \sum_{i=1}^N P_i(\mathbf{v}) - 1 \right) f_V(\mathbf{v}) d\mathbf{v}$$

With respect to  $P_i(\mathbf{v})$ :

$$\begin{aligned} f_V(\mathbf{v}) v_i - \lambda \left( 1 + \log \left( \frac{P_i(\mathbf{v})}{P_i^0} \right) \right) f_V(\mathbf{v}) + \mu(\mathbf{v}) f_V(\mathbf{v}) &= 0 \\ \Rightarrow P_i(\mathbf{v}) &= P_i^0 e^{\frac{v_i + \mu(\mathbf{v}) - \lambda}{\lambda}} = \frac{P_i^0 e^{\frac{v_i}{\lambda}}}{\sum_{j=1}^N P_j^0 e^{\frac{v_j}{\lambda}}} \end{aligned}$$

This is the foundation of MNL model in microeconomics. This is a reasonable model in one stage problem. Will it still be reasonable in multi-stage problem? If yes, I want to show it. If not, I want to derive a new model based on this foundation.

## 2 Multi-stage Choice Model

Xu, Y., & Wang, Z. (2018). Assortment optimization for a multi-stage choice model. Available at SSRN 3243742.

Use MNL model to analyze multi-stage choice model, a consumer makes current decision with considering future impact. (consumer purchase at most one in each stage)

Gao, P., Ma, Y., Chen, N., Gallego, G., Li, A., Rusmevichientong, P., & Topaloglu, H. (2021). Assortment optimization and pricing under the multinomial logit model with impatient customers: Sequential recommendation and selection. *Operations research*, 69(5), 1509-1532.

Dividing products into  $n$  partitions, consumer will purchase once exists a product better than non-purchase (the consumer won't consider the future products). Also use MNL model.

### 3 Model

Set of all products  $A = \{1, \dots, N\}$ . There are at most  $T$  stages, we use  $\{A_t\}_{t=1}^T$  to denote the sequence of assortments over  $T$  stages. Each stage has  $n$  products ( $nT = N$ ). Assortment in different stages are disjoint,  $A_k \cap A_l = \emptyset, \forall k \neq l$ .

We assume a product's profits to the platform doesn't have relation to its values to consumers.

State of nature at stage  $t$  is  $\mathbf{v}^t = \{v_0^t, v_1^t, v_2^t, \dots, v_n^t, v_{n+1}^t\} \in \bar{V}$ , where  $\bar{V} = \{v_0\} \cup V \cup \mathbb{R}$ , where  $v_i^t, i = 1, \dots, m$  denote the payoffs of product  $i$  in  $S_t$ ,  $v_0^t \equiv v_0$  is the value of leaving the platform,  $v_{n+1}^t$  is the expected value of entering next stage.

The setting of beliefs and information strategy is similar to former settings.  $b_{\cdot, \mathbf{v}}^t$  is the belief at stage  $t$ .  $f^t$  is the information strategy at stage  $t$ .

$\pi(b) = \max_i \mathbb{E}_b[v_i]$ .  $i^*(b) \in \operatorname{argmax}_i \mathbb{E}_b[v_i]$ .

$$b_{0, \mathbf{v}}^t = f_V^t(\mathbf{v}) = \int_s f^t(s, \mathbf{v}) ds$$

The updated posterior belief

$$b_{s, \mathbf{v}}^t = f^t(\mathbf{v}|s) = \frac{f^t(s, \mathbf{v})}{f_S^t(s)}$$

The optimization problem for the consumer is

$$\Pi_t = \max_{f^t} \int_s \pi(b_s^t) f_S^t(s) ds - \hat{c}(f^t)$$

$\hat{c}(f^t)$  is the cost of paying attention.

Considering the discount factor  $\gamma \in [0, 1]$ ,  $v_{n+1}^t = \gamma \Pi_{t+1}$ .

For strategy  $f^t$ , the set of signals that lead to purchasing product  $i$  is  $S_i^t = \{s : i^*(b_s^t) = i\}$

Probability of selecting  $i$  condition on state  $\mathbf{v}$

$$P_i^t(\mathbf{v}) = \int_{s \in S_i^t} f^t(s|\mathbf{v}) ds$$

The unconditional probability of selecting  $i$

$$(P_i^t)^0 = \int_{\mathbf{v}} P_i^t(\mathbf{v}) f_V^t(\mathbf{v}) d\mathbf{v} = \int_{\mathbf{v}} \int_{s \in S_i^t} f^t(s|\mathbf{v}) ds f_V^t(\mathbf{v}) d\mathbf{v} = \int_{\mathbf{v}} \int_{s \in S_i^t} f^t(s, \mathbf{v}) ds d\mathbf{v}$$

Then, we can infer that

$$\begin{aligned} \int_s \pi(b_s^t) f_S^t(s) ds &= \sum_{i=0}^{n+1} \int_{s \in S_i^t} \mathbb{E}_{b_s^t}[v_i] f_S^t(s) ds \\ &= \sum_{i=0}^{n+1} \int_{s \in S_i^t} \int_{\mathbf{v}} f^t(\mathbf{v}|s) v_i d\mathbf{v} f_S^t(s) ds \\ &= \sum_{i=0}^{n+1} \int_{s \in S_i^t} \int_{\mathbf{v}} f^t(s|\mathbf{v}) f_V^t(\mathbf{v}) v_i d\mathbf{v} ds \\ &= \sum_{i=0}^{n+1} \int_{\mathbf{v}} P_i^t(\mathbf{v}) f_V^t(\mathbf{v}) v_i d\mathbf{v} \\ &= \sum_{i=1}^n \int_{\mathbf{v}} P_i^t(\mathbf{v}) f_V^t(\mathbf{v}) v_i d\mathbf{v} + (P_0^t)^0 v_0 + (P_{n+1}^t)^0 \gamma \Pi_{t+1} \end{aligned}$$

We still use the entropy based Kullback-Leibler distance to measure the cost,

$$\hat{c}(f^t) \equiv \lambda \sum_{i=0}^{n+1} \int_{\mathbf{v}} P_i^t(\mathbf{v}) \log \left( \frac{P_i^t(\mathbf{v})}{(P_i^t)^0} \right) f_V^t(\mathbf{v}) d\mathbf{v}$$

Then, the optimization problem is

$$\begin{aligned} \Pi_t = & \max_{\{P_i^t(\mathbf{v})\}_{\mathbf{v} \in \bar{V}, i=1, \dots, n+1}} \sum_{i=0}^{n+1} \int_{\mathbf{v}} P_i^t(\mathbf{v}) f_V^t(\mathbf{v}) v_i d\mathbf{v} - \lambda \sum_{i=0}^{n+1} \int_{\mathbf{v}} P_i^t(\mathbf{v}) \log \left( \frac{P_i^t(\mathbf{v})}{(P_i^t)^0} \right) f_V^t(\mathbf{v}) d\mathbf{v} \\ \text{s.t. } & \sum_{i=0}^{n+1} P_i^t(\mathbf{v}) = 1, \forall \mathbf{v} \in \bar{V} \end{aligned}$$

By using Lagrangian, we can solve that  
(with respect to  $P_i^t(\mathbf{v})$ )

$$\begin{aligned} f_V^t(\mathbf{v}) v_i - \lambda (1 + \log \left( \frac{P_i^t(\mathbf{v})}{(P_i^t)^0} \right) f_V^t(\mathbf{v}) + \mu(\mathbf{v}) f_V^t(\mathbf{v})) &= 0 \\ \Rightarrow P_i^t(\mathbf{v}) = (P_i^t)^0 e^{\frac{v_i + \mu(\mathbf{v}) - \lambda}{\lambda}} &= \frac{(P_i^t)^0 e^{\frac{v_i}{\lambda}}}{\sum_{j=0}^{n+1} (P_j^t)^0 e^{\frac{v_j}{\lambda}}} \end{aligned}$$

Since  $v_{n+1} = \gamma \Pi_{t+1}$

$$P_i^t(\mathbf{v}) = \frac{(P_i^t)^0 e^{\frac{v_i}{\lambda}}}{(P_0^t)^0 e^{\frac{v_0}{\lambda}} + \sum_{j=1}^n (P_j^t)^0 e^{\frac{v_j}{\lambda}} + (P_{n+1}^t)^0 e^{\frac{\gamma \Pi_{t+1}}{\lambda}}}$$

It is reasonable to set the prior beliefs of purchasing product  $i = 1, \dots, n$  are the same, i.e.,  $(P_i^t)^0 = \frac{1 - (P_0^t)^0 - (P_{n+1}^t)^0}{n}$ ,  $\forall i = 1, \dots, n$ .  $(P_{n+1}^t)^0$  is the prior belief of entering next stage, which is the expected probability that  $\gamma \Pi_{t+1} > \max\{v_0, v_1, \dots, v_n\}$ ,  $(P_0^t)^0$  is the probability that  $v_0 \geq \max\{\gamma \Pi_{t+1}, v_1, \dots, v_n\}$ . Here we assume when leave the platform and enter the next stage is indifferent for the consumer, i.e.,  $v_0 = \gamma \Pi_{t+1}$ , the consumer prefer to leave the platform. (risk-aversion)

To simplify the form, we set  $q^t$ , where  $q^t = \max\{v_1^t, \dots, v_n^t\}$  follows a distribution  $Q^t$ ,  $Q^t(x) = \int_{\mathbf{v} \in \{\mathbf{v}: \max \mathbf{v} \leq x\}} f_V^t(\mathbf{v}) d\mathbf{v}$

We use  $p_0^t$  to denote  $(P_0^t)^0$ ,  $p_i^t = p^t$  to denote  $(P_i^t)^0$ ,  $i = 1, \dots, n$  and  $p_{n+1}^t = p_*^t$  to denote  $(P_{n+1}^t)^0$  in the following analysis.

$$\begin{aligned} P_i^t(\mathbf{v}) &= \frac{p_i^t e^{\frac{v_i}{\lambda}}}{p_0^t e^{\frac{v_0}{\lambda}} + p^t \sum_{j=1}^n e^{\frac{v_j}{\lambda}} + p_*^t e^{\frac{\gamma \Pi_{t+1}}{\lambda}}}, \quad i = 0, 1, \dots, n+1 \\ p_*^t &= \text{Prob}(\gamma \Pi_{t+1} > v_0 \text{ and } q^t < \gamma \Pi_{t+1}) \\ p_0^t &= \text{Prob}(v_0 \geq \gamma \Pi_{t+1} \text{ and } q^t < v_0) \end{aligned}$$

**Lemma 1.**  $p_0^t$  and  $p_*^t$  can't be nonzero at the same time.

**For**  $\{t : \gamma \Pi_{t+1} > v_0\}$

The prior purchase probability is

$$p_0^t = 0, \quad p^t = \frac{1 - Q^t(\gamma \Pi_{t+1})}{n}, \quad p_*^t = Q^t(\gamma \Pi_{t+1})$$

Then the purchase probability is

$$\begin{aligned}
P_0^t(\mathbf{v}) &= 0 \\
P_i^t(\mathbf{v}) &= \frac{(1 - Q^t(\gamma\Pi_{t+1}))e^{\frac{v_i}{\lambda}}}{(1 - Q^t(\gamma\Pi_{t+1})) \sum_{j=1}^n e^{\frac{v_j}{\lambda}} + nQ^t(\gamma\Pi_{t+1})e^{\frac{\gamma\Pi_{t+1}}{\lambda}}}, \quad i = 1, \dots, n \\
P_{n+1}^t(\mathbf{v}) &= \frac{nQ^t(\gamma\Pi_{t+1})e^{\frac{\gamma\Pi_{t+1}}{\lambda}}}{(1 - Q^t(\gamma\Pi_{t+1})) \sum_{j=1}^n e^{\frac{v_j}{\lambda}} + nQ^t(\gamma\Pi_{t+1})e^{\frac{\gamma\Pi_{t+1}}{\lambda}}}
\end{aligned}$$

**For  $\{t : \gamma\Pi_{t+1} \leq v_0\}$ : Stop at this stage**

The prior purchase probability is

$$p_0^t = Q^t(v_0), \quad p^t = \frac{1 - Q^t(v_0)}{n}, \quad p_*^t = 0$$

Then the purchase probability is

$$\begin{aligned}
P_0^t(\mathbf{v}) &= \frac{nQ^t(v_0)e^{\frac{v_0}{\lambda}}}{(1 - Q^t(v_0)) \sum_{j=1}^n e^{\frac{v_j}{\lambda}} + nQ^t(v_0)e^{\frac{v_0}{\lambda}}} \\
P_i^t(\mathbf{v}) &= \frac{(1 - Q^t(v_0))e^{\frac{v_i}{\lambda}}}{(1 - Q^t(v_0)) \sum_{j=1}^n e^{\frac{v_j}{\lambda}} + nQ^t(v_0)e^{\frac{v_0}{\lambda}}}, \quad i = 1, \dots, n \\
P_{n+1}^t(\mathbf{v}) &= 0
\end{aligned}$$