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A Theory of Monopoly of Complementary Goods

The theory of monopoly of complementary goods raises two important issues. First, when can a monopolist increase his net return by selling at least one of a set of complementary commodities at a price below its marginal cost? This paper shows that this practice is consistent with a maximum monopoly net return and rational customers who calculate correctly the cost of the combination of complementary commodities. Second, it is shown that a monopolist can increase his net return by forming groups of complementary commodities and selling these groups to different types of customers. It is a maintained hypothesis that the total cost of a set of complements is a linear function of the quantities. Hence, the theory relies on the nature of the demand conditions to explain these practices. The theory applies to tie-in sales, block booking, and resale price maintenance.

I. Introduction

Tie-in sales of complementary commodities are a pervasive business practice often subject to different interpretations. It is sometimes easy to explain the presence of tie-ins because the cost of the combination is lower than the sum of the costs of the components sold separately. No doubt this explains why new automobiles come equipped with four wheels instead of having wheels sold separately from the car. There is a theoretical possibility that tie-ins are used by monopolists in order to increase their monopoly return even though the cost of the combination is the same as the sum of the costs of the components. The purpose of this work is to furnish a formal analysis of tie-in sales by a monopolist in order to see if this is so. In addition we shall study under what conditions if any a monopolist who sells the component commodities separately might sell one or more of them at prices below their marginal costs. Thus suppose a vendor of peanuts and beer has an exclusive concession in a sports arena. Would profit maximization lead him to sell the peanuts at a price below their marginal cost and the beer at a price above their marginal cost? Allen (1949, sec. 14.4) studies this

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class of problems without exploiting all of the relevant economic constraints. As an example of tie-in sales there is the well-known practice of IBM who required the rentor of tabulating equipment to buy all of their punch cards from IBM. Professor Aaron Director, in his classes, explains this practice as a form of price discrimination. The usage of punch cards could measure differences among rentors of tabulating equipment that would enable IBM to obtain a larger net return than by the rental of the tabulating equipment at a single price to all of its customers.[1]

II. Formal Theory

Assume there are m commodities such that a customer of type i combines them to form a good using proportions given by the m-vector a_i where $a_i = < a_{i1}, \ldots, a_{im} >$. The components are nonnegative and types of customers differ with respect to the magnitude of these proportions.

Let p_j denote the unit price of commodity j. The cost of one unit of the good i to a customer of type i satisfies the following equation:

$$P_i = a_{i1}p_1 + a_{i2}p_2 + \ldots + a_{im}p_m. \tag{1}$$

Let Q_i denote the number of units of the good i that is bought by type i customers. The demand for good i has a linear approximation given as follows:

$$Q_i = b_{oi} - b_{1i}P_i,\ b_{oi},\ b_{1i} > 0. \tag{2}$$

If the monopolist could sell good i to the type i customers by assembling the components in the proportions given by the vector a_i, then (2) would represent the demand function for good i.

Let q_j denote the number of units of commodity j. Then

$$q_j = \Sigma_i a_{ji} Q_i, \qquad j = 1, \ldots, m, \tag{3}$$

shows the relation between the number of units of the commodity j and the number of units of the n goods, Q_i. To display the demand for the commodities q_j that is implied by the demand for the goods it is

1. Ferguson (1965) gives an excellent survey of some of the problems studied here. Adams and Yellen (1976) treat a special case of monopoly of two complementary commodities and attempt to derive welfare implications. Burstein (1960) seems to believe that a monopoly of one commodity can obtain a higher net return by tying to another even if the second is unrelated to the first. This is implausible. For instance, assume a firm has a monopoly in flashlight batteries, the bread market is competitive, and that consumers regard bread and flashlight batteries as unrelated. How can the firm with the flashlight battery monopoly raise its return by forcing all of its customers to buy all of their bread from it along with flashlight batteries? It would seem more likely that the tie-in would lower rather than raise the monopoly return under these circumstances.

convenient to use a more concise notation. Let $A = [a_{ij}]$ denote an $n \times m$ matrix. The price relations (1) become

$$P = A\,p, \tag{4}$$

and the quantity relations (3) become

$$q = A^T Q \qquad (A^T \text{ is the transpose of } A). \tag{5}$$

The demand relations (2) are given by

$$Q = b - B\,P, \tag{6}$$

where b is an n-vector whose ith element is b_{oi} and B is a diagonal matrix whose ith diagonal entry is b_{1i}. The demand for the m commodities are as follows:

$$q = A^T b - A^T BA\,p. \tag{7}$$

Observe that the matrix of price coefficients, $A^T BA$, is positive definite and that all of its elements are nonnegative. Therefore, the assumptions give the conclusion that the price coefficients satisfy a number of restrictions.

Assume that the cost of producing and selling the m commodities is linear. This assumption has several advantages. It isolates the incentive for tie-in sales to those coming from demand. If the cost were nonlinear then it would be easy to show that the sale of combinations of commodities would be advantageous to both the producers and their customers if it could lower total cost for any given scale of operations. We wish to rule out these incentives for tie-ins so we postulate that: total cost $= \Sigma_j c_j q_j$. The average cost of q_j and the marginal cost of q_j are both equal to c_j. There are no fixed costs.

At this point some observations are helpful. The slopes of the demand for the goods Q_i, b_{1i}, may differ. Therefore, there may be relations between these slopes and customer types that are capable of being revealed by the commodity proportions. This suggests that there may be a larger maximum profit if the monopolist can sell the n goods Q_i separately to each type of customer at a price P_i than if he must sell the m components separately to all types of customers at a common unit price p_j for commodity j. The analysis does not distinguish between the cases where a type of customer may buy all of the commodities in his desired proportions at the same time or at different times. For instance, it makes no difference whether a customer first buys peanuts and then later buys beer or buys his peanuts and beer at the same time. Finally, we shall assume that there are more types of customers than there are component commodities so that $n > m$. The monopolist may try to furnish enough *different* commodities to make $m = n$ in order to discriminate among customer types by the addition of enough distinguishing components. If this were possible then there would be a complete correspondence between the customer type and the commodities, and the problem would be of little interest. The interesting problem arises for $n > m$. We shall assume that the rank of the $n \times m$ matrix A is m.

Let c denote the m-vector giving the unit costs of the m commodities q_j. Let C denote the n-vector giving the unit costs of the n goods, Q_i.

$$C = A\,c \tag{8}$$

relates the two. Define $Y = P - C$, which is the n-vector giving the unit mark-ups for the n goods and let $y = p - c$,[2] the m-vector giving the unit mark-ups for the m commodities. Then

$$Y = A\,y. \tag{9}$$

Let ρ denote the profit from the sale of the n goods and π the profit from the sale of the m commodities. We have

$$\pi = y^T q \quad \text{and} \quad \rho = Y^T Q. \tag{10}$$

$$Q_c = b - BC \quad \text{and} \quad q_c = A^T b - A^T BA\,c \tag{11}$$

give the quantities demanded at prices equal to unit costs. Thus, Q_c and q_c give the quantities that would be demanded in perfect competition. With the aid of (11), we may write the demand relations (6) and (7) as follows:

$$Q = Q_c - B\,Y \quad \text{and} \quad q = q_c - A^T BA\,y. \tag{12}$$

Consequently, in place of (10) we have

$$\rho = Y^T(Q_c - B\,Y) \quad \text{and} \quad \pi = y^T(q_c - A^T BA\,y). \tag{13}$$

Neither Y nor y are subject to nonnegativity constraints. Therefore, the necessary conditions for maximum profits satisfy

$$Q_c - 2\,B\,Y = 0 \quad \text{and} \quad q_c - 2A^T BA\,y = 0. \tag{14}$$

Since B is positive definite and A is of rank m, these equations have solutions, and the sufficient conditions for the existence of a maximum are satisfied.

Assume that $Q_c > 0$ so that positive quantities of the n goods are demanded at prices equal to marginal cost. It is immediate from (14) that

$$Y = \left(\frac{1}{2}\right) B^{-1} Q_c > 0. \tag{15}$$

2. Let x and y denote n-vectors so that

$$x = < x_1, \ldots, x_n >$$

The scalar product is $\Sigma\, x_i\, y_i$. In all sections except for section IV, the notation used for the scalar product is $x^T y$. In Section IV, because it is necessary to have subscripts the notation used is $x \cdot y$. Both are standard notations and should cause the reader no difficulty. It would be awkward to change the notation in Section IV to conform to the rest of the paper. I prefer the notation $x^T y$ where this is convenient.

Therefore, the profit maximizing mark-ups Y are strictly positive and none of the n goods would be sold at a price below their marginal cost. However,

$$A\,y = Y \qquad \text{and} \qquad y = \left(\frac{1}{2}\right)(A^TBA)^{-1}q_c. \tag{16}$$

It does not follow from (16) that $Y > 0$ implies $y > 0$. Since $A^TQ_c = q_c$, we may conclude that $Q_c > 0$ implies $q_c > 0$ by virtue of the hypothesis $A > 0$.

It may not be possible for the monopolist to sell the n goods Q_i to each of the n types of customers. Instead he may find it necessary to sell the m commodities q_j at profit maximizing mark-ups y_j that satisfy (16). Now $q_c > 0$ and $y^Tq_c = 2y^T(A^TBA)y > 0$ do imply that at least one element of the optimal y which satisfies (16), must be positive. More than this cannot be established in general. It may well be that profit maximization with respect to the choice of mark-ups for the m commodities dictates the sale of one or more of them at prices below their marginal costs.

Let ρ^* and π^* denote the respective maxima. Then

$$\rho^* = Y^TBY \qquad \text{and} \qquad \pi^* = y^TA^TBA\,y. \tag{17}$$

We wish to see whether a monopolist can obtain a bigger maximal profit by selling the n goods at the optimal mark-ups Y than he can obtain by selling the m commodities q_j at the optimal mark-ups y_j. Concisely, given $n > m$, is it true that $\rho^* > \pi^*$?

We begin our study of how to answer this question with the observation that there is no $1 \leftrightarrow 1$ relation between the n goods Q_i and the m commodities q_j because

$$A^TQ = A^T(Q + u) = q \qquad \text{if } A^Tu = 0. \tag{18}$$

Now since A has rank m there are indeed nonzero u that can satisfy $A^Tu = 0$. Therefore, there do exist different vectors Q and $Q + u$ that correspond to the same q. This suggests the possibility of finding two different vectors of the n goods Q_i which correspond to the same q but which give two different values of the maximum profit. Hence one of the values would be larger than the other although both would give the same set of optimal q_j.

Reconsider the maximum problem with respect to the choices of the optimal Y and y. The givens of the maximum problems are the demand parameters, b and B together with the cost vector c and the $n \times m$ matrix A. The optimal Y satisfies (15) and the optimal y satisfies (16). Both are uniquely determined by Q_c and q_c, respectively. However, two distinct values of Q_c which differ by the addition of a u such that $A^Tu = 0$ would give rise to the same y and different Y's. Therefore, there would be two different values of the maximum profits with respect to the choice of Y but only a single value of the maximum profit with respect to the choice of y. Nor is this all. A monopoly with respect to the choice of the optimal mark-ups for the m commodities cannot attain the higher maximum profit that is available by a choice of the optimal mark-ups for the n goods. This completes the proof of the following:

Theorem 1. Given q_c there exist two distinct values of the n vectors $Q_c, Q_c + u$ with $u \neq 0$ and $A^T u = 0$ such that the maximal profit for Q_c differs from the maximal profit for $Q_c + u$ while the maximal profit for q_c is the same since $A^T Q_c = A^T(Q_c + u) = q_c$.

This result means that a monopoly of the n goods cannot yield a maximum return that is smaller than the maximum monopoly return that is attainable by a monopoly of the m commodities q_j.

We obtain additional insight into the relation between the two problems with the help of some geometric considerations. We begin with some definitions.

Definition. $N(A^T) = \{u : A^T u = 0\}$ so that $N(A^T)$ is the null space of A^T.

Since A is of rank m, $N(A^T)$ is of dimension $n - m > 0$. Let $N(A^T)^\perp$ denote the orthogonal complement of $N(A^T)$. Thus, $N(A^T)^\perp = \{Z: Z^T u = 0 \text{ for some } u \in N(A^T)\}$. The dimension of $N(A^T)^\perp$ is m. Any vector $Q_c \in R^n$ has a unique representation

$$Q_c = Z + u: \quad u \in N(A^T) \qquad \text{and} \qquad Z \in N(A^T)^\perp. \tag{19}$$

Although $Q_c > 0$ by hypothesis, it does not necessarily follow that $Z > 0$. Observe that since $Z^T Q_c = Z^T Z > 0$, at least one coordinate of Z must be positive. Also,

$$A^T Z = A^T Q_c = q_c.$$
$$\rho_o = Y_o^T B\, Y_o \qquad \text{and} \qquad \rho_1 = Y_1^T \mathrm{B}\, \mathrm{Y}_1.$$
$$2B\, Y_o = Z \qquad \text{and} \qquad 2B\, Y_1 = Q_c = Z + u.$$

Hence Y_o is optimal for Z and Y_1 is optimal for Q_c. Also,

$$\pi_o = y_o^T A^T BA\, y_o \qquad \text{and} \qquad \pi_1 = y_1^T A^T BA\, y_1$$

where

$$2(A^T BA)\, y_o = q_{c_o} \qquad \text{and} \qquad 2(A^T BA)\, y_1 = q_{c_1},$$

where

$$q_{c_o} = A^T Z \qquad \text{and} \qquad q_{c_1} = A^T Q_c = A^T(Z + u) = q_{c_o}.$$

Therefore, $q_{c_o} = q_{c_1} = q_c$ and $y_o = y_1$. Consequently, $\rho_o = \pi_o = \pi_1$. Observe that since $Q_c > 0$ and $A^T > 0$, there is the implication $q_c > 0$. From the necessary conditions for the optimal Y we may conclude that $Y_1 = Y_o + \left(\frac{1}{2}\right) B^{-1} u$. Consequently,

$$\rho_1 = \left[Y_o + \left(\frac{1}{2}\right)B^{-1}u\right]^T B\left[Y_o + \left(\frac{1}{2}\right)B^{-1}u\right].$$

$$= \rho_o + Y_o^T u + \left(\frac{1}{4}\right) u^T B^{-1} u.$$

But $$Y_o = Ay_o ==> Y_o^T u = y_o^T A^T u = 0.$$

Therefore, $$\rho_1 - \rho_o = \left(\frac{1}{4}\right) u^T B^{-1} u > 0.$$

This completes the proof of:

Theorem 2. Given an arbitrary $Q_c > 0$, there is an optimal vector Y_1 such that $\rho^* = \rho_1 > \rho_o = \pi_o = \pi^*$.

This argument has an interesting economic interpretation. Consider the demand relations for the n goods with $P = C$ so that $Q_c = b - B C$. Since $Q_c = Z + u$, we may also write

$$Z = b - u - B C. \tag{20}$$

Thus, u represents a shift of demand among the n goods such that the quantities demanded of the m commodities at $p = c$ remain the same. Therefore, the optimal mark-ups for the m commodities do not change as a result of this shift u while the optimal mark-ups for the n goods would change.

This can be seen in another way. Suppose that the optimal Y is given so that $Y = A y$. We wish to solve this set of equations for y. We have n equations in m unknowns. To find the y that satisfies this system given Y, we may proceed as follows:

$$A^T Y = A^T A y ==> y = (A^T A)^{-1} A^T Y. \tag{21}$$

But the y satisfying (21) is the same for all $Y + v$ such that $A^T v = 0$. This means that the choice of y cannot reach optimal mark-ups Y that differ by the addition of v if $A^T v = 0$. There are not enough degrees of freedom in the choice of y for this purpose.

Reconsider the demand equations for the n goods, Q_i. We have

$$Q = Q_c - BY. \tag{22}$$

Since the profit-maximizing $Y > 0$, it follows from (22) that the quantities demanded of the n goods at the profit maximizing mark-ups are smaller than the quantities that would be demanded at $P = C$. Therefore, if we interpret the sale of the n goods as equivalent to price discrimination, this result means that the price discrimination results in the production and sale of quantities of the n goods that are less than would be bought at prices equal to their marginal costs.

We now return to the study of the signs of the optimal mark-ups of the m commodities. Observe that since $P = Ap$, the price per unit paid by a customer of type i is a linear combination of the prices of the m

commodities forming one unit of Q_i. Also, $Y = Ay$ and $Q_c > 0$ implies $Y > 0$. Therefore the optimal mark-ups for all of the n goods must be positive. There is an assumption of customer rationality by virtue of the relations between P and p. Consequently, even if the customers can obtain some of the m component commodities at prices below their marginal costs, no customer of type i pays a price P_i below C_i, which is the marginal cost of the good he buys.

The matrix $A^TBA \geqslant 0$, has rank m, and is positive definite. In studying the signs of the optimal mark-ups for the m commodities, we may assume without loss of generality that in addition A^TBA is indecomposable. Formally:

Definition. A square $m \times m$ matrix M is said to be indecomposable if there is no permutation matrix V such that

$$V\,M\,V^{-1} = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

where M_{11} and M_{22} are square matrices. (A permutation matrix is one having exactly one nonzero element in each row and column which equals 1. Hence it is a permutation of rows and columns of the identity matrix.) (Debreu and Herstein, 1953, p. 598).

In the present application the matrix M is symmetric. Hence if M were decomposable, the off-diagonal matrix $M_{12} = 0$ and M would be block diagonal. The economic interpretation of decomposability is that there would be a subset of commodities that go into a subset of the goods and the remaining commodities go into the remaining goods. Thus, there would be no overlap of commodities between the two subsets of goods. That is, there would be no commodity that would be an input into *all* of the goods. Consequently some commodities would be specific to some of the goods and the remaining commodities specific to the remaining goods. However, within a subclass of goods there would be indecomposability. This is to say that the matrices M_{ii} would be indecomposable. Some zero elements of A^TBA can be present even with indecomposability. The assumption that A^TBA is of rank m implies that all of the diagonal elements of A^TBA are positive.

A simple example illustrates these assertions. Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix},\ A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix},\ A^TA = \begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$$

if the only positive elements of A are a_{11}, a_{31}, and a_{22}. This would mean that commodity 2 does not appear in goods 1 and 3 and commodity 1 does not appear in good 2. Hence commodity 2 is specific to good 2 and commodity 1 is specific to goods 1 and 3.

The i, j element of A^TBA is given as follows:

$$d_{ij} = \Sigma_{k=1}^{m} a_{ki}\, a_{kj}\, b_{1k}\,, \qquad i, j = 1, \ldots m. \tag{23}$$

We see that $d_{ij} = 0 <==>$ for all k, $a_{ki}\, a_{kj} = 0$. This can happen even if A^TBA is indecomposable. Since A is of rank m, $d_{ii} > 0$.

We may conclude that there is no loss of generality from the assumption that A^TBA is indecomposable. We use this assumption in the next theorem.

Theorem 3.—Let $A^TBA \geqslant 0$ be indecomposable. Then the maximal eigen value of A^TBA is simple and the corresponding eigen vector is strictly positive.

Proof.—This is Debreu and Herstein's theorem 1 (1953).

Call r_o the maximal eigen value of A^TBA and x_o the corresponding eigen vector so that

$$A^TBA\, x_o = r_o\, x_o. \tag{24}$$

The theorem asserts that $x_o > 0$ and that r_o is a simple root of the algebraic equation det $(A^TBA - r\, I) = 0$.

Corollary. Under the hypotheses of the theorem, A^TBA must have at least two distinct eigen values.

Proof. Suppose not. Then if there were not at least two distinct eigen values, the maximal eigen value, r_0, could not be simple. Q.E.D.

Theorem 4. Max $x^T A^TBA\, x$ with respect to x: $x^Tx = 1$ has the value r_o.

Proof. This is a standard proposition of matrix algebra. Q.E.D.

Corollary. The maximum maximorum of π occurs for $q_c = \beta\, x_o$.

Proof. From (17) we have π and (16) gives the relation between y and q_c.

In particular, if $q_c = \beta\, x_o$ then $y = \left(\frac{1}{2}\right)\left(\frac{\beta}{r_o}\right) x_o > 0$. We may choose β so that $y^Ty = 1$. Appealing to the theorem, it follows that $q_c = \beta\, x_o$ can attain the largest possible maximum profit. Q.E.D.

This result means that a necessary condition for attaining the largest possible maximum profit is to have all optimal mark-ups positive so that $y > 0$. But the optimal y satisfies $2\,(A^TBA)\, y = q_c$ and q_c is not under the monopoly control. The best possible q_c for the monopolist is one proportional to x_o. Such a q_c gives the largest possible maximum profit and for such a q_c it is also true that $y > 0$.

Theorem 5. There is a $q_c > 0$ for which the corresponding optimal y must have at least one negative element.

Proof. By the corollary of theorem 3 there are at least two distinct eigen values. Hence there is an eigen value r_1: $0 < r_1 < r_o$. Let x_1 denote an eigen vector for r_1. Necessarily, $x_1^Tx_o = 0$ since eigen vectors for distinct eigen values of a symmetric positive definite matrix are orthogonal. Since $x_o > 0$, at least one element of x_1 is negative. Let

$$y = \beta_o x_o + \beta_1 x_1 ==> q_c = 2\,(r_o\,\beta_o x_o + r_1\,\beta_1 x_1). \qquad (25)$$

Plainly, it is possible to have β_o and β_1 such that $q_c > 0$ and $y \not\geq 0$. Necessarily, $\beta_o > 0$ because $q_c > 0$. Observe that since r_o the maximal eigen value must be a simple root of det $[A^TBA - rI] = 0$, it is not possible to express every $q_c > 0$ so that $\beta_1 = 0$. This is to say that there are $q_c > 0$ such that necessarily $\beta_1 \neq 0$. But then there are nonzero β_o and β_1 capable of giving $q_c > 0$ and $y \not\geq 0$. Q.E.D.

Figure 1 illustrates both theorems 4 and 5. Let $X_o = \beta_o x_o$ and $X_1 = \beta_1 x_1$. Hence $y = X_o + X_1$ and $q_c = 2[r_o X_o + r_1 X_1]$. $\pi = y^T A^T BA\, y ==> \pi = r_o\,\|X_o\|^2 + r_1\,\|X_1\|^2$. If $\|y\|^2 = 1$, then the possible vectors y are represented by the rays of unit length from the origin 0 in figure 1 to the

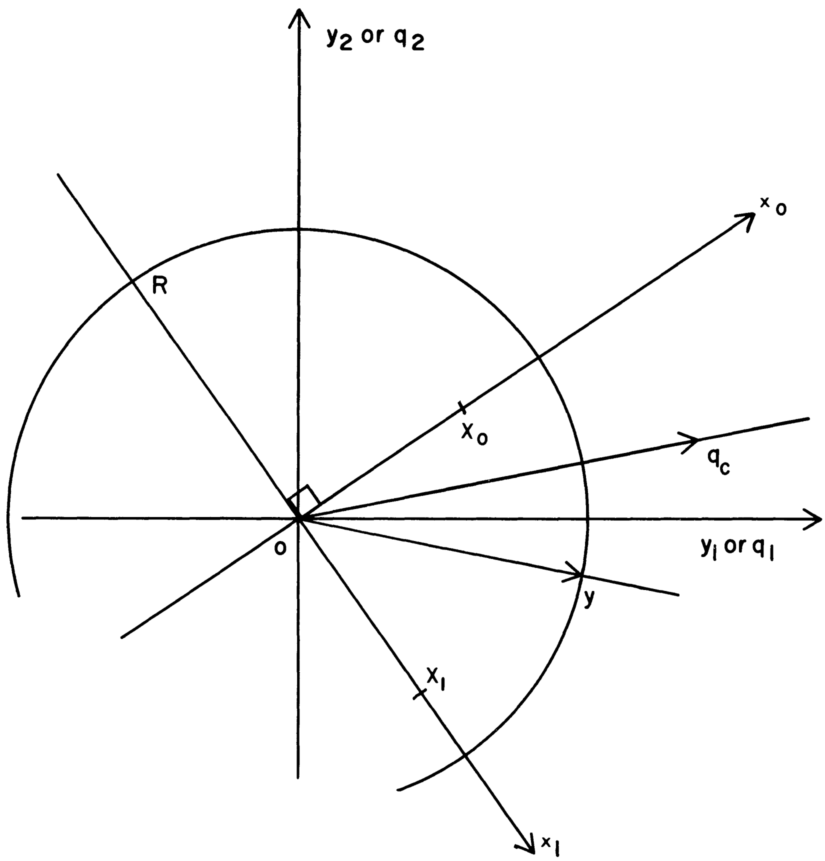

FIG. 1

boundary of the unit circle. For instance $0R$ is a vector of length 1. The vector $0y$ is the sum of the vectors $0X_o$ and $0X_1$ and $0y$ is of length 1. The vector y is not in the positive orthant while the vector $0q_c$ is in the positive orthant. This illustrates theorem 5. To illustrate theorem 4, we see that $\pi = r_o\|X_o\|^2 + r_1\|X_1\|^2 \leqslant r_o$, since $\|X_o\|^2 + \|X_1\|^2 = 1$.

These results show there is a $q_c > 0$ such that $y = \left(\frac{1}{2}\right)(A^TBA)^{-1} q_c \not\geqslant 0$.

Equivalently, the matrix $(A^TBA)^{-1}$ must have at least two negative elements, although, of course, all its diagonal elements and all its principal minors are positive. This is plausible on economic grounds. Reconsider the definition of complementary commodities. Two commodities j and k are complements if $\partial q_j/\partial p_k < 0$ for all j and k. In terms of the demand relations $q = A^Tb - (A^TBA)\,p$, this means that the matrix $A^TBA > 0$, which it is. Alternatively, write

$$p = (A^TBA)^{-1}\,b - (A^TBA)^{-1}\,q. \tag{26}$$

The commodities j and k are complements if

$$\frac{\partial p_j}{\partial q_k}\begin{cases} < 0 & \text{for } j = k, \\ \geqslant 0 & \text{for } j \neq k. \end{cases} \tag{27}$$

In terms of (26), this definition asserts that the diagonal elements of $(A^TBA)^{-1}$ are positive and the off-diagonal elements are nonpositive. Therefore, complementarity requires that not all of the elements of $(A^TBA)^{-1}$ have the same sign. Nor is this all. A standard result in matrix analysis gives

Theorem 6. Let $M = M_d - M_v$, $M_d, M_v > 0$, $M = M^T$ and M positive definite. Then $M^{-1} > 0$.

Proof. See Varga (1962, theorem 3.11, corollary 3).

This result implies that the definition of complementarity given by (27) implies complementarity in the sense that $A^TBA > 0$. Therefore, we cannot expect to have positive optimal mark-ups for all m commodities and arbitrary $q_c > 0$.

There is another way of explaining why complementarity may result in the sale of one or more of the commodities at prices below their marginal costs. For each type of customer the commodities are complements. However, different types of customers combine the commodities into a good in different proportions. Consequently, the aggregate demand for the commodities can give the appearance of substitutability among some so that $\partial p_j/\partial q_k \not\geqslant 0$ for all distinct pairs, j and k. Perhaps across customer types the proportions among a pair of commodities j and k vary inversely. A customer of one type may combine a large quantity of commodity 1 with a small quantity of commodity 2, while a customer of another type may do the reverse.

Depending on how many of these two types there are, which, in turn, depends on the sizes of the slopes and intercepts of the underlying demands for the n goods, there may result the appearance of substitutability between the two commodities so that $\partial p_j/\partial q_k < 0$ for this pair.

III. Applications

The preceding analysis applies to two situations: tie-in sales and loss leaders. In a tie-in sale the seller of one commodity requires his customers to make all of their purchases of at least one of the complementary commodities from him. The seller cannot meet with success in this policy unless he has a monopoly in at least one of the commodities in the tie-in.[3] Otherwise, customers could obtain all of their requirements from other sources of supply. If the seller does have a monopoly in at least one of the commodities in the tie-in, then he can increase his net return if there are different types of customers who use the commodities in different proportions. It may be possible for the seller to furnish each type of customer with a different good. In terms of the above analysis, the seller offers a type i customer a quantity Q_i of good i at a unit price P_i for the package. This enables him to obtain a higher net return than by the sale of the separate component commodities at constant unit prices. The alternative to selling the package is the sale of the m commodities separately in quantities q_j at prices p_j per unit.

For example, IBM required the rentors of its tabulating equipment, over which it held a monopoly by virtue of its patents, to buy all of their punch cards from it. Alternatively, IBM could have rented out its machines without imposing this restriction on the purchase of cards. In principle, IBM could have charged a different rental rate to different customers and could have achieved the same effect without tying the purchase of cards. Perhaps IBM did not have the information necessary to discriminate in rental rates among its customers and it measured the intensity of demand for the machine by the usage of cards. Therefore, the effective price paid by a type i customer is $P_i = p_o + a_{i1} p_1$, where p_0 is the unit rental of the tabulating machine, p_1 is the price per card, and a_{i1} the number of cards he uses per tabulating machine. If the price per card is above its marginal cost to IBM and if IBM charges the

3. This assertion should not be taken in support of a conclusion that a tie-in is an infallible sign of monopoly. The total cost of the tie-in may be lower than the sum of the costs of the components. This may be the reason for the tie-in. The theory here maintains the hypothesis that there are no cost savings owing to the tie-in and it addresses the effects of the demand conditions.

same rental to all customers, then it obtains a higher return from the customers who use the machines more intensively. If the IBM price for punch cards had been less than the price charged by competitors, then IBM would have encountered the problem of preventing its rental customers from selling punch cards to others. It is, therefore, more plausible that the IBM price for punch cards was above its marginal cost.

A necessary condition for the use of tie-ins as a metering device is that customers combine the component commodities in different proportions. Were the seller to offer all customers the same package of component commodities, this cannot be owing to his desire to discriminate in price among them. There is no possibility of metering. Even if the seller had a monopoly over one of the commodities in the package, his incentive for the tie-in must arise from considerations other than price discrimination. For instance, Eastman Kodak required a buyer of its color film simultaneously to buy and pay for developing. Since all customers obtained the same combination at the same price for the two, there was no price discrimination. Perhaps Eastman Kodak sold the processing and the film as a package because it was cheaper to do so than to sell them separately.

A tie-in sale is a means of price discrimination only if different types of customers obtain the component commodities in different proportions. A tie-in that offers all customers the same combination at the same price is probably explained by the lower cost of the combination than the sum of the costs of the separate component commodities.

The sale of one or more commodities at prices below their marginal cost is comprehensible if the seller has a monopoly over *all* of the complementary commodities and cannot offer them in different packages to the different types of buyers. No seller would offer one commodity at a price below its marginal cost if he faces competition in the sale of the complementary commodities. For instance, many cocktail lounges offer their customers free peanuts, potato chips, and so on. They probably hope that this will increase their total return by the sale of their alcoholic beverages. Now imagine that the waiters in a cocktail lounge were in competition so that each waiter could quote different prices for the drinks that he sells his customers. It is hard to believe that in such a cocktail lounge there would be any free peanuts or potato chips. With respect to the customers currently present in a cocktail lounge, the owner has a modicum of monopoly as is shown by his offering free peanuts.

The examples of loss leaders seem to have the common feature that the commodity sold at a price below its marginal cost is bought earlier than the one sold at a price above its marginal cost. Therefore, some may wish to argue that the success of this policy rests on mistaken customer expectations.[4] This is not so. In the formal analysis each customer forms a correct estimate of the price $P_i = \Sigma\, a_{ij} p_j$ that he pays per unit of the good Q_i. To explain the success of this policy it is unnecessary to evoke mistaken expectations. The use of the loss leader depends on the assumption that the seller cannot offer each customer a tailor-made combination of the component commodities. Thus, there would be no free peanuts in a cocktail lounge if the owner could offer customers packages consisting of various assortments of alcoholic beverages plus salty edibles. This should not be taken to imply the absence of competition among cocktail lounges. Customers can look at the posted prices before ordering and go elsewhere if they please.

IV. Some General Remarks on Complements and Substitutes

The preceding material refers to linear demand relations. We now consider nonlinear demand relations. To this end, assume there are two functions as follows:

$$w = G(q) \quad \text{and} \quad u = F(p) \tag{28}$$

such that w gives the maximum amount that the seller can collect from the buyers if he gives them an all-or-none choice of q, and u gives the maximum amount he can collect if he allows them to buy all they want of the m commodities at the constant-unit prices p. The G function extracts a lump sum payment from the buyers for the quantities q. The F function can yield a two-part price for the m commodities such that the seller collects a lump-sum plus revenue obtained from the sale of the commodities at the constant-unit prices p. We do not exclude the possibility that $p = 0$ so that the whole payment is just the lump sum $F(0)$. We may use these two functions to derive the demand for the commodities.

Assume that both $G(.)$ and $F(.)$ are twice differentiable. Using the $G(.)$ function we may write

$$p = G_q(q), \tag{29}$$

where $G_q(q)$ denotes m-vector of partial derivatives of G (so that $\nabla G = G_q$ is the gradient of G). Under the additional assumption that $G(q)$ is a concave function of q, the vector of functions $G_q(.)$ has some of the properties of normal demand functions. In fact,

4. A seller may give away free samples to potential customers. Those who find the commodity satisfactory will subsequently buy it. The firm can be successful only if the price is high enough to cover the cost of the free samples. This is not an example of a sale below cost by a monopolist. It is readily understood as a sensible way of operating in a competitive market where the potential customers initially have less than the optimal stock of knowledge about the product.

Definition. The m commodities are substitutes if

$$\partial^2 G/\partial q_j \partial q_k < 0 \text{ for all } q_j \text{ and } q_k. \tag{30a}$$

The hypothesis of concavity implies in addition that the matrix $[\partial^2 G/\partial q_j \partial q_k]$ is negative semidefinite. Thus, if the m commodities are substitutes,

$$G_q(q + \delta q) < G_q(q) \qquad \text{for all } \delta q \geqslant 0. \tag{30b}$$

Now consider $F(p)$. It is plausible to assume that F is a maximum for $p = 0$. Now write by analogy with (29)

$$F_p(p) = q - q^*, \tag{31}$$

where q^* is a vector of constants. If $p = 0$ gives the maximum of $F(p)$, then $F_p(0) = 0 ===> q^*$ is the maximum quantity demanded. To make sense of q^*, one must assume that it is finite and positive. We take (31) as the definition of the demand curves for the m commodities derived from the function $F(p)$. If $F(p)$ is a concave function of p, then

$$F_p(p) \cdot p \leqslant F(p) - F(0) \leqslant F_p(0) \cdot p = 0. \tag{32}$$

It follows from (31) and (32) that $(q - q^*) \cdot p \leqslant 0$ for all $p \geqslant 0$. We have:

Definition. The m commodities are all complements if

$$F_{pp}(p) < 0. \tag{33}$$

Therefore, for complements,

$$F_p(p + \delta p) < F_p(p) \quad \text{for all } \delta p \geqslant 0.^5 \tag{34}$$

Consequently, $q \leqslant q^*$ for all $p \geqslant 0$. It is possible to prove:

Theorem 7. Let $F(.)$ and $G(.)$ be concave and assume that both $F(0) \geqslant 0$ and $G(0) \geqslant 0$.

If all m commodities are complements, then there is a q:

$$q \cdot p_o = F(p_o) \quad \text{and} \quad q \cdot p \leqslant F(p) \quad \text{for all } p \leqq p_o. \tag{35}$$

If all m commodities are substitutes, then there is a p:

$$p \cdot q_o = G(q_o) \quad \text{and} \quad p \cdot q \leqslant G(q) \quad \text{for all } q \leqq q_o. \tag{36}$$

Proof. See Telser (1978, theorem 4.6). Q.E.D.[6]

Consider first the case of complements. The theorem asserts the existence of a bundle of the m commodities that can be offered to all of the customers at a price vector p_o such that their expenditure equals

5. Neither (33) nor (34) imply convexity.

6. Condition (35) can hold even if the m commodities are not complements. A more general result is as follows:

Theorem 8. Let $F(p) \geqslant F(p_o)$ for all p and p_o such that $p \leqslant p_o$. If there is a $q \geqq 0$ such that $q \cdot p_o = F(p_o)$ then $q\ p \leqslant F(p)$ for all $p \leqq p_o$.

Proof. $p \leqslant p_o$ implies $q \cdot p \leqslant q \cdot p_o = F(p_o) \leqslant F(p)$. Q.E.D.

A similar argument applies to (36).

the most that they would be willing to pay, $F(p_o)$, and for any price vector $p \leqslant p_o$, they would spend less than the maximum $F(p)$. This means that the seller can extract the whole consumer benefit by offering the bundle of complementary commodities q. For complements it is the bundle of commodities offered and it is not the alternative of buying them at given constant unit prices.

To extract the whole consumer benefit from the buyers of m substitutes, Theorem 7 gives different advice to the seller. It tells him that he can find a set of price p such that the buyers would be willing to pay $G(q_o)$ for q_o at these prices. The revenue the seller could obtain for smaller quantities $q < q_o$ at prices p would be less than the maximum amount that the buyers would be willing to pay.

It is helpful to give a less formal description of the distinction between complements and substitutes. For m complementary commodities, a good is an appropriate combination of these m constituents. The good is incomplete without all of these constituents. For m substitute commodities, each one of them contains the same desired good to some degree. Therefore, there is a set of prices, one for each of the m substitute commodities, such that one dollar's worth of each of these m commodities is essentially the same good. It is prices that render the m substitutes into an equivalent good.

The bundle q that extracts the maximum payment from the customers of the m complementary commodities is a solution of the system of equations given as follows:

$$
\begin{aligned}
q_1 p_1^o &= F(p_1^o, 0, \ldots, 0), \\
q_1 p_1^o + q_2 p_2^o &= F(p_1^o, p_2^o, 0, \ldots, 0), \\
&\cdots \\
\Sigma q_j p_j^o &= F(p_1^o, p_2^o, \ldots, p_m^o).
\end{aligned}
\tag{37}
$$

It is instructive to look at the special case of a quadratic $F(p)$.

$$
F(p) = \left(\frac{1}{2}\right)(p_u - p)^T q + p^T q, \tag{38}
$$

where p_u gives the m-vector of prices such that the quantity demanded of each commodity would be zero. We do not require that p_u be nonnegative. The vector q is a linear function of prices, say, $q = A^T b - A^T BA\, p$, where $A^T BA$ and b have the same properties as in Section II. The equations corresponding to (37) are given as follows:

$$
\begin{aligned}
q_1 p_1^o &= \Sigma_j p_{uj} q_j, \\
q_1 p_1^o + q_2 p_2^o &= \Sigma_j p_{uj} q_j, \\
&\cdots \\
\Sigma_j q_j p_j^o &= \Sigma_j p_{uj} q_j.
\end{aligned}
\tag{39}
$$

It is an implication of (39) that

$$q_2 p_2^o = q_3 p_3^o = \dots = q_m p_m^o = 0. \tag{40}$$

Therefore, $q_j \neq 0$ in conjunction with (40) requires that $p_j^o = 0$ for all $j \geqslant 2$. But then from the first equation of (39) we obtain the implication that

$$q_1(p_1^o - p_{u1}) = \Sigma_2^m p_{uj} q_j \tag{41}$$

is the only equation that the q_j's and p_1^o must satisfy. We also wish to restrict the solution to $q_j > 0$. Thus, $p_{uj} > 0$ for all j would require that $p_1^o > p_{u1}$. It is possible to have the q_j's corresponding to the quantities that would be bought at prices equal to the marginal costs of these commodities. If so, this would fix the values of the q_j's in (41). The seller could then choose p_1^o to satisfy (41) and this would give him the maximum amount that the customers would be willing to pay. Observe that in effect every commodity is given away "free" except for the first commodity which is sold at a high enough price p_1^o to satisfy (41).

This theory seems useful in explaining block booking. Block booking was a practice in use by the producers of motion pictures who required exhibitors to take a package of motion pictures of various kinds—good, bad, and average, westerns and musicals, romance and war films—without the alternative of selecting the individual films. Assume that each exhibitor or motion picture theater has a heterogeneous market consisting of customers with different tastes. The demand facing an exhibitor can be regarded as a set of complementary commodities such that each group of films appeals to a different segment of his market. Few of his customers would like to see all of the films in the package. Exhibitor i demands $Q_i = \Sigma_i a_{ij} q_j$, where q_j is the quantity of film j demanded by the national market. If the maximum amount that exhibitors are willing to pay is a concave function of p so that $u = F(p)$ applies, then the motion picture producer can extract the largest amount from the exhibitors by offering them a package of films.

This theory also predicts that a group of competing films would not be offered as a package. Instead the exhibitor would quote prices for the competing or substitute films and would allow the exhibitors to choose from among these at the stipulated prices. For such competing or substitute films, say two westerns, the function $w = G(q)$ applies. Theorem 7 asserts that the seller can extract the maximum amount from his customers by quoting an appropriate set of prices for the substitute commodities.[7] Block booking might result in a package consisting of grade A and grade B movies, and would allow substitutions

7. An audience need not regard two Westerns as substitutes. They may regard them as complements. Some film festivals show nothing but films of the same type. In this case we may say that the audience regards them all as complements.

within the class of grade A movies according to appropriate prices for each of the movies in that class.[8]

V. A Free Rider Problem

A tie-in sale is sometimes a method of preventing a free rider problem. Assume that commodity 1 cannot be sold separately at a positive price. Perhaps commodity 1 is information about another commodity 2 that a potential buyer of commodity 2 can obtain from those who sell it. For instance, commodity 2 may be a new product sold by retailers some of whom furnish information about it to their potential customers. There is a free rider problem if customers can obtain information about the product from one retailer who charges nothing for this service and then they buy the product at a lower price from another retailer who furnishes little or no information about the product. The second retailer can obtain a competitive return at a lower price than the first owing to his lower costs as he does not bear the same expense of supplying customers with information about the product like the first retailer. He can obtain a free ride at the expense of the first retailer. This situation cannot persist. Those retailers who provide information and make no sales would soon desist. Those retailers who provide no information and rely on the benevolence of competing retailers to do so would no longer obtain a free ride. The product, complementary with information, could not be sold without information and would no longer be made available by any retailer. Unless the manufacturer can solve this free rider problem, he cannot induce independent retailers to carry his product.[9]

Formally,

$$Y_i = \Sigma_j a_{ij} y_j, \quad i = 1, \ldots, n, \tag{42}$$

and

$$y_1 = p_1 - c_1 = -c_1 \quad \text{if } p_1 = 0, \tag{43}$$

where commodity 1 is information that by hypothesis cannot be sold separately at a positive price. The manufacturer wishes to have a method of distributing the product, commodity 2, so that no retailer can obtain a free ride at the expense of another. The optimal policy should tie the information and the physical commodity together so that the physical product cannot be sold by a retailer who does not provide the information. A method of solving this problem is to set a minimum price on commodity 2. If the manufacturer requires $p_2 \geqslant p_2^0$ then he

8. Stigler (1968) explains block booking as a method of raising the receipts of the producer. This argument is valid only under special conditions. See Telser (1965) for a discussion of some numerical examples of block booking where these conditions do not hold.

9. For a more complete discussion of the free rider problem and resale price maintenance, see Telser (1960).

prevents a retailer, who does not bear the cost of information, from selling the product, commodity 2, at a lower price than the retailer who does provide the information. Consequently, all retailers have an incentive to furnish the information necessary to sell the product to retail customers.

We now study the resulting equilibrium in more detail. Let a customer of type i desire a combination of the input commodities as expressed by the elements of the m-vector a_i where $a_i = < a_{i1}, \ldots, a_{im} >$, and let retailers specialize in selling good i to type i customers. Thus, Q_i represents the quantity of good i demanded by type i customers from type i retailers at a price of P_i per unit of good i. A retailer does not sell the m components separately. Instead he sells a joint product at a mark-up Y_i subject to the constraint that $p_1 = 0$. Therefore, a type i retailer incurs a fixed cost per unit equal to $a_{i1} c_1$. For the continued supply of the commodity that requires information as a complementary input, $Y \geqslant 0$ is a necessary condition. Competition among the retailer results in $Y = 0$. Since $Q = Q_c - B\,Y$, it follows that $Q = Q_c$, and the equilibrium gives the competitive quantities of the joint goods.

Consider figures 2 and 3. The lines L_iL_i give the locus of points for which $Y_i = -a_{i1}c_1 + a_{i2}y_2 = 0$. Imposing the minimum price p_2 implies a minimum mark-up y_2^0. There are different values of y_2 consistent with $Y_i = 0$ as we can see from the figures. The choice of y_2^0 depends on how much information the manufacturer wishes potential customers to obtain from those retailers who sell his product. Any retailer such that $Y_i = 0 ==> y_2 < y_2^0$ would not carry the product. That is, $Y_i = 0 ==> y_2 = a_{i1}c_1/a_{i2}$. Consequently, $Q_i = 0$ for all i: $a_{i1}c_1/a_{i2} < y_2^0$.

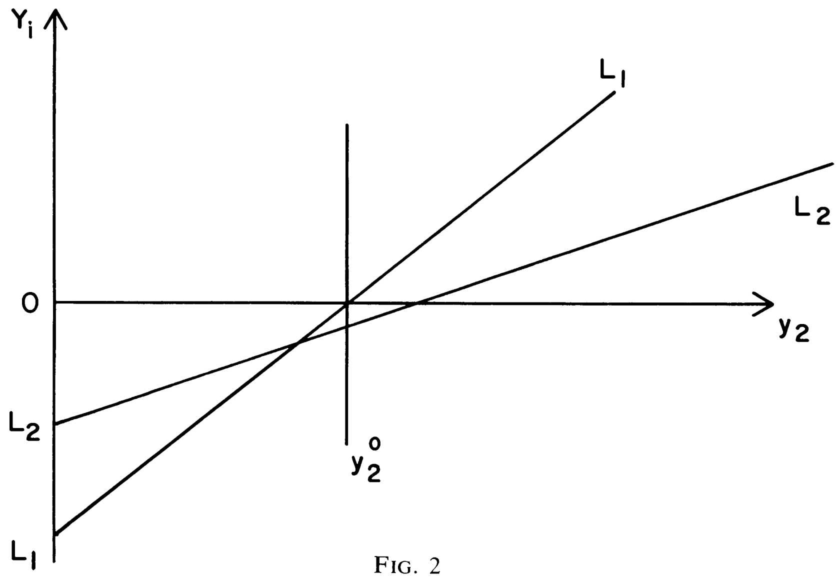

FIG. 2

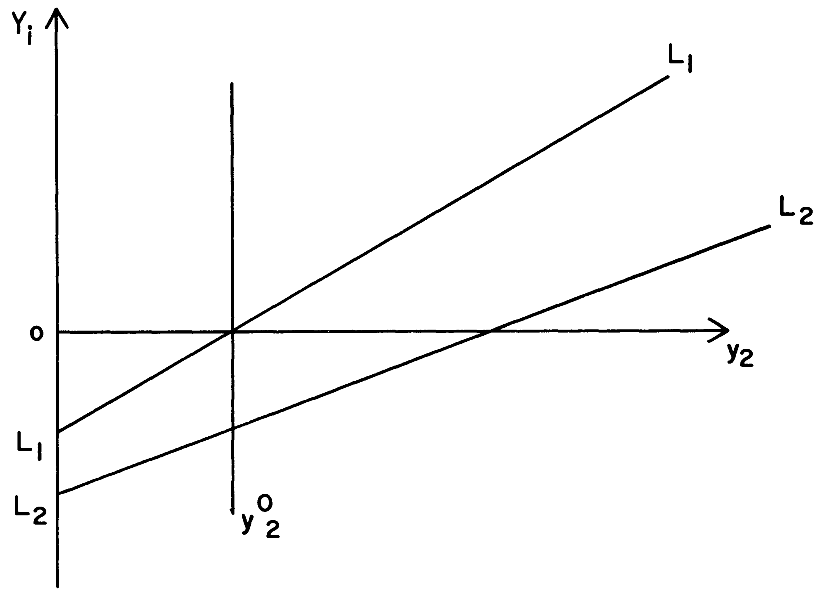

FIG. 3

The retailers may attempt to evade the restriction on the retail price of the second commodity by offering a third commodity jointly with it so that the price of the package is below the sum of the minimum price for commodity 2 and the normal price of commodity 3. The cost of preventing these evasions may be so large that the manufacturer cannot solve the free rider problem by imposing a minimum retail price of commodity 2. Nevertheless there is the possibility of using a minimum retail price to induce retailers to furnish the optimal combination of inputs while preserving a competitive equilibrium among them.

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