改变度量单位

$$y_i^* = c_1 y_i \qquad x_i^* = c_2 x_i$$

$$y_i^* = \hat{\beta}_1^* x_i^* + \hat{\beta}_0^*$$

$$\hat{\beta}_1^* = \frac{\sum (x_i^* - \bar{x}^*)(y_j^* - \bar{y}^*)}{\sum (x_i^* - \bar{x}^*)^2}$$

$$= \frac{\sum (c_2 x_i - c_2 \bar{x})(c_1 y_j - c_1 \bar{y})}{\sum (c_2 x_i - c_2 \bar{x})^2}$$

$$= \frac{c_1}{c_2} \hat{\beta}_1$$

$$\hat{\beta}_0^* = \overline{y_i^*} - \hat{\beta}_i^* \overline{x_i^*}$$

$$= c_1 \hat{\beta}_0$$

Incoporate nonlinearility

$$\log y = \beta_1 x + \beta_0 + u$$

$$\log \frac{y_2}{y_1} = \beta_1 \Delta x$$

$$\log \frac{y_2 - y_1 + y_1}{y_1} = \beta_1 \Delta x$$

$$\log\left(\frac{y_2 - y_1}{y_1} + 1\right) = \beta_1 \Delta x$$

r_y: Percent of changing y

$$\log(r_y + 1) = \beta_1 \Delta x.$$

$\log(z+1) \approx z$ when z is really small.

$$\Rightarrow r_y \simeq \beta_1 \Delta x.$$

$$\Rightarrow \% \Delta y = 100 r_y \simeq 100 \beta_1 \Delta x.$$

Elasticity : $\displaystyle \varepsilon_{y,x} = \frac{\% \Delta y}{\% \Delta x} = \frac{\frac{\Delta y}{y} \times 100\%}{\frac{\Delta x}{x} \times 100\%}$

$$= \underbrace{\frac{\Delta y}{\Delta x}}_{\text{slope}} \cdot \frac{x}{y}$$

$$\log y = \beta_1 \log x + \beta_0 + u$$

$$r_y \simeq \beta_1 r_x$$

$$\beta_1 \simeq \frac{r_y}{r_x} = \varepsilon_{y,x}$$

$$y = \beta_1 \log x + \beta_0 + u.$$

$\Delta y \simeq \beta_1 \% x.$

$\Delta y \simeq \frac{\beta_1}{100} \% \Delta x$

$\widehat{\Delta y} \simeq \frac{\hat{\beta}_1}{100} \% \Delta x.$

Assumption

SLR.1 Linear in parameters

$y = \beta_0 + \beta_1 x + u.$

SLR.2 Random sampling.

$\{(x_i, y_i); i = 1, \ldots, n\}$ The data is a random sample drawn from the population.

$y_i = \beta_0 + \beta_1 x_i + u_i$, each data points follows the equation.

SLR.3 Sample variation in the explanatory variable.

$\sum_{i=1}^{n} (x_i - \bar{x})^2 > 0.$

SLR.4 Zero conditional mean.

$E(u_i | x_i) = 0$

$$E(\hat{\beta}_0) = \beta_0 \ , \ E(\hat{\beta}_1) = \beta_1$$

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(y_i-\bar{y})}{\underbrace{\sum_{i=1}^{n}(x_i-\bar{x})^2}} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i - \overbrace{\sum_{i=1}^{n}(x_i-\bar{x})}^{=0}\bar{y}}{SST_x}$$

$$SST_x = \sum_{i=1}^{n}(x_i-\bar{x})x_i - \sum_{i=1}^{n}\underbrace{(x_i-\bar{x})}_{=0.}\bar{x} = \sum_{i=1}^{n}(x_i-\bar{x})x_i$$

$$= \frac{\sum_{i=1}^{n}(x_i-\bar{x})y_i}{SST_x} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})(\beta_0+\beta_1 x_i+u_i)}{SST_x}.$$

$$= \frac{\sum_{i=1}^{n}\underbrace{(x_i-\bar{x})}_{=0}\beta_0 + \sum_{i=1}^{n}(x_i-\bar{x})(\beta_1 x_i) + \sum_{i=1}^{n}u_i(x_i-\bar{x})}{\sum_{i=1}^{n}(x_i-\bar{x})x_i}$$

$$= \beta_1 + \frac{\sum u_i(x_i-\bar{x})}{SST_x}$$

$$E(\hat{\beta}_1|X) = \beta_1 + E\left[\frac{\sum u_i(x_i-\bar{x})}{SST_x}\Big| X\right]$$

$X = b.$

$x_i = b_i, \ i=1,2,\dots n$

$$= \beta_1 + \frac{1}{SST_x}E\left(\sum u_i(x_i-\bar{x}) / X\right)$$

$$= \beta_1 + \frac{1}{SST_x}\sum(x_i-\bar{x})\underbrace{E(u_i|x_i)}_{=0.}$$

$$= \beta_1$$

$$E(\hat{\beta}_1) = E_x(E(\hat{\beta}_1|X)) = \beta_1$$

SLR.5 Homoscedasticity.

$$Var(u_i|x_i) = \sigma^2$$

$$\Rightarrow Var(Y_i|x_i) = \underbrace{Var(\beta_0|x_i)}_{=0} + \underbrace{Var(\beta_1 x_i|x_i)}_{=0} + Var(u_i|x_i)$$

$= \sigma^2$

$Var(\hat{\beta}_1) = \frac{\sigma^2}{SST_x}$, $Var(\hat{\beta}_0) = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{SST_x}$

$$Var(\hat{\beta}_1 | X) = Var(\beta_1 + \frac{\sum u_i (x_i - \bar{x})}{SST_x} | X)$$
$$= \frac{1}{SST_x^2} \cdot \sum (x_i - \bar{x})^2 \, Var(u_i | x_i)$$
$$= \frac{\sum (x_i - \bar{x})^2}{SST_x^2} \sigma^2 = \frac{\sigma^2}{SST_x}$$

X is given randomly.

$$\Rightarrow Var(\hat{\beta}_1) = Var(\hat{\beta}_1 | X) = \frac{\sigma^2}{SST_x}$$

$$Var(\hat{\beta}_0 | X) = Var(\bar{y} - \hat{\beta}_1 \bar{x} | X)$$
$$= Var(\bar{y}|X) + Var(\hat{\beta}_1 \bar{x} | X)$$
$$= Var(\frac{\sum y_i}{n} | X) + \frac{(\bar{x})^2 \sigma^2}{SST_x}$$
$$= \frac{\sigma^2}{n} + \frac{(\bar{x})^2 \sigma^2}{SST_x} = \frac{\sum x_i^2 \sigma^2}{n SST_x}$$

$$\Longrightarrow Var(\hat{\beta}_0) = \frac{\sum x_i^2 \sigma^2}{n SST_x}$$

$\sigma^2 = Var(u_i) = E(u_i^2)$

$\hat{u}_i = y_i - \hat{y}_i = (\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1)x_i + u_i$

$E(\hat{u}_i) = E(u_i) = 0$

Use $\hat{u}_i$ as a proxy for u_i.

$\Rightarrow Var(\hat{u}_i) = \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{SSR}{n}$

by OLS's Algebra Properties

$\underbrace{\frac{1}{n}\sum_{i=1}^{n}\hat{u}_i}_{=0} = (\beta_0 - \hat{\beta}_0) + (\beta_1 - \hat{\beta}_1)\bar{x} + \bar{u} = 0.$

Subtract

$\hat{u}_i = (u_i - \bar{u}) - (\hat{\beta}_1 - \beta_1)(x_i - \bar{x})$

$\hat{u}_i^2 = (u_i - \bar{u})^2 + (\hat{\beta}_1 - \beta_1)^2(x_i - \bar{x})^2 - 2(u_i - \bar{u})(\hat{\beta}_1 - \beta_1)(x_i - \bar{x})$

$\sum_{i=1}^{n}\hat{u}_i^2 = \sum_{i=1}^{n}(u_i - \bar{u})^2 + (\hat{\beta}_1 - \beta_1)^2 SST_x - 2(\hat{\beta}_1 - \beta_1)\sum(u_i - \bar{u})(x_i - \bar{x}).$

$E(\sum_{i=1}^{n}(u_i - \bar{u})^2) = (n-1)\sigma^2$

$E((\hat{\beta}_1 - \beta_1)^2 SST_x) = \frac{\sigma^2}{SST_x} \cdot SST_x = \sigma^2$

$E(2(\hat{\beta}_1 - \beta_1)\sum(u_i - \bar{u})(x_i - \bar{x})) = E(2(\hat{\beta}_1 - \beta_1)\underbrace{\sum u_i(x_i - \bar{x})})$

$= E(2(\hat{\beta}_1 - \beta_1)^2 SST_x) = 2\sigma^2$ $\qquad \hat{\beta}_1 = \beta_1 + \frac{\sum u_i(x_i - \bar{x})}{SST_x}$

$\Rightarrow E[\sum_{i=1}^{n}\hat{u}_i^2] = (n-1)\sigma^2 + \sigma^2 - 2\sigma^2 = (n-2)\sigma^2$

$\Rightarrow E(\hat{\sigma}^2) = \frac{n-2}{n}\sigma^2$

$\Rightarrow$ the $\hat{\sigma}^2$ is not an unbiased estimator of σ^2

$\Rightarrow$ So we change $\hat{\sigma}^2 = \frac{SSR}{n}$ into $\hat{\sigma}^2 = \frac{SSR}{n-2}$

$$\hat{\sigma}^2 = \frac{1}{n-2}\sum_{i=1}^{n}\hat{u}_i^2 = \frac{SSR}{n-2}$$

$\Rightarrow E(\hat{\sigma}^2) = \sigma^2$. $\hat{\sigma}^2$ is an unbiased estimator of σ^2

$$\widehat{Var}(\hat{\beta}_1) = \frac{\hat{\sigma}^2}{SST_x}$$

$$Sd(\hat{\beta}_1) = \sqrt{Var(\hat{\beta}_1)}$$

$$Se(\hat{\beta}_1) = \sqrt{\widehat{Var}(\hat{\beta}_1)}$$

$$\widehat{Var}(\hat{\beta}_0) = \frac{\hat{\sigma}^2 \sum_{i=1}^{n} x_i^2}{n\, SST_x}$$

$$Se(\hat{\beta}_0) = \sqrt{\frac{\hat{\sigma}^2 \sum_{i=1}^{n} x_i^2}{n\, SST_x}}$$