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IE598GT: Homework 1

Due: March 5, 2021 (5 PM)

Note:

- The due date is strict. Late submissions will not be graded.
- Your answers must be concise and clear. Explain sufficiently so that we can easily determine that you understand.
- Only **electronic submissions** are accepted.
- You are expected to do each homework on your own. You may discuss concepts with your classmates, but there must be no interactions about solutions.
- Plagiarism will be dealt with severely − no credit for the homework.
- **There are 100 points in total, and you need 80 points for the full homework credit.**

1. **(10 points)** Assume that a set M of m items need to be fairly divided among a set N of n agents, and each agent i has an additive valuation function v_i, where $v_i(S) = \sum_{j\in S} v_i(\{j\}), \forall S \subseteq M$. Show that

   a. Envy-freeness (EF) implies proportionality (Prop)

   b. Envy-freeness up to one item (EF1) implies proportionality up to one item (Prop1)

   c. Prop does not imply EF

   d. Prop1 does not imply EF1

2. **(20 points)** Suppose there are n agents with additive valuations. Show that

   a. Show that MMS allocations exist when there are two agents.

   b. EF1 implies $1/n$-MMS.

   c. Show an example where EF1 implies $\Omega(1/n)$-MMS.

   d. Show an example where an MMS allocation is not EF1.

3. **(10 points)**

   a. Show that 6/7-MMS allocation exists for three agents.[1]

   b. Show that 4/5-MMS allocation exists for four agents.

4. **(10 points)**

   a. Show an example for which the envy-cycle procedure does not give an EFX allocation.

   b. Show that the envy-cycle procedure runs in polynomial time.

[1] The best known factor is 8/9 for three agents in *Approximate Maximin Share Allocations in Matroids* by Laurent Gourvès 1 Jérôme Monnot.

5. **(20 points)**

   a. Show all Pareto optimal (PO) allocations in the following example with 2 agents $\{a_1, a_2\}$ with additive valuations and 4 indivisible goods $\{g_1, g_2, g_3, g_4\}$, where the value of each good for each agent is given as follows:

	g_1	g_2	g_3	g_4
a_1	3	1	8	0
a_2	5	0	7	3

   b. How many are EF1?

   c. How many are EFX?

   d. How many are Prop1?

6. **(10 points)** An allocation $A = (A_1, \ldots, A_n)$ is called α-EFX if

$$v_i(A_i) \geq \alpha \cdot v_i(A_j \setminus g), \ \forall g \in A_j, \ \forall i, j \ .$$

   Design an algorithm to obtain $\frac{1}{2}$-EFX allocation when agents have monotone subadditive valuations, where $v(S \cup T) \leq v(S) + v(T), \forall S, T \subseteq M$ and $v(S) \leq v(T), \forall S \subseteq T \subseteq M$.

7. **(10 points)** Suppose we want to fairly allocate a set of indivisible *chores*, for which each agent has negative utility. What is the natural analogue of EF1 allocation in this case? Design an algorithm to obtain an EF1 allocation.

8. **(10 points)** Suppose the set of indivisible items consists of both goods and chores. What is the natural analogue of EF1 allocation in this case? Design a polynomial-time algorithm to obtain an EF1 allocation when agents have additive valuations.

1. (a) EF : $V_i(A_i) \geq V_i(A_j) \quad \forall i, j.$

Prop : $V_i(A_i) \geq \frac{1}{n} V_i(M) \quad \forall i.$

$$EF \Rightarrow \sum_{j \in N} V_i(A_i) \geq \sum_{j \in M} V_i(A_j) \quad \forall i.$$

$$n V_i(A_i) \geq V_i(M) \quad \forall i$$

$$V_i(A_i) \geq \frac{1}{n} V_i(M) \quad \forall i \quad (\text{i.e. Prop}).$$

(b) EF1 : $V_i(A_i) \geq V_i(A_j \setminus g_j) \quad \exists g_j \in A_j, \forall i, j.$

additive valuation $\Rightarrow V_i(A_i) \geq V_i(A_j) - V_i(g_j) \quad \exists g_j \in A_j \quad \forall i, j.$

Let $g^*(i) = \underset{g \in \{M \setminus A_i\}}{\arg\max} \; V_i(g)$

Hence $V_i(g^*(i)) \geq V_i(g_j) \quad \forall g_j \in A_j \quad \forall i, j.$

$$V_i(A_i) \geq V_i(A_j) - V_i(g_j) \geq V_i(A_j) - V_i(g^*(i)).$$

$$\sum_{j \in N} V_i(A_i) = n V_i(A_i) \geq \sum_{j \in N} (V_i(A_j) - V_i(g^*(i)))$$

$$\geq V_i(M) - n V_i(g^*(i))$$

$$\Rightarrow V_i(A_i) \geq \frac{1}{n} V_i(M) - V_i(g^*(i))$$

$$V_i(A_i) + V_i(g^*(i)) \geq \frac{1}{n} V_i(M)$$

$$\Rightarrow V_i(A_i \cup \{g\}) \geq \frac{1}{n} V_i(M) \quad \exists g \in \{M \setminus A_i\}. \forall i.$$

(i.e. Prop1).

(c). an allocation $A = \{A_1, A_2, A_3\}$

$V_1(A_1) = 3 \quad V_1(A_2) = 4 \quad V_1(A_3) = 1.$

$V_2(A_1) = 3 \quad V_2(A_2) = 4 \quad V_2(A_3) = 1.$

$V_3(A_1) = 4 \quad V_3(A_2) = 2 \quad V_3(A_3) = 3.$

$V_i(A_i) \geqslant \frac{1}{3} V_i(M) \ \forall i$, but $V_1(A_1) \leq V_1(A_2)$

This example satisfies Prop, but doesn't satisfy EF.

(d) example:

	g_1	g_2	g_3	g_4
a_1	1	2	4	4
a_2	1	2	4	4.

$A_1 = \{g_1, g_2\}.$

$A_2 = \{g_3, g_4\}.$

$V_1(A_1 \cup g_3) = 7 \geqslant \frac{1}{2} V_1(M) = \frac{11}{2}$

$V_2(A_2 \cup g_2) = 10 \geqslant \frac{1}{2} V_2(M) = \frac{11}{2}$

$\Rightarrow$ satisfy Prop1.

However $V_1(A_1) \leq V_1(A_2 \setminus g) \ \forall g \in A_2$

$\Rightarrow$ doesn't satisfy EF1.

2. m items set M

(a) Set allocation $A=\{A_1, A_2\}$ s.t. $V_1(A_1)=\mu_1$, $V_1(A_2)\geqslant \mu_1$. Hence we can allocate both A_1 or A_2 to agent 1.

Since $\max\{V_2(A_1), V_2(A_2)\}\geqslant \mu_2$,

Let's allocate A_i to angent 2 where $i=\underset{i=1,2}{\arg\max}\{V_2(A_i)\}$.

and allocate the other to agent 1.

$\Longrightarrow$ MMS-allocations exist for two agents.

(b). EF1: $V_i(A_i)\geqslant V_i(A_j\setminus g_j)$ $\exists g_j\in A_j$. $\forall i,j$.

$$\Longrightarrow (n-1)V_i(A_i)\geqslant \sum_{j\neq i\, j\in N}[V_i(A_j)-V_i(g_j)]$$

$$nV_i(A_i)\geqslant V_i(M)-\sum_{j\neq i\, j\in N}V_i(g_j)$$

assume $V_i(M)-\sum_{j\neq i\, j\in N}V_i(g_j)<\mu_i$

$$V_i(M\setminus\{g_1,g_2\ldots g_{i-1},g_{i+1}\ldots g_n\})<\mu_i<V_i(g_j)\ \forall j.$$

it is impossible to get an $A^*\subseteq M$ s.t $V_i(A^*)=\mu_i$

$$\Longrightarrow V_i(M)-\sum_{j\neq i\, j\in N}V_i(g_j)\geqslant \mu_i$$

$$\Longrightarrow nV_i(A_i)\geqslant \mu_i$$

$$V_i(A_i)\geqslant \frac{1}{n}\mu_i.$$

(c) (RRA) let each agent picks item one by one (Reapent 1 to n), every one choose the item with highest value in their eyes. finally the allocation A will be EF1.

according to (b), A will also be $\frac{1}{n}$-MMS.

(d).

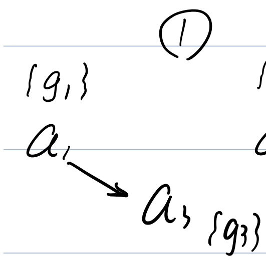
	g_1	g_2	g_3	g_4	g_5	
a_1	5	5	5	9	9	$A_1 = \{g_4\}$.
a_2	1	1	1	1	1	$A_2 = \{g_1, g_2\}$
a_3	1	1	1	1	1	$A_3 = \{g_3, g_5\}$

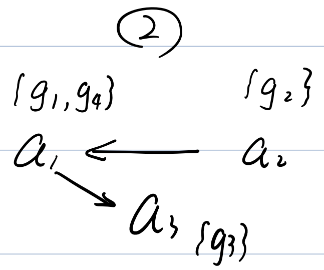
which is MMS but not EF1.

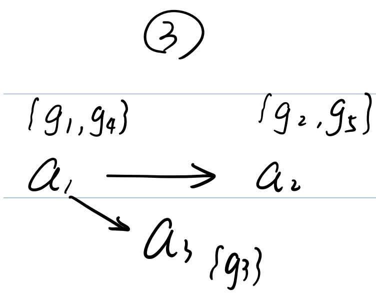
3 a. ① scale valuation $\Rightarrow V_i(M) = 3$ $i = 1, 2, 3$.

4. (a).

	g_1	g_2	g_3	g_4	g_5
a_1	2	1	4	1	100
a_2	5	6	3	4.	5
a_3	1	2	3	1	1







results: $A_1 = \{g_1, g_4\}$

$A_2 = \{g_2, g_5\}$.

$A_3 = \{g_3\}$

which is not EFX.

(b) ① compare n agents' values: $(n-1)+(n-2)+\cdots+1 = \frac{n(n-1)}{2}$

$m-n$ items remain

② $(m-n)$ times compare new with others: $n-1$.

③ Finding cycle: $O(n^2)$

times: $O(n^3 m)$.

5. (a)

	g_1	g_2	g_3	g_4
a_1	3	1	8	0
a_2	5	0	7	3

① $A_1 = \{g_2\}$ $A_2 = \{g_1, g_3, g_4\}$

② $A_1 = \{g_1, g_2\}$ $A_2 = \{g_3, g_4\}$

③ $A_1 = \{g_2, g_3\}$ $A_2 = \{g_1, g_4\}$.

④ $A_1 = \{g_1, g_2, g_3\}$ $A_2 = \{g_4\}$.

(b) EF1: ② ③ $\underline{\underline{2.}}$

(c) EFX: ③ $\underline{\underline{1}}$

(d) Prop1: ① ② ③ ④ $\underline{\underline{4}}$

6. $A = (A_1, \ldots, A_n)$

$\frac{1}{2}$-EFX: $V_i(A_i) \geq \frac{1}{2} \cdot V_i(A_j \setminus g)$ $\forall g \in A_j$ $\forall i,j$.

①: sorted all items from big to small $(g_1', g_2', \ldots g_m')$.

②: allocate g_i' to agent i $(i = 1, 2 \ldots n)$.

one by one from $i=1$ to $i=n$.

③: Pick next g' and allocate it to agent n. $A_i \leftarrow A_i + \{g'\}$

then agent $n-1$, agent $n-2$, …… to $\underline{\underline{\text{agent } k}}$

where $v(A_k) \geq \frac{1}{2} v(A_1)$

and $v(A_{k+1}) < \frac{1}{2} v(A_1)$

④: repeat ③ until the value of agent n's allocation $\geq \frac{1}{2} v(A_1)$

Repeat ② ③ ④ until all items are allocated.

7. A is EF1 if each agent prefers their own allocation after removing some item than any others' allocation.

$V_i(A_i \setminus g) \geq V_i(A_j) \quad \exists g \in A_i \quad \forall i,j.$

(RRA) from i to n (repeat by cycle)

let each agent choose one by one

every one choose the item with hightest value in their eyes.

8. A is EF1 if each agent prefers their own allocation than any other's allocation after removing "good"

from other's bundle or "chore" from their own bundle.

$V_i(A_i | g) \geqslant V_i(A_j | g) \ \exists g \in A_i \cup A_j \ \forall i, j.$

Envy-Cycle-procedure

(1) Start with an empty allocation $A = (\phi, \dots, \phi)$

(2) Initialize $R = M$

(3) Create a graph G where there is a vertex i for each agent i and there is a directed edge from i to j if i envies j i.e. $V_i(A_i) \leq V_i(A_j)$.

(4) Repeat until R is empty

① pick an item g from G, if g is "good", then pick a source i and allocate g to i. if g is "chore", then pick an goal j and allocate g to j.

② If G has no source and goal then there must be a cycle, let's exchange the bundle along the cycle.

(5) Output allocation A.