Maximin Shares

- Suppose we ask some agent i to partition all the items into n bundles (one for each agent), and then all other agent will choose their favorite bundle before i (i.e., i will choose the last remaining bundle)
  - How will i partition?

maximize the least-value bundle

- Example:

$$\max_{P \in \mathbb{P}} \min_{A_k} V_i(A_k).$$

$$\mathbb{P} = \{(A_1, \ldots, A_n) \mid \bigcup_i A_i = M\}.$$

Maximin Shares

- The maximum value, agent I can guarantee in such a process is called i’s maximin share (MMS) value
  - Let us denote i’s MMS value by μ_i
- MMS allocation: We say that an allocation $A = (A_1, \dots, A_n)$ is MMS if $v_i(A_i) \geq \mu_i, \forall i$
- Example:

goods.

agents

	g_1	g_2	g_3	g_4
a_1	5	3	2	0
a_2	2	3	4	1

$\mu_1 = 8$ $\mu_2 = 5$

$\underbrace{\{g_1, g_2\} \quad \{g_3, g_4\}}_{\text{MMS allocation.}}$

Maximin Shares

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- Assume that agents have additive valuations

- Existence? MMS allocation doesn't always exist.

No. (not a simple example). (2015).

Maximin Share Allocation

- α-MMS allocation: We say that an allocation $A = (A_1, \ldots, A_n)$ is α-MMS, for some $\alpha \in (0,1)$, if $v_i(A_i) \geq \alpha\, \mu_i, \forall i$

$\alpha = 1 \Rightarrow$ MMS doesn't always exist.

- 1/2-MMS Allocation? Yes!

~~best~~ $\left(\frac{3}{4} + \frac{1}{12n}\right)$-MMS allocation! $\leftarrow$ [2020]. G.T.
exists.

Open question: existence $\frac{3}{4}$ and non-existence 0.99.

~~1/2-~~MMS ~~Allocation~~ Properties (addictive)

Assume addictive valuation

$g_1 \cdots g_i \cdots g_m$.

$a_1 \vdots a_i \vdots a_n$

$\boxed{v_{ij}}$

- Properties:
  - Scale Invariant

$v_{ij} \leftarrow$ agent i's value for good j.

Scale all valuation.

$v'_{ij} \leftarrow \gamma v_{ij}$. $\forall i, j$. $\boxed{\gamma > 0}$

$\mu_i \leftarrow$ MMS value is given by $\{v_{ij}\}$s.

$\mu'_i \leftarrow$ MMS value is given by $\{v'_{ij}\}$s.

$\mu'_i = \gamma \mu_i$

Proof: (1) $\mu_i' \geq \gamma \mu_i$ (2) $\mu_i' \leq \gamma \mu_i$.

(1) for $\{V_{ij}\}$s agent i, allocation $P = (A_1, \dots, A_n)$

$$\text{s.t. } P \in \operatorname*{argmax}_{P \in \mathbb{P}} \min_{A_k} \sum_{j \in A_k} V_{ij}$$

set A_k is the minimum allocation in P.

$$\sum_{j \in A_k} V_{ij} = \mu_i$$

then if we use the same allocation for $\{V_{ij}'\}$s.

$$\sum_{j \in A_k} V_{ij}' = \sum_{j \in A_k} \gamma V_{ij} = \gamma \mu_i.$$

A_k is still the minimum allocation in P.

However, there might exists better P

Hence $\mu_i' \geq \gamma \mu_i$.

(2). Similarly for $\{V_{ij}'\}$s agent i allocation P'

$$\sum_{j \in A_k'} V_{ij}' = \mu_i' \qquad \sum_{j \in A_k'} V_{ij} = \sum_{j \in A_k'} \frac{1}{\gamma} V_{ij}' = \frac{1}{\gamma} \mu_i'$$

$$\Rightarrow \mu_i \geq \frac{1}{\gamma} \mu_i' \quad \Rightarrow \mu_i' \leq \gamma \mu_i.$$

~~1/2~~-MMS ~~Allocation~~ Properties (additive)

- Properties:
  - Scale Invariant
  - Average is upper bound: $\mu_i \le \frac{v_i(M)}{n}$ $= \sum_{j \in M} V_{ij}$

$$\mu_i = \max_{P \in \mathbb{P}} \min_{A_k} V_k(A_k)$$

given P μ_i must be the smallest one in P.

$$\mu_i = \sum_{j \in A_k} V_{ij} \Rightarrow n\mu_i \le V_i(M)$$

$$\mu_i \le \frac{V_i(M)}{n}$$

~~½~~-MMS ~~Allocation~~ Properties (additive)

- Properties:
  - Scale Invariant
  - Average is upper bound: $\mu_i \leq \frac{v_i(M)}{n}$
  - (Allocating one ~~high-value~~ item to an agent (not i) and remaining. does not harm

Example:

give g_4 to a_3 and let them go.

	g_1	g_2	g_3	~~g_4~~
a_1				
a_2				
~~a_3~~				

$\boxed{\{a_1, a_2, a_3\} \times \{g_1, g_2, g_3, g_4\}}$ μ_i

$\Downarrow$

$\boxed{\{a_1, a_2\} \times \{g_1, g_2, g_3\}}$ μ_i'

$\mu_i' \geq \mu_i$ $\quad i = 1, 2$

<u>does't harm agent i.</u>

Mathematical

A set of N agents and M items.

$\mu_i^n(M)$ = MMS value of agent i if i need to divide M into n bundles.

Suppose we remove one agent i^* and one item j^*.

$N' = N \setminus \{i^*\}$ $\qquad$ $M' = M \setminus \{j^*\}$

Claim: $\mu_i^{n-1}(M') \geq \mu_i^n(M)$.

i 要分配M，系统自动将 item j^* 自动分配给 i^*

另一agent.

① 如果 j^* 小 ⟶ 分配M'的下限大于M.

$V_i(\{j^*\}) \leq \mu_i^n(M)$ 因为i不必被迫选择最小项了.

② 如果 j^* 大 ⟶ 分配M'的下限与M相同

$V_i(\{j^*\}) > \mu_i^n(M)$ 因为i是最后拿的，即使在M中j^*也轮不到他.

allocating one item to another agent and removing doesn't harm agent i.

$$\mu_i^{n-1}(M') \geq \mu_i^n(M).$$

~~½-MMS Allocation~~ Properties (addictive)

- Properties:
  - Scale Invariant
  - Average is upper bound: $\mu_i \leq \frac{v_i(M)}{n}$
  - Allocating one high-value item to an agent does not harm
- We scale the valuations such that $v_i(M) = n$

$\Longrightarrow \mu_i \leq \frac{v_i(M)}{n} = 1. \ \forall i \in N$

example

	g_1	g_2	g_3	
a_1	5	3	1	9
a_2	2	1	3	6

$\times \frac{2}{9}$

$\Longrightarrow$

$\times \frac{2}{6}$

	g_1	g_2	g_3
a_1	$\frac{10}{9}$	$\frac{6}{9}$	$\frac{2}{9}$
a_2	$\frac{4}{6}$	$\frac{2}{6}$	$\frac{6}{6}$

Algorithm $\frac{1}{2}$-MMS Allocation.

Step1: Scale valuation s.t. $V_i(M) = n$, $\forall i \in N$.

$\Rightarrow \mu_i \le 1$. $\forall i \in N$.

Allocation: $A = (A_1, \dots, A_n)$ s.t. $V_i(A_i) \geq \frac{1}{2}$, $\forall i$.

$V_i(A_i) \geq \frac{1}{2} \geq \frac{1}{2}\mu_i \Rightarrow A$ is $\frac{1}{2}$-MMS Allocation.

Step2: If there is a agent i^* and item j^* s.t. $V_{i^*j^*} \geq \frac{1}{2}$

then we can remove i^* with j^*.

need scale again

Step3: Suppose we have some agents N' and some item M' left. $V_{ij} < \frac{1}{2}$, $\forall i, j$.

for agent i. bag B

keep adding item to B until $V_i(B) \geq \frac{1}{2}$

$V_i(B) \in [\frac{1}{2}, 1)$

Bag filling. then assign B to i.

可能最后剩余的 $B_n < \frac{1}{2}$?

Q: Why we will be able to give every agent $\geq \frac{1}{2}$?

由于重复多次的 step 1, 2, M 中已经没有 $V_{ij} \geq \frac{1}{2}$ 的项了.

$V_i(M) = n$ $V_i(M \setminus B) \geq n-1$, $\forall i$. n 项均为 $[0, \frac{1}{2})$ 之间.

M 需要分为 n 组.

$V_i(M \setminus B_1 \setminus B_2) \geq n-2$. ---- $V_i(M \setminus B_1 \setminus \cdots \setminus B_{n-1}) \geq 1$.

Bag Filling

Example