(Nash 1950b) Every finite strategic-form game has a mixed strategy equilibrium.

main idea: Kakutani's fixed point theorem.

Prove: there is a player i's r_i (reaction correspondence) maps each strategy profile $\sigma \in \Sigma = \times_i \Sigma_i$ to the set of mixed strategies ($\in \Sigma_i$) that maximize i's payoff when his opponents play σ_{-i}.

$r: \Sigma \to \Sigma$ be the Cartesian Product of r_i.

i.e. $r = (r_1 \times r_2 \times r_3 \times \cdots \times r_n)$.

Hence $\sigma \in r(\sigma) \Rightarrow \sigma_i \in r_i(\sigma)\ \forall i$.

fixed point $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$.

Proof: According to Kakutani's theorem, the sufficient conditions for $r: \Sigma \to \Sigma$ to have a fixed point:

(1) Σ is a compact, convex, nonempty subset of a (finite-dimensional) Euclidean space.

(2) $r(\sigma)$ is nonempty for all σ.

(3) $r(\sigma)$ is convex for all σ.

用 upper hemi-continuity 也行.

(4) $r(\cdot)$ have a closed graph : If a sequence $(\sigma^n, \hat{\sigma}^n) \longrightarrow (\sigma, \hat{\sigma})$ with $\hat{\sigma}^n \in r(\sigma^n)$, then $\hat{\sigma} \in r(\sigma)$.

(3): assume $\exists$ $\sigma' \in r(\sigma)$, $\sigma'' \in r(\sigma)$

$\lambda \sigma' + (1-\lambda) \sigma_i'' \notin r(\sigma)$

$\left.\begin{array}{l} \sigma_i' \in r_i(\sigma) \\ \sigma_i'' \in r_i(\sigma) \end{array}\right\} \Rightarrow \sigma_i'$ and σ_i'' are best responses to σ_{-i}.

$u_i(\sigma_i', \sigma_{-i}) = u_i(\sigma_i'', \sigma_{-i})$

$\Rightarrow u_i(\lambda \sigma' + (1-\lambda)\sigma_i'', \sigma_{-i})$

$= \lambda u_i(\sigma_i', \sigma_{-i}) + (1-\lambda) u_i(\sigma_i'', \sigma_{-i})$

is also best response to σ_{-i}.

$\Rightarrow$ contradiction.

(4) assume $\exists$ sequence $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$

$\hat{\sigma}^n \in r(\sigma^n)$ but $\hat{\sigma} \notin r(\sigma)$.

Thus, there is a $\varepsilon > 0$ and σ' s.t.

$u_i(\sigma_i', \sigma_{-i}) > u_i(\hat{\sigma}_i, \sigma_{-i}) + 3\varepsilon$.

Since u_i is continous and $(\sigma^n, \hat{\sigma}^n) \to (\sigma, \hat{\sigma})$

for n sufficiently large

$u_i(\sigma_i', \sigma_{-i}^n) > u_i(\sigma_i', \sigma_{-i}) - \varepsilon > u_i(\hat{\sigma}_i, \sigma_{-i}) + 2\varepsilon$

$> u_i(\hat{\sigma}_i^n, \sigma_{-i}^n) + \varepsilon.$

which contradicts to $\hat{\sigma}_i^n \in r(\sigma)$