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IE598 GT: Homework 3

Due: April 21, 2021 (5 PM)

Note:

- The due date is strict. Late submissions will not be graded.
- Your answers must be concise and clear. Explain sufficiently so that we can easily determine that you understand.
- Only electronic submissions are accepted on **gradescope**.
- You are expected to do each homework on your own. You may discuss concepts with your classmates, but there must be no interactions about solutions.
- **There are 80 points in total, and you need 60 points for the full homework credit.**
- Plagiarism will be dealt with severely − no credit for the homework.

1. **(10 points)** Prove that in every network cost-sharing game, the PoA is at most k, where k is the number of players.

2. **(10 points)** Suppose a finite cost-minimization game of k players admits a function Φ with the property that, for every outcome $\mathbf{s} = (s_1, \ldots, s_k)$, every player i, and every deviation s_i' with $C_i(s_i', \mathbf{s_{-i}}) < C_i(\mathbf{s})$, $\Phi(s_i', \mathbf{s_{-i}}) < \Phi(\mathbf{s})$. Is this enough to conclude that the game has at least one Pure NE? (Here, $\mathbf{s_{-i}} = (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_k)$ is strategies of all players except i.)

3. **(10 points)** In nonatomic selfish routing, suppose we modify the Pigou's example (where one unit of flow needs to go from s to t) so that the lower edge has the cost function $c(x) = x^d$ for some $d \geq 1$. What is the price of anarchy of the resulting selfish routing network, as a function of d?

4. **(10 points)** Suppose we modify the following example (done in the class) so that all of the network edges are undirected. In other words, each player i can choose a path from s_i to t that traverses each edge in either direction. What is the price of stability in the resulting game?

5. **(10 points)** Exhibit a network cost sharing game in which the minimizer of the potential function (recall that $\Phi(P) = \sum_{e\in E}\sum_{i=1}^{f_e}\frac{\gamma_e}{i}$, where γ_e is the fixed cost of e) is not the lowest cost PNE.

6. **(10 points)** Consider an atomic selfish routing game with affine cost functions. Let $C(f)$ denote the total cost of a flow f and $\Phi(f)$ the value of Rosenthal's potential function for f. Prove that

$$\frac{1}{2}C(f) \le \Phi(f) \le C(f)$$

for every flow f.

7. **(10 points)** Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \ge 0$, then the Pigou bound $\alpha(\mathcal{C})$ in nonatomic selfish routing is 4/3.

8. **(10 points)** Prove that in an atomic selfish routing network of parallel links, every equilibrium flow minimizes the Rosenthal's potential function.

(1) Suppose the worst equilibrium strategies $P = (P_1, P_2, \ldots, P_k)$
worst equilibrium flow $f = \{f_e\}_{e\in E}$
optimal strategies $P^* = (P_1^*, \ldots P_k^*)$.

if i deviation $P_i \to P_i^*$

$$C_i(P_1, \ldots P_i^* \ldots, P_k) = \sum_{e\in P_i^*}\frac{\gamma_e}{f_e^*} = k\cdot\sum_{e\in P_i^*}\frac{\gamma_e}{f_e^*\cdot k}$$

$$\le k\cdot C_i(P_1^*, \ldots P_i^*, \ldots P_k^*)$$

$$\Rightarrow k\cdot C_i(P_1^* \ldots P_i^*, \ldots, P_k^*) \ge C_i(P_1 \ldots P_i^* \ldots P_k)$$

$$\ge C_i(P_1 \ldots P_k).$$

$$\Rightarrow \sum_{i=1}^{k} k\cdot C_i(P_1^* \ldots P_k^*) \ge \sum_{i=1}^{k} C_i(P_1 \ldots P_k)$$

$$\Rightarrow K\cdot \mathrm{Cost}(P_1^* \ldots P_k^*) \ge \mathrm{Cost}(P_1 \ldots P_k)$$

$$\Rightarrow PoA = \frac{\mathrm{Cost}(P_1 \ldots P_k)}{\mathrm{Cost}(P_1^* \ldots P_k^*)} \le K.$$

2. **(10 points)** Suppose a finite cost-minimization game of k players admits a function Φ with the property that, for every outcome $\mathbf{s} = (s_1, \dots, s_k)$, every player i, and every deviation s_i' with $C_i(s_i', \mathbf{s_{-i}}) < C_i(\mathbf{s})$, $\Phi(s_i', \mathbf{s_{-i}}) < \Phi(\mathbf{s})$. Is this enough to conclude that the game has at least one Pure NE? (Here, $\mathbf{s_{-i}} = (s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_k)$ is strategies of all players except i.)

Enough.

2. Assume there is no Pure NE.

i.e. $\forall\ S=(S_1, S_2, \dots, S_k)\ \exists\ S_i'$ s.t. $C_i(S_i', S_{-i}) < C_i(S)$.

Since $C_i(S_i', S_{i-1}) < C_i(S) \Rightarrow \Phi(S_i', S_{-i}) < \Phi(S)$.

$\forall\ S=(S_1, \dots, S_k)\ \exists\ S_i'$ st. $\Phi(S_i', S_{-i}) < \Phi(S)$.

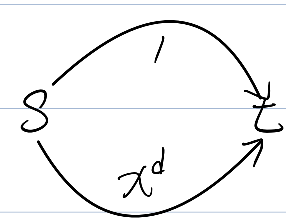
which means Φ doesn't have minimum value.

which contradict to finite game.

Hence there exists at least one Pure NE.

3. **(10 points)** In nonatomic selfish routing, suppose we modify the Pigou's example (where one unit of flow needs to go from s to t) so that the lower edge has the cost function $c(x) = x^d$ for some $d \geq 1$. What is the price of anarchy of the resulting selfish routing network, as a function of d?

3.



Equilibrium: all choose low edge.

Cost = 1.

Optimal: $\min\limits_{x} \{ x \cdot x^d + (1-x) \}$.

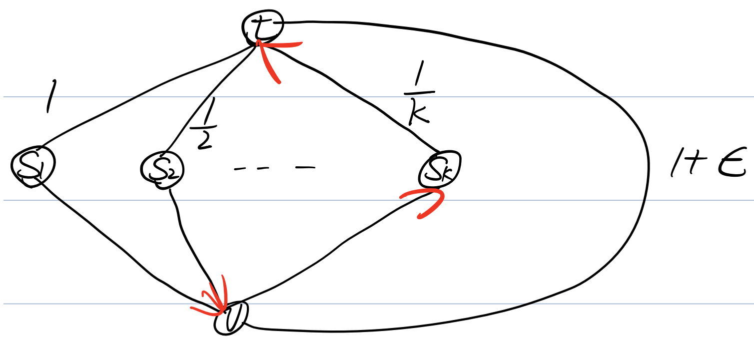
Low edge: $x = \left(\frac{1}{d+1}\right)^{\frac{1}{d}}$

$$\text{Cost} = \left[\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\right]^{d+1} + 1 - \left(\frac{1}{d+1}\right)^{\frac{1}{d}}$$

$$= \left(\frac{1}{d+1}\right)^{\frac{d+1}{d}} + 1 - \left(\frac{1}{d+1}\right)^{\frac{1}{d}}$$

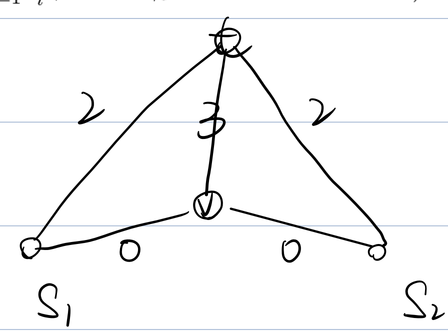
$$PoA = \frac{1}{\left(\frac{1}{d+1}\right)^{\frac{d+1}{d}} + 1 - \left(\frac{1}{d+1}\right)^{\frac{1}{d}}}$$

4. **(10 points)** Suppose we modify the following example (done in the class) so that all of the network edges are undirected. In other words, each player i can choose a path from s_i to t that traverses each edge in either direction. What is the price of stability in the resulting game?



$$PoS = \frac{\frac{1}{k}}{\frac{1}{k}} = 1.$$

5. **(10 points)** Exhibit a network cost sharing game in which the minimizer of the potential function (recall that $\Phi(P) = \sum_{e \in E} \sum_{i=1}^{f_e} \frac{\gamma_e}{i}$, where γ_e is the fixed cost of e) is not the lowest cost PNE.



two players start from S_1, S_2. have same destination t.
they can eith go directly or go together from v.

$\gamma_{s_1 v} = \gamma_{s_2 v} = 0$, $\gamma_{vt} = 3$, $\gamma_{s_1 t} = \gamma_{s_2 t} = 2$.

$P_1 = ((s_1 \to t), (s_2 \to t))$ $\quad \Phi(P_1) = 4$. $\mathrm{Cost}(P_1) = 4$.

$P_2 = ((s_1 \to v \to t), (s_2 \to v \to t))$ $\quad \Phi(P_2) = \frac{9}{2}$, $\mathrm{Cost}(P_2) = 3$.

P_1 is minimizer of Φ, but P_2 has lower cost.

6. **(10 points)** Consider an atomic selfish routing game with affine cost functions. Let $C(f)$ denote the total cost of a flow f and $\Phi(f)$ the value of Rosenthal's potential function for f. Prove that

$$\frac{1}{2}C(f) \le \Phi(f) \le C(f)$$

for every flow f.

$$\Phi(f) = \sum_{e\in E} \sum_{i=1}^{f_e} (ai+b) \le \sum_{e\in E} \sum_{i=1}^{f_e} (af_e+b)$$

$$= \sum_{e\in E} f_e(af_e+b) = C(f).$$

$$\Phi(f) = \sum_{e\in E} \sum_{i=1}^{f_e} (ai+b) = \sum_{e\in E} \frac{(1+f_e)f_e}{2}\cdot a + f_e b$$

$$= \frac{1}{2}\sum_{e\in E} f_e\,(a\cdot(f_e+1)+2b)$$

$$\ge \frac{1}{2}\sum_{e\in E} f_e\,(af_e+b) = \frac{1}{2}C(f)$$

Hence $\frac{1}{2}C(f) \le \Phi(f) \le C(f)$

7. **(10 points)** Prove that if $\mathcal{C}$ is the set of cost functions of the form $c(x) = ax + b$ with $a, b \geq 0$, then the Pigou bound $\alpha(\mathcal{C})$ in nonatomic selfish routing is 4/3.

$$\alpha(C) = \sup_{a,b\geq 0}\ \sup_{r\geq 0}\ \sup_{x\geq 0} \left\{ \frac{r\cdot(ar+b)}{x(ax+b)+(r-x)(ar+b)} \right\}$$

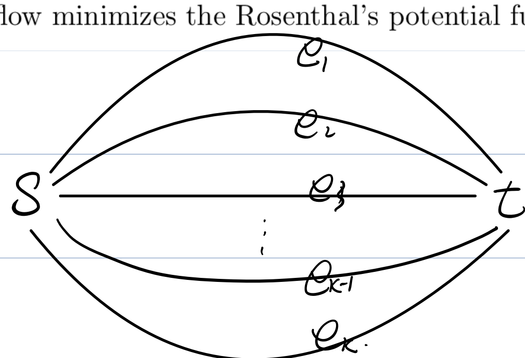
$$\overset{r\neq 0}{=} \sup_{a,b\geq 0}\ \sup_{r>0}\ \sup_{x\geq 0} \left\{ \frac{ar+b}{a(\frac{x^2}{r}+r-x)+b} \right\}.$$

$$\overset{x=\frac{r}{2}}{=} \sup_{a,b\geq 0}\ \sup_{r>0} \left\{ \frac{ar+b}{a\cdot\frac{3}{4}r+b} \right\}$$

$$= \sup_{a,b\geq 0}\ \sup_{r>0} \left\{ \frac{4}{3} - \frac{\frac{1}{3}b}{a\cdot\frac{3}{4}r+b} \right\}.$$

$$\overset{a\neq 0,\, b=0}{=} \frac{4}{3}$$

8. **(10 points)** Prove that in an atomic selfish routing network of parallel links, every equilibrium flow minimizes the Rosenthal's potential function.



$\Phi(f) = \sum_{e\in E} \sum_{i=1}^{f_e} C_e(i)$. Initial strategy: $\mathbb{P}=(P_1, \ldots P_n)$.

Suppose player i deviate to P_i'

$C_i(P_1, \ldots P_i' \ldots P_n) - C_i(P_1, \ldots P_i, \ldots, P_n)$

$$= C_{P_i'}(f'_{P_i'}) - C_{P_i}(f_{P_i})$$

$$= C_{P_i'}(f_{P_i'}+1) - C_{P_i}(f_{P_i})$$

$$\Phi(f') - \Phi(f)$$

$$= \sum_{e\in E}\sum_{i=1}^{f_e'} C_e(i) - \sum_{e\in E}\sum_{i=1}^{f_e} C_e(i).$$

$$= \sum_{i=1}^{f_{P_i'}+1} C_{P_i'}(i) + \sum_{i=1}^{f_{P_i}-1} C_{P_i}(i) + \sum_{e\notin(P_i'\cup P_i)}\sum_{i=1}^{f_e} C_e(i) - \sum_{e\in E}\sum_{i=1}^{f_e} C_e(i)$$

$$= C_{P_i'}(f_{P_i'}+1) - C_{P_i}(f_{P_i})$$

$$= C_i(P_1 \ldots P_i', \ldots P_n) - C_i(P_1 \ldots P_i, \ldots P_n).$$

Obviously a minimizer(s) of Φ f is a equilibrium.

Assume any other equilibrium f'.

Since they are both equilibrium.

$C_{e_j}(f_{e_j}) \leq C_{e_i}(f_{e_i}+1)$. $C_{e_j}(f'_{e_j}) \leq C_{e_i}(f'_{e_i}+1)$. $\forall i \neq j$.

And because it is selfish routing game, the $C_{e_i}(\cdot)$ is nondecreasing.

$$\Phi(f') - \Phi(f) = \sum_{j=1}^{k}\Big(I_{(f'_{e_j} > f_{e_j})} \sum_{i=f_{e_j}+1}^{f'_{e_j}} C_{e_j}(i)$$

$C_e(\cdot)$ is nondecreasing

$$- I_{(f'_{e_j} < f_{e_j})} \sum_{i=f'_{e_j}+1}^{f_{e_j}} C_{e_j}(i)\Big).$$

$$\leq \sum_{j=1}^{k}\Big[I_{(f'_{e_j} > f_{e_j})} (f'_{e_j} - f_{e_j}) \cdot C_{e_j}(f'_{e_j})$$

$$-I_{(f'_{e_j} < f_{e_j})} (f_{e_j} - f'_{e_j}) C_{e_j}(f'_{e_j}+1)].$$

$$\left(\begin{array}{l} \text{Since } C_{e_j}(f'_{e_j}) \le C_{e_i}(f'_{e_i}+1). \ \forall i \ne j. \\ \text{set } C'_{e_0} = \max_i C_{e_i}(f'_{e_i}) \end{array} \right)$$

$$\le \sum_{j=1}^{k} [I_{(f'_{e_j} > f_{e_j})} (f'_{e_j} - f_{e_j}) C_{e_j}(f'_{e_j}) - I_{(f'_{e_j} < f_{e_j})} (f_{e_j} - f'_{e_j}) C'_{e_0}]$$

$$\le \sum_{j=1}^{k} [I_{(f'_{e_j} > f_{e_j})} (f'_{e_j} - f_{e_j}) C'_{e_0} - I_{(f'_{e_j} < f_{e_j})} (f_{e_j} - f'_{e_j}) C'_{e_0}]$$

$$= C'_{e_0} \sum_{j=1}^{k} [I_{(f'_{e_j} > f_{e_j})} (f'_{e_j} - f_{e_j}) - I_{(f'_{e_j} < f_{e_j})} (f_{e_j} - f'_{e_j})]$$

$$\left(\begin{array}{l} \text{Since } \sum_{i=1}^{k} f_{e_i} = \sum_{i=1}^{k} f'_{e_i} = n., \text{ we can know:} \\ \sum_{i=1}^{k} (f'_{e_i} - f_{e_i}) = 0 = \sum_{i=1}^{k} [(f'_{e_i} - f_{e_i}) I_{(f'_{e_j} > f_{e_j})} - (f_{e_i} - f'_{e_i}) I_{(f_{e_i} > f'_{e_i})}] \end{array} \right)$$

$$= C'_{e_0} \cdot 0 = 0.$$

Hence $\Phi(f') \le \Phi(f)$.

Since f is the minimizer of Φ, we can know any other equilibrium f' is also minimizer of Φ. Which means all equilibriums minimizes Φ.