Divide the Dollar Game

- Two player *A* and *B*
- Each player simultaneously demands for their share of 1 dollar
  - If the sum of demands ≤ 1, then both gets their demands
  - Otherwise, (0, 0)

- What are the NE of this game?

$ 100 dollars.

(50, 50). ✓

(25, 75) ✓

$(k, 100-k) \quad \forall k \in [0, 100]$.

Focal Equilibrium

- When there are multiple equilibria, there is one that can be determined depending on environment, culture, etc.

Thomas Schelling:

“It is each person’s expectation of what the other expects him to expect to be expected to do”

Bargaining and Cooperation in Two-Player Game

- This is not to maximize the total welfare
  - Players want to cooperate to maximize their own expected payoffs

- Prisoner's dilemma has very bad NE
- Players can have better equilibria using communication and binding contracts to coordinate their strategies

- There could still be many equilibria

What is the reasonable bargaining solution?

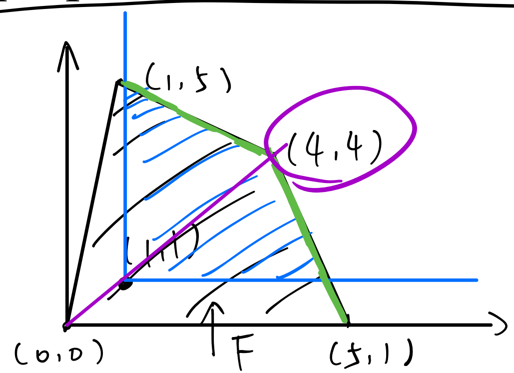
- F: set of all feasible payoffs that players can achieve if they cooperate
- $v = (v_1, v_2)$: disagreement point

- We can assume that F is a closed convex set
- (F, v) defines a two person bargaining problem

- Goal is to find an appropriate solution function $\phi(F, v)$

	C	S
C	(4,4)	(1,5)
S	(5,1)	(0,0)

$v = (1,1)$



What is the reasonable bargaining solution?

- Nash approached this problem axiomatically and generated a list of properties that a reasonable bargaining solution should satisfy:
- 1 Strong efficiency: For any $x \in F$, if $x \geq \phi(F, v)$ then $x = \phi(F, v)$
  - An inefficient outcome leaves space for renegotiation
- 2 Individual Rational (IR): $\phi(F, v) \geq v$
- Scale Invariance: For any $\lambda_1, \lambda_2, r_1, r_2$ such that $\lambda_1, \lambda_2 > 0$,

$G = \{(\lambda_1 x_1 + r_1,\ \lambda_2 x_2 + r_2) \mid (x_1, x_2) \in F\}$ and $w = (\lambda_1 v_1 + r_1, \lambda_2 v_2 + r_2)$ then $\phi(G, w) = (\lambda_1 \phi_1(F, v) + r_1, \lambda_2 \phi_2(F, v) + r_2)$

- Independence of Irrelevant Alternatives: If $G \subseteq F$ is closed convex and $\phi(F, v) \in G$ then $\phi(G, v) = \phi(F, v)$
- 5 Symmetry: If $v_1 = v_2$ and $\{(x_2, x_1) \mid (x_1, x_2) \in F\} = F$ then $\phi_1(F, v) = \phi_2(F, v)$

Nash Bargaining Solution

- There is exactly one function $\phi(F, v)$ that satisfies all axioms

$$\phi(F, v) \in arg \max_{x \in F;\, x \geq v} (x_1 - v_1)(x_2 - v_2)$$

- Always exists
- Unique solution

Examples

What happens when there are more than two players?

- Let's generalize the Nash bargaining solution
  - $(v_1, v_2, \ldots, v_n)$: disagreement point if players didn't cooperate
  - F : set of feasible payoffs that players can get if they cooperate
- (F, v): n-person bargaining problem

- Nash bargaining solution:

$$\phi(F, v) \in arg \max_{x \in F;\, x \geq v} \prod_i (x_i - v_i)$$

Example 1

- Coalition: Any non-empty subset of a set of players
- Grand Coalition: all players

- $N = \{1, 2, 3\}$, \$300 to be divided among 3 players
- Players get 0 unless they propose the same allocation
- Disagreement point $v = ?$
- $F = ?$
- Nash bargaining solution:

Example 2

- $N = \{1, 2, 3\}$, \$300 to be divided among 3 players
- Players get 0 unless player 1 and 2 propose the same allocation
- Disagreement point $v = ?$
- $F = ?$
- Nash bargaining solution:

Example 3

- $N = \{1, 2, 3\}$, \$300 to be divided among 3 players
- Players get 0 unless any two players propose the same allocation
  - What happens here?

Nash bargaining solution for more than two players

- Nash bargaining solution:

$$\phi(F, v) \in arg \max_{x \in F;\, x \geq v} \prod_i (x_i - v_i)$$

- Not widely used for the analysis of cooperative games
- Problem: Completely ignore the possibility of cooperation among subsets of players

Transferable Utility

- A common commodity (e.g., money) that players can freely transfer among themselves
- Assign a number to each coalition
  - $v: 2^N \rightarrow R$
  - $v(S)$ = worth of coalition S (i.e., total amount of transferable utility, S can earn without any help of outside players)
  - $v(\emptyset) = 0$
- v represents a coalition game
- Example: \$300 → all players should propose the same allocation. (A, B, C) to get \$300.

$v(A) = v(B) = v(C) = 0$

$v(\{A, B\}) = v(\{B, C\}) = 0$

$v(\{A, B, C\}) = 300.$

Core

Coalition game.

- An allocation $x = (x_1, x_2, \ldots, x_n)$ is in core of v if

$$\sum_{i \in S} x_i \geq v(S), \qquad \forall S \subseteq N$$

- If an allocation is not in the core then there is some coalition S such that the players in S could all do strictly better than in x by cooperating together and dividing $v(S)$ among themselves
- Core of Example 1: $v(A) = v(B) = v(C) = 0$

$x_1 \geq 0 \quad x_1 + x_3 \geq 0 \quad v(AB) = v(BC) = v(CA) = 0.$

$x_2 \geq 0 \quad x_1 + x_2 + x_3 = 300. \quad v(ABC) = 300.$

$x_3 \geq 0.$

$x_1 + x_2 \geq 0$

$x_2 + x_3 \geq 0$

Example 2:

$v(A) = v(B) = v(C) = 0$

$v(BC) = v(CA) = 0.$

$v(ABC) = 300 = v(AB)$

$x_1 \geq 0$ $\quad x_1 + x_3 \geq 0$

$x_2 \geq 0$ $\quad x_1 + x_2 + x_3 = 300.$ $\Rightarrow$ any allocation that divide 300 among A and B is in the core!

$x_3 \geq 0.$

$x_1 + x_2 \geq 300$

$x_2 + x_3 \geq 0$

$(200, 100, 0).$

Core

- An allocation $x = (x_1, x_2, \dots, x_n)$ is in core of v if

$$\sum_{i \in S} x_i \geq v(S), \qquad \forall S \subseteq N$$

- If an allocation is not in the core then there is some coalition S such that the players in S could all do strictly better than in x by cooperating together and dividing $v(S)$ among themselves

- Problem

Core

- An allocation $x = (x_1, x_2, \ldots, x_n)$ is in core of v if
$$\sum_{i \in S} x_i \geq v(S), \qquad \forall S \subseteq N$$

- If an allocation is not in the core then there is some coalition S such that the players in S could all do strictly better than in x by cooperating together and dividing $v(S)$ among themselves

- Problem: empty or very large
  - makes it difficult as a predictive theory

What we want?

- We would like to identify a mapping $\psi(N, v) \to R^n$, where $\psi_i(N, v)$ is the payoff of player i
- $\boxed{\text{Shapley}}$ approached the problem axiomatically, i.e., what kind of properties we want the solution ψ to satisfy
- Symmetry: If i and j always contribute the same amount to each coalition of other agents (i.e., $\forall S$ such that S doesn't contain i and j, $v(S \cup \{i\}) = v(S \cup \{j\})$) then $\psi_i(N, v) = \psi_j(N, v)$
- Dummy Player: If the amount i contribute to any coalition is exactly that i is able to achieve alone (i.e., $\forall S, i \notin S, v(S \cup \{i\}) - v(S) = v(\{i\})$, then $\psi_i(N, v) = v(\{i\})$
- Additivity: Two coalition games v_1, v_2 involve the same set of players. Suppose we remodel the setting as a single game in which each coalition S receives payoff of $v_1(S) + v_2(S)$, then $\psi(N, v_1 + v_2) = \psi(N, v_1) + \psi(N, v_2)$

Shapley Value

- There is a unique function ψ that satisfy all the axioms

$$\psi_i(N,v) := \sum_{S \subseteq N \setminus \{i\}} \underbrace{\frac{|S|!\,(|N|-|S|-1)!}{|N|!}}_{\text{排列的概率}} \underbrace{(v(S \cup \{i\}) - v(S))}_{\text{有 i 与无 i 的效用区别}}$$

排列的概率

有 i 与无 i 的效用区别. marginal utility.

$|N|-|S|-1$ i S

$\triangle \cdots \triangle$ $\triangle$ $\triangle \cdots \triangle$

$(|N|-|S|-1)!$ $|S|!$

Example1:

$\psi_A(N,v)$.

$= \frac{1}{6} \times 300 + \frac{1}{6} \times 300$

$= 100.$

$\frac{1}{6}$: A B C → 300

$\frac{1}{6}$: B A C → 0

$\frac{1}{6}$: A C B → 300

$\frac{1}{6}$: B C A → 0

$\frac{1}{6}$: C A B → 0

$\frac{1}{6}$: C B A → 0.

Shapley Value

- There is a unique function ψ that satisfy all the axioms

$$\psi_i(N, v) \coloneqq \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N|-|S|-1)!}{|N|!} (v(S \cup \{i\}) - v(S))$$

- It is a powerful tool to evaluate the power structure in a coalitional game

Examples

Voting Game

- A parliament of a country ABCD is made of 4 political parties with 45, 25, 15, 15 representatives
- They want to vote whether to pass a $100M spending bill and how much of this amount should be controlled by each party
- A majority vote (minimum of 51) is required to pass the legislation
- If the bill doesn't pass, each will get 0 to spend

- Winning coalition?
- Core?
- Shapley Value?