Fisher Market Model

- A set N of n agents (buyers), a set M of m divisible items (goods)
- Each buyer i has
  - budget of B_i dollars
  - utility function $v_i: R_+^m \to R_+$ over bundle of goods

- Given prices $p = (p_1, \ldots, p_m)$ of goods, each buyer wants to purchase an optimal bundle that maximizes her utility function
- p is called competitive equilibrium if market clears, i.e., demand of each good meets its supply

Linear Utilities

- Let v_{ij} is the utility of buyer i for one unit of good j
- Utility of buyer i for a bundle $x_i = (x_{i1}, x_{i2}, \dots, x_{im})$ of goods is given by

$$u_i(x_i) := \sum_j v_{ij} x_{ij}$$

Maximum bang-per-buck (MBB) Goods

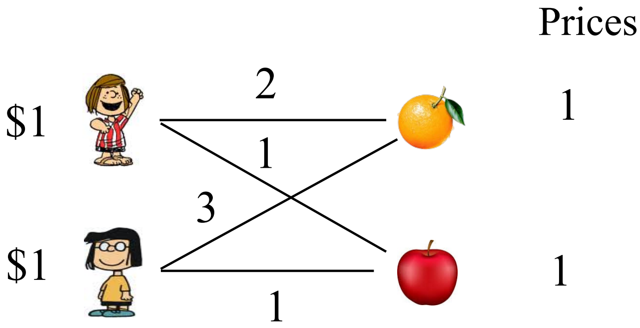
- At prices p, a good j is called MBB for buyer i if

$$\frac{v_{ij}}{p_j} = \max_k \frac{v_{ik}}{p_k}$$

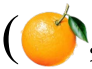
- Each buyer wants only her MBB goods

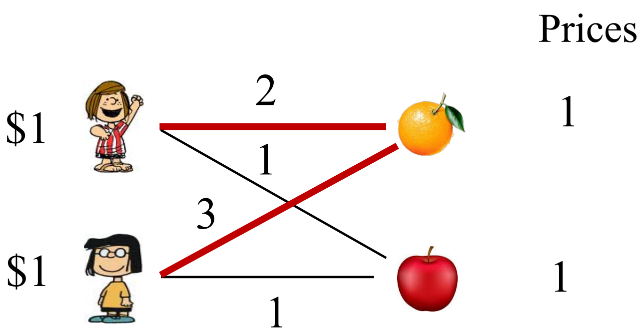
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function



Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function

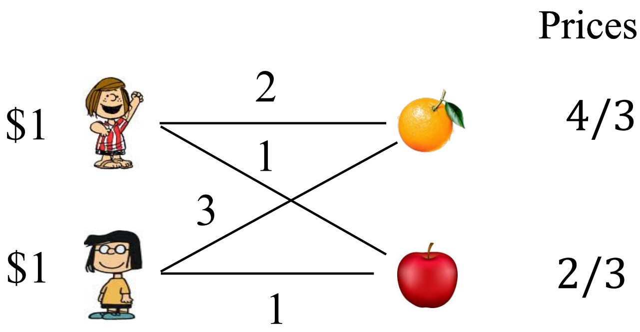


Demand ≠ Supply

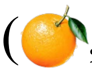
Not an Equilibrium!

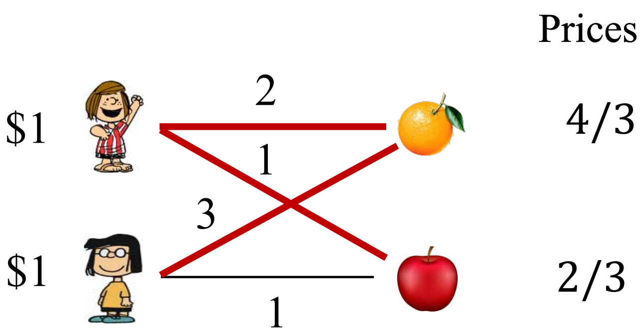
Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of $1 and a linear utility function



Example

- 2 Buyers (, ), 2 Items (, ) with unit supply
- Each buyer has budget of \$1 and a linear utility function



Demand = Supply

Equilibrium!

Existence of Competitive Equilibrium

"including linear"

- Equilibrium exists under a very general class of utility functions
- For linear utilities, the following Eisenberg-Gale convex program gives an equilibrium: always exists.

Budget tie

$$\max \sum_{i \in N} B_i \log \sum_{j \in M} v_{ij} x_{ij}$$

utility.

$$\sum_{i \in N} x_{ij} \leq 1, \qquad \forall j \in M$$

$$x_{ij} \geq 0, \qquad \forall i \in N, j \in M$$

Efficiency: Pareto optimality

- An allocation $y = (y_1, y_2, \dots, y_n)$ Pareto dominates another allocation $x = (x_1, x_2, \dots, x_n)$ if
  - $u_i(y_i) \geq u_i(x_i)$, for all buyers i and
  - $u_k(y_k) > u_k(x_k)$ for some buyer k

Efficiency: Pareto optimality

- An allocation $y = (y_1, y_2, \ldots, y_n)$ Pareto dominates another allocation $x = (x_1, x_2, \ldots, x_n)$ if
  - $u_i(y_i) \geq u_i(x_i)$, for all buyers i and
  - $u_k(y_k) > u_k(x_k)$ for some buyer k

- x is said to be Pareto optimal (PO) if there is no y that Pareto dominates it

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof:

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof: (by contradiction)

- Let (p, x) be equilibrium prices and allocations
- Suppose y Pareto dominates x

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof: (by contradiction)

- Let (p, x) be equilibrium prices and allocations
- Suppose y Pareto dominates x
- $\sum_j p_j \; = ?$

First Welfare Theorem

Theorem: Competitive equilibrium outputs a PO allocation

Proof: (by contradiction)

- Let (p, x) be equilibrium prices and allocations
- Suppose y Pareto dominates x
- $\sum_j p_j \underbrace{\sum_i x_{ij}}_{=1} = ? \sum_i B_i$
- How much money i needs to purchase y_i?

$$\sum_j \left(p_j \sum_i y_{ij}\right) > \sum_i B_i = \sum_j p_j$$

$$\Rightarrow \sum_i y_{ij} > 1$$

Proof: If $\exists\, y$ s.t. $u_i(y_i) \geq u_i(x_i),\ \forall i \in N.$
$u_k(y_k) > u_k(y_k),\ \exists\, k \in N$

then the money i need to purchase $y_i \geq B_i$.

the money k need to purchase $y_k > B_k$.

$$\sum_j \underbrace{\left(p_j \sum_i y_{ij}\right)}_{\text{money}} > \sum_j B_j = \sum_j p_j \geq \sum_j p_j \underbrace{\left(\sum_i y_{ij}\right)}_{\leq 1}$$

according to constraint.

Contradiction.

Competitive Equilibrium with Equal Incomes

- Each buyer has 1 dollar

Theorem: Competitive equilibrium outputs an envy-free allocation

Proof: Assume x is not envy-free allocation.

i.e. $\exists\, k \in N, i \in N$ s.t. $u_k(x_k) < u_i(x_i)$.

$\Rightarrow$ x_k is not an optimal bundle.

$\Rightarrow$ x is not an equilibrium allocation.

Competitive Equilibrium with Equal Incomes

- Each buyer has 1 dollar

Theorem: Competitive equilibrium outputs an envy-free allocation

Proof:

- Let (p, x) be equilibrium prices and allocations
- Did buyer i has an option the bundle x_j of buyer j, for any j?