Inefficiency of Equilibria

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  - What is the meaning of optimal solution?
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- Choice of objective function and choice of equilibrium concept

How to measure the quality of solution?

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- Egalitarian: maximizes the minimum payoff (max min)

Why we want to measure inefficiency of equilibria?

The answer to this may help

- We may take some measure to improve it
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- How to quantify the selfish behavior (equilibrium)?
  - When there are multiple equilibria, which one to choose?

Price of Anarchy (PoA)

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$$\text{PoA} = \frac{\text{worst equilibrium cost}}{\text{optimal cost}} \geq 1 \quad \text{(cost)}$$

Keep it always greater than or equal to 1.

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S C

$$\begin{array}{c} \\ S \\ C \end{array}\begin{bmatrix} 0,0 & 5,1 \\ 1,5 & 2,2 \end{bmatrix}$$

- If PoA is close to 1, then ?

$NE = \{(C,S),(S,C),((\frac{3}{4}S,\frac{1}{4}C),(\frac{3}{4}S,\frac{1}{4}C))\}$

(Utilitarian)

$$\text{PoA} = \frac{6}{\frac{3}{\cancel{16}\,4}\times\cancel{6}\,3\times\cancel{2}+\frac{1}{16}\times\cancel{4}} = \frac{12}{5}$$

Price of Stability (PoS)

$PoS = \frac{6}{6} = 1$

- PoS is defined as the ratio between the value of optimal solution and the value of the best equilibrium solution

$$PoS = \frac{\text{value of optimal solution}}{\text{value of best equilibrium}} \geq 1 \quad \text{(payoff/utility)}$$

$$PoS = \frac{\text{best equilibrium cost}}{\text{optimal cost}} \geq 1 \qquad \text{(cost)}$$

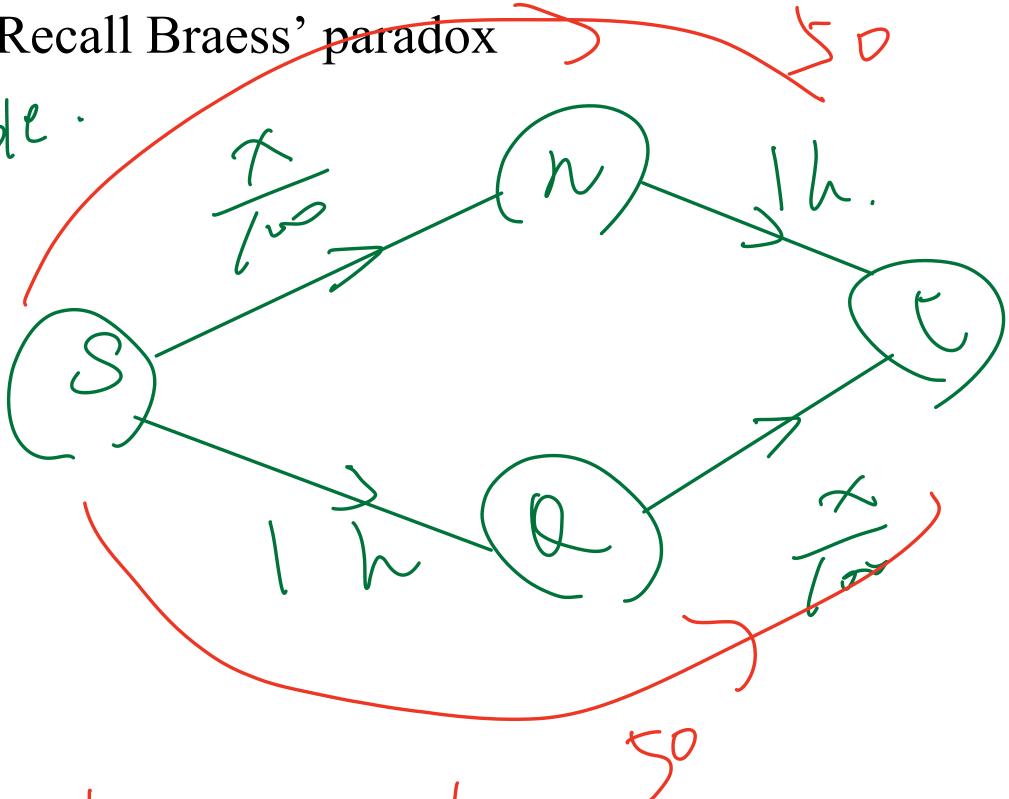
- PoS quantifies the necessary degradation of solution quality due to imposition of the game theoretic constraint of stability
- $1 \leq PoS \leq PoA$

Example

Selfish Routing

- Applications: road traffic, communication networks, etc.
- Recall Braess' paradox

100 people.



1.5 hours each.

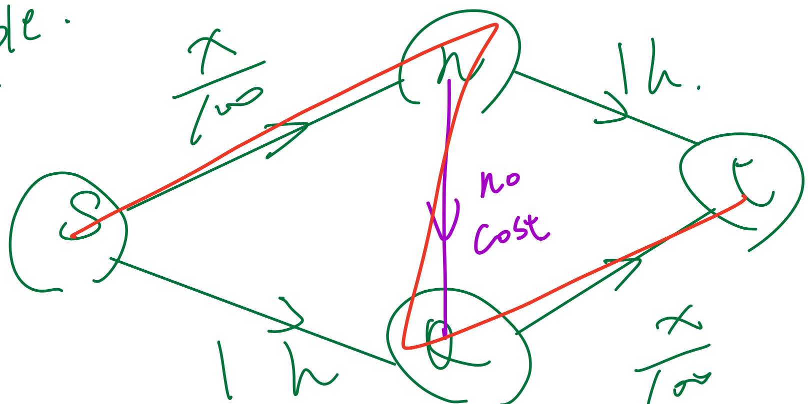
Selfish Routing

- Applications: road traffic, communication networks, etc.
- Recall Braess' paradox
  - Atomic setting ← each player is not negligible
  - Non-atomic setting (each individual has negligible size)

Millions of people.

$PoA = PoS = \frac{2}{1.5} = \frac{4}{3}$

100 people.



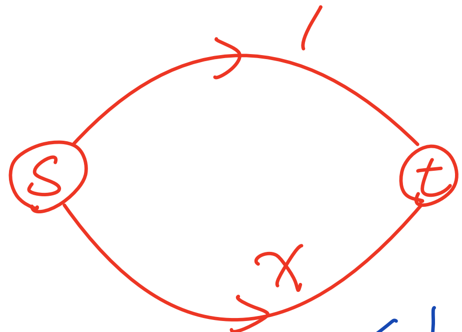
$\frac{x}{100} \le 1 \Rightarrow$ 2 hours each.

Pigou's Example (1920)

Non-atomic setting. (total 1 unit flow).

- A simple selfish routing network
  - Two paths from *s* to *t* with delay (cost) of 1 and *x*

PoA = PoS = $\frac{4}{3}$



$x \le 1$.

Equilibrium solution: $\Rightarrow$ All choose "x". $\Rightarrow x = 1$.

Optimal solution: $\min_x \{x^2 + (1-x)\}$

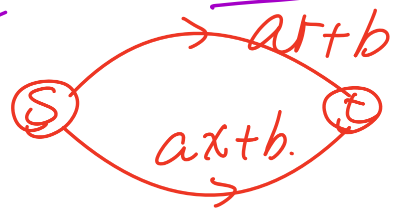
$\Rightarrow x = \frac{1}{2}$

half choose "1", half choose "x".

Pigou's Example (1920)

- A simple selfish routing network
  - What happens when the paths have affine cost function $ax + b$ where $a, b \geq 0$ and r units of flow need to go from s to t

r units



$$PoA = PoS = \frac{ar^2 + br}{\frac{3}{4}ar^2 + br} \leq \frac{4}{3}$$

Equilibrium sol: $ax+b \leq ar+b \Rightarrow$ all choose "$ax+b$"

$\Rightarrow x = r$.

optimal sol: $\min_{x} \{(ar+b)(1-x) + x(ax+b)\}$.

$\Rightarrow x = \frac{r}{2} \Rightarrow$ half "$ar+b$"

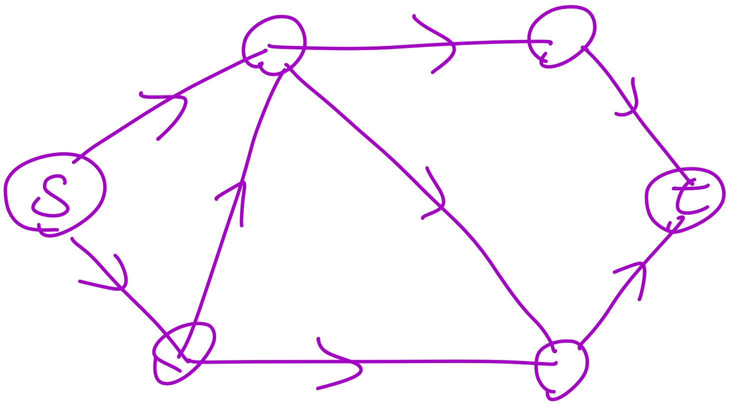
half "$ax+b$".

PoA of General Network

- Let $G = (V, E)$ be network and we have affine cost function on edges, i.e., $c_e(x) = a_e x + b_e$
- Let $f = \{f_e\}_{e \in E}$ be an equilibrium flow

$\sum_{e \in E} f$ = total flow

1 or r in former examples.



Cost of equilibrium: $C(f) = \sum_{e \in E} (a_e f_e + b_e) f_e$.

Cost of optimal: $C(f^*)$.

PoA of General Network

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- Let $C(f)$ be the cost of f
- $C(f) =$?

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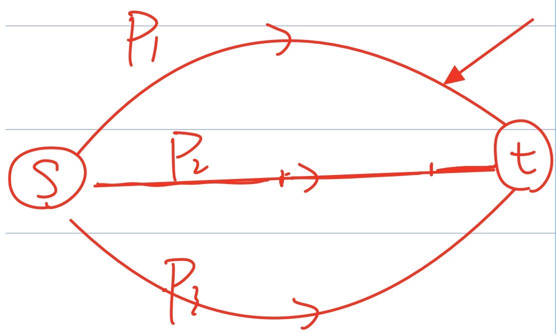
- Let $f = \{f_e\}_{e \in E}$ be an equilibrium flow
- Let $C(f)$ be the cost of f
- $C(f) =$?

- Let $f^* = \{f_e^*\}_{e \in E}$ be an optimal solution

- PoA = ? $\dfrac{C(f)}{C(f^*)} \leq \dfrac{4}{3}$

Equilibrium: $f = \{f_e\}_{e \in E}$

total cost of each path.



$$C_{P_1}(f) = \sum_{e \in P_1} (\underline{a_e f_e + b_e})$$

$a_e f_e + b_e$ 可以表示任一段的 cost.

$$C_{P_2}(f) = \sum_{e \in P_2} (a_e f_e + b_e)$$

$$C_{P_3}(f) = \sum_{e \in P_3} (a_e f_e + b_e)$$

① if there are positive amount of flow both on P_1 and P_2, then $C_{P_1}(f) = C_{P_2}(f)$

Otherwise one would switch from its path to the path with the lowest cost until they are same.

② if positive flow on P_1, zero flow on P_2. then $C_{P_1}(f) \leq C_{P_2}(f)$.

Otherwise one would choose P_2 instead of P_1.

$$PoA = \frac{C(f)}{C(f^*)}.$$

Prove PoA $\le \frac{4}{3}$

Proof: Set Equilibrium flow $f = \{f_e\}_{e\in E}$

Let $X = \{x_e\}_{e\in E}$ be an arbitrary flow.

$$C^f(x) = \sum_{e\in E} (a_e f_e + b_e) x_e$$

$$= \sum (a_e f_e x_e + b_e x_e).$$

$$f_e x_e - x_e^2 \le \frac{1}{4} f_e^2$$

$$\le \sum (a_e x_e^2 + b_e x_e) + \sum \frac{a_e}{4} f_e^2$$

$$\le C(x) + \frac{1}{4} C(f)$$

In equilibrium, the flow only lies on smallest cost paths, hence $C^f(x) \begin{cases} = C^f(f) & \text{still only lies on smallest cost paths} \\ > C^f(f) & \text{some flow lie on other paths.} \end{cases}$

Hence,

$C^f(x) \ge C^f(f) \quad \forall x.$

$$\Longrightarrow C(f^*) + \frac{1}{4} C(f) \ge C^f(f^*) \ge C^f(f) = C(f)$$

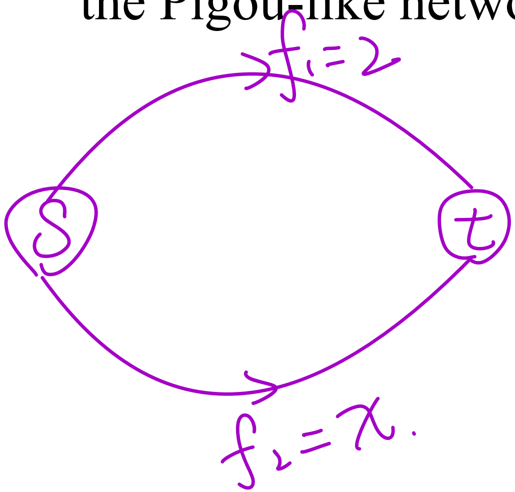
$$\Longrightarrow C(f^*) \ge \frac{3}{4} C(f)$$

$$\Longrightarrow \text{PoA} \le \frac{4}{3}.$$

Atomic Setting

- Few number of players, so the size is not negligible

- Example: Two players want to send 1 unit of flow from *s* to *t* in the Pigou-like network



$$\begin{array}{lccc} & x_1 & x_2 & \text{cost.} \\ \text{Equilibrium} \left\{ \begin{array}{l} \\ \\ \end{array} \right. & (\,0 & ,\ 2\,) & 4 \\ & (\,1 & ,\ 1\,). & 3 \end{array}$$

$$PoA = \frac{\text{worst equilibrium}}{\text{optimal}} = \frac{4}{3}$$

$$PoS = \frac{\text{best equilibrium}}{\text{optimal}} = 1.$$

Atomic Setting

- Few number of players, so the size is not negligible
- Example: Two players want to send 1 unit of flow from s to t in the Pigou-like network
- Equilibrium?

  ① the cost of every player can be different.

  ②. can have multiple equilibriums

- Difference with non-atomic setting?

Atomic Setting

- Few number of players, so the size is not negligible
- Example: Two players want to send 1 unit of flow from s to t in the Pigou-like network
- Equilibrium?

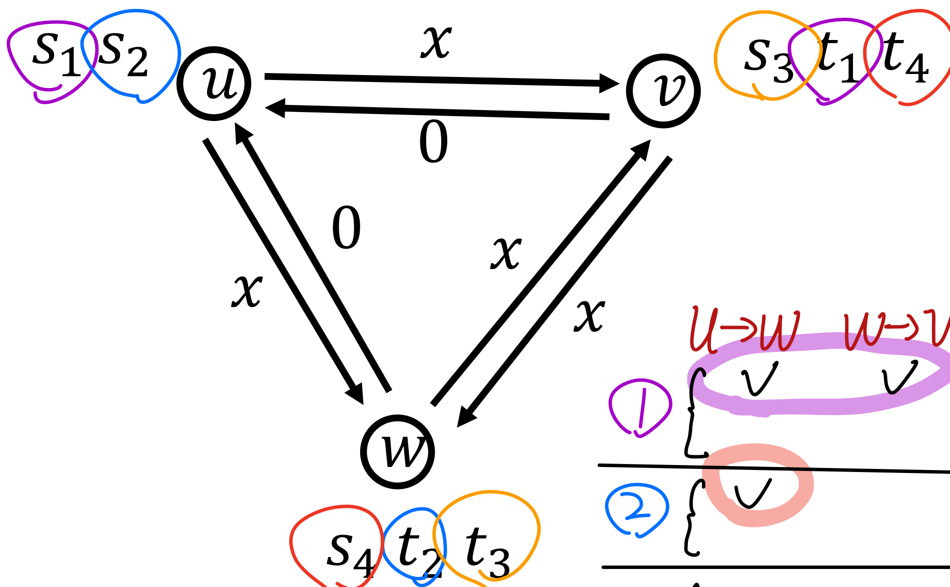
Atomic Setting

- General problem: k players, each player i wants to send 1 unit of flow from s_i to t_i on a single path in a network G

different players have different starts and terminals.

PoA

- PoA can be larger than 4/3



	$u \to w$	$w \to v$	$v \to w$	$u \to v$
①	✓	✓		✓
②	✓		✓	✓
③	✓		✓	
④		✓		✓

- PoA = 5/2

optimal :

worst equilibrium :

$$\text{PoA} = \frac{10}{4} = \frac{5}{2}.$$