General Two-Player Games

- Given a game (A, B), the objective is to find
  - one Nash equilibrium
  - all Nash equilibrium
- Examples Coordinate game.

$$
\begin{array}{c|cc}
 & C & NC \\
\hline
C & 5,5 & 0,1 \\
NC & 1,0 & 1,1
\end{array}
$$

all NE : $\underbrace{(C, C), (NC, NC)}_{\text{two pure equilibrium}}$.

mixed strategy:

$\left(\left(\frac{1}{5}C, \frac{4}{5}NC\right), \left(\frac{1}{5}C, \frac{4}{5}NC\right)\right)$. NE.

No pure equilibrium for R-P-S game.

Simplifications

The set of Nash equilibria doesn't change if we

- multiply A and/or B by a positive constant

Simplifications

The set of Nash equilibria doesn't change if we

- multiply A and/or B by a positive constant
- multiply A and B by two different positive constants

If (x, y) is a NE of (A, B) then $g(x, y)$ is also a NE of $g(\alpha A, \beta B)$. $\alpha, \beta > 0$.

Proof: Defination of NE (x, y)

$\Rightarrow$ x is a best response to y. y is a best response to x. i.e.

$$x^T A y \geq \tilde{x}^T A y. \ \forall \tilde{x} \in ls.$$
$$x^T B y \geq x^T B \tilde{y} \quad \forall \tilde{y} \in ls.$$

$$\Rightarrow \alpha(x^T A y) \geq \alpha(\tilde{x}^T A y) \Rightarrow x^T(\alpha A) y \geq \tilde{x}^T(\alpha A) y$$
$$\beta(x^T B y) \geq \beta(x^T B \tilde{y}) \Rightarrow x^T(\beta B) y \geq \tilde{x}^T(\beta B) y.$$

Simplifications

The set of Nash equilibria doesn't change if we

- add a constant to every entry of *A* and/or *B*

$$x^T A y \geq \tilde{x}^T A y. \ \forall \tilde{x} \in LS_1$$
$$x^T B y \geq x^T B \tilde{y} \quad \forall \tilde{y} \in LS_2$$

$$\alpha = \begin{bmatrix} a & \cdots & a \\ \vdots & & \vdots \\ a & \cdots & a \end{bmatrix}_{m \times n}.$$

$$x = [x_1, x_2, \dots x_m]^T \quad \sum_{i=1}^{m} x_i = 1.$$
$$y = [y_1, y_2, \dots y_n]^T \quad \sum_{i=1}^{n} y_i = 1.$$

$$\Rightarrow \quad x^T \alpha = [a, a \dots a]$$
$$(x^T \alpha) y = a. \quad \forall x, y.$$

$$\Rightarrow \begin{cases} x^T A y + a \geq \tilde{x}^T A y + a \\ x^T B y + a \geq x^T A \tilde{y} + a \end{cases} \Leftrightarrow \begin{array}{l} x^T (A+\alpha) y \geq \tilde{x}^T (A+\alpha) y. \\ x^T (B+\alpha) y \geq x^T (A+\alpha) \tilde{y} \end{array}$$

Simplifications

The set of Nash equilibria doesn't change if we

- add a constant to every entry of A and/or B
- add a constant to a row of $\cancel{A}^{B}$ $B' \leftarrow B + \alpha$.

$$\alpha = \begin{bmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ a & \cdots & a & \cdots & a \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & \cdots & 0 & \cdots & 0 \end{bmatrix}_{m \times n}$$

Constant to a row of B.

$x^T B y \geq x^T B \tilde{y} \quad \forall \tilde{y} \in ls_2$

$\Rightarrow x^T B y + a x_i \geq x^T B \tilde{y} + a x_i$

$\Rightarrow x^T (B+\alpha) y \geq x^T (B+\alpha) \tilde{y}. \quad \forall \tilde{y} \in ls_2.$

$\alpha y = [0, \ldots, a, \ldots, 0]^T$

$x^T(\alpha y) = a x_i \quad \forall x, y.$

Constant to a column of A.

$$\beta = \begin{bmatrix} 0 & \cdots & b & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & b & 0 & & 0 \\ \vdots & & \vdots & \vdots & & \vdots \\ 0 & & b & 0 & & 0 \end{bmatrix}$$

$x^T A y \geq \tilde{x}^T A y \quad \forall \tilde{x} \in ls_1$

$\Rightarrow x^T A y + b y_j \geq \tilde{x}^T A y + b y_j$

$\Rightarrow x^T (A+\beta) y \geq \tilde{x}^T (A+\beta) y. \quad \forall \tilde{x} \in ls.$

$A' \leftarrow A + \beta$

$x^T \beta = [0 \cdots b \; 0 \cdots 0]$

$(x^T \beta) y = b y_j \quad \forall x, y$

Simplifications

The set of Nash equilibria doesn't change if we

- add a constant to every entry of A and/or B
- add a constant to a row of $\cancel{A}$ B
- add a constant to a column of $\cancel{B}$$A$

Observe that

- We can add different constants to different rows of $\cancel{A}$ B and different columns of $\cancel{B}$$A$
- We can assume without loss of generality that

$$0 \le A_{ij}, B_{ij} \le 1, \ \forall i, j$$

- Constant sum games are equivalent to zero-sum games

Support Enumeration Algorithm

- Support of a vector x is defined as: $\{i : x_i > 0\}$
- Example:

$x = \{0.5, 0, 0.3, 0.2, 0\}$

$supp(x) = \{1, 3, 5\}.$

$y = \{1, 0, 0\}.$

$supp(y) = \{1\}$

Support Enumeration Algorithm

- Support of a vector x is defined as: $\{i : x_i > 0\}$

- Suppose we know the support of a Nash equilibrium (x, y), then how to find (x, y)?

Given y, the payoff of player 1 from i^{th} strategies $= A_i y$.

Given x, the payoff of player 2 from j^{th} strategies $= x^T \cdot B^j$

$$\Rightarrow \begin{cases} A_i y \geq A_{i'} y & \forall i \in Supp(x),\ i' \notin Supp(x). \\ A_i y = A_{i'} y & \forall i \neq i' \in Supp(x). \end{cases}$$

$$\begin{cases} x^T \cdot B^j \geq x^T \cdot B^{j'} & \forall j \in Supp(y),\ j' \notin Supp(y). \\ x^T \cdot B^j = x^T \cdot B^{j'} & \forall j \neq j' \in Supp(y). \end{cases}$$

$$\sum_i x_i = 1, \quad x_i = 0 \ \forall i \notin Supp(x) \quad x_i \geq 0 \ \forall i.$$

$$\sum_j y_j = 1, \quad y_j = 0 \ \forall j \notin Supp(y) \quad y_j \geq 0 \ \forall j.$$

↳ linear feasible programming

find feasible (x, y). (if exists).

the (x, y) will be NE.

(at least one because NE always exists.)

Support Enumeration Algorithm

1. ■ Enumerate over all support pairs

for each subset of $\underbrace{\{1, \ldots, m\}}^{u}$. $\quad m \left[\overset{n}{A} \right]_{m \times n}$.

for each subset of $\overset{v}{\{1, \ldots, n\}}$.

let $u = supp(x)$, $v = supp(y)$.

write the linear feasible program.

if there is a feasible (x, y)

then output that!

Support Enumeration Algorithm

- Enumerate over all support pairs

(2) - Running time of the algorithm?

#Subset of $\{1, 2, \ldots, m\} = 2^m$.

#Subset of $\{1, 2, \ldots, n\} = 2^n$

$\leq 2^m \cdot 2^n \cdot (\text{Time to solve an LP})$.

$= 2^{m+n} \cdot \underline{(\text{Time to solve an LP})} \to$ polytime.

$\Rightarrow$ It will give us all the NE of the game!

Examples

Dominant and Dominated Strategies

Dominant and Dominated Strategies

- A pure strategy $i \in S_1$ is strictly dominated in game (A, B) for player 1 if there exists another pure strategy $i' \in S_1$ such that

$$A_{i'j} > A_{ij}, \quad \forall j \in S_2$$

  - We say that i' strictly dominates i

- We can iteratively remove the strictly dominated strategies for both players from the game

Examples

2

1, 3	4, 1
0, 2	3, 4

1

Allowing Mixed Strategies

- A pure strategy $i \in S_1$ is strictly dominated in game (A, B) for player 1 if there exists a mixed strategy $x \in \Delta(S_1)$ such that

$$x^T A^j > A_{ij}, \quad \forall j \in S_2$$

  - We say that x strictly dominates i

- We can iteratively remove the strictly dominated strategies for both players from the game

Examples

10, 1	0, 4
4, 2	4, 3
0, 5	10, 2

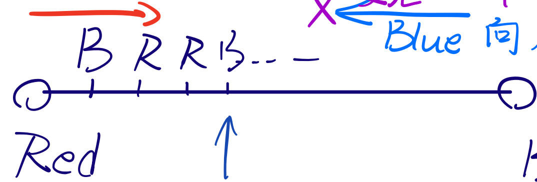
Sperner's Lemma

Red 向右漫延

交汇 one point.

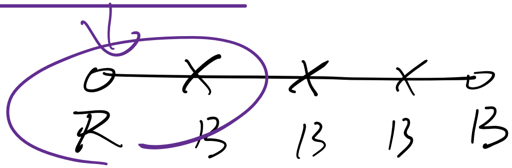
Blue 向左漫延

1 dimension.



each of them, you have a choice to put a color: R/B.

Claim: No matter how you color the intermediate points, there is always a subinterval with both the colors.



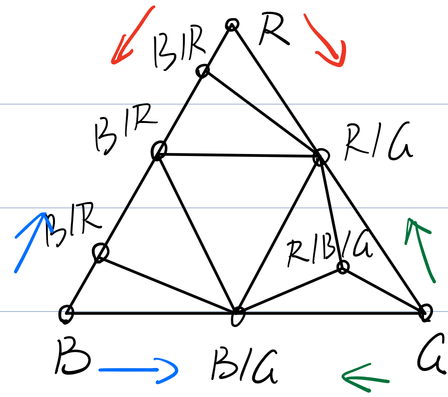
Proof: begin with Red, finally you will get a Blue no matter how.

How many sub-interval with two different color?

odd number.

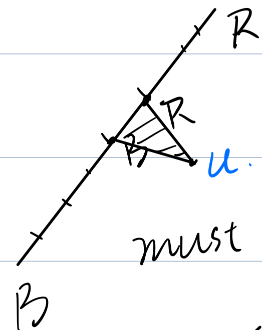
2 dimension

Claim: there will be always a Small - △ with all three colors.



odd number

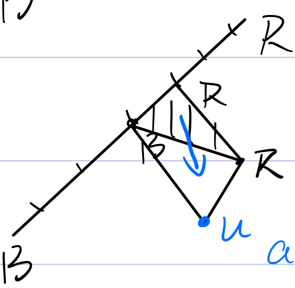
Proof:



must exist BR

if u is G, we are done.

otherwise we will have a new BR.



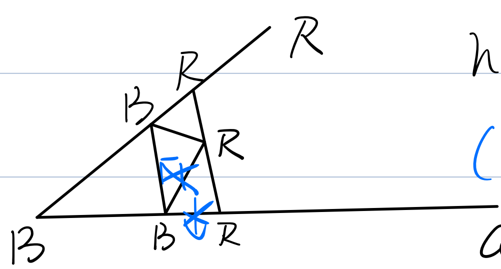
finally, it will move to RG or BG.

and there must have G in RG or BG.

How could it stop?

①. find a △ have three colors. (don't have new BR).

②

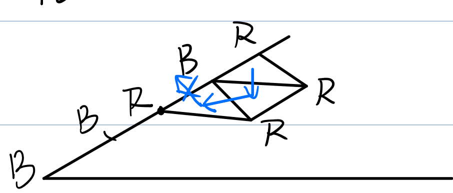


have BR on other edges.

(new BR is on other edge).

(impossible).

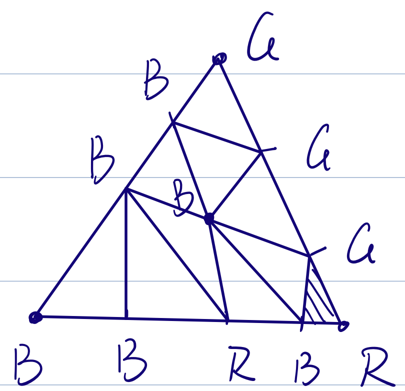
③

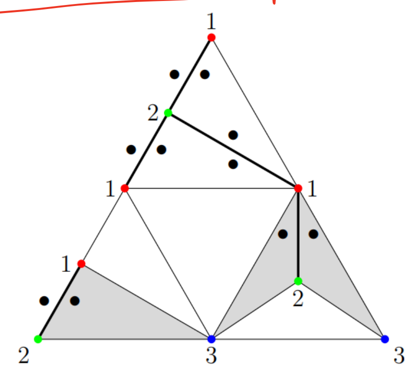


move back to the edge

(we can start again)

Proof of odd number (and exists). (by Moor Xu.)





put a dot on each side of each 12-segment.

① interior segment contributes 0 or 2 dots.

② boundary segment contributes 0 or 1 dots.

number of dots in triangle mod 2 =

number of 1-2 segments on boundary mod 2.

Since # 1-2 segments on boundary must be odd,

$\Longrightarrow$ the # dots in triangle must be odd.

Complete triangle (three colors) only contains 1 dot.

others contain 0 or 2 dots.

$\Longrightarrow$ the number of complete triangle must be odd.

PPAD

In the n-dimension Sperner's Lemma.
there are odd number of panchromatic simplices.

(1-dim $\longrightarrow$ 2-dim $\longrightarrow$ $\cdots$ $\longrightarrow$ n dim).

Linear Complementarity Problem (LCP) formulation.

$$\min_{y} \; y^T(q - My)$$
$$\text{s.t.} \quad q - My \geq 0$$
$$y \geq 0$$

$$\begin{array}{c} \leftarrow n \rightarrow \\ \uparrow n \downarrow \begin{bmatrix} M \end{bmatrix} \end{array} \begin{bmatrix} y \end{bmatrix} \leq \begin{bmatrix} q \end{bmatrix}$$

Given M, q, the problem is
to find y, s.t.

$\underbrace{My \leq q}_{\text{polyhedral}}$ and $\underbrace{y^T(My-q) = 0}_{\text{Complementarity constraints}}$ $\forall i$. $\quad y \geq 0$

LCP formulation for the NE (A, B).

$$\underset{x,y}{\text{minimize}} \quad [x^T y^T] \begin{bmatrix} 1 - Ay \\ 1 - xB^T \end{bmatrix}$$

选择策略使每次都可获得最大回报

与最大回报间的差.

Assume $0 \le A_{ij}, B_{ij} \le 1$

$$\begin{cases} Ay \le 1 \; ; \; x_i \geq 0 \; ; \; x_i(A_i y - 1) = 0, \; \forall i \\ x^T B \le 1 \; ; \; y_j \geq 0 \; ; \; y_j(x^T B^j - 1) = 0, \; \forall j. \end{cases}$$

最大Return

player 1 choose i is best or won't choose.

$$\begin{bmatrix} A_{n\times n} & 0 \\ 0 & B^T_{n\times n} \end{bmatrix} \begin{bmatrix} y_{n\times 1} \\ x_{n\times 1} \end{bmatrix} \le \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$$

Claim: ① All NE of game (A, B) are solutions of LCP

and ② every solutions of LCP is a NE of g(A, B).

PPAD

- Sperner's Lemma is PPAD-complete

Lemke-Howson Algorithm

- Simplex Algorithm for Linear Programming

this path never revisit the same vertex.

this path has to end up at a NE

⇒ the number of NE in a two player game is odd.

Lemke-Howson Algorithm

- Simplex Algorithm for Linear Programming

- Nash equilibrium computation is PPAD-hard