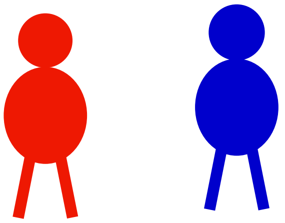
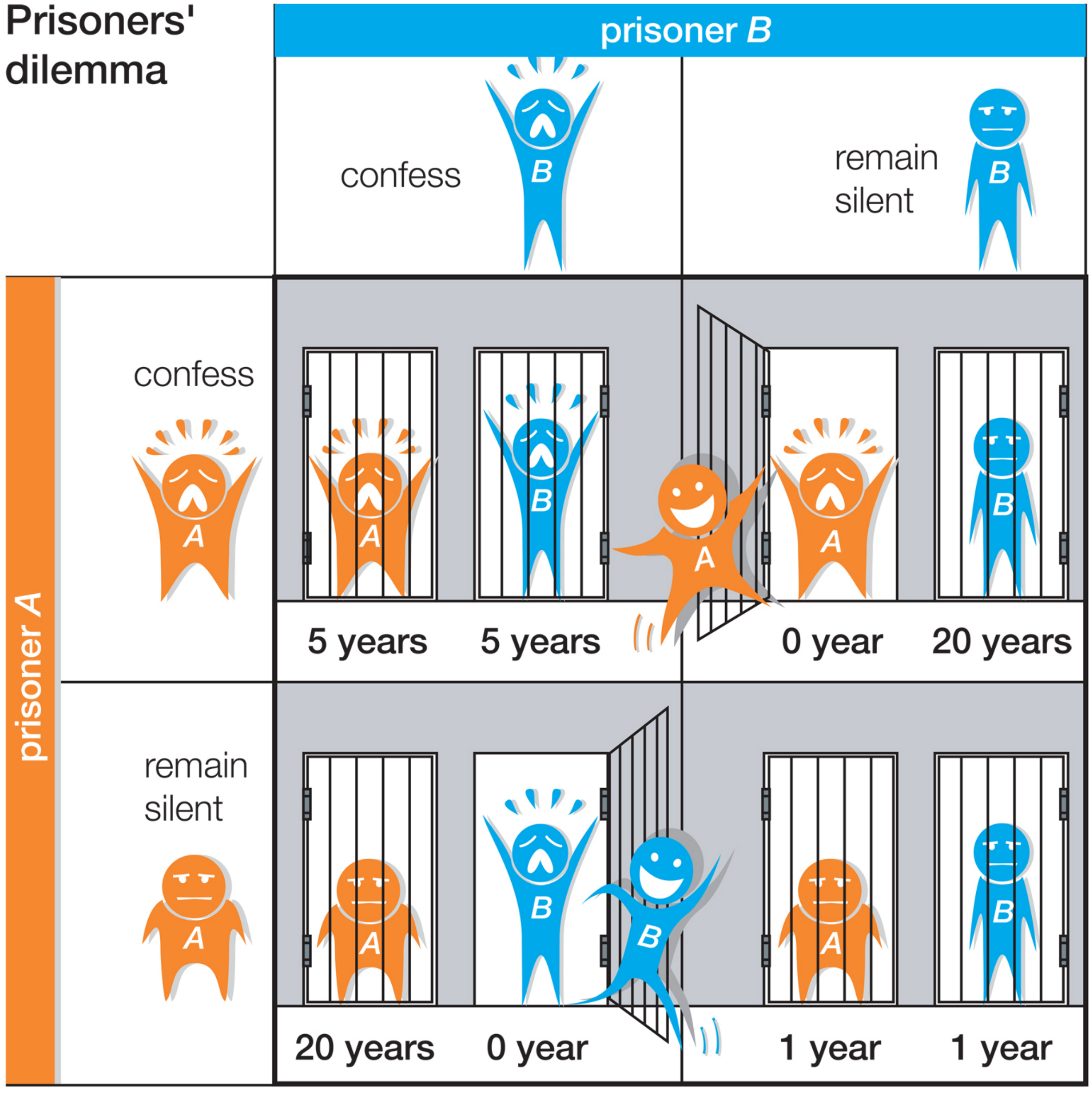
Prisoner's Dilemma



Two thieves caught for burglary

Two options: {confess, remain silent}



Prisoners' dilemma
prisoner B
confess
remain silent
prisoner A
confess
remain silent
5 years
5 years
0 year
20 years
20 years
0 year
1 year
1 year

Prisoner's Dilemma

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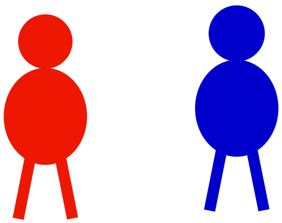

	C	S
C	-5 -5	0 -20
S	-20 0	-1 -1

Prisoner's Dilemma

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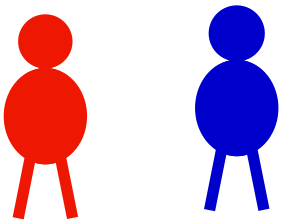

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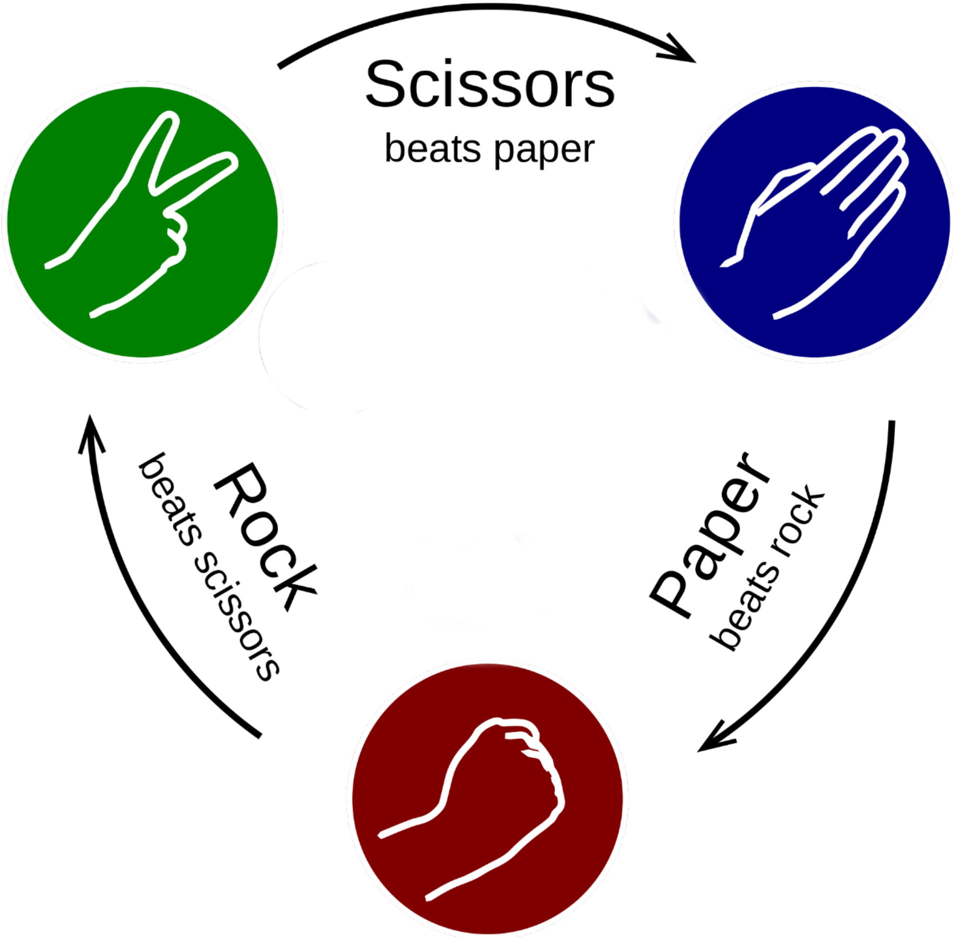
S	-20 0	-1 -1

Only stable state

Rock-Paper-Scissors

Rock-Paper-Scissors



Rock-Paper-Scissors

	R	P	S
R	0 0	-1 1	1 -1
P	1 -1	0 0	-1 1
S	-1 1	1 -1	0 0

No pure stable state!

Both playing (1/3,1/3,1/3) is the only stable state.

Why?

Two Player Finite Games

- Each player has finitely many strategies to play
  - Need not be same

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- Example of infinite set of strategies?

Two Player Finite Games

- Each player has finitely many strategies to play
  - Need not be same
- Example of infinite set of strategies?
- What should be assume about the players?
  - Rational, selfish, intelligent

Two Player Finite Games

- For finite games,
  - $S_1 = \{1, \ldots, m\}$ denote the set of m strategies of player 1
  - $S_2 = \{1, \ldots, n\}$ denote the set of n strategies of player 2
- We can write the payoffs in a matrix form (A, B)

Two Player Finite Games

- For finite games,
  - $S_1 = \{1, \dots, m\}$ denote the set of m pure strategies of player 1
  - $S_2 = \{1, \dots, n\}$ denote the set of n pure strategies of player 2

- We can write the payoffs in a matrix form (A, B)
  - Player 1 is called row player
  - Player 2 is called column player

- Assumption: The payoff matrix is known to both the players

$$A: \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & \cdot A_{ij} & & \\ & & & & \\ & & & & \end{array} \right] \end{array} \qquad B = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & \cdots & n \end{array} \\ \left[\begin{array}{ccccc} & & & & \\ & & \cdot B_{ij} & & \\ & & & & \\ & & & & \end{array} \right] \end{array}$$

Best Response

- Best strategy against the opponent strategy

- Example:

	R	B	S
R	0,0	-1,1	1,-1
B	1,-1	0,0	-1,1
S	-1,1	1,-1	0,0

Equilibrium in Pure Strategies

- $(i, j), i \in S_1, j \in S_2$ is called an equilibrium if i is a best response to j for row player and j is a best response to i for column player

equilibrium

	C	S
C	(-5, 5)	~~(0, -20)~~
~~S~~	~~(-20, 0)~~	~~(-1, 1)~~

Equilibrium in Pure Strategies

- $(i, j), i \in S_1, j \in S_2$ is called an equilibrium if i is a best response to j for row player and j is a best response to i for column player

- Existence? No, Rock-Paper-Scissors.

Mixed Strategies

- Randomly select pure strategies
- $\Delta(S_1) := \{x \in R_+^m \mid x_i \geq 0; \sum_i x_i = 1\}$
- $\Delta(S_2) := \{y \in R_+^n \mid y_j \geq 0; \sum_j y_j = 1\}$

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- Pure strategies are also mixed strategies

- How many mixed strategies? infinite

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- Pure strategies are also mixed strategies

- How many mixed strategies?

- What are the players' expected payoffs when they play (x, y)?

Mixed Strategies

- Randomly select pure strategies
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- $\Delta(S_2) := \{y \in R_+^n \mid y_j \geq 0; \sum_j y_j = 1\}$

- What are the payoffs of players when they play (x, y)?

$(X^T \cdot A \cdot y,\ X^T \cdot B \cdot y)$.

$$A: \begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & \cdots & n \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ \vdots \\ m \end{matrix} & \begin{bmatrix} & & & & \\ & & & \cdot A_{ij} & \\ & & & & \\ & & & & \end{bmatrix} \end{array}$$

player 1: $\underline{\underline{x^T \cdot A \cdot y}}$

player 2: $\underline{\underline{x^T \cdot B \cdot y}}$.

Best Response

- What are the best mixed strategies against the opponent mixed strategy?

Given opponet mixed strategy y.

$$A \cdot y = \begin{bmatrix} A_1 y \\ A_2 y \\ \vdots \\ A_m y \end{bmatrix}$$

find x s.t $\boxed{x \in \arg\max_{x} x^T A y}$

Equilibrium in Mixed Strategies

- $(x, y) \in (\Delta(S_1), \Delta(S_2))$ is called a (mixed) equilibrium if y is a best response to x and x is a best response to y
  - $x^T A y \geq \tilde{x}^T A y, \ \forall \tilde{x} \in \Delta(S_1)$
  - $x^T B y \geq x^T B \tilde{y}, \ \forall \tilde{y} \in \Delta(S_2)$

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- Existence? Yes!

Equilibrium in Mixed Strategies

- $(x, y) \in (\Delta(S_1), \Delta(S_2))$ is called a (mixed) equilibrium if y is a best response to x and x is a best response to y
  - $x^T Ay \ >= \tilde{x}^T Ay, \ \forall \tilde{x} \in \Delta(S_1)$
  - $x^T By \ >= x^T B\tilde{y}, \ \forall \tilde{y} \in \Delta(S_2)$

- John Nash (1951) showed that there always exists a (mixed) equilibrium in any finite game

Equilibrium in Mixed Strategies

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- John Nash (1951) showed that there always exists a (mixed) equilibrium in any finite game
  - We now call this a Nash equilibrium of the game
- Proof? ("fixed point theorem")

: Suppose there are 101 numbers for $\{1, 2, \ldots, 100\}$

→ Can we say that there exists at least two same numbers? Yes.

Equilibrium in Mixed Strategies

- $(x, y) \in (\Delta(S_1), \Delta(S_2))$ is called a (mixed) equilibrium if y is a best response to x and x is a best response to y
  - $x^T Ay \;>= \tilde{x}^T Ay,\; \forall \tilde{x} \in \Delta(S_1)$
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- John Nash (1951) showed that there always exists a (mixed) equilibrium in any finite game
  - We now call this a Nash equilibrium of the game
- Proof? The existence proof doesn't give a clue how
- Computation? to compute a Nash Equilibrium!

Zero-Sum Game

- Rock-Paper-Scissor
- Player 1's profit = Player 2's loss
- $A + B = 0$ or $B = -A$

- Goal: Find an equilibrium (x, y) of game $(A, -A)$

Only need one matrix this time.

Maximin Payoff

- How much payoff row player can ensure to herself?

Maximin Payoff

- How much payoff row player can ensure to herself?

$$\pi_1 = \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} x^T A y$$

- Let x^* be the strategy which gives a payoff of at least π_1 to the row player

Minimax Payoff

- How much payoff column player can ensure to himself?

$$\pi_2 = \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} x^T A y$$

- Let y^* be the strategy which gives a payoff of at least π_2 to the column player

Minimax Theorem

Theorem [von-Neumann'28]

$$\pi_1 = \max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)} x^T A y = \min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)} x^T A y = \pi_2 .$$

This implies that (x^*, y^*) are best response to each other, hence an equilibrium

Proof:

Minimax Theorem

Theorem [von-Neumann'28]

$$\max_{x\in\Delta(S_1)} \min_{y\in\Delta(S_2)} x^T A y = \min_{y\in\Delta(S_2)} \max_{x\in\Delta(S_1)} x^T A y$$

This implies that (x^*, y^*) are best response to each other, hence an equilibrium

Proof:

- $\pi_1 \leq \pi_2$
- $\pi_2 \leq \pi_1$

Minimax Theorem

Theorem [von-Neumann'28]

$$\pi_1 = \max_{\substack{x\in\Delta(S_1) \\ x^*}} \min_{\substack{y\in\Delta(S_2) \\ y_1}} x^T A y = \min_{\substack{y\in\Delta(S_2) \\ y^*}} \max_{\substack{x\in\Delta(S_1) \\ x_2}} x^T A y = \pi_2$$

This implies that (x^*, y^*) are best response to each other, hence an equilibrium

Proof:

- $\pi_1 \leq \pi_2$

minimum about y. maximum about x.

$$\pi_1 = \underbrace{x^{*T} A y_1}_{\text{minimum about } y} \leq x^{*T} A y^* \leq \underbrace{x_2^T A y^*}_{\text{maximum about } x} = \pi_2$$

y_1 minimize $\{x^T A y\}$

a function of x.

肯定比给个y^*小.

x_2 maximize $\{x_2^T A y\}$

a function of y

肯定比给个x^*大.

Minimax Theorem

Theorem [von-Neumann'28]

$$\max_{x\in\Delta(S_1)} \min_{y\in\Delta(S_2)} x^T A y = \min_{y\in\Delta(S_2)} \max_{x\in\Delta(S_1)} x^T A y$$

This implies that (x^*, y^*) are best response to each other, hence an equilibrium

Proof:

- $\pi_1 \le \pi_2$

$$\pi_1 = {x^*}^T A y_1 \le {x^*}^T A y^* \le x_2 A y^* = \pi_2$$

- $\pi_2 \le \pi_1$ (We will use the fact that there exists a NE: say (x', y'))

$$\pi_2 \le \underline{x'^T A y'} \le \pi_1$$

player 2 wants to minimize it.

output

player 1 wants to maximize it.

Minimax Theorem using LP Duality

- Let A_i denote the i^{th} row of A
- Let A^j denote the j^{th} column of A
- Given y, what is the payoff from i^{th} strategy of row player?

Best - Worst - Case.

①.

(can't be mixed strategy. if given x.)

player 2 chooses a pure strategy, decides A_j

Worst case for player 1.

$\min\{(x^T A^1), (x^T A^2), (x^T A^3), \dots (x^T A^n)\} = \pi_1$.

(Given y)

player 1 choose a mixed strategy that maximizes π_1.

Best-worst-case

$$\max_x \pi_1$$

$$\text{s.t. } x^T A^j \geq \pi_1, \ \forall j \in S_2.$$

$$\sum_i x_i = 1 ; \ x_i \geq 0, \ \forall i \in S_1.$$

② Worst-Best-case

player 1 chooses a pure strategy, decides A_i.

Best-case for player 1.

$\max\{(A_1 \cdot y), (A_2 \cdot y), \dots (A_m \cdot y)\}$.

(Given x)

player 2 choose a mixed strategy to minimize π_2.

Worst-Best-case

$$\max_y \pi_2$$

$$\text{s.t. } A_i y \leq \pi_2, \ \forall i \in S_1.$$

$$\sum_j y_j = 1 ; \ y_j \geq 0 \ \forall j \in S_2.$$

Minimax Theorem using LP Duality

- Let A_i denote the i^{th} row of A
- Given y, what is the payoff from i^{th} strategy of row player?
  - $A_i\, y$
- Payoff from x?

Minimax Theorem using LP Duality

- Let A_i denote the i^{th} row of A
- Given y, what is the payoff from i^{th} strategy of row player?
  - $A_i\, y$
- Payoff from x?
  - $x^T Ay$
- Is there any strategy that gives more payoff than $\max_i A_i y$?

Minimax Theorem using LP Duality

$$\underbrace{\max_{x \in \Delta(S_1)} \min_{y \in \Delta(S_2)}}\, x^T A y = \underbrace{\min_{y \in \Delta(S_2)} \max_{x \in \Delta(S_1)}}\, x^T A y$$

- Consider the following linear programs (LP):

$$\begin{aligned} &\max \ \pi_1 \\ &x^T A^j \geq \pi_1, \forall j \in S_2 \\ &\sum_i x_i = 1 \\ &x_i \geq 0, \forall i \in S_1 \end{aligned} \qquad \xleftrightarrow{\text{dual.}} \qquad \begin{aligned} &\min \ \pi_2 \\ &\pi_2 \geq A_i y, \forall i \in S_1 \\ &\sum_j y_j = 1 \\ &y_j \geq 0, \forall j \in S_2 \end{aligned}$$

$\Rightarrow \pi_1 = \pi_2$ for LP duality.

Equivalence of Zero-Sum Games and LP