Polynomial Parity Arguments on Directed Graphs (PPAD)

- End of a Line (EOL)

Given a directed graph $G=(V,E)$ s.t. every vertex has in-degree at most 1 and out-degree at most 1. (Total degree ≥ 1)

if you are given a vertex with in-degree 0.

then there is another vertex with in-degree or out-degree 0.

find it

can't make a cycle.

PPAD-hard

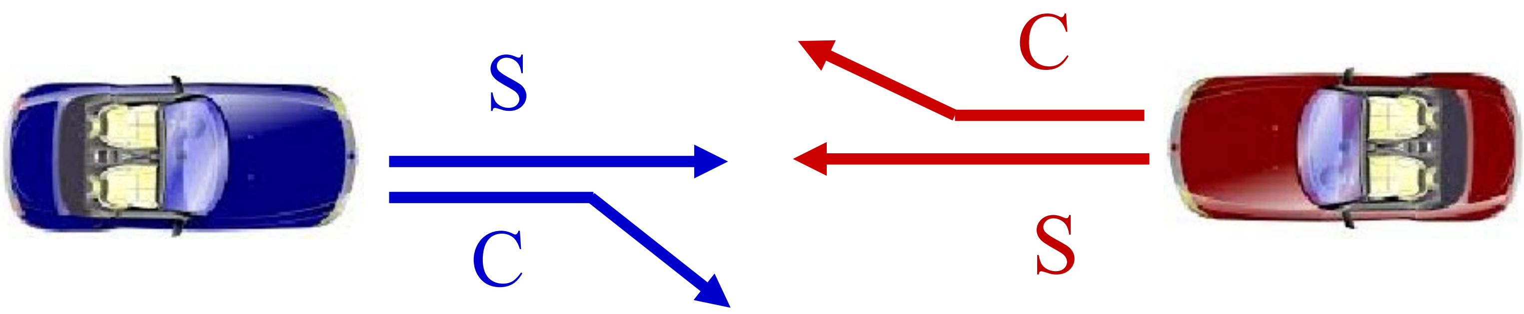
Any problem that can be reduced to EOL is in PPAD

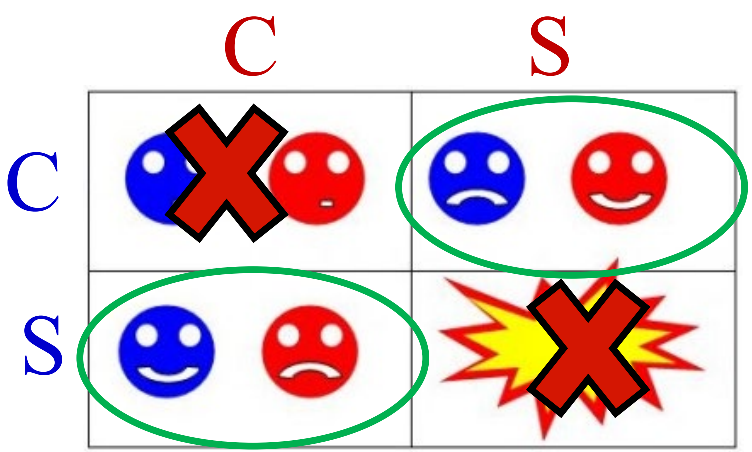
If EOL reduce to problem A, then A is PPAD-Hard.

Claim: Sperner's Lemma is PPAD-Hard.

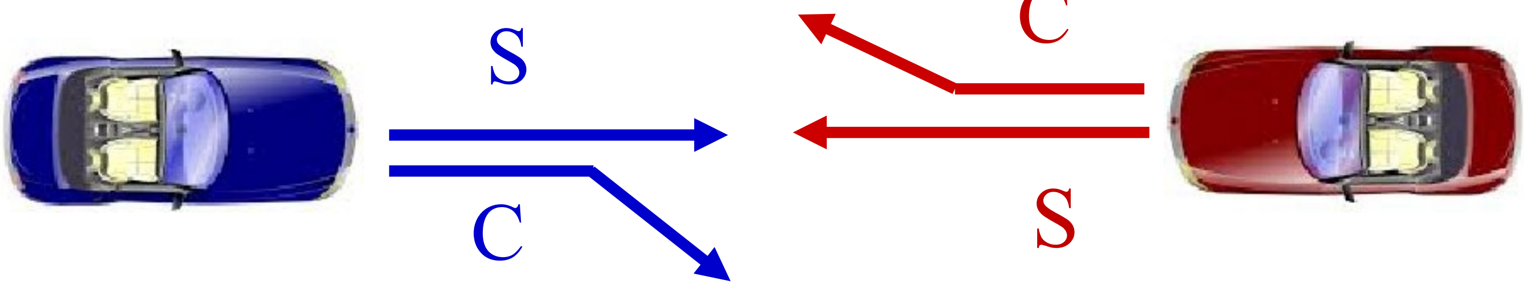
Nash equilibrium is PPAD-Hard.

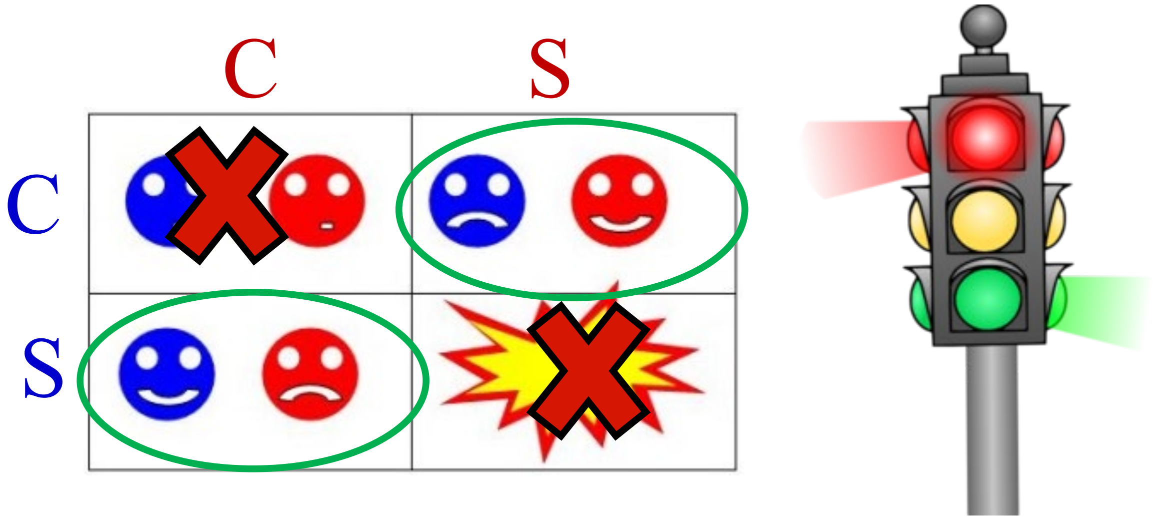
Chicken





Chicken (Traffic Light)





Game of Chicken

		p	$1-p$
		C	S
q	C	4, 4	1, 5
$1-q$	S	5, 1	0, 0

- What are all NE?

pure: (S, C), (C, S).

mixed: $4p + (1-p) = 5p \Rightarrow p = \frac{1}{2}$.

$4q + (1-q) = 5q \Rightarrow q = \frac{1}{2}$.

$[(\frac{1}{2}C, \frac{1}{2}S), (\frac{1}{2}C, \frac{1}{2}S)]$.

Game of Chicken

	C	S
C	4, 4	1, 5
S	5, 1	0, 0

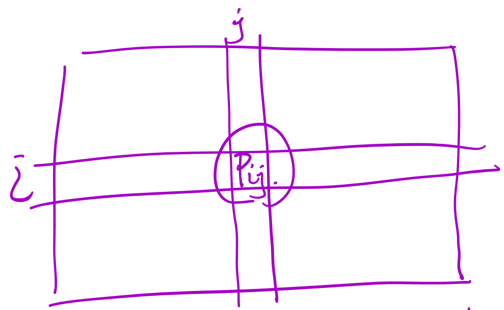
- What are all NE?
- What are the issues with NE?

Trusted Game Coordinator

- In NE, each player choose their strategy $\boxed{\text{independently}}$
- Suppose there is an external correlation device (a game coordinator such as traffic light) that $\boxed{\text{advices}}$ each player to play (i, j) with probability p_{ij}
- The probabilities $p'_{ij}s$ are known to both the players, but not necessarily the advice given to the other player
- Players may choose to not follow the advice given to them

Correlated Equilibrium

- $\{p_{ij}\}$ is a correlated equilibrium (CE) if it is best for each player to follow the advice assuming that the other player is following the advice



$$\left.\begin{array}{l} p_{ij} \geq 0 \quad \forall i, j \\ \sum_{i,j} p_{ij} = 1 \end{array}\right\} \text{Prob distribution.}$$

Given Player 1 played $\underline{i}$. the prob Player 2 plays $\underline{j}$ is: $\dfrac{p_{ij}}{\sum_k p_{ik}}$.

Player 1's expected payoff: $\sum_j A_{ij} \cdot \dfrac{p_{ij}}{\sum_k p_{ik}}$.

$$CE: \left|\begin{array}{l} \sum_j A_{ij} \frac{P_{ij}}{\sum_k P_{ik}} \geq \sum_j A_{i'j} \frac{P_{ij}}{\sum_k P_{ik}}, \quad \forall i, \; i' \in S_1. \\ \text{(follow advice)} \qquad\qquad \text{(not follow)} \\ \sum_i B_{ij} \frac{P_{ij}}{\sum_k P_{kj}} \geq \sum_i B_{ij'} \frac{P_{ij}}{\sum_k P_{kj}}, \quad \forall j, \; j' \in S_2. \\ P_{ij} \geq 0, \; \forall i, j. \qquad \sum_{i,j} P_{ij} = 1. \end{array}\right.$$

$CE: \{P_{ij}\}$. s.t.

$\Rightarrow$ Polyhedron $\Rightarrow$ convex set.

Correlated Equilibrium

- $\{p_{ij}\}$ is a correlated equilibrium (CE) if it is best for each player to follow the advice assuming that the other player is following the advice

- Set of CE is a polyhedral set and hence, convex
- Complexity of finding a CE? = linear feasible program.

polynominal time

Correlated Equilibrium

- How to find the best CE (which maximizes the total payoff)?

- Example

4, 4 P_{11}	1, 5 P_{12}
5, 1 P_{21}	0, 0 P_{22}.

max. $8P_{11} + 6P_{12} + 6P_{21}$

s.t. $4P_{11} + P_{12} \geq 5P_{11} + 0 \cdot P_{12}$ $\quad P_{12} \geq P_{11}$

$5P_{21} + 0P_{22} \geq 4P_{21} + 1P_{22}$ $\quad P_{21} \geq P_{22}$

$4P_{11} + P_{21} \geq 5P_{11} + 0 \cdot P_{21}$ $\quad P_{21} \geq P_{11}$

$5P_{12} + 0 \cdot P_{22} \geq 4P_{12} + P_{22}$ $\quad P_{12} \geq P_{22}$

$P_{11}, P_{12}, P_{21}, P_{22} \geq 0$

$\sum P_{ij} = 1$.

$P_{12} \; ? \; P_{21}$

$\vee$

$P_{11} \; ? \; P_{22}$

$P_{11} = P_{12} = P_{21} = \frac{1}{3}$

$P_{22} = 0$.

Correlated Equilibrium

Claim: NE $\subseteq$ CE

Suppose (x, y) is a NE. Set $P_{ij} = x_i y_j$

$$\sum_j A_{ij} P_{ij} \geq \sum_j A_{i'j} P_{i'j}$$

$$\Rightarrow \sum_j A_{ij} \cancel{x_i} y_j \geq \sum_j A_{i'j} \cancel{x_i} y_j .$$

$$\Rightarrow \sum_j A_{ij} y_j \geq \sum_j A_{i'j} y_j .$$

Correlated Equilibrium

4, 4	1, 5
5, 1	0, 0

- What are the problems with CE?

① not unique. but we can choose particular one.

② it is not achieve the optimal payoff.