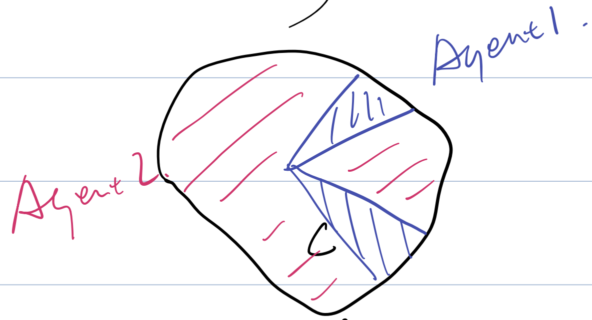
Cake Cutting Problem

- An asymmetric cake C needs to be "fairly" divided among two people (agents)
  - C can be divided among as many pieces as desired
- Let $A = (A_1, A_2)$ denote a partition of the cake (i.e., $A_1 \cup A_2 = C$), where A_i is the piece allocated to agent i
- Each agent i has a valuation function $v_i(.)$ such that $v_i(C) = 1$
- Envy-freeness (EF): We say that A is envy-free if each agent (weakly) prefers their own allocation than any other agents' allocation, i.e., $v_i(A_i) \geq v_i(A_j), \forall i, j$
- Proportionality (Prop): We say that A is proportional if each agent gets at least $1/n$ share of the entire cake, where n is the number of agents, i.e., $v_i(A_i) \geq \frac{1}{n} v_i(C) \geq \frac{1}{n}, \quad \forall i$

Cake Cutting Problem.



$A = (A_1, A_2)$.

$A_1 \cup A_2 = C$.

asymmetric $\xrightarrow{\text{fairly}}$ two agents

value functions : $V_i(\cdot)$, $V_i(C)=1$.

$V_i(A_i) \geq V_i(A_j)$, $\forall i \neq j$. (EF).

$V_i(A_i) \geq \frac{1}{n} V_i(C) \geq \frac{1}{n}$, $\forall i$. (Prop).

n: number of agents.

Q: How to divide the cake between two agents so that the division is envy-free?

Q: EF $\Rightarrow$ Prop.

$$EF \Rightarrow \sum_{j=1}^{n} V_i(A_i) \geq \sum_{j=1}^{n} V_i(A_j) \quad \forall\, i, j$$

$$n\, V_i(A_i) \geq V_i(A_1) + V_i(A_2) + \cdots V_i(A_n) = 1.$$

$$\Rightarrow V_i(A_i) \geq \frac{1}{n}. \quad (\text{Prop})$$

Cut and Choose Protocol

- Pick an agent, say 1, arbitrarily, and let her propose a fair partition (A, A') of the cake to 2
- Agent 2 will choose the part which she like the most and Agent 1 will keep the remaining part

Claim: The cut-and-choose protocol achieves both envy-freeness and proportionality

Fair Division of Indivisible Items

- A set M of m indivisible items (like cell phone, painting, etc.) needs to be fairly divided among a set N of n agents
- Each agent i has a valuation function over the items, denoted by, $v_i : 2^m \rightarrow R_+$
  - We will assume that valuation functions are monotone non-decreasing
- Example:

2 Agents 3 items.

	g_1	g_2	g_3
Agent 1	5	0	10
Agent 2	2	5	10

Additive valuation:

$v_1(\{g_1, g_2\}) = v_1(\{g_1\}) + v_1(\{g_2\})$.

$= 5 + 0 = 5$.

Fair Division of Indivisible Items

- What is a good notion of fairness in case of indivisible items?
  - EF?
  - Prop?

2 Agents - 1 items
hard to achieve
EF or Prop.

- We need to relax these notions to achieve even existence of a fair allocation!

- The questions we will looking are:
  - Existence
  - Uniqueness
  - Algorithm

Fair Division of Indivisible Items

- Envy-freeness up to removal of one item (EF1): We say that A is EF1 if each agent (weakly) prefers their own allocation than after removing some item from any other agents' allocation, i.e., $v_i(A_i) \geq v_i\left(A_j \setminus g_j\right), g_j \in A_j, \ \ \forall i, j$

- Existence?

n agents: $N=\{a_1,\dots,a_n\}$. m items: $M=\{\cancel{g_1}, \times, \dots, g_m\}$.

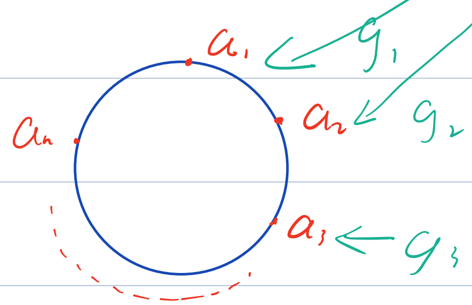
$\Longrightarrow$ find allocation: $A=(A_1,\dots A_n)$.

s.t. A is EFI.

(agents have addictive valuation).

Algorithms:

Round-robin Algorithm: (RRA)



a_i choose g_i one by one:

$g_i \leftarrow \max\limits_{g_i}\{V_i(g_i)\}$

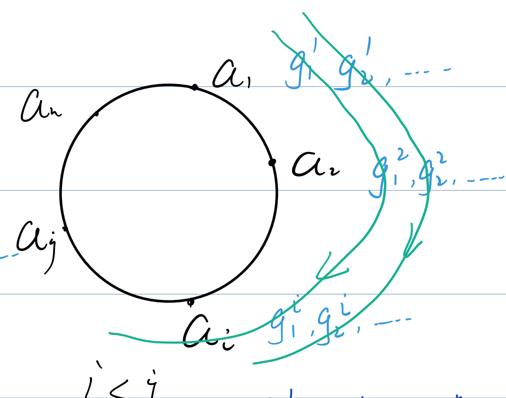
Claim: The output of RRA is EFI.

① $i<j \Rightarrow g_k^i \geq g_k^j$

$V_i(g_k^i) \geq V_i(g_k^j), \ \forall k.$

$\Downarrow$

$V_i(A_i) \geq V_i(A_j)$.



$g_1^1 \geq g_1^2 \geq \cdots g_1^n \geq g_2^1 \geq \cdots \geq g_2^n \geq \cdots$

$i>j \Longrightarrow V_i(A_i) \geq V_i(A_j)$. ①.EF.

② $i \ ? \ j$.

Since $g_k^i \geq g_s^j$, $\forall\, k < s$, $i \neq j$.

$\Longrightarrow V_i(g_k^i) \geq V_i(g_{k+1}^j)$.

$\Longrightarrow V_i(\underbrace{A_i}_{g_1^i + g_2^i + \cdots + g_n^i}) \geq V_i(\underbrace{A_j \setminus g_1^j}_{g_2^j + g_3^j + \cdots + g_n^j})$ ② EF1.

- Example: not additive valuation $\not\Longrightarrow$ EF1.

2 agents, 3 items $\{g_1, g_2, g_3\}$.

$V_i(\cdot)$ is monotone non-decreasing.

$v_1(\{g_1\}) = 10$, $v_1(\{g_2\}) = 5$.

$v_1(\{g_1, g_2\}) = 11$.

Envy-Cycle Procedure

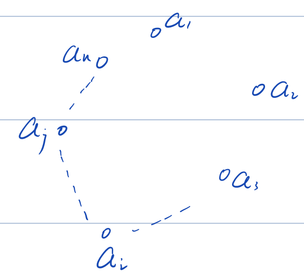
- Start with an empty allocation $A = (\emptyset, \ldots, \emptyset)$ $(A_i = \emptyset, \forall i)$
- Initialize $R = M$ // unallocated items
- Create a graph G where there is a vertex i for each agent i, and there is a directed edge from i to j if i envies j, i.e., $v_i(A_i) \geq v_i(A_j)$
- Repeat until R is empty
  - Pick a source, say i, in G (observe that no agent envies i) and allocate one item g from R to i $(A_i \leftarrow A_i \cup g;\ R \leftarrow R \setminus g)$
  - If G has no source, then observe that there must be a cycle
  - Exchange the bundles along the cycle
- Output allocation A

Algorithm:

(Envy) Graph: $G = (V, E)$.
directed.

m items $\{g_1, g_2, \dots, g_m\}$.

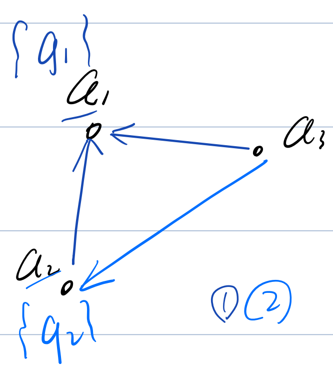
$A \leftarrow \emptyset$ begins with empty

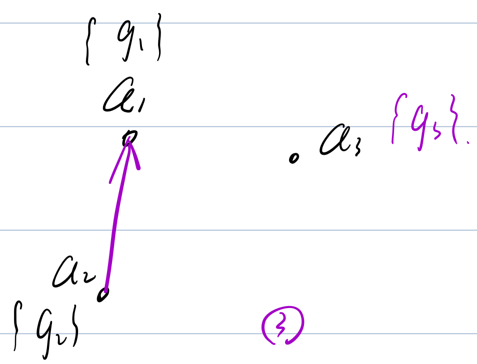


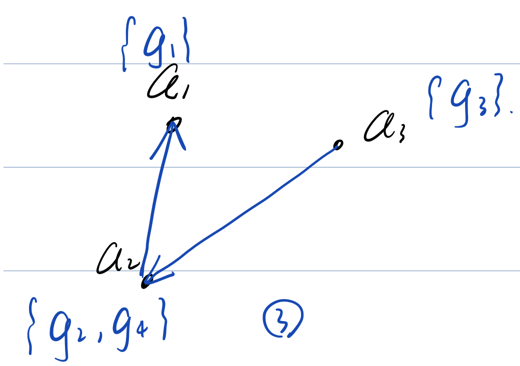
allocate g_i so that no one envy in the graph.

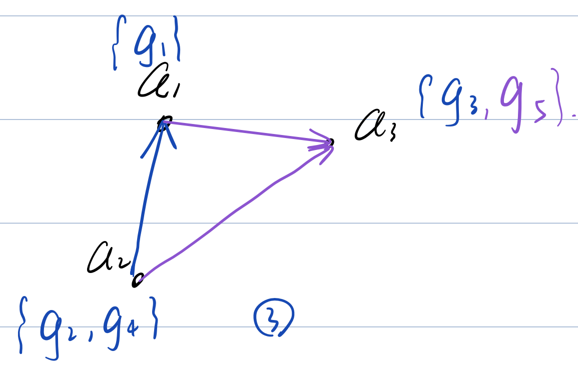
Example:

	$\cancel{g_1}$	$\cancel{g_2}$	$\cancel{g_3}$	$\cancel{g_4}$	$\cancel{g_5}$
a_1	10	5	6	1	11
a_2	5	3	2	1	10
a_3	1	2	3	4	5





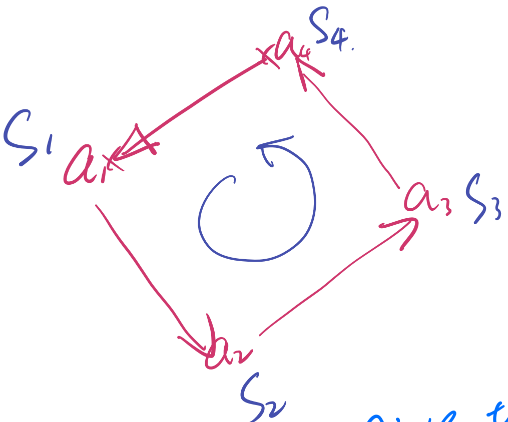




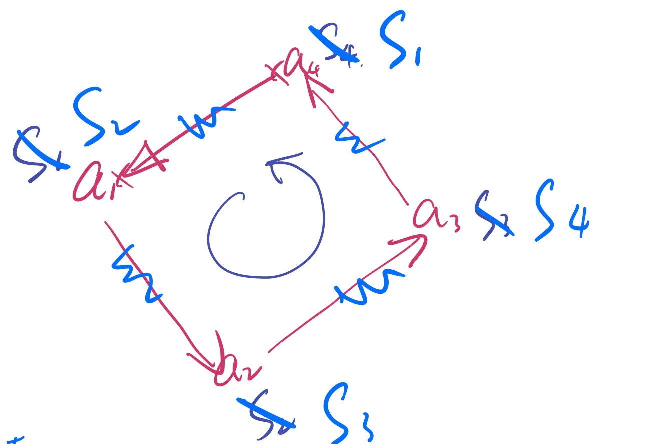
Envy-Cycle Procedure

Claim: The procedure converges to an allocation where all items are allocated

Proof: Why it cannot happen that we never find a source?



$\Longrightarrow$



give them what they want

(break cycle)

Envy-Cycle Procedure

Claim: The final allocation A is $EF1$

Proof: It is based on two observations:

- Valuation of each agent is non-decreasing
- If we remove the last item an agent i received during the procedure, then no other agent envies i's remaining bundle of items

$\Rightarrow$ EF1 allocation always exists.

$\Rightarrow$ There is a efficient algorithm.

Fair Division of Indivisible Items

- Envy-freeness up to removal of any item (EFX): We say that A is EFX if each agent (weakly) prefers their own allocation than after removing any item from any other agents' allocation, i.e., $v_i(A_i) \geq v_i(A_j \setminus g_j), \forall g_j \in A_j, \quad \forall i, j$ (for all).

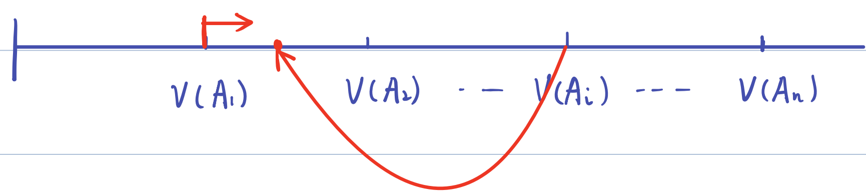
  EF1 only need one item satisfy the ineq. (exist).

- Existence?
  - 2 Agents?
  - 3 Agents?

EF ⇒ EFX ⇒ EF1.

EFX: Identical Agents. $V_1(\cdot) = V_2(\cdot) = \cdots = V_n(\cdot) = V(\cdot)$.

1. Start with an arbitrary allocation $(A_1, A_2, \ldots, A_n)$ s.t. $\cup_i A_i = M$.
2. While A is not EFX.
   - Index bundles so that $v(A_1) \le v(A_2) \le \cdots \le v(A_n)$
   - if A is not EFX then $\exists i.$ $\exists g \in A_i$ s.t. $v(A_1) < v(A_i \setminus g)$.
   - $A_1 \leftarrow A_1 \cup g$; $A_i \leftarrow A_i \setminus g$.



the value of smallest bundle will strictly increase.

Can't continue forever. ⇐ Proof of convergence.

⇒ EFX exists for identicial agents!

Cut and choose protocol: Agent 1 divides agent 2 chooses.

Agent 1: EFX → $V_1(A_1) \geq V_1(A_2 / g)$. $\forall g \in A_2$

→ $V_1(A_2) \geq V_1(A_1 / g)$ $\forall g \in A_1$

Agent 2: choose bigger: $V_2(A_2) \geq V_2(A_1)$.

- Existence?
  - 2 Agents? 2018 paper.
  - 3 Agents?

open question!

addictive valuation ⇒ exists