PoA ≤ 5/2

Different players have different $\underline{\underline{s}}$ and $\underline{\underline{t}}$.

Theorem: In every atomic selfish routing network with affine cost functions, PoA is at most 5/2

Proof: $f = \{f_e\}_{e\in E}$ equilibrium flow.

$P = \{P_1, P_2, \cdots P_n\}$ equilibrium strategy.

$$f_e = \sum_i |e \cap P_i|$$

$$C_i(P_1, P_2, \cdots, P_i', \cdots P_n) \geq C_i(P_1, P_2, \cdots, P_i, \cdots P_n)$$

$\forall P_i', \forall i.$

$$(P_1, P_2, \cdots, P_i', \cdots P_n) \Rightarrow \{f_e'\}_{e\in E}, \quad f_e' = \begin{cases} f_e, & e \notin P_i \cup P_i' \\ f_e, & e \in P_i \cup P_i' \\ f_e - 1, & e \in P_i \setminus P_i' \\ f_e + 1, & e \in P_i' \setminus P_i \end{cases}$$

$$C_i(P_1, P_2, \ldots P_i' \ldots P_n) = \sum_{e \in P_i'} (a_e f_e' + b_e).$$

$$= \sum_{e \in P_i \cap P_i'} (a_e f_e + b_e) + \sum_{e \in P_i' \setminus P_i} (a_e(f_e+1)+b_e).$$

$$\geq \sum_{e \in P_i} (a_e f_e + b_e).$$

$f_e' \leq f_e + 1.$

$$\sum_{e \in P_i'} (a_e(f_e+1) + b_e) \geq \sum_{e \in P_i} (a_e f_e + b_e).$$

Total cost: $\sum_i \sum_{e \in P_i'} (a_e(f_e+1)+b_e) \geq \sum_i \sum_{e \in P_i} (a_e f_e + b_e).$

$$\sum_{e \in E} f_e' (a_e(f_e+1)+b_e) \geq \sum_{e \in E} f_e (a_e f_e + b_e).$$

could be any f_e', let it be f_e^*.

$$\sum_{e \in E} f_e^* (a_e(f_e+1)+b_e) \geq \sum_{e \in E} f_e (a_e f_e + b_e).$$

Lemma: $\forall a, b \in \mathbb{N}.\quad a(b+1) \leq \frac{5}{3}a^2 + \frac{1}{3}b^2$

$$\Longrightarrow \sum_{e \in E} f_e^* (a_e(f_e+1)+b_e) = \sum_{e \in E} a_e \underline{f_e^*(f_e+1)} + f_e^* b_e.$$

$$\leq \sum_{e \in E} a_e \left(\frac{5}{3} f_e^{*2} + \frac{1}{3} f_e^2\right) + f_e^* b_e$$

$$\sum_{e \in E} \frac{5}{3} a_e f_e^{*2} + \frac{1}{3} a_e f_e^2 + f_e^* b_e \geq \sum_{e \in E} a_e f_e^2 + b_e f_e$$

$$\sum_{e \in E} \frac{5}{3} a_e f_e^{*2} + \cancel{\frac{1}{3} a_e f_e^2} + f_e^* b_e - \cancel{\frac{1}{3} a_e f_e^2} \geq \sum_{e \in E} a_e f_e^2 + b_e f_e - \frac{1}{3} a_e f_e^2$$

$$\frac{5}{3} \sum_{e \in E} f_e^* (a_e f_e^* + b_e) \geq \frac{2}{3} \sum_{e \in E} f_e (a_e f_e + b_e)$$

$\Longrightarrow$

$$\text{PoA} = \frac{\text{Worst equilibrium cost.}}{\text{Optimal cost}} = \frac{\sum_{e \in E} f_e (a_e f_e + b_e).}{\sum_{e \in E} f_e^* (a_e f_e^* + b_e)} \leq \frac{5}{2}.$$

PoA $\leq 5/2$

Theorem: In every atomic selfish routing network with affine cost functions, PoA is at most 5/2

Proof:

- (For all $a, b \in \{0, 1, 2, \ldots\}$, $a(b+1) \leq \frac{5}{3}a^2 + \frac{1}{3}b^2$)

$\mathrm{PoA} \leq 5/2$

Theorem: In every atomic selfish routing network with affine cost functions, PoA is at most 5/2

Proof:

- (For all $a, b \in \{0, 1, 2, \dots\}$, $a(b+1) \leq \frac{5}{3}a^2 + \frac{1}{3}b^2$)

- Btw, how do we know whether (pure) equilibrium exists or not?
- If there doesn't exist an equilibrium (like in R-P-S) then the above theorem has no meaning

Rosenthal's Theorem (1973)

Theorem: Every atomic selfish routing game, with arbitrary real valued functions, there exists at least one equilibrium flow

Proof:

- Define a potential function

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

Rosenthal's Theorem (1973)

Theorem: Every atomic selfish routing game, with arbitrary real valued functions, there exists at least one equilibrium flow

Proof:

$f_{e_1} = 0$

$f_{e_2} = 2$

- Define a potential function

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

$$\Phi(f) = \sum_{i=1}^{f_{e_1}} c_{e_1}(i) + \sum_{i=1}^{f_{e_2}} c_{e_2}(i)$$

$$= c_{e_2}(1) + c_{e_2}(2)$$

$$= 1 + 2 = 3.$$

one player

all

- Change in deviator's ~~payoff~~ cost. = change in the potential function

Change in i's cost: $C_i(P_1, \ldots P_i', \ldots P_n) - C_i(P_1, \ldots, P_n)$

$$= \sum_{e \in P_i'} c_e(f_e') - \sum_{e \in P_i} c_e(f_e).$$

$$f'_e = \begin{cases} f_e : e \in P_i \cap P_i' \\ f_e+1 : e \in P_i' \setminus P_i \end{cases} \Rightarrow \quad = \sum_{e \in P_i \cap P_i'} C_e(f_e) + \sum_{e \in P_i' \setminus P_i} C_e(f_e+1) - \sum_{e \in P_i} C_e(f_e)$$

$$= \sum_{e \in P_i' \setminus P_i} C_e(f_e+1) - \sum_{e \in P_i \setminus P_i'} C_e(f_e) = \text{change in } \Phi(f).$$

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

因为只在 f_e+1 或 f_e 变动不影响累加.

Given any strategy profile $P = (P_1, \dots, P_n) \xrightarrow{\text{induce}} \Phi(f)$.

finitely many different strategy profiles are possible.

exists P s.t. $\min_{P \in \mathbb{P}} \Phi(f)$.

Claim: P is a pure Nash Equilibrium.

Proof: Assume P is not a equilibrium, then there exists a player i who wants to deviate to O_i'.

$\Rightarrow$ change in i's cost = change in the potential function value.

lower? $\Longrightarrow$ can't be lower

$\times \rightarrow$ contradiction!

Rosenthal's Theorem (1973)

Theorem: Every atomic selfish routing game, with arbitrary real valued functions, there exists at least one equilibrium flow

Proof:

- Define a potential function

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i)$$

- Change in deviator's payoff = change in the potential function
- Let's take a flow that minimizes the potential function (since there are finitely many flows, such a flow f exists)

Potential Game

Rosenthal's Theorem: Every atomic selfish routing game, with arbitrary real valued functions, there exists at least one equilibrium flow

- Proof works for arbitrary cost functions. We didn't use any conditions on $c_e(.)$!
- We never used any network structure, so valid for abstract game as well!

- **Potential Game:** There exists a potential function such that change in the deviator's payoff is same as the change in potential function

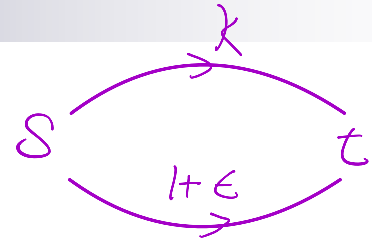
Cost Sharing Games

- So far, we have seen games with negative externalities
- Externality: cost or benefit that affects someone who didn't choose to incur it

Problem:

- Given a network $G = (V, E)$, k players, $\gamma_e \geq 0$ (cost of edge e)
- Player i wants to send 1 unit of flow from s_i to t_i
- Strategies of player i = ?
- $P = (P_1, \ldots, P_k)$ be the strategy profile, then the cost of player i is $C_i(P) = \sum_{e \in P_i} \frac{\gamma_e}{f_e}$, where $f_e = |\{j : e \in P_j\}|$
- Global objective is $C(P) = \sum_{e \in E : f_e > 0} \gamma_e$

Example



- Two players, two edges between s and t with cost k and $1+\epsilon$

$(2, 0).$

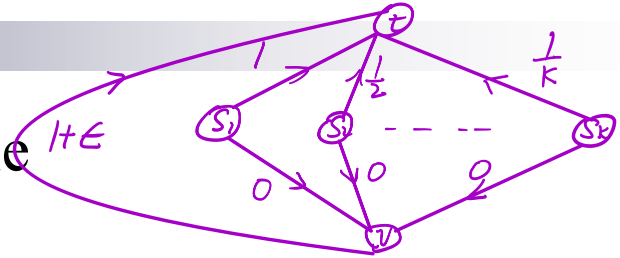
- Equilibrium = ? $(0, 2)$

- Optimal Solution = ? $(0, 2).$

- PoA = ?

- PoS = ?

Example



- k players, different sources s_i but the same destination t
- Each player can either go directly or go together from v
- Cost is $1 + \epsilon$ from v to t and the direct path cost is $1, \frac{1}{2}, \ldots \frac{1}{k}$
- Equilibrium = ? Everyone go directly. $1 + \frac{1}{2} + \cdots + \frac{1}{k}$
- Optimal Solution = ? Everyone go together from v.
- PoA = ? $\frac{1+\epsilon}{k}$ for all.
- PoS = ?

$$PoA = PoS = \frac{1 + \frac{1}{2} + \cdots + \frac{1}{k}}{\frac{1+\epsilon}{\cancel{k}} \cdot \cancel{k}} = \mathcal{H}_k \cdot \frac{1}{1+\epsilon}.$$

PoS Bound

Theorem: In every network cost sharing game with k players, there exists a pure Nash equilibrium (PNE) with cost at most k^{th} harmonic number times that of an optimal solution

Proof: Rosenthal's potential function

$$\Phi(f) = \sum_{e \in E} \sum_{i=1}^{f_e} c_e(i) = \sum_{e \in E} \gamma_e \sum_{i=1}^{f_e} \frac{1}{i} \leq H_k \cdot \gamma_e$$

change in deviator's payoff = change in $\Phi(f)$

$$C_i(P_1, P_2, \dots P_i', \dots P_k) - C_i(P_1, P_2, \dots P_i, \dots P_k).$$

$$= \sum_{e \in P_i' \setminus P_i} C_e(f_e + 1) - \sum_{e \in P_i \setminus P_i'} C_e(f_e)$$

$$= \sum_{e \in P_i' \setminus P_i} \frac{\gamma_e}{f_e + 1} - \sum_{e \in P_i \setminus P_i'} \frac{\gamma_e}{f_e} = \Phi(f') - \Phi(f)$$

find P s.t. minimize $\Phi(f)$. P is NE

$$Cost(P) \leq \Phi(P) \leq H_K \cdot cost(P).$$

Now set optimal strategy P^*

$$Cost(P) \leq \Phi(P) \leq \Phi(P^*) \leq H_K \cdot Cost(P^*).$$

How to Find a PNE?

- Start with arbitrary strategy profile $P = (P_1, P_2, \dots, P_k)$
- Check whether P is a PNE
- If yes, then output P
- Otherwise, there is a player, say i, who has a better strategy (i.e., current strategy is not a best response)
- Change player i's strategy to its best response, and repeat

(Best Response Dynamics) Converges to a PNE

Best Response Dynamics

Theorem: Best response dynamics always converges to a PNE in a finite potential game

Proof:

- Running time?