IE 598 GT : LECTURE 9. FEB 25TH, 2021

__GOALS__ 1) COMPUTING PROP1 + PO ALLOCATION

2) HOW (FISHER) MARKETS CAN HELP IN FAIR DIVISION

__RECAP__

$\boxed{1}$ FISHER MARKET

$U_{ij} \geq 0$

v_{ij} = value of good j for agent i

AGENTS $\to N$: $1, \ldots, i, \ldots, n$

ITEMS/GOODS $\to M$: $1, \ldots, j, \ldots, m$

$$\boxed{\begin{array}{l} \max\ U_i(z) \\ \text{s.t. } \sum_j p_{ij} z_j \leq B_i \end{array}}$$

Budget B_i

Price $p_j \geq 0$

Allocation x

$u_i(x_i) = \sum_j v_{ij} x_{ij}$

(x, p) is a Fisher Market Equilibrium (FME):

(i) Market Clears. $\forall j \in M \quad \sum_i x_{ij} = 1.$

(ii) Budget Exhaustion. $\forall i \in N \quad \sum_j p_j x_{ij} = B_i.$

(iii) Maximum Bang per Buck (MBB).

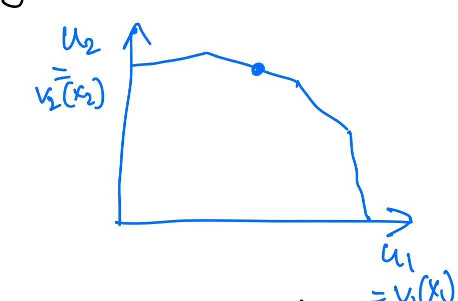
$$x_{ij} > 0 \Rightarrow \frac{v_{ij}}{p_j} = \max_{g \in M} \frac{v_{ig}}{p_g} = \alpha_i$$

$\boxed{2}$ PROPERTIES :

- x is Pareto-optimal (PO) if there is no y which 'dominates' x.

Here, y dominates x if

$$\forall i \quad v_i(y_i) \geq v_i(x_i)$$
$$\exists k \quad v_k(y_k) > v_k(x_k)$$



- x is Envy-Free (EF) if $\forall i, k$: $u_i(x_i) \geq u_i(x_k)$

(1) For every FME (x, p), x is Pareto-Optimal (PO). efficiency ✓

(2) " with 1\$, x is Envy-Free (EF). fairness ✓

(PO + Prop 1)

Key Idea "Spending" acts as a proxy for "value".

Spending $p(x_i) := \sum_j p_j x_{ij}$ amount of money i spends

MBB ratio $\alpha_i := \max_{g \in M} \frac{v_{ig}}{p_g}$

$x_{ij} > 0 \Rightarrow v_{ij}/p_j = \alpha_i$ (Property of FME)

Value $v_i(x_i) = \sum_j v_{ij} x_{ij} = \sum_{j : x_{ij} > 0} \alpha_i p_j x_{ij}$

$= \alpha_i \sum_j p_j x_{ij} = \alpha_i \, p(x_i)$ = α_i Spending

TODAY COMPUTING $\boxed{\text{PROP 1 + PO}}$ ALLOCATIONS.

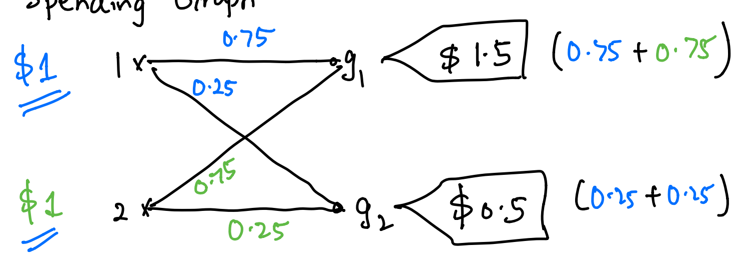
x is Prop 1 if $\forall i \; \exists g \in M$ s.t. $v_i(x_i \cup \{g\}) \geq \frac{v_i(M)}{n}$.

Here, we consider indivisible goods, so x is 'integral'. $\forall i, j \; x_{ij} \in \{0, 1\}$

Consider FME (x, p) with equal (1\$) budgets.

- Divisible ... But x is PO and EF ... so also Prop.
- Can we somehow modify it to make it integral?

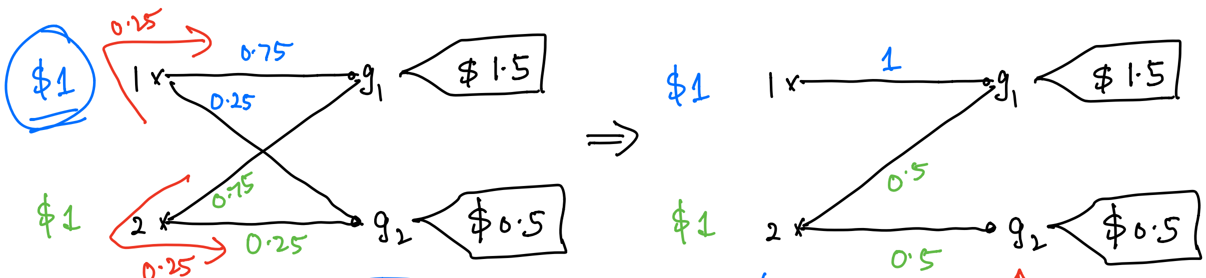
"Spending Graph"



Can we remove the cycle?

Can remove the cycle by circulating money along edges!

Circulating money

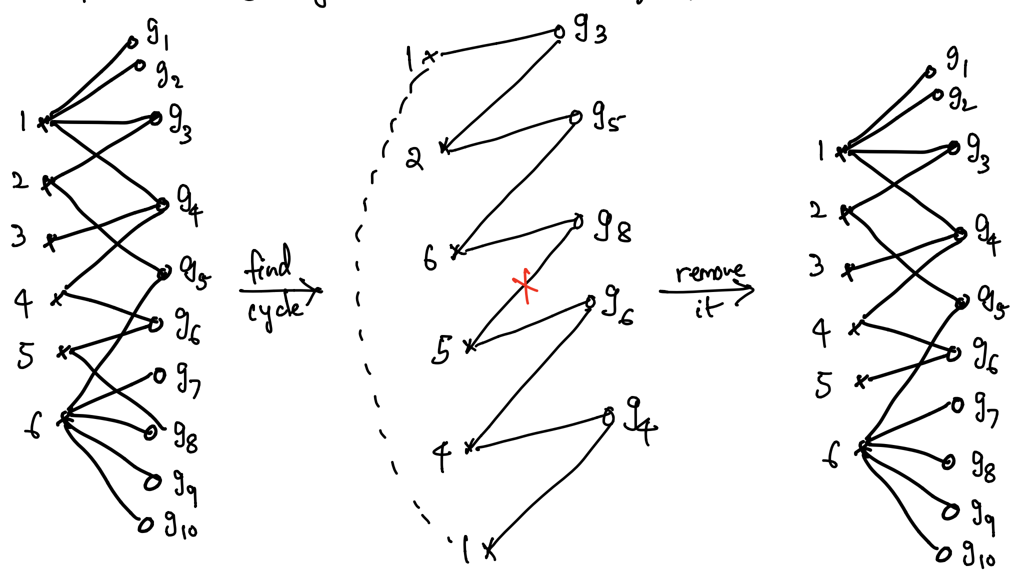


$u_1(x_1) = \alpha_1\, p(x_1) = \alpha_1 \cdot (1\$) = u_1(x_1') = \alpha_1\, p(x_1')$ now integrally allocated!
$= \alpha_1 \cdot 1.$

Does this change agents' value?

No.

Keep removing cycles in Spending graph.

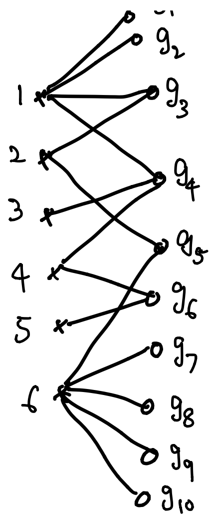


This procedure terminates. (Why?)

⟨ Take-away Can always assume spending Graph of FME (x, p) is a forest (no-cycles). ⟩

We now have an acyclic spending graph with each agent having 1\$.

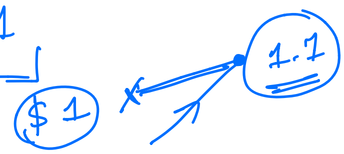
Still not integral. Can we round it?

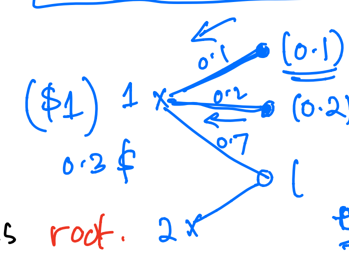


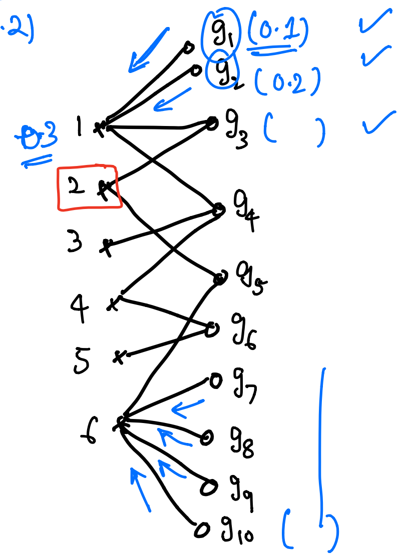
How to round in a manner that gives Prop1 + PO?

$v_i(x_i \cup \{g\}) \geq \frac{1}{n} v_i(M)$

$p(x_i \cup \{g\}) \geq 1$







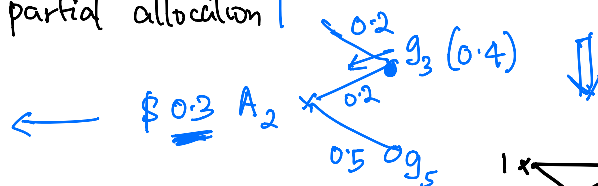
Algorithm

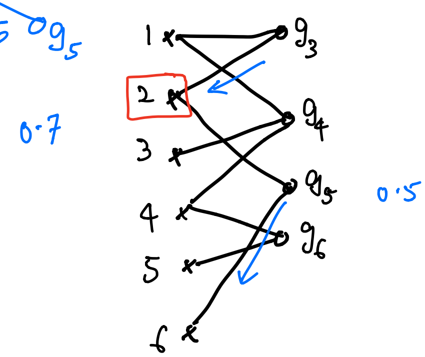
1. Make an agent as root.

   Initialize $x' \leftarrow (\phi, \dots, \phi)$

2. Allocate all leaf goods to parent agents.

   $x' \leftarrow$ updated partial allocation





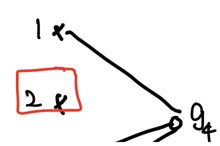
3. while there is a root agent i :

   while there is an edge (i,g)

   s.t. $p(x'_i \cup \{g\}) \leq 1$:

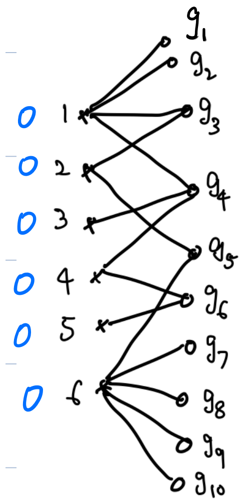
   $x'_i \leftarrow x'_i \cup \{g\}$ (Give g to i)

   Give every remaining child good j of i to any child agent k of j and



$v_i(X_i \cup \{g\}) \geq \frac{1}{n} v_i(M)$: Prop1.

Spending: $p(X_i \cup \{g\}) \geq B_i$ (in this example $B_i = 1\ \forall i$).



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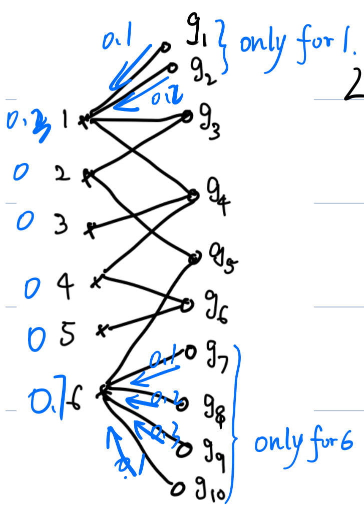

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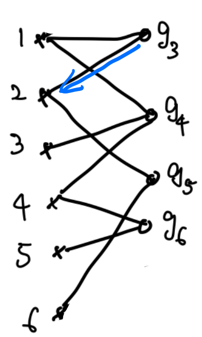

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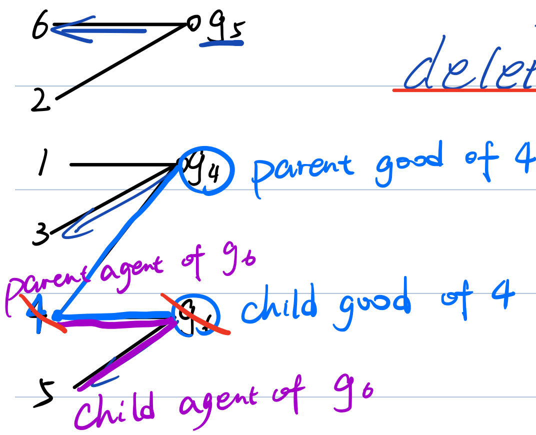
Give every remaining child good j of i to any child agent k of j and delete j. Delete i.

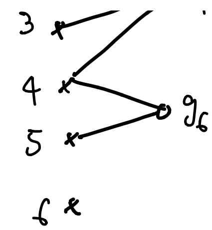


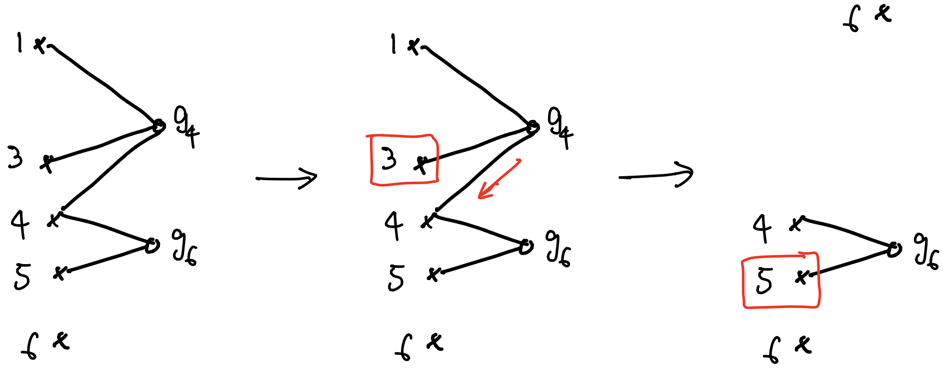
4. Return X'. [$p(X_i') = 1$ is not necessary].

The X' is a Prop1 + PO allocation.

delete j.

Delete i.





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Analysis

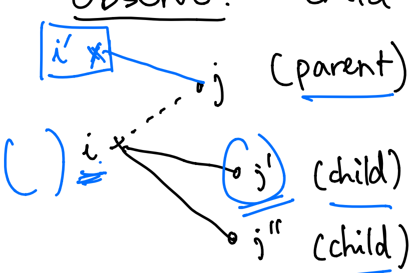
The algorithm returns a Prop1 + PO allocation X'.

Proportional share of agent i = 1\$

We want to show that for all $i \in N$:

$$\exists g \text{ s.t. } p(x'_i \cup \{g\}) \geq 1 .$$

Observe: Child nodes of i are not deleted before i.

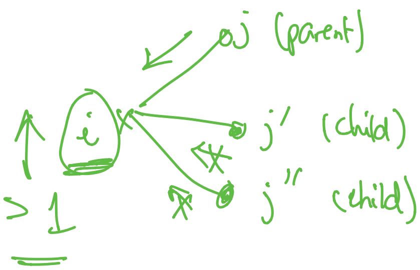


- If i has no children. $p(x'_i \cup \{j\}) \geq 1$.
- If i has children.
  - i doesn't get some child j'. $p(x'_i) + p_{j'} \geq 1$
  - i gets all children. Then $p(x'_i \cup \{j\}) \geq 1$

There is always some g s.t. $p(x'_i) + p_g \geq 1$.

More generally, can replace $1\$$ with B_i. $\hookrightarrow B_i' + p_g \geq B_i$

$$u_i(x_i' \cup \{g\}) = \alpha_i \, p(x_i' \cup \{g\}) \qquad B_i' \leq B_i + p_g$$
$$\geq \alpha_i \cdot 1$$
$$= \alpha_i \cdot \frac{p(M)}{n} \geq \frac{v_i(M)}{n}.$$



$B_i' \leq B_i + p_j$

Therefore for every agent i, $p(x_i' \cup \{g\}) \geq 1$ for some g.
Thus x' is Prop1.

x' is PO since (x', p) is FME of a FM with budgets $p(x_i')$.
not $1\$$

Run-time? poly(m,n) time.

In conclusion:
A Prop1 + PO allocation can be computed in polynomial time.

REMARKS

$\forall i \quad B_i - p_g \leq B_i' \leq B_i + p_g$

(1) Rounding algorithm works with general budgets.
FME (x,p), budgets $B_i \longrightarrow$ Integral FME (x',p), budgets B_i'

(2) Can also get EF_1^1 + PO allocation using this idea.
Relaxation of EF. $\quad u_i(x_i \cup \{j\}) \geq u_i(x_k \setminus \{g\})$

(3) FME also useful in computing $\boxed{\text{EF1+PO allocations.}}$

OPEN: EF1+PO allocations in polynomial time.

$\forall i, k: \quad \exists g \in X_k, \exists j$

$v_i(X_i \cup \{j\}) \geq v_i(X_k \setminus \{g\})$