Fair Allocation of Indivisible Items

- A set N of n agents, a set M of m indivisible items

- Proportionality (Prop): We say that allocation $A = (A_1, \dots, A_n)$ is proportional if each agent gets at least $1/n$ share of the all items, where n is the number of agents, i.e.,

EF ⇒ Prop.

$$v_i(A_i) \geq \frac{1}{n} v_i(M), \qquad \forall i$$

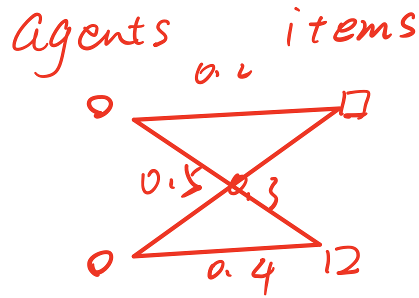
Relax.

- Prop1: We say that A is proportional up to one item if each agent gets at least $1/n$ share of all items after adding one more item from outside, i.e.,

$$v_i(A_i \cup \{g\}) \geq \frac{1}{n} v_i(M), \qquad g \in M \setminus A_i, \forall i$$

Prop1 Allocation

- Let (p, x) be equilibrium prices and allocations
- Since x is an envy-free allocation ⇒ it is also proportional
- We can assume that there is no cycle in the support of x



cycle

Q3. Show that $\frac{6}{7}$-MMS allocation exists for three agents.

$\triangle$ $\frac{3}{4}$-MMS allocation for 3 agents!

Step 1: Normalize valuations $V_i(M)=3$, $i \in \{1,2,3\}$. $\Rightarrow \mu_i \leq 1$.

Step 2: Suppose there is an item j s.t. $V_{ij} \geq \frac{3}{4}$.

↳ if yes, $\Longrightarrow$ only two agents $\Longrightarrow$ exists MMS.

if No $\Longrightarrow V_{ij} < \frac{3}{4}$ $\forall i, j$. $\Rightarrow m > 4$.

Let's order $V_{i1} \geq V_{i2} \geq \cdots \geq V_{im}$ $\forall i$.

Suppose allocation

A_1	A_2	$\boxed{A_3}$
g_1	g_2	g_3
....		g_4

① if $\geq \frac{3}{4}$

medium $\Rightarrow$ doesn't decrease remaing MMS-value.

证明见 $\frac{2}{3}$-MMS.

we can remove it.

$\Longrightarrow$ two agents.

②

if $< \frac{3}{4} \Rightarrow V_{i4} < \frac{3}{8}$

$\boxed{g_6}$

$\boxed{g_5}$

$\boxed{g_4}$

$\boxed{g_3}$

$< \frac{3}{4}$

keep adding until $V_i(A_3) \geq \frac{3}{4}$

$V_i(A_3) < \frac{3}{4} + \frac{3}{8} = \frac{9}{8}$.

$\Downarrow$

$V_i(A_1 \cup A_2) > 3 - \frac{9}{8} = \frac{15}{8}$

Similarly we can adding items to A_2 until $V_i(A_2) \geq \frac{3}{4}$

since $A_2 \in [\frac{3}{4}, \frac{9}{8}]$

then $A_1 \in (\frac{3}{4}, \frac{3}{2}]$

△ $\frac{6}{7}$-MMS allocation for 3 agents.

Step 1: Normalize valuations $V_i(M)=3$ $i=\{1,2,3\} \Rightarrow \mu_i=1$.

Step 2: Suppose there is an agent i and item j

s.t. $V_{ij} \geq \frac{6}{7}$. we can allocate j to i.

then there are two agents which must exist 1-MMS allocation.

if $V_{ij} < \frac{6}{7}$ $\forall i,j. \Longrightarrow m \geq 4$.

Let's order $V_{i1} \geq V_{i2} \geq \cdots \geq V_{im}$ $\forall i$.

Suppose allocation

A_1	A_2	A_3
g_1	g_2	g_3
		g_4

① if $V_i(\{g_3,g_4\}) \geq \frac{6}{7}$, we can allocate $\{g_3,g_4\}$

to i. without decreasing remaining MMS-value

then it's a two agents problem.

② if $V_i(\{g_3,g_4\}) < \frac{3}{7} \Rightarrow V_i(g_j) \leq \frac{3}{14}$ $\forall j \geq 4$.

keep adding items to $A_3=\{g_3,g_4\}$ until $V_i(A_3) \geq \frac{6}{7}$

Finally, we can get $V_i(A_3) \in [\frac{6}{7}, \frac{15}{14}]$.

Similarily, we can add items to $A_2 = \{g_2\}$ until $V_i(A_2) \geq \frac{6}{7}$

$V_i(A_2) \in [\frac{6}{7}, \frac{15}{14}]$. $\Rightarrow$ $V_i(A_3) \in [\frac{6}{7}, \frac{9}{7}]$.

the $A = \{A_1, A_2, A_3\}$ is the $\frac{6}{7}$-MMS allocation.

(3). if $V_i(\{g_3, g_4\}) \in (\frac{3}{7}, \frac{6}{7}) \Rightarrow V_i(g_j) < \frac{3}{7}$ $\forall j \geq 4$.

$V_i(g_j) > \frac{3}{14}$ $j = 1, 2, 3$.

Prop1 Allocation

- Let (p, x) be equilibrium prices and allocations
- Since x is an envy-free allocation $\Rightarrow$ it is also proportional
- We can assume that there is no cycle in the support of x

- Pick an agent arbitrarily and make her the root
- Rounding?