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IE598GT: Homework 4

Due: May 07, 2021 (5 PM)

Note:

- The due date is strict. Late submissions will not be graded.
- Your answers must be concise and clear. Explain sufficiently so that we can easily determine that you understand.
- Only electronic submissions are accepted on **gradescope**.
- You are expected to do each homework on your own. You may discuss concepts with your classmates, but there must be no interactions about solutions.
- **There are 60 points in total, and you need 40 points for the full homework credit.**
- Plagiarism will be dealt with severely – no credit for the homework.

1. **(5 points)** Suppose you want to hire a contractor to perform some task, like remodeling a house. Each contractor has a private cost for performing the task. Give an analog of the Vickrey auction in which contractors report their costs and the auction chooses a contractor and a payment. Truthful reporting should be a dominant strategy in your auction and, assuming truthful bids, your auction should select the contractor with the smallest private cost.

   [Aside: auctions of this type are called *procurement* or *reverse* auctions.]

2. **(5 points)** Suppose there are k identical copies of a good and $n > k$ bidders. Suppose also that each bidder can receive at most one good. What is the analog of the second-price auction? Prove that your auction is DSIC.

3. **(5 points)** Consider the following *random priority* mechanism for the assignment of dorm rooms to college students.

   Random Priority

   initialize R to the set of all rooms
   randomly order the agents
   for $i = 1, 2, \ldots, n$ **do**
       assign the ith agent her favorite room r from among those in R
       delete r from R

   Does this mechanism DSIC, no matter which random ordering is chosen by the mechanism?

4. **(5 points)** Recall the knapsack auction where each bidder i has a publicly known size w_i and a private valuation. Consider a variant of a knapsack auction in which we have two knapsacks, with known capacities W_1 and W_2. Feasible sets of this single-parameter setting now correspond to subsets S of bidders that can be partitioned into sets S_1 and S_2 satisfying $\sum_{i \in S_j} w_i \leq W_j$ for $j = 1, 2$.

Consider the allocation rule that first uses the single-knapsack greedy allocation rule (discussed in the class) to pack the first knapsack, and then uses it again on the remaining bidders to pack the second knapsack. Does this algorithm define a monotone allocation rule? Give either a proof of this fact or an explicit counterexample.

5. **(10 points)** Consider a variant of a knapsack auction in which both the valuation v_i and the size w_i of each bidder i are private. A mechanism now receives both bids $\mathbf{b}$ and reported sizes $\mathbf{a}$ from the bidders. An allocation rule $\mathbf{x}(\mathbf{b}, \mathbf{a})$ now specifies the amount of capacity allocated to each bidder, as a function of the bids and reported sizes. Feasibility dictates that $\sum_{i=1}^{n} x_i(\mathbf{b}, \mathbf{a}) \leq W$ for every $\mathbf{b}$ and $\mathbf{a}$, where W is the total capacity of the shared resource. We define the utility of a bidder i as $v_i - p_i(\mathbf{b}, \mathbf{a})$ if she gets her required capacity (i.e., $x_i(\mathbf{b}, \mathbf{a}) \geq w_i$) and as $-p_i(\mathbf{b}, \mathbf{a})$ otherwise. Note that this is not a single-parameter environment.

   Consider the following mechanism. Given bids $\mathbf{b}$ and reported sizes $\mathbf{a}$, the mechanism runs the greedy knapsack auction, discussed in the class, taking the reported sizes $\mathbf{a}$ at face value, to obtain a subset of winning bidders and prices $\mathbf{p}$. The mechanism concludes by awarding each winning bidder capacity equal to her reported size a_i, at a price of p_i; losing bidders receive and pay nothing. Is this mechanism DSIC? Prove it or give an explicit counterexample.

6. **(10 points)** Consider a mechanism design problem where the set of outcomes is the unit interval $[0, 1]$ and each agent i has *single-peaked preferences*, meaning that there is an agent-specific "peak" $x_i \in [0, 1]$ such that i strictly prefers y to z whenever $z < y \leq x$ or $x \leq y < z$. Thus an agent with single-peaked preferences wants the chosen outcome to be as close to her peak as possible.

   a. Is the mechanism that accepts a reported peak from each agent and outputs the average DSIC?
   b. Is the mechanism that accepts a reported peak from each agent and outputs the median DSIC? Feel free to assume that the number of agents is odd.

7. **(10 points)** Consider the following extension of the sponsored search setting discussed in lecture. Each bidder i now has a publicly known *quality* β_i (in addition to a private valuation v_i per click). As usual, each slot j has a CTR α_j, and $\alpha_1 \geq \alpha_2 \cdots \geq \alpha_k$. We assume that if bidder i is placed in slot j, its probability of a click is $\beta_i \alpha_j$ – thus, bidder i derives value $v_i \beta_i \alpha_j$ from this outcome.

   Describe the surplus-maximizing allocation rule in this generalized sponsored search setting. Argue that this rule is monotone. Give an explicit formula for the per-click payment of each bidder that extends this allocation rule to a DSIC mechanism.

8. **(10 points)** Consider a set M of distinct items. There are n bidders, and each bidder i has a publicly known subset $T_i \subseteq M$ of goods that it wants, and a private valuation v_i for getting them. If bidder i is awarded a set S_i of items at a total price of p, then her utility is $v_i x_i - p$, where x_i is 1 if $S_i \supseteq T_i$ and 0 otherwise. This is a single parameter environment. Since each item can only be awarded to one bidder, a subset W of bidders can all receive their desired subsets simultaneously if and only if $T_i \cap T_j = \emptyset$ for each distinct $i, j \in W$.

   We note that the problem of maximizing the social surplus of bidders, given the v_i's and T_i's as input, is NP-hard.

   (a) Here is a greedy algorithm for the social welfare maximization problem, given bids $\mathbf{b}$ from the bidders.

```
initialize W = ∅ and X = M
sort and re-index the bidders so that
b_1 ≥ b_2 ≥ ··· ≥ b_n
for i = 1, 2, 3, ..., n do
    if T_i ⊆ X then
        remove T_i from X and add i to W
return winning bidders W
```

Does this algorithm define a monotone allocation rule? Prove it or give an explicit counterexample.

(b) Prove that if all bidders report truthfully and have sets T_i of cardinality at most d, then the outcome of the allocation rule in the greedy algorithm has social welfare at least $\frac{1}{d}$ times that of the maximum possible.

1. **(5 points)** Suppose you want to hire a contractor to perform some task, like remodeling a house. Each contractor has a private cost for performing the task. Give an analog of the Vickrey auction in which contractors report their costs and the auction chooses a contractor and a payment. Truthful reporting should be a dominant strategy in your auction and, assuming truthful bids, your auction should select the contractor with the smallest private cost.

   [Aside: auctions of this type are called *procurement* or *reverse* auctions.]

Allocation: Hire the Lowest bidder.

Payment: Pay the second lowest bid

2. **(5 points)** Suppose there are k identical copies of a good and $n > k$ bidders. Suppose also that each bidder can receive at most one good. What is the analog of the second-price auction? Prove that your auction is DSIC.

Allocation: give the items to highest k bidders.

Payment: pay the $(k+1)^{th}$ highest bid.

Prove the auction is DSIC:

1) We want to prove for every bidder $b_i = v_i$ is dominat strategy.

Proof: for a bidder i. set B_i is the k^{th} highest bid in the bidders except i.

① $B_i > V_i$

if $b_i \geq B_i$ win $\Rightarrow$ utility $= V_i - B_i < 0$

if $b_i < B_i$ loss $\Rightarrow$ utility $= 0$.

②. $B_i \leq V_i$.

if $b_i \geq B_i$ win $\Rightarrow$ utility $= V_i - B_i \geq 0$

if $b_i < B_i$ loss $\Rightarrow$ utility $= 0$.

Hence we know $b_i = V_i$ is the best strategy no matter $B_i > V_i$ or $B_i \leq V_i$.

$b_i = V_i$ is a dominant Strategy for every bidder.

2) Since we choose the highest k bidder. we maximize social welfare $\sum_i V_i X_i$.

3) the cost is $O(n)$ which is efficient.

1) 2) 3) $\Longrightarrow$ the auction is DSIC.

3. **(5 points)** Consider the following *random priority* mechanism for the assignment of dorm rooms to college students.

Random Priority

initialize R to the set of all rooms
randomly order the agents
for $i = 1, 2, \ldots, n$ **do**
    assign the ith agent her favorite room r from among those in R
    delete r from R

Does this mechanism DSIC, no matter which random ordering is chosen by the mechanism?

No, it may don't maximize the social welfare.

Example: two room A, B

first agent values A : $V_{1A} = 3$

values B : $V_{1B} = 4$.

while second agent values A : $V_{2A} = 0$

values B : $V_{2B} = 5$.

This example the social welfare is not maximized.

4. **(5 points)** Recall the knapsack auction where each bidder i has a publicly known size w_i and a private valuation. Consider a variant of a knapsack auction in which we have two knapsacks, with known capacities W_1 and W_2. Feasible sets of this single-parameter setting now correspond to subsets S of bidders that can be partitioned into sets S_1 and S_2 satisfying $\sum_{i \in S_j} w_i \leq W_j$ for $j = 1, 2$.

Consider the allocation rule that first uses the single-knapsack greedy allocation rule (discussed in the class) to pack the first knapsack, and then uses it again on the remaining bidders to pack the second knapsack. Does this algorithm define a monotone allocation rule? Give either a proof of this fact or an explicit counterexample.

Yes, it is monotone.

Proof: Assume $\exists\, b_i' < b_i$ s.t. $x_i(b_i) < x_i(b_i')$.

Since $x_i \in \{0,1\}$, $x_i(b_i)=0$, $x_i(b_i')=1$.

$x_i(b_i)=0$ means $\frac{b_i}{w_i}$ is too small, the greedy algorithm stops before it. Assume in the situation the last picked one is $\frac{b_j}{w_j}$, then $\frac{b_j}{w_j} > \frac{b_i}{w_i}$.

it is same in the situation without bidder i.

$x_i(b_i')=1$ means $\frac{b_i'}{w_i}$ is picked by the greedy algorithm. $\frac{b_i'}{w_i} \geqslant \frac{b_j}{w_j}$, otherwise it won't be picked.

$\Rightarrow \frac{b_i'}{w_i} \geqslant \frac{b_j}{w_j} > \frac{b_i}{w_i} \Rightarrow \underline{\underline{b_i' > b_i}}$

which contradicts to our setting.

Hence X is monotone.

5. **(10 points)** Consider a variant of a knapsack auction in which both the valuation v_i and the size w_i of each bidder i are private. A mechanism now receives both bids $\mathbf{b}$ and reported sizes $\mathbf{a}$ from the bidders. An allocation rule $\mathbf{x}(\mathbf{b}, \mathbf{a})$ now specifies the amount of capacity allocated to each bidder, as a function of the bids and reported sizes. Feasibility dictates that $\sum_{i=1}^{n} x_i(\mathbf{b}, \mathbf{a}) \leq W$ for every $\mathbf{b}$ and $\mathbf{a}$, where W is the total capacity of the shared resource. We define the utility of a bidder i as $v_i - p_i(\mathbf{b}, \mathbf{a})$ if she gets her required capacity (i.e., $x_i(\mathbf{b}, \mathbf{a}) \geq w_i$) and as $-p_i(\mathbf{b}, \mathbf{a})$ otherwise. Note that this is not a single-parameter environment.

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No. if $a_i < w_i$ $u_i \equiv -p_i(b,a)$

if $a_i \geqslant w_i$ $\begin{cases} \text{if loss } u_i = -p_i(b,a) \\ \text{if win } u_i = v_i - p_i(b_i, a_i). \end{cases}$

$\Rightarrow$ $a_i = w_i$ is dominat Strategy.

Counterexample: $W = 10$. (Assume bid truthfully).

bidder 1: $v_1 = 6$, $w_1 = 6$ will bid $(6, 6)$.

bidder 2: $v_2 = 4$, $w_2 = 5$ will bid $(4, 5)$

bidder 3: $v_3 = 3$, $w_3 = 5$. will bid $(3, 5)$.

it will just pack bidder 1.

if bidders report truthfully, the allocation still doesn't maximize social welfare.

Hence ① bid truthfully. ② maximize social welfare.

there is at least one doesn't hold.

$\Rightarrow$ not DSIC.

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a. Is the mechanism that accepts a reported peak from each agent and outputs the average DSIC?

b. Is the mechanism that accepts a reported peak from each agent and outputs the median DSIC? Feel free to assume that the number of agents is odd.

a. No: agents won't report honestly.

Proof:
i.e. $b_i = x_i \ \forall i$ is not a NE.

Given $b_j = x_j \ \forall j \neq i$, agent could make profit.

by deviating to $b_i = (n+1)x_i - \sum_{j \neq i} x_j$.

b. No. $b_i = x_i$ is dominat strategy this place.

Proof:
but Even if the agents report honestly, the medium can't maximize the social welfare.

$$x_{(\frac{n}{2})} \notin \arg\max_k \sum_i u_i(k)$$

$\Rightarrow$ not DSIC.

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Allocation: we give jots to highest k bidders.

higher $b_i \Rightarrow$ higher ranking in the bidders.

$\Rightarrow$ the x_i can't decrease i.e. monotone.

Payment: $P_i(v, b_{-i}) = \beta_i \left[v\, x_i(v, b_{-i}) - \int_0^v x_i(z, b_{-i})\,dz \right]$

8. **(10 points)** Consider a set M of distinct items. There are n bidders, and each bidder i has a publicly known subset $T_i \subseteq M$ of goods that it wants, and a private valuation v_i for getting them. If bidder i is awarded a set S_i of items at a total price of p, then her utility is $v_i x_i - p$, where x_i is 1 if $S_i \supseteq T_i$ and 0 otherwise. This is a single parameter environment. Since each item can only be awarded to one bidder, a subset W of bidders can all receive their desired subsets simultaneously if and only if $T_i \cap T_j = \emptyset$ for each distinct $i, j \in W$.

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(b) Prove that if all bidders report truthfully and have sets T_i of cardinality at most d, then the outcome of the allocation rule in the greedy algorithm has social welfare at least $\frac{1}{d}$ times that of the maximum possible.

(a) Yes assume $b_i' > b_i$. b_i' ranks j this time $(j \leq i)$.
Hence, the Set M after allocating b_1 to b_{j-1} must contains all element in the Set M after allocating b_1 to b_{i-1}. So, the X_i won't decrease.
$\Longrightarrow$ X is monotone.

(b). Obviously $\forall i\ T_i \subseteq M$ $SW(ALG) \geqslant \max_i b_i = \max_i v_i$

$SW(OPT) = \max \sum_i v_i x_i \leqslant \sum_i v_i$

$$\leq d \max_i V_i \leq d \; SW(ALG)$$

$$\Rightarrow SW(ALG) \geq \frac{1}{d} SW(OPT).$$