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IE598: Homework 2

Due: March 29, 2021 (5 PM)

Note:

- The due date is strict. Late submissions will not be graded.
- Your answers must be concise and clear. Explain sufficiently so that we can easily determine that you understand.
- Only electronic submissions are accepted on **gradescope**.
- You are expected to do each homework on your own. You may discuss concepts with your classmates, but there must be no interactions about solutions.
- **There are 60 points in total, and you need 50 points for the full homework credit.**
- Plagiarism will be dealt with severely – no credit for the homework.

1. **(3 points)** Show that the Rock-Paper-Scissor game, described in the class, has a unique (mixed strategy) Nash equilibrium.

2. Suppose consumers are uniformly distributed along a sea beach of 1 mile long. Ice-cream prices are regulated, so consumers go to the nearest vendor because they dislike walking. Also, assume that they all purchase an ice cream even if they have to walk a full mile. If more than one vendor is at the same location, then they split the business evenly.

   a. **(3 points)** Consider a game in which two ice-cream vendors pick their locations simultaneously. Show that there exists a unique pure strategy Nash equilibrium.

   b. **(3 points)** Show that with three vendors, no pure Nash Equilibrium exists.

3. **(4 points)** Find all Nash equilibria of the following game:

	a_2	b_2	c_2
a_1	5, 4	-1, 2	7, 3
b_1	-1, 3	0, 7	8, 3

4. **(4 points)** Given a game (A, B) and real numbers $c_1, c_2, \ldots, c_n, d_1, \ldots, d_m$, let $\forall (i,j) \in S,\ A'_{ij} = A_{ij} + c_j$ and $B'_{ij} = B_{ij} + d_i$. Clearly (A', B') gives another game of same dimension as (A, B). Show that the set of NE of game (A, B) is exactly same as the set of NE of game (A', B').

5. **(4 points)** Consider the following (symmetric) 2-player game:

$$A = B^T = \begin{bmatrix} 3 & 3 & 0 \\ 4 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix}$$

   Find all Nash equilibria in this game.

6. Consider the Chicken and Prisoner's dilemma games in Table 1. For each of them,

	C	B
C	(3, 3)	(1, 6)
B	(6, 1)	(0, 0)

	C	D
C	(5, 5)	(0, 6)
D	(6, 0)	(1, 1)

Table 1: Chicken and Prisoner's dilemma Games

a. **(4 points)** find the correlated equilibrium that maximizes the expected sum of the two player's payoffs.

b. **(4 points)** find the correlated equilibrium that minimizes the expected sum of two player's payoffs.

7. **(2 points each)** For each statement below, decide whether you think it is *true* or *false* with a short explanation.

(*a*) In every finite two-player zero-sum game, each player gets 0 payoff at a Nash equilibrium.

(*b*) When there are multiple Nash equilibria, one equilibrium can be strongly Pareto efficient (i.e., both players get strictly better payoff) than the other.

(*c*) Nash bargaining solution gives a unique allocation.

(*d*) A game is called *constant-sum game* if the sum of payoffs of each strategy profile is a constant, that is, $A_{ij} + B_{ij} = c, \forall i, j$ where $c \in \mathbb{R}$. The set of Nash equilibria of a constant-sum game is a convex set.

(*e*) In every 2×2 *win-lose game* (every payoff is either 0 or 1), there always exists a pure Nash equilibrium.

8. Consider the bargaining problem where the feasible set is the convex hull of $\{(0,0),(1,0),(0,1),(1/2,1),(1,1/2)\}$.

a. **(2 points)** Suppose the disagreement point is $(0,0)$. What is the Nash bargaining solution?

b. **(2 points)** If the disagreement point in the above bargaining problem is $(0.5, 0.25)$. What is the Nash bargaining solution?

9. Consider a game in which two players simultaneously choose bids, which must be a non-negative integer at most 10. The higher bidder wins $10. If the bids are equal, then neither player receives anything. Each player must pay his own bid, whether or not he wins. (the loser pays too.) Each player's utility is simply his net winnings.

a. **(4 points)** Construct the matrix representation of this game.

b. **(5 points)** Construct a symmetric mixed strategy equilibrium in which every bid less than 10 has a positive probability.

10. Consider the following four-person game in coalitional form, with transferable utility.

$$
\begin{aligned}
&v(\{i\}) = 0, \ \forall i \in \{1,2,3,4\}, \\
&v(\{1,2\}) = v(\{1,3\}) = v(\{2,4\}) = v(\{3,4\}) = 1, \\
&v(\{1,4\}) = v(\{2,3\}) = 0, \\
&v(\{2,3,4\}) = v(\{1,3,4\}) = v(\{1,2,4\}) = 1, \\
&v(\{1,2,3\}) = 2 = v(\{1,2,3,4\}).
\end{aligned}
$$

(Notice that the worth of each coalition except {1,2,3} is equal to the number of disjoint pairs that consist of one player from {1,4} and one player from {2,3} that can be formed among the coalition's members.)

a. **(2 points)** Show that the core of this game consists of a single allocation vector.

b. **(2 points)** Compute the Shapley value of this game. (Notice the symmetry between players 2 and 3.)

c. **(2 points)** Suppose that the worth of {1,2,3} were changed to $v(\{1,2,3\}) = 1$. Characterize the core of the new game, and show that all of the new allocations in the core are strictly better for player 1 than the single allocation in the core of the original game.

d. **(2 points)** Compute the Shapley value of the new game from (*c*).

1.

	R	P	S
R	(0,0)	(-1,1)	(1,-1)
P	(1,-1)	(0,0)	(-1,1)
S	(-1,1)	(1,-1)	(0,0)

obviously, there are no NE in pure strategies.

mixed strategies:

$X = (x_1, x_2, x_3)$ $\quad y = (y_1, y_2, y_3)$

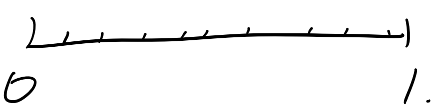
$\sum x_i = 1$ $\quad \sum y_j = 1.$

$$NE \Rightarrow \begin{cases} -x_2 + x_3 = x_1 - x_3 = -x_1 + x_2 \\ -y_2 + y_3 = y_1 - y_3 = -y_1 + y_2 \end{cases} \Rightarrow \underline{x_1 = x_2 = x_3 = y_1 = y_2 = y_3 = \frac{1}{3}}$$

which is unique solution

$\Rightarrow$ unique NE in the game.

2.



① set V_1 at $x \in [0,1]$ $\quad V_2$ at $y \in [0,1]$.

we can set $x \leq y$ which won't influence result.

$(\frac{1}{2}, \frac{1}{2})$ is the unique pure strategy NE

i.e. both locate at middle point.

1) obviously $(\frac{1}{2}, \frac{1}{2})$ is a NE. anyone moves will decrease its profit and increase others

2) assume there is another pure NE: (x_0, y_0) $\quad x_0 < y_0$

profit of $x_0 = x_0 + \frac{y_0 - x_0}{2} = \frac{x_0 + y_0}{2}$

profit of $y_0 = 1 - y_0 + \frac{y_0 - x_0}{2} = 1 - \frac{x_0 + y_0}{2}$

However, in this situation $x = x_0 + \varepsilon < y_0$ $(\varepsilon \to 0)$

can make a larger profit

$\Rightarrow$ contradicts to NE's property.

$\Rightarrow$ $(\frac{1}{2}, \frac{1}{2})$ is a unique pure NE.

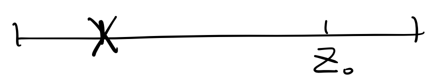
② set V_1 set $x \in [0,1]$, V_2 set $y \in [0,1]$, V_3 set $z \in [0,1]$

assume there is a pure NE (x_0, y_0, z_0)

s.t. $x_0 \le y_0 \le z_0$.

when $x_0 = y_0 < z_0$

profit of $x = \frac{1}{2}(x_0 + \frac{z_0 - x_0}{2}) = \frac{x_0 + z_0}{4}$



profit of $y = \frac{x_0 + z_0}{4}$

profit of $z = 1 - z_0 + \frac{z_0 - y_0}{2} = 1 - \frac{z_0 + y_0}{2}$

when $x_0 < y_0 = z_0$

profit of $x = x_0 + \frac{y_0 - x_0}{2} = \frac{x_0 + y_0}{2}$

profit of $y = \frac{1}{2}(1 - y_0 + \frac{y_0 - x_0}{2}) = \frac{1}{2} - \frac{x_0 + y_0}{4}$

profit of $z = \frac{1}{2} - \frac{x_0 + z_0}{4}$

when $x_0 < y_0 < z_0$

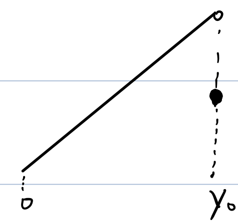
profit of $x = x_0 + \frac{y_0 - x_0}{2} = \frac{x_0 + y_0}{2}$

profit of $y = \frac{y_0 + z_0}{2} - \frac{x_0 + y_0}{2} = \frac{z_0 - x_0}{2}$

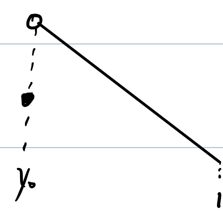
profit of $z = 1 - z_0 + \frac{z_0 - y_0}{2} = 1 - \frac{z_0 + y_0}{2}$

when $x_0 = y_0 = z_0$ profit all equal to $\frac{1}{3}$.

Profit of x (given y_0)



Profit of z (given y_0)



Hence x wants $y_0 - \varepsilon_0$ $\quad$ ($\varepsilon_0 > 0$ $\quad \varepsilon_0 \to 0$)

z wants $y_0 + \varepsilon_0$

However the line $[0,1]$ is continous, we can always find $\varepsilon_0' > 0$ smaller than ε_0 can increase profit of x and z.

$\Rightarrow$ there are no pure NE.

3.

		q	$1-q$	
		a_2	b_2	c_2
p	a_1	5, 4	-1, 2	7, 3
$1-p$	b_1	-1, 3	0, 7	8, 3

pure NE : (a_1, a_2), (b_1, b_2).

mixed NE : $[(\frac{2}{3}a_1, \frac{1}{3}b_1), (\frac{1}{7}a_2, \frac{6}{7}b_2)]$.

$$\begin{cases} 5q - 1 + q = -q \\ 4p + 3 - 3p = 2p + 7 - 7p \end{cases} \qquad \begin{cases} q = \frac{1}{7} \\ p = \frac{2}{3} \end{cases}$$

4. $\forall$ NE of (A, B) : (x, y)

$$\Rightarrow \quad x^T A y \geq \tilde{x}^T A y \quad \forall \tilde{x}$$

$$x^T B y \geq x^T B \tilde{y} \quad \forall \tilde{y}$$

$$C = \begin{bmatrix} c_1 & \cdots & c_n \\ c_1 & \cdots & c_n \\ \vdots & & \vdots \\ c_1 & \cdots & c_n \end{bmatrix} \qquad D = \begin{bmatrix} d_1 & d_1 & \cdots & d_1 \\ d_2 & \cdots & \cdots & \vdots \\ \vdots & & \ddots & \vdots \\ d_m & \cdots & & d_m \end{bmatrix} \qquad \sum x_i = 1 \quad \sum y_j = 1$$

$$\Rightarrow \quad x^T(A+C)y = x^T A y + [c_1 \cdots c_n] \cdot y$$

$$\geq \tilde{x}^T A y + [c_1 \cdots c_n] y = \tilde{x}^T (A+C) y.$$

Similarly $x^T(B+D)y = x^T B y + x^T [d_1 \cdots d_m]$

$$\geq x^T B \tilde{y} + x^T [d_1 \cdots d_m] = x^T (B+D) \tilde{y}$$

$\Rightarrow$ (x, y) is NE of $(A+C, B+D)$

i.e. (A', B').

5. $A = \begin{bmatrix} 3 & 3 & 0 \\ 4 & 0 & 1 \\ 0 & 4 & 5 \end{bmatrix}$ $\qquad B = \begin{bmatrix} 3 & 4 & 0 \\ 3 & 0 & 4 \\ 0 & 1 & 5 \end{bmatrix}$

$$\begin{bmatrix} (3,3) & (3,4) & (0,0) \\ (4,3) & (0,0) & (1,4) \\ (0,0) & (4,1) & (5,5) \end{bmatrix} \quad NE: (x_3, y_3)$$

$$\Rightarrow (5,5).$$

6. a.

	C	B
C	(3, 3) =0	(1, 6) 1-k
B	(6, 1) k	(0, 0) =0

$$\max \ 6P_{CC} + 7(P_{CB} + P_{BC})$$

$$\text{s.t.} \quad 3P_{CC} + P_{CB} \geq 6P_{CC} \qquad P_{CB} \geq 3P_{CC}$$

$$6P_{BC} \geq 3P_{BC} + P_{BB} \qquad \cancel{3P_{BC} \geq P_{BB}}$$

$$3P_{CC} + P_{BC} \geq 6P_{CC} \qquad P_{BC} \geq 3P_{CC}$$

$$6P_{CB} \geq 3P_{CB} + P_{BB} \qquad \cancel{3P_{CB} \geq P_{BB}}.$$

$$P_{ij} \geq 0 \qquad \sum P_{ij} = 1.$$

$$\Rightarrow \begin{cases} P_{CC} = 0 \\ P_{BB} = 0 \\ P_{BC} = k \\ P_{CB} = 1-k \end{cases} \qquad \forall k \in [0,1].$$

b.

	C	D
C	(5, 5) =0	(0, 6) =0.
D	(6, 0) =0	(1, 1) =1

$$\min \ 10P_{CC} + 6(P_{CD} + P_{DC}) + 2P_{DD}.$$

$$\text{s.t.} \quad 5P_{CC} \geq 6P_{CC} + P_{CD}$$

$$6P_{DC} + P_{DD} \geq 5P_{DC}$$

$$5P_{CC} \geq 6P_{CC} + P_{DC} \qquad 0 \geq P_{CC} + P_{DC}$$

$$6P_{CD} + P_{DD} \geq 5P_{CD} \qquad P_{CD} + P_{DD} \geq 0.$$

$$P_{ij} \geq 0 \qquad \sum P_{ij} = 1.$$

$$\Rightarrow \begin{cases} P_{CD} = 0. \\ P_{DD} = 1 \\ P_{CC} = 0 \\ P_{DC} = 0 \end{cases}$$

7. (a) false.

$$\begin{bmatrix} (1,-1) & (1,-1) \\ (2,-2) & (2,-2) \end{bmatrix}$$

like this game. Player 2 always gets negative payoff.

(b) No, if this kind of NE exists, it will be the only NE contradicts to multiple NE.

(c). Yes $\phi(F,v) \in \arg\max_{x \in F : x \geq v} (x_1 - v_1)(x_2 - v_2)$

it is strictly concave.

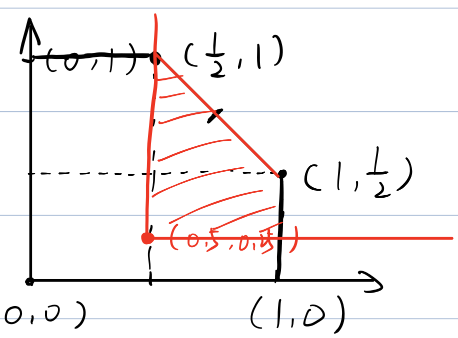
(d). Yes. constant-sum-game can be transformed into a zero-sum-game, zero-sum-game's NE is convex set.

(e). No

$$\begin{bmatrix} (0,1) & (1,0) \\ (1,0) & (0,1) \end{bmatrix}$$

$\Rightarrow$ win-lose game but no pure NE.

8. a.



$\phi_1(F,v) = \phi_2(F,v) = \frac{3}{4}$.

$(\frac{3}{4}, \frac{3}{4})$

b. $\max (x_1 - 0.5)(x_2 - 0.45)$

s.t. $x_2 = \frac{3}{2} - x_1 \quad x_1 \in [0.5, 1]$.

$$\max \left(x_1-\frac{1}{2}\right)\left(\frac{5}{4}-x_1\right)$$

$$\Rightarrow \begin{cases} x_1 = \frac{7}{8} \\ x_2 = \frac{3}{8} \end{cases} \qquad \underline{\left(\frac{7}{8}, \frac{3}{8}\right)}$$

9\. a.

	0	1	2	3	4	5	6	7	8	9	10
0	(0,0)	(0,9)	(0,8)	(0,7)	(0,6)	(0,5)	(0,4)	(0,3)	(0,2)	(0,1)	(0,0).
1	(9,0)	(-1,-1)	(-1,8)	(-1,7)	(-1,6)	(-1,5)	(-1,4)	(-1,3)	(-1,2)	(-1,1)	(-1,0).
2	(8,0)	(8,-1)	(-2,-2)	(-2,7)	(-2,6)	(-2,5)	(-2,4)	(-2,3)	(-2,2)	(-2,1)	(-2,0).
3	(7,0)	(7,-1)	(7,-2)	(-3,-3)	(-3,6)	(-3,5)	(-3,4)	(-3,3)	(-3,2)	(-3,1)	(-3,0)
4	(6,0)	(6,-1)	(6,-2)	(6,-3)	(-4,-4)	(-4,5)	(-4,4)	(-4,3)	(-4,2)	(-4,1)	(-4,0)
5	(5,0)	(5,-1)	(5,-2)	(5,-3)	(5,-4)	(-5,-5)	(-5,4)	(-5,3)	(-5,2)	(-5,1)	(-5,0)
6	(4,0)	(4,-1)	(4,-2)	(4,-3)	(4,-4)	(4,-5)	(-6,-6)	(-6,3)	(-6,2)	(-6,1)	(-6,0)
7	(3,0)	(3,-1)	(3,-2)	(3,-3)	(3,-4)	(3,-5)	(3,-6)	(-7,-7).	(-7,2)	(-7,1)	(-7,0)
8	(2,0)	(2,-1)	(2,-2)	(2,-3)	(2,-4)	(2,-5)	(2,-6)	(2,-7)	(-8,-8)	(-8,1)	(-8,0)
9	(1,0)	(1,-1)	(1,-2)	(1,-3)	(1,-4)	(1,-5)	(1,-6)	(1,-7)	(1,-8)	(-9,-9)	(-9,0)
10	(0,0)	(0,-1)	(0,-2)	(0,-3)	(0,-4)	(0,-5)	(0,-6)	(0,-7)	(0,-8)	(0,-9)	(-10,-10)

b. set $X = Y = [x_0, x_1, \ldots, x_{10}]^T$. $\sum_{i=1}^{10} x_i = 1$.

$$0 = 10x_0 - 1 = 10x_0 + 10x_1 - 2 = 10(x_0+x_1+x_2) - 3.$$

$$= \cdots\cdots = 10(x_0+x_1+\cdots+x_9) - 10.$$

$$\Rightarrow x_0 = \frac{1}{10} \quad x_1 = \frac{1}{10} \quad x_2 = \frac{1}{10} \quad \cdots \quad x_9 = \frac{1}{10}, \quad x_{10} = 0.$$

i.e. $X = Y = \left[\frac{1}{10}, \frac{1}{10}, \ldots, \frac{1}{10}, 0\right]^T$.

10\. a. $x_i \geq 0 \quad i = 1, 2, 3, 4.$

$x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_4 \geq 1$, $x_3 + x_4 \geq 1$.

$\cancel{x_1 + x_4 \geq 0}$, $\cancel{x_2 + x_3 \geq 0}$.

$\cancel{x_2 + x_3 + x_4 \geq 1}$. $\cancel{x_1 + x_3 + x_4 \geq 1}$, $\cancel{x_1 + x_2 + x_4 \geq 1}$.

$x_1 + x_2 + x_3 \geq 2$. $\quad x_1 + x_2 + x_3 + x_4 = 2$.

$\Rightarrow x_4 = 0. \Rightarrow x_2 = 1. \quad x_3 = 1.$

$\Rightarrow x_1 = 0.$

i.e. single allocation $(0, 1, 1, 0)$.

b.

① $1\times\times\times$: $\frac{3!}{4!}$

② $41\times\times$: $\frac{2!}{4!}$

① $\times\times12$, $\times\times13$: $2\times\frac{2!}{4!}$

$\varphi_1(N, v) =$

$\Rightarrow 2\times\frac{2!}{4!}$

$+ 1\times\left(\frac{3!}{4!} + 2\times\frac{2!}{4!}\right)$

$= \frac{7}{12}$

① $2\times\times\times$: $\frac{3!}{4!}$

① $42\times\times$: $\frac{2!}{4!}$

① $32\times\times$: $\frac{2!}{4!}$

① $\times\times21$, $\times\times24$: $2\times\frac{2!}{4!}$

$\varphi_2(N, v)$

$= \varphi_3(N, v) = \frac{7}{12}$

① 14XX: $\frac{2!}{4!}$

① XX42, XX43: $2\times\frac{2!}{4!}$

$\Bigg\}\ \psi_4(N,v) = \frac{1}{4}$

$\psi = [\frac{7}{12}, \frac{7}{12}, \frac{7}{12}, \frac{1}{4}]$.

C. $x_i \geq 0 \quad i=1,2,3,4$.

$x_1+x_2 \geq 1,\ x_1+x_3 \geq 1,\ x_2+x_4 \geq 1,\ x_3+x_4 \geq 1$.

$\cancel{x_1+x_4 \geq 0},\ \cancel{x_2+x_3 \geq 0}$.

$\cancel{x_2+x_3+x_4 \geq 1}.\ \cancel{x_1+x_3+x_4 \geq 1},\ \cancel{x_1+x_2+x_4 \geq 1}$.

$\cancel{x_1+x_2+x_3 \geq 1}$. $\quad x_1+x_2+x_3+x_4 = 2$.

$$\begin{cases} x_1+x_2+x_3+x_4=2 \\ x_1+x_2 \geq 1 \end{cases} \Rightarrow x_3+x_4 \leq 1$$

Since $x_3+x_4 \geq 1 \Rightarrow x_3+x_4 = 1$.

$\Rightarrow x_1+x_2 = x_3+x_4 = 1$.

Similiarly we can get $x_1+x_3 = x_2+x_4 = 1$

$x_1 = t = x_4 \quad x_2 = x_3 = 1-t \quad t \in (0,1]$.

allocation: $\{(t, 1-t, 1-t, t) \mid \forall t \in [0,1]\}$.

$t \geq 0 \Rightarrow$ strictly better than original.

get equality only when $t=0$.

d.

① $1 \times \times \times : \frac{3!}{4!}$

① $4\ 1 \times \times : \frac{2!}{4!}$

① $\times \times 1 2 ,\ \times \times 1 3 : 2 \times \frac{2!}{4!}$

$$\Rightarrow \psi_1(N,v) = \left(1 \times \frac{3!}{4!} + 2 \times \frac{2!}{4!} + \frac{2!}{4!}\right) = \underline{\underline{\frac{1}{2}}}$$

① $2 \times \times \times : \frac{3!}{4!}$

① $3\ 2 \times \times : \frac{2!}{4!}$

① $\times \times 2 1 ,\ \times \times 2 4 : 2 \times \frac{2!}{4!}$

$$\psi_2(N,v) = \psi_3(N,v) = \underline{\underline{\frac{1}{2}}}$$

① $1 4 \times \times : \frac{2!}{4!}$

① $\times \times 4 2 ,\ \times \times 4 3 : 2 \times \frac{2!}{4!}$

$$\psi_4(N,v) = \underline{\underline{\frac{1}{4}}}$$

$$\psi = \left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right].$$