Def^n An inner product $\langle \, , \rangle$ on a vector space is a map. $\langle \, , \rangle : V \times V \to \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$$(x, y) \longrightarrow \langle x, y \rangle$$

such that

$\langle x, y+z \rangle = \langle y+z, x \rangle = \langle y, x \rangle + \langle z, x \rangle = \langle x, y \rangle + \langle x, z \rangle$

i) $\langle x+z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$

ii) $\langle cx, y \rangle = c \langle x, y \rangle$

iii) $\langle x, y \rangle = \overline{\langle y, x \rangle}$

(note if $\mathbb{F} = \mathbb{R} \Rightarrow \langle x, y \rangle = \langle y, x \rangle$)

iv) $\langle x, x \rangle > 0$ if $x \neq \bar{0}$

Example (Frobenious): $V = M_{n \times n}(\mathbb{R})$

$\langle A, B \rangle = tr(B^t A)$

(tr:) $M_{n \times n}(\mathbb{R}) \longrightarrow \mathbb{R}$

is linear. $\quad C \longrightarrow \sum_{j=1}^{n} C_{jj}$

Check: (i) $\langle A+B, C \rangle = tr(C^t(A+B))$

$$= tr(C^t A) + tr(C^t B)$$

$$= \langle A, C \rangle + \langle B, C \rangle.$$

(ii) $\langle cA, B\rangle = tr(B^t(cA)) = tr(cB^tA)$.

$= c\, tr(B^tA) = c\langle A, B\rangle$.

(iii) $\langle A, B\rangle = tr(B^tA) = tr[(B^tA)^t]$

$= tr(A^tB) = \langle B, A\rangle$.

(iv) $\langle A, A\rangle = tr(A^tA) = \sum_j (A^tA)_{jj}$

$(A^tA)_{jj} = \sum_k (A^t)_{jk} A_{kj} = \sum_k A_{kj}^2 > 0$.

unless $A_{1j}, \cdots A_{nj} = 0$

Example: $\langle x, y\rangle = \sum_{i=1}^n a_i \bar{b}_i$, $x = (a_1, \cdots, a_n)$, $y = (b_1, \cdots, b_n)$

Standard inner product.

Inner product space.

Def$^{\underline{n}}$ An inner product space is a vector space V with a fixed inner product.

Let $V, \langle , \rangle$ be an inner product space.

Def$^{\underline{n}}$ The length (or norm) of $v \in V$ is $\|v\| = \sqrt{\langle v, v\rangle}$.

Def$^{\underline{n}}$ The distance between x, y in V is $\|x-y\|$.

Def$^{\underline{n}}$ The sphere of radius r and center $x \in V$

is $\{ y \in V \mid \|x-y\| = r \}$

Theorem For any $x, y \in V$ and $c \in \mathbb{F}$.

(a) $\|cx\| = |c| \, \|x\|$

(b) $\|x\| = 0 \iff x = \bar{0}_V$

(c) $|\langle x, y \rangle| \leq \|x\| \, \|y\|$ Cauchy-Schwarz

(d) $\|x+y\| \leq \|x\| + \|y\|$. triangle inequality.