Math 416 **Name (Print):** Yang Wenxiao
Summer 2020
Final Ver 2

Time Limit: 180 Minutes Proctor ____________

This exam contains 13 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Do not write in the table to the right.

Problem	Points	Score
1	18	
2	18	
3	18	
4	20	
5	16	
6	20	
7	20	
8	24	
9	30	
10	16	
Total:	200	

1. (18 points) For what value(s) of c does the system of linear equations have a solution

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & cx_3 & + & x_4 & = & c \\ & & -x_2 & + & x_3 & + & 2x_4 & = & 0 \\ x_1 & + & 2x_2 & + & x_3 & - & x_4 & = & -c \end{array}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 1 & 2 & 1 & -1 & -c \end{array}\right] \longrightarrow \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & -1 & 1 & 2 & 0 \\ 0 & 1 & 1-c & -2 & -2c \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 1 & c & 1 & c \\ 0 & 1 & 1-c & -2 & -2c \\ 0 & 0 & 2-c & 0 & -2c \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & c+1 & 3 & c \\ 0 & 1 & -1 & -2 & 0 \\ 0 & 0 & 2-c & 0 & -2c \end{bmatrix}$$

$c \neq 2$, $$\begin{cases} x_1 = c - 3t - (c+1)\dfrac{-2c}{2-c} \\ x_2 = 2t + \dfrac{-2c}{2-c} \\ x_3 = \dfrac{-2c}{2-c} \\ x_4 = t \end{cases}$$

$\underline{c \neq 2}$ the system have a solution.

2. (18 points) Let

$$A = \begin{pmatrix} 5 & 2 & 0 & -8 & -8 \\ 4 & 1 & 2 & -8 & -9 \\ 5 & 1 & 3 & 5 & 19 \\ -8 & -5 & 6 & 8 & 5 \end{pmatrix}$$

(a) (9 points) Find the null space of A

$$A x = \bar{0} \qquad \left[\begin{array}{ccccc|c} 5 & 2 & 0 & -8 & -8 & 0 \\ 4 & 1 & 2 & -8 & -9 & 0 \\ 5 & 1 & 3 & 5 & 19 & 0 \\ -8 & -5 & 6 & 8 & 5 & 0 \end{array}\right]$$

$$\rightarrow \begin{bmatrix} 1 & 1 & -2 & 0 & 1 & 0 \\ 0 & -3 & 10 & -8 & -13 & 0 \\ 0 & 0 & 1 & -47 & -94 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{cases} x_1 = -60t_1 - 122t_2 \\ x_2 = 154t_1 + 309t_2 \\ x_3 = 47t_1 + 94t_2 \\ x_4 = t_1 \\ x_5 = t_2 \end{cases}$$

$$N(A) = \operatorname{Span}\left\{ \begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

(b) (9 points) Find a basis for the null space of A

$$\beta = \left\{ \begin{bmatrix} -60 \\ 154 \\ 47 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -122 \\ 309 \\ 94 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

3. (18 points) Prove the following.

(a) (6 points) If A is similar to the identity matrix I, then $A = I$

A is similar to $I \Rightarrow \exists Q.\ A = Q^{-1} I Q$

$\Rightarrow A = Q^{-1} Q = I$

(b) (6 points) If A is invertible, then AB is similar to BA,

A is invertible $\Rightarrow \exists A^{-1},\ AA^{-1} = A^{-1}A = I$

$\Rightarrow AB = A(BA)A^{-1}$

$\Rightarrow AB$ is similar to BA.

(c) (6 points) If A is similar to B, then A^k is similar to B^k for any positive integer k.

A is similar to $B \Rightarrow \exists Q.\ A = Q^{-1} B Q$

$\Rightarrow A^k = (Q^{-1} B Q)^k \quad k \in \mathbb{N}^+$

$= (Q^{-1} B Q) \cdot (Q^{-1} B Q) \cdots (Q^{-1} B Q)$

$= Q^{-1} B (Q Q^{-1}) B (Q Q^{-1}) B \cdots (Q Q^{-1}) B Q$

$= Q^{-1} B^k Q$

$\Rightarrow A^k$ is similar to B^k, $k \in \mathbb{N}^+$

4. (20 points) Circle each set of vectors below that is a subspace of $\mathbb{R}^3$. No justification needed.

   (a) $\{(a, b, c) \in \mathbb{R}^3 : a = b = 0\}$

   (b) $\{(a, b, c) \in \mathbb{R}^3 : b = 0\}$

   (c) $\{(a, b, c) \in \mathbb{R}^3 : ab = 0\}$

   (d) $\{(a, b, c) \in \mathbb{R}^3 : a = b\}$

   (e) $\{(a, b, c) \in \mathbb{R}^3 : ab > 0\}$

5. (16 points) Do ONE of the following.

   (a) Suppose V is a finite dimensional inner product space. Prove that every orthogonal operator T on V, i.e., $\langle T(u), T(v)\rangle = \langle u, v\rangle$, $\forall u, v \in V$, is an isomorphism.

   (b) Prove that if T is a normal operator on an inner product space V, then

$$\|T(x)\| = \|T^*(x)\|$$

   for every $x \in V$.

(b) T is normal $\Rightarrow TT^* = T^*T$

$\| T(x) \| = \sqrt{\langle T(x), T(x) \rangle} = \sqrt{\langle x, T^*T(x) \rangle}$.

$= \sqrt{\langle x, TT^*(x) \rangle} = \sqrt{\langle T^*(x), T^*(x) \rangle}$

$= \| T^*(x) \|$

6. (20 points) Using the following steps find a Jordan Canonical Form of the linear map

$$T\colon P_3(\mathbb{R}) \to P_3(\mathbb{R})$$

that sends a polynomial $a + bx + cx^2 + dx^3$ to the polynomial

$$(3a + b + c - d) + (-a + b - c + d)x + (a + b + 2)x^2 + (2a + 2b + c + d)x^3.$$

Let $\alpha = \{1, x, x^2, x^3\}$ be the standard ordered basis for $P_3(\mathbb{R})$.

Step I: Let $A = [T]_\alpha = \begin{pmatrix} 3 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & 1 & 2 & 0 \\ 2 & 2 & 1 & 1 \end{pmatrix}$ The characteristic polynomial is $(1-t)(2-t)^3$. You *need not* compute it.

Step II: Compute the dimensions of E_1 and E_2, where E_1, E_2 are the eigenspaces corresponding to the eigenvalues 1, 2, respectively. Note $E_1 \subseteq K_1$ and $E_2 \subseteq K_2$, where K_1, K_2 are the generalized eigenspaces corresponding to the eigenvalues 1, 2, respectively.

$$A - I_4 = \begin{bmatrix} 2 & 1 & 1 & -1 \\ -1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 0 \\ 2 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_1 = \operatorname{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$A - 2I_4 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_2 = \operatorname{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Question continued

$\beta_1 = \left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$

Step III: Determine the cycle structure of each of the generalized spaces K_1 and K_2.

$$(A-2I_4)^2 = \begin{bmatrix} -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & 1 \end{bmatrix} \Rightarrow \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$(A-2I_4)^3 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & -1 \end{bmatrix} \Rightarrow \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

$$\gamma_1 = \left\{ (A-2I_4)\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\gamma_2 = \left\{ (A-2I_4)\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} = \left\{ \begin{pmatrix} 0 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$$K_1 = \text{span}(\beta_1) = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$\beta_2 = \gamma_1 \cup \gamma_2 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\} \quad K_2 = \text{span}(\beta_2) = \text{Span}\left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Step IV: Write a Jordan Canonical Form.

$r_{1,1} = 4-3 = 1.$

$r_{2,1} = 4-2 = 2$

$r_{2,2} = 2-1 = 1$

$$\text{JCF}: \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

7. (20 points) Do **FIVE** of the following.

   (a) State the Dimension Theorem (also called Rank-Nullity Theorem).

   (b) State Cramer's rule.

   (c) Define the Frobenius inner product on $M_{n\times n}(\mathbb{F})$.

   (d) State the Cauchy–Schwarz Inequality.

   Let $T : V \to V$ be a linear map.

   (e) Define the generalized eigenspace of an eigenvalue λ of T.

   (f) Define what it mean for x to be a generalized eigenvector corresponding to an eigenvalue λ of T.

   (g) Define the cycle of a generalized eigenvector x corresponding to an eigenvalue λ of T.

   (h) Define what it means for a projection $T : V \to V$ to be an orthogonal projection.

(a) if map $T: V \to T$ is linear, and V is finite dimensional.
then $\dim(N(T)) + \dim(R(T)) = \dim(V)$

(b) In a system of linear equations.
$Ax = b$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & \cdots & a_{nn} \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$

For $k = 1, 2, \dots, n$

$x_k = \dfrac{\det(M_k)}{\det(A)}$, M_k is the matrix A replacing k^{th} colum by b.

(d) $\langle , \rangle : V \times V \to \mathbb{F}$, $\forall x, y \in V$

$|\langle x, y\rangle| \leq \|x\| \|y\|$

(e) generalized eigenspace of λ is
$K_\lambda = \{ x \in V : (T - \lambda I_V)^p (x) = \bar{0}_V$, for some integer $p > 0 \}$

(f) $x \in V$ is a generalized eigenvector corresponding to λ
if $(T - \lambda I_V)^p (x) = \bar{0}_V$ for some integer $p > 0$.

8. (24 points) Mark each of the following as True or False

Let S be a set of m vectors in $\mathbb{R}^n$

(a) If $m > n$ then S is linearly independent

False

(b) If S is linearly independent and T is a subset of S, then T is linearly independent

True

(c) If S is linearly dependent and T is a subset of S, then T is linearly dependent

False

(d) If $m = n$ then S forms a basis for $\mathbb{R}^n$

False

Let A and B be n by n matrices

(e) $\det(AB) = \det(B)\det(A)$

True

(f) $\det(A^T) = \det(A)$

True

(g) $\det(A^{-1}) = \det(A)$

False

(h) $\det(kA) = k^n \det(A)$

True

9. (30 points) Assume that the vector space $P_2(\mathbb{R})$ is equipped with the inner product

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)dt.$$

(a) (10 points) Applying Gram-Schmidt process to $\beta = \{1,\ 2x,\ x^2\}$ to find an orthonormal basis α for the subspace $P_2(\mathbb{R})$.
Hint: Draw a picture and use the subspace $W = \text{span}\{1,\ 2x\}$ to find a vector orthogonal to W. You may also want to use the fact that the integral of any odd function over $[-1,1]$ is 0.

$$u_1 = \frac{1}{\sqrt{\langle 1, 1\rangle}} = 1$$

$$2x - \langle 2x, 1\rangle \cdot 1 = 2x$$

$$u_2 = \frac{2x}{\sqrt{\langle 2x, 2x\rangle}} = \sqrt{3}\,x$$

$$x^2 - \langle x^2, \sqrt{3}x\rangle \cdot \sqrt{3}x - \langle x^2, 1\rangle \cdot 1 = x^2 - \frac{1}{3}$$

$$u_3 = \frac{x^2 - \frac{1}{3}}{\sqrt{\langle x^2 - \frac{1}{3},\ x^2 - \frac{1}{3}\rangle}} = \frac{3\sqrt{5}}{2}x^2 - \frac{\sqrt{5}}{2}$$

$$\alpha = \left\{ 1,\ \sqrt{3}x,\ \frac{3\sqrt{5}}{2}x^2 - \frac{\sqrt{5}}{2} \right\}$$

(b) (5 points) Compute the coordinate vector $[f]_\alpha$ for $f(x) = x - x^2$. (*Express the vector f in terms of the orthonormal vectors you have found in part (a).*)

$$[f]_\alpha = (x_1,\ x_2,\ x_3)$$

$$\begin{cases} x_1 - \frac{\sqrt{5}}{2}x_3 = 0 \\ \sqrt{3}x_2 = 1 \\ \frac{3\sqrt{5}}{2}x_3 = -1 \end{cases} \Rightarrow \begin{cases} x_1 = -\frac{1}{3} \\ x_2 = \frac{\sqrt{3}}{3} \\ x_3 = -\frac{2\sqrt{5}}{15} \end{cases}$$

$$[f]_\alpha = \left(-\frac{1}{3},\ \frac{\sqrt{3}}{3},\ -\frac{2\sqrt{5}}{15}\right).$$

Question continued

Now consider $P_2(\mathbb{R})$ as a subspace of $P_3(\mathbb{R})$ equipped with the inner product

$$\langle f, g\rangle = \frac{1}{2}\int_{-1}^{1} f(t)g(t)dt.$$

(c) (10 points) Find the orthogonal projection of $h(x) = x^3$ onto $P_2(\mathbb{R})$.

$\forall y \in P_2(\mathbb{R})$ Let $y = a + bx + cx^2$

$$\|x^3 - y\|^2 = \langle x^3 - a - bx - cx^2, x^3 - a - bx - cx^2\rangle$$

$$= \frac{1}{7} + a^2 + \frac{b^2}{3} - \frac{2b}{5} + \frac{c^2}{5} + \frac{2ac}{3}$$

$$= \frac{1}{7} + (a + \frac{1}{3}c)^2 + \frac{1}{3}(b - \frac{3}{5})^2 - \frac{3}{25} + \frac{4}{45}c^2$$

$$\min_y \|x^3 - y\| \Rightarrow a = 0 \quad b = \frac{3}{5} \quad c = 0$$

$$\Rightarrow y = \frac{3}{5}x$$

$$\Rightarrow \mathrm{proj}_{P_2(\mathbb{R})}(x^3) = \frac{3}{5}x$$

(d) (5 points) Using part (c) find $P_2(\mathbb{R})^\perp$ as a subspace of $P_3(\mathbb{R})$.

$P_2(\mathbb{R})^\perp = \{ s \in P_3(\mathbb{R}) \mid \langle s, y\rangle = 0 \quad \forall y \in P_2(\mathbb{R})\}$

from (c): $\langle x^3, y\rangle = \langle \frac{3}{5}x, y\rangle$

$\langle a_1 + a_2x + a_3x^2 + a_4x^3, b_1 + b_2x + b_3x^2\rangle$.

$= \langle a_1 + (a_2 + \frac{3}{5}a_4)x + a_3x^2, b_1 + b_2x + b_3x^2\rangle$

$\Rightarrow a_1 = 0, \ a_2 + \frac{3}{5}a_4 = 0, \ a_3 = 0$

$$P_2(\mathbb{R})^\perp = \{-\frac{3}{5}tx + tx^3 \mid t \in \mathbb{R}\}$$

10. (16 points) or the following n by n matrix, find all the eigenvalues

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\det(A - tI_n) = \det\begin{bmatrix} 1-t & 1 & \cdots & 1 \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1-t \end{bmatrix} = \det\begin{bmatrix} n-t & n-t & \cdots & n-t \\ 1 & 1-t & \cdots & 1 \\ \vdots & & \ddots & \vdots \\ 1 & \cdots & & 1-t \end{bmatrix}$$

$$= (n-t)\det\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1-t & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1-t \end{bmatrix} = (n-t)\det\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & -t & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & & -t \end{bmatrix}$$

$$= (n-t)(-t)^{n-1}$$

$\lambda_1 = n$ $\qquad\qquad$ $\lambda_2 = 0$