1. Vector Space example

addition
multiplication

$C^0(\mathbb{R}) = \{\text{continuous function}\}$.

$f(x) = x^2$, $g(x) = \sin x$.

$(f+g)(x) = x^2 \sin x$

$(cf)(x) = c\,x^2$

2. Linear maps. example

① $\mathbb{R} \to C^0(\mathbb{R})$ $\quad y \to f_y(x) = y$

<real number> produces <a function>

② $C^1(\mathbb{R}) \to C^0(\mathbb{R})$ $\quad f \to f'$

$C^1(\mathbb{R}) = \{\text{continous functions with continous derivatives}\}$.

③ An $m \times n$ matrix A defines a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$.

example: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $L_A : \mathbb{R}^2 \to \mathbb{R}^2$ $\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}$

maps like L_A are used to study linear equations.

3. Systems of linear equations.

Variables $x_1, x_2 \cdots x_n$

Equations (linear) m of them.

$a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$

....

$a_{m1}x_1 + a_{m2}x_2 + \cdots a_{mn}x_n = b_m$

find all solutions.

Strategy:

Manipulate equations to get to simpler system.

elementary row operations.

① $E_i \longleftrightarrow E_j$ ② $E_i \longleftrightarrow cE_i \ (c \neq 0)$ ③ $E_i \longleftrightarrow E_i + cE_j$

$$\begin{array}{c}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}\end{array}\begin{array}{c} R_1 \\ \cdots \\ \\ R_m \end{array}$$

Theorem 1: None of these change the set of solutions.

Row Echelon Form:

i.
Def^n the leading entry in a row is the leftmost nonzero entry

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}$$

ii.
Def^n A matrix is in Row Echelon Form (REF)

if (1) All zero rows are below all nonzero rows.

(2) the leading entry of each row to be to the right of the leading entry of the row above

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iii. Def^n A matrix is in Reduced Row Echelon Form (RREF)

if it is in REF and

(1) the leading entries are all 1.

(2) the leading entries are the only nonzero enery in the columns

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem 2: Every matrix can be put in RREF by a finite sequence of elementary row operations.

Theorem 3: The RREF of a matrix is unique.

Theorem 4: The solution set of a linear system with augmented matrix in RREF is easily described in a standard way.

example: Consider the system with augmented matrix.

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{LS} \begin{aligned} x_1 \quad -x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 1 \end{aligned}$$

This system has no solutions. (It is inconsistent).

(Moral: A leading entry in the rightmost colum of REF $\Rightarrow$ system is inconsistent.)

example:

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{LS} \begin{aligned} x_1 \quad -x_3 &= 0 \\ x_2 \quad &= 0 \\ 0 &= 0 \end{aligned}$$ infinitely solutions.

(Convention: use variable corresponding to colums in RREF with no leading entries to parameterize solution set.)

Set $x_3 = t \Rightarrow x_1 = t \quad x_2 = 0.$

$(x_1, x_2, x_3) = \{(t, 0, t) \mid t \in \mathbb{R}\}$

example:

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{bmatrix}$$

Set $x_2 = t_1 \quad x_4 = t_2$

$x_1 = 2 - 2t_1 - t_2 \quad x_3 = 3 - 3t_2$

$x_5 = 4$

The rank of a matrix.

Get a k parameter solutions, where k equals to the number of columns (besides leftmost) without leading entries.

$n-k$ is the rank of the matrix.

Corallary: Any system have

① infinitely many solutions: $\left[\begin{array}{ccccc:c} 1 & 2 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow k > 0$

② exacely one solution: $\left[\begin{array}{ccc:c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 4 \end{array}\right] \Rightarrow k = 0.$

③ no solutions:
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right] \Rightarrow k > 0.$$

Theorem 2: Every matrix can be put in RREF by a finite sequence of elementary row operations.

① $R_i \leftrightarrow R_j$ ② $R_i \leftrightarrow cR_i$ $(c \neq 0)$ ③ $R_j \leftrightarrow R_i + cR_j$

Proof: by construction (Gaussian Elimination)

(A) forward pass to REF

$$\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \\ 3 & 2 & 1 & 2 \end{bmatrix} \xrightarrow[R_1 \leftrightarrow R_2]{\text{Use ①}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 3 & 2 & 1 & 2 \end{bmatrix}$$

(i) (put nonzero entry at top of leftmost nonzero column)

$$\xrightarrow[R_3 \to R_3 - 3R_1]{\text{Use ③}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & -1 \end{bmatrix}$$

(ii) (get rid of all the nonzero entries below this entry).

$$\xrightarrow[R_3 \to R_3 + R_2]{\text{Use ③}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

(ignore row with leftmost leading entry and repeat (i) (ii))

Repeat until at bottom row.

Now in REF ☆.

B). Backward pass from REF to RREF

(i).

Using ② makes rightmost leading entry 1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix} \xrightarrow{R_3 \to \frac{1}{2}R_3} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(ii) Use ③ to put zeros above this 1.

$$\xrightarrow[R_2 \to R_2 - 3R_1]{R_1 \to R_1 - R_3} \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(iii) Repeat for the next leading entry.

$$\xrightarrow{R_1 \to R_1 - R_2} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$