**Math 416**
**Summer 2020**
**Exam 1, Ver 1 (AM)**

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**Time Limit: 90 Minutes**

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This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

| Problem | Points | Score |
|---|---|---|
| 1 | 15 | |
| 2 | 15 | |
| 3 | 18 | |
| 4 | 20 | |
| 5 | 17 | |
| 6 | 15 | |
| Total: | 100 | |

Do not write in the table to the right.

1. (15 points) Find an equation involving $a, b$ and $c$ so that the augmented matrix below produces at least one solution to the system of linear equations it represents.

$$\left(\begin{array}{ccc|c} 2 & 0 & 4 & a \\ 2 & -2 & 2 & b \\ 4 & 2 & 10 & c \end{array}\right)$$

$$\rightarrow \begin{bmatrix} 2 & 0 & 4 & a \\ 0 & -2 & -2 & b-a \\ 0 & 2 & 2 & c-2a \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & \frac{a}{2} \\ 0 & 1 & 1 & \frac{a-b}{2} \\ 0 & 1 & 1 & \frac{c-2a}{2} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 2 & \frac{a}{2} \\ 0 & 1 & 1 & \frac{a-b}{2} \\ 0 & 0 & 0 & \frac{c-3a+b}{2} \end{bmatrix}$$

at lent one solution $\Rightarrow \frac{c-3a+b}{2} = 0.$

$\Rightarrow \underline{c-3a+b=0}$

2. (15 points) Using elementary row operations find the inverse of the matrix $B = \begin{pmatrix} -4 & 0 & 5 \\ -3 & 3 & 5 \\ -1 & 2 & 2 \end{pmatrix}$.

$$[B \mid I] = \left[\begin{array}{ccc|ccc} -4 & 0 & 5 & 1 & 0 & 0 \\ -3 & 3 & 5 & 0 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow{C_2 \to C_2 - C_1} \left[\begin{array}{ccc|ccc} -4 & 0 & 5 & 1 & 0 & 0 \\ 1 & 3 & 0 & -1 & 1 & 0 \\ -1 & 2 & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[C_3 \to C_3 + C_2]{C_1 \to C_1 + 4C_2} \left[\begin{array}{ccc|ccc} 0 & 12 & 5 & -3 & 4 & 0 \\ 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 5 & 2 & -1 & 1 & 1 \end{array}\right] \xrightarrow{C_1 \to C_1 - \frac{12}{5}C_3} \begin{bmatrix} 0 & 0 & \frac{1}{5} & -\frac{3}{5} & \frac{8}{5} & -\frac{12}{5} \\ 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 5 & 2 & -1 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow[C_3 \to \frac{1}{5}C_3]{C_1 \to 5C_1} \begin{bmatrix} 0 & 0 & 1 & -3 & 8 & -12 \\ 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 1 & \frac{2}{5} & -\frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{bmatrix} \xrightarrow{C_3 \to C_3 - \frac{2}{5}C_1} \begin{bmatrix} 0 & 0 & 1 & -3 & 8 & -12 \\ 1 & 3 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & -3 & 5 \end{bmatrix}$$

$$\xrightarrow{C_2 \to C_2 - 3C_3} \begin{bmatrix} 0 & 0 & 1 & -3 & 8 & -12 \\ 1 & 0 & 0 & -4 & 10 & -15 \\ 0 & 1 & 0 & 1 & -3 & 5 \end{bmatrix} \xrightarrow[\substack{C_2 \to C_1 \\ C_3 \to C_1}]{C_1 \to C_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 10 & -15 \\ 0 & 1 & 0 & 1 & -3 & 5 \\ 0 & 0 & 1 & -3 & 8 & -12 \end{array}\right]$$

$$= [I \mid B^{-1}]$$

$$B^{-1} = \begin{bmatrix} -4 & 10 & -15 \\ 1 & -3 & 5 \\ -3 & 8 & -12 \end{bmatrix}$$

3. (18 points) (a) (6 points) What does it mean for a set of vectors $\{v_1, v_2, v_2\}$ to be Linearly independent

$v_3$ print error ?

$a_1 = a_2 = a_3 = 0$ is the only solution for the equation:

$$a_1 v_1 + a_2 v_2 + a_3 v_3 = \bar{0}$$

(b) (6 points) What is the span of a set of vectors $\{v_1, v_2, v_2\}$

print error ?

$$\mathrm{Span}\{v_1, v_2, v_3\} = \{a_1 v_1 + a_2 v_2 + a_3 v_3 \mid a_1, a_2, a_3 \in \mathbb{R}\}$$

(if it is not a print error:

$$\mathrm{Span}\{v_1, v_2, v_2\} = \{a_1 v_1 + a_2 v_2 \mid a_1, a_2 \in \mathbb{R}\}$$ )

(c) (6 points) Give a basis for $P_2(\mathbb{R})$

$$\{1, x, x^2\}$$

4. (20 points) Let $T$ be a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ real matrix. Prove that the null space $N(T)$ is a subspace of the vector space $\mathbb{R}^n$

$T$ is linear $\Rightarrow T(0^n) + T(0^n) = T(0^n) \Rightarrow T(0^n) = 0^m$

(a) $T(0^n) = 0^m$, so $0^n \in N(T)$

(b) $\forall\, x, y \in N(T)$. i.e. $T(x) = T(y) = 0^m$

$T$ is linear $\Rightarrow T(x+y) = T(x) + T(y) = 0^m$

$\Rightarrow x+y \in N(T)$

(c) $\forall\, x \in N(T)$ i.e. $T(x) = 0^m$

$\forall c \in \mathbb{R}$

$T$ is linear $\Rightarrow T(cx) = cT(x) = 0^m$

$\Rightarrow cx \in N(T)$

from (a) (b) (c)

we know $N(T)$ is a subspace of $\mathbb{R}^n$

5. (17 points) Consider the linear map $T : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(A) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (A^t + A)$$

(a) (12 points) Compute $[T]_\alpha$, where
$\alpha = \left\{ E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$

Let $A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}$ $A^t = \begin{bmatrix} a_1 & a_3 \\ a_2 & a_4 \end{bmatrix}$

$$T(A) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2a_1 & a_2+a_3 \\ a_2+a_3 & 2a_4 \end{bmatrix} = \begin{bmatrix} a_2+a_3 & 2a_4 \\ 2a_1 & a_2+a_3 \end{bmatrix}$$

$[T(A)]_\alpha = \begin{bmatrix} a_2+a_3 \\ 2a_4 \\ 2a_1 \\ a_2+a_3 \end{bmatrix}$ $[A]_\alpha = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix}$

$4 \times 1$

$$[T]_\alpha = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

(b) (5 points) Determine if $T$ is invertible.

$T(A) = 0_{2\times 2} \Rightarrow \begin{cases} a_2+a_3 = 0 \\ a_1 = 0 \\ a_4 = 0 \end{cases}$ $\Rightarrow T(A) \neq \{0_{2\times 2}\}$

$\Rightarrow$ T is not <u>onto</u>

$\Rightarrow$ T is not invertible

6. (15 points) **TRUE** / **FALSE**. Give a short justification.

(a) If $A \neq B$ than $\det(A) \neq \det(B)$

F : $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ $A \neq B$

but $\det(A) = \det(B)$
$= 0$.

(b) If a system of linear equations has two different solutions, then it has infinitely many solutions.

T Linear equations has infinitely solutions.
or zero solutions
or exactly one solutions

(c) If $A$ and $B$ are both invertible than $AB$ is invertible

T $A, B$ invertible $\Rightarrow$ exist $A^{-1}, B^{-1}$ $AA^{-1} = BB^{-1} = I$
$= A^{-1}A = B^{-1}B$

$\Rightarrow (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = I$ $\Rightarrow AB$ invertible
$(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = I$

(d) $(AB)^{-1} = A^{-1}B^{-1}$

F from c we know $B^{-1}A^{-1} = (AB)^{-1}$
and the inverse of matrix
is unique

(e) It is not possible for $Span(v_1, v_2, v_3) = Span(v_3)$

F $v_1 = (1, 0, 0)$ $v_2 = (2, 0, 0)$ $v_3 = (3, 0, 0)$

$Span(v_1, v_2, v_3) = Span(v_3)$.