Homework 8 MATH 416: ABSTRACT LINEAR ALGEBRA

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(Exercises are taken from *Linear Algebra, **Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. §5.1 #2 For each of the following linear operators T on a vector space V and ordered bases β, compute $[T]_\beta$, and determine whether β is a basis consisting of eigenvectors of T.

a. §5.1 #2 (a) $V = \mathbb{R}^2$, $T\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 10a - 6b \\ 17a - 10b \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\}$

b. §5.1 #2 (c) $V = \mathbb{R}^3$, $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 3a + 2b - 2c \\ -4a - 3b + 2c \\ -c \end{pmatrix}$, and $\beta = \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\}$

(Use the blank space in the following page to write your answers.)

(a.) $[T]_\beta = \left[\begin{bmatrix} -2 \\ -3 \end{bmatrix}_\beta, \begin{bmatrix} 2 \\ 4 \end{bmatrix}_\beta \right] = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$

$\det([T]_\beta - tI_2) = \det\begin{bmatrix} -t & 2 \\ -1 & -t \end{bmatrix} = \det\begin{bmatrix} -t & 2 \\ 0 & -t-\frac{2}{t} \end{bmatrix} = t^2+2 \neq 0$

$\Rightarrow$ no

(c) $[T]_\beta = \left[\begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix}_\beta, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}_\beta, \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}_\beta \right] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$

$\Rightarrow$ yes.

2. §5.1 #3 For each of the following matrices $A \in M_{n\times n}(\mathbb{F})$,

(i) Determine all the eigenvalues of A.

(ii) For each eigenvalue λ of A, find the set of eigenvectors corresponding to λ.

(iii) If possible, find a basis for $\mathbb{F}^n$ consisting of eigenvectors of A.

(iv) If successful in finding such a basis, determine an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

a. §5.1 #3 (a) $\begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ for $\mathbb{F} = \mathbb{R}$

b. §5.1 #3 (b) $\begin{pmatrix} 0 & -2 & -3 \\ -1 & 1 & -1 \\ 2 & 2 & 5 \end{pmatrix}$ for $\mathbb{F} = \mathbb{R}$

(Use the blank space in the following page to write your answers.)

a. (i) $\det[A - tI_n] = \begin{vmatrix} 1-t & 2 \\ 3 & 2-t \end{vmatrix} = (1-t)(2-t) - 6$

$= t^2 - 3t - 4$

$= (t-4)(t+1)$.

$\underline{\lambda_1 = 4 \quad \lambda_2 = -1.}$

(ii) $A - \lambda_1 I_n = \begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{span}\{\begin{pmatrix} 2 \\ 3 \end{pmatrix}\}$

$A - \lambda_2 I_n = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda} = \text{Span}\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$

(iii) $\beta = \{\begin{pmatrix} 2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}\}$

(iv) $Q = [I]_\beta^\alpha = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$

Continued from Question 2.

$$D = Q^{-1} A Q = \begin{bmatrix} 4 & 0 \\ 0 & -1 \end{bmatrix}$$

b. (i)

$$\det(A - tI_n) = \begin{vmatrix} -t & -2 & -3 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{vmatrix} = \begin{vmatrix} 1-t & 1-t & 1-t \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{vmatrix}$$

$$(1-t) \begin{vmatrix} 1 & 1 & 1 \\ -1 & 1-t & -1 \\ 2 & 2 & 5-t \end{vmatrix} = (1-t) \begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-t & 2 \\ 0 & 0 & 3-t \end{vmatrix}$$

$$= (1-t)(2-t)(3-t) \Rightarrow \lambda_1 = 1 \quad \lambda_2 = 2 \quad \lambda_3 = 3$$

(ii) $$A - \lambda_1 I_n = \begin{bmatrix} -1 & -2 & -3 \\ -1 & 0 & -1 \\ 2 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & -2 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \right\}$$

$$A - \lambda_2 I_n = \begin{bmatrix} -2 & -2 & -3 \\ -1 & -1 & -1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}$$

$$A - \lambda_3 I_n = \begin{bmatrix} -3 & -2 & -3 \\ -1 & -2 & -1 \\ 2 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_3} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

Continued from Question 2.

(iii) $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$

(iv) $Q = [I]_\beta^\alpha = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

3. §5.1 #4 For each linear operator T on V, find the eigenvalues of T and an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

a. §5.1 #4 (b) $V = \mathbb{R}^3$, $T(a,b,c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$

b. §5.1 #4 (h) $V = M_{2\times 2}(\mathbb{R})$, $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}$

a. $$[T]_\alpha^\alpha = \begin{bmatrix} 7 & -4 & 10 \\ 4 & -3 & 8 \\ -2 & 1 & -2 \end{bmatrix}$$

$$\det\left[[T]_\alpha^\alpha - tI_3\right] = \begin{vmatrix} 7-t & -4 & 10 \\ 4 & -3-t & 8 \\ -2 & 1 & -2-t \end{vmatrix} = (t+1)(t-1)(2-t)$$

$$\Rightarrow \underline{\lambda_1 = -1 \quad \lambda_2 = 1 \quad \lambda_3 = 2}$$

$$[T]_\alpha^\alpha - \lambda_1 I_3 = \begin{bmatrix} 8 & -4 & 10 \\ 4 & -2 & 8 \\ -2 & 1 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -1 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{span}\{(1,2,0)\}$$

$$[T]_\alpha^\alpha - \lambda_2 I_3 = \begin{bmatrix} 6 & -4 & 10 \\ 4 & -4 & 8 \\ -2 & 1 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{span}\{(1,-1,-1)\}$$

$$[T]_\alpha^\alpha - \lambda_3 I_3 = \begin{bmatrix} 5 & -4 & 10 \\ 4 & -5 & 8 \\ -2 & 1 & -4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_3} = \text{span}\{(2,0,-1)\}$$

$$\underline{\beta = \{(1,2,0), (1,-1,-1), (2,0,-1)\}}$$

Continued from Question 3.

b. $[T]_\alpha^\alpha = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

$\det[[T]_\alpha^\alpha - tI_4] = \begin{vmatrix} -t & 0 & 0 & 1 \\ 0 & 1-t & 0 & 0 \\ 0 & 0 & 1-t & 0 \\ 1 & 0 & 0 & -t \end{vmatrix} = (-t)(1-t)^2(-t+\frac{1}{t})$

$= -(t+1)(1-t)^3$

$\lambda_1 = -1 \quad \lambda_2 = 1$

$[T]_\alpha^\alpha - \lambda_1 I_4 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$

$[T]_\alpha^\alpha - \lambda_2 I_4 = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\right\}$

$\beta = \left\{ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$

4. §5.2 #2 For each of the following matrices $A \in M_{n\times n}(\mathbb{R})$, test A for diagonalizability, and if A is diagonalizable, find an invertible matrix Q and a diagonal matrix D such that $Q^{-1}AQ = D$.

a. §5.2 #2 (e) $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}$

b. §5.2 #2 (g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

(e)

$$\det(A - tI_3) = \begin{vmatrix} -t & 0 & 1 \\ 1 & -t & -1 \\ 0 & 1 & 1-t \end{vmatrix} = t^2 - t^3 - t + 1$$

$$= (1-t)(t^2+1)$$

$$= (1-t)(t-i)(t+i)$$

doesn't split over $\mathbb{R}$.

$\Rightarrow$ Not diagonalizable.

(g) $$\det(A - tI_3) = \begin{vmatrix} 3-t & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{vmatrix} = (4-t)\begin{vmatrix} 1 & 1 & 1 \\ 2 & 4-t & 2 \\ -1 & -1 & 1-t \end{vmatrix}$$

$$= (4-t)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2-t & 0 \\ 0 & 0 & 2-t \end{vmatrix} = (4-t)(2-t)^2$$

$$\Rightarrow \lambda_1 = 4 \quad \lambda_2 = 2$$

$$A - \lambda_1 I_3 = \begin{bmatrix} -1 & 1 & 1 \\ 2 & 0 & 2 \\ -1 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{span}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}\right)\}$$

$$A - \lambda_2 I_3 = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\}$$

Continued from Question 4.

$$\beta = \left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right\}$$

$$Q = [I]_{\beta}^{\alpha} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

5. §5.2 #3 For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[T]_\beta$ is a diagonal matrix.

a. §5.2 #3 (a) $V = P_3(\mathbb{R})$ and T is defined by $T(f(x)) = f'(x) + f''(x)$

b. §5.2 #3 (d) $V = P_2(\mathbb{R})$ and T is defined by $T(f(x)) = f(0) + f(1)(x + x^2)$

a. $\alpha = \{1, x, x^2, x^3\}$

$$[T]_\alpha^\alpha = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\det\left[[T]_\alpha^\alpha - t I_4\right] = \begin{vmatrix} -t & 1 & 2 & 0 \\ 0 & -t & 2 & 6 \\ 0 & 0 & -t & 3 \\ 0 & 0 & 0 & -t \end{vmatrix} = t^4 \Rightarrow \lambda_1 = 0.$$

$$[T]_\alpha^\alpha - \lambda_1 I_4 = \begin{bmatrix} 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_\lambda = \operatorname{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\right\}$$

algebra multi. of $\lambda_1 = 4 \neq 1 =$ geom. multi of $\lambda_1 = \dim E_\lambda$ $\Rightarrow$ not diagonalizable

b. $\alpha = \{1, x, x^2\}$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\det\{[T]_\alpha^\alpha - t I_3\} = \begin{vmatrix} 1-t & 0 & 0 \\ 1 & 1-t & 1 \\ 1 & 1 & 1-t \end{vmatrix} = (1-t)t(t-2)$$

$$\Rightarrow \lambda_1 = 1 \quad \lambda_2 = 0 \quad \lambda_3 = 2$$

$$[T]_\alpha^\alpha - \lambda_1 I_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \operatorname{span}\left\{\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}\right\}$$

$$[T]_\alpha^\alpha - \lambda_2 I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \Rightarrow E_{\lambda_2} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right\}$$

$$[T]_\alpha^\alpha - \lambda_3 I_3 = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \Rightarrow E_{\lambda_3} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

$$\beta = \{-x^2 - x + 1,\ -x^2 + x,\ x^2 + x\}$$

6. §5.2 #7 For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in M_{2\times 2}(\mathbb{R}),$$

find an expression for A^n, where n is an arbitrary positive integer.

$$\det(A - tI_2) = \begin{vmatrix} 1-t & 4 \\ 2 & 3-t \end{vmatrix} = t^2-4t-5 = (t-5)(t+1)$$

$$\Rightarrow \lambda_1 = 5 \quad \lambda_2 = -1.$$

$$A - \lambda_1 I_2 = \begin{bmatrix} -4 & 4 \\ 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

$$A - \lambda_2 I_2 = \begin{bmatrix} 2 & 4 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \quad Q = [I]_\beta^\alpha = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{3} & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} & -\frac{1}{3} \end{array}\right]$$

$$\Rightarrow Q^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix}$$

$$A^n = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 5^n & 0 \\ 0 & (-1)^n \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 5^n + 2(-1)^n & 2(5^n) - 2(-1)^n \\ 5^n - (-1)^n & 2(5^n) + (-1)^n \end{bmatrix}$$

Problems **not** from the textbook exercises.

7. In class we proved the following statement.

Thm *Suppose $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of A. If w is a nonzero element of E_{λ_2}, then w is not contained in E_{λ_1}.*

Use this to prove the following.

Cor *Suppose $\lambda_1 \neq \lambda_2$ are distinct eigenvalues of A. If β_1 is a basis of E_{λ_1} and β_2 is a basis of E_{λ_2}, then $\beta_1 \cup \beta_2$ is linearly independent.*

Assume $\beta_1 \cup \beta_2$ is linearly dependent.

suppose $\beta_1 = \{x_1, x_2 \cdots x_n\}$, $\beta_1 \cup \beta_2 = \{x_1, x_2 \cdots x_n, y_1, y_2 \cdots y_n\}$

$y_1, y_2 \cdots y_m \in \beta_2$

So exsits $a_1, a_2 \cdots, a_n, b_1, b_2 \cdots b_m$ not all equal to 0.

$a_1 x_1 + a_2 x_2 + \cdots a_n x_n + b_1 y_1 + b_2 y_2 + \cdots b_m y_m = 0$.

i.e. $a_1 x_1 + a_2 x_2 + \cdots a_n x_n = (-b_1) y_1 + (-b_2) y_2 \cdots + (-b_m) y_m$.

Since β_1 is a basis of E_{λ_1}, β_2 is a basis of E_{λ_2}

β_1, β_2 are lin. indep.

So $a_1 = a_2 = \cdots = a_n = 0 \Leftrightarrow b_1 = b_2 = \cdots = b_m = 0$

Hence there must have $a_i \neq 0$ and $b_j \neq 0$

then $a_1 x_1 + a_2 x_2 + \cdots a_n x_n \in E_{\lambda_1}$

$a_1 x_1 + \cdots + a_n x_n = (-b_1) y_1 + (-b_2) y_2 + \cdots (-b_m) y_m \in E_{\lambda_2}$

from **Thm** above it is impossible

$\Rightarrow \beta_1 \cup \beta_2$ is lin. indep.

(Question 8 and 9) Determine if the following statements are TRUE or FALSE. If TRUE give a proof, if FALSE give a counterexample.

8. *If v_1 and v_2 are both eigenvectors of A, then so is $v_1 + v_2$.*

False. $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ $\lambda_1 = 4$ $V_1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$\lambda_2 = -1$ $V_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$V_1 + V_2 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$

$A(V_1 + V_2) = \lambda (V_1 + V_2)$

$\begin{bmatrix} 7 \\ 13 \end{bmatrix} = \lambda \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

$\Rightarrow$ no solutions

9. *If $D \in M_{n\times n}$ is diagonal, then for any $A \in M_{n\times n}$ we have $DA = AD$.*

False $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$DA = \begin{bmatrix} 1 & 2 \\ 6 & 8 \end{bmatrix} \neq AD = \begin{bmatrix} 1 & 4 \\ 3 & 8 \end{bmatrix}$