# Homework 10 
Math 416: Abstract Linear Algebra

Name: Date:

(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

Here are theorems you may want to use.

**Theorem 6.5** Let $V$ be a nonzero finite-dimensional inner product space. Then $V$ has a orthonormal basis $\beta$. Furthermore, if $\beta = \{v_1, v_2, ..., v_n\}$ and $x \in V$, then

$$x = \sum_{i=1}^{n} \langle x, v_i \rangle v_i.$$

**Theorem 6.6** Let $W$ be a nonzero finite-dimensional subspace of an inner product space $V$, and let $y \in V$. Then there exist unique vectors $u \in W$ and $z \in W^{\perp}$ such that $y = u + z$. Furthermore, if $\{v_1, v_2, ..., v_k\}$ is an orthonormal basis for $W$, then

$$u = \sum_{i=1}^{k} \langle y, v_i \rangle v_i.$$

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**1. §6.2 #2** In each part, *(i)* apply the Gram-Schmidt process to the given subset $S$ of the inner product space $V$ to obtain an orthogonal basis for span$(S)$. Then normalize the vectors in this basis to obtain an orthonormal basis $\beta$ for span$(S)$, and *(ii)* compute the Fourier coefficients of the given vector relative to $\beta$. Finally, *(iii)* use Theorem 6.5 to verify your result.

**a. §6.2 #2 (b)** $V = \mathbb{R}^3$, $S = \{(1,1,1), (0,1,1), (0,0,1)\}$, and $x = (1,0,1)$

**b. §6.2 #2 (c)** $V = P_2(\mathbb{R})$ with the inner product $\langle f, g \rangle = \int_0^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$, and $h(x) = 1 + x$

**c. §6.2 #2 (g)** $V = M_{2\times 2}(\mathbb{R})$(with $\langle A, B \rangle = \text{tr}(B^* A)$), $S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$

(Use the blank spaces in the following two pages to write your solutions.)

Continued from Question 1.

**a. §6.2 #2 (b)** $V = \mathbb{R}^3$, $S = \{(1,1,1),(0,1,1),(0,0,1)\}$, and $x = (1,0,1)$

$$u_1 = \frac{1}{\sqrt{1+1+1}}(1,1,1) = \frac{\sqrt{3}}{3}(1,1,1).$$

$$v_2 = w_2 - \langle w_2, u_1\rangle u_1 = (0,1,1) - \frac{2\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{3}(1,1,1) = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$u_2 = \frac{v_2}{\|v_2\|} = \frac{\sqrt{6}}{6}(-2,1,1).$$

$$v_3 = w_3 - \langle w_3, u_1\rangle u_1 - \langle w_3, u_2\rangle u_2 = (0,0,1) - \frac{1}{3}(1,1,1) - \frac{1}{6}(-2,1,1).$$

$$= (0, -\frac{1}{2}, \frac{1}{2})$$

$$u_3 = \frac{v_3}{\|v_3\|} = \frac{\sqrt{2}}{2}(0,-1,1)$$

$$\beta = \{\frac{\sqrt{3}}{3}(1,1,1), \frac{\sqrt{6}}{6}(-2,1,1), \frac{\sqrt{2}}{2}(0,-1,1)\}.$$

$$\langle x, u_1\rangle = \frac{2\sqrt{3}}{3} \quad \langle x, u_2\rangle = -\frac{\sqrt{6}}{6} \quad \langle x, u_3\rangle = \frac{\sqrt{2}}{2}$$

Fourier coefficient.

$$\frac{2\sqrt{3}}{3}\times\frac{\sqrt{3}}{3}(1,1,1) - \frac{\sqrt{6}}{6}\times\frac{\sqrt{6}}{6}(-2,1,1) + \frac{\sqrt{2}}{2}\times\frac{\sqrt{2}}{2}(0,-1,1)$$

$$= (1,0,1) = x.$$

**b. §6.2 #2 (c)** $V = P_2(\mathbb{R})$ with the inner product $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$, $S = \{1, x, x^2\}$, and $h(x) = 1 + x$

$u_1 = 1.$ $\quad v_2 = x - \langle x, 1\rangle\cdot 1 = x - \int_0^1 t\,dt = x - \frac{1}{2}$

$$\|v_2\| = \sqrt{\langle v_2, v_2\rangle} = \sqrt{\int_0^1 (t-\frac{1}{2})^2 dt} = \frac{1}{2\sqrt{3}}$$

$$u_2 = \frac{v_2}{\|v_2\|} = 2\sqrt{3}\,(x - \frac{1}{2}).$$

$$v_3 = x^2 - \langle x^2, 1\rangle\cdot 1 - \langle x^2, 2\sqrt{3}(x-\frac{1}{2})\rangle\, 2\sqrt{3}(x-\frac{1}{2})$$

$$= x^2 - \int_0^1 t^2 dt - \left(\int_0^1 2\sqrt{3}t^3 - \sqrt{3}t^2 dt\right) 2\sqrt{3}(x - \frac{1}{2}) = x^2 - x + \frac{1}{6}$$

$$\|v_3\| = \sqrt{\int_0^1 (x^2 - x + \frac{1}{6})^2 dx} = \sqrt{\frac{1}{5} - \frac{1}{2} + \frac{4}{9} - \frac{1}{6} + \frac{1}{36}} = \frac{1}{6\sqrt{5}}$$

$$u_3 = \frac{v_3}{\|v_3\|} = 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right).$$

$$\beta = \left\{ 1,\ 2\sqrt{3}\left(x - \frac{1}{2}\right),\ 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) \right\}.$$

Continued from Question 1.

$$\underbrace{\langle h(x), 1\rangle = \int_0^1 1 + t\,dt = \frac{3}{2},\ \langle h(x), 2\sqrt{3}(x-\tfrac{1}{2})\rangle = \frac{\sqrt{3}}{6},\ \langle h(x), 6\sqrt{5}(x^2-x+\tfrac{1}{6})\rangle = 0.}_{\text{Fourier coefficients.}}$$

$$h(x) = x + 1 = \frac{3}{2} + \frac{\sqrt{3}}{6} \times 2\sqrt{3}\left(x - \frac{1}{2}\right) + 0$$

**c. §6.2 #2 (g)** $V = M_{2\times 2}(\mathbb{R})$(with $\langle A, B\rangle = \mathrm{tr}(B^*A)$), $S = \left\{ \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix}, \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} \right\}$, and $A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix}$

$$\|w_1\| = \sqrt{\mathrm{tr}(A^*A)} = 6. \Rightarrow u_1 = \frac{w_1}{\|w_1\|} = \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix}.$$

$$v_2 = w_2 - \langle w_2, u_1\rangle u_1 = \begin{pmatrix} -1 & 9 \\ 5 & -1 \end{pmatrix} - \begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}.$$

$$\|v_2\| = 6\sqrt{2} \Rightarrow u_2 = \frac{v_2}{\|v_2\|} = \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}.$$

$$v_3 = w_3 - \langle w_3, u_1\rangle u_1 - \langle w_3, u_2\rangle u_2 = \begin{pmatrix} 7 & -17 \\ 2 & -6 \end{pmatrix} + \begin{pmatrix} 6 & 10 \\ -2 & 2 \end{pmatrix} + \begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix}$$

$$= \begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$$

$$\|v_3\| = 9\sqrt{2} \Rightarrow u_3 = \frac{v_3}{\|v_3\|} = \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$$

$$\beta = \left\{ \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix},\ \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix},\ \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix} \right\}.$$

$$\underbrace{\langle A, u_1\rangle = 24 \quad \langle A, u_2\rangle = 6\sqrt{2} \quad \langle A, u_3\rangle = -9\sqrt{2}}_{}\ \text{Fourier coefficients.}$$

$$A = \begin{pmatrix} -1 & 27 \\ -4 & 8 \end{pmatrix} = 24 \times \frac{1}{6}\begin{pmatrix} 3 & 5 \\ -1 & 1 \end{pmatrix} + 6\sqrt{2} \cdot \frac{1}{6\sqrt{2}}\begin{pmatrix} -4 & 4 \\ 6 & -2 \end{pmatrix} - 9\sqrt{2} \cdot \frac{1}{9\sqrt{2}}\begin{pmatrix} 9 & -3 \\ 6 & -6 \end{pmatrix}$$

**2. §6.2 #6** Let $V$ be an inner product space, and let $W$ be a finite-dimensional subspace of $V$. If $x \notin W$, prove that there exists $y \in V$ such that $y \in W^\perp$, but $\langle x, y\rangle \neq 0$. *Hint: Use Theorem 6.6. See the first page of this homework sheet or refer to the textbook.*

By Theoren 6.6, we know any $x \in V$, exists $u \in W$ and $v \in W^\perp$, $x = u + v$. where $u \in W$, $v \in W^\perp$

Since $x \notin W$, ~~$\langle x, v\rangle \neq 0 \Rightarrow \langle u+v, v\rangle \neq 0 \Rightarrow$~~ $v \neq \vec{0}$

Assume for all $y \in W^\perp$ $\langle x, y\rangle = 0$

$\Rightarrow \langle u+v, y\rangle = \langle u, y\rangle + \langle v, y\rangle = \langle v, y\rangle = 0$. $\forall y \in W^\perp$

$\Rightarrow v = \vec{0}$ which contradicts to the conclution above.

Set $y = v$, we have $\langle x, y\rangle = \langle x, v\rangle = \langle v, v\rangle \neq 0$.

**3. §6.2 #7** Let $\beta$ be a basis for a subspace $W$ of an inner product space $V$, and let $z \in V$. Prove that $z \in W^\perp$ if and only if $\langle z, v\rangle = 0$ for every $v \in \beta$.

"$\Rightarrow$": $v \in \beta \Rightarrow v \in W$

hence $\langle \underset{W^\perp}{z}, \underset{W}{v}\rangle = 0$.

"$\Leftarrow$": $\langle z, v\rangle = 0$, $\forall v \in \beta$

$\Rightarrow \langle z, x\rangle = 0 \quad \forall x \in \operatorname{span}(\beta)$.

$\overset{W=\operatorname{span}(\beta)}{\Rightarrow} \langle z, x\rangle = 0 \quad \forall x \in W$.

$\Rightarrow z \in W^\perp$

**4. §6.2 #9** Let $W = \text{span}(\{(i,0,1)\})$ in $\mathbb{C}^3$. Find orthonormal bases for $W$ and $W^\perp$.

$\langle (i,0,1),(i,0,1)\rangle = i(-i)+1 = 2.$

$\beta_W = \{\frac{1}{\sqrt{2}}(i,0,1)\}.$

$\langle (x,y,z),(i,0,1)\rangle = x(-i)+z = 0. \Rightarrow z = xi.$

$\beta_{W^\perp} = \{(0,1,0), \frac{1}{\sqrt{2}}(1,0,i).\}.$

**5. §6.2 #13 (c)** Let $V$ be an inner product space and $W$ be a finite-dimensional subspace of $V$. Prove the following result.

$W = (W^\perp)^\perp$ (*Hint: Use the previous question 2. §6.2 #6.*)

i.e.

$W = \{x \in V \mid \langle x,y\rangle = 0, \ \forall y \in W^\perp\}.$

Proof: Let $\{x \in V \mid \langle x,y\rangle = 0, \ \forall y \in W^\perp\} = K.$

by 6.2.6 if $x \notin W$, exist $y \in W^\perp$ $\langle x,y\rangle \neq 0$

and the $x \notin K. \Rightarrow K \subseteq W.$

$\forall x \in W$, satisfies $\langle x,y\rangle = 0$ for any $y \in W^\perp$

$\Rightarrow W \subseteq K$

$\Rightarrow W = K \Rightarrow W = \{x \in V \mid \langle x,y\rangle = 0, \ \forall y \in W^\perp\}$

i.e. $W = (W^\perp)^\perp$

**6. §6.3 # 3** For each of the following inner product spaces $V$ and linear operators $T$ on $V$, evaluate $T^*$ at the given vector in $V$.

**a. §6.3 #3 (a)** $V = \mathbb{R}^2,\ T(a,b) = (2a+b, a-3b),\ x = (3,5)$

**b. §6.3 #3 (c)** $V = P_1(\mathbb{R})$ with $\langle f, g\rangle = \int_{-1}^{1} f(t)g(t)dt,\ T(f) = f' + 3f,\ f(t) = 4-2t$

a. $\beta = \{(1,0),(0,1)\}$.

$[T]_\beta^\beta = [[(2,1)]_\beta, [1,-3]_\beta] = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

$[T^*]_\beta^\beta = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

$[T^*(x)]_\beta = \begin{bmatrix} 11 \\ -12 \end{bmatrix} \Rightarrow T^*(x) = (11, -12)$.

b. $\alpha = \{1, t\}$. $V_1 = 1$ $\|V_1\| = \sqrt{\int_{-1}^{1} 1\,dt} = \sqrt{2}$ $u_1 = \frac{1}{\sqrt{2}}$

$V_2 = t - \langle t, 1\rangle 1 = t$. $u_2 = \frac{V_2}{\|V_2\|} = \frac{\sqrt{3}}{\sqrt{2}} t$.

$\beta = \{\frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} t\}$. $T(\frac{1}{\sqrt{2}}) = \frac{3}{\sqrt{2}}$ $T(\sqrt{\frac{3}{2}} t) = \sqrt{\frac{3}{2}} + 3\sqrt{\frac{3}{2}} t$

$[T]_\beta^\beta = \begin{bmatrix} 3 & \sqrt{3} \\ 0 & 3 \end{bmatrix}$ $[T^*]_\beta^\beta = \begin{bmatrix} 3 & 0 \\ \sqrt{3} & 3 \end{bmatrix}$ $\begin{bmatrix} 4\sqrt{2} \\ -2\sqrt{\frac{2}{3}} \end{bmatrix}$.

$[T^*(f(t))]_\beta = \begin{bmatrix} 3 & 0 \\ \sqrt{3} & 3 \end{bmatrix} \begin{bmatrix} 4\sqrt{2} \\ -2\sqrt{\frac{2}{3}} \end{bmatrix} = \begin{bmatrix} 12\sqrt{2} \\ 2\sqrt{6} \end{bmatrix}$

$\Rightarrow T^*(f(t)) = 12\sqrt{2} \times \frac{1}{\sqrt{2}} + 2\sqrt{6} \times \sqrt{\frac{3}{2}} t = 12 + 6t$

**7. §6.3 #12** Let $V$ be an inner product space, and let $T$ be a linear operator on $V$. Prove the following results.

**a. §6.3 #12 (a)** $R(T^*)^\perp = N(T)$

**b. §6.3 #12 (b)** If $V$ is finite-dimensional, then $R(T^*) = N(T)^\perp$. *Hint: Use the previous question 5( §6.2 #13 (c)).*

a. $R(T^*) = \{ y \mid y = T^*(x),\ \forall x \in V \}$.

$N(T) = \{ x \in V \mid T(x) = 0 \}$.

$\langle v, T^*(x) \rangle = 0 \quad \forall x \in V \iff \langle T(v), x \rangle = 0 \quad \forall x \in V$

$\iff T(v) = 0 \iff v \in N(T)$

$\therefore R(T^*)^\perp = N(T)$.

b. by $ 6.2 # 13(c)

$N(T)^\perp = (R(T^*)^\perp)^\perp = R(T^*)$

**8. §6.3 #14** Let $V$ be an inner product space, and let $y, z \in V$. Define $T : V \to V$ by $T(x) = \langle x, y\rangle z$ for all $x \in V$. First prove that $T$ is linear. Then show that $T^*$ exists and find an explicit expression for it.

$$T(x_1 + c x_2) = \langle x_1 + c x_2, y\rangle z$$
$$= (\langle x_1, y\rangle + \langle c x_2, y\rangle) z$$
$$= (\langle x_1, y\rangle + c\langle x_2, y\rangle) z$$
$$= \langle x_1, y\rangle z + c\langle x_2, y\rangle z$$
$$= T(x_1) + c\,T(x_2)$$

$\Leftrightarrow$ T is linear

$$\langle T(x_1), x_2\rangle = \langle \langle x_1, y\rangle z, x_2\rangle.$$
$$= \langle x_1, y\rangle \langle z, x_2\rangle.$$
$$= \overline{\langle y, x_1\rangle}\,\overline{\langle x_2, z\rangle}$$
$$= \overline{\langle \langle x_2, z\rangle y, x_1\rangle}$$
$$= \langle x_1, \langle x_2, z\rangle y\rangle.$$

$\Rightarrow$ $T^*$ exists and $T^*(x) = \langle x, z\rangle y.$

**9. §6.4 #2** For each linear operator $T$ on an inner product space $V$, *(i)* determine whether $T$ is normal, self-adjoint, or neither. *(ii)* If possible, produce an orthonormal basis of eigenvectors of $T$ for $V$ and list the corresponding eigenvalues.

**a. §6.4 #2 (a)** $V = \mathbb{R}^2$ and $T$ is defined by $T(a, b) = (2a - 2b, -2a + 5b)$

**b. §6.4 #2 (c)** $V = \mathbb{C}^2$ and $T$ is defined by $T(a, b) = (2a + ib, a + 2b)$

**c. §6.4 #2 (e)** $V = M_{2\times 2}(\mathbb{R})$ and $T$ is defined by $T(A) = A^t$

(Use the blank space in the following page to write your solutions.)

a. (a) $\alpha = \{(1,0),(0,1)\}$.

$[T]_\alpha^\alpha = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$ $\quad [T^*]_\alpha^\alpha = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$

$\Rightarrow$ T is normal and self-adjoint.

$\det([T]_\alpha^\alpha - tI_2) = \begin{vmatrix} 2-t & -2 \\ -2 & 5-t \end{vmatrix} = 4-(2-t)(5-t)$.

$= (-t+1)(t-6)$

$\lambda_1 = 1 \quad \lambda_2 = 6$

$[T]_\alpha^\alpha - \lambda_1 I_2 = \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \operatorname{span}\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$.

$[T]_\alpha^\alpha - \lambda_2 I_2 = \begin{bmatrix} -4 & -2 \\ -2 & -1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \operatorname{span}\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\}$.

$\beta = \{\frac{1}{\sqrt{5}}(2,1), \frac{1}{\sqrt{5}}(1,-2)\}$

b. (c). $\alpha = \{(1,0),(0,1)\}$.

$[T]_\alpha^\alpha = \begin{bmatrix} 2 & i \\ 1 & 2 \end{bmatrix}$ $\quad [T^*]_\alpha^\alpha = \begin{bmatrix} 2 & 1 \\ -i & 2 \end{bmatrix}$.

$[T]_\alpha^\alpha [T^*]_\alpha^\alpha = \begin{bmatrix} 5 & 2+2i \\ 2-2i & 5 \end{bmatrix} = [T^*]_\alpha^\alpha [T]_\alpha^\alpha$

Continued from 9.

T is normal but not self-adjoint.

$$\det([T]_\alpha^\alpha - tI_2) = \begin{vmatrix} 2-t & i \\ 1 & 2-t \end{vmatrix} = t^2+4-4t+i$$
$$= \left(t-2+\frac{1+i}{\sqrt{2}}\right)\left(t-2-\frac{1+i}{\sqrt{2}}\right)$$

$\lambda_1 = 2-\frac{1+i}{\sqrt{2}}$ $\quad \lambda_2 = 2+\frac{1+i}{\sqrt{2}}$

$$[T]_\alpha^\alpha - \lambda_1 I_2 = \begin{bmatrix} \frac{1+i}{\sqrt{2}} & i \\ 1 & \frac{1+i}{\sqrt{2}} \end{bmatrix} \Rightarrow E_{\lambda_1} = \mathrm{Span}\left\{\begin{pmatrix} 1+i \\ -\sqrt{2} \end{pmatrix}\right\}.$$

$$[T]_\alpha^\alpha - \lambda_2 I_2 = \begin{bmatrix} -\frac{1+i}{\sqrt{2}} & i \\ 1 & \frac{1+i}{\sqrt{2}} \end{bmatrix} \Rightarrow E_{\lambda_2} = \mathrm{Span}\left\{\begin{pmatrix} 1+i \\ \sqrt{2} \end{pmatrix}\right\}.$$

$$\beta = \left\{\frac{1}{2}(1+i, -\sqrt{2}), \frac{1}{2}(1+i, \sqrt{2})\right\}.$$

c.(e). $\alpha = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad [T^*]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

T is normal and self-adjoint.

$$\det([T]_\alpha^\alpha - tI_4) = \begin{vmatrix} 1-t & 0 & 0 & 0 \\ 0 & -t & 1 & 0 \\ 0 & 1 & -t & 0 \\ 0 & 0 & 0 & 1-t \end{vmatrix} = (1-t)^2(-t)\left(-t+\frac{1}{t}\right)$$
$$= (1-t)^2(t^2-1)$$
$$= (t-1)^3(t+1)$$

$\lambda_1 = 1$ $\quad \lambda_2 = -1$

$$[T]_\alpha^\alpha - \lambda_1 I_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

$$[T]_\alpha^\alpha - \lambda_2 I_4 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \Rightarrow E_{\lambda_2} = \mathrm{Span}\left\{\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}\right\}.$$

$$\beta = \left\{\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

**10. §6.4 #4** Let $T$ and $U$ be self-adjoint operators on an inner product space $V$. Prove that $TU$ is self-adjoint if and only if $TU = UT$.

$T, U$ are sel-adjoint $\Rightarrow T = T^*$, $U = U^*$.

Proof: "$\Rightarrow$": $TU$ is self-adjoint. $\Rightarrow TU = (TU)^* = U^* T^*$

$= UT$

"$\Leftarrow$": $TU = UT \Rightarrow TU = T^* U^* = (UT)^*$

$= (TU)^*$

$\Rightarrow TU$ is self-adjoint.