Homework 7 MATH 416: ABSTRACT LINEAR ALGEBRA

NAME: Yang Wenxiao DATE: 2020.7.14.

(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. Exercise §3.1 #1 Label the following statements as true or false (Answer is back, give a short explanation!).

(a) An elementary matrix is always square. T

(b) The only entries of an elementary matrix are zeros and ones. F type II: $R_i \to \lambda R_i$ λ is also an entry. type III: $R_j \to R_j + \lambda R_i$

(c) The $n \times n$ identity matrix is an elementary matrix. T

(d) The product of two $n \times n$ elementary matrices is an elementary matrix. F $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & \lambda \\ 1 & 0 \end{bmatrix}$ it's not an elementary matrix

(e) The inverse of an elementary matrix is an elementary matrix. T

(f) The sum of two $n \times n$ elementary matrices is an elementary matrix. F $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & \lambda \end{bmatrix}$ is not

(g) The transpose of an elementary matrix is an elementary matrix. T

(h) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then B can also be obtained by performing an elementary *column* operation on A. F

$B = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}$

(i) If B is a matrix that can be obtained by performing an elementary row operation on a matrix A, then A can also be obtained by performing an elementary row operation on B. T

2. §3.1 #8 Prove that if a matrix Q can be obtained from a matrix P by an elementary row operation, then P can be obtained from Q by an elementary row operation of the same type. *Hint: Treat each type of elementary row operation separately.*

type I: $P \xrightarrow{R_i \leftrightarrow R_j} Q$ $\quad Q \xrightarrow{R_j \leftrightarrow R_i} P$

type II: $P \xrightarrow{R_i \to cR_i} Q$ $\quad Q \xrightarrow{R_i \to \frac{1}{c} R_i} P$

type III: $P \xrightarrow{R_i \to R_i + cR_j} Q$ $\quad Q \xrightarrow{R_i \to R_i - cR_j} P$

3. §3.2 #2 Find the rank of the following matrices.

a. **§3.2 #2 (c)** $\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \end{bmatrix}$ Rank = 2

b. **§3.2 #2 (d)** $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ Rank = 1

c. **§3.2 #2 (e)** $\begin{pmatrix} 1 & 2 & 3 & 1 & 1 \\ 1 & 4 & 0 & 1 & 2 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$

$$\xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 3 & 1 & 1 \\ 0 & 2 & -3 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = 3

4. §3.2 #5 (e) For the following matrix, compute the rank and the inverse if it exists.

$$\begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & -2 & 0 \end{bmatrix}$$

Rank $= 3$.

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right]$$

the inverse is $\begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$

5. §3.2 #6 (a) For the following linear transformation T, determine whether T is invertible, and compute T^{-1} if it exists.

$$T : P_2(\mathbb{R}) \to P_2(\mathbb{R}) \text{ defined by } T(f(x)) = f''(x) + 2f'(x) - f(x)$$

$\alpha = \{1, x, x^2\}$ is a basis of $P_2(\mathbb{R})$.

$T(ax^2+bx+c) = -ax^2+(4a-b)x+(2a+2b-c) = \bar{0}$

$\Rightarrow \begin{cases} 2a+2b-c=0 \\ -a=0 \\ 4a-b=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases} \Rightarrow N(T)=\{\bar{0}\}$

So T is 1 to 1.

$\forall k(x) = a_0x^2+b_0x+c_0 \in P_2(\mathbb{R})$, $\exists f(x) = -a_0x^2-(4a_0+b_0)x - (10a_0+2b_0+c_0) \in P_2(\mathbb{R})$ such that $T(f(x)) = k(x)$

$\Rightarrow T$ is onto

$\Rightarrow T$ is invertible

and $T^{-1}(ax^2+bx+c) = -ax^2-(4a+b)x - (10a+2b+c)$

6. §3.2 #7 Express the invertible matrix

$$\begin{pmatrix}1&2&1\\1&0&1\\1&1&2\end{pmatrix}$$

as a product of elementary matrices.

$$\begin{bmatrix}1&2&1\\1&0&1\\1&1&2\end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}1&0&1\\1&2&1\\1&1&2\end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix}1&0&1\\0&2&0\\1&1&2\end{bmatrix}$$

$$\xrightarrow{R_2 \to \frac{1}{2} R_2} \begin{bmatrix}1&0&1\\0&1&0\\1&1&2\end{bmatrix} \xrightarrow{R_3 \to R_3 - R_1} \begin{bmatrix}1&0&1\\0&1&0\\0&1&1\end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - R_2} \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix} \xrightarrow{R_1 \to R_1 - R_3} \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$I_{3\times 3} \xrightarrow{R_1 \to R_1 + R_3} \xrightarrow{R_3 \to R_3 + R_2} \xrightarrow{R_3 \to R_3 + R_1} \xrightarrow{R_2 \to 2R_2} \xrightarrow{R_2 \to R_2 + R_1} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix}1&2&1\\1&0&1\\1&1&2\end{bmatrix}$$

$$\therefore \begin{bmatrix}1&2&1\\1&0&1\\1&1&2\end{bmatrix} = \begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\1&1&0\\0&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\0&2&0\\0&0&1\end{bmatrix}$$

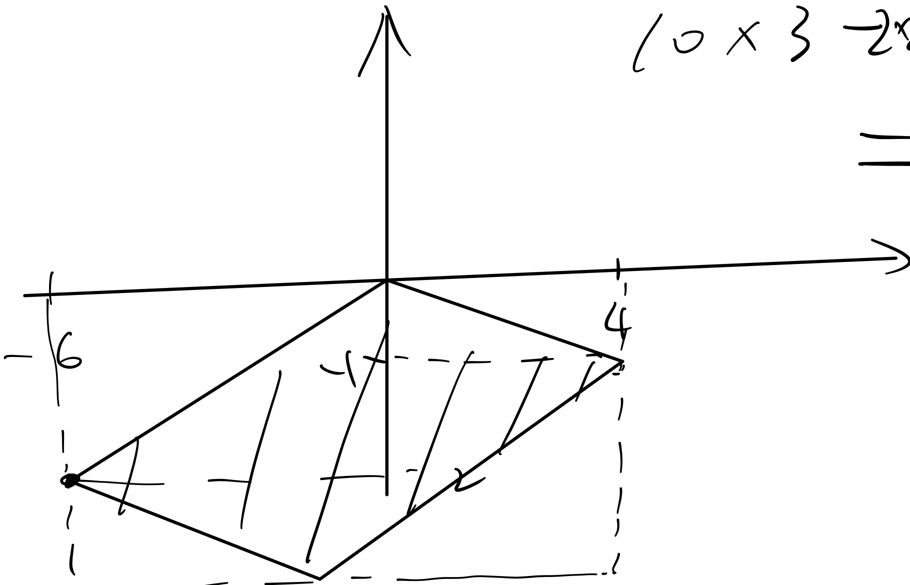
$$\begin{bmatrix}1&0&0\\0&1&0\\1&0&1\end{bmatrix} \begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix} \begin{bmatrix}1&0&1\\0&1&0\\0&0&1\end{bmatrix}$$

7. §4.1 #4 (c) For the following pairs of vectors u and v in $\mathbb{R}^2$, compute the area of the parallelogram determined by u and v.

$$u = (4, -1) \text{ and } v = (-6, -2).$$

$$10 \times 3 - 2 \times 2 \times 6 \times \frac{1}{2} - 2 \times 4 \times 1 \times \frac{1}{2}$$

$$= 14$$



8. §4.1 #7 Prove that $\det(A^t) = \det(A)$ for any $A \in M_{2\times 2}(F)$.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^t = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

$$\det(A) = a \det[d] - b \det[c]$$

$$= ad - bc$$

$$= a \det[d] - c \det[b] = \det(A^t)$$

9. §4.2 #2 Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix}3a_1 & 3a_2 & 3a_3\\ 3b_1 & 3b_2 & 3b_3\\ 3c_1 & 3c_2 & 3c_3\end{pmatrix} = k\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}.$$

(Fully explain why.)

$$\det\begin{bmatrix}3a_1 & 3a_2 & 3a_3\\ 3b_1 & 3b_2 & 3b_3\\ 3c_1 & 3c_2 & 3c_3\end{bmatrix} = 3\det\begin{bmatrix}a_1 & a_2 & a_3\\ 3b_1 & 3b_2 & 3b_3\\ 3c_1 & 3c_2 & 3c_3\end{bmatrix}$$

$$= 3\times 3\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ 3c_1 & 3c_2 & 3c_3\end{bmatrix}$$

$$= 3\times 3\times 3\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix} = 27\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$$

$$\Rightarrow k = 27$$

10. §4.2 #3 Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix}2a_1 & 2a_2 & 2a_3\\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3\\ 7c_1 & 7c_2 & 7c_3\end{pmatrix} = k\det\begin{pmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{pmatrix}.$$

(Fully explain why.)

$$\det\begin{bmatrix}2a_1 & 2a_2 & 2a_3\\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix} = \det\begin{bmatrix}2a_1 & 2a_2 & 2a_3\\ 3b_1 & 3b_2 & 3b_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix} + \frac{5}{7}\det\begin{bmatrix}2a_1 & 2a_2 & 2a_3\\ 7c_1 & 7c_2 & 7c_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix}$$

same $\Rightarrow = 0$

$$= \det\begin{bmatrix}2a_1 & 2a_2 & 2a_3\\ 3b_1 & 3b_2 & 3b_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix} = 2\det\begin{bmatrix}a_1 & a_2 & a_3\\ 3b_1 & 3b_2 & 3b_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix}$$

$$= 2\times 3\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ 7c_1 & 7c_2 & 7c_3\end{bmatrix} = 2\times 3\times 7\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix}$$

$$= 42\det\begin{bmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{bmatrix} \Rightarrow k = 42$$

11. §4.2 #4 Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

(Fully explain why.)

$$\det\begin{bmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} = \det\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix} + \det\begin{bmatrix} c_1 & c_2 & c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}$$

$$= \det\begin{bmatrix} b_1 & b_2 & b_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1 & a_2 & a_3 \end{bmatrix} + \det\begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{bmatrix}$$

$$= \det\begin{bmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{bmatrix} + \det\begin{bmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}$$

$$= -1\times(-1)\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} + (-1)\times(-1)\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix}$$

$$= 2\det\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \Rightarrow k=2.$$

12. §4.2 #8 Evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$$

along the third row

$$\det\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{bmatrix} = -1\det\begin{bmatrix} 0 & 2 \\ 1 & 5 \end{bmatrix} - 3\det\begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$$

$$= 2 - 15 = -13$$

13. §4.2 #21 Evaluate the determinant of the given matrix by any legitimate method.

$$\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$$

$$\det\begin{bmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 4 & -1 & 1 \\ 0 & 3 & 4 & -5 \end{bmatrix}$$

$$= \det\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 3 & 4 & -5 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 19 & -38 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & -5 & 11 \\ 0 & 0 & 19 & -43 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$= 1 \times 1 \times 19 \times 5 = 95$$