Homework 6 MATH 416: ABSTRACT LINEAR ALGEBRA

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(Exercises are taken from *Linear Algebra, **Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

Here are definitions you may want to use.

Definition Let V be a vector space and W_1 and W_2 be subspaces of V such that $V = W_1 \oplus W_2$. (Recall the definition of direct sum given in the exercises of Section 1.3.). A function $T : V \to V$ is called the **projection on** W_1 **along** W_2 if, for $x = x_1 + x_2$ with $x_1 \in W_1$ and $x_2 \in W_2$, we have $T(x) = x_1$. (Textbook, Section 2.1, Page. 76)

Definition If S_1 and S_2 are nonempty subsets of a a vector space V , then the **sum** of S_1 and S_2, denoted $S_1 + S_2$, is the set $\{x + y : x \in S_1 \text{ and } y \in S_2\}$. (Textbook, Section 1.3, Page. 22)

Definition A vector space V is called the **direct sum** of W_1 and W_2 if W_1 and W_2 are subspaces of V such that $W_1 \cap W_2 = \{0_V\}$ and $W_1 + W_2 = V$. We denote that V is the direct sum of W_1 and W_2 by writing $V = W_1 \oplus W_2$. (Textbook, Section 1.3, Page. 22)

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1. Exercise §2.4 #14 Let

$$V = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{R} \right\}.$$

Construct an isomorphism from V to $\mathbb{R}^3$. (Show that the map you have constructed is an isomorphism.)

$T: V \to \mathbb{R}^3 \quad T(X) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} X, \quad X \in V$

need to prove T is linear.

Let $X_0 = \begin{bmatrix} X_a & X_a + X_b \\ 0 & X_c \end{bmatrix} \in V$

$T(X_0) = \begin{bmatrix} X_a & X_a + X_b \\ 0 & X_c \end{bmatrix} = 0 \Rightarrow \begin{cases} X_a = 0 \\ X_b = 0 \\ X_c = 0 \end{cases}$ i.e. $X_0 = 0 \Rightarrow N(T) = \{0\}$

$\Rightarrow$ 1 to 1

$\forall y_0 = \begin{bmatrix} y_1 & y_2 \\ 0 & y_3 \end{bmatrix} \in \mathbb{R}^3 \quad \exists X = \begin{bmatrix} y_1 & y_1 + (y_2 - y_1) \\ 0 & y_3 \end{bmatrix} \in V$ s.t. $T(X) = y_0$

$\Rightarrow$ onto

T is 1 to 1 and onto $\Rightarrow T$ is an isomorphism from V to $\mathbb{R}^3$

2. §2.4 #16 Let B be an $n \times n$ invertible matrix. Define $\Phi : M_{n\times n}(\mathbb{F}) \to M_{n\times n}(\mathbb{F})$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

$\phi(A) = 0 = B^{-1}AB \Rightarrow A = 0 \Rightarrow N(\phi) = \{0\} \Rightarrow$ 1 to 1.

$\forall C \in M_{n\times n}(\mathbb{F})$, $\exists A = BCB^{-1} \in M_{n\times n}(F)$ s.t. $\phi(A) = C \Rightarrow$ onto

ϕ is 1 to 1 and onto $\Rightarrow \phi$ is an isomorphism.

need to check ϕ is linear.

3. §2.5 #1 Label the following statements as true or false (Answer is back, give a short explanation!).

(a) Suppose that $\beta = \{x_1, x_2, ..., x_n\}$ and $\beta' = \{x'_1, x'_2, ..., x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$. F $[x'_j]_\beta$ $[v]_\beta = [I]_{\beta'}^{\beta} [v]_{\beta'}$

(b) Every change of coordinate matrix is invertible. T

(c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[T]_\beta = Q[T]_{\beta'}Q^{-1}$. T

(d) The matrices $A, B \in M_{n\times n}(\mathbb{F})$ are called similar if $B = \cancel{Q^t AQ}$ $Q^{-1}AQ$ for some $Q \in M_{n\times n}(\mathbb{F})$. F

(e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[T]_\beta$ is similar to $[T]_\gamma$. T

4. §2.5 #2 For each of the following pairs of ordered bases β and β' for $\mathbb{R}^2$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

a. §2.5 #2 (a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$

b. §2.5 #2 (b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$

c. §2.5 #2 (c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$

d. §2.5 #2 (d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

(a). $(I_{\mathbb{R}^2})_{\beta'}^{\beta} = \left[[(a_1, a_2)]_\beta , [(b_1, b_2)]_\beta \right] = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

(b) $(I_{\mathbb{R}^2})_{\beta'}^{\beta} = \left[[(0,10)]_\beta , [(5,0)]_\beta \right] = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$

(c) $(I_{\mathbb{R}^2})_{\beta'}^{\beta} = \left[[(1,0)]_\beta , [(0,1)]_\beta \right] = \begin{bmatrix} 3 & -1 \\ 5 & -2 \end{bmatrix}$

(d) $(I_{\mathbb{R}^2})_{\beta'}^{\beta} = \left[[(2,1)]_\beta , [(-4,1)]_\beta \right] = \begin{bmatrix} 2 & -1 \\ 5 & -4 \end{bmatrix}$

5. §2.5 #3 For each of the following pairs of ordered bases β and β' for $P_2(\mathbb{R})$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.

a. §2.5 #12 (c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$

b. §2.3 #12 (d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$

(a) $$\left[I_{P_2(\mathbb{R})}\right]_{\beta'}^{\beta} = \left[(1)_\beta \quad (x)_\beta \quad (x^2)_\beta\right] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ -3 & 2 & 1 \end{bmatrix}$$

(b) $$\left[I_{P_2(\mathbb{R})}\right]_{\beta'}^{\beta} = \left[(x^2+x+4)_\beta \quad (4x^2-3x+2)_\beta \quad (2x^2+3)_\beta\right]$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 3 & -2 & 1 \\ -1 & 3 & 1 \end{bmatrix}$$

6. §2.5 #6 For each matrix A and ordered basis β, find $[L_A]_\beta$. Also, find an invertible matrix Q such that $[L_A]_\beta = Q^{-1}AQ$.

a. §2.5 #6 (a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$

b. §2.3 #6 (c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$

(a)

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$[L_A]_{\beta'} = A \qquad [L_A]_\beta = [I]_{\beta'}^{\beta} \, [L_A]_{\beta'} \, [I]_{\beta}^{\beta'}$$

$$Q = [I]_\beta^{\beta'} = \left[\left[\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]_{\beta'} \; \left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\beta'} \right] = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[L_A]_\beta = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 11 \\ -2 & -4 \end{bmatrix}$$

(b)

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$Q = [I]_\beta^{\beta'} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$[L_A]_\beta = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

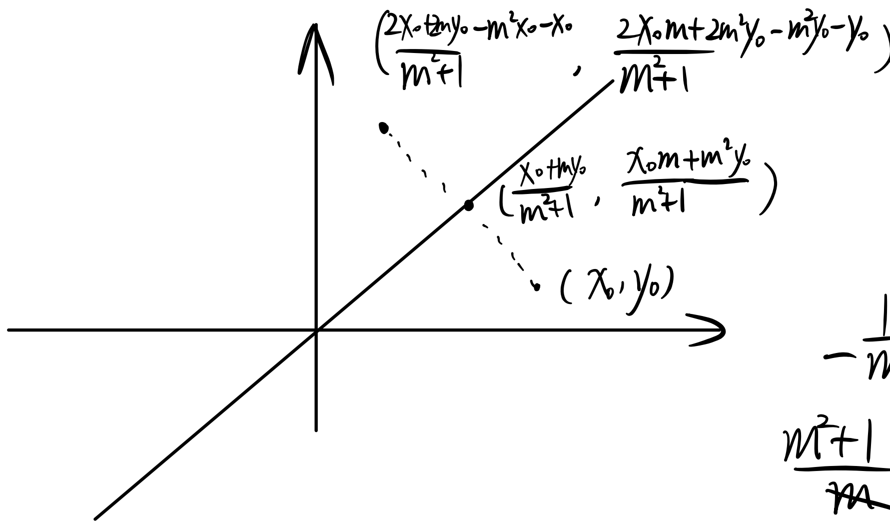
$$= \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 2 \\ -2 & -3 & -4 \\ 1 & 1 & 2 \end{bmatrix}$$

7. §2.5 #7 In $\mathbb{R}^2$, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $T(x, y)$, where

a. §2.5 #7 (a) T is the reflection of $\mathbb{R}^2$ about L.

b. §2.5 #7 (b) T is the projection on L along the line perpendicular to L. (See the definition of *projection* in the exercises of Section 2.1.)

(a).



$$y = -\frac{1}{m}x + \frac{x_0}{m} + y_0$$

$$-\frac{1}{m}x + \frac{x_0}{m} + y_0 = mx.$$

$$\frac{m^2+1}{\cancel{m}}x = \frac{x_0+my_0}{\cancel{m}}$$

$$x = \frac{x_0+my_0}{m^2+1}$$

$$T(x, y) = \left(\frac{x+2my-m^2x}{m^2+1}, \frac{m^2y+2mx-y}{m^2+1}\right)$$

(b)

$$T(x, y) = \left(\frac{x+my}{m^2+1}, \frac{xm+m^2y}{m^2+1}\right)$$

Problems **not** from the textbook exercises.

8. Compute the inverse of each of the following matrices, if possible.

a. $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}$ **b.** $B = \begin{pmatrix} 1 & 2 & 1 \\ -1 & 1 & 2 \\ 1 & 0 & 1 \end{pmatrix}$ **c.** $C = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix}$

a.

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 1 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

b.

$$[B \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ 0 & 1 & 0 & \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & \frac{1}{2} \end{bmatrix}$$

c.

$$C = \begin{bmatrix} 1 & 2 & 6 \\ 1 & 0 & 2 \\ 1 & 1 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 6 \\ 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$\dim C = 2 < 3 \Rightarrow C$ has no inverse.