Math 416 Practice Exam 2

1. Find the determinant of $B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + y, 2y, -x + y + 2z).$$

(1) Find the eigenvalues and corresponding eigenvectors of T.

(2) Find a basis for each of the eigenspaces.

(3) Is T diagonalizable? If so, diagonalize it.

3. Prove $\det(E) = \det(E^t)$ when E is an elementary matrix.

4. Let T be a linear map on a finite dimensional vector space V.

(1) Describe the eigenspace E_λ of T corresponding to an eigenvalue λ in terms of the null space of a certain linear map on V.

(2) Prove that E_λ is a T-invariant subspace of V.

5. Find the matrix power $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}^k$ for $k \in \mathbb{N}$.

6. **True / False**. Justify your answer.

(a) If $\langle\ ,\ \rangle$ is an inner product on a vector space V and $\langle x, y \rangle = 0$ for all $y \in V$, then $x = 0$.

(b) If T is a linear operator on a vector space V, then the T-cyclic subspace generated by $v \in V$ is T-invariant.

(c) If A is a transition matrix, then $\lim_{k\to\infty} A^k$ exists.

(d) Any linear operator on a finite dimensional vector space that has only one eigenvalue is not diagonalizable.

(e) If a linear operator on $\mathbb{R}^n$ has n distinct eigenvalues, then it is diagonalizable.

(f) If v is an eigenvector of a linear operator T corresponding to λ, then either $T(v) = 0$ or $T(v)$ is an eigenvector corresponding to λ.

1. Find the determinant of $B = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{pmatrix}$.

$$\det B = \begin{vmatrix} 3 & 3 & 3 & 1 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 1 & 1 & 2 & 3 \end{vmatrix} = 6 \begin{vmatrix} 1 & 1 & 1 & \frac{1}{3} \\ 0 & -\frac{1}{2} & -1 & -\frac{1}{3} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & \frac{2}{3} \end{vmatrix} = -2.$$

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T(x, y, z) = (x + y, 2y, -x + y + 2z).$$

(1) Find the eigenvalues and corresponding eigenvectors of T.

(2) Find a basis for each of the eigenspaces.

(3) Is T diagonalizable? If so, diagonalize it.

$\alpha = \{(1,0,0), (0,1,0), (0,0,1)\}.$

$$[T]_\alpha^\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix}$$

$$\det([T]_\alpha^\alpha - tI_3) = \begin{vmatrix} 1-t & 1 & 0 \\ 0 & 2-t & 0 \\ -1 & 1 & 2-t \end{vmatrix} = (1-t)(2-t)^2$$

$\lambda_1 = 1 \qquad \lambda_2 = 2$

$$[T]_\alpha^\alpha - \lambda_1 I_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}^T\right\}$$

eigenvectors are $a(1,0,1)$, $a \neq 0$.

$$[T]_\alpha^\alpha - \lambda_2 I_3 = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T\right\}$$

$$\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}^T, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}^T, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}^T \right\}.$$

eigenvectors are.
$a(1,1,0) + b(0,0,1)$
$a \neq 0$ or $b \neq 0$.

$\dim E_1 = 1 =$ alg. mult. of 1. $\dim E_2 = 2 =$ alg. mult. of 2.

$\Rightarrow$ Yes, $[T]_\beta^\beta = [I_3]_\alpha^\beta [T]_\alpha^\alpha [I_3]_\beta^\alpha$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

3. Prove $\det(E) = \det(E^t)$ when E is an elementary matrix.

elementary matrix. E

type I : $R_i \leftrightarrow R_j$ $E = E^t \Rightarrow \det E = \det E^t$

type II : $R_i \to \lambda R_i$ $E = E^t$

type III : $R_i \to R_i + \lambda R_j$ $\det E = \det E^t = 1$.

4. Let T be a linear map on a finite dimensional vector space V.

(1) Describe the eigenspace E_λ of T corresponding to an eigenvalue λ in terms of the null space of a certain linear map on V.

(2) Prove that E_λ is a T-invariant subspace of V.

(1) $Tv = \lambda v.$

$(T-\lambda I)(v) = \bar{0} \Rightarrow E_\lambda = N(T-\lambda I).$

(2). $\forall v \in E_\lambda,\ T(v) = \lambda v \in E_\lambda \Rightarrow E_\lambda$ is T-invariant

$$\left.\begin{array}{l} (T-\lambda I)\bar{0} = \bar{0} \Rightarrow \bar{0} \in E_\lambda. \\ \forall v_1, v_2 \in E_\lambda. \\ (T-\lambda I)(v_1+v_2) = \bar{0}. \\ (T-\lambda I)(c v_1) = \bar{0} \end{array}\right\} \Rightarrow E_\lambda \text{ is subspace of } V.$$

$v_1+v_2,\ cv_1 \in E_\lambda.$

5. Find the matrix power $\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}^k$ for $k \in \mathbb{N}$.

$A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} \quad \det(A - tI_2) = \begin{vmatrix} 3-t & 1 \\ 0 & 1-t \end{vmatrix} = (3-t)(1-t)$

$\lambda_1 = 3 \quad \lambda_2 = 1. \quad E_{\lambda_1} = \operatorname{span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\}$

$E_{\lambda_2} = \operatorname{span}\{\begin{pmatrix} 1 \\ -2 \end{pmatrix}\}$

$\beta = \{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \end{pmatrix}\}.$

$Q = [I]_\alpha^\beta = \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$

$$A = \begin{bmatrix} 3 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} \\ 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 3^k & \frac{1}{2}3^k - \frac{1}{2} \\ 0 & 1 \end{bmatrix}$$

6. **True / False**. Justify your answer.

(a) If $\langle\ ,\ \rangle$ is an inner product on a vector space V and $\langle x, y\rangle = 0$ for all $y \in V$, then $x = 0$.

(b) If T is a linear operator on a vector space V, then the T-cyclic subspace generated by $v \in V$ is T-invariant.

(c) If A is a transition matrix, then $\lim_{k\to\infty} A^k$ exists.

(d) Any linear operator on a finite dimensional vector space that has only one eigenvalue is not diagonalizable.

(e) If a linear operator on $\mathbb{R}^n$ has n distinct eigenvalues, then it is diagonalizable.

(f) If v is an eigenvector of a linear operator T corresponding to λ, then either $T(v) = 0$ or $T(v)$ is an eigenvector corresponding to λ.

(a) True $\langle x, x\rangle = 0 = \|x\|^2 \Rightarrow x = 0$.

(b) True. $W = \text{Span}\{v, T(v), T^2(v), \dots\} \subset V$.

$T(T^k(v)) = T^{k+1}(v) \Rightarrow \{v, T(v), \dots\}$ is T-invariant

$\Rightarrow \forall w \in W \quad T(w) \in W$

(c) False $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$

(d) False $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(e) True. $\dim E_\lambda = 1 = $ alg. mult. of λ

$1 \leq$ geom. mult. $\leq$ alg. mult. $= 1$.

(f) True. if $T(v) \neq 0$

$T(T(v)) = T(\lambda v) = \lambda T(v)$.