Homework 3 — MATH 416: ABSTRACT LINEAR ALGEBRA

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(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. Exercise §1.6 #1 Label the following statements as true or false (Answer is back, give a short explanation!).

(a) The zero vector space has no basis. F ~~$\vec{0}$~~ empty set is a basis of itself.

(b) Every vector space that is generated by a finite set has a basis. T

(c) Every vector space has a finite basis. F $\{\{2^n\}\}$

(d) A vector space cannot have more than one basis. F $\{1\}$ and $\{2\}$ both are basis of $\mathbb{R}$

(e) If a vector space has a finite basis, then the number of vectors in every basis is the same. T

(f) The dimension of $P_n(\mathbb{F})$ is n. F, $n+1$

(g) The dimension of $M_{m\times n}(\mathbb{F})$ is $m+n$. F, mn

(h) Suppose that V is a finite dimensional vector space, that S_1 is a linearly independent subset of V, and that S_2 is a subset of V that generates V. Then S_1 cannot contain more vectors than S_2. T

(i) If S generates the vector space V, then every vector in V can be written as a linear combination of vectors in S in only one way. F, $S=\{u_1=1, u_2=2\}$ $V=\mathbb{R}$ $3=3u_1=\frac{3}{2}u_2$

(j) Every subspace of a finite-dimensional vector space is finite-dimensional. T

(k) If V is a vector space having dimension n, then V has exactly one subspace with dimension 0 and exactly one subspace with dimension n. T

(l) If V is a vector space having dimension n, and if S is a subset of V with n vectors, then S is linearly independent if and only if S spans V. T

2. §1.6 #2 Determine which of the following sets are bases for $\mathbb{R}^3$.

a. §1.6 #2 (b) { (2, -4, 1), (0, 3, -1), (6, 0, -1) }

b. §1.6 #2 (d) { (-1, 3, 1), (2, -4, -3), (-3, 8, 2) }

a. $a_1(2,-4,1)+a_2(0,3,-1)+a_3(6,0,-1)=0$

$$\begin{bmatrix} 2 & 0 & 6 & 0 \\ -4 & 3 & 0 & 0 \\ 1 & -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{cases} a_1 = -3t \\ a_2 = -4t \\ a_3 = t \end{cases} (t \in \mathbb{R})$$

So, No

b. $b_1(-1,3,1)+b_2(2,-4,-3)+b_3(-3,8,2)=0$

$$\begin{bmatrix} -1 & 2 & -3 & 0 \\ 3 & -4 & 8 & 0 \\ 1 & -3 & 2 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & 3 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{cases} b_1 = 0 \\ b_2 = 0 \\ b_3 = 0 \end{cases}$$

So, Yes

3. §1.6 #7 The vectors $u_1 = (2, -3, 1), u_2 = (1, 4, -2), u_3 = (-8, 12, -4), u_4 = (1, 37, -17)$, and $u_5 = (-3, -5, 8)$ generate $\mathbb{R}^3$. Find a subset of the set $\{u_1, u_2, u_3, u_4, u_5\}$ that is a basis for $\mathbb{R}^3$.

$a_1u_1+a_2u_2+a_3u_3+a_4u_4+a_5u_5=0$

$$\begin{bmatrix} 2 & 1 & -8 & 1 & -3 & 0 \\ -3 & 4 & 12 & 37 & -5 & 0 \\ 1 & -2 & -4 & -17 & 8 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -4 & -3 & 0 & 0 \\ 0 & 1 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

$$\begin{cases} a_1 = 4s+3t \\ a_2 = -7t \\ a_3 = s \\ a_4 = t \\ a_5 = 0 \end{cases}$$

$s=0$: $\{u_1, u_2, u_4\}$ linearly dependent

$t=0$: $\{u_1, u_3\}$ linearly dependent

$\{u_1, u_2, u_5\}$ is a basis for $\mathbb{R}^3$

Let $S = \{u_a, u_b, u_c, u_d\}$, $(u_i = i_0 + i_1 x + i_2 x^2 + \cdots + i_7 x^7)$ $i = a, b, c, d$

Assume S is linearly dependent, i.e. exists k_a, k_b, k_c, k_d such that are not all zero and $k_a u_a + k_b u_b + k_c u_c + k_d u_d = 0$

Suppose the degree of u_i is t_i, $k_a a_t + k_b b_t + k_c c_t + k_d d_t = 0$ $(t_a < t_b < t_c < t_d)$ $(i_t = (i_{ta}, i_{tb}, i_{tc}, i_{td}))$ $i = a, b, c, d$

, so $\begin{bmatrix} a_{ta} & b_{ta} & c_{ta} & d_{ta} & 0 \\ a_{tb} & b_{tb} & c_{tb} & d_{tb} & 0 \\ a_{tc} & b_{tc} & c_{tc} & d_{tc} & 0 \\ a_{td} & b_{td} & c_{td} & d_{td} & 0 \end{bmatrix}$ $i_n = 0$, $n > i_{t_i}$, $i = a, b, c, d$, $n \le 7$ $\begin{bmatrix} a_{ta} & b_{ta} & c_{ta} & d_{ta} & 0 \\ 0 & b_{tb} & c_{tb} & d_{tb} & 0 \\ 0 & 0 & c_{tc} & d_{tc} & 0 \\ 0 & 0 & 0 & d_{td} & 0 \end{bmatrix}$

has solucions other than $\bar{0}$ which is impossible. So S is linearly independent

Problems not from the textbook exercises.

not good.

4. Let $S \subset P_7$ be a set of 4 polynomials such that no two of them have the same degree (and all degrees are nonnegative). Show that S is linearly independent.

Let $S = \{u_a, u_b, u_c, u_d\}$, the degree of u_i is i, $-1 \le a \le b \le c \le d \le 7$.

$k_a u_a + k_b u_b + k_c u_c + k_d u_d = 0$.

① $u_d = -\frac{k_a}{k_d} u_a - \frac{k_b}{k_d} u_b - \frac{k_c}{k_d} u_c \Rightarrow$ no solutions. $\Rightarrow k_d = 0$.

② $\Rightarrow k_c = 0$ ③ $\Rightarrow k_b = 0$ ④ $k_a = 0$

$\Rightarrow$ Lin. indep

X.

~~Let $S = \{s_1, s_2, \cdots, s_n, \cdots\}$~~ X can't assume its countable

Since v is in $Span(S)$, $v = a_1 s_1 + a_2 s_2 + \cdots + a_n s_n + \cdots$ $(a_1, a_2 \cdots a_n, \cdots$ are not all zero)

$S \cup \{v\} = \{v, s_1, s_2, \cdots, s_n, \cdots\} \supseteq S$, so $span(S) \subseteq span(S \cup \{v\})$

any $m \in span(S \cup \{v\})$

$m = b_0 v + b_1 s_1 + b_2 s_2 + \cdots + b_n s_n + \cdots$

$= (a_1 b_0 + b_1) s_1 + (a_2 b_0 + b_2) s_2 + \cdots + (a_n b_0 + b_n) s_n + \cdots \in span(S)$

so $span(S \cup \{v\}) \subseteq span(S)$

so $span(S \cup \{v\}) = span(S)$

5. Let S be a subset of the vector space V. Show that if v is in $span(S)$ then $span(S \cup \{v\}) = span(S)$.

$\Rightarrow$

$Span(S)$ is a subspace of V. $\Rightarrow$ $\forall x_1, x_2 \in span(S)$, $a \in \mathbb{R}$ $x_1 + x_2 \in span(S)$. $a x_1 \in span(S)$

$\forall x \in Span(S \cup \{v\})$. $\exists y \in Span(S)$ such that

$x = y + av$,

Because $y, v \in Span(S)$, so $av \in Span(S)$.

$y + av \in Span(S)$.

therefore $x \in Span(S)$.

$\therefore Span(S \cup \{v\}) \subseteq Span(S)$.

Obviously $S \subseteq S \cup \{v\}$ $Span(S) \subseteq Span(S \cup \{v\})$ $\therefore Span(S \cup \{v\}) = Span(S)$

V

6. Recall that the set of all 3×3 symmetric matrices is a vector space. Compute its dimension. (You must find a candidate basis and then verify that it is a basis.)

dimension is 6.

$S = \{S_1, S_2, S_3, S_4, S_5, S_6\}$

$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $S_4 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$,

$S_5 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $S_6 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,

① $b_1S_1 + b_2S_2 + \cdots + b_6S_6 = O_{3\times3} \Rightarrow \begin{cases} b_1 = 0 \\ b_2 = 0 \\ \cdots \\ b_6 = 0 \end{cases}$ so S is linearly independent.

② any 3×3 symmetric matrices

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} = a_{11}S_1 + a_{12}S_2 + a_{22}S_3 + a_{13}S_4 + a_{23}S_5 + a_{33}S_6 \in \mathrm{Span}(S)$$

So $V \subseteq \mathrm{Span}(S)$

and any linear combination of S $\quad C_1S_1 + C_2S_2 + \cdots C_6S_6 = \begin{bmatrix} C_1 & C_2 & C_4 \\ C_2 & C_3 & C_5 \\ C_4 & C_5 & C_6 \end{bmatrix} \in V$

So $\mathrm{Span}(S) \subseteq V$

So $\mathrm{Span}(S) = V$

Thus S is a basis of V.

7. Determine if $\{\sin^2 x, \cos^2 x, 1\}$ is a linearly dependent subset of $\mathcal{F}(\mathbb{R})$.

$\sin^2 x + \cos^2 x - 1 \equiv 0$ for any $x \in \mathbb{R}$

So $\{\sin^2 x, \cos^2 x, 1\}$ is a linearly dependent subset of $\mathcal{F}(\mathbb{R})$.