Homework 1

MATH 416: ABSTRACT LINEAR ALGEBRA

NAME: DATE:

(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. Exercise §3.4 #2 Use Gaussian elimination to solve the following systems of linear equations.

a. §3.4 #2 (a)
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -1 \\ 2x_1 & + & 2x_2 & + & x_3 & = & 1 \\ 3x_1 & + & 5x_2 & - & 2x_3 & = & -1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 3 & 5 & -2 & -1 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & -1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 - \frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & -2 & 3 & 3 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \xrightarrow[R_3 \to -2R_3]{R_2 \to -\frac{1}{2}R_2} \begin{bmatrix} 1 & 2 & -1 & -1 \\ 0 & 1 & -\frac{3}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\begin{cases} x_1 + 2x_2 - x_3 = -1 \\ x_2 - \frac{3}{2}x_3 = -\frac{3}{2} \\ x_3 = -1 \end{cases} \Rightarrow \begin{cases} x_1 = 4 \\ x_2 = -3 \\ x_3 = -1 \end{cases}$$

$$\begin{bmatrix} 4 \\ -3 \\ -1 \end{bmatrix}$$

b. §3.4 #2 (c)

$$\begin{array}{rcrcrcrcr} x_1 & + & 2x_2 & & & + & 2x_4 & = & 6 \\ 3x_1 & + & 5x_2 & - & x_3 & + & 6x_4 & = & 17 \\ 2x_1 & + & 4x_2 & + & x_3 & + & 2x_4 & = & 12 \\ 2x_1 & & & - & 7x_3 & + & 11x_4 & = & 7 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 0 & 2 & 6 \\ 3 & 5 & -1 & 6 & 17 \\ 2 & 4 & 1 & 2 & 12 \\ 2 & 0 & -7 & 11 & 7 \end{bmatrix} \xrightarrow[\substack{R_3-2R_1 \\ R_4-2R_1}]{R_2-3R_1} \begin{bmatrix} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & -4 & -7 & 7 & -5 \end{bmatrix}$$

$$\xrightarrow{R_4-4R_2} \begin{bmatrix} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -3 & 7 & -1 \end{bmatrix} \xrightarrow{R_4+3R_3} \begin{bmatrix} 1 & 2 & 0 & 2 & 6 \\ 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_2 \to -R_2} \begin{bmatrix} 1 & 2 & 0 & 2 & 6 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{cases} x_1 + 2x_2 + 2x_4 = 6 \\ x_2 + x_3 = 1 \\ x_3 - 2x_4 = 0 \\ x_4 = -1 \end{cases}$$

$$\begin{cases} x_1 = 2 \\ x_2 = 3 \\ x_3 = -2 \\ x_4 = -1 \end{cases} \qquad \begin{bmatrix} 2 \\ 3 \\ -2 \\ -1 \end{bmatrix}$$

c. §3.4 #2 (f)
$$\begin{array}{rcrcrcrcl} x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 2 \\ 2x_1 & + & 4x_2 & - & x_3 & + & 6x_4 & = & 5 \\ & & x_2 & & & + & 2x_4 & = & 3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 2 & 4 & -1 & 6 & 5 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & -1 & 3 & 2 \\ 0 & 1 & 0 & 2 & 3 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Set $X_4 = t$

$$\begin{cases} X_1 + 2X_2 - X_3 + 3t = 2 \\ X_2 + 2t = 3 \\ X_3 = 1 \end{cases}$$

$$\begin{cases} X_1 = -3 + t \\ X_2 = 3 - 2t \\ X_3 = 1 \\ X_4 = t \end{cases}$$

$$(X_1, X_2, X_3, X_4) = \{ (-3+t, 3-2t, 1, t) \mid t \in \mathbb{R} \}$$

2. Exercise §3.4 #5 Let the reduced echelon form of A be

$$\begin{pmatrix} 1 & 0 & 2 & 0 & -2 \\ 0 & 1 & -5 & 0 & -3 \\ 0 & 0 & 0 & 1 & 6 \end{pmatrix}.$$

Determine A if the first, second, and fourth columns of A are

$$\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix},$$

respectively.

$$\begin{bmatrix} 1 & 0 & t_1 & 1 & S_1 \\ -1 & -1 & t_2 & -2 & S_2 \\ 3 & 1 & t_3 & 0 & S_3 \end{bmatrix} \xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + R_1} \begin{bmatrix} 1 & 0 & t_1 & 1 & S_1 \\ 0 & -1 & (t_1+t_2) & -1 & (S_1+S_2) \\ 0 & 1 & (t_3-3t_1) & -3 & (S_3-3S_1) \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & 0 & t_1 & 1 & S_1 \\ 0 & -1 & (t_1+t_2) & -1 & (S_1+S_2) \\ 0 & 0 & (t_2+t_3-2t_1) & -4 & (S_2+S_3-2S_1) \end{bmatrix}$$

$$\begin{cases} t_1 = 2 \\ t_1 + t_2 = 5 \\ t_2 + t_3 - 2t_1 = 0 \end{cases} \qquad \begin{cases} t_1 = 2 \\ t_2 = 3 \\ t_3 = 1 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 2 & 1 & S_1 \\ 0 & -1 & 5 & -1 & S_1+S_2 \\ 0 & 0 & 0 & -4 & S_2+S_3-2S_1 \end{bmatrix}$$

$$\xrightarrow[R_3 \to -\frac{1}{4}R_3]{R_2 \to -R_2} \begin{bmatrix} 1 & 0 & 2 & 1 & S_1 \\ 0 & 1 & -5 & 1 & -(S_1+S_2) \\ 0 & 0 & 0 & 1 & -\frac{1}{4}(S_2+S_3-2S_1) \end{bmatrix} \xrightarrow[R_2 - R_3]{R_1 - R_3}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & (\frac{1}{2}S_1 + \frac{1}{4}S_2 + \frac{1}{4}S_3) \\ 0 & 1 & -5 & 0 & (-\frac{3}{2}S_1 - \frac{3}{4}S_2 + \frac{1}{4}S_3) \\ 0 & 0 & 0 & 1 & -\frac{1}{4}(S_2+S_3-2S_1) \end{bmatrix} \qquad \begin{cases} \frac{1}{2}S_1 + \frac{1}{4}S_2 + \frac{1}{4}S_3 = -2 \\ -\frac{3}{2}S_1 - \frac{3}{4}S_2 + \frac{1}{4}S_3 = -3 \\ -\frac{1}{4}(S_2+S_3-2S_1) = 6 \end{cases}$$

$$\begin{cases} S_1 = 4 \\ S_2 = -7 \\ S_3 = -9 \end{cases} \qquad \begin{bmatrix} 1 & 0 & 2 & 1 & 4 \\ -1 & -1 & 3 & -2 & -7 \\ 3 & 1 & 1 & 0 & -9 \end{bmatrix}$$

3. Exercise §1.2 #1 Label the following statements as true or false (Answer is back, give a short justification!).

(a) Every vector space contains a zero vector. T

(b) A vector space may have more than one zero vector. F $\bar{0}' = \bar{0}' + \bar{0} = \bar{0} + \bar{0}' = \bar{0}$

(c) In any vector space, $ax = bx$ implies $a = b$. F $x = \bar{0}$

(d) In any vector space, $ax = ay$ implies $x = y$. F $a = 0$

(e) A vector in $\mathbb{F}^n$ may be regarded as a matrix in $M_{n\times 1}(\mathbb{F})$ T

(f) An $m \times n$ matrix has m columns and n rows. F m rows n columns.

(g) In $P(\mathbb{F})$, only polynomials of the same degree may be added. F n degree add n-1 degree is n degree still belongs to $P(F)$

(h) If f and g are polynomials of degree n, then $f + g$ is a polynomial of degree n.

F $f(x) = -x^n$ $g(x) = x^n$

(i) If f is a polynomial of degree n and c is a nonzero scalar, then cf is a polynomial of degree n.

T

(j) A nonzero scalar of $\mathbb{F}$ may be considered to be a polynomial in $P(\mathbb{F})$ having degree zero.

T

(k) Two functions in $\mathcal{F}(S, \mathbb{F})$ are equal if and only if they have the same value at each element of S.

T

4. Exercise §1.2 #12 A real-valued function f defined on the real line is called an *even function* if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition and scalar multiplication defined in Example 3 (Page 9.) is a vector space. (Example 3: Let $S \neq \emptyset$, and let $\mathcal{F}(S, \mathbb{F}) = \{f | f : S \to \mathbb{F} \text{ function}\}$. The set $\mathcal{F}(S, \mathbb{F})$ is a vector space under the operations, for $f, g \in \mathcal{F}(S, \mathbb{F}), c \in \mathbb{F}$, $(f + g)(s) = f(s) + g(s)$, $(cf)(s) = cf(s)$ $\forall s \in S$.)

Let the set of even functions be M, $f(t), g(t) \in M$

(a) $\bar{0}(t) = 0 = \bar{0}(-t)$, $\bar{0}(t) \in M$

(b) Addition: $(f+g)(t) = f(t) + g(t) = f(-t) + g(-t) = (f+g)(-t)$

$\Rightarrow (f+g)(t) \in M$

(c) Scalar multiplication:

$(cf)(t) = cf(t) = cf(-t) = (cf)(-t)$

$\Rightarrow (cf)(t) \in M$

Since $M \subseteq \mathcal{F}(S,\mathbb{F})$, we can conclude M is a subspace of $\mathcal{F}(S,\mathbb{F})$

then M is a vector space.

5. Exercise §1.2 #18 Let $V = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\}$. For $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$, define

$$(a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2) \text{ and } c(a_1, a_2) = (ca_1, ca_2).$$

No

Is V a vector space over $\mathbb{R}$ with these operations? Justify your answer.

$\because (a_1, a_2) + (b_1, b_2) = (a_1 + 2b_1, a_2 + 3b_2)$

$(b_1, b_2) + (a_1, a_2) = (b_1 + 2a_1, b_2 + 3a_2)$

$\therefore (a_1, a_2) + (b_1, b_2) \neq (b_1, b_2) + (a_1, a_2)$

6. Exercise §1.3 #1 Label the following statements as true or false (Answer is back, give a short justification!).

F different operations

(a) If V is a vector space and W is a subset of V that is a vector space, then W is a subspace of V.

(b) The empty set is a subspace of every vector space. F empty set doesn't have $\vec{0}$.

(c) If V is a vector space other than the zero vector space, then V contains a subspace W such that $W \neq V$.

T

(d) The intersection of any two subsets of V is a subspace of V. F $[1,2] \cap [2,3] = \{2\}$ doesn't have $\vec{0}$

(e) An $n \times n$ diagonal matrix can never have more than n nonzero entries. T

(f) The trace of a square matrix is the product of its diagonal entries. F

sum

(g) Let W be the xy-plane in $\mathbb{R}^3$; that is $\{(a_1, a_2, 0) : a_1, a_2 \in \mathbb{R}\}$. Then $W = \mathbb{R}^2$. F

$\mathbb{R}^2 = \{(a_1, a_2) : a_1, a_2 \in \mathbb{R}\} \neq W$

7. Exercise §1.3 #8 Determine whether the following sets are subspaces of $\mathbb{R}^3$ under the operations of addition and scalar multiplication defined on $\mathbb{R}^3$. Justify your answers.

$(3a_2, a_2, -a_2)$

(a) $W_1 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = 3a_2 \text{ and } a_3 = -a_2\}$

(b) $W_2 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 = a_3 + 2\}$ Let $(a_1, a_2, a_3), (b_1, b_2, b_3) \in W_i$

(c) $W_3 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 2a_1 - 7a_2 + a_3 = 0\}$

(d) $W_4 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 - 4a_2 - a_3 = 0\}$

(e) $W_5 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : a_1 + 2a_2 - 3a_3 = 1\}$

$\sqrt{6}$ $\sqrt{12}$ 1

(f) $W_6 = \{(a_1, a_2, a_3) \in \mathbb{R}^3 : 5a_1^2 - 3a_2^2 + 6a_3^2 = 0\}$

(a) Yes i. $\bar{0} \in W_1$

ii. $\because a_1 + b_1 = 3a_2 + 3b_2 = 3(a_2 + b_2)$

$a_3 + b_3 = -a_2 - b_2 = -(a_2 + b_2)$

$\therefore (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \in W_1$

iii. $\because ca_1 = c3a_2 = 3(ca_2)$, $ca_3 = -ca_2$

$\therefore c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_1$

(b) No $\because a_1 + b_1 = a_3 + 2 + b_3 + 2 = a_3 + b_3 + 4$

$\therefore (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \notin W_2$

(c) Yes i. $\bar{0} \in W_1$

ii. $\because 2(a_1+b_1) - 7(a_2+b_2) + (a_3+b_3) = (2a_1 - 7a_2 + a_3) + (2b_1 - 7b_2 + b_3) = 0$

$\therefore (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \in W_3$

iii. $\because 2ca_1 - 7ca_2 + ca_3 = c(2a_1 - 7a_2 + a_3) = 0$

$\therefore c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_3$

(d) Yes i. $\bar{0} \in W_1$

ii. $\because (a_1+b_1) - 4(a_2+b_2) - (a_3+b_3) = (a_1 - 4a_2 - a_3) + (b_1 - 4b_2 - b_3) = 0$

$\therefore (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \in W_4$

iii. $\because ca_1 - 4ca_2 - ca_3 = c(a_1 - 4a_2 - a_3) = 0$

$\therefore c(a_1, a_2, a_3) = (ca_1, ca_2, ca_3) \in W_4$

(e) No. $\because (a_1+b_1) + 2(a_2+b_2) - 3(a_3+b_3) = 2$

$\therefore (a_1, a_2, a_3) + (b_1, b_2, b_3) = (a_1+b_1, a_2+b_2, a_3+b_3) \notin W_5$

(f) No. $(\sqrt{12}, \sqrt{22}, 1), (\sqrt{6}, \sqrt{12}, 1) \in W_6$

$(\sqrt{12}, \sqrt{22}, 1) + (\sqrt{6}, \sqrt{12}, 1) = (\sqrt{12}+\sqrt{6}, \sqrt{22}+\sqrt{12}, 2) \notin W_6$

Problems not from the textbook exercises.

8. For what value(s) of c is the following system inconsistent?

$$\begin{array}{ccccccc} x_1 & + & cx_2 & & & = & 0 \\ x_1 & + & 2x_2 & - & x_3 & = & 0 \\ & & x_2 & + & x_3 & = & 2 \end{array} .$$

$$\begin{bmatrix} 1 & c & 0 & 0 \\ 1 & 2 & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1} \begin{bmatrix} 1 & c & 0 & 0 \\ 0 & 2-c & -1 & 0 \\ 0 & 1 & 1 & 2 \end{bmatrix}$$

$$\xrightarrow{R_3 \to R_3 + R_2} \begin{bmatrix} 1 & c & 0 & 0 \\ 0 & 2-c & -1 & 0 \\ 0 & 3-c & 0 & 2 \end{bmatrix}$$

$$3 - c = 0 \Rightarrow c = 3$$

9. Show that if matrices A and B have the same $RREF$, then A can be turned into B by a finite sequence of elementary row operations.
Hint: Show that any elementary row operation can be undone (inverted).

Let's see the elementary row operations:

① $R_i \leftrightarrow R_j$ $\qquad R_j \leftrightarrow R_i$ (inverted)

② $R_i \to cR_i \ (c \neq 0)$ $\qquad R_i \to \frac{1}{c} R_i \ (c \neq 0)$ (inverted)

③ $R_i \to R_i + cR_j$ $\qquad R_i \to R_i - cR_j$ (inverted).

So any elementary row operation is inverted.

Since A, B can be turned into RREF by a finite sequence of elementary row operations, then RREF can also be turned into A or B by a finite sequence of elementary row operations.

the steps number from A to RREF is S_A, and from RREF to B is S_B, so the steps number from A to $B \leq S_A + S_B$

$S_A + S_B$ is a finite number.

Hence the state is proved.