Homework 2 MATH 416: ABSTRACT LINEAR ALGEBRA

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(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. Exercise §1.3 #20 Prove that if W is a subspace of a vector space V and $w_1, w_2, ..., w_n$ are in W, then $a_1w_1 + a_2w_2 + ... + a_nw_n \in W$ for any scalars $a_1, a_2, ..., a_n$.

Since W is a subspace of a vector space V, and $w_1, w_2, \cdots, w_n$ are in W.

from (C). we know a_1w_1, a_2w_2 --- $a_nw_n \in W$

then from (b). we know $a_1w_1 + a_2w_2 + \cdots + a_nw_n \in W$

better use mathmatical induction.

2. §1.4 #1 Label the following statements as true or false (Answer is back, give a short justification!).

(a) The zero vector is a linear combination of any nonempty set of vectors. T

(b) The span of $\emptyset$ is $\emptyset$. F $\quad span\{\emptyset\} = \{\bar{0}\}$

(c) If S is a subset of a vector space V, then $span(S)$ equals the intersection of all subspaces of V that contains S. T

(d) In solving a system of linear equations, it is permissible to multiply an equation by any constant. F, except 0.

(e) In solving a system of linear equations, it is permissible to add any multiple of one equation to another. T

(f) Every system of linear equations has a solution.

F $\quad \begin{cases} x + y = 1 \\ 2x + 2y = 0 \end{cases}$ has no solution

3. §1.4 #3 (f) For the following vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two.

$$(-2, 2, 2), (1, 2, -1), (-3, -3, 3)$$

$$(-2, 2, 2) = a_1(1, 2, -1) + a_2(-3, -3, 3)$$

$$\begin{cases} a_1 - 3a_2 = -2 \\ 2a_1 - 3a_2 = 2 \\ -a_1 + 3a_2 = 2 \end{cases} \quad \begin{bmatrix} 1 & -3 & -2 \\ 2 & -3 & 2 \\ -1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_1 = 4 \\ x_2 = 2 \end{cases} \to \underline{\text{Yes}}$$

4. §1.4 #4 (f) For the following polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.

$$6x^3 - 3x^2 + x + 2, x^3 - x^2 + 2x + 3, 2x^3 - 3x + 1$$

$$[6, -3, 1, 2] = a_1[1, -1, 2, 3] + a_2[2, 0, -3, 1]$$

$$\begin{cases} a_1 + 2a_2 = 6 \\ -a_1 = -3 \\ 2a_1 - 3a_2 = 1 \\ 3a_1 + a_2 = 2 \end{cases} \quad \begin{bmatrix} 1 & 2 & 6 \\ -1 & 0 & -3 \\ 2 & -3 & 1 \\ 3 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 6 \\ 0 & 2 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

No solution

So $\underline{\text{NO}}$

5. §1.4 #10 Show if

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } M_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

then the span of $\{M_1, M_2, M_3\}$ is the set of all symmetric 2×2 matrices.

Yes.

① Obviously. any symmetric 2x2 matrices $\begin{bmatrix} a & b \\ b & c \end{bmatrix} =$ $aM_1 + cM_2 + bM_3$ is a linear combination of $\{M_1, M_2, M_3\}$

So the set of all symmetric 2x2 matrices $\subseteq$ span$\{M_1, M_2, M_3\}$

② Prove there is no X, such that $X \in$ span$\{M_1, M_2, M_3\}$, but $X \notin$ the set of all symmetric 2x2 matries.

Proof: Assume not: there is a linear combination

$a_1M_1 + a_2M_2 + a_3M_3$ is not symmetric.

i.e. $\begin{bmatrix} a_1 & a_3 \\ a_3, & a_2 \end{bmatrix}$ is not symmetric which can't be true

So span$\{M_1, M_2, M_3\}$ is the set of all symmetric 2x2 matrices.

6. Exercise §1.5 #1 Label the following statements as true or false (Answer is back, give a short explanation!).

(a) If S is a linearly dependent set, then each vector in S is a linear combination of other vectors in S.

F, $\{(1,0),(0,1),(2,0)\}$ is a linearly dependent set, but $(0,1)$ is not a linear combination of $(1,0)$ and $(2,0)$

(b) Any set containing the zero vector is linearly dependent.

T

(c) The empty set is linearly dependent.

F ~~$0\cdot\phi=\bar{0}$, $a\phi=\phi$~~ by definition $\emptyset$ is linearly independent.

(d) Subsets of linearly dependent sets are linearly dependent.

F $\{(1,0),(0,1),(2,0)\}$ is linearly dependent $\{(1,0),(0,1)\}$ is linearly independent.

(e) Subsets of linearly independent sets are linearly independent.

T

(f) If $a_1x_1+a_2x_2+...+a_nx_n=0$ and $x_1,x_2,...,x_n$ are linearly independent, then all scalars a_i are zero.

T

7. Exercise §1.5 #2 Determine whether the following sets are linearly dependent, or linearly independent.

a. §1.5 #2(c) $\{x^3+2x^2, -x^2+3x+1, x^3-x^2+2x-1\}$ in $P_3(\mathbb{R})$

b. §1.5 #2(d) $\{x^3-x, 2x^2+4, -2x^3+3x^2+2x+6\}$ in $P_3(\mathbb{R})$

c. §1.5 #2(e) $\{(1,-1,2),(1,-2,1),(1,1,4)\}$ in $\mathbb{R}^3$

a. $\begin{cases} a_1 + a_3 = 0 \\ 2a_1 - a_2 - a_3 = 0 \\ 3a_2 + 2a_3 = 0 \\ a_2 - a_3 = 0 \end{cases}$ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 2 & -1 & -1 & 0 \\ 0 & 3 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$ is the only solution, so linearly independent

Continued from problem 7. Use the following blank space to write your solutions.

b. $\begin{cases} a_1 \quad -2a_3=0 \\ 2a_2+3a_3=0 \\ -a_1 \quad +2a_3=0 \\ 4a_2+6a_3=0 \end{cases}$ $\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 4 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} a_1 = 2t \\ a_2 = -\frac{3}{2}t \\ a_3 = t \end{cases}$ $t \in \mathbb{R}$ so <u>linearly dependent</u>

c. $\begin{cases} a_1+a_2+a_3=0 \\ -a_1-2a_2+a_3=0 \\ 2a_1+a_2+4a_3=0 \end{cases}$ $\begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & -2 & 1 & 0 \\ 2 & 1 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{cases} a_1 = -3t \\ a_2 = 2t \\ a_3 = t \end{cases}$ $t \in \mathbb{R}$ so <u>linearly depondent</u>

Problems not from the textbook exercises.

8. Let $\mathcal{F}(\mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R}\}$ be the vector space of functions on $\mathbb{R}$.

a. Show that the subset $W = \{f \in \mathcal{F}(\mathbb{R}) | f(1) = 0\}$ is a subspace of $\mathcal{F}(\mathbb{R})$.

b. Show that the subset $W = \{f \in \mathcal{F}(\mathbb{R}) | f(0) = 1\}$ is not a subspace of $\mathcal{F}(\mathbb{R})$.

$\bar{0}_{\mathcal{F}(\mathbb{R})}(s) = 0 \quad \forall s \in \mathbb{R}$

$f, g \in W$

a. (a) $\bar{0}_{\mathcal{F}(\mathbb{R})}(1) = 0$, so $\bar{0}_{\mathcal{F}(\mathbb{R})} \in W$ (b) $f(1) + g(1) = 0 = (f+g)(1)$, so $(f+g) \in W$

(c) $c f(1) = c \cdot 0 = 0 = (cf)(1)$, so $(cf) \in W$

So W is a subspace of $\mathcal{F}(\mathbb{R})$

b. $\bar{0}_{\mathcal{F}(\mathbb{R})}(0) = 0$, so $\bar{0}_{\mathcal{F}(\mathbb{R})} \notin W$

$f(0) + g(0) = 2 = (f+g)(0)$, so $(f+g) \notin W$

$cf(0) = c = (cf)(0)$, so $(cf) \notin W$

$(c \neq 1)$

So W is not a subspace of $\mathcal{F}(\mathbb{R})$

9. Prove the following statement or give a counterexample.
"If u_1, u_2, u_3 are three vectors in $\mathbb{R}^3$ and none is a scalar multiple of another, then they are linearly independent."

$u_1 = (1, 0, 1)$, $u_2 = (0, 1, 0)$, $u_3 = (1, 1, 1)$

$10 u_1 + 10 u_2 - 10 u_3 = 0$

none is a scalar multiple of another but they are linearly dependent.