Fact performing an elementary row operation on $A \in M_{m\times n}$ can be described using matrix multiplication.

Def^n $I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$I_{2\times 2} \xrightarrow{\text{type I}: R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1$$

$$\xrightarrow{\text{type II}: R_2 \to \lambda R_2 \ \lambda \neq 0} \begin{bmatrix} 1 & 0 \\ 0 & \lambda \end{bmatrix} = E_2$$

$$\xrightarrow{\text{type III}: R_2 \to R_2 + \lambda R_1 \ \lambda \neq 0} \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} = E_3$$

Theorem Let E be the elementary matrix obtained from I_n by performing row operation R ($E = E(R)$).

For any $A \in M_{m\times n}$ the product $E(R)\cdot A$ is equal to the matrix obtained from A by performing R.

$$E_1 \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

$$E_2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \lambda c & \lambda d \end{bmatrix}$$

$$E_3 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c+\lambda a & d+\lambda b \end{bmatrix}$$

Cor Each elementary matrix is invertible.

$\mathcal{R} : R_i \leftrightarrow R_j \qquad \mathcal{R}^{-1} : R_j \leftrightarrow R_i.$

$\mathcal{R} : R_i \to \lambda R_i \qquad \mathcal{R}^{-1} : R_i \to \frac{1}{\lambda} R_i$

$\mathcal{R} : R_i \to R_i + \lambda R_j \quad \mathcal{R}^{-1} : R_i \to R_i - \lambda R_j.$

Proof: $E(\mathcal{R})E(\mathcal{R}^{-1}) = I_n$

Theorem For every $A \in M_{m\times n}$, there is a finite set of elementary matrices $E_1 \cdots E_k \in M_{m\times m}$ such that $E_k \cdot E_{k-1} \cdots E_1 A$ is in RREF.

Theorem $A \in M_{n\times n}$ is invertible <u>if and only if</u> there is a finite set of elementary matrices $E_1, \ldots, E_k \in M_{n\times n}$ such that $E_k \cdots E_1 A = I_n$

$A^{-1} = E_k \cdots E_1$ and $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$

Cor A is invertible if and only if it can be written as a product of elementary matrices.

Recall: rank of $T: V \to W$ is $\dim(R(T))$

Def$^{\underline{n}}$ The rank of $A \in M_{m\times n}$, $\operatorname{rank}(A)$, is the rank of $L_A: \mathbb{R}^n \to \mathbb{R}^m$ $\dim(R(L_A)) = \operatorname{rank}(A)$

Prop if $B \in M_{m\times m}$ is invertible, then $\operatorname{rank}(BA) = \operatorname{rank}(A)$

Proof: $\operatorname{rank}(BA) = \operatorname{rank}(L_{BA})$

$$= \dim(R(L_{BA}))$$

$$R(L_{BA}) = \{L_{BA}(v) \mid v \in \mathbb{R}^n\} \subset \mathbb{R}^m$$

$$= \{(BA)(v) \mid v \in \mathbb{R}^n\}$$

$$= \{B(A(v)) \mid v \in \mathbb{R}^n\}$$

$$= L_B(\underbrace{\{A(v) \mid v \in \mathbb{R}^n\}}_{R(L_A)})$$

$$L_B: \mathbb{R}^m \to \mathbb{R}^m$$

Define $\tilde{L}_B : R(L_A) \to L_B(R(L_A))$ $R(L_{BA})$

$\tilde{L}_B$ is invertible since (B is invertible. $\Rightarrow L_B$ is invertible)

So $\dim(R(L_A)) = \dim(R(L_{BA}))$

$$\text{rank}(A) = \text{rank}(BA)$$

Cor Elementary row operations don't change rank.

Cor $\text{rank}(A) = \text{rank}(\text{RREF}(A))$