Markov Chain.

(i) Set of states $S_1, \dots, S_n$.

<u>ex</u> people live in Chicago $\begin{cases} S_1 : \text{living in city.} \\ S_2 : \text{living in a subrub.} \end{cases}$

(ii) time step.

(iii) probability vector at time k.

$\mathbb{R}^n \ni \bar{p}_k = \begin{pmatrix} (\bar{p}_k)_1 \\ \vdots \\ (\bar{p}_k)_n \end{pmatrix}$ ← probability of being in state 1 at time k.

$(\bar{p}_k)_j \geq 0 \ , \quad \sum_{j=1}^{n} (\bar{p}_k)_j = 1$

(iv) Transition matrix $A \in M_{n \times n}$

$A = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & A_{ij} & \vdots \\ A_{n1} & \cdots & A_{nn} \end{bmatrix}$ A_{ij} = probability of moving from <u>state j to state i</u> in one time step.

$A_{ij} \geq 0 \ , \quad \sum_{i=1}^{n} A_{ij} = 1.$

$\bar{p}_0$

$\bar{p}_1 = A\bar{p}_0$

$\bar{p}_n = A^n \bar{p}_0$

$\lim_{n\to\infty} \bar{p}_n = (\lim_{n\to\infty} A^n)\, \bar{p}_0$

Theorem

Suppose for some $d \geq 1$, A^d has all positive entries.

Then, (a) 1 is an eigenvalue of A.

$\dim E_1 = 1$ and $E_1 = \mathrm{Span}\{\bar{u}\}$,

where $\bar{u}$ is a prob vector.

(b) For any other eigenvalue λ, $|\lambda| < 1$.

(c) $\lim_{n\to\infty} A^n = (\bar{u}, \bar{u}, \cdots \bar{u})$

$\lim_{n\to\infty} \bar{p}_n = (\bar{u}, \bar{u}, \cdots \bar{u})\, \bar{p}_0 = \bar{u}$ (for any $\bar{p}_0$)

Proof: (a) $\det(B) = \det(B^t) \Rightarrow \det(A - tI_n) = \det(A^t - tI_n)$

Consider A^t rows add to 1.

$$\mathrm{Sum}\begin{bmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & \ddots & & \vdots \\ \vdots & & \ddots & \\ a_{1n} & \cdots & & a_{nn} \end{bmatrix} = 1$$

Let $\bar{v} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

$A^t \bar{v} = (a_1^t \cdots a_n^t) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = a_1^t + \cdots + a_n^t = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \bar{v}$

So $\bar{v}$ is an eigenvector of A^t with eigenvalue 1.

So 1 is an eigenvalue of A.

(b) λ is an eigenvalue of A^t

$\Rightarrow A^t v = \lambda v$ and $v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} \neq \bar{0}$

Let $|v_k| = \max_j \{|v_j|\}$

$$\begin{cases} (A^t v)_k = \sum_j (A^t)_{kj} v_j \\ (A^t v)_k = (\lambda v)_k = \lambda v_k \end{cases}$$

$$|\lambda v_k| = |\sum_j (A^t)_{kj} v_j|$$

$$= |\sum_j A_{jk} v_j|$$

$$\leq \sum_j |A_{jk} v_j|$$

$$|\lambda v_k| \leq |v_k| \underbrace{\sum_j A_{jk}}_{\text{(Column sums of } A = 1)} = |v_k|$$

$\Rightarrow |\lambda| \leq 1.$

(a) (b)

- If $A_{ij} > 0 \ \forall i, j$, then $(\dim E_1 = 1)$ and $(\lambda \neq 1 \Rightarrow |\lambda| < 1)$

Proof: Suppose $|\lambda| = 1$ and $A^t v = \lambda v$ for $v \neq \bar{0}$

It suffices to show $v = c \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$

This implies $\lambda = 1$ and $\dim E_1 = 1$

$$|\lambda v_k| = |\sum_j ((A^t)_{kj} v_j)|$$

all $A_{jk} v_j$ have same sign. (no cancellation) $\overset{=}{\leq} \sum_j (A_{jk} |v_j|)$

$\overset{=}{\leq} |v_k| \sum_j A_{jk} = |v_k|$

the inequalities must be <u>equalities</u>.

- $A_{jk} v_j$ all have same sign

$|v_j| = |v_k|$ for all

Since $A_{jk} > 0 \ \forall j,k \Rightarrow v_j$ are all ≥ 0 or ≤ 0

- $|v_j| = |v_k|$ for $j = 1, \dots, n$

$$v = \begin{pmatrix} v_k \\ \vdots \\ v_k \end{pmatrix} = v_k \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

<u>Remark</u>: Since $\dim E_1 = 1$, we <u>can</u> write $E_1 = \mathrm{Span}\{\bar{u}\}$ where sum entries of $\bar{u}$ add to 1.

(c): Suppose A is a transition matrix such that $A_{ij} > 0$ for all i, j and A is <u>diagonalizable</u>. Then $\lim_{k \to \infty} A^k = [\bar{u}, \dots, \bar{u}]$. where $\bar{u}$ is prob vector and $A\bar{u} = \bar{u}$. (i.e. $\bar{u} \in E_1$).

Proof: from (a) (b), we know $\lambda = 1$ is an eigenvalue, $\dim E_1 = 1$, and any other λ has $|\lambda| < 1$.

$$A = QDQ^{-1} = Q\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & \vdots \\ \vdots & \vdots & \ddots & \\ 0 & \cdots & & \lambda_n \end{bmatrix} \quad |\lambda_j| < 1.$$

$$A^k = Q\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n^k \end{bmatrix} Q^{-1}$$

$$\lim_{k\to\infty} A^k = Q\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{bmatrix} Q^{-1} = L.$$

$$AL = \lim_{k\to\infty} A^{k+1} = L.$$

$L = [l_1, l_2, \cdots, l_n]$.

$[Al_1, Al_2, \cdots, Al_n] = [l_1, \cdots, l_n]$.

$Al_j = l_j$ for $j = 1, 2, \cdots, n$

So, $l_j \in E_1$. Since $\dim E_1 = 1$, we know

$l_j = c_j \bar{u}$ for some c_j. ($\bar{u}$ is the basis that the sum of its entries is 1)

Because the sum of all entries $l_j = 1$. , the $c_j = 1$. $j = 1, 2, \cdots, n$

Hence, $\lim_{k\to\infty} A^k = [\bar{u}, \cdots, \bar{u}]$.

$T(cv) = cT(v) = c(\lambda v) = (c\lambda)v$.

Def$^{\underline{n}}$ $T: V \to V$
a subspace $W \subset V$ is T-invariant if
$T(W) \subset W$. i.e. $T(w) \in W$, $\forall w \in W$
$T_W: W \to W$.

Theorem Suppose $T: V \to V$ is linear and $\dim V < \infty$, if W is T-invariant then the char. poly. of T_W divides char. poly. of T

Proof: Let $\beta_W = \{v_1, \ldots, v_k\}$ be a basis of W. Extend this to a basis for V. $\beta = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\}$

$$[T]_\beta^\beta = \left([T(v_1)]_\beta \cdots\cdots [T(v_n)]_\beta\right)$$

$$= \begin{array}{c} \\ k \end{array}\!\!\begin{array}{c} k \\ \left(\begin{array}{c|c} B_1 & B_2 \\ \hline 0 & B_3 \end{array}\right) \end{array}$$

$$B_1 = [T_W]_{\beta_W}^{\beta_W}$$

$$\det([T]_\beta^\beta - tI_n) = \det \left(\begin{array}{c|c} [T_W]_{\beta_W}^{\beta_W} - tI_k & B_2 \\ \hline 0 & B_3 - tI_{n-k} \end{array}\right)$$

$$= \det([T_W]_{\beta_W}^{\beta_W} - tI_k)\, g(t)$$

T-cyclic subspace

$T: V \to V$

Def^n For $v \in V$, the T-cyclic subspace generated by v.

is $W = \operatorname{Span}\{v, T(v), T^2(v) \ldots\} \subset V$

Observe that W is T-invariant

$T(a_0 v + a_1 T(v) + \cdots + a_k T^k(v)) \in W$

$= a_0 T(v) + a_1 T^2(v) + \cdots a_k T^{k+1}(v) \in W$

Theorem Suppose $T: V \to V$ is linear and $\dim V < \infty$

Let W be a T-cyclic subspace generated by v.

Set $\dim W = k \leq \dim V$. Then

(if $T^{x+1}(v) = \sum_{i=0}^{x} a_i T^i(v) + c_i$, $T^y(v) = \sum_{i=0}^{x} a_i T^i(v) + c_i$ $(y \geq x)$.) $\Rightarrow \dim \leq x$.

(a) $\{v, T(v), \cdots T^{k-1}(v)\}$ is a basis of W.

(b) If $T^k(v) = a_0 v + \cdots + a_{k-1} T^{k-1}(v)$, then the char. poly. of T_W is $(-1)^{k+1}(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} - t^k)$

Proof: Given (a) prove (b):

$$[T_W]_\beta^\beta = [T_W(v)_\beta \quad T_W(T(v))_\beta \quad \cdots \quad T_W(T^{k-1}(v))_\beta]$$

$$= \begin{bmatrix} 0 & 0 & \cdots & 0 & a_0 \\ 1 & 0 & \cdots & 0 & a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & 0 & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1} \end{bmatrix}$$

$$\det([T_W]_\beta^\beta - t I_{t-1}) = \begin{vmatrix} -t & 0 & \cdots & 0 & a_0 \\ 1 & -t & \cdots & 0 & a_1 \\ \vdots & 1 & \ddots & & \vdots \\ \vdots & \vdots & & -t & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1}-t \end{vmatrix}$$

$$= \begin{vmatrix} -t & 0 & \cdots & 0 & a_0 \\ 0 & -t & \cdots & 0 & a_1 + \frac{1}{t}a_0 \\ \vdots & 1 & \ddots & & \vdots \\ \vdots & \vdots & & -t & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1}-t \end{vmatrix} = \begin{vmatrix} -t & 0 & \cdots & 0 & a_0 \\ 0 & -t & \cdots & 0 & a_1 + \frac{1}{t}a_0 \\ 0 & 0 & -t & & a_2 + \frac{1}{t}a_1 + \frac{1}{t^2}a_0 \\ \vdots & \vdots & & -t & \vdots \\ 0 & 0 & \cdots & 1 & a_{k-1}-t \end{vmatrix}$$

$$= \cdots\cdots = (-t)^{k-1}\left(\frac{1}{t^{k-1}}a_0 + \frac{1}{t^{k-2}}a_1 + \cdots + a_{k-1} - t\right)$$

$$= (-1)^{k-1}\left(a_0 + t a_1 + \cdots t^{k-1} a_{k-1} - t^k\right)$$