Math 416 Practice Final

1. Solve

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & & & x_4 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 1 \\ x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & -1 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 3. \end{array}$$

2. Consider the linear map $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = \begin{pmatrix} a & c \\ b-a & a \end{pmatrix}.$$

Let $\alpha = \{1 + x, x + x^2, 1 + x + x^2\}$ and $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the bases for $P_2(\mathbb{R}), M_{2\times 2}(\mathbb{R})$, respectively, where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) Compute the coordinate vector $[f]_\alpha$ for $f(x) = 1 - x + 2x^2$.

(2) Compute the matrix representation $[T]_\alpha^\beta$ with respect to α, β.

(3) Is T injective?

3. Consider a 3×4 matrix $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ -2 & -2 & 1 & 1 \end{pmatrix}$. Let $T = L_A : \mathbb{R}^4 \to \mathbb{R}^3$.

(1) Find a basis β of the null space $N(T)$ so that $N(T) = \text{span}(\beta)$.

(2) Find a basis γ of the range $R(T)$ so that $R(T) = \text{span}(\gamma)$.

(3) Verify the dimension theorem (the rank-nullity theorem).

4.

(1) Compute the determinant of $\begin{pmatrix} 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 1 \end{pmatrix}$ and verify that $\det(A) \neq 0$.

(2) Compute the inverse of A.

5. **True / False**. Justify your answer.

(a) Every square matrix over the complex numbers $\mathbb{C}$ has a Jordan Form.

(b) Every nonzero finite dimensional inner product space has an orthonormal basis.

(c) If S is an orthogonal set of nonzero vectors, then S is linearly independent.

(d) A linear operator T on a finite dimensional inner product space V over $\mathbb{R}$ is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(e) Every symmetric matrix over $\mathbb{R}$ is diagonalizable.

(f) Every symmetric matrix over $\mathbb{C}$ is diagonalizable.

(g) If a linear operator T on a finite dimensional inner product space V over $\mathbb{R}$ is normal, then V admits an orthonormal basis which consists of the eigenvectors of T.

(h) If a linear operator T on a finite dimensional inner product space V is self-adjoint, then it is normal.

(i) If T is a linear operator on a finite dimensional vector space V whose characteristic polynomial splits, then the dimension of the generalized eigenspace K_λ corresponding to an eigenvalue λ is equal to the algebraic multiplicity of λ.

6. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}$.

(1) Find all the eigenvalues of A. 2, 1.

(2) Show that A is diagonalizable. Splits

for λ_1, λ_2 $\dim E_{\lambda_i} =$ alg. mult. of λ_i.

(3) Find a diagonal matrix D and an invertible matrix Q such that $Q^{-1}AQ = D$.

(4) Find the matrix power A^k for $k \in \mathbb{N}$.

7. For a transition matrix $A = \begin{pmatrix} 0.2 & 0.8 \\ 0.8 & 0.2 \end{pmatrix}$, explain why the limit $\lim_{k\to\infty} A^k$ exists, and compute it. If you use Theorems from the text, explicitly state the statements.

A only has positive entries. (A is regular).

$\int_{-1}^{1} \frac{1}{2\sqrt{2}} x^2 \, dx$

8. Consider a subspace $W = \text{span}\left(\{1, \frac{1}{2}x^2\}\right) \subset P_2(\mathbb{R})$ equipped with the inner product

$\frac{1}{6\sqrt{2}} x^3 \Big|_{-1}^{1}$

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)dx.$$

$u_1 = \frac{1}{\sqrt{\int_{-1}^{1} 1 \times 1 \, dx}} = \frac{1}{\sqrt{2}}$

(1) Show that $\{1, \frac{1}{2}x^2\}$ is linearly independent. $\frac{1}{2}x^2 - \langle \frac{1}{2}x^2, \frac{1}{\sqrt{2}} \rangle \frac{1}{\sqrt{2}} = \frac{1}{2}x^2 - \frac{1}{6}$

(2) Using the Gram-Schmidt process to find an *orthonormal* basis for W.

(3) Show that $h(x) = x \notin W$, and compute the orthogonal projection of h onto W.

$\langle x, \frac{1}{\sqrt{2}} \rangle = 0$ $\quad \langle x, \frac{\sqrt{65}}{13}(3x^2-1) \rangle = 0$ $\quad$ min $\langle x - a - bx^2, x - a - bx^2 \rangle$

(4) Is $h(x) = x$ in the orthogonal complement $W^\perp$ of W? Explain.

Yes $\forall y \in W$ $\langle x, y \rangle = 0 \Rightarrow x \in W^\perp$ $a = 0$ $b = 0 \Rightarrow 0.$

9. Let V be the vector space $M_{2\times 2}(\mathbb{R})$ of all 2×2 matrices over $\mathbb{R}$ equipped with an inner product $\langle A, B \rangle = \text{tr}(B^t A)$ (Frobenius inner product). Consider the linear operator T on the inner product space defined by

$$T(A) = \frac{1}{2}(A^t + A).$$

Let $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be a basis for $M_{2\times 2}(\mathbb{R})$, where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad T(A) = \frac{1}{2}\begin{bmatrix} 2a & b+c \\ b+c & 2d \end{bmatrix} = \begin{bmatrix} a & \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) & d \end{bmatrix}$

$\begin{bmatrix} a \\ \frac{1}{2}(b+c) \\ \frac{1}{2}(b+c) \\ d \end{bmatrix}$ $\quad \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$

(1) Find the matrix representation $[T]_\beta$ of T.

(2) Find the adjoint operator T^* of T.

(3) Is T self-adjoint? Is T diagonalizable?

10. Consider the 4×4 matrix

$$A = \begin{pmatrix} 4 & -2 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}.$$

(1) Compute the eigenvalues of A. Does the characteristic polynomial of A split?

(2) Determine if A is diagonalizable.

(3) Find a Jordan basis β and its corresponding Jordan form J so that $Q^{-1}AQ = J$, where Q is an invertible matrix whose columns consist of the Jordan basis.

$\det(A - tI) = \det\begin{bmatrix} 4-t & -2 & 0 & 1 \\ 0 & 2-t & 0 & 2 \end{bmatrix} = (4-t)^2(1-t)(2-t)$

$$\begin{bmatrix} 0 & -1 & 1-t & 1 \\ 0 & 0 & 0 & 4-t \end{bmatrix}$$

$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_2 = 2.$

$m_1 = 2 \quad m_2 = 1 \quad m_3 = 1.$

$$A - 4I = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 0 & -2 & 0 & 2 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$(A-4I)^2 = \begin{bmatrix} 0 & 4 & 0 & -4 \\ 0 & 4 & 0 & -4 \\ 0 & 5 & 9 & -5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & -1 \\ 0 & 0 & 9 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$[A]_\alpha^\alpha = [I]_\beta^\alpha [A]_\beta^\beta$$

$$[I]_\alpha^\beta$$

$$\beta = \left\{ \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

$$\begin{bmatrix} 4 & 1 & & \\ & 4 & & \\ & & 1 & \\ & & & 2 \end{bmatrix} = \qquad [I]_\beta^\alpha$$