Determinant is a map.

$$\det : M_{n\times n} \to \mathbb{R}$$
$$A \to \det(A)$$

Properties:

① A is invertible if and only if $\det(A) \neq 0$.

② $\det(A)$ has geometric meaning.

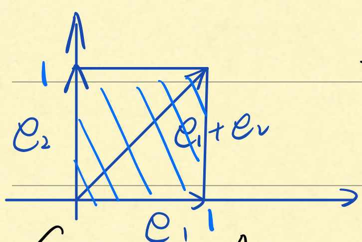
无数个点的集合

Let $[0,1]^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n \mid x_i \in [0,1]\}$

$=$ cube determined by $\{e_1, \cdots, e_n\}$.

$e_1 = (1, 0, \cdots, 0)$

$e_n = (0, \cdots, 1)$.

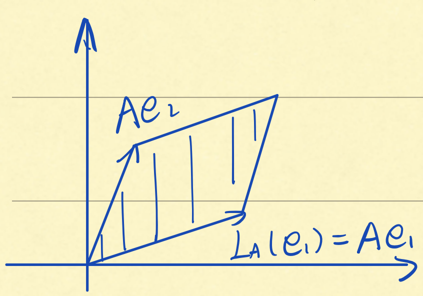


(when $n=2$)

Given $A \in M_{n\times n}$ consider

$L_A([0,1]^n) =$ parallelepiped determined by $\{Ae_1, \cdots, Ae_n\}$

$\text{volume}(L_A([0,1]^n)) = |\det(A)|$.



(when $n=2$)

③ det is not linear except for $n=1$

(It is linear in the rows of A).

④ $\det AB = \det A \det B$

$\det : M_{n\times n} \to \mathbb{R}$ is inductive on n.

Define $\tilde{A}_{ij}$ is the $(n-1)\times(n-1)$ matrix obtained from A by deleting its i^{th} row and j^{th} column.

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1j} \det(\tilde{A}_{1j})$$

Theorem $\det(A)$ is linear in rows of A.
Suppose A, B and C in $M_{n\times n}$ are equal in all rows but the r^{th}. For the r^{th} row suppose that $a_r = b_r + k c_r$

$$A = \begin{bmatrix} 1+k\pi & 1+k\pi \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} \pi & \pi \\ 1 & 1 \end{bmatrix}$$

$\det(A) = \det(B) + k\det(C)$ Theorem 1.

Theorem 2. : $\det(A) = \sum_{j=1}^{n} (-1)^{r+j} A_{rj} \det(\tilde{A}_{rj})$

Cor 1: if A has a row of all zeros then $\det A = 0$

Cor 2: if A has two identical rows, then $\det A = 0$.

Theorem:

(i) $A \xrightarrow{R_i \leftrightarrow R_j} B$: $\det B = -\det A$

(ii) $A \xrightarrow{R_i \leftrightarrow cR_i} B$: $\det B = c \det A$

(iii) $A \xrightarrow{R_i \leftrightarrow R_i + cR_j} B$: $\det B = \det A$.

Upper and lower triangular matrices.

Def$^{\underline{n}}$ $A \in M_{n \times n}$ is upper (lower) triangular if all entries below (above) diagonal are zero.

$\begin{bmatrix} a & d \\ 0 & b \end{bmatrix}$ upper triangular $\begin{bmatrix} a & 0 \\ d & b \end{bmatrix}$ lower triangular.

Theorem: if $A \in M_{n\times n}$ is upper or lower triangular then $\det(A) = A_{11} A_{22} \cdots A_{nn}$

$\det(A) \neq 0 \iff \det(\text{REF of } A) \neq 0$

$\iff$ REF of A has n leading entries

$\iff \operatorname{rank}(A) = n$

$\iff A$ is invertible.

$\det(AB) = \det(A)\det(B)$

$A \xrightarrow{R_k, \cdots, R_2, R_1} B$ REF

$B_{11} B_{22} \cdots B_{nn} = \det B = \varepsilon(R_k) \cdots \varepsilon(R_1) \det(A)$

where $\varepsilon(R) = \begin{cases} -1, & R \text{ type I} \\ c, & R \text{ type II} \\ +1, & R \text{ type III.} \end{cases}$

$\Longrightarrow \det A = \dfrac{\det B}{\varepsilon(R_k) \cdots \varepsilon(R_1)}$

Theorem A, B invertible $\Longleftrightarrow$ AB invertible

Proof: A, B invertible $\Rightarrow (AB)^{-1} = B^{-1}A^{-1}$

AB invertible $\Rightarrow L_{AB} = L_A \cdot L_B$ is invertible

$L_{AB}(x) = L_A(L_B(x))$ $\Rightarrow L_B$ is 1-1, L_A is onto

$L_A: \mathbb{R}^n \to \mathbb{R}^n$, $L_B: \mathbb{R}^n \to \mathbb{R}^n$ $\Rightarrow L_A, L_B$ both are onto and 1-1.

$\Rightarrow L_B, L_A$ both invertible

$\Rightarrow B, A$ are invertible.

Cramer's Rule. $A\bar{x} = \bar{b} \Rightarrow \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \bar{x} = A^{-1}\bar{b}$.

For each $k = 1, \dots, n$

$x_k = \dfrac{\det(M_k)}{\det(A)}$, where M_k is obtained from A by replacing k^{th} column by $\bar{b}$.

Proof: $A = [a_1, \dots, a_k \dots, a_n]$

$M_k = [a_1, \dots \bar{b}_k \dots a_n]$

$I_n = [e_1, \dots e_k \dots e_n]$

$X_k = [e_1, \dots \bar{x} \dots e_n]$

$AX_k = [Ae_1 \dots A\bar{x} \dots Ae_n]$

$$= [a_1 \cdots \vec{b} \cdots a_n]$$

$$= M_k$$

$$x_k = \det X_k = \frac{\det M_k}{\det A}$$

Theorem Suppose $F : M_{n\times n} \to \mathbb{R}$ such that

(1) F is linear in rows.

(2) $F(A) = 0$ if A has two identical rows

(3) $F(I_n) = 1$

Then $F = \det$.