$$\begin{bmatrix} 1 & 1 & 1 & 3 & 6 \\ 1 & -1 & 1 & -1 & -2 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 0 & 2 & 4 \\ 2 & 0 & 2 & 0 & 0 \\ 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Math 416 Practice Exam 1

1. Solve the system of linear equations.

$$\begin{array}{ccccccccc} x_1 & + & x_2 & + & x_3 & + & 3x_4 & = & 6 \\ x_1 & - & x_2 & + & x_3 & - & x_4 & = & -2 \\ x_1 & + & x_2 & + & x_3 & + & x_4 & = & 2 \\ & & x_2 & + & & & x_4 & = & 2. \end{array}$$

$$\begin{cases} x_1 = t_1 \\ x_2 = 0 \\ x_3 = -t_1 \\ x_4 = 2 \end{cases}$$

2. Prove that $W = \{A \in M_{2\times 2}(\mathbb{F}) | \mathrm{tr}(A) = 0\}$ is a subspace of $M_{2\times 2}(\mathbb{F})$ the vector space of all 2×2 matrices over $\mathbb{F}$.

(a) $\bar{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in W$

(b) $\forall\ x_1 = \begin{bmatrix} a_1 & b_1 \\ c_1 & -a_1 \end{bmatrix}, x_2 = \begin{bmatrix} a_2 & b_2 \\ c_2 & -a_2 \end{bmatrix} \in W$ $x_1 + x_2 = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & -a_1-a_2 \end{bmatrix} \in W$

(c) $k x_1 = \begin{bmatrix} ka_1 & kb_1 \\ kc_1 & -ka_1 \end{bmatrix} \in W$

3. Consider $\beta = \{1, 1-2x, 1+x^2\} \subset P_2(\mathbb{R})$. Here $P_2(\mathbb{R})$ is the vector space of polynomials of degree ≤ 2 over $\mathbb{R}$.
(1) Prove that β is linearly independent.
(2) Prove that β is a basis for $P_2(\mathbb{R})$ so that $\dim(P_2(\mathbb{R})) = 3$.

(1) $a_1 \cdot 1 + a_2(1-2x) + a_3(1+x^2) = \bar{0}$
$\Rightarrow a_1 = a_2 = a_3 \Rightarrow$ Linearly independent.

(2). any $y = ax^2 + bx + c \in P_2(\mathbb{R})$
$y = a(1+x^2) - \frac{b}{2}(1-2x) + (\frac{b}{2} - a + c) \cdot 1$
$\in$ span($\mathbb{R}$).
Thus span($\mathbb{R}$) $= P_2(\mathbb{R})$

4. Consider ~~a linear~~ transformation $T : V \to W$.
(1) Prove that $N(T)$ (the null space) is a subspace of V.
(2) Prove that $R(T)$ (the range) is a subspace of W.
(3) If $A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & 2 & 4 \end{pmatrix}$, then find a basis for $N(L_A)$.
Remember $L_A : \mathbb{F}^3 \to \mathbb{F}^3$ is the linear transformation defined by $L(v) = Av$ for every $v \in \mathbb{F}^3$.

$T(0) = 0$. $T(X) = 0$ $T(y) = 0 \Rightarrow T(x+y) = T(x) + T(y) = 0$
$T(cx) = cT(x) = 0$

$T(0) = 0 \in R(T)$ $T(x) \in R(T)$
$T(x) + T(y) = T(x+y) \in R(T)$
$c(T(x)) = T(cx) \in R(T)$

5. Consider the linear map $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_2 + (a_1 - a_0)x + a_3x^2.$$

Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, 1+x, 1-x^2\}$ be bases of $P_3(\mathbb{R})$ and $P_2(\mathbb{R})$, respectively. Consider the polynomial $f(x) = 1 + x + x^3$.
(1) Compute the coordinate vector $[f]_\alpha$. $[1\ 1\ 0\ 1]$
(2) Find the matrix representation $[T]_\alpha^\beta$ of T with respect to α, β.
(3) Find $[T(f)]_\beta$, and verify that $[T(f)]_\beta = [T]_\alpha^\beta [f]_\alpha$.

6. **True / False**. Justify your answer.

(a) If V and W are subspaces of a vector space, then $V \cup W$ is also a subspace. F

(b) A system of linear equations can have only two solutions. F

(c) The set $\{f \in C^0(I,\mathbb{R}) | f(1/2) = 0\}$ is a subspace of the vector space $C^0(I,\mathbb{R})$ of all continuous functions from $I = [0,1]$ to $\mathbb{R}$. T

(d) If S is a linearly independent subset of a vector space, then the zero vector is **not** contained S. T

(e) If $A \in M_{m\times n}(\mathbb{F})$ and $B \in M_{n\times p}(\mathbb{F})$, then $L_A \circ L_B = L_{AB}$. T

(f) If $A \in M_{m\times n}(\mathbb{F})$, then $[L_A]_\alpha^\beta = A$ for any ordered bases α of $\mathbb{F}^n$ and β of $\mathbb{F}^m$. F

$m=2. \quad n=2.$

$(L_A \cdot L_B)(v) = L_A(L_B \cdot v) = ABv$
$= L_{AB}\, v.$

$\alpha = (1,0), (0,1)$

$\beta = (0,1), (1,0)$

$A = I_2 \qquad [L_A]_\alpha^\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq I_2$