Math 416
Summer 2020
Exam 2, Ver 1 (AM)

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Time Limit: 90 Minutes

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may *not* use your books, notes, or any calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **If you use a "fundamental theorem" you must indicate this** and explain why the theorem may be applied.
- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	12	
3	18	
4	15	
5	20	
6	20	
Total:	100	

Do not write in the table to the right.

1. (15 points) Find the Jordan normal form of the following matrix A, and use that to find a formula for A^k

$$\begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}$$

$\alpha = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$. $\quad [A]_\alpha^\alpha = \begin{bmatrix} -5 & 8 \\ -4 & 7 \end{bmatrix}$.

$\det(A - tI_2) = \begin{vmatrix} -5-t & 8 \\ -4 & 7-t \end{vmatrix} = (t-7)(t+5)+32$
$= (t-3)(t+1)$

$\lambda_1 = 3 \quad \lambda_2 = -1$.

$E_{\lambda_1} = \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\} \quad E_{\lambda_2} = \operatorname{span}\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

$\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

$Q = [I_2]_\alpha^\beta = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix} \quad Q^{-1} = [I_2]_\beta^\alpha = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$

$A = Q\,[A]_\beta^\beta\,Q = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$A^k = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -3^k + 2(-1)^k & 2 \cdot 3^k + 2(-1)^{k+1} \\ -3^k + (-1)^k & 2 \cdot 3^k + (-1)^{k+1} \end{bmatrix}$

2. (12 points) Let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ and assume that $det(A) = 5$. Compute the following

(a) (2 points) $det\,(3A)$

$det(3A) = 15$

(b) (2 points) $det\,(2A^{-1})$

$det(2A^{-1}) = \frac{2}{5}$

(c) (2 points) $det\,(2A^2)$

$det(2A^2) = 2\,det A \cdot det A = 50$

(d) (3 points) $det\,\left(3(A^T)^{-1}\right)$

$det(3(A^T)^{-1}) = 3\,det((A^T)^{-1}) = 3\,\frac{1}{det(A^T)}$

$= 3 \cdot \frac{1}{det A} = \frac{3}{5}$

(e) (3 points) $det\left(\begin{pmatrix} a & g & d \\ b & h & e \\ g & i & f \end{pmatrix}\right)$

$det\left(\begin{bmatrix} a & g & d \\ b & h & e \\ g & i & f \end{bmatrix}\right) = det\begin{bmatrix} a & b & g \\ g & h & i \\ d & e & f \end{bmatrix} = -\,det\begin{bmatrix} a & b & g \\ d & e & f \\ g & h & i \end{bmatrix}$

$= -\left(det A + det\begin{pmatrix} 0 & 0 & g-c \\ d & e & f \\ g & h & i \end{pmatrix}\right) = -5 - (g-c)(dh-ge)$

3. (18 points) Let $A = \begin{pmatrix} 1 & 3 \\ -2 & 8 \end{pmatrix}$.

   Write A as a product of elementary matrices.

$$A = \begin{bmatrix} 1 & 3 \\ -2 & 8 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 \\ 0 & 14 \end{bmatrix} \xrightarrow{R_2 \to \frac{1}{14}R_2} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1 \to R_1 - 3R_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

$$I_2 \xrightarrow{R_1 \to R_1 + 3R_2} \xrightarrow{R_2 \to 14R_2} \xrightarrow{R_2 \to R_2 - 2R_1} A$$

$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 14 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}$$

4. (15 points) Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \\ -3 & -3 & 1 \end{pmatrix}.$$

$$\det(A - tI_3) = \begin{vmatrix} 1-t & 1 & -1 \\ -3 & -3-t & 1 \\ -3 & -3 & 1-t \end{vmatrix} = t(1-t)(2+t)$$

$\lambda_1 = 0 \quad \lambda_2 = 1 \quad \lambda_3 = -2$

$$A - 0I_3 = \begin{bmatrix} 1 & 1 & -1 \\ -3 & -3 & 1 \\ -3 & -3 & 1 \end{bmatrix} \Rightarrow E_{\lambda_1} = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

$$A - I_3 = \begin{bmatrix} 0 & 1 & -1 \\ -3 & -4 & 1 \\ -3 & -3 & 0 \end{bmatrix} \Rightarrow E_{\lambda_2} = \text{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$$A + 2I_3 = \begin{bmatrix} 3 & 1 & -1 \\ -3 & -1 & 1 \\ -3 & -3 & 3 \end{bmatrix} \Rightarrow E_{\lambda_3} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

$\lambda_1 = 0$ eigenvectors: $a \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $a \neq 0$.

$\lambda_2 = 1$ eigenvectors: $b \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ $b \neq 0$.

$\lambda_3 = -2$ eigenvectors: $c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ $c \neq 0$.

$a, b, c \in \mathbb{R}$.

5. (20 points) Let A be an $n \times n$ matrix with entries in $\mathbb{R}$.

(a) (4 points) Define what it means for λ to be an eigenvalue of A.

$\lambda \in \mathbb{R}$. such that exist vector (eigenvector) v, satisfies $Av = \lambda v$. $\left((L_A - \lambda I_n)(v) = \bar{0} \right)$.

(b) (4 points) Define what it means for E_λ to be an eigenspace.

the vector space of all vectors v such that $Av = \lambda v$.

(the null space of $L_A - \lambda I$)

$E_\lambda = N(L_A - \lambda I)$

(c) (6 points) Prove that E_λ is a subspace of $\mathbb{R}^n$ and it is L_A-invariant.

$\forall v_1, v_2 \in E_\lambda$

① $(A - \lambda I)\bar{0} = \bar{0} \Rightarrow \bar{0} \in E_\lambda$.

② $(A - \lambda I)(v_1 + v_2) = (A - \lambda I)v_1 + (A - \lambda I)v_2 = \bar{0}$ $\Rightarrow$ E_λ is a subspace of $\mathbb{R}^n$

③ $(A - \lambda I)(cv_1) = c(A - \lambda I)v_1 = \bar{0}$

$L_A(v_1) = \lambda v_1 \in E_\lambda \Rightarrow E_\lambda$ is L_A-invariant.

(d) (6 points) Prove that if λ and μ are eigenvalues and $\lambda \neq \mu$, then the intersection of corresponding eigenspaces is $\{0\}$, i.e., $E_\lambda \cap E_\mu = \{0\}$

$v_3 \in E_\lambda \wedge E_\mu$

$L_A(v_3) = \lambda v_3 = \mu v_3$

$\Rightarrow \bar{0} = (\lambda - \mu) v_3$

Since $\lambda \neq \mu$ we know $\lambda - \mu \neq 0$.

so the $v_3 = \bar{0}$ i.e. $E_\lambda \cap E_\mu = \{\bar{0}\}$.

6. (20 points) **TRUE** / **FALSE**. Give a short justification.

(a) If A is an $n \times n$ matrix and A^3 is *not* invertible, then A is *not* invertible.

True. assume A is invertible, i.e. exists A^{-1}, such that $AA^{-1} = A^{-1}A = I$.

$A^3 = A \cdot A \cdot A$ $\quad A^3 \cdot (A^{-1} \cdot A^{-1} \cdot A^{-1}) = I = (A^{-1} \cdot A^{-1} \cdot A^{-1}) \cdot A^3$

$\Rightarrow A^{-1} \cdot A^{-1} \cdot A^{-1}$ is the invert matrix of A^3 which contradicts to A^3 is not invertible

(b) If A is a square matrix, then A and A^t have the same eigenvalues.

True. $\det(A^t - tI_n) = \det\left((A - tI_n)^t\right)$

$= \det(A - tI_n)$

$\Rightarrow A, A^t$ have same eigenvalues.

(c) If an $n \times n$ matrix is diagonalizable, then there are n distinct eigenvalues.

False $n=2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is diagonalizable

but only have 1 eigenvalue.

(d) If T is a linear transformation and λ is an eigenvalue of T, then λ^n is an eigenvalue of T^n for every positive integer n.

True exists β_1 $[T]_\alpha^\alpha = [I]_{\beta_1}^\alpha \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} [I]_\alpha^{\beta_1}$

$[T^n]_\alpha^\alpha = [I]_{\beta_1}^\alpha \begin{bmatrix} \lambda_1^n & & \\ & \lambda_2^n & \\ & & \ddots & \\ & & & \lambda_n^n \end{bmatrix} [I]_\alpha^{\beta_1}$ exists $\beta_2 = \beta_1$ $[T^n]_{\beta_2}^{\beta_2} = \begin{bmatrix} \lambda_1^n & & \\ & \ddots & \\ & & \lambda_i^n \end{bmatrix}$ i.e. λ^n is an eigenvalue of T^n

(e) The vector space $\mathbb{R}^n$ ($n \geq 1$) has infinitely many inner products.

True $\forall x = (a_1, a_2, \cdots, a_n), y = (b_1, b_2, \cdots, b_n) \in \mathbb{R}^n$

Let $\langle x, y \rangle = k \sum_{i=1}^{n} a_i b_i$

$k \in \mathbb{R}^+$ since the number of k can be infinite

the vector space $\mathbb{R}^n$ has infinitely many inner products.