1. elementary matrix. $E(R)$

$E(R) \cdot A$ is equal to A performs R.

$$\begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \lambda a + c & \lambda b + d \end{bmatrix}.$$

2. Rank of $L_A : \mathbb{R}^n \to \mathbb{R}^m$ is the rank of $A \in M_{m \times n}$: $\dim(R(L_A)) = \operatorname{ran}(A)$.

① B is invertible $\Longrightarrow$ $\operatorname{rank}(BA) = \operatorname{rank}(A)$

②. $\operatorname{rank}(A) = \operatorname{rank}(RREF(A))$.

3. A, B invertible $\Longleftrightarrow$ AB invertible.

4. Cramer's Rule $\quad x_k = \dfrac{\det(M_k)}{\det(A)}$.

5. $T: V \to V$ is diagonalizable if there is a basis $\beta = \{v_1, \ldots, v_n\}$ of V, s.t. $[T]_\beta^\beta = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}$

T is diagonalizable $\Longleftrightarrow$ $\exists$ a basis β consisting of enigenvectors.

3. L_A diagonalizable (A is diagonalizable)

$\Longleftrightarrow$ $\exists\, Q$, $Q^{-1}AQ$ is diagonal.

$$[L_A]_\beta^\beta = [I]_\alpha^\beta [L_A]_\alpha^\alpha [I]_\beta^\alpha$$

4. eigenvalue of A are the (real) roots of char. poly. of A: $\det(A - tI_n) = 0$.

5. $\begin{cases} \text{algebraic multiplicity. } (\lambda - t) \text{ 次数} \\ \text{geometric multiplicity } (\dim(E_\lambda)) \end{cases}$ $\geq$

A is diagonalizable $\iff$ i. char. poly. of A splits over $\mathbb{R}$.

ii. geom. mult = alg. mult.

A: transition matrix.

6. 1 is an eigenvalue of A.

If $A_{ij} > 0$, $\forall i, j$, then $\dim E_1 = 1$

$\lambda \neq 1 \Rightarrow |\lambda| < 1$.

A is diagonalizable, then $\lim_{k\to\infty} A^k = [\bar{u}, \dots, \bar{u}]$ where $\bar{u}$ is prob. vector and $A\bar{u} = \bar{u}$. (i.e. $\bar{u} \in E_1$).

$T: V \to V$

7. W is T-invariant if $T(W) \subset W$

T is linear and $\dim T < \infty$: if W is T-invariant then the char. poly. of T_W divides char. poly. of T.

8. for $v \in V$, $W = \mathrm{Span}\{v, T(v), T^2(v), \ldots\ldots\} < V$. is the T-cyclic subspace generated by v.

W is T-invariant.

T is linear and $\dim T < \infty$, set $\dim W = k \leq \dim V$.

(a) $\{v, T(v), \ldots, T^{k-1}(v)\}$ is a basis of W.

(b) if $T^k(v) = a_0 v + \cdots + a_{k-1} T^{k-1}(v)$, then

$$\det([T_W]_\beta^\beta - t I_{t-1}) = (-1)^{k+1}(a_0 + a_1 t + \cdots + a_{k-1} t^{k-1} - t^k).$$

9. inner product $\langle\ ,\ \rangle : V \times V \to \mathbb{F} = \mathbb{R}$ or $\mathbb{C}$

$\langle x, y \rangle = \overline{\langle y, x \rangle}$.

$\langle x, x \rangle > 0$ if $x \neq \bar{0}$.

$\|v\| = \sqrt{\langle v, v \rangle}$.

$\|x\| = 0 \iff x = \bar{0}_v$.

$|\langle x, y \rangle| \leq \|x\| \|y\|$ Cauchy - Schwarz

$\|x+y\| \leq \|x\| + \|y\|$. triangle inequility.