Homework 9 MATH 416: ABSTRACT LINEAR ALGEBRA

NAME: DATE:

(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. §5.3 #1 Label the following statements as true or false (Answer is back, give a short explanation!).

a. §5.3 #1 (c) Any vector

F ✓

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

$x_i \geq 0$

such that $x_1 + x_2 + ... + x_n = 1$ is a probability vector.

b. §5.3 #1 (d) The sum of the entries of each row of a transition matrix equals 1. F ✓

column

c. §5.3 #1 (e) The product of a transition matrix and a probability vector is a probability vector. T ✓

d. §5.3 #1 (g) Every transition matrix has 1 as an eigenvalue. T ✓

e. §5.3 #1 (h) No transition matrix can have −1 as an eigenvalue. F $\cancel{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$ has −1 as an eigenvalue

not a transition matrix.

f. §5.3 #1 (i) If A is a transition matrix, then $\lim_{m\to\infty} A^m$ exists and has rank 1.

F

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{rank } 2$$

e. $A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\det[A - tI_n] = \begin{vmatrix} -t & 1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 1-t \end{vmatrix} = (1-t)(t^2-1)$

$\Rightarrow$ has −1 as an eigenvalue.

f. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\lim_{m\to\infty} A^m = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank = 3.

2. §5.3 #6 A hospital trauma unit has determined that 30% of its patients are ambulatory and 70% are bedridden at the time of arrival at the hospital. A month after arrival, 60% of the ambulatory patients have recovered, 20% remain ambulatory, and 20% have become bedridden. After the same amount of time, 10% of bedridden patients have recovered, 20% have become ambulatory, 50% remain bedridden, and 20% have died. Determine the percentages of patients who have recovered, are ambulatory, are bedridden, and have died 1 month after arrival. Also determine the eventual percentages of patients of each type.

$$P_0 = \begin{bmatrix} 0.3 \\ 0.7 \\ 0 \\ 0 \end{bmatrix} \qquad A = \begin{bmatrix} 0.2 & 0.2 & 0 & 0 \\ 0.2 & 0.5 & 0 & 0 \\ 0.6 & 0.1 & 1 & 0 \\ 0 & 0.2 & 0 & 1 \end{bmatrix}$$

$$P_1 = A\,P_0 = \begin{bmatrix} 0.2 \\ 0.41 \\ 0.25 \\ 0.14 \end{bmatrix}$$

20% ambulatory.
41% bedridden
25% recovered.
14% died.

$$\det(A - tI_n) = \begin{vmatrix} 0.2-t & 0.2 & 0 & 0 \\ 0.2 & 0.5-t & 0 & 0 \\ 0.6 & 0.1 & 1-t & 0 \\ 0 & 0.2 & 0 & 1-t \end{vmatrix} = (1-t)^2(0.5-t)\cdot\left(\frac{-0.2}{0.5-t} \times 0.2 + 0.2 - t\right).$$

$$= (1-t)^2 (t-0.1)(t-0.6)$$

$\lambda_1 = 1 \qquad \lambda_2 = 0.1 \qquad \lambda_3 = 0.6.$

$$A - I_n = \begin{bmatrix} -0.8 & 0.2 & 0 & 0 \\ 0.2 & -0.5 & 0 & 0 \\ 0.6 & 0.1 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow E_{\lambda_1} = \operatorname{span}\left\{\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\}$$

Continued from Question 2.

$$A-0.1I_n=\begin{bmatrix}0.1&0.2&0&0\\0.2&0.4&0&0\\0.6&0.1&0.9&0\\0&0.2&0&0.9\end{bmatrix}\to\begin{bmatrix}1&2&0&0\\0&1&-\frac{9}{11}&0\\0&0&1&\frac{11}{2}\\0&0&0&0\end{bmatrix}\Rightarrow E_{\lambda_2}=\text{span}\left\{\begin{pmatrix}9\\-\frac{9}{2}\\-\frac{11}{2}\\1\end{pmatrix}\right\}$$

$$A-0.6I_n=\begin{bmatrix}-0.4&0.2&0&0\\0.2&-0.1&0&0\\0.6&0.1&0.4&0\\0&0.2&0&0.4\end{bmatrix}\to\begin{bmatrix}2&-1&0&0\\0&1&1&0\\0&0&1&-2\\0&0&0&0\end{bmatrix}\Rightarrow E_{\lambda_3}=\text{span}\left\{\begin{pmatrix}-1\\-2\\2\\1\end{pmatrix}\right\}$$

$$\beta=\left\{\begin{pmatrix}0\\0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\0\\1\end{pmatrix},\begin{pmatrix}18\\-9\\-11\\2\end{pmatrix},\begin{pmatrix}-1\\-2\\2\\1\end{pmatrix}\right\}$$

$$Q=[I]_\beta^\alpha=\begin{bmatrix}0&0&18&-1\\0&0&-9&-2\\1&0&-11&2\\0&1&2&1\end{bmatrix}\qquad Q^{-1}=[I]_\alpha^\beta=\begin{bmatrix}\frac{8}{9}&\frac{5}{9}&1&0\\\frac{1}{9}&\frac{4}{9}&0&1\\\frac{2}{45}&-\frac{1}{45}&0&0\\-\frac{1}{5}&-\frac{2}{5}&0&0\end{bmatrix}$$

$$\lim_{k\to\infty}A^k=Q\begin{bmatrix}1&&&\\&1&&\\&&0&\\&&&0\end{bmatrix}Q^{-1}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\\frac{8}{9}&\frac{5}{9}&1&0\\\frac{1}{9}&\frac{4}{9}&0&1\end{bmatrix}$$

$$P_s=\lim_{k\to\infty}A^kP_0=\begin{bmatrix}0\\0\\\frac{59}{90}\\\frac{31}{90}\end{bmatrix}\Rightarrow$$

recovered: $\frac{59}{90}$

died: $\frac{31}{90}$

3. §5.3 #7 A player begins a game of chance by placing a marker in box 2, marked *Start*. (See figure below.) A die is rolled, and the marker is moved one square to the left if a 1 or a 2 is rolled and one square to the right if a 3, 4, 5 or 6 is rolled. This process continues until the marker lands in square 1, in which case the player wins the game, or in square 4, in which case the player loses the game. What is the probability of winning this game?

Win	Start		Lose
1	2	3	4

Hint: Instead of diagonalizing the appropriate transition matrix A, it is easier to represent e_2 as a linear combination of eigenvectors of A and then apply A^n to the result.

$$A = \begin{bmatrix} 1 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 1 \end{bmatrix} \qquad P_0 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\det(A - tI_n) = \begin{vmatrix} 1-t & \frac{1}{3} & 0 & 0 \\ 0 & -t & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -t & 0 \\ 0 & 0 & \frac{1}{3} & 1-t \end{vmatrix} = (t-\tfrac{\sqrt{2}}{3})(t+\tfrac{\sqrt{2}}{3})(t-1)^2$$

$\lambda_1 = \frac{\sqrt{2}}{3}$ $\quad \lambda_2 = -\frac{\sqrt{2}}{3}$ $\quad \lambda_3 = 1.$

$$P_S = \lim_{k\to\infty} A^k P_0 = [l_1\ l_2\ l_3\ l_4] \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = l_2$$

$$\lim_{k\to\infty} A^k = [V_1, V_2, V_3, V_4] \begin{bmatrix} 0 & & & \\ & 0 & & \\ & & 1 & \\ & & & 1 \end{bmatrix} [V_1\ V_2\ V_3\ V_4]^{-1}$$

$$= [\bar{0}, \bar{0}, V_3, V_4]\ [V_1\ V_2\ V_3\ V_4]^{-1}$$

$$l_2 = [\bar{0}, \bar{0}, V_3, V_4]\, m_2$$

Continued from Question 3.

$$[V_1, V_2, V_3, V_4]\, m_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$A - \lambda_1 I_n = \begin{bmatrix} 1-\frac{\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -\frac{\sqrt{2}}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 1-\frac{\sqrt{2}}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 3-\sqrt{2} & 1 & & \\ & -\sqrt{2} & 1 & \\ & & 1 & 3-\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow V_1 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{2-3\sqrt{2}}{2} \\ \sqrt{2}-3 \\ 1 \end{pmatrix}$$

$$A - \lambda_2 I_n = \begin{bmatrix} 1+\frac{\sqrt{2}}{3} & \frac{1}{3} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & \frac{\sqrt{2}}{3} & 0 \\ 0 & 0 & \frac{1}{3} & 1+\frac{\sqrt{2}}{3} \end{bmatrix} \longrightarrow \begin{bmatrix} 3+\sqrt{2} & 1 & 0 & 0 \\ 0 & \sqrt{2} & 1 & 0 \\ 0 & 0 & 1 & 3+\sqrt{2} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow V_2 = \begin{pmatrix} -\frac{\sqrt{2}}{2} \\ \frac{2+3\sqrt{2}}{2} \\ -3-\sqrt{2} \\ 1 \end{pmatrix}$$

$$A - \lambda_3 I_n = \begin{bmatrix} 0 & \frac{1}{3} & 0 & 0 \\ 0 & -1 & \frac{1}{3} & 0 \\ 0 & \frac{2}{3} & -1 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Longrightarrow V_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad V_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$m_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \left[\begin{array}{cccc|c} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 1 & 0 & 0 \\ \frac{2-3\sqrt{2}}{2} & \frac{2+3\sqrt{2}}{2} & 0 & 0 & 1 \\ \sqrt{2}-3 & -3-\sqrt{2} & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{array}\right] \Rightarrow \begin{cases} x_1 = \frac{1}{2-3\sqrt{2}} \\ x_2 = \frac{1}{2+3\sqrt{2}} \\ x_3 = \frac{3}{7} \\ x_4 = \frac{2}{7} \end{cases}$$

$$l_2 = \begin{bmatrix} \frac{3}{7} \\ 0 \\ 0 \\ \frac{2}{7} \end{bmatrix} \quad \therefore \quad P(win) = \frac{3}{7}$$

4. §5.4 #2 For each of the following linear operators T on the vector space V, determine whether the given subspace W is a T-invariant subspace of V.

a. §5.4 #4 (c) $V = \mathbb{R}^3$, $T(a,b,c) = (a+b+c, a+b+c, a+b+c)$, and $W = \{(t,t,t) : t \in \mathbb{R}\}$

b. §5.4 #4 (d) $V = C([0,1])$, $T(f(t)) = \left(\int_0^1 f(t)dt\right) t$, and $W = \{f \in V : f(t) = at+b \text{ for some } a \text{ and } b\}$

a. $\forall w \in W$

$$T(w) = T(t,t,t) = (3t, 3t, 3t) \in W$$

$t \in \mathbb{R}$

$\therefore$ W is T-invariant subspace ✓

b. $\forall f \in W$

$$T(f) = \left(\int_0^1 f(t)dt\right)t = \left(\int_0^1 at+b\,dt\right)t$$

$$= \left(\frac{1}{2}at^2+bt\,\Big|_0^1\right)t = \left(\frac{1}{2}a+b\right)t \in W$$

$\therefore$ W is T-invariant subspace ✓

5. §5.4 #3 Let T be a linear operator on a finite-dimensional vector space V. Prove that the following subspaces are T-invariant.

a. §5.4 #3 (a) $\{0\}$ and V

b. §5.4 #3 (b) $N(T)$ and $R(T)$

c. §5.4 #3 (c) E_λ, for any eigenvalue λ of T

a. T is linear $\Rightarrow T(0) = 0 \in \{0\}$

T is $V \to V \Rightarrow T(V) \subset V$

$\therefore \{0\}$ and V are T-invariant

b. $\forall t \in N(T), T(t) = 0 \in N(T)$

$\forall r \in R(T) \subset V, \ T(r) \in T(V) = R(T)$.

$\therefore N(T)$ and $R(T)$ are T-invariant.

c. $\forall v \in E_\lambda, Tv = \lambda v \in E_\lambda$

$\therefore E_\lambda$ is T-invariant.

6. §5.4 #6 (c) For the following linear operator T on the vector space V, find an ordered basis for the T-cyclic subspace generated by the vector z.

$$V = M_{2\times 2}(\mathbb{R}),\ T(A) = A^t,\ \text{and } z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Since $T\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

so the T-cyclic subspace generated by the vector Z, $W = \text{Span}\{Z, T(Z) \cdots\}$

$$= \text{Span}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$$

$\therefore$ the ordered basis is $\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}$

7. §5.4 #16 Let T be a linear operator on a finite-dimensional vector space V.

a. §5.4 #16 (a) Prove that if the characteristic polynomial of T splits, then so does the characteristic polynomial of the restriction of T to any T-invariant subspace V.

b. §5.4 #16 (b) Deduce that if the characteristic polynomial of T splits, then any nontrivial T-invariant subspace of V contains an eigenvector of T.

(a) W is T-invariant

$\Rightarrow$ char. poly. of T_W divides char. poly. of V.
(proved in class).

$$\det(T - tI_n) = \det(T_W - tI_k)\, g(t).$$

Since $\det(T - tI_n)$ splits.
We can know $\det(T_W - tI_k)$ splits too.

(b). eigenvectors are the solutions of
$$\det(T - tI_n) = 0$$

Since $\det(T - tI_n) = \det(T_W - tI_k)\, g(t)$

eigevectors of $T = \{t \mid \det(T_W - tI_k) = 0\}$
$\cup \{t \mid g(t) = 0\}$

Since T_W is nontrival $\# \{t \mid \det(T_W - tI_k) = 0\} > 0$.

$\therefore$ Contains an eigenvector of T.

8. §6.1 #3 In $C([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$ (as defined in Example 3 of the textbook, i.e., $\langle f, g\rangle = \int_0^1 f(t)g(t)dt$), $\|f\|, \|g\|$, and $\|f+g\|$. Then verify both the Cauchy-Schwarz inequality and the triangle inequality.

$$\langle f, g\rangle = \int_0^1 t e^t dt = te^t\Big|_0^1 - e^t\Big|_0^1 = e - e + 1 = 1.$$

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_0^1 t^2 dt} = \frac{\sqrt{3}}{3}$$

$$\|g\| = \sqrt{\langle g, g\rangle} = \sqrt{\int_0^1 e^{2t} dt} = \sqrt{\frac{e^2-1}{2}}$$

$$\|f+g\| = \sqrt{\langle f+g, f+g\rangle} = \sqrt{\int_0^1 (t+e^t)^2 dt} = \sqrt{\frac{11+3e^2}{6}}$$

$$|\langle f, g\rangle| = 1 < \|f\|\,\|g\| = \sqrt{\frac{e^2-1}{6}}$$

$$\|f+g\| = \sqrt{\frac{11+3e^2}{6}} < \|f\| + \|g\| = \frac{\sqrt{3}}{3} + \sqrt{\frac{e^2-1}{2}}$$

9. §6.1 #8 Provide reasons why each of the following is not an inner product on the given vector spaces.

a. §5.4 #8 (a) $\langle (a,b),(c,d)\rangle = ac - bd$ on $\mathbb{R}^2$

b. §5.4 #8 (b) $\langle A, B\rangle = \mathrm{tr}(A+B)$ on $M_{2\times 2}(\mathbb{R})$

(a) $<(a,b),(a,b)> = a^2-b^2$ when $(a,b)=(1,2)$

$<(a,b),(a,b)> = -3 < 0$.

(b) $A=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $B=\begin{bmatrix}1&0\\0&2\end{bmatrix}$

$<A,B> = tr\left(\begin{bmatrix}2&0\\0&3\end{bmatrix}\right) = 5$

$<2A,B> = tr\left(\begin{bmatrix}3&0\\0&4\end{bmatrix}\right) = 7 \neq 2<A,B>$

10. §6.1 #9 Let β be a basis for a finite-dimensional inner product space.

a. §6.1 #9 (a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$. x may not be in β.

b. §6.1 #9 (b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

(a) Let $z = x$. $<x,x>=0 \Rightarrow x=0$

(b) $0 = <x,z> - <y,z> = <x-y,z>$

from a we know $x-y=0$

$\Rightarrow x=y$.

(a). $\beta = \{v_1, \ldots, v_n\}$.

any $v \in V$, $v = \sum_{i=1}^{n} a_i v_i$.

$<x,v> = <x, \sum_{i=1}^{n} a_i v_i> = \sum_{i=1}^{n} \bar{a}_i <x, v_i> = 0 \Longleftrightarrow x=0.$

11. §6.1 #10 Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in $\mathbb{R}^2$.

$$\|x+y\|^2 = \langle x+y, x+y \rangle = \langle x+y, x \rangle + \langle x+y, y \rangle$$

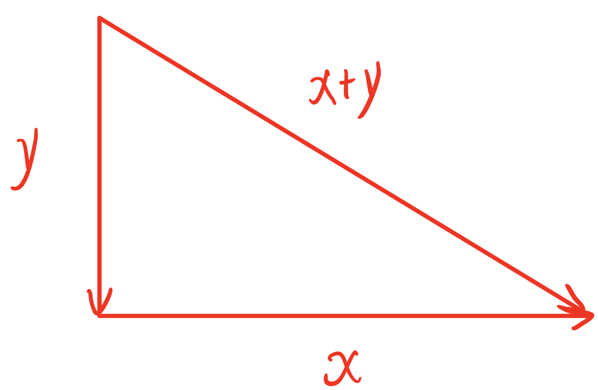
$$= \langle x, x \rangle + \langle y, x \rangle + \langle x, y \rangle + \langle y, y \rangle.$$

$$= \|x\|^2 + \|y\|^2 + \langle x, y \rangle + \langle y, x \rangle.$$

Since x, y are orthogonal vectors in V

$$\langle x, y \rangle = \langle y, x \rangle = 0$$

So $\|x+y\|^2 = \|x\|^2 + \|y\|^2$



$\|x\|$ is the length of x.

$\|y\|$ is the length of y.

$\|x+y\|$ is the length of diagonal connecting x and y.

Problems **not** from the textbook exercises.

(Questions 12 $\sim$ 14) Determine if the following statements are TRUE or FALSE. If TRUE give a proof, if FALSE give a counterexample.

12. *Every one dimensional subspace of $\mathbb{R}^n$ can be spanned by a probability vector.*

False $\quad \text{span}\{(1,-1,0,0\cdots 0)\}$

13. *The matrix $A = \begin{pmatrix} .9 & .02 \\ .1 & .98 \end{pmatrix}$ is a diagonalizable transition matrix.*

True

$$\det(A - tI_2) = \begin{vmatrix} 0.9-t & 0.02 \\ 0.1 & 0.98-t \end{vmatrix} = (0.9-t)(0.98-t) - 0.002$$

$$= 0.88 - 1.88t + t^2 \quad \lambda_1 = 1 \quad \lambda_2 = 0.88$$

$$E_{\lambda_1} = \text{span}\left\{\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right\} \quad E_{\lambda_2} = \text{span}\left\{\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}$$

exists $\beta = \left\{\begin{pmatrix} 1 \\ 5 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix}\right\}, \; [A]_\beta^\beta = \begin{bmatrix} 1 & 0 \\ 0 & 0.88 \end{bmatrix}$

14. *If A is a square matrix, then A and A^t have the same eigenvectors.*

~~True~~

$\det(A - tI_n) = \det(A^t - tI_n)$

$\Rightarrow$ the solutions of $\det(A - tI_n) = 0$

$=$ the solutions of $\det(A^t - tI_n) = 0$.

$\Rightarrow$ A and A^t have same eigen~~vectors~~ eigenvalues.

False

$Av = \lambda v \qquad A^t v' = \lambda v'$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \quad \lambda = 1. \quad \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$

$E_1 = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \qquad E_1' = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$

15. A square matrix A is block upper triangular if it has the form

$$A = \begin{pmatrix} B & C \\ O & D \end{pmatrix},$$

where O is a zero matrix and B and D are square matrices.

$$\text{ex. } A = \begin{pmatrix} 1 & 2 & a & b \\ 3 & 4 & c & d \\ 0 & 0 & 7 & 8 \\ 0 & 0 & 9 & 10 \end{pmatrix}$$

If A is block upper triangular prove that $\det(A) = \det(B)\det(D)$.
(*Hint: You can argue by induction on the size of D.*)

$A \in M_{n\times n}$

①. $D \in M_{1\times 1}$ $\quad A = \begin{bmatrix} B_{(n-1)\times(n-1)} & C \\ 0 & D \end{bmatrix}$

$\det(A) = D \det B = \det(B)=\det(D)$
equation holds.

②. assume when $D \in M_{i\times i}$ the equation holds

i.e. $A = \begin{bmatrix} B_{(n-i)\times(n-i)} & C \\ 0 & D_{i\times i} \end{bmatrix}$ $\quad \det(A) = \det(B)\det(D)$

③. when $D \in M_{(i+1)\times(i+1)}$ $\quad A = \begin{bmatrix} B_{(n-i-1)\times(n-i-1)} & C \\ 0 & D_{(i+1)\times(i+1)} \end{bmatrix}$

$$D_{(i+1)\times(i+1)} = \begin{bmatrix} d_{11} & \cdots & d_{1,(i+1)} \\ \vdots & \ddots & \vdots \\ d_{(i+1),1} & \cdots & d_{(i+1),(i+1)} \end{bmatrix}$$

set $D^k_{i\times i}$ be the matrix that $D_{(i+1)\times(i+1)}$ delete k^{th} column and $(i+1)^{th}$ row

$$\det(A) = \sum_{k=1}^{i+1} (-1)^{k+i+1} d_{(i+1),k} \det \begin{bmatrix} B_{(n-i-1)\times(n-i-1)} & C \\ 0 & D^k_{i\times i} \end{bmatrix}$$

$$= \sum_{k=1}^{i+1} (-1)^{k+i+1} d_{(i+1),k} \det(B)\det(D^k)$$

$$= \det(B) \cdot \left(\sum_{k=1}^{i+1} (-1)^{k+i+1} d_{(i+1),k} \det(D^k) \right).$$

$= \det(B)\det(D)$, so equation holds on $D \in M_{(i+1)\times(i+1)}$ when holds on $D \in M_{i\times i}$

$\therefore \det(A) = \det(B)\det(D)$ holds for all $D \in M_{i\times i}$ $\quad i = 1, 2, \cdots, n.$