Def^n $T: V \to V$ is <u>isometry</u>.

if $\langle T(x), T(y) \rangle = \langle x, y \rangle$.

V over $\mathbb{C}$: T is isometry is also called unitary.

V over $\mathbb{R}$: T is isometry is also called orthogonal.

Theorem: following are equivalent.

a) $T: V \to V$ is an isometry.

b) $TT^* = T^*T = I_V$

c) If β is an orthonormal basis, then so is $T(\beta)$.

d) $T(\beta)$ is orthonormal for some orthonormal β

e) $\|T(x)\| = \|x\| \quad \forall x \in V$

Lemma: Suppose $S: V \to V$ is self-adjoint if $\langle S(x), x \rangle = 0$ $\forall x \in V$, then $S = T_{0_V}$.

Proof: S is self-adjoint $\Rightarrow$ exists $\beta = \{v_1, \dots, v_n\}$ be an orthonormal basis consisting of eigenvectors of S.

$$S(v_j) = \lambda_j v_j$$

$$0 = \langle S(v_j), v_j \rangle = \langle \lambda_j v_j, v_j \rangle = \lambda_j \|v_j\|^2$$

So, $\lambda_j = 0$ for $j = 1, \dots, n$.

Cor: V has an orthonormal basis β with corresponding eigenvectors of absolute 1 $\Longleftrightarrow$ T is self-adjoint and orthogonal

Proof of Thm 3:

a) $\Rightarrow$ b): $(T^*T)^* = T^*T \Rightarrow T^*T$ is self-adjoint.

$\Rightarrow T^*T - I_V$ is self-adjoint

$$\langle (T^*T - I_V)(x), x\rangle = \langle T^*T(x) - x, x\rangle.$$

$$= \langle T(x), T(x)\rangle - \langle x, x\rangle.$$

$= 0$. Since T is isometry.

$\Rightarrow T^*T - I_V = T_{0_V}$

$\Rightarrow T^*T = I_V$

$\Rightarrow T^* = T^{-1}$

b) $\Rightarrow$ c): $\beta = \{v_1, \dots, v_n\}$ $\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases}$

$T(\beta) = \{T(v_1), \dots, T(v_n)\}$.

$$\langle T(v_i), T(v_j)\rangle = \langle v_i, (T^*T)(v_j)\rangle = \langle v_i, v_j\rangle = \delta_{ij}.$$

c) $\Rightarrow$ d) Obviously.

d) $\Rightarrow$ e): $\langle T(x), T(x)\rangle = \langle \sum_j a_j v_j, \sum_j a_j v_j\rangle$

$$= \sum_{i,j} a_i \bar{a}_j \langle v_i, v_j\rangle = \sum_i |a_i|^2 = \langle x, x\rangle.$$

e) $\Rightarrow$ a): $\|T(x)\| = \|x\|$

$$\Longrightarrow \langle T(x), T(x)\rangle = \langle x, x\rangle.$$

FACT: $\|\cdot\| \Rightarrow \langle\cdot\rangle$ (polarization)

$$\|x+y\|^2 = \langle x+y, x+y\rangle = \|x\|^2 + \langle x,y\rangle + \langle y,x\rangle + \|y\|^2$$

$$= \|x\|^2 + \underbrace{\langle x,y\rangle + \overline{\langle x,y\rangle}}_{2\mathrm{Re}(\langle x,y\rangle)} + \|y\|^2$$

Theorem Isometries are rare.

Jordan Canonical Form.

Theorem (Thm JCF).

If the char. poly. of $T: \mathbb{V} \to \mathbb{V}$ splits.

then there is a basis β of $\mathbb{V}$.

Such that $[T]_\beta^\beta$ is in Jordan Canonical Form.

Def^n

Jordan Block $\begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix}$

Jordan Canonical Form $\begin{bmatrix} A_1 & & \\ & A_2 & \\ & & \ddots & \\ & & & A_l \end{bmatrix}$ where A_j is Jordan block.

$$\det\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} = \det\begin{bmatrix} \lambda & & \\ & \ddots & \\ & & \lambda \end{bmatrix}$$

FACT: $A = \begin{bmatrix} \lambda & 1 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \Rightarrow A^k = \begin{bmatrix} \lambda^k & k\lambda^{k-1} & \cdots & \frac{k(k-1)\cdots(k-n+2)}{(n-1)!}\lambda^{k-n+1} \\ 0 & \ddots & & \vdots \\ \vdots & & \ddots & k\lambda^{k-1} \\ 0 & \cdots & 0 & \lambda^k \end{bmatrix}$

$$\begin{bmatrix} A_1 & & \\ & \ddots & \\ & & A_l \end{bmatrix}^k = \begin{bmatrix} A_1^k & & \\ & \ddots & \\ & & A_l^k \end{bmatrix}$$

Def^n $x \in V$ is a generalized eigenvector of $T: V \to V$ corresponding to λ if $(T - \lambda I_V)^p(x) = \bar{0}_V$ for some integer $p > 0$.

Let x be a gen. eigenvector. p is the smallest integer s.t. $(T - \lambda I_V)^p(x) = \bar{0}_V$

$\Rightarrow y = (T - \lambda I_V)^{p-1}(x) \neq \bar{0}_V$ and $(T - \lambda I_V)^p(y) = (T - \lambda I_V)(y) = \bar{0}_V$

$\Rightarrow y$ is an eigenvector of T.

Def^n generalized eigenspace of λ is

$K_\lambda = \{ x \in V : (T - \lambda I_V)^p(x) = \bar{0}_V$ for some $p > 0 \}$

Theorem 1.1 a): K_λ is a T-invariant subspace of V containing E_λ eigenvectors of λ.

b) : For $\mu \neq \lambda$, the restriction of $T-\mu I_V$ to K_λ is one-to-one. (i.e. $(T-\mu I_V)(x)=\bar{0}_V$, $x \in K_\lambda \Rightarrow x=\bar{0}_V$)

$$\underset{K_\lambda}{(T-\mu I_V)(\underset{\Uparrow}{x})} = \underset{K_\lambda}{\underset{\Uparrow}{T(x)}} - \underset{K_\lambda}{\underset{\Uparrow}{\mu x}} \in K_\lambda$$

Assume $x \neq \bar{0}_V$. $(T-\mu I_V)(x)=\bar{0}_V$.

$(T-\lambda I_V)^p(x)=\bar{0}_V$

$y=(T-\lambda I_V)^{p-1}(x) \neq \bar{0}_V$

$(T-\mu I_V)(y)=(T-\lambda I_V)^{p-1}(T-\mu I_V)(x)$
$=\bar{0}_V$

$y \in E_\lambda$ and $y \in E_\mu$ $\mu \neq \lambda$

so $y \in E_\lambda \cap E_\mu$ and $y \neq \bar{0}_V$ which is impossible.

Thm (JCF)

$$A_j = \begin{bmatrix} \lambda_j & 1 & \cdots & 0 \\ \vdots & \lambda_j & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \lambda_j \end{bmatrix}$$

$\underset{n_1 \times n_1}{A_1} \Rightarrow$

$T(v_1)=\lambda_1 v_1$ $\qquad (T-\lambda_1 I_V)(v_1)=\bar{0}_V$

$T(v_2)=v_1+\lambda_1 v_2$ $\qquad (T-\lambda_1 I_V)(v_2)=v_1$

$\cdots$ $\qquad \cdots$

$T(v_{n_1})=v_{n_1-1}+\lambda_1 v_{n_1}$ $\qquad (T-\lambda_1 I_V)(v_{n_1})=v_{n_1-1}$

$\{v_1, v_2, \ldots, v_{n_1}\} = \{(T-\lambda_1 I_V)^{n_1-1}(v_{n_1}), \ldots, (T-\lambda_1 I_V)(v_{n_1}), v_{n_1}\}$

Theorem 1.2. if λ has alg. mult. m, then $K_\lambda = N((T-\lambda I_V)^m)$.

Theorem 1.3 Let $\lambda_1, \ldots \lambda_k$ be the distinct eigenvalues of $T: V \to V$ For any $x \in V$, there are $v_j \in K_{\lambda_j}$ s.t. $x = v_1 + \cdots + v_k$. ($\operatorname{Span}(K_{\lambda_1} \cup \cdots \cup K_{\lambda_k}) = V$)

Thm 1.4 Let β_j be a basis for K_{λ_j}. Then

a) $\beta_i \cap \beta_j = \phi$ for $i \neq j$.

b) $\beta = \beta_1 \cup \cdots \cup \beta_k$ is a basis for V

c) $\dim(K_{\lambda_j})$ = alg. mult. for λ_j.

Proof: $\dim(K_{\lambda_j}) \leq$ alg. mult of λ_j, $\sum_{j=1}^{k} \dim(K_{\lambda_j}) = \dim V$ $\sum_{j=1}^{k}$ alg. mult of λ_j

$\Rightarrow \dim K_{\lambda_j}$ = alg mult. for λ_j.

Def^n Let $x \in K_\lambda$, p is the smallest integer s.t.

$(T - \lambda I_V)^p(x) = \bar{0}_V$

Let $\gamma = \{(T-\lambda I_V)^{p-1}(x), \ldots, (T-\lambda I_V)(x), x\}$.

γ is the cycle of generalized eigenvectors generated by x.

Thm 2.1 a) γ is linearly indep.

b) $W = \mathrm{Span}(\gamma)$ is T-invariant

c) $[T_W]_\gamma^\gamma$ is a Jordan block

Thm 2.2 Let $\gamma_1, \ldots, \gamma_m$ be cycles for λ with linearly indep. initial vectors. Then $\gamma = \bigcup_{j=1}^{m} \gamma_j$ is lin. indep.

Thm 2.3 Each K_λ has a basis consisting of disjoint cycles.

Thm λ is an eigenvalue of T with alg. mult. m

$r_1 = \dim(V) - \mathrm{rank}(T - \lambda I_V)$.

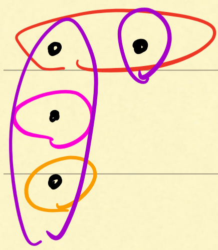
$r_i = \mathrm{rank}((T-\lambda I_V)^{i-1}) - \mathrm{rank}((T-\lambda I_V)^i)$

$i = 2, \ldots$ until $r_1 + r_2 + \cdots + r_i = m$.

$r_1 = \#$ of Jordan Blocks for λ.

ex: $\lambda = 2$

$r_1 = 2$, $r_2 = 1$. $r_3 = 1$.



\# of r_1

\# of r_2

\# of r_3

Size of Jordan block of λ.

$$\hookrightarrow \begin{bmatrix} \boxed{\begin{matrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{matrix}} & \\ & 2 \end{bmatrix}$$