Def$^{\underline{n}}$ V is <u>finite dimensional</u> (otherwise: infinite dimensional.) if it has a finite basis.

The number of elements in any basis of V is the <u>dimension</u> of V, <u>$\dim V$.</u>

Theorem: Let W be a subspace of V. if <u>V is finite dimensional,</u> then $\dim V \leq \dim W$.

($\dim V = \dim W$ if and only if $V = W$).

Proof: $\dim V = n$.

Let $\beta_V = \{u_1, \cdots, u_n\}$ be a basis for V.

Let $\mathcal{U} = \{w_1, \cdots, w_k\} \subset W$ lin. indep

Repl. Thm $(S = \beta_V, U = \mathcal{U}) \Rightarrow k \leq n$

we can construct a basis of W, recursively.

① if $W = \{\bar{0}_V\}$ then done

otherwise choose a $w_1 \neq \bar{0}_V$ in W

② if $W = \text{span}(\{w_1\})$, then done

otherwise choose a $w_2 \notin \text{Span}(\{w_1\})$

$\Rightarrow w_1, w_2$ is lin. indep.

- - - - - - This process must stop at some.

such that $W = \mathrm{Span}(\underbrace{\{w_1, \dots, w_k\}}_{\text{lin. indep}})$

$$\Downarrow$$

$$\underline{k \le n.}$$

Linear Transformations (Maps)

Def^n A map T from a vector space V to a vector space W ($T: V \to W$) is linear for all $v_1, v_2 \in V$ and $c \in \mathbb{R}$

(a) $T(v_1+v_2) = T(v_1) + T(v_2)$ $\xleftrightarrow{\text{Rmk 1}}$ $T(v_1+cv_2)$

(b) $T(cv_1) = cT(v_1)$ $= T(v_1) + cT(v_2)$

Null Space and range

Def^n Suppose $T: V \to W$ is linear.

The <u>null space</u> (kernel) of T is $N(T) = \{v \in V \mid T(v) = \bar{0}_W\} \subset V$

The <u>range</u> (image) of T is $R(T) = \{T(v) \mid v \in V\} \subset W$

Theorem: $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Dimension Theorem

if $T: V \to W$ is linear and V is finite dimensional, then $\dim(N(T)) + \dim(R(T)) = \dim(V)$

Ex: $T: \mathbb{R}^n \to \mathbb{R}^n$

$(x_1, \cdots, x_n) \to (x_1, \cdots, \underline{x_m}, 0, \cdots, 0)$ $(m < n)$

$N(T) = \{(x_1, \cdots, x_n) \mid x_1 = 0, \cdots, x_m = 0\}$

$\dim N(T) = n - m$

$\dim R(T) = m$.

Proof: Set $\dim V = n$

$N(T)$ is a subspace of V of $\dim k \le n$.

Let $\beta_N = \{u_1, \cdots, u_k\}$ be a basis of $N(T)$.

① Let $\beta = \{u_1, \cdots, u_n\}$ be a basis for V

Repl. Thm $(S = \beta, \mathcal{U} = \beta_N\} \Rightarrow$

we can add $n-k$ elements of β to β_N so that the resulting set generates V.

$$\mathrm{Span}(\underbrace{\{u_1, \cdots, u_k, v_1, \cdots, v_{n-k}\}}_{\beta_V}) = V$$

β_V is a basis of V (by Refinement Theorem)

$\mathrm{Span}(\{\underbrace{T(u_1), \cdots T(u_k)}_{\bar{0}_W}, T(v_1) \cdots T(v_{n-k})\}) = R(T)$

② $\{T(v_1), \cdots, T(v_{n-k})\} \subset R(T)$, $T(v_1) \cdots T(v_{n-k})$ linear indep $\Rightarrow$ is a basis of $R(T)$

$\Rightarrow \dim R(T) = n-k$

1-1 linear transformation

Def^n $T: V \to W$ is onto (surjective) if $R(T) = W$ or $\forall w \in W, \exists v \in V$ s.t. $T(v) = w$

Def^n $T: V \to W$ is 1-1 (injective) if

$$T(v_1) = T(v_2) \Rightarrow v_1 = v_2$$

Theorem if $T: V \to W$ is linear then T is 1-1 if and only if $N(T) = \{\bar{0}_V\}$

Equivalent (T is a linear map and between vector spaces of sam finite dimension)

(1) T is 1-1 , (2) T is onto , (3) $\underset{\underset{\dim(R(T))}{\shortparallel}}{\text{rank}}(T) = \dim V$

Matrix.

$\beta = \{v_1, \dots, v_n\}$ is a basis of V, then $v \in V$ can be expressed uniquely in the form : $v = a_1 v_1 \cdots + a_n v_n$

$a_1, \dots, a_n$ are unique for v.

Def^n $[v]_\beta = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} \in \mathbb{R}^n$

$[\ ]_\beta : V \to \mathbb{R}^n$ is linear, 1-1 and onto

$\beta = \{v_1, \dots, v_n\}$ be a basis for V.

$\gamma = \{w_1, \dots, w_m\}$ be a basis for W.

for v_j, we have

$T(v) = a_1 w_1 + \cdots a_m w_m$

$T(v_j) = a_{1j} w_1 + \cdots + a_{mj} w_m$

$[T]_\beta^\gamma = (a_{ij})$

Def$^{\underline{n}}$ $[T]_\beta^\gamma$ is the matrix representation of T with respect to the bases β, γ.

$$[T]_\beta^\gamma = \begin{bmatrix} | & & | \\ [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \\ | & & | \end{bmatrix}_{m\times n}$$

Thm For all $v \in V$, $[T(v)]_\gamma = [T]_\beta^\gamma [v]_\beta$

Ex: $T: P_3 \to P_2$ $\quad \beta = \{1, x, x^2, x^3\}$

$f \to f'$ $\quad \gamma = \{1, x, x^2\}$

$$[T_d]_\beta^\gamma = \begin{bmatrix} | & & | \\ [T(1)]_\beta & \cdots & [T(x^3)]_\beta \\ [0]_\beta & & [3x^2]_\beta \\ | & & | \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

More precisely

Theorem 1: $T: V \to W$, β, γ are fixed basis of V, W

for all $v \in V$, $[T(v)]_\gamma = \underbrace{[T]_\beta^\gamma [v]_\beta}$

acts like matrix multiplication.

Proof: $\beta = \{v_1, \ldots, v_n\}$, $\gamma = \{w_1, \ldots, w_m\}$

$$[T(v)]_\gamma = [T(a_1 v_1 + \cdots + a_n v_n)]_\gamma$$

$$= [a_1 T(v_1) + \cdots + a_n T(v_n)]_\gamma$$

$$= a_1 [T(v_1)]_\gamma + \cdots + a_n [T(v_n)]_\gamma$$

(The map $\begin{array}{c} W \xrightarrow{[\ ]_\gamma} R^m \\ w \longrightarrow [w]_\gamma \end{array}$ is linear)

$$= \begin{bmatrix} [T(v_1)]_\gamma & \cdots & [T(v_n)]_\gamma \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

$$= [T]_\beta^\gamma [v]_\beta$$

Vector Space of linear transformations.

$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ linear}\}$

$S \in \mathcal{L}(X, Y)$, $T \in \mathcal{L}(Y, Z)$ $\quad X \xrightarrow{S} Y \xrightarrow{T} Z$

$$T \circ S : X \to Z$$
$$x \to T(S(x)).$$

Theorem 2: $T \circ S$ is linear

Proof: $(T \circ S)(x_1 + c x_2) = T(S(x_1 + c x_2))$

$$= T(S(x_1) + c S(x_2))$$

$$= T(S(x_1)) + c T(S(x_2))$$

$$= (T \circ S)(x_1) + c(T \circ S)(x_2)$$

Theorem 3 $[T \circ S]_\alpha^\gamma = \underbrace{[T]_\beta^\gamma [S]_\alpha^\beta}$

matrix multiplication.

Example: $A \in M_{m \times n} \to L_A : \underset{\alpha}{\mathbb{R}^n} \to \underset{\beta}{\mathbb{R}^m}$ standard bases

$x \to Ax$ → Left multiplication transformation.

$B \in M_{p \times m} \to L_B : \underset{\beta}{\mathbb{R}^m} \to \underset{\gamma}{\mathbb{R}^p}$

$$[L_B \circ L_A]_\alpha^\gamma = [L_B]_\beta^\gamma [L_A]_\alpha^\beta$$

$$= BA = [L_{BA}]_\alpha^\gamma$$

$$L_B \circ L_A = L_{BA}$$

Proof: $\alpha = \{x_1, \dots, x_n\}$

$$[T \circ S]_\alpha^\gamma = ([(T \circ S)(x_1)]_\gamma \cdots\cdots [(T \circ S)(x_n)]_\gamma)$$

$$= ([T(S(x_1))]_\gamma \cdots\cdots [T(S(x_n))]_\gamma)$$

by Theorem 1.

$$= ([T]_\beta^\gamma [S(x_1)]_\beta \cdots [T]_\beta^\gamma [S(x_n)]_\beta)$$

$$= [T]_\beta^\gamma \left([S(x_1)]_\beta \cdots [S(x_n)]_\beta \right)$$

$$= [T]_\beta^\gamma [S]_\alpha^\beta$$