Homework 4 MATH 416: ABSTRACT LINEAR ALGEBRA

NAME: Yang Wenxiao DATE: 2020. 6. 30

(Exercises are taken from *Linear Algebra, **Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

1. Exercise §2.1 #1 Label the following statements as true or false. In each part, V and W are finite-dimensional vector spaces (over $\mathbb{F}$), and T is a function from V to W (Answer is back, give a short explanation!).

(a) If T is linear, then T preserves sums and scalar products. T

(b) If $T(x+y) = T(x) + T(y)$, then T is linear. F also need $T(cx) = cT(x)$

(c) T is one-to-one if and only if the only vector x such that $T(x) = 0_W$ is $x = 0_V$. F T need to be linear

(d) If T is linear, then $T(0_V) = 0_W$. T

(e) If T is linear, then $\text{nullity}(T) + \text{rank}(T) = \dim(W)$. F V need to be finite dimensional

(f) If T is linear, then T carries linearly independent subsets of V onto linearly independent subsets of W F $P_2 \to P_1$ $f \to f'$ $\{1, x, x^2\} \to \{0, 1, 2x\}$ not lin. indep.

(g) If $T, U : V \to W$ are both linear and agree on a basis for V, then $T = U$. T

(h) Given $x_1, x_2 \in V$ and $y_1, y_2 \in W$, there exists a linear transformation $T : V \to W$ such that $T(x_1) = y_1$ and $T(x_2) = y_2$. F when $x_1 = x_2$ $y_1 \neq y_2$

$T(x_1 - x_2) = T(0)$

$= T(x_1) - T(x_2) = y_1 - y_2$

$T(x_2 - x_1) = T(0)$

$= T(x_2) - T(x_1) = y_2 - y_1$

$\Rightarrow y_1 = y_2$ contradiction.

For the following two problems (**Problem 2, Problem 3**), *(i)* prove that T is a linear transformation, and *(ii)* find bases for $N(T)$ and $R(T)$. Then *(iii)* compute the nullity and rank of T, and verify the dimension theorem. Finally, *(iv)* use the appropriate theorems in this section to determine whether T is one-to-one or onto.

2. §2.1 #2 $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (a_1 - a_2, 2a_3)$.

(i) $T(x_1+cy_1,\ x_2+cy_2,\ x_3+cy_3)$

$= ((x_1-x_2)+c(y_1-y_2),\ 2x_3+c2y_3) = (x_1-x_2,\ 2x_3)+c(y_1-y_2,\ 2y_3)$

$= T(x_1, x_2, x_3) + c(y_1, y_2, y_3) \Rightarrow$ linear.

(ii) bases for $N(T)$: $\{(1,1,0)\}$

bases for $R(T)$: $\{(1,0),(0,1)\}$

(iii) nullity $= 1$, rank $= 2$, $\dim V = 3$

nullity + rank $= \dim V$

(iv) $N(T) \neq \{\vec{0}\} \Rightarrow$ not 1-1

$\forall w = (x_1, x_2) \in W$, $\exists v = (x_1, 0, \frac{x_2}{2})$

$\Rightarrow$ onto

Continued from the previous question

3. **§2.1 #5** $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ defined by $T(f(x)) = xf(x) + f'(x)$.

(i) $T(f(x)+cp(x)) = x(f(x)+cp(x)) + f'(x) + cp'(x)$

$= xf(x) + f'(x) + c(xp(x) + p'(x))$

$= T(f(x)) + cT(p(x)) \Rightarrow$ linear

(ii) $xf(x) + f'(x) = 0$

$ax^2+bx+c \quad ax^3+bx^2+cx+2ax+b=0.$

$\Rightarrow a=0 \quad b=0 \quad c=0$

bases for $N(T) = \{ 0 \}$

bases for $R(T) = \{1, x, x^2, x^3\}$

(iii) nullity $=-1$ rank $=4$ $\dim V = 3$

nullity + rank = dim V

(iv) $N(T) = \{0\} \Rightarrow$ 1-1

$T(ax^2+bx+c) = ax^3+bx^2+(2a+c)x+b$ can't express any $y \in P_3$

$\Rightarrow$ not onto

4. §2.1 #9 In this exercise, $T : \mathbb{R}^2 \to \mathbb{R}^2$ is a function. For each of the following parts, state why T is *not* linear.

a. §2.1 #9 (b) $T(a_1, a_2) = (a_1, a_1^2)$

b. §2.1 #9 (e) $T(a_1, a_2) = (a_1 + 1, a_2)$

a.

$$T(x_1+cy_1, x_2+cy_2) = (x_1+cy_1, (x_1+cy_1)^2)$$

$$\neq (x_1+cy_1, x_1^2+cy_1^2) = T(x_1,x_2)+cT(y_1,y_2)$$

So not linear

b.

$$T(x_1+cy_1, x_2+cy_2) = (x_1+cy_1+1, x_2+cy_2)$$

$$\neq (x_1+cy_1+2, x_2+cy_2) = T(x_1,x_2)+cT(y_1,y_2)$$

So not linear

5. §2.1 #11 Prove that there exists a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $T(1,1) = (1,0,2)$ and $T(2,3) = (1,-1,4)$. What is $T(8,11)$?

$$(8,11) = a_1(1,1) + a_2(2,3)$$

$$\begin{bmatrix} 1 & 2 & 8 \\ 1 & 3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 8 \\ 0 & 1 & 3 \end{bmatrix} \qquad \begin{cases} a_1 = 2 \\ a_2 = 3 \end{cases}$$

$$T(8,11) = (5,-3,16)$$

6. §2.1 #18 Give an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $N(T) = R(T)$.

$T(x_1, x_2) = (x_1 - x_2, x_1 - x_2)$

$N(T) = (x, x) \quad x \in R$

$R(T) = (x, x) \quad x \in R$

7. §2.2 #2 Let β and γ be the standard ordered bases for $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. For each linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, compute $[T]_\beta^\gamma$.

a. §2.2 #2 (a) $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(a_1, a_2) = (2a_1 - a_2, 3a_1 + 4a_2, a_1)$

b. §2.2 #2 (b) $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(a_1, a_2, a_3) = (2a_1 + 3a_2 - a_3, a_1 + a_3)$

a. $\beta = \{(1,0), (0,1)\} \quad \gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$

$[T(1,0)]_\beta = (2, 3, 1)_\beta \quad [T(0,1)]_\beta = (-1, 4, 0)_\beta$

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & -1 \\ 3 & 4 \\ 1 & 0 \end{bmatrix}$$

b. $\beta = \{(1,0,0), (0,1,0), (0,0,1)\} \quad \gamma = \{(1,0), (0,1)\}$

$[T(1,0,0)]_\beta = (2, 1)_\beta \quad [T(0,1,0)]_\beta = (3, 0)_\beta \quad [T(0,0,1)]_\beta = (-1, 1)_\beta$

$$[T]_\beta^\gamma = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

8. §2.2 #3 Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(a_1, a_2) = (a_1 - a_2, a_1, 2a_1 + a_2)$. Let β be the standard ordered basis for $\mathbb{R}^2$ and $\gamma = \{(1,1,0),(0,1,1),(2,2,3)\}$. Compute $[T]_\beta^\gamma$. If $\alpha = \{(1,2),(2,3)\}$, compute $[T]_\alpha^\gamma$.

$\beta = \{(1,0),\ (0,1)\}$

$[T(1,0)]_\beta = (1,1,2)_\gamma = \begin{bmatrix} -\frac{1}{3} \\ 0 \\ \frac{2}{3} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$[T(0,1)]_\beta = (-1,0,1)_\gamma = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 0 \\ 0 & 1 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

$$[T]_\beta^\gamma = \begin{bmatrix} -\frac{1}{3} & -1 \\ 0 & 1 \\ \frac{2}{3} & 0 \end{bmatrix}$$

$\alpha = \{(1,2),(2,3)\}$

$[T(1,2)]_\alpha = (-1,1,4)_\gamma = \begin{bmatrix} -\frac{7}{3} \\ 2 \\ \frac{2}{3} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 1 \\ 0 & 1 & 3 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix}$$

$[T(2,3)]_\alpha = (-1,2,7)_\gamma = \begin{bmatrix} -\frac{11}{3} \\ 3 \\ \frac{4}{3} \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 1 & 2 & 2 \\ 0 & 1 & 3 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}$$

$$[T]_\alpha^\gamma = \begin{bmatrix} -\frac{7}{3} & -\frac{11}{3} \\ 2 & 3 \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix}$$

9. §2.2 #5 Let

$$\alpha = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

$$\beta = \{1, x, x^2\}$$

and

$$\gamma = \{1\}.$$

a. §2.2 #5 (a) Define $T : M_{2\times 2}(\mathbb{F}) \to M_{2\times 2}(\mathbb{F})$ by $T(A) = A^t$. Compute $[T]_\alpha$.

$$T\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\alpha = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_\alpha = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$T\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_\alpha = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_\alpha = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$[T]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Continued from Question 9.

b. §2.2 #5 (b) Define

$$T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R}) \quad \text{by} \quad T(f(x)) = \begin{pmatrix} f'(0) & 2f(1) \\ 0 & f''(3) \end{pmatrix},$$

where $'$ denotes differentiation. Compute $[T]_\beta^\alpha$.

$$T(1)_\alpha = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} \qquad T(x) = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}_\alpha = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$$

$$T(x^2)_\alpha = \begin{pmatrix} 0 & 2 \\ 0 & 2 \end{pmatrix}_\alpha = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix}$$

$$[T]_\beta^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

c. §2.2 #5 (e) If

$$A = \begin{pmatrix} 1 & -2 \\ 0 & 4 \end{pmatrix},$$

compute $[A]_\alpha$.

$$[A]_\alpha = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 4 \end{bmatrix}$$

Problems not from the textbook exercises.

10. Let $T : V \to W$ be a linear map.

a. Prove that $N(T)$ is a subspace of V.

b. Prove that T is 1-1 iff $N(T) = \{\bar{0}_V\}$.

a. $N(T) = \{ v \in V \mid T(v) = \bar{0}_W \} \subset V$

(a) T is linear so $T(\bar{0}_V) = \bar{0}_W \rightarrow \bar{0}_V \in N(T)$

(b) Let $x_1, x_2 \in N(T)$ i.e. $T(x_1) = T(x_2) = \bar{0}_W$

$T(x_1 + x_2) = T(x_1) + T(x_2) = \bar{0}_W$

$\Rightarrow x_1 + x_2 \in N(T)$

(c) $T(c\, x_1) = c\, T(x_1) = \bar{0}_W \Rightarrow c x_1 \in N(T)$

$\therefore N(T)$ is a subspcace of V

b. "$\Rightarrow$": Since T is 1-1, then if $T(v_1) = T(v_2)$, $\Longrightarrow v_1 = v_2$

assume there is a $v_0 \in V$ such that $T(v_0) = \bar{0}_W$ and $v_0 \neq \bar{0}_V$

$T(v_1 + v_0) = T(v_1) + T(v_0) = T(v_1)$, so $v_1 + v_0 = v_1$

$v_0 = \bar{0}_V$ contradiction

So $N(T) = \bar{0}_V$ if T is 1-1

"$\Leftarrow$": $N(T) = \bar{0}_V$, assume there is $x_1, x_2 \in V$, $T(x_1) = T(x_2)$

but $x_1 - x_2 = x_0 \neq 0$

$T(x_1) = T(x_2) = T(x_1 - x_0) = T(x_1) - T(x_0)$

so $T(x_0) = 0$ which contradicts to $N(T) = \bar{0}_V$

11. For an $m \times n$ matrix A we define the map

$$\begin{aligned} L_A : \mathbb{R}^n &\longrightarrow \mathbb{R}^m \\ \bar{x} &\longmapsto A\bar{x}. \end{aligned}$$

a. Using the definition of the product $A\bar{x}$ prove that the map $\bar{x} \mapsto A\bar{x}$ is linear.

b. Find a basis for $N(L_A)$, where

$$A = \begin{pmatrix} 2 & 3 & 1 & 4 & -9 \\ 1 & 1 & 1 & 1 & -3 \\ 1 & 1 & 1 & 2 & -5 \end{pmatrix}$$

(Hint: $\bar{x} \in N(L_A) \Leftrightarrow A\bar{x} = \bar{0}$ and you can find these solutions.)

a. $L_A: \bar{X} \to A\bar{X}$

$L_A(\bar{X}_1 + c\bar{X}_2) = A(\bar{X}_1 + c\bar{X}_2) = A\bar{X}_1 + cA\bar{X}_2$

$= L_A(\bar{X}_1) + cL_A(\bar{X}_2)$

$\therefore$ $\bar{X} \to A\bar{X}$ is linear.

b. $\left[\begin{array}{ccccc|c} 2 & 3 & 1 & 4 & -9 & 0 \\ 1 & 1 & 1 & 1 & -3 & 0 \\ 1 & 1 & 1 & 2 & -5 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 \end{array}\right]$

$$\begin{cases} X_1 = -2t_1 + 2t_2 \\ X_2 = t_1 - t_2 \\ X_3 = t_1 \\ X_4 = 2t_2 \\ X_5 = t_2 \end{cases}$$

$t_1, t_2 \in \mathbb{R}$

basis: $\{(-2, 1, 1, 0, 0), (2, -1, 0, 2, 1)\}$