Homework 5 MATH 416: ABSTRACT LINEAR ALGEBRA

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(Exercises are taken from *Linear Algebra,* ***Fourth Edition*** by Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence)

Here are Theorems you may want to use.

Theorem 2.11 Let V, W, and Z be finite-dimensional vector spaces with ordered bases α, β, and γ, respectively. Let $T : V \to W$ and $U : W \to Z$ be linear transformations. Then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

Theorem 2.14 Let V and W be finite-dimensional vector spaces having ordered bases β and γ, respectively, and let $T : V \to W$ be linear. Then, for each $u \in V$, we have

$$[T(u)]_\gamma = [T]_\beta^\gamma [u]_\beta.$$

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1. Exercise §2.3 #1 Label the following statements as true or false. In each part, $\overset{\alpha}{V}, \overset{\beta}{W}$, and $\overset{\gamma}{Z}$ are vector spaces with ordered (finite) bases α, β, and γ, respectively; $T : V \to W$ and $U : W \to Z$ denote linear transformations; and A and B denote matrices (Answer is back, give a short explanation!).

(a) $[UT]_\alpha^\gamma = [T]_\alpha^\beta [U]_\beta^\gamma$. F

(b) $[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha$ for all $v \in V$. T

(c) $[U(w)]_{\cancel{\beta}\,\gamma} = [U]_{\alpha}^{\cancel{\beta}\,\gamma}[w]_{\cancel{\beta}\,\alpha}$ for all $w \in W$. F

(d) $[I_V]_\alpha = I$. T

(e) $[T^2]_\alpha^\beta = ([T]_\alpha^\beta)^2$. F $T: V \to W$, no T^2 if ~~$\dim V \neq \dim W$~~ V is not equal to W.

(f) $A^2 = I$ implies that $A = I$ or $A = -I$ F exist $A = A^{-1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

(g) $T = L_A$ for some matrix A. ~~unique~~ F and α, β may not be standard ordered the answer teacher gives: maps have different inputs.

(h) $A^2 = O$ implies that $A = O$, where O denotes the zero matrix. F $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

(i) $L_{A+B} = L_A + L_B$. T

(j) If A is square and $A_{ij} = \delta_{ij}$ for all i and j, then $A = I$. T

(g) if A is an $m \times n$ matrix with entries from field F and if α, β are the standard ordered basis vectors for F^n, F^m, then we know $A = [L_A]_\alpha^\beta$. Consider the example. Let $V = W = \mathbb{R}^2$, $\alpha = \{(1,0),(0,1)\}$, $\beta = \{(0,1),(1,0)\}$ and $A = I_2$, then $[L_{I_2}]_\alpha^\beta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$

2. a. §2.3 #2 (a) Let

$$A = \begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -3 \\ 4 & 1 & 2 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 1 & 4 \\ -1 & -2 & 0 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 2 \\ -2 \\ 3 \end{pmatrix}.$$

Compute $A(2B+3C), (AB)D$, and $A(BD)$.

b. §2.3 #2 (b) Let

$$A = \begin{pmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 4 & 0 & 3 \end{pmatrix}.$$

Compute A^t, A^tB, BC^t, CB, and CA.

(a) $A(2B+3C) = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 5 & 3 & 6 \\ 5 & -4 & 4 \end{bmatrix}$

$= \begin{bmatrix} 20 & -9 & 18 \\ 5 & 10 & 8 \end{bmatrix}$

$(AB)D = \begin{bmatrix} 13 & 3 & 3 \\ -2 & -1 & -8 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 29 \\ -26 \end{bmatrix} = A(BD)$

(b) $A^t = \begin{bmatrix} 2 & -3 & 4 \\ 5 & 1 & 2 \end{bmatrix}$, $A^tB = \begin{bmatrix} 23 & 19 & 0 \\ 26 & -1 & 10 \end{bmatrix}$

$BC^t = \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 12 \\ 16 \\ 29 \end{bmatrix}$

$CB = \begin{bmatrix} 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & 0 \\ 1 & -1 & 4 \\ 5 & 5 & 3 \end{bmatrix} = \begin{bmatrix} 27 & 7 & 9 \end{bmatrix}$

Continued from the previous question. Use the following blank page to write your solutions.

$$CA = \begin{bmatrix} 4 & 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & 1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 20 & 26 \end{bmatrix}$$

3. §2.3 #3 Let $g(x) = 3 + x$. Let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ and $U : P_2(\mathbb{R}) \to \mathbb{R}^3$ be the linear transformations respectively defined by

$$T(f(x)) = f'(x)g(x) + 2f(x) \quad \text{and} \quad U(a + bx + cx^2) = (a + b, c, a - b).$$

Let β and γ be the standard ordered bases of $P_2(\mathbb{R})$ and $\mathbb{R}^3$, respectively.

$\beta = \{1, x, x^2\} \qquad \gamma = \{(1,0,0), (0,1,0), (0,0,1)\}$

a. §2.3 #3 (a) Compute $[U]_\beta^\gamma$, $[T]_\beta$, $[UT]_\beta^\gamma$ directly. Then use Theorem 2.11 to verify your result.

b. §2.3 #3 (b) Let $h(x) = 3 - 2x + x^2$. Compute $[h(x)]_\beta$ and $[U(h(x))]_\gamma$. Then use $[U]_\beta^\gamma$ from **a** and Theorem 2.14 to verify your result.

a. $[U]_\beta^\gamma = \left[[U(1)]_\gamma \; [U(x)]_\gamma \; [U(x^2)]_\gamma \right] = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}$

$[T]_\beta^\beta = [\, 2 \quad 3+3x \quad 4x^2+6x \,]_\beta = \begin{bmatrix} 2 & 3 & 0 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$

$[UT]_\beta^\gamma = \left[[U(T(1))]_\gamma \; [U(T(x))]_\gamma \; [U(T(x^2))]_\gamma \right]$

$= \left[[U(2)]_\gamma \; [U(3+3x)]_\gamma \; [U(4x^2+6x)]_\gamma \right]$

$= \begin{bmatrix} 2 & 6 & 6 \\ 0 & 0 & 4 \\ 2 & 0 & -6 \end{bmatrix}$

$[UT]_\beta^\gamma = [U]_\beta^\gamma \, [T]_\beta^\beta$

b. $[h(x)]_\beta = \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}$

$[U(h(x))]_\gamma = [1, 1, 5]_\gamma = \begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$

$[U(h(x))]_\gamma = [U]_\beta^\gamma \, [h(x)]_\beta$

4. §2.3 #4 For each of the following parts, let T be the linear transformation defined in the corresponding part of Exercise 5 of Section 2.2 (See HW#4 sheet). Use Theorem 2.14 to compute the following vectors.

a. §2.3 #4 (a) $[T(A)]_\alpha$, where $A = \begin{pmatrix} 1 & 4 \\ -1 & 6 \end{pmatrix}$.

b. §2.3 #4 (b) $[T(f(x))]_\alpha$, where $f(x) = 4 - 6x + 3x^2$.

a. $[A]_\alpha = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 6 \end{bmatrix}$ $[T]_\alpha^\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$[T(A)]_\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ -1 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \\ 6 \end{bmatrix}$$

b. $[f(x)]_\beta = \begin{bmatrix} 4 \\ -6 \\ 3 \end{bmatrix}$ $[T]_\beta^\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$$[T(f(x))]_\alpha = \begin{bmatrix} 0 & 1 & 0 \\ 2 & 2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -6 \\ 3 \end{bmatrix} = \begin{bmatrix} -6 \\ 2 \\ 0 \\ 6 \end{bmatrix}$$

5. §2.3 #12 Let V, W, and Z be vector spaces, and let $T : V \to W$ and $U : W \to Z$ be linear.

a. §2.3 #12 (a) Prove that if UT is one-to-one, then T is one-to-one. Must U also be one-to-one?

b. §2.3 #12 (b) Prove that if UT is onto, then U is onto. Must T also be onto?

c. §2.3 #12 (c) Prove that if U and T are one-to-one and onto, then UT is also.

No

$UT : V \to Z$

a. UT is 1 to 1 $\Rightarrow N(UT) = \{\bar{0}\}$

assume T is not 1 to 1. i.e. $\exists\ x \neq \bar{0}$ $T(x) = \bar{0}$

$(UT)(x) = U(T(x)) = U(\bar{0}) \overset{U \text{ is linear}}{=} \bar{0}$ which contradicts to $N(UT) = \{\bar{0}\}$

So T is 1 to 1.

and U doesn't need to be 1 to 1.

b. No

UT is onto $\Rightarrow R(UT) = Z$

$\forall z \in Z\ \exists\ v \in V\ (UT)(v) = z = U(T(v))$

So $\exists\ w = T(v) \in W\ U(w) = z$ thus U is onto

and T doesn't need to be 1 to 1.

c. 1 to 1: $N(U) = \{\bar{0}\}, N(T) = \{\bar{0}\}$

$(UT)(x) = \cancel{\{}\bar{0}\cancel{\}} \Rightarrow U(T(x)) = \{\bar{0}\} \Rightarrow T(x) = \bar{0}$

$\Rightarrow x = \bar{0}$

$\therefore N(UT) = \{\bar{0}\}$ i.e. UT is 1 to 1.

onto: $\forall w \in W\ \exists\ v \in V\ T(v) = w$

Since $\forall z \in Z\ \exists\ w \in W\ U(w) = U(T(v)) = (UT)(v) = z$

$\therefore \forall z \in Z\ \exists\ v \in V\ (UT)(v) = z$

6. §2.3 #18 Using only the definition of matrix multiplication, prove that multiplication of matrices is associative.

$A = [a_1 \cdots a_n]$ $B = [b_1 \cdots b_m]$ $C = [c_1 \cdots c_k]$.

$(AB)C = (Ab_1 + \cdots Ab_m)[c_1 \cdots c_k]$.

$= \underbrace{[Ab_1 \cdots Ab_m]}_{D}[c_1 \cdots c_k]$.

$= [Dc_1 \cdots Dc_k]$.

$A(BC) = A\,[Bc_1 \cdots Bc_k]$.

$= [ABc_1 \cdots ABc_k]$.

$AB = [Ab_1 \cdots Ab_m] = D$.

$\therefore (AB)C = A(BC)$

7. §2.4 #1 Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $T : V \to W$ is linear and A and B denote matrices.

~~A invertible~~

(a) $([T]_\alpha^\beta)^{-1} = [T^{-1}]_\alpha^\beta$. α β F

(b) T is invertible if and only if T is one-to-one and onto. T

same as 1.(g).

(c) $T = L_A$, where $A = [T]_\alpha^\beta$. F α, β may not be standard ordered

(d) $M_{2\times 3}(\mathbb{F})$ is isomorphic to $\mathbb{F}^5$. F $\dim M_{2\times 3}(\mathbb{F}) \neq \dim \mathbb{F}^5$

(e) $P_n(\mathbb{F})$ is isomorphic to $P_m(\mathbb{F})$ if and only if $n = m$. T

(f) $AB = I$ implies that A and B are invertible. F still need $BA = I$

$A = (0, 1)$ $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

(g) If A is invertible, then $(A^{-1})^{-1} = A$. T

(h) A is invertible if and only if L_A is invertible. T

(i) A must be square in order to possess an inverse. T

8. §2.4 #2 For each of the following linear transformations T, determine whether T is invertible and justify your answer.

a. §2.4 #2 (d) $T : P_3(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T(p(x)) = p'(x)$

b. §2.4 #2 (e) $T : M_{2\times 2}(\mathbb{R}) \to P_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c + d)x^2$

(d) $\dim P_3(\mathbb{R}) = 4 \neq 3 = \dim P_2(\mathbb{R})$ so No

(e) $\dim M_{2\times 2}(\mathbb{R}) = 4 \neq 3 = \dim P_2(\mathbb{R})$ so No

9. §2.4 #3 Which of the following pairs of vector spaces are isomorphic? Justify your answers.

(a) $\mathbb{F}^3$ and $P_3(\mathbb{F})$

(b) $\mathbb{F}^4$ and $P_3(\mathbb{F})$

(c) $M_{2\times 2}(\mathbb{R})$ and $P_3(\mathbb{R})$

(d) $V = \{A \in M_{2\times 2}(\mathbb{R}) : \text{tr}(A) = 0\}$ and $\mathbb{R}^4$

(a). No $\dim \mathbb{F}^3 = 3 \neq 4 = \dim P_3(\mathbb{F})$

(b) Yes $\dim \mathbb{F}^4 = 4 = \dim P_3(\mathbb{F})$

(c) Yes $\dim M_{2\times 2}(\mathbb{R}) = 4 = \dim P_3(\mathbb{R})$

(d) No $V = \left\{ \begin{matrix} a & b \\ c & -a \end{matrix} \right\}$

$\dim V = 3 \neq 4 = \dim \mathbb{R}^4$

Problems not from the textbook exercises.

10. Let A, B be $n \times n$ matrices.

a. Prove that if A and B are invertible, then so is AB and $(AB)^{-1} = B^{-1}A^{-1}$.

b. Prove that if AB is invertible then so are A and B.

c. Prove that if A is invertible then so is A^t and $(A^t)^{-1} = (A^{-1})^t$.

a. A and B are invertible.

$\Rightarrow$ exists A^{-1}, B^{-1} such that $AA^{-1} = BB^{-1} = I$
$= A^{-1}A = B^{-1}B$

So $AB \cdot B^{-1}A^{-1} = I = B^{-1}A^{-1} \cdot AB$

thus AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

b. AB is invertible

$\Rightarrow$ exists $(AB)^{-1}$ such that $AB(AB)^{-1} = (AB)^{-1}AB$
$= I$

$A\left(B(AB)^{-1}\right) = (AB)(AB)^{-1}$
$= (AB)^{-1}(AB) = \left((AB)^{-1}A\right)B = I$

Let $(B(AB)^{-1})A = K$

$A(B(AB)^{-1})A = AK$

$A = AK$

$\Rightarrow K = I$

$\therefore (B(AB)^{-1})A = I$
$= B\left((AB)^{-1}A\right)$

So A, B are invertible

$A^{-1} = B(AB)^{-1}$

$B^{-1} = (AB)^{-1}A$

c. $A^t(A^{-1})^t = (\cancel{AA^{-1}})^t = I^t = I = (A^{-1})^tA^t$
$= (\cancel{A^{-1}A})^t$

So A^t is invertible AA^{-1}

and $(A^t)^{-1} = (A^{-1})^t$