Linear Dependence and Linear independence.

Def^n A subset $\{u_1, \cdots, u_k\} \subset V$ is <u>linearly dependent</u> if there are scalars $a_1 \cdots a_k$ not all zero such that $a_1u_1 + \cdots + a_ku_k = \bar{0}$

(not linearly dependent = linearly independent.)

Example

If $\{u_1, u_2, u_3\} = \{(-1,1,2),(1,2,1),(5,1,-4)\}$ is linearly dependent?

$$a_1u_1 + a_2u_2 + a_3u_3 = \bar{0}$$

$$\begin{cases} -a_1 + a_2 + 5a_3 = 0 \\ a_1 + 2a_2 + a_3 = 0 \\ 2a_1 + a_2 - 4a_3 = 0 \end{cases} \Rightarrow \begin{cases} a_1 = 0 \\ a_2 = 0 \\ a_3 = 0 \end{cases}$$

$\therefore \{u_1, u_2, u_3\}$ is linearly independent.

Example. $\{\sin x, \cos x\} \subset \mathcal{F}(\mathbb{R})$ is linearly independent.

there is no $a \neq 0$ such that $\sin x = a\cos x$.

Theorem if $\{u_1, \cdots, u_k\} \subset V$ is linearly dependent, then one u_j can be expressed as linear combination of the others.

Proof: linearly dependent

$\Rightarrow \exists\ a_1, \cdots, a_k \in \mathbb{R}$ not all zero, $a_1u_1 + \cdots + a_ku_k = \bar{0}$

Assume $a_j \neq 0$.

$\Rightarrow \frac{1}{a_j}(a_1u_1 + \cdots + a_ku_k) = \frac{1}{a_j}\bar{0}$

$\Rightarrow u_j = -\frac{a_1}{a_j}u_1 - \cdots - \frac{a_k}{a_j}u_k$

Theorem Suppose $\{u_1 \dots u_k\} \subset V$ is linearly dependent. There is a subset $\{u_{i_1}, \dots u_{i_l}\}$ which is linearly independent and satisfies $\mathrm{Span}(\{u_1 \dots u_k\}) = \mathrm{Span}(\{u_{i_1}, \cdots, u_{i_l}\})$.

Proof: Just delete the u_j in $\{u_1, \cdots u_k\}$ which can be expressed as linear combination of the others.

$\Rightarrow$ get $\{u_{i_1}, \cdots, u_{i_l}\}$.

Def^n $\beta \subset V$ is a basis of V if

(i) β is linearly independent.

(ii) $\mathrm{Span}(\beta) = V$. 用 $\forall v \in V$, $v \in \mathrm{Span}(\beta)$ 证明

(every vector space has a basis).

Theorem if $\beta \subset V$ is a basis then every $u \in V$ can be expressed in a unique way as an element of $\mathrm{Span}(\beta)$.

Proof: Assume not: u can be expressed in two ways.

① $u = a_1u_1 + a_2u_2 + \cdots + a_ku_k$

② $u = b_1u_1 + b_2u_2 + \cdots + b_ku_k + b_{k+1}u_{k+1} + \cdots + b_lu_l$.

$(u_i \in \beta,\ i = 1, 2, \cdots, l)$.

Hence $\bar{0} = u - u$

$$= (a_1-b_1)u_1 + (a_2-b_2)u_2 + \cdots + (a_k-b_k)u_k - b_{k+1}u_{k+1} - \cdots - b_lu_l$$

Since β is linearly independent, all the coefficients are 0.

So $a_1 = b_1,\ a_2 = b_2,\ \cdots,\ a_k = b_k,\ b_{k+1} = b_{k+2} = \cdots = b_l = 0$.

Example: ① P_n: $\beta = \{1, x, x^2 \cdots x^n\}$

② $V = \{ \underbrace{\{a_n\}}_{\text{sequence}} \}$ is a vector space.

Let e_j be the sequence $0, 0, \cdots, 0, \underset{j^{th}\text{ term}}{1}, 0, \cdots$

$\beta = \{ e_1, e_2, \cdots \}$

③ $P = \{$ polynomials of all degrees $\}$

$\beta = \{ 1, x, x^2, x^3, \cdots \}$

④ $\mathcal{F}(\mathbb{R})$ has basis

(if $\mathrm{Span}\{S\} = V$, then S is the generating set of V.)

Theorem If V has a finite generating set, then V has a finite basis.

Theorem if V has a finite basis, then any basis of V has the same number of elements.

Theorem (Replacement Theorem) Suppose $S = \{ s_1, \cdots, s_n \}$ generates V. 假设

: if $\mathcal{U} = \{ u_1, \cdots, u_k \}$ is linearly independent, 条件

(then ① $k \le n$ and ② there is a subset $T \subset S$ of size $n-k$ such that $\mathrm{Span}(\mathcal{U} \cup T) = V$.)

Proof (induction)

Base case: $k=0$, i.e. $\mathcal{U}=\phi$. take $T=S$

Inductive step: Assume true for j. need to show true for $j+1$.

$\mathcal{U}_j=\{u_1,\dots,u_j\}$ is linearly independent.

there is a $T_j=\{s_1,\dots,s_{n-j}\}$ such that $\mathrm{Span}(\{u_1,\dots,u_j,s_1,\dots,s_{n-j}\})$

$=V\ni u_{j+1}$ ($\mathcal{U}_{j+1}=\{u_1,\dots,u_j,u_{j+1}\}$ is linearly independent)

$u_{j+1}=a_1u_1+a_2u_2\cdots+a_ju_j+b_1s_1+\dots+b_{n-j}s_{n-j}$

($b_1,\dots b_{n-j}$ are not all zero)

Assume $b_{n-j}\neq 0$ take $T_{j+1}=\{s_1,\dots,s_{n-j-1}\}$

$\mathrm{Span}(\{u_1,\dots u_j,u_{j+1},s_1,\dots,s_{n-j-1}\})=V$

Theorem if V has a finite basis, then any basis of V has the same number of elements.

Proof: Let β be a finite basis with n elements

Assume $\tilde{\beta}$ be another basis

①. Prove $\tilde{\beta}$ is finite: assume not, $\tilde{\beta}$ contains a set $\tilde{U}$ of $n+1$ linearly independent vectors.

Apply Replacement Theorem ($S=\beta$, $\mathcal{U}=\tilde{U}$) $\Rightarrow n+1 \leq n$,

so contradiction.

② Prove size of $\tilde{\beta}=n$.

Replacement Theorem ($S=\beta$, $\mathcal{U}=\tilde{\beta}$) $\Rightarrow$ size of $\tilde{\beta} \leq n$

($S=\tilde{\beta}$, $\mathcal{U}=\beta$) $\Rightarrow$ size of $\tilde{\beta} \geq n$.

$\Rightarrow$ size of $\tilde{\beta}=n$.