Math 416 Practice Final

1. Solve

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & + & & & x_4 & = & 2 \\ 2x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 1 \\ x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & -1 \\ x_1 & + & 2x_2 & + & 3x_3 & + & 4x_4 & = & 3. \end{array}$$

$$\left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 2 & -1 & 1 & 3 & 1 \\ 1 & -2 & 1 & 2 & -1 \\ 1 & 2 & 3 & 4 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cccc:c} 1 & 1 & 0 & 1 & 2 \\ 0 & -3 & 1 & 1 & -3 \\ 0 & -3 & 1 & 1 & -3 \\ 0 & 1 & 3 & 3 & 1 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{ccccc} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & 3 & 3 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 = 1-t \\ x_2 = 1 \\ x_3 = -t \\ x_4 = t \end{cases}$$

2. Consider the linear map $T : P_2(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by

$$T(a + bx + cx^2) = \begin{pmatrix} a & c \\ b-a & a \end{pmatrix}.$$

Let $\alpha = \{1 + x, x + x^2, 1 + x + x^2\}$ and $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ be the bases for $P_2(\mathbb{R}), M_{2\times 2}(\mathbb{R})$, respectively, where $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$.

(1) Compute the coordinate vector $[f]_\alpha$ for $f(x) = 1 - x + 2x^2$.

(2) Compute the matrix representation $[T]_\alpha^\beta$ with respect to α, β.

(3) Is T injective?

(1) $[f]_\alpha =$

$$\begin{cases} x_1 + \qquad x_3 = 1 \\ x_1 + x_2 + x_3 = -1 \\ \qquad x_2 + x_3 = 2 \end{cases} \Rightarrow \begin{cases} x_1 = -3 \\ x_2 = -2 \\ x_3 = 4 \end{cases}$$

$$[f]_\alpha = (-3, -2, 4)$$

(2) $$[T]_\alpha^\beta = \left[\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)_\beta \left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)_\beta \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right)_\beta \right]$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

(3) $$T(a+bx+cx^2) = O_{2\times 2} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

$\Rightarrow N(T) = \{\bar{0}\}$

$\Rightarrow T$ is injective.

3. Consider a 3×4 matrix $A = \begin{pmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ -2 & -2 & 1 & 1 \end{pmatrix}$. Let $T = L_A : \mathbb{R}^4 \to \mathbb{R}^3$.

(1) Find a basis β of the null space $N(T)$ so that $N(T) = \text{span}(\beta)$.

(2) Find a basis γ of the range $R(T)$ so that $R(T) = \text{span}(\gamma)$.

(3) Verify the dimension theorem (the rank-nullity theorem).

(1)

$$T(x) = \begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 1 & 1 & 0 \\ -2 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 0 & 4 & -5 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -\frac{3}{4}t_1 - \frac{1}{4}t_2 \\ x_2 = \frac{5}{4}t_1 + \frac{3}{4}t_2 \\ x_3 = t_1 \\ x_4 = t_2 \end{cases}$$

$$\beta = \{(-\tfrac{3}{4}, \tfrac{5}{4}, 1, 0), (-\tfrac{1}{4}, \tfrac{3}{4}, 0, 1)\}.$$

(2) $\gamma = \{(2,1,1), (1,0,1)\}.$

(3) $\dim N(T) = 2 \quad \dim R(T) = 2$

$$\dim N(T) + \dim R(T) = 4 = \dim \mathbb{R}^4$$

4.

(1) Compute the determinant of $\begin{pmatrix} 3 & 2 & -1 & 0 \\ 2 & -1 & 0 & 1 \\ 2 & -2 & 0 & 2 \\ 1 & 0 & -1 & 1 \end{pmatrix}$ and verify that $\det(A) \neq 0$.

(2) Compute the inverse of A.

(1).

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 2 & -1 \\ \cdots \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 & 1 \\ 0 & -1 & 2 & -1 \\ \cdots \end{vmatrix} = -2. \neq 0.$$

$$\begin{vmatrix} 0 & 0 & -2 & 2 \\ 1 & 0 & -1 & 1 \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

(2). $[A \,\vdots\, I] = \left[\begin{array}{cccc:cccc} 3 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 2 & -2 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$

$$\to \begin{bmatrix} 0 & 2 & 2 & -3 & 1 & 0 & 0 & -3 \\ 0 & -1 & 2 & -1 & 0 & 1 & 0 & -2 \\ 0 & -2 & 2 & 0 & 0 & 0 & 1 & -2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 6 & -5 & 1 & 2 & 0 & -7 \\ 0 & -1 & 2 & -1 & 0 & 1 & 0 & -2 \\ 0 & 0 & -2 & 2 & 0 & -2 & 1 & 2 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & -4 & 3 & -1 \\ 0 & 1 & 0 & 0 & 1 & -3 & \frac{7}{2} & -1 \\ 0 & 0 & 1 & 0 & 1 & -3 & \frac{5}{2} & -2 \\ 1 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \end{bmatrix}$$

$$\to \left[\begin{array}{cccc:cccc} 1 & 0 & 0 & 0 & 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 1 & -3 & \frac{7}{2} & -1 \\ 0 & 0 & 1 & 0 & 1 & -3 & \frac{5}{2} & -2 \\ 0 & 0 & 0 & 1 & 1 & -4 & 3 & -1 \end{array}\right]$$

5. **True** / **False**. Justify your answer.

(a) Every square matrix over the complex numbers $\mathbb{C}$ has a Jordan Form.

(b) Every nonzero finite dimensional inner product space has an orthonormal basis.

(c) If S is an orthogonal set of nonzero vectors, then S is linearly independent.

(d) A linear operator T on a finite dimensional inner product space V over $\mathbb{R}$ is self-adjoint if and only if there exists an orthonormal basis for V consisting of eigenvectors of T.

(e) Every symmetric matrix over $\mathbb{R}$ is diagonalizable.

(f) Every symmetric matrix over $\mathbb{C}$ is diagonalizable. $\begin{bmatrix} i & 1 \\ 1 & -i \end{bmatrix}$ $(i-t)(-i-t)-1$. $t^2=0$. $t=0$.

(g) If a linear operator T on a finite dimensional inner product space V over $\mathbb{R}$ is normal, then V admits an orthonormal basis which consists of the eigenvectors of T. $T=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

(h) If a linear operator T on a finite dimensional inner product space V is self-adjoint, then it is normal.

(i) If T is a linear operator on a finite dimensional vector space V whose characteristic polynomial splits, then the dimension of the generalized eigenspace K_λ corresponding to an eigenvalue λ is equal to the algebraic multiplicity of λ.

(a) Every square matrix. A over the complex numbers

$\Longleftrightarrow$ det A splits. $\Longrightarrow$ A has a Jordan Form.

(b) Use Gram-Schmidt Process.

(c) Assume not $S_1 = a_2S_2 + \cdots + a_nS_n$, $a_i \neq 0$.

$\langle S_1, S_i \rangle = a_i \| S_i \|^2 \neq 0$

(d) self-adjoint $\Longleftrightarrow$ matrix is symmetric in $\mathbb{R}$ $\Longleftrightarrow$

matrix have a othonormal basis for $\mathbb{R}^n$ consisting of eigenvectors.