Groups

Hw3	2-4
Hw4	2-3
Hw5	3-6
Hw6	1-4
Hw7	1-3

Modular arithmetic, Zn

Hw3	1
Hw4	1, 4
Hw5	2
Hw8	1-2, 4

Rings

Hw7	4
Hw8	3
Hw10	3-4
Hw11	1-3

Group actions

Hw9	3-4
Hw10	1-2

A selection of practice exercises from Fraleigh.

4 : 3, 6, 24, 30, 37
5 : 10, 12, 23
6 : 15, 16, 18, 28, 46
8 : 5, 17, 36
9 : 11, 15, 27ab, 31, 36
10 : 4, 15, 22, 24, 32
11 : 2, 5, 8, 11, 18

13: 3, 10, 19, 20, 28, 49
14: 5, 14, 27, 34
15: 5, 14, 29, 30, 31

More (some overlap)

13 : 3, 4, 18, 19, 37, 38
14 : 5, 10, 15, 39
15 : 5, 6, 8, 15
18 : 9, 18, 25, 40, 44, 46
19 : 14, 18, 23
20 : 8, 9, 10, 15, 18
21 : -
22 : 6, 17, 24, 25

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 3
Due Friday February 4

---

1. Determine (in a systematic way) the smallest integer x with the property that $x \equiv 4 \pmod 7$ and $x \equiv 9 \pmod{17}$.

2. Write the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 7 & 6 & 4 & 8 & 5 & 3 & 9 & 2 & 1 \end{pmatrix}.$$

as a product of disjoint cycles. Determine the smallest positive integer n such that the permutation $\sigma^n = \sigma \circ \sigma \circ \cdots \circ \sigma$ (n times) is the identity.

3. Find a permutation σ of $X = \{1, 2, 3, 4, 5, 6\}$ such that

$$\sigma \circ (1\,2\,6\,5) = (2\,3\,5\,4) \circ \sigma.$$

4. Determine the sign of each of the permutations

$$\begin{aligned} \sigma &= (5\,1\,4)(5\,2\,3\,1)(4\,3\,2) \\ \tau &= (5\,1)(4\,5\,2)(3\,1\,4)(3\,2) \end{aligned}$$

Conclude that $\sigma \neq \tau$. Provide two witnesses i and j such that $\sigma(i) \neq \tau(i)$ and $\sigma(j) \neq \tau(j)$ to verify your conclusion.

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 4
Due Friday February 11

---

1. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ be the set of remainders modulo 5. Label the vertices of a regular pentagon clock-wise with $0, 1, 2, 3, 4$.

   a) Describe the permutation $\rho_3 : \mathbb{Z}_5 \to \mathbb{Z}_5$ with $\rho(x) = x + 3$ geometrically as a transformation of the pentagon and give its disjoint cycle form as a permutation of the set $\mathbb{Z}_5$.

   b) Describe the permutation $\sigma_1 : \mathbb{Z}_5 \to \mathbb{Z}_5$ with $\sigma(x) = 2 - x$ geometrically as a transformation of the pentagon and give its disjoint cycle form as a permutation of the set $\mathbb{Z}_5$.

   c) Show that the permutations $\{x \mapsto x + b : b \in \mathbb{Z}_5\} \cup \{x \mapsto 2a - x : a \in \mathbb{Z}_5\}$ form a group of size 10.

2. Fraleigh Section 5 Problem 47: Prove that if G is an abelian group, written multiplicatively, with identity element e, then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G.

3. Fraleigh Section 5 Problem 51: Let G be a group and let a be one fixed element of G. Show that $H_a = \{x \in G \mid xa = ax\}$ is a subgroup of G.

4. Let $M_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$. Then M_n is a group under multiplication (the multiplicative operation is associative, 1 is an identity element, and $1 = sa + tn$ shows that a has an inverse s).

   a) For a prime p what is the size of the group M_{p^2}? Is M_9 a cyclic group?

   b) For distinct primes p and q what is the size of the group M_{pq}? Is M_{15} a cyclic group?

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 5
Due Friday February 18

---

1. Fraleigh Section 5 Problem 39: Answer the ten True or False questions.
2. Determine all elements $a \in \mathbb{Z}_{24}$ such that $\langle a \rangle \leq \mathbb{Z}_{24}$ is a subgroup of order 8.
3. The regular 9-gon with vertices 1 to 9 has 18 symmetries that include $\rho = (1\,2\,3\,4\,5\,6\,7\,8\,9)$ and $\sigma = (1\,2)(3\,9)(4\,8)(5\,7)$. Give the disjoint cycle form for the permutations $\sigma\rho, \sigma\rho^2$ and $\rho\sigma, \rho^2\sigma$.
4. Recall that an isomorphism $\phi : G \to G'$ between groups G and G' is a bijection such that $\phi(xy) = \phi(x)\phi(y)$. Let G be a group and, for $g \in G$, let $\phi_g : G \to G$ be the map $\phi_g(x) = gxg^{-1}$.
   a) Show that ϕ_g is a bijection, for all $g \in G$.
   b) Show that ϕ_g is an isomorphism, for all $g \in G$.
5. Let S_6 be the group of permutations of $\{1, 2, 3, 4, 5, 6\}$. [1]
   a) Show that there are as many permutations in S_6 with cycle structure $(1\,2)$ as there are permutations with cycle structure $(1\,2)(3\,4)(5\,6)$.
   b) Show that there are as many permutations in S_6 with cycle structure $(1\,2\,3\,4\,5\,6)$ as there are permutations with cycle structure $(1\,2\,3)(4\,5)$.
6. Find the maximal possible order for an element in S_9.

---

[1] In general, an isomorphism $\phi : S_n \to S_n$ is of the form ϕ_g as defined in the previous problem. The group S_6 is an exception. The map that sends $(1\,2) \mapsto (1\,2)(3\,4)(5\,6)$ and $(1\,2\,3\,4\,5\,6) \mapsto (1\,2\,3)(4\,5)$ gives an isomorphism that does not preserve the cycle structure.

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 6
Due Friday March 4

---

1. Give an example of a group G, a subgroup $H \leq G$, and elements $a, b \in G$ such that $aH \cap bH = \emptyset$ but $Ha \cap Hb \neq \emptyset$.

2. Let H_1 be the group of permutations of $\{1, 2, 3\}$ and let H_2 be the group of permutations of $\{4, 5, 6\}$. Let $H = H_1 \times H_2$ be the direct product. So that $H \leq S_6$ is a subgroup of permutations of $\{1, 2, 3, 4, 5, 6\}$.

   (a) Find the index $[S_6 : H]$.

   (b) Let $A_1 \neq A_2$ be two different 3-subsets of $\{1, 2, 3, 4, 5, 6\}$ and let $\sigma_1, \sigma_2 \in S_6$ be such that $\sigma_1(\{1, 2, 3\}) = A_1$ and $\sigma_2(\{1, 2, 3\}) = A_2$. Show that $\sigma_1 H \cap \sigma_2 H = \emptyset$.

   c) How many 3-subsets are there in $\{1, 2, 3, 4, 5, 6\}$? Explain the relation between your answers for part a) and part b).

3. Prove that the only homomorphism $\phi : S_3 \to \mathbb{Z}_3$ is the zero map.

4. The subgroup $H = \{(1), (12)(34), (13)(24), (14)(23)\} \leq S_4$ contains the identity, all three permutations with cycle structure $(ab)(cd)$, and no other permutations. Explain how this implies directly that H is a normal subgroup, i.e. that $\sigma H = H\sigma$ or $\sigma H \sigma^{-1} = H$ for all $\sigma \in S_4$.

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 7
Due Friday March 11

---

1. Let $G = D_{12} = \{1, \rho, \dots, \rho^5, \sigma, \sigma\rho, \dots, \sigma\rho^5\}$ be the group of symmetries of a regular hexagon and let $H = [G, G] = \{1, \rho^2, \rho^4\}$ be its commutator subgroup.

   a) Give the partition of G into left cosets of H

   b) Determine the structure of the group G/H as abelian group.

2. Determine the structure as abelian group for the following factor groups.

   a) $G = (\mathbb{Z}_8 \times \mathbb{Z}_{12})/\langle (2,2) \rangle$.

   b) $G = (\mathbb{Z}_8 \times \mathbb{Z}_{12})/\langle (3,3) \rangle$.

3. Let $N \lhd G$ be a normal subgroup. Show that the commutator $[a, b] \in N$ for all $a \in N$ and $b \in G$.

4. Show that the four binary matrices

$$O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

   form a ring under matrix addition and multiplication. Decide if the ring is a field.

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 8
Due Friday March 25

---

1. Find all solutions for each of the following congruences.

   a) $6x \equiv 3 \pmod{12}$

   b) $3x \equiv 6 \pmod{12}$

2. Find the values of $x^6 \pmod{91}$ as x ranges over all integers. Explain your answer.

3. (Fraleigh Exercise 19.26) Let R be a ring that contains at least two elements. Suppose for each nonzero $a \in R$, there exists a unique $b \in R$ such that $aba = a$.

   a) Show that R has no zero divisors.

   b) Show that $bab = b$.

   c) Show that R has unity.

   d) Show that R is a division ring.

4. For each of the given polynomials find all zeros in the indicated ring.

   a) $x^3 + 7x + 4$ in $\mathbb{Z}_{13}$.

   b) $(x + 3)(x + 1)(x - 1)$ in $\mathbb{Z}_{15}$.

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 9
Due Friday April 8

---

1. Find $a, b, c, u, v, w \in \mathbb{Z}_7$ such that
$$(x^3 + 3x^2 + bx + 1)(x^3 + ax^2 + 3x + c) = x^6 + ux^5 + 2x^4 + vx^3 + 2x^2 + wx + 2$$
as polynomials in $\mathbb{Z}_7[x]$

2. Determine the number of irreducible polynomials $x^2+bx+c \in \mathbb{Z}_5[x]$. A quadratic polynomial in $\mathbb{Z}_5[x]$ is irreducible if it can not be written as a product of two linear polynomials in $\mathbb{Z}_5[x]$.

3. Let X be the set of bracelets with 6 red beads and 6 blue beads. Find five different bracelets with nontrivial isotropy subgroup. A bracelet has nontrivial isotropy subgroup if some nontrivial rotation of it has the same colors in the same positions.

4. Let $G = S_6$ act on itself by conjugation. The transpositions $\{(12), \ldots, (56)\}$ form a single orbit under conjugation. Let $H = \{g \in G : g(12)g^{-1} = (12)\}$ be the stabilizer of (12).

   a) Give the number of transpositions.

   b) Use part a) to determine the size of H.

   c) Describe the group H as a subgroup of S_6. (Hint: look for a description as a direct product of two subgroups)

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 10
Due Friday April 15

---

1. The cyclic group of order 12 acts on $\{1, 2, \ldots, 12\}$ with the following cycle structure.

$$\begin{array}{cc} (1) & 1\times \\ (**)(**)(**)(**)(**)(**) & 1\times \\ (***)(***)(***)(***) & 2\times \\ (****)(****)(****) & 2\times \\ (******)(******) & 2\times \\ (************) & 4\times \end{array}$$

Determine the number of different colored necklaces with 12 beads when each bead is colored with one of 3 colors.

2. The symmetry group of the dodecahedron (the platonic solid with 12 regular pentagons as faces) is the group A_5. The 60 symmetries divide into the identity, 24 rotations with axis of rotation through the midpoint of two opposite faces, 20 rotations with axis of rotation through a pair of opposite vertices, and 15 rotations with axis of rotation through the midpoints of two opposite edges. The cycle structures for the rotations acting on the 12 faces are

$$\begin{array}{cc} (1) & 1\times \\ (**)(**)(**)(**)(**)(**) & 15\times \\ (***)(***)(***)(***) & 20\times \\ (*****)(*****) & 24\times \end{array}$$

Determine the number of different colored dodecahedrons if each face is colored with one of 3 colors.

3. Fraleigh 26 Exercises 13-14.

a) Give an example to show that a factor ring of an integral domain may have zero divisors.

b) Give an example to show that a factor ring of a ring with zero divisors may be an integral domain.

4. Fraleigh 26 Exercise 30. An element a of a ring R is nilpotent if $a^n = 0$ for some integer $n > 0$. Show that the collection of all nilpotent elements in a commutative ring R is an ideal (called the nilradical of R).

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 11 (problem 4 is practice problem for midterm 3, not part of homework)
Due Friday April 22

---

1. Let R be the ring of 2×2 matrices over the integers. Let $S = \{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\} \subset R$ be the subring of upper triangular matrices.

   a) Is S an R-ideal?

   b) Let $I = \{\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\} \subset R$. Is I an R-ideal? Is I an S-ideal?

   c) Let $J = \{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\} \subset R$. Is J an R-ideal? Is J an S-ideal?

2. Let $\phi : R \longrightarrow S$ be a ring homomorphism. Let $Q \subset S$ be a prime ideal in S. Show that $\phi^{-1}(Q) = \{x \in R : \phi(x) \in Q\}$ is a prime ideal in R.

3. Fraleigh 27 Exercises 15-17. Let R be the ring $\mathbb{Z}\times\mathbb{Z}$.

   a) Find a maximal ideal of R.

   b) Find a prime ideal of R that is not maximal.

   c) Find a nontrivial proper ideal of R that is not prime.

---

4. Let $R = \mathbb{Z}I + \mathbb{Z}A$, for matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

   a) Show that R is a ring and that A is a unit in R.

   b) Let $x_0 = 0, x_1 = 1,\ x_{i+1} = x_i + x_{i-1}$ for $i \geq 1$ be the Fibonacci sequence. Show that

$$A^i = \begin{pmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{pmatrix}.$$

   c) Show that $3R \subset R$ is a maximal ideal. Hint: Verify, by considering the Fibonacci sequence modulo 3, that the matrices $I, A, \ldots, A^7$ are distinct modulo 3 and that $I+3R, A+3R, \ldots, A^7+3R$ represent the eight different nonzero elements in the ring $R/3R$.

   d) Show that $5R \subset R$ is not a prime ideal. Hint: Find $x \in \mathbb{Z}$ such that $(I + xA)^2 \in 5R$.

MATH 417 - Introduction to Abstract Algebra

Midterm1 - Five questions

---

1. Let G be a group with identity e and suppose that $a * b * c = e$ for $a, b, c \in G$. Show that $b * c * a = e$ also.

2. Let $G = S_3$ be the group of permutations of the set $\{1, 2, 3\}$ and let H be the subgroup $\{(1), (2\,3)\}$. Describe G as a union of disjoint left cosets of H.

3. The groups A_4 (the subgroup of even permutations of $\{1, 2, 3, 4\}$) and D_{12} (the group of symmetries of a regular hexagon) are of the same size.

   (a) Give the definition of an isomorphism $\phi : G \to G'$ between two groups G and G'.

   (b) Decide if the groups A_4 and D_{12} are isomorphic. Motivate your answer.

4. Under the right condition the cyclic group $\mathbb{Z}_{mn}$ can be written as a direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ of two cyclic groups.

   (a) Write each of the cyclic groups $\mathbb{Z}_{40}$, $\mathbb{Z}_{45}$ and $\mathbb{Z}_{48}$ as a direct product of two cyclic groups.

   (b) Write the group $G = \mathbb{Z}_{40} \times \mathbb{Z}_{45} \times \mathbb{Z}_{48}$ as a direct product $\mathbb{Z}_m \times \mathbb{Z}_n$ such that $n|m$.

5. Let S_6 be the group of permutations of the set $\{1, 2, 3, 4, 5, 6\}$.

   (a) Give a permutation $\sigma \in S_6$ such that $(1\,3\,5\,2\,4\,6) = \sigma(3\,1\,2\,4\,6\,5)\sigma^{-1}$.

   (b) Give a transposition $\sigma \in S_6$ with the given property.

MATH 417 - Introduction to Abstract Algebra

Midterm2 - Five questions

---

1. Let $\phi : \mathbb{Z}_6 \times \mathbb{Z}_{12} \longrightarrow \mathbb{Z}_3 \times \mathbb{Z}_6$ be the group homomorphism that maps $(a, b) \mapsto (a, b - a)$. Let $H = \ker \phi$ be the kernel.

   (a) List all the elements of H.

   (b) Decide, with motivation, if H is cyclic.

2. Let G be a finite group. Show that an element $a \in G$ of order two belongs to the center $Z(G)$ of G if and only if $H = \{1, a\}$ is a normal subgroup of G.

3. (a) Determine the commutator $[(123), (145)]$ for the permutations $(123), (145) \in S_5$.
   (b) Determine the order of $6 + \langle 20 \rangle \in \mathbb{Z}_{80}/\langle 20 \rangle$.

4. Determine the zeros in the ring $\mathbb{Z}_7$ of the polynomial $x^{90} + 2x^{60} - 3x^{30} - 4x^{20} + x^{10}$.

5. Let R be an integral domain. For each of the rings R_1 and R_2 decide if it is an integral domain. Motivate your choice by giving a proof in case it is or by providing an argument why not in case it is not.

   (a) $R_1 = M_2(R)$ is the ring of 2×2 matrices with entries in R.

   (b) $R_2 = R[x]$ is the ring of polynomials in x with coeffciients in R.

MATH 417 - Introduction to Abstract Algebra

Midterm3 - Four questions

---

1. Let R be a commutative ring with $1 \neq 0$.

(a) Give the definition of a maximal ideal.

(b) Give an example of a ring R and a maximal ideal in R.

(c) Give the definition of a prime ideal.

(d) Give an example of a ring R and a prime ideal in R that is not a maximal ideal.

2. The dihedral group G of order 16 acts on $X = \{1, 2, \dots, 8\}$. The group G consists of 8 rotations and 8 reflections. As elements of S_8, the rotations (left) and the reflections (right) have the following cycle structure.

$(*)(*)(*)(*)(*)(*)(*)(*)$	$\# = 1$
$(**)(**)(**)(**)$	$\# =$
$(****)(****)$	$\# =$
$(********)$	$\# =$
	total $\# = 8$

$(**)(**)(**)(**)$	$\# =$
$(**)(**)(**)(*)(*)$	$\# =$
	total $\# = 8$

(a) Fill in the frequencies for each of the cycle structures.

(b) Verify that G acts transitively on X by computing the average number of fixed points.

(c) Use the cycle stuctures and their frequencies to determine the number of different colored bracelets with 8 beads when each bead is colored with one of 2 colors.

3. For each of the following rings give the number of units and the number of zero divisors. (Short answers without detailed proofs are fine but include the steps that you use to arrive at your answer)

(a) $R = \mathbb{Z}_3[x]/(x^2 - 1)$.

(b) $R = \mathbb{Z}_3[x]/(x^2 + 1)$.

(c) $R = \mathbb{Z}_4[x]/(x^2)$.

4. Let the group G act on the set X.

(a) Define the stabilizer G_x of an element $x \in X$.

(b) Define the orbit O_x of an element $x \in X$.

(c) Prove that for elements $x, y \in X$ with $O_x = O_y$ the stabilizers G_x and G_y are conjugate, i.e., there exists $a \in G$ such that $G_y = aG_xa^{-1} = \{aga^{-1} : g \in G_x\}$.