Lecture 38 04/29

p-groups and Sylow Theorem (math 500)

Application of group action

[Defn] A finite group of order p^α, p a prime, some $\alpha \geq 0$ is called a p-group.

[First Sylow Theorem]

If G is a finite group of order $p^\alpha m$, $\gcd(p,m)=1$, then it contains a subgroup H of order p^α. H is called a Sylow p-subgroup.

[Second Sylow Theorem]

Any two sylow p-subgroups of group G are conjugate.

[Third Sylow Theorem]

The number of Sylow p-subgroups of a group G is 1 modulo p.

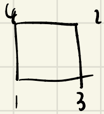
⊛ The classical proof proves the theorem in the order 1, 2, 3.

We prove a version of 2, then 3, ...

[Ex] $G = S_4$ $|G| = 4! = 2^3 \cdot 3$

(1). First Sylow Thm $\Rightarrow$ Contains subgroup of order 8. (D_8 ✓)

(2) 2nd Thm $\Rightarrow$ 3 copies of D_8 that are conjugate to each other

①


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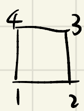

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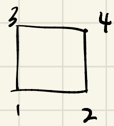

(3). $3 \equiv 1 \pmod 2$

[Preliminaries]

For subgroups $H, K \le G$, define $HK = \{hk : h \in H, k \in K\}$

⊛ Prop: If H, K are finite, then $|HK| = \dfrac{|H||K|}{|H \cap K|}$

[Proof] HK is a disjoint union of the left cosets of K.

$\Rightarrow \underline{\underline{HK = h_1K \cup h_2K \cup \cdots \cup h_rK.}}$ $(*)$

For any two $h, h' \in H$, $hK = h'K \Leftrightarrow h^{-1}h' \in K \Leftrightarrow h^{-1}h' \in H \cap K$.

$\Leftrightarrow h(H \cap K) = h'(H \cap K)$

Therefore, $H = h_1(H \cap K) \cup \cdots \cup h_r(H \cap K)$ for the same $h_1, \cdots, h_r$ as in $(*)$

$r = \dfrac{|HK|}{|K|}$, $r = \dfrac{|H|}{|H \cap K|}$ $\Rightarrow |HK| = \dfrac{|H||K|}{|H \cap K|}$

[Remark] ⊛ In general, HK is just a set and need not be a group.

[Group action by conjugation]

Let X be the set of all subgroups of a group G. G acts on X by conjugation $(\underset{\in G}{g}, \underset{\in X}{H}) \longmapsto gHg^{-1} \in X$.

The <u>stabilizer for this</u> particular action is called the normalizer of H in G.

$N_G(H) = \{g \in G : gHg^{-1} = H\} = \{g \in G : gH = Hg\}$

[Lemma]

If $K \leq N_G(H)$, then HK is a subgroup of G (and not just a set)

[Proof] Let $a = h_1 k_1$, $b = h_2 k_2$

then $ab = h_1 k_1 h_2 k_2 = h_1 \underbrace{k_1 h_2 k_1^{-1}}_{\in H} k_1 k_2 \in HK$.

$a^{-1} = (h_1 k_1)^{-1} = \underbrace{k_1^{-1} h_1^{-1} k_1}_{\in H} k_1^{-1} \in HK$.

$\Rightarrow$ HK is a subgroup of G.

By the Orbit-Stabilizer Thm, if $H \triangleleft N_G(H) \leq G$, then the number of subgroups in G conjugate to H is $[G : N_G(H)]$.

[Ex] $H = \langle (1\,2\,3\,4) \rangle \triangleleft D_8 \leq S_4$, $[S_4 : D_8] = 3$.

$\Rightarrow$ S_4 has 3 subgroups conjugate to H (3 cyclic subgroups of order 4:

$\langle (1\,2\,3\,4) \rangle$ $\langle (1\,3\,4\,2) \rangle$ $\langle (1\,4\,2\,3) \rangle$

[Defn] The center of a group G is $Z(G) = \{ a \in G : \underbrace{ag = ga}_{\Updownarrow \; gag^{-1} = a} \text{ for all } g \in G \}$

A group G acts on itself $(X = G)$ by conjugation $(\underset{\in G}{g}, \underset{\in X}{h}) \longmapsto ghg^{-1} \in X$

The orbit of $a \in G$ is of size $1 \Leftrightarrow a \in Z(G)$

Thm (Class Equation)

Let G act on itself by conjugate and let $O(g_1), \dots, O(g_r)$ be the orbits of size > 1

Then $$|G| = |Z(G)| + \sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|}$$

where $C_G(g_i)$ is the stabilizer of $g_i \in G$ under conjugation

$$C_G(g_i) = \{ g \in G : g g_i g^{-1} = g_i \}$$

[Proof] $$G = \underset{\text{union of orbits of 1}}{Z(G)} \cup \underbrace{O(g_1) \cup \cdots \cup O(g_r)}_{\text{orbits of size} > 1}$$

$$\Rightarrow |G| = |Z(G)| + \sum_{i=1}^{r} |O(g_i)| = |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)]$$

$$= |Z(G)| + \sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|}$$