Wednesday, April 27, 2022 11:39 AM

HW 11

4. Let $R = \mathbb{Z}I + \mathbb{Z}A$, for matrices

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

a) Show that R is a ring and that A is a unit in R.

b) Let $x_0 = 0, x_1 = 1,\ x_{i+1} = x_i + x_{i-1}$ for $i \geq 1$ be the Fibonacci sequence. Show that

$$A^i = \begin{pmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{pmatrix}.$$

c) Show that $3R \subset R$ is a maximal ideal. Hint: Verify, by considering the Fibonacci sequence modulo 3, that the matrices $I, A, \ldots, A^7$ are distinct modulo 3 and that $I+3R, A+3R, \ldots, A^7+3R$ represent the eight different nonzero elements in the ring $R/3R$.

d) Show that $5R \subset R$ is not a prime ideal. Hint: Find $x \in \mathbb{Z}$ such that $(I + xA)^2 \in 5R$.

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, ...

These are the first a few terms of Fibonacci sequence.

a) Check.

b) $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, prove by induction

$$A^i = \begin{pmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{pmatrix}$$

$$A^{i+1} = A^i \cdot A = \begin{pmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_i & x_{i-1}+x_i \\ x_{i+1} & x_i + x_{i+1} \end{pmatrix}$$

$$= \begin{pmatrix} x_i & x_{i+1} \\ x_{i+1} & x_{i+2} \end{pmatrix}. \checkmark$$

c) To show $R/3R$ is a field. Note that $|R/3R| = 3 \times 3 = 9$.

If we can show that $\underbrace{0+3R,\ I+3R,\ A+3R,\ \ldots A^7+3R}_{9 \text{ elements}}$ are distinct, then these are all elements of $R/3R$.

Just check by yourself that they're distinct making use of the Fibonacci sequence. Note that

$$A^8 + 3R = \begin{pmatrix} x_7 & x_8 \\ x_8 & x_9 \end{pmatrix} = \begin{pmatrix} 13 & 21 \\ 21 & 34 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 21 \\ 21 & 21 \end{pmatrix}}_{\in 3R} + 13 I$$

$$= 13 I + 3R$$

$$= I + 3R$$

So all nonzero element $A^i + 3R$, $0 \leq i \leq 7$ are invertible.

So all nonzero elements $A + 3R$, $0 \leq i \leq 7$ are invertible.

$\Rightarrow$ $R/3R$ is a field.

d). To show $R/5R$ is not an integral domain. Find a zero divisor.

Use the hint

$$(I + xA)^2 = I + 2xA + x^2(I + A) = (1+x^2)I + (2x+x^2)A$$

Want $(I + xA)^2 \in 5R$, want to have x such that

$$5 \mid 1+x^2, \quad 5 \mid 2x + x^2, \quad \text{taking } x = \pm 3 \text{ works.}$$