MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 6
Due Friday March 4

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1. Give an example of a group G, a subgroup $H \leq G$, and elements $a, b \in G$ such that $aH \cap bH = \emptyset$ but $Ha \cap Hb \neq \emptyset$.

2. Let H_1 be the group of permutations of $\{1,2,3\}$ and let H_2 be the group of permutations of $\{4,5,6\}$. Let $H = H_1 \times H_2$ be the direct product. So that $H \leq S_6$ is a subgroup of permutations of $\{1,2,3,4,5,6\}$.

   (a) Find the index $[S_6 : H]$.

   (b) Let $A_1 \neq A_2$ be two different 3-subsets of $\{1,2,3,4,5,6\}$ and let $\sigma_1, \sigma_2 \in S_6$ be such that $\sigma_1(\{1,2,3\}) = A_1$ and $\sigma_2(\{1,2,3\}) = A_2$. Show that $\sigma_1 H \cap \sigma_2 H = \emptyset$.

   c) How many 3-subsets are there in $\{1,2,3,4,5,6\}$? Explain the relation between your answers for part a) and part b).

3. Prove that the only homomorphism $\phi : S_3 \to \mathbb{Z}_3$ is the zero map.

4. The subgroup $H = \{(1), (12)(34), (13)(24), (14)(23)\} \leq S_4$ contains the identity, all three permutations with cycle structure $(ab)(cd)$, and no other permutations. Explain how this implies directly that H is a normal subgroup, i.e. that $\sigma H = H\sigma$ or $\sigma H \sigma^{-1} = H$ for all $\sigma \in S_4$.