Def
$H \leq G$.

$aH = \{ah \mid h \in H\} \leq G$ is called left coset of H.

Thm Let $H \leq G$, $a, b \in G$. $aH \cap bH = \phi$ or $aH = bH$.
Moreover, $aH = bH \iff a^{-1}b \in H$.

Prof: Assume that $aH \cap bH \neq \phi$, and let $ah = bk \in aH \cap bH$.
with $h, k \in H$.

$ah = bk \iff h = a^{-1}bk \iff a^{-1}b = h \cdot k^{-1} \in H$.

thus $a^{-1}b \in H$.

if $aH \cap bH \neq \phi$.

$bH = (a \cdot a^{-1}b)H = a(\underbrace{a^{-1}b}_{\in H} H) \Rightarrow bH = aH$.

permutation

A partition of group: $G = a_1H \cup a_2H \cup \cdots \cup a_rH$.

Define a left coset $a+H = b+H$ if $b-a \in H$.

right is the same $a+H = H+a$.

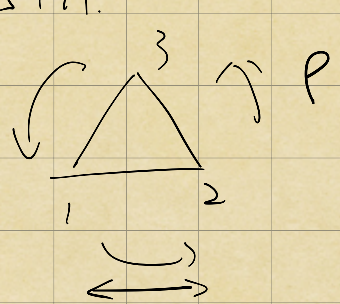
Ex: $G = \mathbb{Z}_6$, $H = \{0, 3\}$. $0+H = \{0, 3\}$. $3+H = \{3, 0\}$.

$1+H$ $4+H$

$2+H$ $5+H$.

Ex. $G = S_3$ $H = \{(1), (\sigma_2)\}$.

$G = \{1, \rho, \rho^2, \sigma, \sigma\rho, \sigma\rho^2\}$.



$\rho H = \{\rho, \rho\sigma\}$ $H\rho = \{\rho, \sigma\rho\}$.
$= \{\rho, \sigma\rho^2\}$

$\sigma = (1\ 2)$.

$\rho^2 H = \{\rho^2, \rho^2\sigma\}$ $\quad H\rho^2 = \{\rho, \sigma\rho^2\}$.
$= \{\rho^2, \sigma\rho\}$.

two different partition of G.

Theom: (Lagrange's Theorem)
Let $H \leq G$ be a subgroup. The order $|H|$ divides the order $|G|$

Proof: Consider the left coset of H. Then give a partition $G = a_1H \cup a_2H \cdots \cup a_rH$ of G.

$$G = |a_1H| + |a_2H| + \cdots + |a_rH|.$$
$$= r|H| \to |H| \mid |G|$$

Theom: for $a \in G$. the order of a (the smallest m such that $a^m = e$) divides $|G|$.

Proof: For $a \in G$. $H = \{a^n, n \in \mathbb{Z}\} \leq G$.
H is the size of m. $(H = \{e, a, a^2, \ldots a^{m-1}\})$.
$a^m = e$.

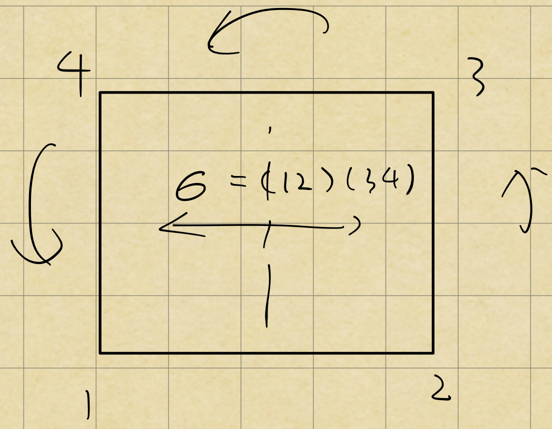
With Lagrange's Theom.
$$|H| = m \mid |G|.$$

Remark: A subgroup of $\mathbb{Z}_n$ is of order $\frac{n}{d}$.

Ex: $|S_3| = 6$. S_3 contains element of order $1, 2, 3$ but without 6.

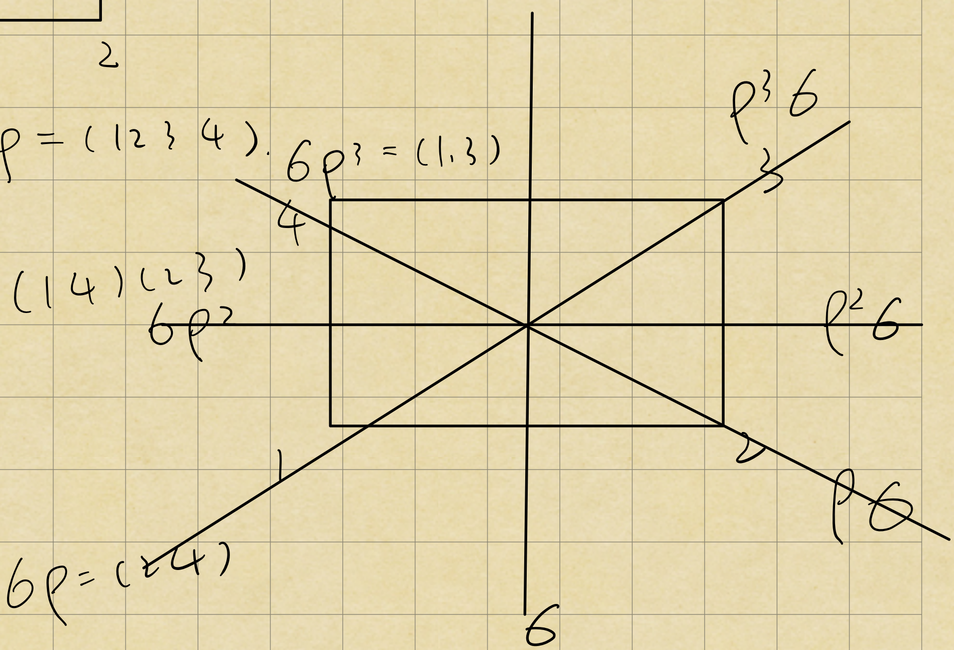
$|A_4| = 12$ but no subgroup of order $\underline{6}$. $(6 \mid 12)$.

(Fig 8.13 Page 80. Subgroup of D_8).



$D_8 = \{1, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}.$

$\rho = (1234).$



$H = \{1, \rho^2\}.$

$\rho H = \{\rho, \rho^3\}.$ $H\rho = \{\rho, \rho^3\}.$

$\sigma H = \{\sigma, \sigma\rho^2\}.$ $H\sigma = \{\sigma, \rho^2\sigma\} = \{\sigma, \sigma\rho^2\}.$

$\sigma\rho H = \{\sigma\rho, \sigma\rho^3\}$ $H\sigma\rho = \{\sigma\rho, \rho^2\sigma\rho\} = \{\sigma\rho, \sigma\rho^3\}.$

Even G is immutable.

Left and Right coset are same

(subgroups of D_8).

D_8

$H \cup \sigma H$ $\quad H \cup \rho H$ $\quad H \cup \sigma\rho H.$

$\{1, \sigma\}$ $\quad \{1, \sigma\rho^2\}.$