MATH 417 Lec06-15

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目录

1 Integers

1.1 Proposition 1.4.1: Properties of integers $\mathbb{Z}$

Proposition 1 (Proposition 1.4.1.)**.** *The following hold in the integers $\mathbb{Z}$:*

(i) *Addition* and *multiplication* are *commutative* and *associative* operations in $\mathbb{Z}$.
(ii) $0 \in \mathbb{Z}$ is an identity element for addition; that is, $\forall a \in \mathbb{Z}, 0 + a = a$.
(iii) Every $a \in \mathbb{Z}$ has an additive inverse, denoted $-a$ and given by $-a = (-1)a$, satisfying $a + (-a) = 0$.
(iv) $1 \in \mathbb{Z}$ is an identity element for multiplication; that is, for all $a \in \mathbb{Z}$, $1a = a$.
(v) The *distributive* law holds: $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$.
(vi) Both $\mathbb{N} = \{x \in Z | x \geq 0\}$ and $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$ are *closed* under *addition* and *multiplication*. That is, if x and y are in one of these sets, then $x + y$ and xy are also in that set.
(vii) For any two nonzero integers $a, b \in \mathbb{Z}$, $|ab| \geq \max\{|a|, |b|\}$. Strict inequality holds if $|a| > 1$ and $|b| > 1$.

From this we get cancellation.

$$ab = ac \Rightarrow b = c \ or \ a = 0$$

1.2 Definition: Divide

Suppose $a, b \in \mathbb{Z}, b \neq 0$, <u>$b$ divides a</u> if $\exists m \in \mathbb{Z}$, so that $a = bm, b|a$. Otherwise, write $b \nmid a$.

1.3 Proposition 1.4.2: properties of integer division

Proposition 2 (Proposition 1.4.2)**.** $\forall a, b \in \mathbb{Z}$

(i) if $a \neq 0$, then $a|0$
(ii) if $a|1$, then $a = \pm 1$
(iii) if $a|b$ & $b|a$, then $a = \pm b$
(iv) if $a|b$ & $b|c$, then $a|c$
(v) if $a|b$ & $a|c$, then $a|(mc + nb) \forall m, n \in \mathbb{Z}$

1.4 Definitions: Prime, The Greatest common divisor $gcd(a, b)$

$p > 1, p \in \mathbb{Z}$ is called <u>*prime*</u> if the only divisors are $\pm 1, \pm p$.
Given $a, b \in \mathbb{Z}, a, b \neq 0$, the <u>*greatest common divisor*</u> of a and b is $c \in \mathbb{Z}, c > 0$ s.t.
(1) $c|a$ and $c|b$; (2) if $d|a, d|b$, then $d|c$
The c is unique, we write it $gcd(a, b)$.

1.5 Euclidean Algorithm

Proposition 3 (Proposition 1.4.7(Euclidean Algorithm))**.** *Given $a, b \in \mathbb{Z}, b \neq 0$, then $\exists q, r \in \mathbb{Z}$ s.t. $a = qb + r, 0 \leq r \leq |b|$.*

Example 1 (Exercise 1.4.3)**.** *For the pair $(a, b) = (130, 95)$, find $gcd(a, b)$ using the Euclidean Algorithm and express it in the form $gcd(a, b) = sa + tb$ for $s, t \in Z$.*

$$\begin{aligned} &130 = 95 + 35; \quad 95 = 2 \times 35 + 25 \\ &35 = 25 + 10; \quad\quad 25 = 2 \times 10 + 5 \\ &10 = 2 \times 5 + 0 \end{aligned}$$

$$\begin{aligned} 5 &= 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35 \\ &= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130 \end{aligned}$$

$$gcd(130, 95) = gcd(95, 35) = gcd(35, 25) = gcd(25, 10) = gcd(10, 5) = gcd(5, 0) = 5$$

We can also express it by matrix

	q	r	s	t
-1		130	1	0
0	1	95	0	1
1	2	35	1	-1
2	1	25	-2	3
3	2	10	3	-4
4	2	5	-8	11

Hence $gcd(130, 95) = 5 = -8 \cdot 130 + 11 \cdot 95$

1.6 Proposition: $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$

Theorem 1. *$d = gcd(a, b)$ is of the form $sa + tb$*

证明. We may assume $0 \leq a \leq b$

For $a = 0$, $d = b = 0 \cdot a + 1 \cdot b$.

For $a > 0$, let $b = q \cdot a + r$ with $0 \leq r < a \leq b$. Then

$$\begin{aligned} \{sa + tb : s, t \in \mathbb{Z}\} &= \{sa + t(q \cdot a + r) : s, t \in \mathbb{Z}\} = \{tr + ua : t, u \in \mathbb{Z}\} \\ &= \ldots \{x \cdot 0 + y \cdot d : x, y \in \mathbb{Z}\} = \{..., -2d, -d, 0, d, 2d, ...\} \end{aligned}$$

□

Proposition 4 (第二种表示，第二种证明)**.** *$\forall a, b \in \mathbb{Z}$, not both 0, $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$. i.e. $\exists m_0, n_0 \in \mathbb{Z}$ s.t. $gcd(a, b) = m_0 a + n_0 b$.*

证明. Let c be the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$. $c = m_0 a + n_0 b > 0$. Let $d = ma + nb \in M$, $d = qc + r$ where $0 \le r < c$ (by Euclidean Algorithm).

$$r = d - qc = (m - qm_0)a + (n - qn_0)b \in M$$

Since c is the smallest integer in M and $r \in [0, c)$, so $r = 0$. $\Rightarrow d = qc$. So $c|d$.
$a = 1a + 0b \in M \Rightarrow c|a$, $b = 0a + 1b \in M \Rightarrow c|b$.
If $t|a, t|b$ then $t|m_0 a + n_0 b$ i.e. $t|c$. $\Rightarrow c = gcd(a, b)$. □

1.7 Well-Ordering Principle (Least Integer Axiom)

There is a smallest integer in every nonempty subset S of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$

1.8 Proposition 1.4.10: $gcd(b, c)$, $b|ac \Rightarrow b|a$

Proposition 5 (Proposition 1.4.10)**.** *Suppose $a, b, c \in \mathbb{Z}$. If b, c are relatively prime i.e. $gcd(b, c) = 1$ and $b|ac$, then $b|a$.*

证明. $gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$ s.t. $1 = mb + nc \Rightarrow a = amb + anc$. Since $b|nac, b|amb \Rightarrow b|a$. □

1.8.1 Corollary: $p|ab \Rightarrow p|a$ or $p|b$

Corollary 1 (Corollary of Prop 1.4.10)**.** *$a, b, p \in \mathbb{Z}, p > 1$ prime. If $p|ab$, then $p|a$ or $p|b$.*

证明. If $p|b$, done. Otherwise, $gcd(p, b) = 1$. By Prop 1.4.10, $p|a$. □

1.9 Fundamental Theorem of Arithmetic: Any integer $a \ge 2$ has a unique prime factorization

1.9.1 Existence

Lemma 1. *Any integer $a \ge 2$ is either a prime or a product of primes.*

证明. Set $S \subset \mathbb{N}$ be the set of all n without the given property.
Assume that S in nonempty and m is the least element in S.
Since m is not a prime, it can be written as $m = ab$ with $1 < a, b < m$. Since m is the least element in S, $a, b \notin S$. Then m is a product of primes. Contradiction. Thus, $S = \emptyset$. □

1.9.2 Uniqueness

Theorem 2 (Fundamental Theorem of Arithmetic)**.**

Any integer $a > 1$ has a unique prime factorization: $a = p_1^{k_1} \cdot p_2^{k_2} \cdot ... p_n^{k_n}$ where $p_i > 1$ is prime, $k_i \in \mathbb{Z}_+, \forall i = 1, ..., n, p_i \neq p_j, \forall i \neq j$.

证明.

a) Existence: (Previous Lemma)

b) Uniqueness:

1) Method 1:

Suppose $a = p_1^{n_1} \cdot p_2^{n_2} \cdot ...p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot ...q_j^{r_j}$. Where $p_1 > p_2 > ... > p_k, q_1 > q_2 > ... > q_j, n_i, r_i \geq 1$.

$p_1 | a \Rightarrow \exists\ q_i\ s.t.\ p_1 | q_i$. Similarly, $\exists\ q_i\ s.t.\ q_1 | p_{i'}$.

$q_1 \leq p_{i'} \leq p_1 \leq q_i \Rightarrow q_1 = p_{i'} = p_1 = q_i$

We can also know $n_1 = r_1$, otherwise we would have two prime factorization of the quotient where the largest primes are different by dividing $p_1^{\min\{n_1, r_1\}}$.

Then we can get $b = p_2^{n_2} \cdot ...p_k^{n_k} = q_2^{r_2} \cdot ...q_j^{r_j}$. Then prove it by induction.

2) Method 2:

Suppose $a = p_1 \cdot p_2 \cdot ...p_k = q_1 \cdot q_2 \cdot ...q_t$. For a p_i, there must exist a q_j s.t. $p_i = q_j$:

Assume that $p_i \neq q_t$, $gcd(p_i, q_t) = 1$. Then $\exists a, b$ such that $1 = ap_i + bq_t$. Multiplying both sides by $q_1 \cdot q_2 \cdot ...q_{t-1}$:

$$q_1 \cdot q_2 \cdot ...q_{t-1} = ap_i q_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t$$

Since $p_i | q_1 \cdot q_2 \cdot ...q_t$, we can conclude that $p_i | (ap_i q_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t)$

$$\text{i.e. } p_i | q_1 \cdot q_2 \cdot ...q_{t-1} \text{ if } p_i \neq q_t$$

Then prove by induction.

□

2 Modular arithmetic

2.1 Congruences

2.1.1 Congruent modulo m: $a \equiv b$ mod m

Given $m \in \mathbb{Z}_+$, define a relation on $\mathbb{Z}$: **congruence modulo** m

$$a \equiv b \text{ mod } m, \text{ if } m | (a - b)$$

Read as "a is congruent to b mod n"; Notation: $a \equiv b$ mod m.

Equivalent to: a, b have the same remainder after division by m.

2.1.2 Proposition: For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" is an <u>equivalence relation</u>

Proposition 6 (Proposition 1.5.1)**.** *For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b \bmod m$" is an <u>equivalence relation</u>*

证明.

1) <u>Reflexive</u>: $\forall a \in \mathbb{Z}, m|0 = (a-a)$, so $a \equiv a \bmod m$ i.e. $a \sim a$.

2) <u>Symmetric</u>: $\forall a, b \in \mathbb{Z}$, $a \equiv b \bmod m$, then $m|(a-b) \Rightarrow m|(b-a) \Rightarrow b \equiv a \bmod m$. i.e. $a \sim b \Rightarrow b \sim a$.

3) <u>Transitive</u>: $\forall a, b, c \in \mathbb{Z}$, $a \equiv b \bmod m$, $b \equiv c \bmod m$. Then $m|(a-b), m|(b-c) \Rightarrow m|(a-b)+(b-c) = (a-c) \Rightarrow a \equiv c \bmod m$.

□

2.1.3 Theorem: the equivalence relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$

Theorem 3. *the equivalence relation "$a \sim b \Leftrightarrow a \equiv b \bmod m$" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$*

证明. Prove any $a \in \mathbb{Z}$ belongs to a unique Ω_i.

a) Existence: Division Algorithm $\Rightarrow a = qm + r$, $0 \leq r < m$. $a \in \Omega_r$.

b) Uniqueness: Assume a in two sets, $a \in \Omega_r \cap \Omega_{r^1}$, $0 \leq r^1 < r < m$.

   Then $m|a-r$ and $m|a-r^1 \Rightarrow m|r-r^1$, which is impossible because $0 < r-r^1 < m$. Contradiction.

□

2.1.4 Proposition: Addition and Mutiplication of Congruences

Proposition 7. *Fix integer $m \geq 2$. If $a \equiv r \bmod m$ and $b \equiv s \bmod m$, then $a+b \equiv r+s \bmod m$ and $ab \equiv rs \bmod m$*

证明.

a) Addition: $m|(a-r), m|(b-s) \Rightarrow m|(a-c)+(b-d) \Rightarrow m|(a+b)-(c-d)$.

b) Mutiplication: $m|(a-r)b + r(b-s) \Rightarrow m|ab - rs$.

□

2.2 Solving Linear Equations on Modular m

2.2.1 Theorm: unique solution of $aX \equiv b$ mod m if $gcd(a, m) = 1$

Theorem 4. *If $gcd(a, m) = 1$, then $\forall b \in \mathbb{Z}$ the congruence $aX \equiv b$ mod m has a unique solution.*

证明.

1) Existence: Since $gcd(a, m) = 1$, $\exists s, t$ such that

$$1 = sa + tm$$

$$\text{(Version 1)}$$

$$\text{(Mutiplying } X\text{)}$$

$$X = saX + tmX$$

$$aX \equiv b \text{ mod } m \Leftrightarrow aX = km + b$$

$$\Leftrightarrow X \equiv sb \text{ mod } m$$

$$\text{(Version 2)}$$

$$\text{(Mutiplying } s\text{)}$$

$$saX \equiv sb \text{ mod } m$$

$$(1 - tm)X \equiv sb \text{ mod } m$$

$$X \equiv sb \text{ mod } m$$

$X \equiv sb$ mod m is the solution to $aX \equiv b$ mod m.

2) Uniqueness: Assume x, y are two solutions,

$$ax \equiv b \text{ mod } , ay \equiv b \text{ mod } m \Rightarrow a(x - y) \equiv 0 \text{ mod } m$$

Since $gcd(a, m) = 1$, $m|(x - y) \Rightarrow x = y$, $(x, y \in \{0, 1, ..., m - 1\})$

Example 2. *Solve $3X \equiv 5$ mod 11.*

$gcd(3, 11) = 1$, $1 = 4 * 3 - 1 * 11$,

$$X \equiv 9$$

□

2.3 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

将给定 n，相同余数的数分为一组

Fix $n \in \mathbb{Z}_+$, we call $[a]_n = [a]$ the **congruence class** of a modulo n.

$$[a] = \{b \in \mathbb{Z} | b \equiv a \text{ mod } n\} = \{a + kn | k \in \mathbb{Z}\}$$

2.3.1 Set of congruence classes of mod n: $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], ..., [n-1]\}$

The set of *congruence classes* of mod n is denoted $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

Proposition 8 (Proposition 1.5.2.). *For any $n \geq 1$ there are exactly n congruences classes modulo n, which we may write as*

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

证明.
For any $a \in \mathbb{Z}$. By Euclidean algorithm, $a = qn + r$, $q, r \in \mathbb{Z}$, $0 \leq r < n \Rightarrow a \in [r]$. So, $\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$.
When $0 \leq a < b \leq n-1$, $n \nmid (b-a)$, so $[a] \neq [b]$ the n congruence classes listed are all distinct. Hence, there are exactly n congruence classes. □

2.3.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

Fix $n \in \mathbb{Z}$, we define addition+ and multiplication$\cdot$ on $\mathbb{Z}_n$:

$$[a] + [b] = [a+b] = \{a + b + (k+j)n | k, j \in \mathbb{Z}\}$$
$$[a] \cdot [b] = [ab] = \{ab + (aj + bk + kjn)n | k, j \in \mathbb{Z}\}$$

This is well defined, follows Lemma 1.5.3.

Proposition 9 (Proposition 1.5.5.). *Let $a, b, c, d, n \in \mathbb{Z}, n \geq 1$, then*
(i) Addition and multiplication are commutative and associative operations in $\mathbb{Z}_n$.
(ii) $[a] + [0] = [a]$.
(iii) $[-a] + [a] = [0]$.
(iv) $[1][a] = [a]$.
(v) $[a]([b] + [c]) = [a][b] + [a][c]$.

证明.

□

2.3.3 Units(i.e. invertible) in Congruence Classes

将与 n 互质的数分为一组
Say $[a] \in \mathbb{Z}_n$ is a **unit** or is **invertible** if $\exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [1]$.

2.3.4 Proposition 1.5.6: Set of units in congruence classes: $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \textit{ is a unit}\} = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$

The set of **invertible** elements in $\mathbb{Z}_n$ will be denoted $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \textit{ is a unit}\}$.

Proposition 10 (Proposition 1.5.6.). *For all $n \geq 1$, we have $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$.*

证明.
By Proposition 1.4.8, we know there exists b, c s.t. $ab + cn = 1$. So, $ab \equiv 1 \ mod \ n$, $[1] = [ab] = [a][b]$. So, $\{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\} \subset \mathbb{Z}_n^{\times}$
$[a]$ *is a unit* $\Rightarrow \exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow gcd(a, n) = 1$. So, $\mathbb{Z}_n^{\times} \subset \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$. □

Note 1. *Inverse of* $[a]$ *is unique, i.e.* $[b] = [a]^{-1}$ *is unique.*

$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$

2.3.5 Corollary 1.5.7: if p is prime, $\varphi(p) = \mathbb{Z}_p^{\times} = \{[1], [2], ..., [p-1]\}$

Corollary 2 (Corollary 1.5.7)**.** *If* $p \geq 2$ *is prime,* $\mathbb{Z}_p^{\times} = \{[1], [2], ..., [p-1]\}$.

2.4 Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^{\times}|$

Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^{\times}|$.
p prime, $\varphi(p) = p - 1$.

2.4.1 $m|n$, $\pi_{m,n}([a]_n) = [a]_m$

Example 3 (Exercise 1.5.4)**.** *If* $m|n$*, we can define* $\pi_{m,n} : \mathbb{Z}_n \to \mathbb{Z}_m$ *by* $\pi_{m,n}([a]_n) = [a]_m$*. Prove it is well-defined.*

证明.
We write $[a]_n = [c]_n$, verify that $[a]_m = [c]_m$.
Since $m|n$, there exists $k \in \mathbb{Z}$ s.t. $n = km$.
$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z}$ s.t. $c = a + jn$.
$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m$ □

2.5 Theorem 1.5.8(Chinese Remainder Theorem): $n = mk, gcd(m, k) = 1$, $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

Theorem 5 (Theorem 1.5.8(Chinese Remainder Theorem))**.** *If* $m, n, k > 0, n = mk, gcd(m, k) = 1$*, then* $F : \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_k$ *which is given by* $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$*, then* F *is a bijection.*

证明.
(1)Injective: $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$ i.e. $a \equiv b \ mod \ m, a \equiv b \ mod \ n$. $\exists i, j \in \mathbb{Z}$ s.t. $b = a + im = a + jk \Rightarrow k|im$. Since $gcd(m, k) = 1$, $k|i \Rightarrow n = mk|im$. Then $[b]_n = [a]_n + [im]_n = [a]_n$.
(2)Surjective: prove $\forall u, v \in \mathbb{Z}$, $\exists a\mathbb{Z}$ s.t. $[a]_m = [u]_m, [a]_k = [v]_k$.

Since $gcd(m,k)=1$, $\exists s,t\in\mathbb{Z}$ so that $1=sm+tk$.
Let $a=(1-tk)u+(1-sm)v$, $[a]_m=[(u-v)sm+v]_m=[v]_m$, $[a]_k=[(v-u)tk+u]_k=[u]_k$. $\square$

Note 2. $F([a]_n[b]_n)=F([ab]_n)=([ab]_m,[ab]_k)=([a]_m[b]_m,[a]_k[b]_k)$

Since F is a bijection, $[ab]_n=[1]_n$ iff $([a]_m[b]_m,[a]_k[b]_k)=([1]_m,[1]_k)$.

2.5.1 Proposition 1.5.9+Corollary 1.5.10: $m,n,k>0, n=mk, gcd(m,k)=1$, then $F(\mathbb{Z}_n^\times)=\mathbb{Z}_m^\times\times\mathbb{Z}_k^\times$, then $\varphi(n)=\varphi(m)\varphi(k)$

Proposition 11 (Proposition 1.5.9+Corollary 1.5.10)**.** *If* $m,n,k>0, n=mk, gcd(m,k)=1$*, then* $F(\mathbb{Z}_n^\times)=\mathbb{Z}_m^\times\times\mathbb{Z}_k^\times$*, then* $\varphi(n)=\varphi(m)\varphi(k)$*.*

2.6 prime factorization: $n=p_1^{r_1}...p_k^{r_k}$, then $\varphi(n)=(p_1-1)p_1^{r_1-1}...(p_k-1)p_k^{r_k-1}$

Proposition 12. *If* $n\in\mathbb{Z}$ *is positive integre with prime factorization* $n=p_1^{r_1}...p_k^{r_k}$*, then* $\varphi(n)=(p_1-1)p_1^{r_1-1}...(p_k-1)p_k^{r_k-1}$

证明.
$\mathbb{Z}_{p^r}=\{[0],[1],...,[p^r-1]\}$, the number of multiples of p is $\frac{p^r}{p}=p^{r-1}$. Then $\varphi(p^r)=|\mathbb{Z}_{p^r}^\times|=p^r-p^{r-1}=(p-1)p^{r-1}$. So,

$$\varphi(n)=\varphi(p_1^{r_1})...\varphi(p_k^{r_k})=(p_1-1)p_1^{r_1-1}...(p_k-1)p_k^{r_k-1}$$

$\square$

3 Complex numbers

$\mathbb{C}=\{a+bi|a,b\in\mathbb{R}\}$, $\mathbb{R}=\{a+0i|a\in\mathbb{R}\}\subset\mathbb{C}$
Addition & multiplication

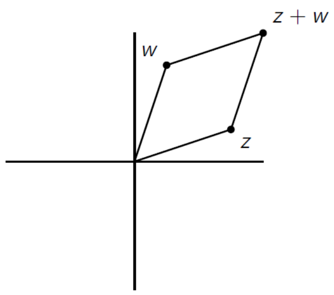
$$\begin{aligned}(a+bi)+(c+di)&=(a+c)+(b+d)i\\(a+bi)(c+di)&=ac+bci+adi+bdi^2\\&=(ac-bd)+(bc+ad)i\end{aligned}$$

<u>**Complex conjugation**</u>: $z=a+bi, \bar{z}=a-bi$, $\overline{zw}=\bar{z}\bar{w}$
<u>**Absolute value**</u>: $|z|=\sqrt{a^2+b^2}$, $|z|^2=z\bar{z}$
<u>**Additive inverse**</u>: $-z=-a-bi$
<u>**Multiplicative inverse**</u>: $z^{-1}=\frac{1}{z}=\frac{1}{a+bi}=\frac{a-bi}{a^2+b^2}=\frac{\bar{z}}{|z|^2}$

$$z\in\mathbb{C},\overline{z+\bar{z}}=\bar{z}+\bar{\bar{z}}=z+\bar{z}$$
$$\textit{Real part: } Re(z)=\frac{z+\bar{z}}{2}$$
$$\textit{Imaginary part: } Im(z)=\frac{z-\bar{z}}{2i}$$

3.1 Geometric Meaning of Addition and Multiplication

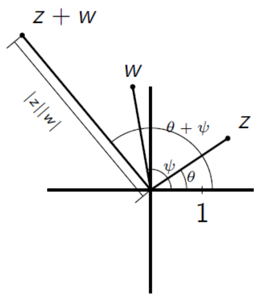
Addition: parallelogram law



Multiplication:

$$\begin{aligned} z &= a + bi \neq 0 \\ &= r\cos\theta + r\sin\theta i \\ &= r(\cos\theta + i\sin\theta) \\ |z|^2 &= a^2 + b^2 = r^2 \end{aligned}$$

$$\begin{aligned} z =& r(\cos\theta + i\sin\theta) \\ w =& s(\cos\phi + i\sin\phi) \\ zw =& rs[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)] \\ =& rs[cos(\theta+\phi) + i\sin(\theta+\phi)] \\ =& |z||w|[cos(\theta+\phi) + i\sin(\theta+\phi)] \end{aligned}$$



We will write,

$$\begin{aligned} &\cos\theta + i\sin\theta = e^{i\theta} \\ &e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \\ &z = |z|e^{i\theta} \end{aligned}$$

3.2 Theorem 2.1.1: $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$

Theorem 6 (Theorem 2.1.1)**.** *Supose a nonconstant polynomial $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$.*

3.2.1 Corollary 2.1.2: $f(x) = a_n \prod_{i=1}^{n}(x-k_i) = a_n(x-k_1)(x-k_2)...(x-k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$

Corollary 3 (Corollary 2.1.2)**.** *Every nonconstant polynomial with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$ can be factored as $f(x) = a_n \prod_{i=1}^{n}(x - k_i) = a_n(x - k_1)(x - k_2)...(x - k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$.*

3.2.2 Corollary 2.1.3: $a_i \in \mathbb{R}$, f can be expresses as a product of linear and quadratic polynomials

Corollary 4 (Corollary 2.1.3)**.** *If $f(x) = a_0+a_1x+...+a_nx^n$ is a nonconstant polynomial $a_0, a_1, ..., a_n \in \mathbb{R}, a_n \neq 0$. Then f can be expresses as a product of linear and quadratic polynomials.*

这里 $a_0, a_1, ..., a_n$ 是实数!

证明.
(1)Obviously, the corollary holds at $n = 1$ and $n = 2$.
(2)Suppose the corollary holds for all situations that $n < k$.
When $n = k$, $f(x) = a_0 + a_1x + ... + a_kx^k, a_k \neq 0$.
By F.T.A., f has a root α in $\mathbb{C}$.
If $\alpha \in \mathbb{R}$, long division $f(x) = q(x)(x - \alpha)$. q has real coefficients, *degree of* $q = k - 1$. Since the corollary holds at $n = k - 1$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.
If $\alpha \notin \mathbb{R}$

$$0 = f(\alpha) = a_0 + a_1\alpha + ... + a_k\alpha^k$$
$$0 = \overline{f(\alpha)} = a_0 + a_1\bar{\alpha} + ... + a_n\bar{\alpha}^n = f(\bar{\alpha})$$

Since $\bar{\alpha} \neq \alpha$, $(x - \alpha)(x - \bar{\alpha})|f$.
$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2$ is a polynomial with coefficients in $\mathbb{R}$. So $f(x) = q(x)(x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2)$, q has real coefficients with degree $k - 2$. The corollary also holds at $n = k - 2$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

□

4 Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))

Definition: A field is a nonempty set $\mathbb{F}$ with two operations:

1. addition, written $a+b, \forall a, b \in \mathbb{F}$;
2. multiplication, written $a \cdot b = ab, \forall a, b \in \mathbb{F}$.

such that:

(i) *addition* and *multiplication* are associative and commutative

(ii) *multiplication* distributes over *addition*: $a(b+c) = ab + ac, \forall a, b, c \in \mathbb{F}$

(iii) $\exists$ an additive identity $0 \in \mathbb{F}$ s.t. $0 + a = a, \forall a \in \mathbb{F}$.

(iv)$\forall a \in \mathbb{F}$, $\exists$ an additive inverse $-a$ s.t. $a + (-a) = 0, \forall a \in \mathbb{F}$.

(v) $\exists$ a multiplicative identity: $1 \in \mathbb{F}$ s.t. $1a = a, \forall a \in \mathbb{F}, 1 \neq 0$.

(vi) $\forall a \in \mathbb{F}, a \neq 0$, a has a multiplicative inverse $a^{-1} = \frac{1}{a} \in \mathbb{F} : a \cdot \frac{1}{a} = 1$.

Proposition 13 (Proposition 2.2.2)**.** *$\mathbb{F}$ a field, $a, b \in \mathbb{F}$, then*

(i) If $a + b = b$ then $a = 0$

(ii) If $ab = b$ and $b \neq 0$, then $a = 1$

(iii) $0a = 0$

(iv) If $a + b = 0$, then $b = -a$

(v) If $a \neq 0$ and $ab = 1$, then $b = a^{-1}$

Example 4. *$\mathbb{Z}_4$ is not a field. Because $[2]_4$ doesn't have multiplicative inverse in $\mathbb{Z}_4$.*

4.1 Subfield $(\mathbb{K}, +, \cdot)$: $\mathbb{K} \subseteq \mathbb{F}$, closed under $+, \cdot$ and inverse

Definition:Suppose $\mathbb{F}$ is a field and $\mathbb{K} \subseteq \mathbb{F}$ s.t.

$$0, 1 \in \mathbb{K}$$
$$\forall a, b \in \mathbb{K}, a + b, ab, -a, a^{-1}(\textit{if } a \neq 0) \in \mathbb{K}$$

We call $\mathbb{K}$ a subfield of $\mathbb{F}$.

Example 5. $\mathbb{Q} \subseteq \mathbb{R}, \mathbb{R} \subseteq \mathbb{C}, \mathbb{Q} \subseteq \mathbb{C}$

Example 6. *$\mathbb{K} \subseteq \mathbb{Z}_p$ a subfield $\Rightarrow \mathbb{K} = \mathbb{Z}_p$. Prove by induction.*

4.1.1 Proposition 2.2.3: Subfield 继承 operations 自成一 field

Proposition 14 (Proposition 2.2.3)**.** *Suppose $\mathbb{K} \subset \mathbb{F}$ is a subfield of a field $\mathbb{F}$ Then the operations of $\mathbb{F}$ make $\mathbb{K}$ into a field.*

$\Rightarrow$We can prove a set is a field by proving it is a subfield of a known field.

5 Polynomials

Let $\mathbb{F}$ be any field. A polynomial over $\mathbb{F}$ in variable x is a formal sum:

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n = \sum_{i=0}^{n} a_ix^i$$

where $n \geq 0$ is an integer, $a_1, a_1, ..., a_n \in \mathbb{F}$.
Polynomial is a squence $\{a_k\}_{k=0}^{\infty}$ with $a_m = 0, \forall m > n$.

5.1 $\mathbb{F}[x]$: Polynomial ring 在一个 field 上形成的所有多项式 (方程) 的集合

Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients in the field $\mathbb{F}$.

$$\mathbb{F}[x] = \{\sum_{i=0}^{n} a_ix^i | n \geq 0, n \in \mathbb{Z}, a_0, ..., a_n \in \mathbb{F}\}$$

We call the $\mathbb{F}[x]$ *polynomial ring* over the field $\mathbb{F}$.

$$f = \sum_{i=0}^{n} a_ix^i, g = \sum_{j=0}^{n} a_jx^j \in \mathbb{F}[x]$$
$$f + g = \sum_{i=0}^{n}(a_i + b_i)x^i \in \mathbb{F}[x]$$
$$fg(\sum_{i=0}^{n} a_ix^i)(\sum_{j=0}^{n} a_jx^j) = \sum_{i=0}^{2n}(\sum_{j=0}^{i} a_jb_{i-j})x^i$$

5.1.1 Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

Proposition 15 (Proposition 2.3.2)**.** *Suppose* $\mathbb{F}$ *is any field. Then,*

(i) Addition and multiplication are commutative & associative operations on $\mathbb{F}[x]$
(ii) Multiplication distributes over addition
(iii) $0 \in \mathbb{F}$, is additive identity in $F[x] : \forall f \in \mathbb{F}[x], f + 0 = 0$
(iv) $\forall f \in \mathbb{F}[x], f = (-1)f$ is the additive inverse: $f + (-1)f = 0$.
(v) $1 \in \mathbb{F}$, is the multiplicative identity in $\mathbb{F}[x]:\ 1f = f,\ \forall f \in \mathbb{F}[x]$

5.2 Degree of a Polynomial: $deg(f)$

$f = \sum_{i=0}^{n} a_ix^i$, $deg(f)$ = degree of f is,

$$deg(f) = \begin{cases} 0 & \textit{if } f \textit{ is constant, } f \neq 0 \\ n & \textit{if } a_n \neq 0 \textit{ in above } (a_n = \textit{leading coefficent}) \\ -\infty & \textit{if } f = 0 \end{cases}$$

Define $-\infty + a = a + (-\infty) = -\infty\ \forall a \in \mathbb{Z} \cup \{-\infty\}$

5.2.1 Lemma 2.3.3: $deg(fg) = deg(f) + deg(g),\ deg(f+g) \leq \max\{deg(f), deg(g)\}$

Lemma 2 (Lemma 2.3.3)**.** *For any field $\mathbb{F}$ and f , $g \in \mathbb{F}[x]$,*

$$deg(fg) = deg(f) + deg(g)$$
$$deg(f+g) \leq \max\{deg(f), deg(g)\}$$

5.3 Corollary 2.3.5: Unit(invertible) in $\mathbb{F}[x]$: constant$\neq 0$ iff $deg(f) = 0$

Corollary 5 (Corollary 2.3.5)**.** *For any field $\mathbb{F}$ and $f \in \mathbb{F}[x]$, Then f is a unit(i.e. invertible) in $\mathbb{F}[x]$ iff $deg(f) = 0$.*

证明.
Obviously, $deg(f) = 0 \Rightarrow\ f$ is a unit.
Suppose f is a unit, i.e. $\exists g \in \mathbb{F}[x]$ s.t. $fg = 1$.
$0 = deg(fg) = deg(f) + deg(g) \Rightarrow deg(f), deg(g) \geq 0 \Rightarrow deg(f) = 0, deg(g) = 0.$ □

5.4 Irreducible Polynomials: "无法分解为两个 $degree \geq 1$ 的多项式积"的多项式: 至少一个是 constant (i.e. $degree = 0$)

A nonconstant polynomial f is irreducible if $f = uv,\ u, v \in \mathbb{F}[x]$, then either u or v is a unit(i.e., constant$\neq 0$)

5.5 Theorem 2.3.6: nonconstant polynomials 可以被唯一地分解

Theorem 7 (Theorem 2.3.6)**.** *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is any nonconstant. Then $f = ap_1p_2 \ldots p_k$ where $a \in \mathbb{F},\ p_1, \ldots p_k \in \mathbb{F}[x]$ are irreducible monic polynomials (monic = i.e. leading coeff. 1). If $f = bq_1q_2 \ldots q_r$ with $b \in \mathbb{F}$ and $q_1, q_2, \ldots, q_r \in \mathbb{F}[x]$ monic irreducible, then $a = b, k = r$, and after reindexing $p_i = q_i,\ \forall i$*

Lemma 3 (Lemma 2.3.7)**.** *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is nonconstant monic polynomial. Then $f = p_1p_2 \ldots p_k$ where each p_i is monic irreducible.*

证明.
Prove it by induction. When $deg(f) = 1,\ f = uv,\ u, v \in \mathbb{F}[x]$, $deg(f) = deg(u) + deg(v) \Rightarrow$ one of these is 0.
Suppose the lemma holds for all degree$< n$. When $deg(f) = n$,
Either f is irreducible, done.
Suppose $f = uv$ with/ $deg(u), deg(v) \geq 1$
$\Rightarrow deg(u), deg(v) < n \Rightarrow u = p_1p_2 \ldots p_k, v = q_1q_2 \ldots q_j$ So, $f = p_1p_2 \ldots p_kq_1q_2 \ldots q_j$. □

Example 7. $x^2 - 1 \in \mathbb{Q}[x]$ *reducible*
$x - 1, x + 1 \in \mathbb{Q}[x]$ *irreducible*
$x^2 + 1 \in \mathbb{Q}[x]$ *irreducible*

$x^2 + 1 \in \mathbb{C}[x]$ *reducible*
$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$ *reducible*

5.6 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$, f divides g, $f|g$ means $\exists u \in \mathbb{F}[x]$ s.t. $g = fu$.

Proposition 16 (Proposition 2.3.8)**.** *$f, h, g \in \mathbb{F}[x]$, then*

(i) If $f \neq 0, f|0$
(ii) If $f|1$, f is nonzero constant
(iii) If $f|g$ and $g|f$, then $f = cg$ for some $c \in \mathbb{F}$
(iv) If $f|g$ and $g|h$, then $f|h$
(v) If $f|g$ and $f|h$, then $f|(ug + vh)$ for all $u, v \in \mathbb{F}[x]$.

5.6.1 Greatest common divisor of f and g: is not unique, we denote monic Greatest common divisor as $gcd(f, g)$

If $f, g \in \mathbb{F}[x]$ are nonzero polynomials, a greatest common divisor of f and g is a polynomial $h \in \mathbb{F}[x]$ such that
(i) $h|f$ and $h|g$, and
(ii) if $k \in \mathbb{F}[x]$ and $k|f$ and $k|g$, then $k|h$.
the *gcd* is not unique, but the monic *gcd* is unique. We call it **the monic greatest common divisor**, denote it $gcd(f, g)$.

Example 8.

$$x^2 - 1, x^2 - 2x + 1 \in \mathbb{Q}[x]$$
$$(x-1)(x+1), (x-1)^2 \in \mathbb{Q}[x]$$
$$x - 1 = gcd(x^2 - 1, x^2 - 2x + 1)$$

5.6.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

Proposition 17 (Proposition 2.3.9)**.** *Given $f, g \in \mathbb{F}[x]$, $g \neq 0$, then $\exists q, r \in \mathbb{F}[x]$ s.t. $deg(r) < deg(g)$ and $f = qg + r$*

Example 9.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$

5.6.3 Proposition 2.3.10: $gcd(f, g)$ 是 degree 最小的 f,g 的线性组合

Proposition 18 (Proposition 2.3.10)**.** *Any 2 nonzero polynomials $f, g \in \mathbb{F}[x]$ have a gcd in $\mathbb{F}[x]$. In fact among all polynomials in the set $M = \{uf + vg|u, v \in \mathbb{F}[x]\}$ any nonconstant of minimal degree are gcds.*

证明.
$h \in M$, $deg(h) = d$ minimal. Let $k|f$ and $k|g \Rightarrow k|uf + vg,\ \forall u, v \Rightarrow k|h$.
Suppose $h' \in M$ is any nonzero element. $deg(h') \geq deg(h) \Rightarrow\ \exists q, r \in \mathbb{F}, deg(r) < deg(h)\ h' = qh + r$. $r = h' - qh \in M$. Since $deg(h) = d$ is nonconstant minimal degree, $r = 0 \Rightarrow h' = qh$. So $\exists q_1, q_2 \in \mathbb{F}[x],\ 1f + 0g = q_1 h, 0f + 1g = q_2 h \Rightarrow h|g, h|f$. □

Example 10.

$$
\begin{aligned}
&f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x] \\
&f = 3g + x^2 - 3x + 2 \\
&g = (x + 1)(x^2 - 3x + 2) + x - 1 \\
&x^2 - 3x + 2 = (x - 2)(x - 1) \\
&\Rightarrow gcd(f, g) = x - 1 \\
&x - 1 = g - (x + 1)(x^2 - 3x + 2) = g - (x + 1)(f - 3g) = (3x + 4)g - (x + 1)f
\end{aligned}
$$

Example 11. *Find a greatest common divisor of $f = x^3 - x^2 - x + 1$ and $g = x^2 - 3x + 2$ in $\mathbb{Q}[x]$, and express it in form $uf + vg,\ u, v \in \mathbb{Q}[x]$.*

$$
\begin{aligned}
&f = (x + 2)g + 3x - 3 \\
&g = \frac{1}{3}(x - 2)(3x - 3) \\
&gcd(f, g) = 3x - 3 \\
&3x - 3 = f - (x + 2)g
\end{aligned}
$$

5.6.4 Proposition 2.3.12: $gcd(f, g) = 1, f|gh \Rightarrow f|h$

Proposition 19 (Proposition 2.3.12)**.** *If $f, g, h \in \mathbb{F}[x]$, $gcd(f, g) = 1$, and $f|gh$, then $f|h$.*

5.6.5 Corollary 2.3.13: irreducible f, $f|gh \Rightarrow f|g$ or $f|h$

Corollary 6 (Corollary 2.3.13)**.** *If $f \in \mathbb{F}[x]$ is irreducible, and $f|gh$, then $f|g$ or $f|h$.*

Since f is irreducible, we have two possible situations:
1. $gcd(f, g) = f$, i.e. $f|g$ done.
2. $gcd(f, g) = 1$, then according to Prop 2.3.12, we can know $f|h$.

5.7 Roots

Root:$\alpha \in \mathbb{F}$ is a root of f if $f(\alpha) = 0$.

5.7.1 Corollary 2.3.16(of Euclidean Algorithm): f 可被分为 $(x-\alpha)q + f(\alpha)$i.e. if α is a root, then $(x-\alpha)|f$

Corollary 7 (Corollary 2.3.16(of Euclidean Algorithm))**.** *$\forall f \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$, there exists a polynomial $q \in \mathbb{F}[x]$ s.t. $f = (x-\alpha)q + f(\alpha)$. In particular , if α is a root, then $(x-\alpha)|f$.*

5.8 Multiplicity

If α is a root of f , say its *multiplicity* is m, if $x - \alpha$ appears m times in irreducible factorization.

5.8.1 *Sum of multiplicity* $\leq deg(f)$

Proposition 20 (Proposition 2.3.17)**.** *Given a nonconstant polynomial $f \in \mathbb{F}[x]$, the number of roots of f, counted with multiplicity, is at most $deg(f)$.*

5.9 Roots in a filed may not in its subfield

Note if $\mathbb{F} \subset \mathbb{K}$, then $\mathbb{F}[x] \subset \mathbb{K}$. $f \in \mathbb{F}[x]$ may have no roots in $\mathbb{F}$, but could have roots in $\mathbb{K}$

Example 12. *$x^n - 1 \in \mathbb{Q}[x]$ has a root in $\mathbb{Q}$: 1; has 2 roots if n even: ± 1*
roots in $\mathbb{C}$ $: \zeta_n = e^{\frac{2\pi i}{n}}$, *then* $\zeta_n^n = e^{2\pi i} = 1$; $(\zeta_n^k)^n = e^{2\pi k i} = 1$ *So, the roots:* $\{e^{\frac{2\pi k i}{n}} | k = 0, ..., n-1\}$
The roots of $x^n - d$: $\{e^{\frac{2\pi k i}{n}} \sqrt{d} | k = 0, ..., n-1\}$

6 Linear Algebra

6.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$)

A vector space over a field $\mathbb{F}$ is a set V w/ an operation addition $+ : V \times V \to V$ and an operation scalar multiplication $\mathbb{F} \times V \to V$

(1) Addition is associative & commutative
(2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$
(3) $1v = v \forall v \in V$(where $1 \in \mathbb{F}$ is multi. id. in $\mathbb{F}$)
(4) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ \alpha(\beta v) = (\alpha\beta)v$
(5) $\forall v \in V,\ (-1)v = -v$ we have $v + (-v) = 0$
(6) $\forall \alpha \in \mathbb{F},\ v, u \in V,\ \alpha(v + u) = \alpha v + \alpha u$
(7) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ (\alpha + \beta)v = \alpha v + \beta v$

6.1.1 A field is a vector space over its subfield

Example 13. $\mathbb{K} \subset \mathbb{F}$ *is a subfield of a field* $\mathbb{F}$*. Then* $\mathbb{F}$ *is a vector space over* $\mathbb{K}$*. (Since* $\mathbb{F} \subset \mathbb{F}[x]$*, then* $\mathbb{F}[x]$ *is a vector space over* $\mathbb{F}$*.)*

6.1.2 Vector subspace

Suppose that V is a vector space over $\mathbb{F}$. A vector subspace or just subspace is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W,\ av \in W,\ \forall v, w \in W,\ a \in \mathbb{F}$.

Example 14. $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$*, then* $\mathbb{L}$ *is a subspace of* $\mathbb{F}$ *over* $\mathbb{K}$*.*

6.2 Linear independent, Linear combination

6.3 span V, basis, dimension, Proposition 2.4.10

A set of elements $v_1, ..., v_n \in V$ is said to **span** V if every vector $v \in V$ can be expressed as a linear combination of $v_1, ..., v_n$. If $v_1, ..., v_n$ spans and is linearly independent, then we call the set a ***basis*** for V.

Proposition 21 (Proposition 2.4.10.)**.** *Suppose* V *is a vector space over a field* $\mathbb{F}$ *having a basis* $\{v_1, ..., v_n\}$ *with* $n \geq 1$*.*

(i) For all $v \in V$, $v = a_1v_1 + ... + a_nv_n$ for exactly one $(a_1, ..., a_n) \in \mathbb{F}^n$.
(ii) If $w_1, ..., w_n$ span V , then they are linearly independent.
(iii)If $w_1, ..., w_n$ are linearly independent, then they span V.
If a vector space V over $\mathbb{F}$ has a basis with n vectors, then V is said to be n-dimensional (over $\mathbb{F}$) or is said to have **dimension** n.

6.3.1 Standard basis vectors

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1) \in \mathbb{F}^n$$

are a basis for $\mathbb{F}^n$ called the **standard basis vectors**.

6.4 Linear transformation

Given two vector spaces V and W over $\mathbb{F}$ a **linear transformation** is a function $T : V \to W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$, we have

$$T(av) = aT(v) \text{ and } T(v + w) = T(v) + T(w)$$

Proposition 22 (Proposition 2.4.15.)**.** *If* V *and* W *are vector spaces and* $v_1, ..., v_n$ *is a basis for* V *then any function from* $\{v_1, ..., v_n\} \to W$ *extends uniquely to a linear transformation* $V \to W$*.*

Any $v \in V$, $\exists(a_1, ..., a_n)$ s.t. $v = a_1v_1 + ... + a_nv_n$. Then $T(v) = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$

6.4.1 Corollary 2.4.16: 一个线性变换对应一个矩阵 *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$

Corollary 8 (Corollary 2.4.16.). *If $v_1, ..., v_n$ is a basis for a vector space V and $w_1, ..., w_n$ is a basis for a vector space W (both over $\mathbb{F}$), then any linear transformation $T : V \to W$ determines (and is determined by) the $m \times n$ matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & ... & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}^T = A \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from V to W; $M_{m\times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in $\mathbb{F}$. $T \to A(T)$ defines a *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** T.

6.4.2 Proposition 2.4.19: 线性变换矩阵相乘仍为线性变换矩阵

Proposition 23 (Proposition 2.4.19). *Suppose that V , W, and U are vector spaces over $\mathbb{F}$, with fixed chosen bases. If $T : V \to W$ and $S : W \to U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \to U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

6.5 GL(V): invertible(bijective) linear transformations $V \to V$

Given a vector space V over F, we let $GL(V) \subset \mathcal{L}(V, V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \textit{ is a bijection}\} = \mathcal{L}(V, V) \cap Sym(V)$$

7 Euclidean geometry basics

7.1 Euclidean distance, inner product

Euclidean distance on $\mathbb{R}^n$:

$$|x - y| = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

Euclidean inner product:

$$x \cdot y = x_1y_1 + \cdots + x_ny_n = x^T y$$

7.2 Isometry of $\mathbb{R}^n$: a bijection $\mathbb{R}^n \to \mathbb{R}^n$ preserves distance

An **isometry** of $\mathbb{R}^n$ is a bijection $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x - y|,\ \forall x, y \in \mathbb{R}^n$$

7.2.1 $Isom(\mathbb{R}^n)$: set of all isometries of $\mathbb{R}^n$

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of $\mathbb{R}^n$,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n | |\Phi(x) - \Phi(y)| = |x - y|,\ \forall x, y \in \mathbb{R}^n\}$$

7.2.2 $Isom(\mathbb{R}^n)$ is closed under $\circ$ and inverse

Proposition 24. *$\Phi, \Psi \in Isom(\mathbb{R}^n)$, then $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$*

证明.
Since Φ, Ψ are bijections, so is $\Phi \circ \Psi$. Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x - y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x - y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

□

7.3 $A \in GL(n, \mathbb{R})$, $T_A(v) = Av$: $A^t A = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a *invertible linear transofrmations* $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t (Aw) = v^t A^t A w$$

$$A^t A = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)}\ T_A \in Isom(\mathbb{R}^n)$$

7.4 Linear isometries i.e. orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\}$

We define the all isometries in *invertible linear transofrmations* $\mathbb{R}^n \to \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\} \subset GL(n, \mathbb{R})$$

7.4.1 Special orthogonal group $SO(n) = \{A \in O(n) | det(A) = 1\}$: orthogonal group with $det(A) = 1$

$O(n)$ are the matrices representing linear isometries of $\mathbb{R}^n$. $1 = det(I) = det(A^t A) = det(A^t) det(A) = det(A)^2 \Rightarrow det(A) = 1$ or $det(A) = -1$. We use **special orthogonal group** represents A with $det(A) = 1$,

$$SO(n) = \{A \in O(n) | det(A) = 1\}$$

7.5 translation: $\tau_v(x) = x + v$

Define a *translation* by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \to \mathbb{R}^n,\ \tau_v(x) = x + v$$

7.5.1 translation is an isometry

Note 3 (Exercise 2.5.3). *$\forall v \in \mathbb{R}^n, \tau_v$ is an isometry.*

证明. $|\tau_v(x) - \tau_v(y)| = |(x+v) - (y+v)| = |x - y|$ □

7.6 The composition of a translation and an orthogonal transformation is an isometry $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since *the composition of isometries is an isometry,* $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries**.

7.6.1 Theorem 2.5.3: All isometries can be represented by a composition of ***a translation*** and ***an orthogonal transformation***, $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

Theorem 8 (Theorem 2.5.3). $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

8 Group $(G, *)$: a set with an operation(associative, identity, inverse)

A *group* is a nonempty set G with an operation $* : G \times G \to G$ s.t. the following holds
(1) $*$ is associative
(2) $\exists e \in G$ an identity element $e * g = g * e = g\ \forall g \in G$
(3) $\forall g \in G, \exists g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$

8.1 $(Sym(X), \circ)$ symmetric/permutation group of X

Example 15. *If X is any nonempty set, permutation group of X : $\{\sigma : X \to X | \sigma$ is a bijection$\}$, then*

1. $\circ$ is associative;
2. $id : X \to X,\ id(x) = x\ \forall x \in X$ is the idenity;
3. $\sigma \in Sym(X), \sigma^{-1} \in Sym(X)$ is the inverse function.

$(Sym(X), \circ)$ **is a group called the symmetric group of** X

8.2 Order of a group $(G, *)$

$|G|$ =Order of a group $(G, *)$

8.3 Abelian group (group that $*$ satisfies commutative)

$(\mathbb{Z}, +)$ is a group and $+$ is commutative, we call this kind of groups(statify commutative) *abelian group.*

Example 16. *If $\mathbb{F}$ is a field, then $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \cdot)$ are abelian group.*

Example 17. *If V is a vector space over $\mathbb{F}$, then $(V, +)$ abelian group.*

As we know a V is a vector space over $\mathbb{F}$ means V is a field whose subfields include $\mathbb{F}$.

8.4 H: Subgroup of $(G, *)$ ($H \neq \emptyset \subset G$ closed under $*$; inverse $g^{-1} \in H$), written as $H < G$

$H \neq \emptyset \subset G$ is a *subgroup* of $(G, *)$ if,
1. $\forall\ g, h \in H$, $g * h \in H$.
2. $\forall\ g \in H$, $g^{-1} \in H$.
write $H < G$ if H is a subgroup of $(G, *)$.

8.4.1 Proposition 2.6.8: $H < G$, $(H, *)$ is a group: A group's operation with its any subgroup is also a group

Proposition 25 (Proposition 2.6.8)**.** *If $(G, *)$ is a group, $H \subset G$ is a subgroup, then $(H, *)$ is a group.*

Example 18. *$(G, *)$ is a group, then $e < G,\ G < G$.*

Example 19. *$\mathbb{K} \subset \mathbb{F}$ is a subfield, then $\mathbb{K} < \mathbb{F},\ \mathbb{K}^\times < \mathbb{F}^\times$.*

Example 20. *$W \subset V$ is a vector subspace, $W < V$.*

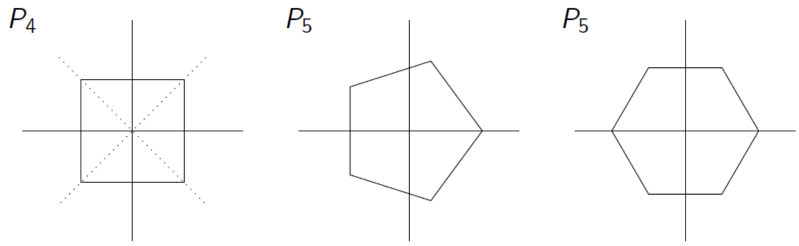
Example 21. *$1 \in S^1 \subset \mathbb{C}^\times$, $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. S^1 is a subgroup.*

证明.
$S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. For any $e^{i\theta}, e^{i\psi} \in S^1$, $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1, e^{-i\theta} \in S^1$. □

Example 22. *$Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$*

Example 23. *If $\mathbb{F}$ is a field, $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \to \mathbb{F} \in Sym(\mathbb{F}) | \sigma(a + b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$*

Example 24. *Dihedral Groups:*



保留多边形

Let $P_n \subset \mathbb{R}^2$ be a regular $n - gon$

$D_n < Isom(\mathbb{R}^2)$, $D_n = \{\Phi \in Isom(\mathbb{R}^2) | \Phi(P_n) = P_n\}$

9 Ring $(R, +, \cdot)$: $+$ is associative, commutative, identity, inverse $\in R$; $\cdot$ is associative, distributes over $+$

Definition 1. *A ring is a nonempty set with two operations, called addition and multiplication,* $(R, +, \cdot)$ *such that*

(1): $(R, +)$ is an ablian group: i.e. $+$ is associatve and commucative. $0, -a \in R$
(2): $\cdot$ is associative.
(3): $\cdot$ distributes over $+$: $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

9.1 Commutative ring: ring's $\cdot$ is commutative

If "$\cdot$" is commutative, we call $(R, +, \cdot)$ a commutative ring.

9.2 Ring with 1: exists multiplication identity $1 \in R$

If there exists an element $1 \in R \backslash \{0\}$ such that $a1 = 1a = a$, $\forall a \in R$,then we say that R is a ring with 1.

9.3 Field $\mathbb{F}$ is a commutative ring with 1; $\mathbb{F}[x]$ is also a commutative ring with 1

Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))
Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

9.4 $S \subset R$: Subring (closed under $+$ and $\cdot$; addictive inverse $-a \in S$)

9.4.1 Proposition 2.6.27: $(S, +, \cdot)$ is a ring

Proposition 26 (Proposition 2.6.27). *If $S \subset R$ is a subring, then $+, \cdot$ make S into a ring.*

10 Group theory

10.1 Properties of Group Operation

10.1.1 Proposition 3.1.1: $g * h = h$ or $h * g = h$, then $g = e$; $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

Proposition 27 (Proposition 3.1.1). *Let $(G, *)$ be a group with identity $e \in G$, then*

(1) if $g, h \in G$ and either $g * h = h$ or $h * g = h$, then $g = e$
(2) if $g, h \in G$ and $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

10.1.2 Corollary 3.1.: $e^{-1} = e,\ (g^{-1})^{-1} = g,\ (g * h)^{-1} = h^{-1} * g^{-1}$

Corollary 9 (Corollary 3.1.2). $e^{-1} = e,\ (g^{-1})^{-1} = g,\ (g * h)^{-1} = h^{-1} * g^{-1}$

10.1.3 Proposition 3.1.3: $g * h = k * h$ or $h * g = h * k$, then $g = k$

Proposition 28 (Proposition 3.1.3). *If $g * h = k * h$ or $h * g = h * k$, then $g = k$.*

10.1.4 Proposition 3.1.4: $g * x = h$ and $x * g = h$ have unique solutions $x \in G$.

Proposition 29 (Proposition 3.1.4). *$g * x = h$ and $x * g = h$ have unique solutions $x \in G$.*

10.2 Power of an Element

We define g^n recursively for $n \geq 0$ by setting $g^0 = e$ and for $n \geq 1$, we set $g^n = g^{n-1} * g$. For $n \leq 0$, we define $g^n = (g^{-1})^{-n}$.

10.2.1 Proposition 3.1.5: $g^n * g^m = g^{n+m}$, $(g^n)^m = g^{nm}$

Proposition 30 (Proposition 3.1.5). *(1) $g^n * g^m = g^{n+m}$; (2) $(g^n)^m = g^{nm}$*

10.3 $(G \times H, \circledast)$: <u>Direct Product</u> of G and H

$(G, *)$ a group $(H, \star)$ a group. Define an operation on $G \times H$, $\circledast$:

$$(h, k) \circledast (h', k') = (h * h', k * k')$$

10.3.1 Proposition 3.1.7: $(G \times H, \circledast)$ is a group

Proposition 31 (Proposition 3.1.7). *$(G \times H, \circledast)$ is a group. The identity is (e_G, e_H), inverse is (g^{-1}, h^{-1})*

usually written as

$$(h, k)(h', k') = (hh', kk')$$

10.4 Subgroups and cyclic groups

10.4.1 Proposition 3.2.2: Intersection of a Collection of Subgroups is a group

Proposition 32 (Proposition 3.2.2)**.** *Let G be a group and suppose $\mathcal{H}$ is any collection of subgroups of G. Then $K = \cap_{H\in\mathcal{H}} H < G$ is a subgroup of G.*

10.4.2 Subgroup Generated by A: $\langle A\rangle = \cap_{H<G;A\subset H} H$

We define **Subgroup Generated by A:**

$$\langle A\rangle = \cap_{H\in\mathcal{H}(A)} H$$

where $\mathcal{H}(A)$ is the set of all subgroups of G containing the set A:

$$\mathcal{H}(A) = \{H < G | A \subset H \textit{ and } H \textit{ is a subgroup of } G\}$$

10.4.3 Cyclic Subgroup generated by a: $\langle a\rangle = \cap_{H<G;a\in H} H$ (G is cyclic if exists g, $\langle g\rangle = G$)

If $A = \{a\}$, then $\langle a\rangle$ $(= \langle\{a\}\rangle)$ = the cyclic subgroup generated by a
Say G is cyclic if $\exists g \in G$, s.t. $G = \langle g\rangle$; g is called a generator for G in this case.

10.4.4 Proposition 3.2.3: $\langle g\rangle = \{g^n | n \in \mathbb{Z}\}$

Proposition 33 (Proposition 3.2.3)**.** *Let G be a group, $g \in G$. Then*

$$\langle g\rangle = \{g^n | n \in \mathbb{Z}\}$$

10.4.5 Corollary 3.2.4: G is a cyclic group $\Rightarrow$ G is abelian

Corollary 10 (Corollary 3.2.4)**.** *If G is a cyclic group (i.e. exits $g \in G$ s.t. $\langle g\rangle = G$), then G is abelian (i.e. commutative).*

10.4.6 Equivalent properties of order of g: $|g| = |\langle g\rangle| < \infty$

Proposition 34 (Proposition 3.2.6)**.** *Let G be a group for $g \in G$, the following are equivalent:*

(i) $|g| < \infty$
(ii) $\exists n \neq m$ in $\mathbb{Z}$ so that $g^n = g^m$
(iii) $\exists n \in \mathbb{Z},\ n \neq 0$ so that $g^n = e$
(iv) $\exists n \in \mathbb{Z}_+$so that $g^n = e$
If $|g| < \infty$, then $|g|$ = smallest $n \in \mathbb{Z}_+$so that $g^n = e$, and $\langle g\rangle = \left\{e, g, g^2, \ldots, g^{n-1}\right\} = \{g^n \mid n = 0, \ldots, n-1\}$

10.4.7 $(\mathbb{Z},+)$ Theorem 3.2.9: $H < \mathbb{Z}$ is a subgroup $\Rightarrow H = \{0\}$ or $H = \langle d\rangle$; $\langle a\rangle < \langle b\rangle$ if and only if $b|a$

Theorem 9 (Theorem 3.2.9)**.** *If $H < \mathbb{Z}$ is a subgroup, then either $H = \{0\}$, or else $H = \langle d\rangle$, where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently, $a \to \langle a\rangle$ defines a **bijection** from $N = \{0, 1, 2, ...\}$ to the set of subgroups of $\mathbb{Z}$. Furthermore, for $a, b \in \mathbb{Z}_+$, we have $\langle a\rangle < \langle b\rangle$ if and only if $b|a$.*

10.4.8 $(\mathbb{Z}_n,+)$ Theorem 3.2.10: $H < \mathbb{Z}_n$ is a subgroup $\Rightarrow H = \langle [d]\rangle$; $\langle [d]\rangle < \langle [d']\rangle$ if and only if $d'|d$

Theorem 10 (Theorem 3.2.10)**.** *For any $n \geq 2$, if $H < \mathbb{Z}_n$ is a subgroup, then there is a positive divisor d of n so that*

$$H = \langle [d]\rangle$$

Furthermore, this defines a bijection between divisors of H and subgroups of $\mathbb{Z}_n$. Furthermore, if $d, d' > 0$ are two divisors of n, then $\langle [d]\rangle < \langle [d']\rangle$ if and only if $d'|d$.

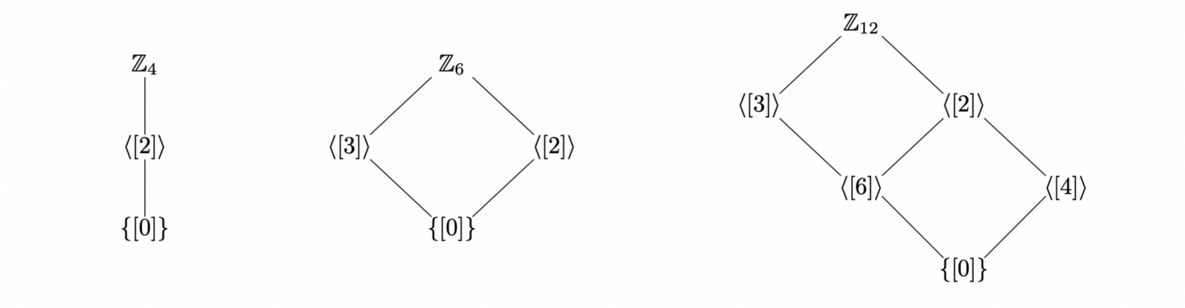
If $H = \langle [d]\rangle$ is a subgroup of H, then $[n] \in H$, so $d|n$. And $|H| = |\langle [d]\rangle| = \frac{n}{d}$, so $|H||d$

10.4.9 Subgroup Lattice

The set of all subgroups of a group of G, together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup $\{e\}$ at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.

参考文献

[1] Christopher J Leininger Introduction to Abstract Algebra (Draft) 2017.



Writing down the subgroup lattice is as easy as writing down the divisibility lattice in which n is placed at the bottom, 1 at the top, and all intermediate divisors in between, connected by edges when there is divisibility. The congruence class of the divisor generates the corresponding subgroup in the subgroup lattice.

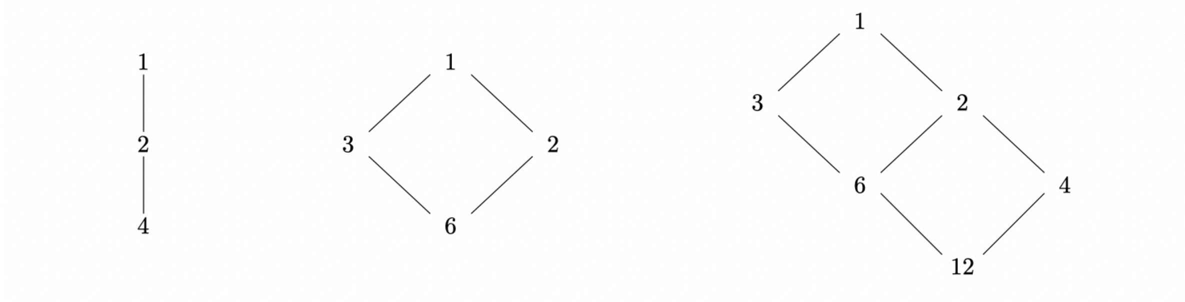


图 1: