Lecture 28 04/01 (or to say, G acts on X).

Recap of Thm 16.3: If there's a group action $*: G \times X \to X$, then
① $\phi: G \to S_G$, $g \mapsto \sigma_g$ is well defined ② ϕ is a homomorphism.

Ex: Let $G = L_3(2)$ be 3×3 invertible matrices over $\mathbb{Z}_2$.

$|G| = 7\cdot 6\cdot 4 = 168$ multiply any nonzero $\vec{x}$ gives us a non zero $\vec{y}$. $(A\vec{x} = \vec{y})$

and let $X = \left\{ \underset{1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} \underset{2}{\begin{pmatrix}0\\1\\0\end{pmatrix}} \underset{3}{\begin{pmatrix}1\\1\\0\end{pmatrix}} \underset{4}{\begin{pmatrix}0\\0\\1\end{pmatrix}} \underset{5}{\begin{pmatrix}1\\0\\1\end{pmatrix}} \underset{6}{\begin{pmatrix}0\\1\\1\end{pmatrix}} \underset{7}{\begin{pmatrix}1\\1\\1\end{pmatrix}} \right\}$

then G acts on X.

For $g = \begin{pmatrix}0&1&0\\1&0&1\\0&1&1\end{pmatrix} \in G$. $g\overset{1}{\begin{pmatrix}1\\0\\0\end{pmatrix}} = \overset{2}{\begin{pmatrix}0\\1\\0\end{pmatrix}}$ $g\overset{2}{\begin{pmatrix}0\\1\\0\end{pmatrix}} = \overset{5}{\begin{pmatrix}1\\0\\1\end{pmatrix}}$

We can find $\sigma_g \in S_X$ is the permutation $\sigma_g = (1\ 2\ 5\ 4\ 6\ 3\ 7) \in S_X$.

the assignment $g \mapsto \sigma_g$ gives the map $\phi: G \to S_7$.

① The map is injective: if $\sigma_g = id \in S_7$ then $\begin{pmatrix}1\\0\\0\end{pmatrix} \to \begin{pmatrix}1\\0\\0\end{pmatrix} = \vec{e}_1$, $\begin{pmatrix}0\\1\\0\end{pmatrix} \to \begin{pmatrix}0\\1\\0\end{pmatrix} = \vec{e}_2$, $\begin{pmatrix}0\\0\\1\end{pmatrix} \to \begin{pmatrix}0\\0\\1\end{pmatrix} = \vec{e}_3$ $\Rightarrow g\vec{x} = \vec{x}$ $\forall \vec{x} = \sum_i m_i \vec{e}_i$

and thus $g = \begin{pmatrix}1&&\\&1&\\&&1\end{pmatrix}$, which means $\ker\phi = \{e\} \Rightarrow$ injective

With the injective map $\phi: L_3(2) \to S_7$.

we can intepret $L_3(2)$ as a subgroup of S_7. $\boxed{L_3(2) \le S_7}$

and Cayley's Thm gives us $L_3(2) \le S_{168}$. (Because G acts on G. and $\phi(G) \le S_{|G|}$).

[Remark]

In general the hom $\phi: G \to S_X$. is not injective.

We say that G acts faithfully on X if ϕ is injective.

(Different elements $g \in G$ are mapped into different permutations)

Important examples of group actions

(Let $H \leq G$ be a subgroup of G)

1. $G \times G \longrightarrow G.$ $(g_1, g_2) \longmapsto g_1 g_2$

2. $G \times G \longrightarrow G$ $(g_1, g_2) \longmapsto g_1 g_2 g_1^{-1}$ conjugation

3. $G \times G/H \longrightarrow G/H$ $(g, aH) \longmapsto gaH.$

(When H is not normal, $X = G/H$ is just a set.)

[Thm] For G acting on X, define a relation $\sim$ on X via

$$x_1 \sim x_2 \iff x_2 = g x_1 \text{ for some } g \in G.$$

Then $\sim$ is an equivalence relation.

[Proof] 1. Reflecxive $x_1 \sim x_1$ $(x_1 = e x_1)$

2. Symmetric $x_1 \sim x_2 \Rightarrow x_2 \sim x_1$ $(x_1 = g^{-1} x_2)$

3. Transitive $x_1 \sim x_2,\ x_2 \sim x_3 \Rightarrow x_1 \sim x_3$ (using $g_1(g_2 x_1) = (g_1 g_2) x_1 = x_3$.)

[Def] For a group action G on X, X partitions into equivalence classes. Denote the class containing x by Gx. Gx is called the orbit of x. $(Gx = \{ gx \mid g \in G \}.)$

[Remark] The partition of X as equivalence classes takes the form $X = (Gx_1) \cup (Gx_2) \cup \cdots \cup (Gx_r).$

r disjoint orbits

[Def] Let G act on X, for $x \in X$, define

$G_x = \{ g \in G \mid gx = x \}$, then

G_x is a subgroup of G called the stabilizer of x.
(or the isotropy subgroup of x)

[Thm] (Orbit-Stabilizer Theorem !) (prove it next class)

Let G act on X, and let $x \in X$, then $|Gx| = [G : G_x] = |G/G_x|$.

[Example] Let S_4 act on itself by conjugation $((g,h) \mapsto ghg^{-1})$

Two elements $x_1, x_2 \in S_4$ are conjugate if and only if they have the same cycle structure. Possible cycle structures are :

$x_1 = (1)$ $\quad |Gx_1| = 1$

$x_2 = (13)(24)$ $\quad |Gx_2| = 3 = \frac{4!}{2!2!} \frac{1}{2!}$

$x_3 = (12)$ $\quad |Gx_3| = 6 = \binom{4}{2}$

$x_4 = (123)$ $\quad |Gx_4| = 8 = \binom{4}{3} \frac{3!}{3}$

$x_5 = (1234)$ $\quad |Gx_5| = 6 = \frac{4!}{4}$

$X = Gx_1 \cup Gx_2 \cup Gx_3 \cup Gx_4 \cup Gx_5$, $\quad 24 = 1+3+6+8+6$.

With the theorem we see that

$|G_{x_1}| = 24$ $\quad G_{x_1} = S_4$

$|G_{x_2}| = 8$ $\quad G_{x_2} = D_8$ $\longrightarrow$ $x_2 = \rho^2$ commutes with any element in D_8

$|G_{x_3}| = 4$ $\quad G_{x_3} = \langle (12), (34) \rangle$ $\quad \{1, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$

$|G_{x_4}| = 3$ $\quad G_{x_4} = \langle (123) \rangle$ $\quad \forall a \in D_8,\ a\rho^2 a^{-1} = aa^{-1}\rho^2 = \rho^2$.

$|G_{x_5}| = 4$ $\quad G_{x_5} = \langle (1234) \rangle$