

# MATH 417

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# 1 Function and Set

## 1.1 Function

$A \times B = \{(a, b) | a \in A, b \in B\}$ .

Function is a rule  $\sigma$  that assigns an element  $B$  to *every* element of  $A$ .

$$\sigma : A \rightarrow B$$

$$\forall a \in A, \sigma(a) \in B.$$

$$\sigma(a) = \text{value of } \sigma \text{ at } a. \text{ (the image of } a \text{)}$$

A set  $C \subset B$ , we call  $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$  as the preimage of  $a$ .

An element  $b \in B$ , we call  $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$  as the fiber of  $b$ .

$A$  is the domain of  $\sigma$ ,  $B$  is the range of  $\sigma$ .

### 1.1.1 Composition of functions

$\sigma : A \rightarrow B, \tau : B \rightarrow C$ . The function  $\tau \circ \sigma : A \rightarrow C$  is  $\forall a \in A, (\tau \circ \sigma)(a) = \tau(\sigma(a))$

### 1.1.2 Proposition 1.1.3: Associativity of Functions

**Proposition 1** (Proposition 1.1.3).  $\sigma : A \rightarrow B, \tau : B \rightarrow C, \rho : C \rightarrow D$  functions then,

$$\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$$

### 1.1.3 Injective, surjective, bijective

A function  $\sigma : A \rightarrow B$  is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, \text{ s.t. } \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

### 1.1.4 Lemma 1.1.7: 两个 injective/surjective/bijective 的方程的 composition 保留性质

**Lemma 1** (Lemma 1.1.7). Suppose  $\sigma : A \rightarrow B, \tau : B \rightarrow C$  are functions,

If  $\sigma, \tau$  are injective, then  $\tau \circ \sigma$  is injective.

If  $\sigma, \tau$  are surjective, then  $\tau \circ \sigma$  is surjective.

If  $\sigma, \tau$  are bijective, then  $\tau \circ \sigma$  is bijective.

### 1.1.5 Proposition 1.1.8: Inverse of Function

**Proposition 2** (Proposition 1.1.8). *A function  $\sigma : A \rightarrow B$  is a bijection if  $\exists$  a function  $\tau : B \rightarrow A$  such that*

$$\sigma \circ \tau = id_B = \text{identity on } B (id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$

Such  $\tau$  is unique, called inverse of  $\sigma$ ,  $\tau = \sigma^{-1}$ .

## 1.2 Set

### 1.2.1 Well Defined Set

**Definition 1.** *A set  $S$  is **well defined** if an object  $a$  is either  $a \in S$  or  $a \notin S$ .*

### 1.2.2 Power Set

**Definition 2.** *For any set  $A$ , we denote by  $\mathcal{P}(A)$  the collection of all subsets of  $A$ .  $\mathcal{P}(A)$  is the **power set** of  $A$ .*

### 1.2.3 Cardinalities of Sets, Pigeonhole Principle

**Definition 3.** *If  $A$  is a set,  $|A|$  = cardinality of  $A$  = # of elements*

$n \in \mathbb{N}$ ,  $|\{1, \dots, n\}| = n$ ;  $|\emptyset| = 0$  ( $\emptyset$  = empty set ).

$|A| = |B|$  if there is a bijection  $\sigma : A \rightarrow B$ .

If there is an *injection*  $\sigma : A \rightarrow B$ , we can write  $|A| \leq |B|$ ;

If there is a *surjection*  $\sigma : A \rightarrow B$ , we can write  $|A| \geq |B|$ .

**Theorem 1** (Pigeonhole Principle). *If  $A$  and  $B$  are sets and  $|A| > |B|$ , then there is no injective function  $\sigma : A \rightarrow B$ .*

### 1.2.4 $B^A$ : Sets of Function

If  $A, B$  are sets, then  $B^A = \{\sigma : A \rightarrow B | \sigma \text{ a function}\}$ .

**Example 1.**  $n \in \mathbb{Z}$ , we define a function  $f : B^{\{1, \dots, n\}} \rightarrow B^n (= B \times B \times B \times \dots \times B)$  by the equation  $f(\sigma) = \{\sigma(1), \dots, \sigma(n)\}$ , where  $\sigma : \{1, \dots, n\} \rightarrow B$ . The  $f$  is a bijection.

证明.

1. *Injective:*

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), \dots, \sigma_1(n)\} = \{\sigma_2(1), \dots, \sigma_2(n)\}$$

Since  $\sigma : \{1, \dots, n\} \rightarrow B$ , it is sufficient to prove  $\sigma_1 = \sigma_2$ .

2. *Surjective*:

$$\forall \{b_1, \dots, b_n\}, \text{ we have } \sigma^*(x) = b_x, x = 1, \dots, n. \text{ s.t. } f(\sigma^*) = \{b_1, \dots, b_n\}$$

□

**Example 2.**

$$C(\mathbb{R}, \mathbb{R}) = \{\text{continuous functions } \sigma : \mathbb{R} \rightarrow \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

### 1.2.5 Operation definitions

**Definition 4.** A binary operation on a set  $A$  is a function  $*$  :  $A \times A \rightarrow A$ .

The operation is associative if  $a * (b * c) = (a * b) * c, \forall a, b, c \in A$ .

The operation is commutative if  $a * b = b * a, \forall a, b \in A$ .

**Example 3.**  $+, \circ$  are both associative and commutative operations on  $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$ ;  $-$  is a neither associative nor commutative operation on  $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ , but not  $\mathbb{N}$ .

**Definition 5.** A subset  $H \subset S$  is closed under  $*$  if  $a * b \in H$  for all  $a, b \in H$ .

**Definition 6.**  $*$  has identity element  $e \in S$  if  $a * e = e * a = a$  for all  $s \in S$ .

## 2 Equivalence relations and Partition

### 2.1 Equivalence relations (理性等价的定义)

理性的等价需要满足: (1)Reflexive, (2)Symmetric, (3)Transitive. Given a set  $X$ , a relation on  $X$  is a subset of  $R \subset X \times X$ . We write  $a \sim b$ .

A relation  $\sim$  is said to be

1. *Reflexive* if  $\forall x \in X$ , we have  $x \sim x$ .
2. *Symmetric* if  $\forall x, y \in X, x \sim y \Rightarrow y \sim x$ .
3. *Transitive* if  $\forall x, y, z \in X, x \sim y, y \sim z \Rightarrow x \sim z$ .

The *sim* is called **equivalence relation** if it is *reflexive*, *Symmetric* and *Transitive*.

**Example 4.** Set  $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$ , satisfies  $(a, b) \sim (c, d)$  if  $ad = bc$ .

1. *Reflexive*:  $(a, b) \sim (a, b), \forall (a, b) \in \mathbb{Z}^2$ .
2. *Symmetric*:  $\forall (a, b), (c, d) \in \mathbb{Z}^2, (a, b) \sim (c, d) \Rightarrow (c, d) \sim (a, b)$ .
3. *Transitive*:  $\forall (a, b), (c, d), (u, v) \in \mathbb{Z}^2, (a, b) \sim (c, d), (c, d) \sim (u, v) \Rightarrow ad = bc, cv = du \Rightarrow acv = adu = bcu \Rightarrow av = bu \Rightarrow (a, b) \sim (u, v)$ .

So this is an equivalence relation.

**Example 5.**  $f : X \rightarrow Y$  is a function, define  $\sim_f$  on  $X$  by  $a \sim_f b$  if  $f(a) = f(b)$ .

1. *Reflexive*:  $a \sim a, \forall a \in X$ .
  2. *Symmetric*:  $a, b \in X, a \sim b \Rightarrow b \sim a$ .
  3. *Transitive*:  $\forall a, b, c \in X, a \sim b, b \sim c \Rightarrow f(a) = f(b) = f(c) \Rightarrow a \sim c$ .
- So  $\sim_f$  is an equivalence relation.

## 2.2 Partition (满足不重叠, 无剩余的 set 拆分结果)

$X$  a set, a partition of  $X$  is a collection  $\omega$  of subsets of  $X$  s.t.

- 1)  $\forall A, B \in \omega$  either  $A = B$  or  $A \cap B = \emptyset$ .
- 2)  $\cup_{A \in \omega} A = X$ .

The subsets are the **cells** of partition.

## 2.3 Equivalence class

### 2.3.1 $[x]$ : equivalence class

Define the **equivalence class** of  $x$  to be the subset  $[x] \subset X$ :

$$[x] = \{y \in X | y \sim x\}$$

Where  $\sim$  is an equivalence relation.

$\sim$  is reflexive  $\Rightarrow x \in [x]$ . We say that any  $y \in [x]$  as a **representative** of the equivalence class.

### 2.3.2 $X/\sim$ : set of equivalence classes

Set of equivalence classes 是一个 **set** 被某种 *equivalence relation* 分类的结果

We write the set of equivalence classes

$$X/\sim = \{[x] | x \in X\}$$

## 2.4 Relationship of Equivalence relation, Set of equivalence classes and Partitions

给定  $X$ , Equivalence relation  $\sim$  与 Set of equivalence classes  $X/\sim$  具有相同的信息量; 包含所有 Partitions 的集合与包含所有 Set of equivalence classes 的集合相同。

### 2.4.1 Theorem 1.2.7: Equivalence relation $\sim \Leftrightarrow$ Set of equivalence classes $X/\sim$ ; {all Sets of equivalence classes} = {all Partitions}

**Theorem 2** (Theorem 1.2.7).  $X/\sim$  is a partition of  $X$ . Conversely, given a partition  $\omega$  of  $X$ , there exists a unique equivalence relation  $\sim_\omega$  s.t.  $X/\sim_\omega = \omega$ .

(1) Equivalence relation  $\sim$  生成一个对应的 Set of equivalence classes  $X/\sim$ , 该  $X/\sim$  就是一个 Partition。(可以看作 1. 所有 Set of equivalence classes 都是 Partitions; 2.  $\sim \Rightarrow X/\sim$  由方式推结果)  
(2) 反之, 我们也可以根据已有的 Partition  $\omega$ , 将其作为一种分类方式  $\sim_\omega$  的 (i.e.  $X/\sim_\omega = \omega$ ) 这个对应的  $\sim_\omega$  存在且是唯一的。(可以看作 1. 所有 Partitions 都是 Set of equivalence classes; 2.  $X/\sim \Rightarrow \sim$  由结果推方式)

证明.

(1)  $X/\sim$  is a partition of  $X$ :

$$\forall x, y \in X \text{ s.t. } [x] \cap [y] \neq \emptyset$$

$$\text{Let } z \in [x] \cap [y] \Rightarrow z \sim x, z \sim y$$

$$\forall w \in [x] \Rightarrow w \sim x \Rightarrow x \sim w \Rightarrow z \sim w \Rightarrow w \sim z \Rightarrow w \sim y \Rightarrow [x] \subset [y]$$

$$\text{Similarly we can prove } [y] \subset [x] \Rightarrow [x] = [y]$$

(2) Given a partition  $\omega$  of  $X$ , there exists a unique equivalence relation  $\sim_\omega$  s.t.  $X/\sim_\omega = \omega$ :

(2.1) Prove there exists an equivalence relation s.t.  $X/\sim_\omega = \omega$ :

We define a relation:  $x \sim_\omega y$  if there exists  $A \in \omega$  s.t.  $x, y \in A \Rightarrow \sim_\omega$  is symmetric and transitive.

Since  $\cup_{A \in \omega} A = X$ , we know  $\forall x \in X, \exists A \in \omega$  s.t.  $x \in A \Rightarrow \sim_\omega$  is reflexive. So  $\sim_\omega$  is an equivalence relation.

We know  $A = [x], \forall A \in \omega, \forall x \in A$  (by  $\sim_\omega$ ), then  $X/\sim_\omega = \{[x] | x \in \cup_{A \in \omega} A\} = \{\{A^* | x \in A^*\} | A^* \in \omega\} = \omega$ .

(2.2) Prove the equivalence relation is unique:

Set  $\sim$  be any equivalence relation that make  $X/\sim = \omega$ , then we know  $\forall A \in \omega, \exists x \in X$  s.t.  $[x] = A$ . According to the definition of  $[x]$ , if  $x \in A, y \sim x$  if and only if  $y \in [x] = A$ . Which is exactly the  $\sim_\omega$ .  $\square$

**Example 6** (the same as example 5).  $f : X \rightarrow Y$  is a function, define  $\sim_f$  on  $X$  by  $a \sim_f b$  if  $f(a) = f(b)$ . In this example the **equivalence classes** are precisely the fibers  $[x] = f^{-1}(f(x))$ .  
 $y \sim_f x \Rightarrow y \in f^{-1}(f(x))$

**Example 7** (the same as example 4). Set  $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$ , satisfies  $(a, b) \sim (c, d)$  if  $ad = bc$ . i.e. we write the equivalence of  $(a, b)$  as  $\frac{a}{b} = [(a, b)]$ . Then  $X/\sim = \mathbb{Q}$ .

**2.4.2 Proposition 1.2.12:** 根据结果  $X/\sim = \{[x] | x \in X\}$  推断的  $\sim_\pi$  equals to  $\sim$ .

**Proposition 3** (Proposition 1.2.12). If  $\sim$  is an equivalence relation on  $X$ , define a surjective function  $\pi : X \rightarrow X/\sim$  by  $\pi(x) = [x]$ . Then  $\sim_\pi = \sim$  (the definition of  $\sim_f$  in example 6.)

证明.

(1) Surjective:

$X/\sim = \{[x] | x \in X\} = \{\pi(x) | x \in X\}$ , so  $\forall y \in X/\sim, y \in \{\pi(x) | x \in X\}$ , there exists  $x \in X$  s.t.  $\pi(x) = y$ .

(2)  $\sim_\pi = \sim$

$a \sim_\pi b$  if  $\pi(a) = \pi(b) \Leftrightarrow [a] = [b]$ , which is exactly the definition of  $\sim$ . □

逻辑:

1. Given  $\sim$ ;
2. Get the corresponding  $X/\sim = \{[x] | x \in X\}$ ;
3.  $\pi(x) = [x]$ ;
4.  $\sim_\pi$ :  $a \sim_\pi b$  iff  $\pi(a) = \pi(b)$
5.  $\sim_\pi = \sim$

根据结果  $X/\sim = \{[x] | x \in X\}$  推断的  $\sim_\pi$  equals to  $\sim$ .

**2.4.3 Proposition 1.2.13:** 给  $X$  标记  $Y: f$ , 给  $X/\sim$  标记  $Y: \tilde{f}$ ,; 两函数之间一一对应

**Proposition 4** (Proposition 1.2.13). *Given any function  $f: X \rightarrow Y$  there exists a unique function  $\tilde{f}: X/\sim \rightarrow Y$  such that  $\tilde{f} \circ \pi = f$ , where  $\pi: X \rightarrow X/\sim$  in proposition 3. Furthermore,  $\tilde{f}$  is a bijection onto the image  $f(X)$ .*

证明.

(1) Existence:

We define  $x_1 \sim_f x_2$  if  $f(x_1) = f(x_2)$ . Set  $\tilde{f}: X/\sim_f \rightarrow Y$ ,  $\tilde{f}([x]) = f(x)$ . Then  $\tilde{f}[\pi(x)] = \tilde{f}([x]) = f(x)$ . Exactly what we require.

(2) Uniqueness:

Set any  $\tilde{f}'$  s.t.  $\tilde{f}' \circ \pi = f$ , then  $\tilde{f}'[\pi(x)] = \tilde{f}'([x]) = f(x)$ , i.e. the  $\tilde{f}$  is unique.

(3) Bijection:

*Surjective*, which we proved before  $\forall f, \exists \tilde{f}$  s.t.  $\tilde{f} \circ \pi = f$ ;

*Injective*, we also have proved the uniqueness  $f = \tilde{f} \circ \pi = \tilde{f}' \circ \pi \Rightarrow \tilde{f} = \tilde{f}'$ . □

### 3 Permutations 改变位置

**Definition 7.** Let  $X$  be a finite set, a permutation is bijection  $\sigma: X \rightarrow X$ .

**Definition 8.** Let  $S_X(Sym(X))$  be the set of all bijection  $\sigma: X \rightarrow X$ .

If  $|X| = n$ ,  $|S_X| = n!$ .

**3.1**  $Sym(X) = \{\sigma: X \rightarrow X | \sigma \text{ is a bijection}\}$ : permutation group of  $X$ ; elements in  $Sym(X)$ : permutations of  $X$

We set  $Sym(X) = \{\sigma: X \rightarrow X | \sigma \text{ is a bijection}\} \subset X^X$ . We call it **symmetric group of  $X$**  or **permutation group of  $X$** . We call the elements in  $Sym(X)$  the **permutations of  $X$**  or the **symmetries of  $X$** .

### 3.1.1 Properties of $\circ$ on $Sym(X)$

**Proposition 5** (Proposition 1.3.1.). *For any nonempty set  $X$ ,  $\circ$  is an operation on  $Sym(X)$  with the following properties:*

- (i)  $\circ$  is associative.
- (ii)  $id_X \in Sym(X)$ , and for all  $\sigma \in Sym(X)$ ,  $id_X \circ \sigma = \sigma \circ id_X = \sigma$ , and
- (iii) For all  $\sigma \in Sym(X)$ ,  $\sigma^{-1} \in Sym(X)$ .

Permutations 类似于 rearrangement, 交换  $X$  中元素的排序。

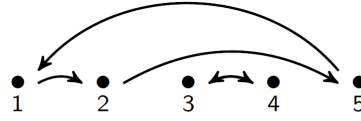
### 3.1.2 $S_n$ : Permutation group on $n$ elements, $\sigma^i$

**Note 1.** When  $X = \{1, \dots, n\}$ ,  $n \in \mathbb{Z}$ , write  $S_n = Sym(X)$  **symmetric/permutation group on  $n$  elements**.

**Note 2.**  $\sigma \in Sym(X)$ , write  $\sigma^n = \sigma \circ \sigma \circ \dots \circ \sigma$ ,  $\sigma^0 = id_X$ ,  $\sigma^{-1} = \text{inverse}$ ,  $r > 0$ ,  $\sigma^{-r} = (\sigma^{-1})^r$ . So,  $r, s \in \mathbb{Z}$ ,  $\sigma^{r+s} = \sigma^r \circ \sigma^s = \sigma^s \circ \sigma^r$ .

### 3.1.3 $k$ -cycle, cyclically permute/fix

**Example 8.**



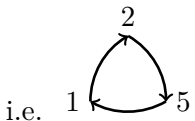
$$1 \xrightarrow{\sigma} 2 \xrightarrow{\sigma} 5 \xrightarrow{\sigma} 1, \quad \tau_1$$

$$3 \xrightarrow{\sigma} 4 \xrightarrow{\sigma} 3, \quad \tau_2$$

图 1: Example of Cycle

In the example of *Figure 1*,  $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$ ,  $\sigma = \tau_1 \circ \tau_2$ , where  $\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}$ ,

$\tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$ .  $\tau_1$  is 3-cycle,  $\tau_2$  is 2-cycle. We could represent  $\tau_1 = (1 \ 5 \ 2) = (5 \ 2 \ 1) = (2 \ 1 \ 5)$ ,



i.e.  $1 \xrightarrow{\tau_1} 2 \xrightarrow{\tau_1} 5 \xrightarrow{\tau_1} 1$ . Similarly, we can represent  $\tau_2 = (3 \ 4) = (4 \ 3)$ , i.e.  $3 \longleftrightarrow 4$

We can find that  $\forall x \in \{1, 2, 3, 4, 5\}$ ,  $\tau_1^3(x) = x$ ,  $\tau_2^2(x) = x$ , so we write  $\tau_1$  as a **3-cycle** in  $S_5$ ,  $\tau_2$  as a

**2-cycle** in  $S_5$ .

Given  $k \geq 2$ , a **k-cycle** in  $S_n$  is a permutation  $\sigma$  with the property that  $\{1, \dots, n\}$  is the union of two disjoint subsets,  $\{1, \dots, n\} = Y \cup Z$  and  $Y \cap Z = \emptyset$ , such that

1.  $\sigma(x) = x$  for every  $x \in Z$ , and
2.  $|Y| = k$ , and for any  $x \in Y, Y = \{\sigma(x), \sigma^2(x), \sigma^3(x) \dots \sigma^k(x) = x\}$ .

We say that  $\sigma$  **cyclically permutes** the elements of  $Y$  and **fixes** the elements of  $Z$ .

$\tau_1 = (1\ 2\ 5)$  **cyclically permutes** the elements of  $Y = \{1, 2, 5\}$  and **fixes** the elements of  $Z = \{3, 4\}$ .

$\tau_2 = (3\ 4)$  **cyclically permutes** the elements of  $Y = \{3, 4\}$  and **fixes** the elements of  $Z = \{1, 2, 5\}$ .

### 3.2 Disjoint cycles

Since the sets are cyclically permuted by  $\tau_1, \tau_2$  (i.e.  $Y$ ) are disjoint. We call the **disjoint cycle notation**  $\sigma = \tau_1 \circ \tau_2 = (1\ 2\ 5)(3\ 4)$ . (Commute the order is irrelevant)

#### 3.2.1 Theorem: Every permutation is a union of disjoint cycles, uniquely.

Given  $\sigma \in S_n$ , there exists a unique (possibly empty) set of pairwise disjoint cycles.

**Theorem 3.** *Let  $X$  be a finite set, the graph of permutation  $\sigma \in S_X$  is a union of disjoint cycle.*

证明. Prove by induction:



If  $|X| = 1$ , the graph is circle:

For  $|X| > 1$ , let  $i_1 \in X$  and let  $\mathcal{O}(i_1) = \{\sigma^r(i_1), r \geq 0\} = \{i_1, \sigma(i_1), \sigma^2(i_1), \dots\}$ .  $\mathcal{O}(i_1)$  is finite, and there is a smallest  $r$  s.t.  $\sigma^r(i_1) \in \{i_1, \sigma(i_1), \sigma^2(i_1), \dots, \sigma^{r-1}(i_1)\}$ . Then  $\sigma^r(i_1) = i_1$  because other elements already have a pre-change under  $\sigma$ .

Then  $i_1 \rightarrow \sigma(i_1) \rightarrow \sigma^2(i_1) \rightarrow \dots \rightarrow \sigma^{r-1}(i_1) \rightarrow i_1$  is a cycle of length  $r$ .

For  $j \notin \mathcal{O}(i_1), \sigma(j) \notin \mathcal{O}(i_1), \sigma^{-1}(j) \notin \mathcal{O}(i_1)$ . Let  $Y = X/\mathcal{O}(i_1)$  then  $\sigma : Y \rightarrow Y$  is a bijection. Then prove by induction.

□

**Example 9.**  $\sigma_1 = (1\ 2\ 6\ 5)(3)(4)$ , can be written by  $\sigma_1 = (1\ 2\ 6\ 5), \sigma_2 = (2\ 3\ 5\ 4)$



$$\sigma_1 \circ \sigma_2 = (1 \ 2 \ 6 \ 5) \circ (2 \ 3 \ 5 \ 4)$$

$$\begin{array}{ccc} 1 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 1 \xrightarrow{(1 \ 2 \ 6 \ 5)} 2 \\ 2 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 3 \xrightarrow{(1 \ 2 \ 6 \ 5)} 3 \\ 3 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 5 \xrightarrow{(1 \ 2 \ 6 \ 5)} 1 \\ 4 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 2 \xrightarrow{(1 \ 2 \ 6 \ 5)} 6 \\ 5 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 4 \xrightarrow{(1 \ 2 \ 6 \ 5)} 4 \\ 6 & \xrightarrow{(2 \ 3 \ 5 \ 4)} & 6 \xrightarrow{(1 \ 2 \ 6 \ 5)} 5 \end{array}$$

$$\text{Then } \sigma_1 \circ \sigma_2 = (1 \ 2 \ 3) \circ (4 \ 6 \ 5)$$

$$\sigma_2 \circ \sigma_1 = (2 \ 3 \ 5 \ 4) \circ (1 \ 2 \ 6 \ 5)$$

$$\begin{array}{ccc} 1 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 2 \xrightarrow{(2 \ 3 \ 5 \ 4)} 3 \\ 2 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 6 \xrightarrow{(2 \ 3 \ 5 \ 4)} 6 \\ 3 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 3 \xrightarrow{(2 \ 3 \ 5 \ 4)} 5 \\ 4 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 4 \xrightarrow{(2 \ 3 \ 5 \ 4)} 2 \\ 5 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 1 \xrightarrow{(2 \ 3 \ 5 \ 4)} 1 \\ 6 & \xrightarrow{(1 \ 2 \ 6 \ 5)} & 5 \xrightarrow{(2 \ 3 \ 5 \ 4)} 4 \end{array}$$

$$\text{Then } \sigma_2 \circ \sigma_1 = (1 \ 3 \ 5) \circ (2 \ 6 \ 4)$$

$$\text{Note: } \sigma_1 \circ \sigma_2 \neq \sigma_2 \circ \sigma_1$$

**Example 10** (Exercise 1.3.2.). Consider  $\sigma = (3 \ 4 \ 8)(5 \ 7 \ 6 \ 9)$  and  $\tau = (1 \ 9 \ 3 \ 5)(2 \ 7 \ 4)$  in  $S_9$  expressed in disjoint cycle notation. Compute  $\sigma \circ \tau$  and  $\tau \circ \sigma$  expressing both in disjoint cycle notation.

$$\begin{aligned} 1 &\rightarrow \sigma(\tau(1)) = \sigma(9) = 5; \quad 2 \rightarrow \sigma(\tau(2)) = \sigma(7) = 6; \\ 3 &\rightarrow \sigma(\tau(3)) = \sigma(5) = 7; \quad 4 \rightarrow \sigma(\tau(4)) = \sigma(2) = 2; \\ 5 &\rightarrow \sigma(\tau(5)) = \sigma(1) = 1; \quad 6 \rightarrow \sigma(\tau(6)) = \sigma(6) = 9; \\ 7 &\rightarrow \sigma(\tau(7)) = \sigma(4) = 8; \quad 8 \rightarrow \sigma(\tau(8)) = \sigma(8) = 3; \\ 9 &\rightarrow \sigma(\tau(9)) = \sigma(3) = 4; \\ \Rightarrow \sigma \circ \tau &= (1 \ 5)(2 \ 6 \ 9 \ 4)(3 \ 7 \ 8) \end{aligned}$$

$$\begin{aligned} 1 &\rightarrow \tau(\sigma(1)) = \tau(1) = 9; \quad 2 \rightarrow \tau(\sigma(2)) = \tau(2) = 7; \\ 3 &\rightarrow \tau(\sigma(3)) = \tau(4) = 2; \quad 4 \rightarrow \tau(\sigma(4)) = \tau(8) = 8; \\ 5 &\rightarrow \tau(\sigma(5)) = \tau(7) = 4; \quad 6 \rightarrow \tau(\sigma(6)) = \tau(9) = 3; \\ 7 &\rightarrow \tau(\sigma(7)) = \tau(6) = 6; \quad 8 \rightarrow \tau(\sigma(8)) = \tau(3) = 5; \\ 9 &\rightarrow \tau(\sigma(9)) = \tau(5) = 1; \\ \Rightarrow \tau \circ \sigma &= (1 \ 9)(2 \ 7 \ 6 \ 3)(4 \ 8 \ 5) \end{aligned}$$

**Example 11.** Let  $\sigma, \tau \in S_7$ , given in disjoint cycle, notation by  $\sigma = (1\ 5\ 4)(3\ 7), \tau = (1\ 3\ 2\ 6\ 4)$ , Compute  $\sigma^2, \sigma^{-1}, \tau \circ \sigma$

$$\begin{aligned}\sigma^2 &= (1\ 4\ 5), & \sigma^{-1} &= (4, 5, 1)(3, 7), \\ 1 \rightarrow \tau(\sigma(1)) &= \tau(5) = 5, & 2 \rightarrow \tau(\sigma(2)) &= \tau(2) = 6, \\ 3 \rightarrow \tau(\sigma(3)) &= \tau(7) = 7, & 4 \rightarrow \tau(\sigma(4)) &= \tau(1) = 3, \\ 5 \rightarrow \tau(\sigma(5)) &= \tau(4) = 1, & 6 \rightarrow \tau(\sigma(6)) &= \tau(6) = 4, \\ 7 \rightarrow \tau(\sigma(7)) &= \tau(3) = 2, \\ \Rightarrow \tau \circ \sigma &= (1, 5)(2, 6, 4, 3, 7)\end{aligned}$$

### 3.2.2 Cycle Structure

- How many permutation  $\sigma \in S_{12}$  has cycle structure  $(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9\ 10)(11\ 12)$ ?

$$\frac{12!}{3^2 2^3 (2!)(3!)}$$

12!: 每个位置的排列

3<sup>2</sup>: 每个长度 3 的 cycle 的每种情况会被重复计算 3 次

2<sup>3</sup>: 每个长度 2 的 cycle 的每种情况会被重复计算 2 次

(2!): 2 个长度 3 的 cycle 具有不同位置的排列

(3!): 3 个长度 2 的 cycle 具有不同位置的排列

- $(1\ 2\ 3)(4\ 5)(6) \in S_6$ ?

$$\frac{6!}{3 \times 2} = 120$$

- General situation:  $\sigma \in S_n$ ,  $r_i$  category of length  $i$ ,  $i = 1, 2, \dots$

$$\frac{n!}{[1^{r_1} 2^{r_2} 3^{r_3} \dots][(r_1!)(r_2!)(r_3!) \dots]}$$

### 3.3 Transposition

**Definition 9.** A **transposition** is a cycle of length 2:  $\sigma = (i\ j)$ .

**3.3.1 Theorem:** 每个 permutation 可以由若干个 (可能不 disjoint 的) transposition 表示

**Theorem 4.** Every permutation  $\sigma$  of  $X$  is a product of transposition. (the product is not unique)

**Equivalent:** Given  $n \geq 2$ , any  $\sigma \in S_n$  can be expressed as a composition of 2-cycles. (not require disjoint)

证明.

Version 1:

$$\begin{aligned}
(x_1 \ x_k)(x_1 \ x_2, \dots, x_{k-1} \ x_k) &= (x_1 \ x_2 \ \dots \ x_{k-1}) \\
(x_1 \ x_2 \ \dots \ x_{k-1} \ x_k) &= (x_1 \ x_k)(x_1, x_2 \ \dots \ x_{k-1}) \\
&= (\mathbf{x}_1 \ \mathbf{x}_k)(\mathbf{x}_1 \ \mathbf{x}_{k-1})(\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_{k-2}) \\
&\dots \\
&= (\mathbf{x}_1 \ \mathbf{x}_k)(\mathbf{x}_1 \ \mathbf{x}_{k-1})(\mathbf{x}_1 \ \mathbf{x}_{k-2}) \dots (\mathbf{x}_1 \ \mathbf{x}_2)
\end{aligned}$$

Version 2:

$$\begin{aligned}
(x_1 \ x_2, \dots, x_{k-1} \ x_k)(x_1 \ x_k) &= (x_2 \ x_3 \ \dots \ x_k) \\
(x_1 \ x_2 \ \dots \ x_{k-1} \ x_k) &= (x_2 \ x_3 \ \dots \ x_k)(x_1 \ x_k) \\
&\dots \\
&= (\mathbf{x}_{k-1} \ \mathbf{x}_k)(\mathbf{x}_{k-2} \ \mathbf{x}_k) \dots (\mathbf{x}_2 \ \mathbf{x}_k)(\mathbf{x}_1 \ \mathbf{x}_k)
\end{aligned}$$

□

**Claim 1.** Cycle of length  $k$  can be written as a product of  $k - 1$  transpositions.

### 3.3.2 Sign of Permutation

**Theorem 5.** Although the product of transposition of a permutation is not unique, the parity (odd or even) of the number of transpositions in a product is unique. We call it the **sign** of permutation.

$$\begin{aligned}
\text{sign}(\sigma) &= (-1)^{(\# \text{ even-length cycles in } \sigma)} \\
&= (-1)^{(\# \text{ transpositions in } \sigma)}
\end{aligned}$$

**Example 12.**

$\sigma_1 = (1 \ 4 \ 7 \ 9)(2 \ 8)(6 \ 10)$ :  $N = 3 + 1 + 1 = 5$  is odd.

$\sigma_2 = (1 \ 2 \ 3 \ 4 \ 5)(6 \ 7 \ 8 \ 9 \ 10)$ :  $N = 4 + 4 = 8$  is even

What happens to a permutation  $\sigma$ 's cycles if  $\sigma \rightarrow (i \ j) \circ \sigma$ ?

1.  $i$  and  $j$  are not contained in  $\sigma$ .
2.  $i$  and  $j$  appear in the same cycle of  $\sigma$ .
3.  $i$  and  $j$  appear in disjoint cycles.

$$\begin{aligned}
(i \ j) \circ (i - -j \sim \sim) &= (i - -) \circ (j \sim \sim) \\
(i \ j) \circ (i - -) \circ (j \sim \sim) &= (i - -j \sim \sim)
\end{aligned}$$

**Proposition 6.**  $\text{sign}((i \ j) \circ \sigma) = -1 \cdot \text{sign}(\sigma)$

证明.

Suppose  $\sigma = (a_1 \ a_2 \ \cdots a_k \ b_1 \ b_2 \ \cdots b_l)$

Then  $(a_1 \ b_1) \circ \sigma = (a_1 \ a_2 \ \cdots a_k)(b_1 \ b_2 \ \cdots b_l)$

$$\text{sign}(\sigma) = \begin{cases} +1 & \text{if } k+l \text{ is odd} \\ -1 & \text{if } k+l \text{ is even} \end{cases}$$

$$\text{sign}((a_1 \ b_1) \circ \sigma) = \begin{cases} -1 & \text{if } k+l \text{ is odd} \\ +1 & \text{if } k+l \text{ is even} \end{cases}$$

□

## 4 Integers

### 4.1 Proposition 1.4.1: Properties of integers $\mathbb{Z}$

**Proposition 7** (Proposition 1.4.1.). *The following hold in the integers  $\mathbb{Z}$ :*

- (i) *Addition and multiplication are commutative and associative operations in  $\mathbb{Z}$ .*
- (ii)  $0 \in \mathbb{Z}$  is an identity element for addition; that is,  $\forall a \in \mathbb{Z}, 0 + a = a$ .
- (iii) Every  $a \in \mathbb{Z}$  has an additive inverse, denoted  $-a$  and given by  $-a = (-1)a$ , satisfying  $a + (-a) = 0$ .
- (iv)  $1 \in \mathbb{Z}$  is an identity element for multiplication; that is, for all  $a \in \mathbb{Z}, 1a = a$ .
- (v) The *distributive* law holds:  $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$ .
- (vi) Both  $\mathbb{N} = \{x \in \mathbb{Z} | x \geq 0\}$  and  $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$  are *closed* under *addition* and *multiplication*. That is, if  $x$  and  $y$  are in one of these sets, then  $x + y$  and  $xy$  are also in that set.
- (vii) For any two nonzero integers  $a, b \in \mathbb{Z}, |ab| \geq \max\{|a|, |b|\}$ . Strict inequality holds if  $|a| > 1$  and  $|b| > 1$ .

From this we get cancellation.

$$ab = ac \Rightarrow b = c \text{ or } a = 0$$

### 4.2 Definition: Divide

Suppose  $a, b \in \mathbb{Z}, b \neq 0$ ,  $b$  divides  $a$  if  $\exists m \in \mathbb{Z}$ , so that  $a = bm, b|a$ . Otherwise, write  $b \nmid a$ .

### 4.3 Proposition 1.4.2: properties of integer division

**Proposition 8** (Proposition 1.4.2).  $\forall a, b \in \mathbb{Z}$

- (i) if  $a \neq 0$ , then  $a|0$
- (ii) if  $a|1$ , then  $a = \pm 1$
- (iii) if  $a|b$  &  $b|a$ , then  $a = \pm b$
- (iv) if  $a|b$  &  $b|c$ , then  $a|c$

(v) if  $a|b$  &  $a|c$ , then  $a|(mc + nb) \forall m, n \in \mathbb{Z}$

#### 4.4 Definitions: Prime, The Greatest common divisor $\gcd(a, b)$

$p > 1, p \in \mathbb{Z}$  is called prime if the only divisors are  $\pm 1, \pm p$ .

Given  $a, b \in \mathbb{Z}, a, b \neq 0$ , the greatest common divisor of  $a$  and  $b$  is  $c \in \mathbb{Z}, c > 0$  s.t.

(1)  $c|a$  and  $c|b$ ; (2) if  $d|a, d|b$ , then  $d|c$

The  $c$  is unique, we write it  $\gcd(a, b)$ .

#### 4.5 Euclidean Algorithm

**Proposition 9** (Proposition 1.4.7(Euclidean Algorithm)). *Given  $a, b \in \mathbb{Z}, b \neq 0$ , then  $\exists q, r \in \mathbb{Z}$  s.t.  $a = qb + r, 0 \leq r < |b|$ .*

**Example 13** (Exercise 1.4.3). *For the pair  $(a, b) = (130, 95)$ , find  $\gcd(a, b)$  using the Euclidean Algorithm and express it in the form  $\gcd(a, b) = sa + tb$  for  $s, t \in \mathbb{Z}$ .*

$$130 = 95 + 35; \quad 95 = 2 \times 35 + 25$$

$$35 = 25 + 10; \quad 25 = 2 \times 10 + 5$$

$$10 = 2 \times 5 + 0$$

$$\begin{aligned} 5 &= 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35 \\ &= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130 \end{aligned}$$

$$\gcd(130, 95) = \gcd(95, 35) = \gcd(35, 25) = \gcd(25, 10) = \gcd(10, 5) = \gcd(5, 0) = 5$$

We can also express it by matrix

	$q$	$r$	$s$	$t$
-1		130	1	0
0	1	95	0	1
1	2	35	1	-1
2	1	25	-2	3
3	2	10	3	-4
4	2	5	-8	11

Hence  $\gcd(130, 95) = 5 = -8 \cdot 130 + 11 \cdot 95$

**4.6 Proposition:**  $\gcd(a, b)$  exists and is the smallest positive integer in the set  $M = \{ma + nb | m, n \in \mathbb{Z}\}$

**Theorem 6.**  $d = \gcd(a, b)$  is of the form  $sa + tb$

证明. We may assume  $0 \leq a \leq b$

For  $a = 0$ ,  $d = b = 0 \cdot a + 1 \cdot b$ .

For  $a > 0$ , let  $b = q \cdot a + r$  with  $0 \leq r < a \leq b$ . Then

$$\begin{aligned}\{sa + tb : s, t \in \mathbb{Z}\} &= \{sa + t(q \cdot a + r) : s, t \in \mathbb{Z}\} = \{tr + ua : t, u \in \mathbb{Z}\} \\ &= \dots \{x \cdot 0 + y \cdot d : x, y \in \mathbb{Z}\} = \{\dots, -2d, -d, 0, d, 2d, \dots\}\end{aligned}$$

□

**Proposition 10** (第二种表示, 第二种证明).  $\forall a, b \in \mathbb{Z}$ , not both 0,  $\gcd(a, b)$  exists and is the smallest positive integer in the set  $M = \{ma + nb | m, n \in \mathbb{Z}\}$ . i.e.  $\exists m_0, n_0 \in \mathbb{Z}$  s.t.  $\gcd(a, b) = m_0a + n_0b$ .

证明. Let  $c$  be the smallest positive integer in the set  $M = \{ma + nb | m, n \in \mathbb{Z}\}$ .  $c = m_0a + n_0b > 0$ .

Let  $d = ma + nb \in M$ ,  $d = qc + r$  where  $0 \leq r < c$  (by Euclidean Algorithm).

$$r = d - qc = (m - qm_0)a + (n - qn_0)b \in M$$

Since  $c$  is the smallest integer in  $M$  and  $r \in [0, c)$ , so  $r = 0$ .  $\Rightarrow d = qc$ . So  $c | d$ .

$a = 1a + 0b \in M \Rightarrow c | a$ ,  $b = 0a + 1b \in M \Rightarrow c | b$ .

If  $t | a, t | b$  then  $t | m_0a + n_0b$  i.e.  $t | c$ .  $\Rightarrow c = \gcd(a, b)$ .

□

## 4.7 Well-Ordering Principle (Least Integer Axiom)

There is a smallest integer in every nonempty subset  $S$  of the natural numbers  $\mathbb{N} = \{0, 1, 2, \dots\}$

### 4.8 Proposition 1.4.10: $\gcd(b, c), b | ac \Rightarrow b | a$

**Proposition 11** (Proposition 1.4.10). Suppose  $a, b, c \in \mathbb{Z}$ . If  $b, c$  are relatively prime i.e.  $\gcd(b, c) = 1$  and  $b | ac$ , then  $b | a$ .

证明.  $\gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$  s.t.  $1 = mb + nc \Rightarrow a = amb + anc$ . Since  $b | nac, b | amb \Rightarrow b | a$ .

□

#### 4.8.1 Corollary: $p | ab \Rightarrow p | a$ or $p | b$

**Corollary 1** (Corollary of Prop 1.4.10).  $a, b, p \in \mathbb{Z}, p > 1$  prime. If  $p | ab$ , then  $p | a$  or  $p | b$ .

证明. If  $p | b$ , done. Otherwise,  $\gcd(p, b) = 1$ . By Prop 1.4.10,  $p | a$ .

□

## 4.9 Fundamental Theorem of Arithmetic: Any integer $a \geq 2$ has a unique prime factorization

### 4.9.1 Existence

**Lemma 2.** Any integer  $a \geq 2$  is either a prime or a product of primes.

证明. Set  $S \subset \mathbb{N}$  be the set of all  $n$  without the given property.

Assume that  $S$  is nonempty and  $m$  is the least element in  $S$ .

Since  $m$  is not a prime, it can be written as  $m = ab$  with  $1 < a, b < m$ . Since  $m$  is the least element in  $S$ ,  $a, b \notin S$ . Then  $m$  is a product of primes. Contradiction. Thus,  $S = \emptyset$ .  $\square$

#### 4.9.2 Uniqueness

**Theorem 7** (Fundamental Theorem of Arithmetic).

Any integer  $a > 1$  has a unique prime factorization:  $a = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_n^{k_n}$  where  $p_i > 1$  is prime,  $k_i \in \mathbb{Z}_+, \forall i = 1, \dots, n, p_i \neq p_j, \forall i \neq j$ .

证明.

a) Existence: (Previous Lemma)

b) Uniqueness:

1) Method 1:

Suppose  $a = p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot \dots \cdot q_j^{r_j}$ . Where  $p_1 > p_2 > \dots > p_k, q_1 > q_2 > \dots > q_j, n_i, r_i \geq 1$ .

$p_1 | a \Rightarrow \exists q_i \text{ s.t. } p_1 | q_i$ . Similarly,  $\exists q_i \text{ s.t. } q_1 | p_{i'}$ .

$q_1 \leq p_{i'} \leq p_1 \leq q_i \Rightarrow q_1 = p_{i'} = p_1 = q_i$

We can also know  $n_1 = r_1$ , otherwise we would have two prime factorization of the quotient where the largest primes are different by dividing  $p_1^{\min\{n_1, r_1\}}$ .

Then we can get  $b = p_2^{n_2} \cdot \dots \cdot p_k^{n_k} = q_2^{r_2} \cdot \dots \cdot q_j^{r_j}$ . Then prove it by induction.

2) Method 2:

Suppose  $a = p_1 \cdot p_2 \cdot \dots \cdot p_k = q_1 \cdot q_2 \cdot \dots \cdot q_t$ . For a  $p_i$ , there must exist a  $q_j$  s.t.  $p_i = q_j$ :

Assume that  $p_i \neq q_t, \gcd(p_i, q_t) = 1$ . Then  $\exists a, b$  such that  $1 = ap_i + bq_t$ . Multiplying both sides by  $q_1 \cdot q_2 \cdot \dots \cdot q_{t-1}$ :

$$q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} = ap_i q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} + bq_1 \cdot q_2 \cdot \dots \cdot q_t$$

Since  $p_i | q_1 \cdot q_2 \cdot \dots \cdot q_t$ , we can conclude that  $p_i | (ap_i q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} + bq_1 \cdot q_2 \cdot \dots \cdot q_t)$

i.e.  $p_i | q_1 \cdot q_2 \cdot \dots \cdot q_{t-1}$  if  $p_i \neq q_t$

Then prove by induction.

$\square$

## 5 Modular arithmetic

### 5.1 Congruences

#### 5.1.1 Congruent modulo $m$ : $a \equiv b \pmod{m}$

Given  $m \in \mathbb{Z}_+$ , define a relation on  $\mathbb{Z}$ : congruence modulo  $m$

$$a \equiv b \pmod{m}, \text{ if } m|(a - b)$$

Read as " $a$  is congruent to  $b \pmod{m}$ "; Notation:  $a \equiv b \pmod{m}$ .

Equivalent to:  $a, b$  have the same remainder after division by  $m$ .

#### 5.1.2 Proposition: For fixed $m \geq 2$ , the relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " is an equivalence relation

**Proposition 12** (Proposition 1.5.1). *For fixed  $m \geq 2$ , the relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " is an equivalence relation*

证明.

- 1) Reflexive:  $\forall a \in \mathbb{Z}, m|0 = (a - a)$ , so  $a \equiv a \pmod{m}$  i.e.  $a \sim a$ .
- 2) Symmetric:  $\forall a, b \in \mathbb{Z}, a \equiv b \pmod{m}$ , then  $m|(a - b) \Rightarrow m|(b - a) \Rightarrow b \equiv a \pmod{m}$ . i.e.  $a \sim b \Rightarrow b \sim a$ .
- 3) Transitive:  $\forall a, b, c \in \mathbb{Z}, a \equiv b \pmod{m}, b \equiv c \pmod{m}$ . Then  $m|(a - b), m|(b - c) \Rightarrow m|(a - b) + (b - c) = (a - c) \Rightarrow a \equiv c \pmod{m}$ .

□

#### 5.1.3 Theorem: the equivalence relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " partitions the integers into $m$ disjoint sets $\Omega_i = \{a | a \sim i\}, i = 0, 1, \dots, m - 1$

**Theorem 8.** *the equivalence relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " partitions the integers into  $m$  disjoint sets  $\Omega_i = \{a | a \sim i\}, i = 0, 1, \dots, m - 1$*

证明. Prove any  $a \in \mathbb{Z}$  belongs to a unique  $\Omega_i$ .

- a) Existence: Division Algorithm  $\Rightarrow a = qm + r, 0 \leq r < m. a \in \Omega_r$ .
- b) Uniqueness: Assume  $a$  in two sets,  $a \in \Omega_r \cap \Omega_{r^1}, 0 \leq r^1 < r < m$ .

Then  $m|a - r$  and  $m|a - r^1 \Rightarrow m|r - r^1$ , which is impossible because  $0 < r - r^1 < m$ . Contradiction.

□



### 5.1.4 Proposition: Addition and Mutiplication of Congruences

**Proposition 13.** *Fix integer  $m \geq 2$ . If  $a \equiv r \pmod{m}$  and  $b \equiv s \pmod{m}$ , then  $a + b \equiv r + s \pmod{m}$  and  $ab \equiv rs \pmod{m}$*

证明.

a) Addition:  $m|(a - r), m|(b - s) \Rightarrow m|(a - r) + (b - s) \Rightarrow m|(a + b) - (r + s)$ .

b) Mutiplication:  $m|(a - r)b + r(b - s) \Rightarrow m|ab - rs$ .

□

## 5.2 Solving Linear Equations on Modular $m$

### 5.2.1 Theorm: unique solution of $aX \equiv b \pmod{m}$ if $\gcd(a, m) = 1$

**Theorem 9.** *If  $\gcd(a, m) = 1$ , then  $\forall b \in \mathbb{Z}$  the congruence  $aX \equiv b \pmod{m}$  has a unique solution.*

证明.

1) Existence: Since  $\gcd(a, m) = 1$ ,  $\exists s, t$  such that

$$1 = sa + tm$$

(Version 1)

(Mutiplying  $X$ )

$$X = saX + tmX$$

$$aX \equiv b \pmod{m} \Leftrightarrow aX = km + b$$

$$\Leftrightarrow X = s(km + b) + b$$

$$\Leftrightarrow X \equiv sb \pmod{m}$$

(Version 2)

(Mutiplying  $s$ )

$$saX \equiv sb \pmod{m}$$

$$(1 - tm)X \equiv sb \pmod{m}$$

$$X \equiv sb \pmod{m}$$

$X \equiv sb \pmod{m}$  is the solution to  $aX \equiv b \pmod{m}$ .

2) Uniqueness: Assume  $x, y$  are two solutions,

$$ax \equiv b \pmod{m}, ay \equiv b \pmod{m} \Rightarrow a(x - y) \equiv 0 \pmod{m}$$

Since  $\gcd(a, m) = 1$ ,  $m|(x - y) \Rightarrow x = y$ , ( $x, y \in \{0, 1, \dots, m - 1\}$ )

**Example 14.** Solve  $3X \equiv 5 \pmod{11}$ .

$$\gcd(3, 11) = 1, 1 = 4 * 3 - 1 * 11,$$

$$X \equiv 4 * 5$$

$$X \equiv 9$$

□

### 5.3 Chinese Remaindar Theorem (CRT): unique solution for $x$ modulo $mn$

**Theorem 10** (Chinese Remaindar Theorem (CRT)).

If  $\gcd(m, n) = 1$ . Then  $\begin{cases} x \equiv r \pmod{m} & (1) \\ x \equiv s \pmod{n} & (2) \end{cases}$  have a unique solution for  $x$  modulo  $mn$ .

证明.

$$(1) \Rightarrow x = km + r \text{ for some } k \in \mathbb{Z}.$$

$$\text{substitute (2)} \Rightarrow km + r \equiv s \pmod{n}$$

$$\Leftrightarrow mk \equiv s - r \pmod{n} \quad (3)$$

According to previous theorem,  $\gcd(m, n) = 1$ , (3) has a **unique** solution.

We say  $k \equiv t \pmod{n}$ ,  $k = ln + t$  for some  $l \in \mathbb{Z}$

$$\Rightarrow x = (ln + t)m + r = lnm + tm + r, \text{ where } tm + r \text{ is the unique solution to } x \pmod{mn}.$$

□

**Example 15.** (Similar to CRT) Find the smallest integer  $x$  such that

$$x \equiv 1 \pmod{11} \text{ and } x \equiv 9 \pmod{13}$$

$$\gcd(11, 13) = 1 \text{ and } 1 = 6 * 11 - 5 * 13$$

Write  $x = 11k + 1$ . Substitute in  $x \equiv 9 \pmod{13}$ :

$$11k \equiv 8 \pmod{13}$$

$$6 * 11k \equiv 6 * 8 \equiv 9 \pmod{13}$$

$$(1 + 5 * 13)k \equiv 9 \pmod{13}$$

$$k \equiv 9 \pmod{13}$$

Then  $x = 11k + 1 = 100$ .

### 5.4 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

将给定  $n$ , 相同余数的数分为一组

Fix  $n \in \mathbb{Z}_+$ , we call  $[a]_n = [a]$  the **congruence class** of  $a$  modulo  $n$ .

$$[a] = \{b \in \mathbb{Z} | b \equiv a \pmod{n}\} = \{a + kn | k \in \mathbb{Z}\}$$

#### 5.4.1 Set of congruence classes of mod $n$ : $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], \dots, [n-1]\}$

The set of *congruence classes* of mod  $n$  is denoted  $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

**Proposition 14** (Proposition 1.5.2.). *For any  $n \geq 1$  there are exactly  $n$  congruences classes modulo  $n$ , which we may write as*

$$\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$$

证明.

For any  $a \in \mathbb{Z}$ . By Euclidean algorithm,  $a = qn + r$ ,  $q, r \in \mathbb{Z}$ ,  $0 \leq r < n \Rightarrow a \in [r]$ . So,  $\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$ .

When  $0 \leq a < b \leq n-1$ ,  $n \nmid (b-a)$ , so  $[a] \neq [b]$  the  $n$  congruence classes listed are all distinct. Hence, there are exactly  $n$  congruence classes.  $\square$

#### 5.4.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

Fix  $n \in \mathbb{Z}$ , we define addition $+$  and multiplication $\cdot$  on  $\mathbb{Z}_n$ :

$$\begin{aligned} [a] + [b] &= [a+b] = \{a+b+(k+j)n | k, j \in \mathbb{Z}\} \\ [a] \cdot [b] &= [ab] = \{ab+(aj+bk+kjn)n | k, j \in \mathbb{Z}\} \end{aligned}$$

This is well defined, follows Lemma 1.5.3.

**Proposition 15** (Proposition 1.5.5.). *Let  $a, b, c, d, n \in \mathbb{Z}, n \geq 1$ , then*

(i) *Addition and multiplication are commutative and associative operations in  $\mathbb{Z}_n$ .*

(ii)  $[a] + [0] = [a]$ .

(iii)  $[-a] + [a] = [0]$ .

(iv)  $[1][a] = [a]$ .

(v)  $[a]([b] + [c]) = [a][b] + [a][c]$ .

证明.

$\square$

#### 5.4.3 Units(i.e. invertible) in Congruence Classes

将与  $n$  互质的数分为一组

Say  $[a] \in \mathbb{Z}_n$  is a **unit** or is **invertible** if  $\exists [b] \in \mathbb{Z}_n$  so that  $[a][b] = [1]$ .

**5.4.4 Proposition 1.5.6: Set of units in congruence classes:**  $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \text{ is a unit}\} = \{[a] \in \mathbb{Z}_n | \gcd(a, n) = 1\}$

The set of **invertible** elements in  $\mathbb{Z}_n$  will be denoted  $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \text{ is a unit}\}$ .

**Proposition 16** (Proposition 1.5.6.). *For all  $n \geq 1$ , we have  $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | \gcd(a, n) = 1\}$ .*

证明.

By Proposition 1.4.8, we know there exists  $b, c$  s.t.  $ab + cn = 1$ . So,  $ab \equiv 1 \pmod n$ ,  $[1] = [ab] = [a][b]$ . So,  $\{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\} \subset \mathbb{Z}_n^\times$ .  
 $[a]$  is a unit  $\Rightarrow \exists [b] \in \mathbb{Z}_n$  so that  $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow \gcd(a, n) = 1$ . So,  $\mathbb{Z}_n^\times \subset \{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$ .  $\square$

**Note 3.** Inverse of  $[a]$  is unique, i.e.  $[b] = [a]^{-1}$  is unique.

$$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$$

**5.4.5 Corollary 1.5.7:** if  $p$  is prime,  $\varphi(p) = \mathbb{Z}_p^\times = \{[1], [2], \dots, [p-1]\}$

**Corollary 2** (Corollary 1.5.7). If  $p \geq 2$  is prime,  $\mathbb{Z}_p^\times = \{[1], [2], \dots, [p-1]\}$ .

**5.5 Euler phi-function:**  $\varphi(n) = |\mathbb{Z}_n^\times|$

Euler phi-function:  $\varphi(n) = |\mathbb{Z}_n^\times|$ .

$p$  prime,  $\varphi(p) = p - 1$ .

**5.5.1**  $m \mid n$ ,  $\pi_{m,n}([a]_n) = [a]_m$

**Example 16** (Exercise 1.5.4). If  $m \mid n$ , we can define  $\pi_{m,n} : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$  by  $\pi_{m,n}([a]_n) = [a]_m$ . Prove it is well-defined.

证明.

We write  $[a]_n = [c]_n$ , verify that  $[a]_m = [c]_m$ .

Since  $m \mid n$ , there exists  $k \in \mathbb{Z}$  s.t.  $n = km$ .

$$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z} \text{ s.t. } c = a + jn.$$

$$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m \quad \square$$

**5.6 Theorem 1.5.8 (Chinese Remainder Theorem):**  $n = mk, \gcd(m, k) = 1$ ,  $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

**Theorem 11** (Theorem 1.5.8 (Chinese Remainder Theorem)). If  $m, n, k > 0, n = mk, \gcd(m, k) = 1$ , then  $F : \mathbb{Z}_n \rightarrow \mathbb{Z}_m \times \mathbb{Z}_k$  which is given by  $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$ , then  $F$  is a bijection.

证明.

(1) Injective:  $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$  i.e.  $a \equiv b \pmod m, a \equiv b \pmod n$ .  $\exists i, j \in \mathbb{Z}$  s.t.  $b = a + im = a + jk \Rightarrow k \mid im$ . Since  $\gcd(m, k) = 1$ ,  $k \mid i \Rightarrow n = mk \mid im$ . Then  $[b]_n = [a]_n + [im]_n = [a]_n$ .

(2) Surjective: prove  $\forall u, v \in \mathbb{Z}, \exists a \in \mathbb{Z}$  s.t.  $[a]_m = [u]_m, [a]_k = [v]_k$ .

Since  $\gcd(m, k) = 1$ ,  $\exists s, t \in \mathbb{Z}$  so that  $1 = sm + tk$ .

Let  $a = (1 - tk)u + (1 - sm)v$ ,  $[a]_m = [(u - v)sm + v]_m = [v]_m$ ,  $[a]_k = [(v - u)tk + u]_k = [u]_k$ .  $\square$

**Note 4.**  $F([a]_n[b]_n) = F([ab]_n) = ([ab]_m, [ab]_k) = ([a]_m[b]_m, [a]_k[b]_k)$

Since  $F$  is a bijection,  $[ab]_n = [1]_n$  iff  $([a]_m[b]_m, [a]_k[b]_k) = ([1]_m, [1]_k)$ .

**5.6.1 Proposition 1.5.9+Corollary 1.5.10:**  $m, n, k > 0, n = mk, \gcd(m, k) = 1$ , then  $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$ , then  $\varphi(n) = \varphi(m)\varphi(k)$

**Proposition 17** (Proposition 1.5.9+Corollary 1.5.10). *If  $m, n, k > 0, n = mk, \gcd(m, k) = 1$ , then  $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$ , then  $\varphi(n) = \varphi(m)\varphi(k)$ .*

**5.7 prime factorization:**  $n = p_1^{r_1} \dots p_k^{r_k}$ , then  $\varphi(n) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$

**Proposition 18.** *If  $n \in \mathbb{Z}$  is positive integer with prime factorization  $n = p_1^{r_1} \dots p_k^{r_k}$ , then  $\varphi(n) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$*

证明.

$\mathbb{Z}_{p^r} = \{[0], [1], \dots, [p^r - 1]\}$ , the number of multiples of  $p$  is  $\frac{p^r}{p} = p^{r-1}$ . Then  $\varphi(p^r) = |\mathbb{Z}_{p^r}^\times| = p^r - p^{r-1} = (p - 1)p^{r-1}$ . So,

$$\varphi(n) = \varphi(p_1^{r_1}) \dots \varphi(p_k^{r_k}) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$$

$\square$

## 6 Group

**6.1 Group  $(G, *)$ : a set with a binary operation (associative, identity, inverse)**

### 6.1.1 Definition

A *group* is a nonempty set  $G$  with a binary operation  $*$  :  $G \times G \rightarrow G$  s.t.

(1) Binary operation on  $G$ ,  $*$  :  $G \times G \rightarrow G$

(2)  $*$  is **associative**

(3)  $G$  contains an **identity** element  $e$  for  $*$ :  $\exists e \in G$  s.t.  $e * g = g * e = g \forall g \in G$

(4) Each element  $a \in G$  has an **inverse**  $b \in G$  s.t.  $a * b = b * a = e$ .

A Group is **abelian** if moreover

(5)  $*$  is **commutative**.

$|G|$  = Order of a group  $(G, *)$

$(\mathbb{Z}, +)$  is a group and  $+$  is commutative, we call this kind of groups (satisfy commutative) *abelian group*.

**Example 17.** If  $\mathbb{F}$  is a field, then  $(\mathbb{F}, +)$  and  $(\mathbb{F}^\times, \cdot)$  are abelian group.

**Example 18.** If  $V$  is a vector space over  $\mathbb{F}$ , then  $(V, +)$  abelian group.

As we know a  $V$  is a vector space over  $\mathbb{F}$  means  $V$  is a field whose subfields include  $\mathbb{F}$ .

### 6.1.2 Uniqueness of identity and inverse

**Lemma 3.** 1. Identity of a group is unique. 2. Inverse of any element in a group is also unique.

证明.

1. Let  $e, e'$  be two identities in  $G$ , then  $e * e' = e = e'$ .
2. Suppose  $b, c$  are both inverse of  $a$ , then

$$b = b * e = b * (a * c) = (b * a) * c = e * c = c$$

□

### 6.1.3 Examples: Permutation group $Sym(X)$ , Klein 4-group, alternating group $A_n$ , Dihedral group

**Example 19.** If  $X$  is any nonempty set, permutation group of  $X : \{\sigma : X \rightarrow X | \sigma \text{ is a bijection}\}$ , then

1.  $\circ$  is associative;
2.  $id : X \rightarrow X$ ,  $id(x) = x \forall x \in X$  is the identity;
3.  $\sigma \in Sym(X)$ ,  $\sigma^{-1} \in Sym(X)$  is the inverse function.

$(Sym(X), \circ)$  is a group called the symmetric group of  $X$

**Example 20.** The Klein four-group is a group with four elements, in which each element is self-inverse (composing it with itself produces the identity) and in which composing any two of the three non-identity elements produces the third one. For example,  $K \leq S_4$

$$K = \{(1), (12)(34), (13)(24), (14)(23)\}$$

**Example 21.** An alternating group is the group of even permutations of a finite set. An alternating group of degree  $n$ ,  $A_n$ .

The cycle structure of  $A_5$ ,

- (1)  $(abcde)$  - even
- (3)  $(abc)$  - even
- (4)  $(ab)(cd)$  - even (odd permutation  $\times$  odd permutation)
- (6)  $e$  - even

**Example 22** (Dihedral group).

The dihedral group of order  $2n$ , denoted  $D_{2n}$ , is the group of symmetries of a regular  $n$ -gon  $A_1A_2 \dots A_n$ , which includes rotations and reflections. It consists of the  $2n$  elements

$$\{1, \rho, \rho^2, \dots, \rho^{n-1}, \sigma, \sigma\rho, \sigma\rho^2, \dots, \sigma\rho^{n-1}\}.$$

The element  $\rho$  corresponds to rotating the  $n$ -gon by  $\frac{2\pi}{n}$ , while  $\sigma$  corresponds to reflecting it across the line  $OA_1$  (here  $O$  is the center of the polygon). So  $\rho\sigma$  mean "reflect then rotate" (like with function composition, we read from right to left). In particular,  $\rho^n = \sigma^2 = 1$ . You can also see that  $\rho^k\sigma = \sigma\rho^{-k} = \sigma\rho^{n-k}$ .

#### 6.1.4 Cancellation Laws

**Theorem 12.** *Let  $G$  be a group. The left and right cancelation laws hold in  $G$ :*

1.  $a * x = a * y \Rightarrow x = y$
2.  $x * a = y * a \Rightarrow x = y$

证明.

Let  $a*x = a*y$ .  $\exists a'$  s.t.  $a'*a = e$ .  $a'*(a*x) = a'*(a*y) \Rightarrow (a'*a)*x = (a'*a)*y \Rightarrow e*x = e*y \Rightarrow x = y$   
Similar for the right cancel law. □

#### 6.1.5 Unique Solution of Linear Equation

**Theorem 13.** *The linear equation  $a * x = b$  and  $y * a = b$  has unique solution.*

证明.

1. Existence: Multiply by  $a'$ :  $a'*(a*x) = a'*b \Rightarrow x = a'*b$  is a solution.
2. Uniqueness: if  $x'$  is another,  $a*x = a*x' = b \Rightarrow x = x'$

□

### 6.2 Subgroup: $H \leq G$

**Definition 10.** *A subset  $H \subseteq G$  is a subgroup of  $G$  if  $H$  is itself a group.*

write  $H \leq G$ ,  $H < G$  if  $H$  is a subgroup of  $(G, *)$ . (If  $H = G$ ,  $H$  is an improper subgroup. If  $H \subsetneq G$ ,  $H$  is a proper subgroup.)

If  $H = \{e\}$ , then  $H$  is a trivial subgroup.

If  $H \neq \{e\}$ , then  $H$  is a nontrivial subgroup.

**Theorem 14.** *A subset  $H \subseteq G$  is a subgroup of  $G$  if and only if*

1.  $H$  is closed under  $*$ . ( $\forall g, h \in H, g * h \in H$ )
2. identity  $e \in H$ .
3. Each  $a \in H$ , the inverse  $a' \in H$

证明.

" $\Rightarrow$ ": if  $H \leq G$  be a subgroup.

1.  $H$  is a group  $\Rightarrow *$  is a binary operation on  $H$ ,  $*$  :  $H \times H \rightarrow H$  i.e.  $H$  is closed under  $*$ .
2. Identity of  $H$ ,  $e_H$  is also a identity of  $G$ , due to the uniqueness of identity,  $e_H = e_G$ .
3.  $a \in H$ ,  $a$ 's inverse  $a'_H \in H$  is also an inverse in  $G$ , due to the uniqueness of identity,  $a'_H = a'_G$ .

" $\Leftarrow$ ":

1.  $H$  is closed under  $*$   $\Rightarrow *$  is a binary operation on  $H$ .
2. 2,3 fulfill the requirement of identity and inverse.
3.  $*$  is operation of group  $G \Rightarrow *$  is associative.

Hence  $H$  is itself a group.

4.  $H$  is a subset of  $G$ , then  $H$  is a subgroup of  $G$ .

□

**6.2.1 Proposition 2.6.8:**  $H < G$ ,  $(H, *)$  is a group: A group's operation with its any subgroup is also a group

不同的 definition.

**Proposition 19** (Proposition 2.6.8). If  $(G, *)$  is a group,  $H \subset G$  is a subgroup, then  $(H, *)$  is a group.

**Example 23.**  $(G, *)$  is a group, then  $e < G$ ,  $G < G$ .

**Example 24.**  $\mathbb{K} \subset \mathbb{F}$  is a subfield, then  $\mathbb{K} < \mathbb{F}$ ,  $\mathbb{K}^\times < \mathbb{F}^\times$ .

**Example 25.**  $W \subset V$  is a vector subspace,  $W < V$ .

**Example 26.**  $1 \in S^1 \subset \mathbb{C}^\times$ ,  $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ .  $S^1$  is a subgroup.

证明.

$S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$ . For any  $e^{i\theta}, e^{i\psi} \in S^1$ ,  $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1$ ,  $e^{-i\theta} \in S^1$ .

□

**Example 27.**  $Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$

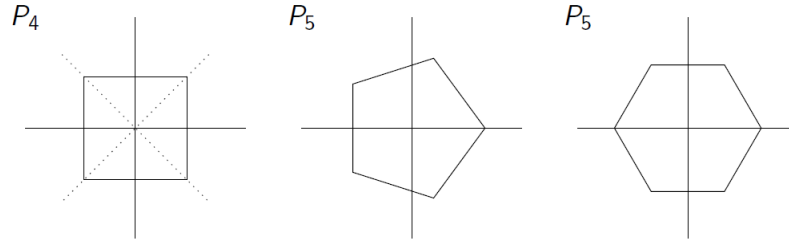


**Example 28.** If  $\mathbb{F}$  is a field,  $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \rightarrow \mathbb{F} \in Sym(\mathbb{F}) | \sigma(a+b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$

**Example 29.** *Dihedral Groups:*

保留多边形

Let  $P_n \subset \mathbb{R}^2$  be a regular  $n$ -gon



$$D_n < Isom(\mathbb{R}^2), D_n = \{\Phi \in Isom(\mathbb{R}^2) | \Phi(P_n) = P_n\}$$

### 6.3 Some Properties of Group Operation

**Proposition 20** (Proposition 3.1.1). *Let  $(G, *)$  be a group with identity  $e \in G$ , then*

- (1) if  $g, h \in G$  and either  $g * h = h$  or  $h * g = h$ , then  $g = e$
- (2) if  $g, h \in G$  and  $g * h = e$  then  $g = h^{-1}$  and  $h = g^{-1}$

**Corollary 3** (Corollary 3.1.2).  $e^{-1} = e$ ,  $(g^{-1})^{-1} = g$ ,  $(g * h)^{-1} = h^{-1} * g^{-1}$

### 6.4 Power of an Element

We define  $g^n$  recursively for  $n \geq 0$  by setting  $g^0 = e$  and for  $n \geq 1$ , we set  $g^n = g^{n-1} * g$ . For  $n \leq 0$ , we define  $g^n = (g^{-1})^{-n}$ .

**Proposition 21** (Proposition 3.1.5). (1)  $g^n * g^m = g^{n+m}$ ; (2)  $(g^n)^m = g^{nm}$

### 6.5 $(G \times H, \otimes)$ : Direct Product of $G$ and $H$

$(G, *)$  a group  $(H, \star)$  a group. Define an operation on  $G \times H$ ,  $\otimes$ :

$$(h, k) \otimes (h', k') = (h * h', k * k')$$

### 6.5.1 Proposition 3.1.7: $(G \times H, \otimes)$ is a group

**Proposition 22** (Proposition 3.1.7).  $(G \times H, \otimes)$  is a group. The identity is  $(e_G, e_H)$ , inverse is  $(g^{-1}, h^{-1})$

usually written as

$$(h, k)(h', k') = (hh', kk')$$

## 6.6 Subgroups and Cyclic Groups

### 6.6.1 Intersection of Subgroups is a Subgroup

**Proposition 23** (Proposition 3.2.2). Let  $G$  be a group and suppose  $\mathcal{H}$  is any collection of subgroups of  $G$ . Then  $K = \cap_{H \in \mathcal{H}} H < G$  is a subgroup of  $G$ .

### 6.6.2 Subgroup Generated by $A$ : $\langle A \rangle$

We define **Subgroup Generated by  $A$** :

$$\langle A \rangle = \cap_{H \in \mathcal{H}(A)} H$$

where  $\mathcal{H}(A)$  is the set of all subgroups of  $G$  containing the set  $A$ :

$$\mathcal{H}(A) = \{H < G \mid A \subset H \text{ and } H \text{ is a subgroup of } G\}$$

### 6.6.3 Cyclic Group: group generated by an element

A group  $G$  is cyclic if exists  $g$  (an element),  $\langle g \rangle = G$ .

$g$  is called a generator for  $G$  in this case.

Easy to prove

$$G = \langle g \rangle = \{\dots g^{-2}, g^{-1}, e, g^1, g^2 \dots\}$$

### 6.6.4 Cyclic Subgroup

If  $A$  is a subgroup of  $G$ , and  $A = \langle \{a\} \rangle = \langle a \rangle$ . Then  $A$  is the cyclic subgroup generated by  $a$ :

$$A = \langle a \rangle \leq G$$

$$\langle a \rangle = \{\dots a^{-2}, a^{-1}, e, a^1, a^2 \dots\}$$

### 6.6.5 Subgroups of a Cyclic Group must be Cyclic

**Theorem 15.** A subgroup of a cyclic group is cyclic.

証明.

Let  $G = \{a^n : n \in \mathbb{Z}\}$  be a cyclic group. Let  $H \leq G$  be a subgroup.

1. If  $H = \{e\}$ , then  $H$  is cyclic.
2. If  $H \neq \{e\}$ , then  $a^n \in H$  for some  $n > 0$ . Check  $m$  be the minimal among all  $n$ .

Claim:  $H = \langle a^m \rangle$

Proof: Clearly  $\langle a^m \rangle \subset H$ .  $\forall a^n \in H$ ,  $n = qm + r$ ,  $0 \leq r < m$ . Then  $a^r = a^n(a^m)^{-q}$ . Since  $m$  is the minimal positive integer s.t.  $a^m \in H$ ,  $r = 0$ .  $\Rightarrow n = qm \Rightarrow a^n \in \langle a^m \rangle$ . Hence  $H = \langle a^m \rangle$  which is cyclic. □

**Example 30** (Subgroups of  $(\mathbb{Z}, +)$ ).

$\mathbb{Z}$  is a cyclic group  $\langle 1 \rangle$ . Its subgroups are  $\langle n \rangle \leq \mathbb{Z}$  for some  $n \geq 0$ . (which is a multiplier of  $n$ .  $(n\mathbb{Z})$ )  
 $n = 0, H = \{0\}; n = 1, H = \mathbb{Z}; n = 2, H = 2\mathbb{Z}$

**6.6.6 Theorem:**  $\langle a^v \rangle < \langle a^n \rangle \Rightarrow \langle a^v \rangle = \langle a^d \rangle, d = \gcd(v, n), |\langle a^v \rangle| = \frac{n}{d}$

**Theorem 16.** Let  $G$  be a cyclic group of order  $n$ . ( $G = \{1, a, a^2, \dots, a^{n-1}\}$ , where  $a^n = 1$ ). Let  $H = \langle a^v \rangle$  be a subgroup of  $G$ . Then  $H$  is generated by  $a^d$  (i.e.  $H = \langle a^d \rangle$ ),  $d = \gcd(v, n)$  and  $|H| = \frac{n}{d}$ .

证明.

Let  $H' = \langle a^d \rangle$ , we need to show that  $H = H'$ .  $d = \gcd(v, n) = d|v \Rightarrow a^v \in \langle a^d \rangle \Rightarrow H \subset H'$ .

While  $d = sv + tn$  for some  $s, t$ .  $\Rightarrow a^d = (a^v)^s(a^n)^t$ . Since  $a^n = 1$ ,  $a^d = (a^v)^s \Rightarrow H' \subset H$ .

Hence,  $H = H' = \langle a^v \rangle$ .  $H = \{1, a^d, a^{2d}, \dots, a^{n-d}\}, |H| = \frac{n}{d}$  □

**6.6.7 Corollary 3.2.4:**  $G$  is a cyclic group  $\Rightarrow G$  is abelian

**Corollary 4** (Corollary 3.2.4). If  $G$  is a cyclic group (i.e. exists  $g \in G$  s.t.  $\langle g \rangle = G$ ), then  $G$  is abelian (i.e. commutative).

**6.6.8 Equivalent properties of order of  $g$ :**  $|g| = |\langle g \rangle| < \infty$

**Proposition 24** (Proposition 3.2.6). Let  $G$  be a group for  $g \in G$ , the following are equivalent:

- (i)  $|g| < \infty$
- (ii)  $\exists n \neq m$  in  $\mathbb{Z}$  so that  $g^n = g^m$
- (iii)  $\exists n \in \mathbb{Z}$ ,  $n \neq 0$  so that  $g^n = e$
- (iv)  $\exists n \in \mathbb{Z}_+$  so that  $g^n = e$

If  $|g| < \infty$ , then  $|g| = \text{smallest } n \in \mathbb{Z}_+ \text{ so that } g^n = e$ , and  $\langle g \rangle = \{e, g, g^2, \dots, g^{n-1}\} = \{g^n \mid n = 0, \dots, n-1\}$

**6.6.9  $(\mathbb{Z}, +)$  Theorem 3.2.9:  $\langle a \rangle < \langle b \rangle$  if and only if  $b|a$**

**Theorem 17** (Theorem 3.2.9). *If  $H < \mathbb{Z}$  is a subgroup, then either  $H = \{0\}$ , or else  $H = \langle d \rangle$ , where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently,  $a \rightarrow \langle a \rangle$  defines a **bijection** from  $N = \{0, 1, 2, \dots\}$  to the set of subgroups of  $\mathbb{Z}$ . Furthermore, for  $a, b \in \mathbb{Z}_+$ , we have  $\langle a \rangle < \langle b \rangle$  if and only if  $b|a$ .*

**6.6.10  $(\mathbb{Z}_n, +)$  Theorem 3.2.10:  $\langle [d] \rangle < \langle [d'] \rangle$  if and only if  $d'|d$**

**Theorem 18** (Theorem 3.2.10). *For any  $n \geq 2$ , if  $H < \mathbb{Z}_n$  is a subgroup, then there is a positive divisor  $d$  of  $n$  so that*

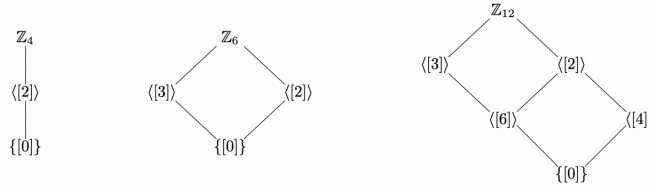
$$H = \langle [d] \rangle$$

*Furthermore, this defines a bijection between divisors of  $H$  and subgroups of  $\mathbb{Z}_n$ . Furthermore, if  $d, d' > 0$  are two divisors of  $n$ , then  $\langle [d] \rangle < \langle [d'] \rangle$  if and only if  $d'|d$ .*

If  $H = \langle [d] \rangle$  is a subgroup of  $H$ , then  $[n] \in H$ , so  $d|n$ . And  $|H| = |\langle [d] \rangle| = \frac{n}{d}$ , so  $|H||d$

**6.6.11 Subgroup Lattice**

The set of all subgroups of a group of  $G$ , together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup  $\{e\}$  at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.



Writing down the subgroup lattice is as easy as writing down the divisibility lattice in which  $n$  is placed at the bottom, 1 at the top, and all intermediate divisors in between, connected by edges when there is divisibility. The congruence class of the divisor generates the corresponding subgroup in the subgroup lattice.

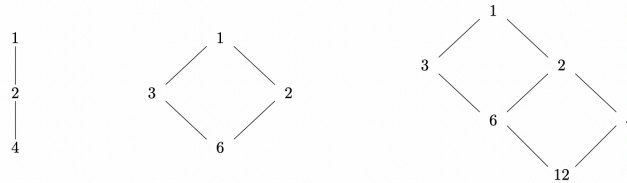


图 2:

## 6.7 Homomorphism

### 6.7.1 Def: Homomorphism, Image

**Definition 11.** If  $(G, *)$  and  $(H, \circ)$  are groups, then a function  $f : G \rightarrow H$  is a **homomorphism** if

$$f(x * y) = f(x) \circ f(y), \quad \forall x, y \in G$$

If  $f$  is also a bijection, then  $f$  is called an **isomorphism**.

**Example 31.** Let  $S_n$  be the symmetric group on  $n$  letters, and let  $\phi : S_n \rightarrow \mathbb{Z}_2$  be defined by

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation,} \\ 1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Show that  $\phi$  is a homomorphism.

**Example 32.** Let  $GL(n, \mathbb{R})$  be the multiplicative group of all invertible  $n \times n$  matrices. Recall that a matrix  $A$  is invertible if and only if its determinant,  $\det(A)$ , is nonzero. Recall also that for matrices  $A, B \in GL(n, \mathbb{R})$  we have

$$\det(AB) = \det(A)$$

**Example 33.**

1.  $\phi : (\mathbb{R}, +) \rightarrow (\mathbb{R}^*, \cdot)$   $\phi(x) = 2^x$ . Then

$$\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$$

$\phi$  is a homomorphism.

2.  $\phi : G \rightarrow G$   $\phi(g) = g^{-1}$ . Then

$$\phi(gh) = (gh)^{-1} = h^{-1}g^{-1} = \phi(h)\phi(g)$$

$\phi$  is not a homomorphism in general; but it is homomorphism if  $G$  is abelian.

**Definition 12.** Let  $\phi$  be a mapping of a set  $X$  into a set  $Y$ , and let  $A \subseteq X$  and  $B \subseteq Y$ . The image  $\phi[A]$  of  $A$  in  $Y$  under  $\phi$  is  $\{\phi(a) \mid a \in A\}$ . The set  $\phi[X]$  is the range of  $\phi$ . The inverse image  $\phi^{-1}[B]$  of  $B$  in  $X$  is  $\{x \in X \mid \phi(x) \in B\}$

### 6.7.2 Properties of Homomorphism

**Theorem 19.** Let  $\phi$  be a homomorphism of a group  $G$  into a group  $G'$ , then

1. if  $e \in G$  is an identity in  $G$ , then  $\phi(e) \in G'$  is the identity in  $G'$ .
2. if  $a \in G$  has inverse  $a' \in G$ , then  $\phi(a) \in G'$  has inverse  $\phi(a') \in G'$ .
3. if  $H \leq G$  is a subgroup of  $G$ , then the image  $\phi(H) = \{\phi(h) : h \in H\} \leq G'$  is a subgroup of  $G'$ .
4. if  $K' \leq G'$  then the inverse image  $\phi^{-1}(K') = \{x \in G : \phi(x) \in K'\} \leq G$ .

### 6.7.3 Kernel of Homomorphism

**Definition 13.** Let  $\phi : G \rightarrow G'$  be a homomorphism of groups. The subgroup  $\phi^{-1}(e') = \{x \in G : \phi(x) = e'\}$  is the kernel of  $\phi$ , denoted by  $\text{Ker}(\phi)$ .

$$\text{Ker}(\phi) \stackrel{\text{def}}{=} \phi^{-1}(e') = \{x \in G : \phi(x) = e'\}$$

**Theorem 20** ( $\text{Ker}\phi$  is normal). Let  $\phi : G \rightarrow G'$  be a homomorphism.  $H = \text{Ker}\phi$ , then for all  $a \in G$ ,  $\phi^{-1}[\phi(a)] = \{x \in G : \phi(x) = \phi(a)\}$  is the left coset  $aH$  of  $H$ , and is also the right coset  $Ha$  of  $H$ .

$$aH = Ha = \{x \in G : \phi(x) = \phi(a)\}$$

证明.

$$\begin{aligned} \phi(x) &= \phi(a) \\ \Leftrightarrow \phi(x)\phi(a)^{-1} &= e' \\ \Leftrightarrow \phi(x)\phi(a^{-1}) &= e' \\ \Leftrightarrow \phi(xa^{-1}) &= e' \\ \Leftrightarrow xa^{-1} &\in H \\ \Leftrightarrow x &\in Ha \end{aligned}$$

Similarity, we can prove  $x \in aH$ . □

**Theorem 21.** A homomorphism is injective if and only if  $\text{Ker}(\phi) = \{e\}$ .

证明.

$$\begin{aligned} \phi(x) = \phi(y) &\Leftrightarrow \phi(x)\phi^{-1}(y) = e' \\ &\Leftrightarrow \phi(x)\phi(y^{-1}) = e' \\ &\Leftrightarrow \phi(xy^{-1}) = e' \\ &\Leftrightarrow xy^{-1} \in \text{Ker}(\phi) \end{aligned}$$

Hence, we can also prove that

$$xy^{-1} \in \text{Ker}(\phi) \Leftrightarrow x = y \text{ if and only if } \text{Ker}(\phi) = \{e\}$$

□

## 6.8 Isomorphism

### 6.8.1 Definition: Isomorphism

**Definition 14.** We say that  $G$  and  $H$  are **isomorphic** if exists an **isomorphism**  $f$ , denoted by  $G \cong H$  or  $G \simeq H$ . (since  $f$  is bijection,  $G \cong H \Leftrightarrow H \cong G$ )

Isomorphic means these two pathes are the same.

$$\begin{array}{ccc} G \times G & \xrightarrow{*} & G \xrightarrow{f} H \\ G \times G & \xrightarrow{(f,f)} & H \times H \xrightarrow{\circ} H \end{array}$$

**Example 34.**  $(\mathbb{Z}_2, +)$ ,  $(\{-1, 1\}, \times)$  and  $\phi : 0 \rightarrow 1; 1 \rightarrow -1$ .

$$\begin{aligned} \phi(0 + 0) &= 1 = \phi(0) \times \phi(0) \\ \phi(0 + 1) &= -1 = \phi(0) \times \phi(1) \\ \phi(1 + 1) &= 1 = \phi(1) \times \phi(1) \end{aligned}$$

**6.8.2 Theorem:**  $\left\{ \begin{array}{l} \sigma : G \rightarrow G' \text{ injective} \\ \sigma(xy) = \sigma(x)\sigma(y) \quad \forall x, y \in G \end{array} \right\} \Rightarrow \sigma(G) \leq G', G \text{ is isomorphic to } \sigma(G)$

**Theorem 22.** Let  $\sigma : G \rightarrow G'$  be an injective map s.t.

$$\sigma(xy) = \sigma(x)\sigma(y), \quad \forall x, y \in G$$

Then the image  $\sigma(G) = \{\sigma(x) : x \in G\}$  is a subgroup of  $G'$  that is isomorphic to  $G$ .

证明.

1. Closed:  $\forall a = \sigma(x), b = \sigma(y) \in \sigma(G)$ , then  $ab = \sigma(x)\sigma(y) = \sigma(xy) \in \sigma(G)$ .
2. Identity:  $\sigma(e) \in \sigma(G)$  is an identity for  $\sigma(G)$ :  $\sigma(e)\sigma(x) = \sigma(ex) = \sigma(x) = \sigma(xe) = \sigma(x)\sigma(e)$
3. Inverse:  $\sigma(x^{-1})$  is an inverse in  $\sigma(G)$  for  $\sigma(x)$ :  $\sigma(x^{-1})\sigma(x) = \sigma(e) = \sigma(x)\sigma(x^{-1})$

□

**6.8.3 Cayley Theorem:**  $G$  is isomorphic to a subgroup of  $S_G$

**Theorem 23** (Cayley Theorem). Let  $G$  be a group and  $S_G$  is the symmetric group of  $G$  (the group of all permutation of  $G$ :  $S_G = \{\text{Bijection } \sigma : G \rightarrow G\}$ ) Then  $G$  is isomorphic to a subgroup of  $S_G$ .

证明.

Set a bijection  $\phi : G \rightarrow S_G$  such that  $\phi(g) = \lambda_g, \forall g \in G$ , where  $\lambda_g$  is a permutation  $\lambda_g : x \rightarrow gx$ .

Claim:  $\lambda_g \in S_G$  (i.e.  $\lambda_g$  is a permutation of  $G$ , a bijection  $G \rightarrow G$ ).

1.  $\lambda_g : G \rightarrow G$  is injective

$$\begin{aligned} \lambda_g(x) &= \lambda_g(y) \\ \Leftrightarrow gx &= gy \\ \Leftrightarrow x &= y \end{aligned}$$

2.  $\lambda_g : G \rightarrow G$  is surjective. Let  $y \in G$

$$\begin{aligned}\lambda_g(x) &= y \\ \Leftrightarrow gx &= y \\ \Leftrightarrow x &= g^{-1}y\end{aligned}$$

Claim:  $\phi(x)\phi(y) = \phi(xy)$

$$\begin{aligned}\phi(x)\phi(y) &= \lambda_x \circ \lambda_y \\ (\lambda_x \circ \lambda_y)(z) &= \lambda_x(yz) = xyz = \lambda_{xy}(z), \forall z \in G \\ \Rightarrow \phi(x)\phi(y) &= \phi(xy)\end{aligned}$$

According to previous theorem,  $\phi(G) \leq G$  and  $G$  is isomorphic to  $\phi(G)$ .

□

## 6.9 Coset and Order

**Definition 15.** If  $H$  is a subgroup of a group  $G$  and  $a \in G$ , then  $aH = \{ah | h \in H\} \leq G$  is called left coset of  $H$ .

**Theorem 24.** Let  $H \leq G$ ,  $a, b \in G$ ,

1.  $aH = bH$  if and only if  $a^{-1}b \in H$
2.  $aH \cap bH = \emptyset$  or  $aH = bH$
3.  $|aH| = |H| \forall a \in G$

证明.

1. Assume that  $aH \cap bH \neq \emptyset$  and let  $ah = bk \in aH \cap bH$  with  $h, k \in H$ .  
 $ah = bk \Leftrightarrow h = a^{-1}bk \Leftrightarrow a^{-1}b = hk^{-1} \in H$ , thus  $a^{-1}b \in H$ .
2. When  $aH \cap bH \neq \emptyset \exists k_1, h \in H$  such that  $ak_1 = bh \in bH$ . Then  $\forall k_2 \in H$   $a = bhk_1^{-1} \Rightarrow ak_2 = bhk_1^{-1}k_2$  where  $hk_1^{-1}k_2 \in H$  so  $ak_2 \in bH$ ,  $\forall k_2 \in H$ .
3.  $x \rightarrow ax$  is bijection  $\Rightarrow |aH| = |H|$ .

□

**Claim 2.** Coset can generate a partition of group:

$$G = a_1H \cup a_2H \cup \cdots \cup a_rH$$



### 6.9.1 index of a subgroup

**Definition 16.** Let  $H$  be a subgroup of a group  $G$ . The number of left cosets of  $H$  in  $G$  is the **index**.

**Note:** Since  $|aH| = |H| \forall a \in G$ , the index of a subgroup is the number of subgroups which have order  $|H|$ .

### 6.9.2 Lagrange Theorem: Order of subgroup divides the order of group

**Theorem 25** (Lagrange Theorem). Let  $H \leq G$  be a subgroup of finite group  $G$ . Then the order  $|H|$  divides the order  $|G|$ .

证明.

Give a partition

$$\begin{aligned} G &= a_1H \cup a_2H \cup \cdots \cup a_rH \\ |G| &= |a_1H| + |a_2H| + \cdots + |a_rH| \\ &= r|H| \rightarrow |H| \mid |G| \end{aligned}$$

□

### 6.9.3 Theorem: Order of element $a \in G = |\langle a \rangle|$ divides $|G|$

**Theorem 26** (Order of element/cyclic subgroup). For  $a \in G$ , the order of  $a$  (the smallest  $m$  such that  $a^m = e$ ) divides  $|G|$ . The order of  $a$  is the order of cyclic subgroup  $\langle a \rangle$  with generator  $a$ .

证明.

For  $a \in G$ ,  $H = \{a^n, n \in \mathbb{Z}\} \leq G$ .  $H$  is the size of  $m$ . With lagrange theorem,  $|H| = m \mid |G|$

□

**Corollary 5.** Every group of prime order is cyclic.

### 6.9.4 Theorem: Order $n$ cyclic group is isomorphic to $(\mathbb{Z}_n, +_n)$

**Theorem 27.** Let  $G$  be a cyclic group with generator  $a$ . If the order of  $G$  is infinite, then  $G$  is isomorphic to  $(\mathbb{Z}, +)$ . If  $G$  has finite order  $n$ , then  $G$  is isomorphic to  $(\mathbb{Z}_n, +_n)$ .

## 6.10 Direct Products

### 6.10.1 Cartesian product

Let  $G_1, G_2, \dots, G_n$  be  $n$  groups. Let  $G = G_1 \times G_2 \times \cdots \times G_n$  be the Cartesian product.

For  $g \in G$ ,  $g = (g_1, \dots, g_n)$ ,  $g_i \in G_i$ .

**Theorem 28.** Then  $(G, *)$  becomes a group with operation  $*$  defined as

$$a * b = (a_1, \dots, a_n) * (b_1, \dots, b_n) = (a_1b_1, \dots, a_nb_n) \quad a, b \in G$$

证明.

(1) Binary operation  $*$ :  $G \times G \rightarrow G$ .

(2)  $*$  is associative:

$$(a * b) * c = a * (b * c) = (a_1 b_1 c_1, \dots, a_n b_n c_n)$$

(3) Identity:  $e = (e_1, \dots, e_n) \in G$

$$e * a = a = a * e$$

(4) Inverse:  $a^{-1} = (a_1^{-1}, \dots, a_n^{-1}) \in G$

$$a * a^{-1} = a^{-1} * a = e$$

□

**6.10.2 Theorem:**  $\mathbb{Z}_m \times \mathbb{Z}_n$  is cyclic and is isomorphic to  $\mathbb{Z}_{mn} \Leftrightarrow \gcd(m, n) = 1$

**Theorem 29.** The group  $\mathbb{Z}_m \times \mathbb{Z}_n$  is cyclic and is isomorphic to  $\mathbb{Z}_{mn}$  if and only if  $\gcd(m, n) = 1$ .

证明.

**Claim:**  $(1, 1)$  generate  $\mathbb{Z}_m \times \mathbb{Z}_n$

$k(1, 1) = (k, k) = (0, 0)$  if and only if  $m|k$  and  $n|k$ . The smallest such  $k$  is  $k = \text{lcm}(m, n) = mn$ .

Hence,  $\mathbb{Z}_m \times \mathbb{Z}_n$  is a cyclic group with order  $mn$ . Then  $\mathbb{Z}_m \times \mathbb{Z}_n$  is isomorphic to  $\mathbb{Z}_{mn}$ .

We can define an isomorphism

$$\phi : \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow \mathbb{Z}_{mn}$$

and its inverse

$$\psi : \mathbb{Z}_{mn} \rightarrow \mathbb{Z}_m \times \mathbb{Z}_n$$

Since  $\mathbb{Z}_{mn} \langle 1 \rangle$ ,  $\mathbb{Z}_m \times \mathbb{Z}_n = \langle (1, 1) \rangle$ , we can write

$$\psi(x \bmod mn) = (x \bmod m, x \bmod n)$$

$\psi$  is well-defined.

To describe  $\phi : \mathbb{Z}_m \times \mathbb{Z}_n \rightarrow \mathbb{Z}_{mn}$  at  $1 = sm + tn$  and let

$$\phi(a \bmod m, b \bmod n) = (atn + bsm \bmod mn)$$

$$\begin{aligned} \psi(atn + bsm \bmod mn) &= (atn + bsm \bmod m, atn + bsm \bmod n) \\ &= (atn \bmod m, bsm \bmod n) \\ &= (a(1 - sm) \bmod m, b(1 - tn) \bmod n) \\ &= (a \bmod m, b \bmod n) \end{aligned}$$

Hence  $\psi$  is the inverse of  $\phi$ .

□

**Corollary 6.** *The group  $\prod_{i=1}^n \mathbb{Z}_{m_i}$  is cyclic and is isomorphic to  $\mathbb{Z}_{m_1 m_2 \dots m_n}$  if and only if the numbers  $m_i$  for  $i = 1, \dots, n$  are such that the gcd of any two of them is 1.*

**Example 35.** *If  $n$  is written as a product of powers of distinct prime numbers, as it*

$$n = (p_1)^{n_1} (p_2)^{n_2} \dots (p_r)^{n_r}$$

*then  $\mathbb{Z}_n$  is isomorphic to*

$$\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \dots \times \mathbb{Z}_{(p_r)^{n_r}}$$

### 6.10.3 Finitely Generated Abelian Groups

**Theorem 30** (Primary Factor Version of the Fundamental Theorem of Finitely Generated Abelian Groups). *Every finitely generated abelian group  $G$  is isomorphic to a direct product of cyclic groups in the form*

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \dots \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \dots \times \mathbb{Z}$$

*where the  $p_i$  are primes, not necessarily distinct, and the  $r_i$  are positive integers. The number of factors of  $\mathbb{Z}$  and the prime powers  $(p_i)^{r_i}$  are unique.*

- $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$  if  $\gcd(m, n) = 1$ .
- Abelian  $\Leftrightarrow \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_n \times \mathbb{Z}_m$

**Example 36.** *Find all abelian group of order 16*

5 nonisomorphic abelian group.

$$\left\{ \begin{array}{l} \mathbb{Z}_{16} \\ \mathbb{Z}_8 \quad \times \mathbb{Z}_2 \\ \mathbb{Z}_4 \quad \times \mathbb{Z}_4 \\ \mathbb{Z}_4 \quad \times \mathbb{Z}_2 \quad \times \mathbb{Z}_2 \\ \mathbb{Z}_2 \quad \times \mathbb{Z}_2 \quad \times \mathbb{Z}_2 \quad \times \mathbb{Z}_2 \end{array} \right.$$

**Example 37.**

$$\mathbb{Z}_6 \times \mathbb{Z}_{40} \times \mathbb{Z}_{49} \simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_{49}$$

$$\mathbb{Z}_{210} \times \mathbb{Z}_{56} \simeq \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_8$$

**6.11 Def: Normal Subgroup**  $H \triangleleft G : aH = Ha, \forall a \in G$

**Definition 17.** *A subgroup  $H \leq G$  is **normal** if its left and right cosets coincide, that is, if*

$$aH = Ha, \quad \forall a \in G$$

*Notation:*  $H \triangleleft G$

**Note** that all subgroups of abelian groups are normal.

### 6.11.1 Thm: Three ways to check if $H$ is normal

**Theorem 31.** " $H < G$  is a normal subgroup of  $G$  ( $H \triangleleft G$ )" is equivalent to

(1)  $ghg^{-1} \in H$  for all  $g \in G$  and  $h \in H$

(2)  $gHg^{-1} = H$  for all  $g \in G$

(3)  $gH = Hg$  for all  $g \in G$

### 6.11.2 Thm: A subgroup is "Well-defined Left Cosets Multiplication" $\Leftrightarrow$ "Normal"

**Theorem 32.** Let  $H$  be a subgroup of a group  $G$ . Then left coset multiplication is well defined by the equation

$$(aH)(bH) = (ab)H$$

if and only if  $H \triangleleft G$  ( $H$  is a normal subgroup of  $G$ ).

i.e. ' $x \in aH$  and  $y \in bH \Rightarrow xy \in abH$ ' if and only if ' $aH = Ha$ ,  $\forall a \in G$ '

证明.

- " $\Rightarrow$ ":  $\forall x \in aH, a^{-1} \in a^{-1}H \Rightarrow xa^{-1} \in H \Leftrightarrow x \in Ha \Rightarrow aH \subset Ha$ ;

Similarly  $a^{-1}H \subset Ha^{-1} \Leftrightarrow Ha \subset aH \Rightarrow aH = Ha$

- " $\Leftarrow$ ": Let  $x \in aH, y \in bH$ . Say  $x = ah_1, y = bh_2$

$$\begin{aligned} xy &= (ah_1)(bh_2) \\ &= a(h_1b)h_2 \\ &= a(bh_3)h_2 \quad (\text{Since } bH = Hb) \\ &= (ab)(h_3h_2) \in abH \end{aligned}$$

□

## 6.12 Factor Group $G/H = \{aH : a \in G\}$

**Definition 18.** The group  $G/H = \{aH : a \in G\}$  with  $(aH)(bH) = abH$  is the factor group (or quotient group) of  $G$  by  $H$ .

### 6.12.1 Def: kernel $H$ forms a factor group $G/H$

**Definition 19.** Let  $\phi : G \rightarrow G'$  be a homomorphism of groups with kernel  $H$ . Then the cosets of  $H$  form a **factor group**,  $G/H = \{aH : a \in G\}$ . where  $(aH)(bH) = (ab)H$ .

Also, the map  $\mu : G/H \rightarrow \phi[G]$  defined by  $\mu(aH) = \phi(a)$  is an isomorphism. Both coset multiplication and  $\mu$  are well defined, independent of the choices  $a$  and  $b$  from the cosets.

### 6.12.2 Cor: $\ker\phi$ is a normal subgroup

**Corollary 7.**  $\ker\phi$  is a normal subgroup:  $\ker\phi \triangleleft G$  for all homomorphisms.

### 6.12.3 Corollary: normal subgroup $H$ forms a group $G/H$

By the Thm: A subgroup is "Well-defined Left Cosets Multiplication"  $\Leftrightarrow$  "Normal".

**Corollary 8.** Let  $H \triangleleft G$  be a **normal subgroup** of  $G$ . Then the cosets of  $H$  form a group  $G/H = \{aH : a \in G\}$  under the binary operation  $(aH)(bH) = (ab)H$ .

证明.

(1)  $*$  is associative.

(2)  $G/H$  has an identity  $H$ .

$$H * aH = aH * H = aH$$

(3)  $aH \in G/H$  has inverse  $a^{-1}H$

□

**Note:** This corollary contains the definition because kernel is normal subgroup (kernel  $\Rightarrow$  normal subgroup). (We can then prove they are exactly the same in the next theorem (kernel  $\Leftarrow$  normal subgroup))

### 6.12.4 Thm: normal subgroup is a kernel of a surjective homomorphism $\gamma : G \rightarrow G/H$

For any normal subgroup  $H \triangleleft G$ , we can define  $\gamma(x) = xH$  which is surjective with  $\ker\gamma = H$

**Theorem 33.** Let  $H \triangleleft G$  be a normal subgroup of  $G$ . Define  $\gamma : G \rightarrow G/H$ ,  $\gamma(x) = xH$ . Then  $\gamma$  is a surjective homomorphism with  $\ker\gamma = H$ .

证明.

1.  $\gamma$  is surjective homomorphism:  $\gamma(ab) = abH = (aH)(bH) = \gamma(a)\gamma(b)$

2.  $\ker\gamma = H$ : The identity in  $G/H$  is the coset  $H$ .

$$\begin{aligned}\ker\gamma &= \gamma^{-1}(H) = \{a \in G : \gamma(a) = aH = H\} \\ &= \{a \in G : a \in H\} = H\end{aligned}$$

□

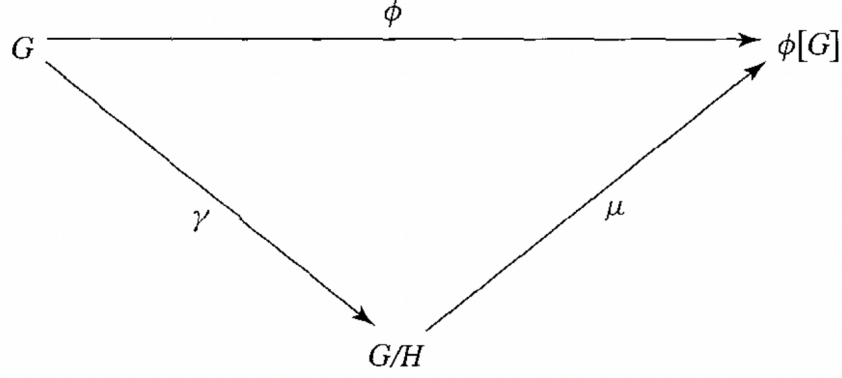


图 3: The Fundamental Homomorphism Theorem

**6.12.5 The Fundamental Homomorphism Theorem:** Every homomorphism  $\phi$  can be factored to a homomorphism  $\gamma : G \rightarrow G/H$  and isomorphism  $\mu : G/H \rightarrow \phi[G]$

**Theorem 34** (The Fundamental Homomorphism Theorem).

Homomorphism  $\phi : G \rightarrow G'$  with kernel  $H$  can be **factored**

$$\phi = \mu\gamma$$

where  $\gamma : G \rightarrow G/H$  is a homomorphism,  $\mu : G/H \rightarrow \phi[G]$  is an isomorphism  
where  $\gamma(g) = gH$ ,  $\mu(gH) = \phi(g)$

Let  $\phi : G \rightarrow G'$  be a group homomorphism with kernel  $H$ .

Then  $\phi[G]$  is a group isomorphic to  $G/H$ , and  $\mu : G/H \rightarrow \phi[G]$  given by  $\mu(gH) = \phi(g)$  is an isomorphism. (If  $\gamma : G \rightarrow G/H$  is the homomorphism given by  $\gamma(g) = gH$ , then  $\phi(g) = \mu\gamma(g)$  for each  $g \in G$ .)

证明. i.e. prove  $\mu$  is (1) well-defined, (2) isomorphism.

(1) well-defined: if  $aH = bH$ , then  $a^{-1}b \in H$ ,

$$\mu(bH) = \mu((a(a^{-1}b))H) = \phi(a(a^{-1}b)) = \phi(a)\phi(a^{-1}b) = \phi(a) = \mu(aH)$$

(2) homomorphism:

$$\mu(aHbH) = \mu(abH) = \phi(ab) = \phi(a)\phi(b) = \mu(aH)\mu(bH)$$

(3) isomorphism i.e. prove  $\ker(\mu)$  is exactly the identity in  $G/H$ :

$$\begin{aligned} \mu(aH) = e' = \phi(a) &\Leftrightarrow a \in \ker(\mu), a \in \ker(\phi) = H \\ &\Leftrightarrow aH = H, \quad aH \text{ is the identity in } G/H \end{aligned}$$

□

**Corollary 9.** Let  $\phi : G \rightarrow G'$  be a homomorphism for finite group  $G, G'$ .

Then (1).  $|\phi(G)| \mid |G|$ ; (2).  $|\phi(G)| \mid |G'|$

证明.

(1) According to the Fundamental Homomorphism theorem,  $\phi(G)$  is one-to-one correspondence to  $G/H$  ( $H$  is the kernel of  $G$ ), then  $|\phi(G)| = |G/H| = |\{aH : a \in G\}| \Rightarrow |\phi(G)| = |G|/|H|$

(2) Proved by Lagrange theorem.

□

**6.12.6 Thm:**  $(H \times K)/(H \times e) \simeq K$  and  $(H \times K)/(e \times K) \simeq H$

**Theorem 35.** Let  $G = H \times K$  be the direct product of groups  $H$  and  $K$ . Then  $\bar{H} = \{(h, e) \mid h \in H\}$  is a normal subgroup of  $G$ . Also  $G/\bar{H}$  is isomorphic to  $K$  in a natural way. Similarly,  $G/\bar{K} \simeq H$  in a natural way.

证明.  $\pi : H \times K \rightarrow K$  where  $\pi(h, k) = k$  has kernel  $\bar{H} = \{(h, e) \mid h \in H\}$ , then  $H \times K/\bar{H}$  is isomorphic to  $K$ . Prove  $G/\bar{K} \simeq H$  in the same way. □

**6.12.7 Thm:** factor group of a cyclic group is cyclic  $[a]/N = [aN]$

**Theorem 36.** A factor group of a cyclic group is cyclic.  $[a]/N = [aN]$

**6.12.8 Ex: 15.11 example**  $\mathbb{Z}_4 \times \mathbb{Z}_6 / (\langle (2, 3) \rangle) \simeq \mathbb{Z}_4 \times \mathbb{Z}_3$  or  $\mathbb{Z}_{12}$

**6.12.9 Thm:** Homomorphism  $\phi : G \rightarrow G'$  preserves normal subgroups between  $G$  and  $\phi[G]$ .

**Theorem 37.** Let  $\phi : G \rightarrow G'$  be a group homomorphism. If  $N$  is a normal subgroup of  $G$ , then  $\phi[N]$  is a normal subgroup of  $\phi[G]$ . Also, if  $N'$  is a normal subgroup of  $\phi[G]$ , then  $\phi^{-1}[N']$  is a normal subgroup of  $G$ .

Note:  $\phi[N]$  is a normal subgroup of  $\phi[G]$  not  $G'$ . Counterexample:  $\phi : \mathbb{Z}_2 \rightarrow S_3$ , where  $\phi(0) = \rho_0$  and  $\phi(1) = \mu_1$  is a homomorphism, and  $\mathbb{Z}_2$  is a normal subgroup of itself, but  $\{\rho_0, \mu_1\}$  is not a normal subgroup of  $S_3$ .

**6.13 Def:** automorphism, inner automorphism

**Definition 20.**

An isomorphism  $\phi : G \rightarrow G$  of a group  $G$  with itself is an automorphism of  $G$ .

The automorphism  $\phi_g : G \rightarrow G$ , where  $\phi_g(x) = gxg^{-1}$  for all  $x \in G$ , is the inner automorphism of  $G$  by  $g$ . Performing  $\phi_g$  on  $x$  is called conjugation of  $x$  by  $g$ .

## 6.14 Simple Groups

**Definition 21.** A group  $G$  is simple if it is nontrivial ( $G \neq \{e\}$ ) and has no proper nontrivial normal subgroups. ( $\nexists H \neq \{e\} \triangleleft G$ )

**Theorem 38.** The alternating group  $A_n$  is simple for  $n \geq 5$   
(alternating group is a group of even permutations on a set of length  $n$ )

## 6.15 The Center and Commutator Subgroups

### 6.15.1 Def: center and commutator subgroup

**Theorem 39.** All finite subgroup  $G$  have two normal subgroups,

- (1) The *center* of  $G$ ,  $Z(G) = \{z \in G : za = az, \forall a \in G\} \triangleleft G$
- (2) The *commutator* subgroup of  $G$ ,  $C(G) = [G, G] = \{[a, b] : a, b \in G\}$ .

**Definition 22.**  $[a, b] = aba^{-1}b^{-1}$  is the commutator of  $a$  and  $b$ .  $[a, b] \in G$  is the unique element such that  $ab = [a, b]ba$ .

### 6.15.2 Thm: commutator subgroup is normal

**Theorem 40.**  $[G, G] \triangleleft G$

証明. Consider  $[a, b] \in [G, G]$ , prove that  $\forall g \in G, g[a, b]g^{-1} \in [G, G]$

$$\begin{aligned} g[a, b]g^{-1} &= g(aba^{-1}b^{-1})g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) \\ &= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} = [gag^{-1}, gbg^{-1}] \in [G, G] \end{aligned}$$

□

**Example 38.**

- (1) For abelian group,  $Z(G) = G$ ,  $C(G) = \{e\}$
- (2)  $G = S_6$ ,  $Z(G) = \{e\}$ ,  $C(G) = \{1, \rho, \rho^2\}$
- (3)  $G = D_8 = \{1, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$ ,  $Z(G) = \{1, \rho^2\}$ ,  $C(G) = \{1, \rho^2\}$
- (4)  $G = D_{12}$ ,  $Z(G) = \{1, \rho^3\}$ ,  $C(G) = \{1, \rho^2, \rho^4\}$
- (5)  $G = A_4$ ,  $Z(G) = \{(1)\}$ ,  $C(G) = \{(1), (12)(34), (13)(24), (14)(23)\}$
- (6)  $G = S_4$ ,  $Z(G) = \{(1)\}$ ,  $C(G) = A_4$

Commutator subgroup of  $S_n$  is  $A_n$ .

Commutator subrgoup of  $D_{2n}$  is  $\{1, \rho^2, \dots, \rho^{n-2}\}$

$\sigma\rho^a = \rho^{n-a}\sigma = \rho^{n-2a}(\rho^a\sigma) \Rightarrow \rho^{n-2a}$  is a commutator  $\forall a \in \mathbb{Z} \Rightarrow C(D_{2n}) = \{1, \rho^2, \dots, \rho^{n-2}\}$  if  $n$  is even.



**6.15.3 Thm:** if  $N \triangleleft G$ , " $G/N$  is abelian"  $\Leftrightarrow$  " $[G, G] < N$ "

**Theorem 41.** If  $N$  is a normal subgroup of  $G$ , then  $G/N$  is abelian if and only if  $[G, G] < N$ .

证明.

If  $N$  is a normal subgroup of  $G$  and  $G/N$  is abelian, then  $(a^{-1}N)(b^{-1}N) = (b^{-1}N)(a^{-1}N)$ ; that is,  $aba^{-1}b^{-1}N = N$ , so  $aba^{-1}b^{-1} \in N$ , and  $C \leq N$ . Finally, if  $C \leq N$ , then

$$\begin{aligned}(aN)(bN) &= abN = ab(b^{-1}a^{-1}ba)N \\ &= (abb^{-1}a^{-1})baN = baN = (bN)(aN)\end{aligned}$$

□

## 6.16 Group Action on a Set

### 6.16.1 Def: action of group $G$ on set $X$

**Definition 23.** Let  $X$  be a set and  $G$  a group. An **action of  $G$  on  $X$**  is a map  $*$  :  $G \times X \rightarrow X$  such that

- (1)  $ex = x$  for all  $x \in X$ .
- (2)  $(g_1g_2)(x) = g_1(g_2x)$  for all  $x \in X$  and all  $g_1, g_2 \in G$ .

Under these conditions,  $X$  **is a  $G$ -set**.

### 6.16.2 Thm: $\phi : G \rightarrow S_X$ as $\phi(g) = \sigma_g$ is a homomorphism

**Theorem 42.** Let group  $G$  act on the set  $X$ ,

- (1)  $\phi : G \rightarrow S_X$  defined by  $\phi(g) = \sigma_g$  is well-defined.  
( $\sigma_g : X \rightarrow X$  defined by  $\sigma_g(x) = gx$  for  $x \in X$  is a permutation of  $X$ )
- (2)  $\phi : G \rightarrow S_X$  defined by  $\phi(g) = \sigma_g$  is a homomorphism with the property that  $\phi(g)(x) = gx$ .

Special case: Let  $G$  act on itself, we get the **Cayley Theorem**:  $G$  is isomorphic to a subgroup of  $S_G$ . In general, for a group  $G$  act on the set  $X$ , the homomorphism  $\phi : G \rightarrow S_X$  is not injective. We say that  $G$  acts faithfully on  $X$  if  $\phi$  is injective.

### 6.16.3 Examples of Group Actions

(Let  $H \leq G$  be a subgroup of  $G$ )

- (1)  $G \times G \rightarrow G, (g_1, g_2) \rightarrow g_1g_2$
- (2)  $G \times G \rightarrow G, (g_1, g_2) \rightarrow g_1g_2g_1^{-1}$  (conjugation)
- (3)  $G \times G/H \rightarrow G/H, (g, aH) \rightarrow gaH$  (when  $H$  is not normal,  $X = G/H$  is just a set.)

## 6.17 Orbits

**6.17.1 Thm: Equivalence Relation**  $x_1 \sim x_2 \Leftrightarrow x_2 = gx_1, \exists g \in G$

**Theorem 43.** For  $G$  acting on  $X$ , define a relation  $\sim$  on  $X$  via

$$x_1 \sim x_2 \Leftrightarrow x_2 = gx_1 \quad \text{for some } g \in G$$

**Definition 24.** A group  $G$  is **transitive** on a  $G$ -set  $X$  if for each  $x_1, x_2 \in X$ , there exists  $g \in G$  such that  $gx_1 = x_2$ .

**6.17.2 Def:**  $Gx = \{gx | g \in G\}$  is the orbit of  $x$

**Definition 25.** For a group action  $G$  on  $X$ ,  $X$  partitions into equivalence classes. Denote the class containing  $x$  by  $Gx$ .  $Gx = \{gx | g \in G\}$  is called the orbit of  $x \in X$ .

**Denote:** the partition of  $X$  as equivalence classes takes the form

$$X = Gx_1 \cup Gx_2 \cup \cdots \cup Gx_r$$

$r$  disjoint orbits.

**6.17.3 Def:**  $G_x = \{g \in G | gx = x\}$  is the stabilizer of  $x$

**Definition 26.** Let  $G$  act on  $X$ , for  $x \in X$ , define  $G_x = \{g \in G | gx = x\}$ , then  $G_x$  is a subgroup of  $G$  called the **stabilizer** of  $x$ . (or the **isotropy subgroup** of  $x$ )

## 6.17.4 Orbit-Stabilizer Theorem

**Theorem 44.** Let  $G$  act on  $X$ , and let  $x \in X$ , then  $|Gx| = [G : G_x] = |G/G_x| = \frac{|G|}{|G_x|}$

**Example 39.**

## 6.18 Applications of $G$ -sets to Counting

**6.18.1 Thm: Burnside's Formula**

**Theorem 45.** Let  $G$  be a finite group and  $X$  a finite  $G$ -set. If  $r$  is the number of orbits in  $X$  under  $G$ , then

$$r = \frac{1}{|G|} \sum_{g \in G} |X^g|$$

i.e.  $r$  equals to the average  $|X^g|$ , where  $X^g = \{x : gx = x\}$

证明.

□

## 7 Ring and Field

**7.1 Ring  $(R, +, \cdot)$ :**  $+$  is associative, commutative, identity, inverse  $\in R$ ;  $\cdot$  is associative, distributes over  $+$

### 7.1.1 Def, Prop

**Definition 27.** A ring is a nonempty set with two operations, called addition and multiplication,  $(R, +, \cdot)$  such that

- (1):  $(R, +)$  is an abelian group: i.e.  $+$  is associative and commutative.  $0, -a \in R$
- (2):  $\cdot$  is associative.
- (3):  $\cdot$  distributes over  $+$ :  $\forall a, b, c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$  and  $(b + c) \cdot a = b \cdot a + c \cdot a$

**Theorem 46.** If  $R$  is a ring with additive identity  $0$ , then for any  $a, b \in R$  we have

- 1.  $0a = a0 = 0$ ,
- 2.  $a(-b) = (-a)b = -(ab)$ ,
- 3.  $(-a)(-b) = ab$ .

### 7.1.2 $S \subset R$ : Subring (closed under $+$ and $\cdot$ ; additive inverse $-a \in S$ )

**Proposition 25** (Proposition 2.6.27). If  $S \subset R$  is a subring, then  $+, \cdot$  make  $S$  into a ring.

### 7.1.3 Def: Commutative ring: ring's $\cdot$ is commutative

If " $\cdot$ " is commutative, we call  $(R, +, \cdot)$  a commutative ring.

### 7.1.4 Def: A ring with 1: the ring exists multiplication identity $1 \in R$

If there exists an element  $1 \in R \setminus \{0\}$  such that  $a1 = 1a = a, \forall a \in R$ , then we say that  $R$  is a ring with 1 (a ring with unity).

Note: We usually discuss  $1 \neq 0$ . If  $1 = 0, a = 1a = 0 \Rightarrow R = \{0\}$ .

### 7.1.5 Def: In a ring $R$ with 1, $u$ is a unit if $\exists v \in R$ s.t. $uv = vu = 1$

**Definition 28.** In a ring  $R$  with 1,  $u$  is a unit if it has a multiplicative inverse in  $R$  i.e.  $\exists v \in R$  s.t.  $uv = vu = 1$

**Example 40.** units in  $\mathbb{Z}$  are  $\{-1, +1\}$ ; in  $\mathbb{Z}_n$  are  $\{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$

### 7.1.6 Def: A ring with 1, $R$ is a division ring if every nonzero element of $R$ is a unit

**Definition 29.** A ring with 1,  $R$  is a division ring if every nonzero element of  $R$  is a unit. This is equivalent to  $R$  has identity and inverse in multiplication.

**7.1.7 Def: Ring Homomorphism:**  $\phi(a + b) = \phi(a) + \phi(b)$ ,  $\phi(ab) = \phi(a)\phi(b)$

**Definition 30.** Let  $R, R'$  be rings. A map  $\phi : R \rightarrow R'$  is a ring homomorphism if

$$\phi(a + b) = \phi(a) + \phi(b)$$

$$\phi(ab) = \phi(a)\phi(b)$$

**7.1.8 Def: zero divisor:** a  $a \neq 0 \in R$  if  $\exists b \neq 0 \in R$  s.t.  $ba = 0$  or  $ab = 0$

**Definition 31.** A nonzero element  $a \in R$  is called a zero divisor if there exists a nonzero  $b \in R$  s.t.  $ba = 0$  or  $ab = 0$

Note: Multiplication cancellation law holds when no zero divisors.

**7.1.9 Remark:** In  $\mathbb{Z}_n$ , an element is either 0 or unit or zero divisor

Remark: In  $\mathbb{Z}_n$ , an element is either (1) 0, (2) a unit, (3) a zero divisor.

$$0 \neq a \in \mathbb{Z}_n \text{ is a } \begin{cases} \text{unit} & \text{if } \gcd(a, n) = 1 \\ \text{zero divisor} & \text{if } \gcd(a, n) \neq 1 \end{cases}$$

$$\text{In } M_n(R) \begin{cases} \text{unit} & \text{if } \text{rank}(A) = n \\ \text{zero divisor} & \text{if } \text{rank}(A) < n \end{cases}$$

In  $R = \mathbb{Z}$ ,  $a \notin \{0, +1, -1\}$  is neither unit nor zero divisor.

**7.1.10 Thm:**  $a \in \mathbb{Z}_n$  is a zero divisor  $\Leftrightarrow \gcd(a, n) \neq 1$ .

**Theorem 47.** In the ring  $\mathbb{Z}_n$ , the zero divisors are precisely those nonzero elements that are not relatively prime to  $n$ .

**7.1.11 Cor:**  $\mathbb{Z}_p$  has no zero divisors if  $p$  is prime.

**7.1.12 Def:** An integral domain is a commutative ring with  $1 \neq 0$  that has no zero divisors

**Definition 32.** An integral domain is a commutative ring with  $1 \neq 0$  that has no zero divisors.

$\mathbb{Z}$  and  $\mathbb{Z}_p$  for any prime  $p$  are integral domains, but  $\mathbb{Z}_n$  is not an integral domain if  $n$  is not prime.

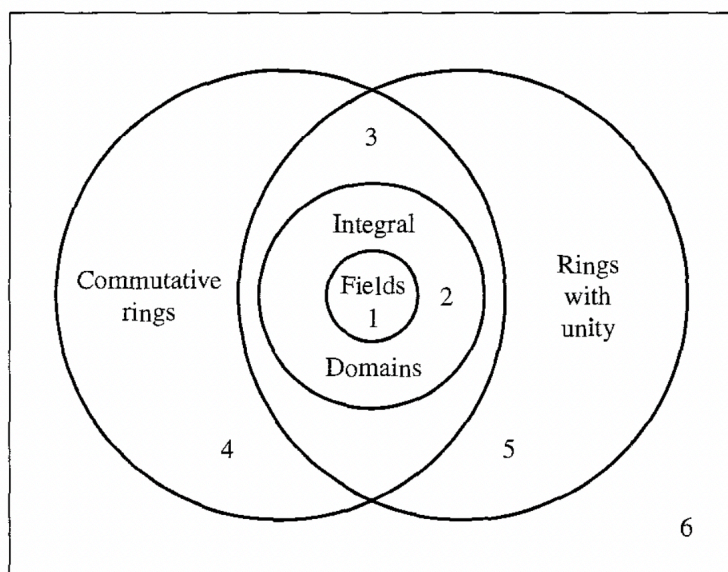
## 7.2 Field $\mathbb{F}$

**7.2.1 Def:** A field is a commutative division ring.

**Definition 33.** A field is a commutative division ring.

Which is equal to a ring satisfies identity, inverse and commutative in multiplication. Field  $(\mathbb{F}, +, \cdot)$  (close, associative, commutative, distributive(M over A), identity & inverse(M,A))

**Note:** nonzero elements of a finite field can form a cyclic (sufficient for abelian) multiplication group.



**19.10 Figure** A collection of rings.

图 4: example: 1.  $\mathbb{Z}_2, \mathbb{Q}$ , 2.  $\mathbb{Z}$ , 3.  $\mathbb{Z}_4$ , 4.  $2\mathbb{Z}$  5.  $M_2(\mathbb{Z}), M_2(\mathbb{R})$ , 6. upper-triangular matrices with integer entries and all zeros on the main diagonal

### 7.2.2 Differences between "Field" and "Integral Domain"

Def: An integral domain is a commutative ring with  $1 \neq 0$  that has no zero divisors

Def: A field is a commutative ring with  $1 \neq 0$  that every nonzero element of  $R$  is a unit.

### 7.2.3 Lemma: A unit is not zero divisor

证明.  $a \in R$  is a unit and  $\frac{1}{a}$  is its inverse.

Assume there exists  $b \neq 0$  s.t.  $ab = 0$ , then

$$\begin{aligned} \frac{1}{a}(ab) &= \frac{1}{a}0 = 0 \\ &= \left(\frac{1}{a}a\right)b = b \end{aligned}$$

Contradiction!

Assume there exists  $b \neq 0$  s.t.  $ba = 0$ , then

$$\begin{aligned} (ba)\frac{1}{a} &= 0\frac{1}{a} = 0 \\ &= b\left(a\frac{1}{a}\right) = b \end{aligned}$$

Contradiction!

□

### 7.2.4 Lemma: A field doesn't has zero divisors

Since a field is a division ring, its nonzero elements are unit which is not zero divisor.

### 7.2.5 Thm: Every field is an integral domain

**Theorem 48.** *Every field is an integral domain.*

prove by previous lemma.

### 7.2.6 Thm: Every finite integral domain is a field

**Theorem 49.** *Every finite integral domain is a field.*

证明. The only thing we need to show is that a typical element  $a \neq 0$  has a multiplicative inverse. Consider  $a, a^2, a^3, \dots$ . Since there are only finitely many elements we must have  $a^m = a^n$  for some  $m < n$ .

Then  $0 = a^m - a^n = a^m(1 - a^{n-m})$ . Since there are no zero-divisors we must have  $a^m \neq 0$  and hence  $1 - a^{n-m} = 0$  and so  $1 = aa^{n-m-1}$  and we have found a multiplicative inverse for  $a$ .  $\square$

### 7.2.7 Note: Finite Integral Domain $\subset$ Field $\subset$ Integral Domain

$\mathbb{Z}_p$  is a field.

$\mathbb{Z}$  is an integral domain but not a field.

## 7.3 The Characteristic of a Ring

**7.3.1 Def: characteristic  $n$  is the least positive integer s.t.  $n \cdot a = 0, \forall a \in R$**

**Definition 34.** *If for a ring  $R$  a positive integer  $n$  exists such that  $n \cdot a = 0$  for all  $a \in R$ , then the least such positive integer is the characteristic of the ring  $R$ . If no such positive integer exists, then  $R$  is of characteristic 0.*

**Example 41.** *The ring  $\mathbb{Z}_n$  is of characteristic  $n$ , while  $\mathbb{Z}, \mathbb{Q}, \mathbb{M}$ , and  $\mathbb{C}$  all have characteristic 0.*

**7.3.2 Thm: In a ring with 1, characteristic  $n \in \mathbb{Z}^+$  s.t.  $n \cdot 1 = 0$**

**Theorem 50.** *Let  $R$  be a ring with 1. If  $n \cdot 1 \neq 0$  for all  $n \in \mathbb{Z}^+$ , then  $R$  has characteristic 0. If  $n \cdot 1 = 0$  for some  $n \in \mathbb{Z}^+$ , then the smallest such integer  $n$  is the characteristic of  $R$ .*

## 8 The Ring $\mathbb{Z}_n$ (Fermat's and Euler's Theorems)

### 8.1 Fermat's Theorem

**8.1.1 Thm: nonzero elements in  $\mathbb{Z}_p$  ( $p$  is prime) form a group under multiplication**

**Theorem 51.** *The nonzero elements in  $\mathbb{Z}_p$  ( $p$  is prime) form a group under multiplication.*

证明.  $\mathbb{Z}_p$  is a finite field. □

**8.1.2 Cor: (Little Theorem of Fermat)**  $a \in \mathbb{Z}$  and  $p$  is prime not dividing  $a$ , then  $a^{p-1} \equiv 1 \pmod{p}$  ( $p$  divides  $a^{p-1} - 1$ )

**Corollary 10** (Little Theorem of Fermat).  $a \in \mathbb{Z}$  and  $p$  is prime not dividing  $a$ , then  $a^{p-1} \equiv 1 \pmod{p}$  ( $p$  divides  $a^{p-1} - 1$ )

证明. Let  $G_p = \{a \in \mathbb{Z}_p : a \neq 0\}$ , by previous theorem, we know the  $G_p$  is a group under multiplication of size  $|G_p| = p - 1$ .

Then the order of  $a$  should divide  $|G_p| = p - 1$ , then

$$a^{p-1} = 1 \in G_p \Rightarrow a^{p-1} \equiv 1 \pmod{p}$$

□

**8.1.3 Cor: (Little Theorem of Fermat)** If  $a \in \mathbb{Z}$ , then  $a^p \equiv a \pmod{p}$  for any prime  $p$

## 8.2 Euler's Theorem

Euler's Theorem is more general form of Fermat's Theorem.

**8.2.1 Thm:**  $G_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$  forms a group under multiplication

**Theorem 52.** The set  $G_n$  of nonzero elements of  $\mathbb{Z}_n$  that are not zero divisors ( $G_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$ ) forms a group under multiplication modulo  $n$ .

**8.2.2 Def: Euler phi function**  $\phi(n) = |G_n|$ , where  $G_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$

More generally, any  $n \in \mathbb{Z}^+$ ,  $a^{p-1} \equiv 1 \pmod{p}$ . Then  $G_n$  is a group under multiplication of size  $|G_n| = \phi(n)$ , we set  $\phi(n)$  be the Euler phi function. E.g.

$$\phi(8) = \#\{a \in \mathbb{Z}_8 : \gcd(a, 8) = 1\} = 4$$

$$\phi(15) = \#\{1, 2, 4, 7, 8, 11, 13, 14\} = 8$$

**8.2.3 Thm: (Euler's Theorem)** If  $a \in \mathbb{Z}$ ,  $n \geq 2$  s.t.  $\gcd(a, n) = 1$  then  $a^{\phi(n)} \equiv 1 \pmod{n}$

**Theorem 53.** If  $a$  is an integer relatively prime to  $n$ , then  $a^{\phi(n)} - 1$  is divisible by  $n$ , that is  $a^{\phi(n)} \equiv 1 \pmod{n}$ .

证明. order of  $a$  should divide  $|G_n| = \phi(n)$  then  $a^{\phi(n)} = 1 \in G_n \Rightarrow a^{\phi(n)} \equiv 1 \pmod{n}$  □

### 8.3 Application to $ax \equiv b \pmod{m}$

#### 8.3.1 Thm: find solution of $ax \equiv b \pmod{m}$ , $\gcd(a, m) = 1$

**Theorem 54.**  $a, b \in \mathbb{Z}_m, \gcd(a, m) = 1$ , then  $ax = b$  has a unique solution in  $\mathbb{Z}_m$

证明. By Euler's Theorem,  $a^{\phi(m)} \equiv 1 \pmod{m}$ , which means  $a$  is a unit of  $\mathbb{Z}_m$ , there exists a unique  $a^{-1} \in \mathbb{Z}_m$ .

Multiply  $a^{-1} \in \mathbb{Z}_m$  on both side, we can get  $x = a^{-1}b$  is the solution.  $\square$

#### 8.3.2 Thm: $ax \equiv b \pmod{m}$ , $d = \gcd(a, m)$ has solutions if $d|b$ , the number of solutions is $d$

**Theorem 55.** Let  $m$  be a positive integer and let  $a, b \in \mathbb{Z}_m$ . Let  $d = \gcd(a, m)$ . The equation  $ax = b$  has a solution in  $\mathbb{Z}_m$  if and only if  $d$  divides  $b$ . When  $d$  divides  $b$ , the equation has exactly  $d$  solutions in  $\mathbb{Z}_m$ .

#### 8.3.3 Cor: $ax \equiv b \pmod{m}$ , $d = \gcd(a, m)$ , $d|b$ , then solutions are $((\frac{a}{d})^{\phi(\frac{m}{d})-1} \frac{b}{d} + k\frac{m}{d}) + (m\mathbb{Z})$ , $k = 0, 1, \dots, d-1$

**Corollary 11.** Let  $d = \gcd(a, m)$ . The congruence  $ax \equiv b \pmod{m}$  has a solution if and only if  $d$  divides  $b$ . When this is the case, the solutions are the integers in exactly  $d$  distinct residue classes modulo  $m$ .

Steps:

(1) let  $a_1 = a/d$ ,  $b_1 = b/d$ ,  $m_1 = m/d$ , solve

$$a_1 s \equiv b_1 \pmod{m_1} \Rightarrow s = a_1^{-1} b_1$$

$$\text{where } a_1^{-1} = a_1^{\phi(m_1)-1}$$

(2) Solutions are

$$(s + km_1) + (m\mathbb{Z}), \quad k = 0, 1, \dots, d-1$$

**Example 42.** Find all solutions of  $12x \equiv 27 \pmod{18}$

$d = \gcd(12, 18) = 6$ ,  $d \nmid 27 \Rightarrow$  no solutions.

**Example 43.** Find all solutions of  $15x \equiv 27 \pmod{18}$

$d = \gcd(15, 18) = 3$ ,  $a_1 = 5, b_1 = 9, m_1 = 6$ . Then  $s = a_1^{-1} b_1 = 5 \cdot 9 = 3$ , then solutions are  $3 + 18\mathbb{Z}$ ,  $9 + 18\mathbb{Z}$ ,  $15 + 18\mathbb{Z}$

## 9 The Field of Quotients of an Integral Domain

Let  $D$  be an integral domain (a ring with 1 has no zero divisors) that we desire to enlarge to a field of quotients  $F$ . A coarse outline of the steps we take is as follows:



### 9.1 Step 1. Define what the elements of $F$ are to be. (Define $S/\sim$ )

$D$  is the given integral domain,  $S = \{(a, b) | a, b \in D, b \neq 0\} \subset D \times D$

#### 9.1.1 Def: equivalent relation $(a, b) \sim (c, d) \Leftrightarrow ad = bc$

**Definition 35.** Two elements  $(a, b)$  and  $(c, d)$  in  $S$  are equivalent, denoted by  $(a, b) \sim (c, d)$ , if and only if  $ad = bc$ .

Note: we can image it as  $\frac{a}{b} = \frac{c}{d}$ , but don't use this form.

**Lemma 4.**  $\sim$  defines an equivalence relation on  $S$ .

证明. easy to prove (1) reflexive, (2) symmetric, (3) transitive. □

### 9.2 Step 2. Define the binary operations of addition and multiplication on $S/\sim$ .

The relation  $\sim$  can define a set of all equivalence classes on  $[(a, b)]$ ,  $(a, b) \in S$ ,  $S/\sim = \{[(a, b)] | (a, b) \in S\}$

#### 9.2.1 lemma: well-defined operations $+, \times$

**Lemma 5.** For  $[(a, b)]$  and  $[(c, d)]$  in  $S/\sim$ , the equations

$$[(a, b)] + [(c, d)] = [(ad + bc, bd)]$$

and

$$[(a, b)][(c, d)] = [(ac, bd)]$$

give well-defined operations of addition and multiplication on  $S/\sim$ .

证明. Assume  $(a_1, b_1) \sim (a, b)$ ,  $(c_1, d_1) \sim (c, d)$ .

Verify  $+$ :  $(ad + bc, bd) \sim (a_1d_1 + b_1c_1, b_1d_1)$

□

### 9.3 Step 3. Check all the field axioms to show that $F$ is a field under these operations.

#### 9.3.1 Thm: $S/\sim$ is a field with $+, \times$

**Theorem 56.** With operation  $+, \times$ .  $S/\sim$  is a field.

证明. Check all field axioms:

$$\text{Associative } + : \quad \checkmark \quad \times : \checkmark$$

$$\text{Identity } + : \quad [(0, 1)] \quad \times : [(1, 1)]$$

$$[(a, b)] + [(0, 1)] = [(a, b)], [(a, b)][(1, 1)] = [(a, b)]$$

$$\text{Inverse } :+ : \quad [(-a, b)] \quad \times : [(b, a)], \forall a \neq 0$$

$$[(a, b)] + [(-a, b)] = [(0, b^2)] = [(0, 1)], \text{ where } (0, b^2) \sim (0, 1) \Leftrightarrow 0 * 1 = b^2 * 0;$$

$$[(a, b)][(b, a)] = [(ab, ab)] = [(1, 1)]$$

$$\text{Commutative } :+ : \quad \checkmark \quad \times : \checkmark$$

$$\text{Distributive laws :} \quad \checkmark$$

□

**9.4 Step 4. Show that  $F$  can be viewed as containing  $D$  as an integral subdomain.**

**9.4.1 Lem:**  $\phi(a) = [(a, 1)]$  is an isomorphism between  $D$  and  $\{[(a, 1)] | a \in D\}$

**Lemma 6.** The map  $\phi : D \rightarrow F = S / \sim$  given by  $\phi(a) = [(a, 1)]$  is an isomorphism of  $D$  with a subring of  $F (= S / \sim)$ .

证明.

$$\phi(a + b) = [(a + b, 1)] = [(a, 1)] + [(b, 1)]$$

$$\phi(ab) = [(ab, 1)] = [(a, 1)][(b, 1)]$$

Injective: assume  $\phi(a) = \phi(b)$ , then

$$[(a, 1)] = [(b, 1)] \Leftrightarrow (a, 1) \sim (b, 1) \Leftrightarrow a = b$$

Surjective:  $\forall [(a, 1)]$  is mapped from  $a$

We prove that  $\phi$  is an isomorphism between  $D$  and  $\{[(a, 1)] | a \in D\}$ . □

**9.4.2 Thm: every element of  $F$  can be expressed as a quotient of two elements of  $D$ :**

$$[(a, b)] = \frac{\phi(a)}{\phi(b)}$$

$\forall [(a, b)] \in F$ ,

$$[(a, b)] = [(a, 1)][(1, b)] = \frac{[(a, 1)]}{[(1, b)]^{-1}} = \frac{[(a, 1)]}{[(b, 1)]} = \frac{\phi(a)}{\phi(b)}$$

**Theorem 57.** Any integral domain  $D$  can be enlarged to (or embedded in) a field  $F = S / \sim$  such that every element of  $F$  can be expressed as a quotient of two elements of  $D$ . (Such a field  $F$  is a **field of quotients** of  $D$ .)

## 10 Polynomials

### 10.1 Def: Polynomials

Let  $R$  be any field. A polynomial over  $R$  in variable  $x$  is a formal sum:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = \sum_{i=0}^n a_i x^i$$

where  $n \geq 0$  is an integer,  $a_1, a_1, \dots, a_n \in \mathbb{F}$ .

Polynomial is a sequence  $\{a_k\}_{k=0}^{\infty}$  with  $a_m = 0, \forall m > n$ .

**Remark:**  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_dx^d$  If  $a_d \neq 0$  and  $a_i = 0, \forall i > d$ ,  $d$  is the degree of  $f(x)$ .

## 10.2 Rings of Polynomials

### 10.2.1 Thm: $R[x]$ is a ring under addition and multiplication

**Theorem 58.** *The set  $R[x]$  of all polynomials in an indeterminate  $x$  with coefficients in a ring  $R$  is a ring under polynomial addition and multiplication.*

*Note: If  $R$  is commutative, then so is  $R[x]$ , and if  $R$  has unity  $1 \neq 0$ , then  $1$  is also unity for  $R[x]$ .*

Let  $R[x]$  denote the set of all polynomials with coefficients in the ring  $R$ .

$$R[x] = \left\{ \sum_{i=0}^n a_i x^i \mid n \geq 0, n \in \mathbb{Z}, a_0, \dots, a_n \in R \right\}$$

We call the  $R[x]$  *polynomial ring* over the ring  $R$ .

$$\begin{aligned} f &= \sum_{i=0}^n a_i x^i, g = \sum_{j=0}^n a_j x^j \in R[x] \\ f + g &= \sum_{i=0}^n (a_i + b_i) x^i \in R[x] \\ fg &= \left( \sum_{i=0}^n a_i x^i \right) \left( \sum_{j=0}^n a_j x^j \right) = \sum_{i=0}^{2n} \left( \sum_{j=0}^i a_j b_{i-j} \right) x^i \end{aligned}$$

### 10.2.2 Def: evaluation homomorphism

**Definition 36.** Let  $F$  be a field, and let  $\alpha \in F$ . Define an evaluation map.  $EV_{x=\alpha} : F[x] \rightarrow F$ ,  $\phi_\alpha(\sum_{i=0}^{\infty} a_i x^i) = \sum_{i=0}^{\infty} a_i \alpha^i$ . Then,

$$\begin{aligned} \phi_\alpha(f(x) + g(x)) &= \phi_\alpha(f(x)) + \phi_\alpha(g(x)) \\ \phi_\alpha(f(x)g(x)) &= \phi_\alpha(f(x))\phi_\alpha(g(x)) \end{aligned}$$

$\phi_\alpha$  is a ring homomorphism. We call it evaluation homomorphism.

**Example 44.** Consider  $EV_{x=2} : \mathbb{Q}[x] \rightarrow \mathbb{Q}$ .  $EV_{x=2}$  is a ring homomorphism. In particular it is a group homomorphism for addition.

$$\phi_2(a_0 + a_1x + \dots + a_nx^n) = a_0 + a_12 + \dots + a_n2^n$$

Note that

$$\phi_2(x^2 + x - 6) = 2^2 + 2 - 6 = 0.$$

Thus  $x^2 + x - 6$  is in the kernel  $N$  of  $\phi_2$ . Of course,

$$x^2 + x - 6 = (x - 2)(x + 3),$$

and the reason that  $\phi_2(x^2 + x - 6) = 0$  is that  $\phi_2(x - 2) = 2 - 2 = 0$ .

**Example 45.** Compute  $EV_{x=4}(3x^{106} + 5x^{99} + 2x^{53}) \in \mathbb{Z}_7[x]$

$$EV_{x=4}(3x^{106} + 5x^{99} + 2x^{53}) =$$

According to the little Theorem of Fermat,  $x^6 \equiv 1 \pmod{7}$ .

$$= 3x^4 + 5x^3 + 2x^5 = 0 \in \mathbb{Z}_7$$

**10.2.3 Def:**  $\alpha$  is zero if  $EV_{x=\alpha}(f(x)) = 0$

**Definition 37.** We say that  $\alpha$  is a zero of  $f(x)$  if  $EV_{x=\alpha}(f(x)) = 0$ .

**Example 46.** Find all zeros of  $f(x) = x^3 + 2x + 2$  in  $\mathbb{Z}_7$ .

Solve by checking all value  $f(x), x = 0, 1, \dots, 6 \Rightarrow$  zeros are  $x = 2, x = 3$ .

**10.3 Degree of a Polynomial:**  $\deg(f)$

$f = \sum_{i=0}^n a_i x^i$ ,  $\deg(f) =$  degree of  $f$  is,

$$\deg(f) = \begin{cases} 0 & \text{if } f \text{ is constant, } f \neq 0 \\ n & \text{if } a_n \neq 0 \text{ in above } (a_n = \text{leading coefficient}) \\ -\infty & \text{if } f = 0 \end{cases}$$

Define  $-\infty + a = a + (-\infty) = -\infty \forall a \in \mathbb{Z} \cup \{-\infty\}$

**10.3.1 Lemma 2.3.3:**  $\deg(fg) = \deg(f) + \deg(g)$ ,  $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$

**Lemma 7** (Lemma 2.3.3). For any field  $\mathbb{F}$  and  $f, g \in \mathbb{F}[x]$ ,

$$\deg(fg) = \deg(f) + \deg(g)$$

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

**10.4 Corollary 2.3.5: Unit(invertible) in  $\mathbb{F}[x]$ : constant  $\neq 0$  iff  $\deg(f) = 0$**

**Corollary 12** (Corollary 2.3.5). For any field  $\mathbb{F}$  and  $f \in \mathbb{F}[x]$ , Then  $f$  is a unit (i.e. invertible) in  $\mathbb{F}[x]$  iff  $\deg(f) = 0$ .

证明.

Obviously,  $\deg(f) = 0 \Rightarrow f$  is a unit.

Suppose  $f$  is a unit, i.e.  $\exists g \in \mathbb{F}[x]$  s.t.  $fg = 1$ .

$$0 = \deg(fg) = \deg(f) + \deg(g) \Rightarrow \deg(f), \deg(g) \geq 0 \Rightarrow \deg(f) = 0, \deg(g) = 0. \quad \square$$

**10.5 Irreducible Polynomials:** “无法分解为两个  $\deg \geq 1$  的多项式积”的多项式: 至少一个 is constant (i.e.  $\deg = 0$ )

A nonconstant polynomial  $f$  is irreducible if  $f = uv$ ,  $u, v \in \mathbb{F}[x]$ , then either  $u$  or  $v$  is a unit (i.e., constant  $\neq 0$ )

## 10.6 Theorem 2.3.6: nonconstant polynomials 可以被唯一地分解

**Theorem 59** (Theorem 2.3.6). Suppose  $\mathbb{F}$  is a field and  $f \in \mathbb{F}[x]$  is any nonconstant. Then  $f = ap_1p_2 \dots p_k$  where  $a \in \mathbb{F}$ ,  $p_1, \dots, p_k \in \mathbb{F}[x]$  are irreducible monic polynomials (monic = i.e. leading coeff. 1). If  $f = bq_1q_2 \dots q_r$  with  $b \in \mathbb{F}$  and  $q_1, q_2, \dots, q_r \in \mathbb{F}[x]$  monic irreducible, then  $a = b, k = r$ , and after reindexing  $p_i = q_i, \forall i$

**Lemma 8** (Lemma 2.3.7). Suppose  $\mathbb{F}$  is a field and  $f \in \mathbb{F}[x]$  is nonconstant monic polynomial. Then  $f = p_1p_2 \dots p_k$  where each  $p_i$  is monic irreducible.

证明.

Prove it by induction. When  $\deg(f) = 1$ ,  $f = uv$ ,  $u, v \in \mathbb{F}[x]$ ,  $\deg(f) = \deg(u) + \deg(v) \Rightarrow$  one of these is 0.

Suppose the lemma holds for all degree  $< n$ . When  $\deg(f) = n$ ,

Either  $f$  is irreducible, done.

Suppose  $f = uv$  with  $\deg(u), \deg(v) \geq 1$

$\Rightarrow \deg(u), \deg(v) < n \Rightarrow u = p_1p_2 \dots p_k, v = q_1q_2 \dots q_j$  So,  $f = p_1p_2 \dots p_kq_1q_2 \dots q_j$ . □

**Example 47.**  $x^2 - 1 \in \mathbb{Q}[x]$  reducible

$x - 1, x + 1 \in \mathbb{Q}[x]$  irreducible

$x^2 + 1 \in \mathbb{Q}[x]$  irreducible

$x^2 + 1 \in \mathbb{C}[x]$  reducible

$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$  reducible

## 10.7 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$ ,  $f$  divides  $g$ ,  $f|g$  means  $\exists u \in \mathbb{F}[x]$  s.t.  $g = fu$ .

**Proposition 26** (Proposition 2.3.8).  $f, h, g \in \mathbb{F}[x]$ , then

- (i) If  $f \neq 0, f|0$
- (ii) If  $f|1$ ,  $f$  is nonzero constant
- (iii) If  $f|g$  and  $g|f$ , then  $f = cg$  for some  $c \in \mathbb{F}$
- (iv) If  $f|g$  and  $g|h$ , then  $f|h$
- (v) If  $f|g$  and  $f|h$ , then  $f|(ug + vh)$  for all  $u, v \in \mathbb{F}[x]$ .

**10.7.1 Greatest common divisor of  $f$  and  $g$ : is not unique, we denote monic Greatest common divisor as  $\gcd(f, g)$**

If  $f, g \in \mathbb{F}[x]$  are nonzero polynomials, a greatest common divisor of  $f$  and  $g$  is a polynomial  $h \in \mathbb{F}[x]$  such that

(i)  $h|f$  and  $h|g$ , and

(ii) if  $k \in \mathbb{F}[x]$  and  $k|f$  and  $k|g$ , then  $k|h$ .

the  $gcd$  is not unique, but the monic  $gcd$  is unique. We call it **the monic greatest common divisor**, denote it  $gcd(f, g)$ .

**Example 48.**

$$\begin{aligned}x^2 - 1, x^2 - 2x + 1 &\in \mathbb{Q}[x] \\(x - 1)(x + 1), (x - 1)^2 &\in \mathbb{Q}[x] \\x - 1 &= gcd(x^2 - 1, x^2 - 2x + 1)\end{aligned}$$

### 10.7.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

**Proposition 27** (Proposition 2.3.9). *Given  $f, g \in \mathbb{F}[x]$ ,  $g \neq 0$ , then  $\exists q, r \in \mathbb{F}[x]$  s.t.  $deg(r) < deg(g)$  and  $f = qg + r$*

**Example 49.**

$$\begin{aligned}f &= 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x] \\f &= 3g + x^2 - 3x + 2\end{aligned}$$

### 10.7.3 Proposition 2.3.10: $gcd(f, g)$ 是 degree 最小的 $f, g$ 的线性组合

**Proposition 28** (Proposition 2.3.10). *Any 2 nonzero polynomials  $f, g \in \mathbb{F}[x]$  have a  $gcd$  in  $\mathbb{F}[x]$ . In fact among all polynomials in the set  $M = \{uf + vg | u, v \in \mathbb{F}[x]\}$  any nonconstant of minimal degree are  $gcds$ .*

证明.

$h \in M$ ,  $deg(h) = d$  minimal. Let  $k|f$  and  $k|g \Rightarrow k|uf + vg$ ,  $\forall u, v \Rightarrow k|h$ .

Suppose  $h' \in M$  is any nonzero element.  $deg(h') \geq deg(h) \Rightarrow \exists q, r \in \mathbb{F}, deg(r) < deg(h)$   $h' = qh + r$ .  $r = h' - qh \in M$ . Since  $deg(h) = d$  is nonconstant minimal degree,  $r = 0 \Rightarrow h' = qh$ . So  $\exists q_1, q_2 \in \mathbb{F}[x]$ ,  $1f + 0g = q_1h$ ,  $0f + 1g = q_2h \Rightarrow h|g, h|f$ .  $\square$

**Example 50.**

$$\begin{aligned}f &= 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x] \\f &= 3g + x^2 - 3x + 2 \\g &= (x + 1)(x^2 - 3x + 2) + x - 1 \\x^2 - 3x + 2 &= (x - 2)(x - 1) \\\Rightarrow gcd(f, g) &= x - 1 \\x - 1 &= g - (x + 1)(x^2 - 3x + 2) = g - (x + 1)(f - 3g) = (3x + 4)g - (x + 1)f\end{aligned}$$

**Example 51.** Find a greatest common divisor of  $f = x^3 - x^2 - x + 1$  and  $g = x^2 - 3x + 2$  in  $\mathbb{Q}[x]$ , and express it in form  $uf + vg$ ,  $u, v \in \mathbb{Q}[x]$ .

$$f = (x + 2)g + 3x - 3$$

$$g = \frac{1}{3}(x - 2)(3x - 3)$$

$$\gcd(f, g) = 3x - 3$$

$$3x - 3 = f - (x + 2)g$$

**10.7.4 Proposition 2.3.12:**  $\gcd(f, g) = 1, f|gh \Rightarrow f|h$

**Proposition 29** (Proposition 2.3.12). *If  $f, g, h \in \mathbb{F}[x]$ ,  $\gcd(f, g) = 1$ , and  $f|gh$ , then  $f|h$ .*

**10.7.5 Corollary 2.3.13:** *irreducible  $f$ ,  $f|gh \Rightarrow f|g$  or  $f|h$*

**Corollary 13** (Corollary 2.3.13). *If  $f \in \mathbb{F}[x]$  is irreducible, and  $f|gh$ , then  $f|g$  or  $f|h$ .*

Since  $f$  is irreducible, we have two possible situations:

1.  $\gcd(f, g) = f$ , i.e.  $f|g$  done.
2.  $\gcd(f, g) = 1$ , then according to Prop 2.3.12, we can know  $f|h$ .

## 10.8 Roots

Root:  $\alpha \in \mathbb{F}$  is a root of  $f$  if  $f(\alpha) = 0$ .

**10.8.1 Corollary 2.3.16(of Euclidean Algorithm):**  $f$  可被分为  $(x - \alpha)q + f(\alpha)$  i.e. if  $\alpha$  is a root, then  $(x - \alpha)|f$

**Corollary 14** (Corollary 2.3.16(of Euclidean Algorithm)).  $\forall f \in \mathbb{F}[x]$  and  $\alpha \in \mathbb{F}$ , there exists a polynomial  $q \in \mathbb{F}[x]$  s.t.  $f = (x - \alpha)q + f(\alpha)$ . In particular, if  $\alpha$  is a root, then  $(x - \alpha)|f$ .

## 10.9 Multiplicity

If  $\alpha$  is a root of  $f$ , say its *multiplicity* is  $m$ , if  $x - \alpha$  appears  $m$  times in irreducible factorization.

**10.9.1 Sum of multiplicity  $\leq \deg(f)$**

**Proposition 30** (Proposition 2.3.17). *Given a nonconstant polynomial  $f \in \mathbb{F}[x]$ , the number of roots of  $f$ , counted with multiplicity, is at most  $\deg(f)$ .*

## 10.10 Roots in a field may not in its subfield

Note if  $\mathbb{F} \subset \mathbb{K}$ , then  $\mathbb{F}[x] \subset \mathbb{K}[x]$ .  $f \in \mathbb{F}[x]$  may have no roots in  $\mathbb{F}$ , but could have roots in  $\mathbb{K}$

**Example 52.**  $x^n - 1 \in \mathbb{Q}[x]$  has a root in  $\mathbb{Q}$ : 1; has 2 roots if  $n$  even:  $\pm 1$

roots in  $\mathbb{C}$ :  $\zeta_n = e^{\frac{2\pi i}{n}}$ , then  $\zeta_n^n = e^{2\pi i} = 1$ ;  $(\zeta_n^k)^n = e^{2\pi k i} = 1$  So, the roots:  $\{e^{\frac{2\pi k i}{n}} | k = 0, \dots, n-1\}$

The roots of  $x^n - d$ :  $\{e^{\frac{2\pi k i}{n}} \sqrt[n]{d} | k = 0, \dots, n-1\}$

## 11 Linear Algebra

### 11.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$ )

A vector space over a field  $\mathbb{F}$  is a set  $V$  w/ an operation addition  $+: V \times V \rightarrow V$  and an operation scalar multiplication  $\mathbb{F} \times V \rightarrow V$

- (1) Addition is associative & commutative
- (2)  $\exists 0 \in V$ , additive identity:  $0 + v = v \forall v \in V$
- (3)  $1v = v \forall v \in V$  (where  $1 \in \mathbb{F}$  is multi. id. in  $\mathbb{F}$ )
- (4)  $\forall \alpha, \beta \in \mathbb{F}, v \in V, \alpha(\beta v) = (\alpha\beta)v$
- (5)  $\forall v \in V, (-1)v = -v$  we have  $v + (-v) = 0$
- (6)  $\forall \alpha \in \mathbb{F}, v, u \in V, \alpha(v + u) = \alpha v + \alpha u$
- (7)  $\forall \alpha, \beta \in \mathbb{F}, v \in V, (\alpha + \beta)v = \alpha v + \beta v$

#### 11.1.1 A field is a vector space over its subfield

**Example 53.**  $\mathbb{K} \subset \mathbb{F}$  is a subfield of a field  $\mathbb{F}$ . Then  $\mathbb{F}$  is a vector space over  $\mathbb{K}$ . (Since  $\mathbb{F} \subset \mathbb{F}[x]$ , then  $\mathbb{F}[x]$  is a vector space over  $\mathbb{F}$ .)

#### 11.1.2 Vector subspace

Suppose that  $V$  is a vector space over  $\mathbb{F}$ . A vector subspace or just subspace is a nonempty subset  $W \subset V$  closed under addition and scalar multiplication. i.e.  $v + w \in W, av \in W, \forall v, w \in W, a \in \mathbb{F}$ .

**Example 54.**  $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$ , then  $\mathbb{L}$  is a subspace of  $\mathbb{F}$  over  $\mathbb{K}$ .

### 11.2 Linear independent, Linear combination

#### 11.3 span $V$ , basis, dimension, Proposition 2.4.10

A set of elements  $v_1, \dots, v_n \in V$  is said to **span**  $V$  if every vector  $v \in V$  can be expressed as a linear combination of  $v_1, \dots, v_n$ . If  $v_1, \dots, v_n$  spans and is linearly independent, then we call the set a **basis** for  $V$ .

**Proposition 31** (Proposition 2.4.10.). Suppose  $V$  is a vector space over a field  $\mathbb{F}$  having a basis  $\{v_1, \dots, v_n\}$  with  $n \geq 1$ .



- (i) For all  $v \in V$ ,  $v = a_1v_1 + \dots + a_nv_n$  for exactly one  $(a_1, \dots, a_n) \in \mathbb{F}^n$ .
- (ii) If  $w_1, \dots, w_n$  span  $V$ , then they are linearly independent.
- (iii) If  $w_1, \dots, w_n$  are linearly independent, then they span  $V$ .

If a vector space  $V$  over  $\mathbb{F}$  has a basis with  $n$  vectors, then  $V$  is said to be  $n$ -dimensional (over  $\mathbb{F}$ ) or is said to have **dimension**  $n$ .

### 11.3.1 Standard basis vectors

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1) \in \mathbb{F}^n$$

are a basis for  $\mathbb{F}^n$  called the **standard basis vectors**.

## 11.4 Linear transformation

Given two vector spaces  $V$  and  $W$  over  $\mathbb{F}$  a **linear transformation** is a function  $T : V \rightarrow W$  such that for all  $a \in \mathbb{F}$  and  $v, w \in V$ , we have

$$T(av) = aT(v) \text{ and } T(v + w) = T(v) + T(w)$$

**Proposition 32** (Proposition 2.4.15.). *If  $V$  and  $W$  are vector spaces and  $v_1, \dots, v_n$  is a basis for  $V$  then any function from  $\{v_1, \dots, v_n\} \rightarrow W$  extends uniquely to a linear transformation  $V \rightarrow W$ .*

Any  $v \in V$ ,  $\exists (a_1, \dots, a_n)$  s.t.  $v = a_1v_1 + \dots + a_nv_n$ . Then  $T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1T(v_1) + \dots + a_nT(v_n)$

**11.4.1 Corollary 2.4.16:** 一个线性变换对应一个矩阵 **bijection**  $\mathcal{L}(V, M) \rightarrow M_{m \times n}(\mathbb{F})$

**Corollary 15** (Corollary 2.4.16.). *If  $v_1, \dots, v_n$  is a basis for a vector space  $V$  and  $w_1, \dots, w_m$  is a basis for a vector space  $W$  (both over  $\mathbb{F}$ ), then any linear transformation  $T : V \rightarrow W$  determines (and is determined by) the  $m \times n$  matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \dots & w_m \end{bmatrix}^T = A \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$  denotes the set of all linear transformations from  $V$  to  $W$ ;  $M_{m \times n}(\mathbb{F})$  the set of  $m \times n$  matrix with entries in  $\mathbb{F}$ .  $T \rightarrow A(T)$  defines a **bijection**  $\mathcal{L}(V, M) \rightarrow M_{m \times n}(\mathbb{F})$ .  $A(T)$  **represents the linear transformation**  $T$ .

### 11.4.2 Proposition 2.4.19: 线性变换矩阵相乘仍为线性变换矩阵

**Proposition 33** (Proposition 2.4.19). *Suppose that  $V$ ,  $W$ , and  $U$  are vector spaces over  $\mathbb{F}$ , with fixed chosen bases. If  $T : V \rightarrow W$  and  $S : W \rightarrow U$  are linear transformations represented by matrices  $A = A(T)$  and  $B = B(S)$ , then  $ST = S \circ T : V \rightarrow U$  is a linear transformation represented by the matrix  $BA = B(S)A(T)$ .*

## 11.5 $GL(V)$ : invertible(bijective) linear transformations $V \rightarrow V$

Given a vector space  $V$  over  $F$ , we let  $GL(V) \subset \mathcal{L}(V, V)$  denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \text{ is a bijection}\} = \mathcal{L}(V, V) \cap Sym(V)$$

## 12 Euclidean geometry basics

### 12.1 Euclidean distance, inner product

**Euclidean distance** on  $\mathbb{R}^n$ :

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

**Euclidean inner product**:

$$x \cdot y = x_1 y_1 + \dots + x_n y_n = x^T y$$

### 12.2 Isometry of $\mathbb{R}^n$ : a bijection $\mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves distance

An **isometry** of  $\mathbb{R}^n$  is a bijection  $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$  that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x - y|, \quad \forall x, y \in \mathbb{R}^n$$

#### 12.2.1 $Isom(\mathbb{R}^n)$ : set of all isometries of $\mathbb{R}^n$

We use  $Isom(\mathbb{R}^n)$  denotes the set of all isometries of  $\mathbb{R}^n$ ,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n | |\Phi(x) - \Phi(y)| = |x - y|, \quad \forall x, y \in \mathbb{R}^n\}$$

#### 12.2.2 $Isom(\mathbb{R}^n)$ is closed under $\circ$ and inverse

**Proposition 34.**  $\Phi, \Psi \in Isom(\mathbb{R}^n)$ , then  $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$

证明.

Since  $\Phi, \Psi$  are bijections, so is  $\Phi \circ \Psi$ . Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x - y|$$

Since  $id \in Isom(\mathbb{R}^n)$ ,

$$|x - y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

□

**12.3**  $A \in GL(n, \mathbb{R})$ ,  $T_A(v) = Av$ :  $A^t A = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix  $A \in GL(n, \mathbb{R})$  i.e. a *invertible linear transformations*  $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is given by  $T_A(v) = Av$ .

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t(Aw) = v^t A^t A w$$

$$A^t A = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)} T_A \in Isom(\mathbb{R}^n)$$

**12.4 Linear isometries i.e. orthogonal group**  $O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\}$

We define the all isometries in *invertible linear transformations*  $\mathbb{R}^n \rightarrow \mathbb{R}^n$  as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\} \subset GL(n, \mathbb{R})$$

**12.4.1 Special orthogonal group**  $SO(n) = \{A \in O(n) | \det(A) = 1\}$ : **orthogonal group with**  
 $\det(A) = 1$

$O(n)$  are the matrices representing linear isometries of  $\mathbb{R}^n$ .  $1 = \det(I) = \det(A^t A) = \det(A^t) \det(A) = \det(A)^2 \Rightarrow \det(A) = 1$  or  $\det(A) = -1$ . We use **special orthogonal group** represents  $A$  with  $\det(A) = 1$ ,

$$SO(n) = \{A \in O(n) | \det(A) = 1\}$$

**12.5 translation:**  $\tau_v(x) = x + v$

Define a *translation* by  $v \in \mathbb{R}^n$ ,

$$\tau_v : \mathbb{R}^n \rightarrow \mathbb{R}^n, \tau_v(x) = x + v$$

**12.5.1 translation is an isometry**

**Note 5** (Exercise 2.5.3).  $\forall v \in \mathbb{R}^n, \tau_v$  is an isometry.

证明.  $|\tau_v(x) - \tau_v(y)| = |(x + v) - (y + v)| = |x - y|$

□

**12.6 The composition of a translation and an orthogonal transformation is an isometry**  $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since *the composition of isometries is an isometry*,  $\forall A \in O(n)$  and  $v \in \mathbb{R}^n$ , the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries.**

**12.6.1 Theorem 2.5.3:** All isometries can be represented by a composition of a *translation* and an *orthogonal transformation*,  $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

**Theorem 60** (Theorem 2.5.3).  $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

## 13 Complex numbers

$$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}, \mathbb{R} = \{a + 0i | a \in \mathbb{R}\} \subset \mathbb{C}$$

Addition & multiplication

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bci + adi + bdi^2$$

$$= (ac - bd) + (bc + ad)i$$

**Complex conjugation:**  $z = a + bi, \bar{z} = a - bi, \overline{z\bar{w}} = \bar{z}\bar{w}$

**Absolute value:**  $|z| = \sqrt{a^2 + b^2}, |z|^2 = z\bar{z}$

**Additive inverse:**  $-z = -a - bi$

**Multiplicative inverse:**  $z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

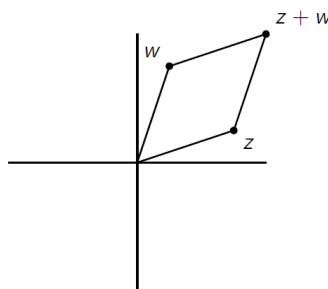
$$z \in \mathbb{C}, \overline{z + \bar{z}} = \bar{z} + \bar{\bar{z}} = z + \bar{z}$$

$$\text{Real part: } \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\text{Imaginary part: } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

### 13.1 Geometric Meaning of Addition and Multiplication

Addition: parallelogram law



Multiplication:

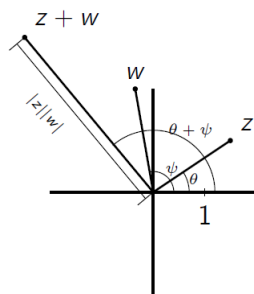
$$z = a + bi \neq 0$$

$$= r \cos \theta + r \sin \theta i$$

$$= r(\cos \theta + i \sin \theta)$$

$$|z|^2 = a^2 + b^2 = r^2$$

$$\begin{aligned}
z &= r(\cos \theta + i \sin \theta) \\
w &= s(\cos \phi + i \sin \phi) \\
zw &= rs[\cos \theta \cos \phi - \sin \theta \sin \phi + i(\cos \theta \sin \phi + \cos \phi \sin \theta)] \\
&= rs[\cos(\theta + \phi) + i \sin(\theta + \phi)] \\
&= |z||w|[\cos(\theta + \phi) + i \sin(\theta + \phi)]
\end{aligned}$$



We will write,

$$\begin{aligned}
\cos \theta + i \sin \theta &= e^{i\theta} \\
e^{i\theta} e^{i\phi} &= e^{i(\theta+\phi)} \\
z &= |z|e^{i\theta}
\end{aligned}$$

**13.2 Theorem 2.1.1:**  $f(x) = a_0 + a_1x + \dots + a_nx^n$  with coefficients  $a_0, a_1, \dots, a_n \in \mathbb{C}$ . Then  $f$  has a root in  $\mathbb{C}$ :  $\exists \alpha \in \mathbb{C}$  s.t.  $f(\alpha) = 0$

**Theorem 61** (Theorem 2.1.1). Suppose a nonconstant polynomial  $f(x) = a_0 + a_1x + \dots + a_nx^n$  with coefficients  $a_0, a_1, \dots, a_n \in \mathbb{C}$ . Then  $f$  has a root in  $\mathbb{C}$ :  $\exists \alpha \in \mathbb{C}$  s.t.  $f(\alpha) = 0$ .

**13.2.1 Corollary 2.1.2:**  $f(x) = a_n \prod_{i=1}^n (x - k_i) = a_n(x - k_1)(x - k_2) \dots (x - k_n)$ , where  $k_1, k_2, \dots, k_n$  are roots of  $f(x)$

**Corollary 16** (Corollary 2.1.2). Every nonconstant polynomial with coefficients  $a_0, a_1, \dots, a_n \in \mathbb{C}$  can be factored as  $f(x) = a_n \prod_{i=1}^n (x - k_i) = a_n(x - k_1)(x - k_2) \dots (x - k_n)$ , where  $k_1, k_2, \dots, k_n$  are roots of  $f(x)$ .

**13.2.2 Corollary 2.1.3:**  $a_i \in \mathbb{R}$ ,  $f$  can be expressed as a product of linear and quadratic polynomials

**Corollary 17** (Corollary 2.1.3). If  $f(x) = a_0 + a_1x + \dots + a_nx^n$  is a nonconstant polynomial  $a_0, a_1, \dots, a_n \in \mathbb{R}, a_n \neq 0$ . Then  $f$  can be expressed as a product of linear and quadratic polynomials.

这里  $a_0, a_1, \dots, a_n$  是实数!

证明.

(1) Obviously, the corollary holds at  $n = 1$  and  $n = 2$ .

(2) Suppose the corollary holds for all situations that  $n < k$ .

When  $n = k$ ,  $f(x) = a_0 + a_1x + \dots + a_kx^k$ ,  $a_k \neq 0$ .

By F.T.A.,  $f$  has a root  $\alpha$  in  $\mathbb{C}$ .

If  $\alpha \in \mathbb{R}$ , long division  $f(x) = q(x)(x - \alpha)$ .  $q$  has real coefficients, *degree of  $q = k - 1$* . Since the corollary holds at  $n = k - 1$ ,  $q(x)$  is a product of linear and quadratics. Then, the corollary also holds at  $n = k$ .

If  $\alpha \notin \mathbb{R}$

$$\begin{aligned} 0 &= f(\alpha) = a_0 + a_1\alpha + \dots + a_k\alpha^k \\ 0 &= \overline{f(\alpha)} = a_0 + a_1\bar{\alpha} + \dots + a_n\bar{\alpha}^n = f(\bar{\alpha}) \end{aligned}$$

Since  $\bar{\alpha} \neq \alpha$ ,  $(x - \alpha)(x - \bar{\alpha})|f$ .

$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2$  is a polynomial with coefficients in  $\mathbb{R}$ . So  $f(x) = q(x)(x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2)$ ,  $q$  has real coefficients with degree  $k - 2$ . The corollary also holds at  $n = k - 2$ ,  $q(x)$  is a product of linear and quadratics. Then, the corollary also holds at  $n = k$ .

□

## 参考文献

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