

MATH 417 Lec06-15

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1 Integers

1.1 Proposition 1.4.1: Properties of integers \mathbb{Z}

Proposition 1 (Proposition 1.4.1.). *The following hold in the integers \mathbb{Z} :*

- (i) *Addition* and *multiplication* are *commutative* and *associative* operations in \mathbb{Z} .
- (ii) $0 \in \mathbb{Z}$ is an identity element for addition; that is, $\forall a \in \mathbb{Z}, 0 + a = a$.
- (iii) Every $a \in \mathbb{Z}$ has an additive inverse, denoted $-a$ and given by $-a = (-1)a$, satisfying $a + (-a) = 0$.
- (iv) $1 \in \mathbb{Z}$ is an identity element for multiplication; that is, for all $a \in \mathbb{Z}, 1a = a$.
- (v) The *distributive* law holds: $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$.
- (vi) Both $\mathbb{N} = \{x \in \mathbb{Z} | x \geq 0\}$ and $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$ are *closed* under *addition* and *multiplication*. That is, if x and y are in one of these sets, then $x + y$ and xy are also in that set.
- (vii) For any two nonzero integers $a, b \in \mathbb{Z}, |ab| \geq \max\{|a|, |b|\}$. Strict inequality holds if $|a| > 1$ and $|b| > 1$.

From this we get cancellation.

$$ab = ac \Rightarrow b = c \text{ or } a = 0$$

1.2 Definition: Divide

Suppose $a, b \in \mathbb{Z}, b \neq 0$, b divides a if $\exists m \in \mathbb{Z}$, so that $a = bm, b|a$. Otherwise, write $b \nmid a$.

1.3 Proposition 1.4.2: properties of integer division

Proposition 2 (Proposition 1.4.2). $\forall a, b \in \mathbb{Z}$

- (i) if $a \neq 0$, then $a|0$
- (ii) if $a|1$, then $a = \pm 1$
- (iii) if $a|b$ & $b|a$, then $a = \pm b$
- (iv) if $a|b$ & $b|c$, then $a|c$
- (v) if $a|b$ & $a|c$, then $a|(mc + nb) \forall m, n \in \mathbb{Z}$

1.4 Definitions: Prime, The Greatest common divisor $\gcd(a, b)$

$p > 1, p \in \mathbb{Z}$ is called prime if the only divisors are $\pm 1, \pm p$.

Given $a, b \in \mathbb{Z}, a, b \neq 0$, the greatest common divisor of a and b is $c \in \mathbb{Z}, c > 0$ s.t.

- (1) $c|a$ and $c|b$; (2) if $d|a, d|b$, then $d|c$

The c is unique, we write it $\gcd(a, b)$.

1.5 Euclidean Algorithm

Proposition 3 (Proposition 1.4.7(Euclidean Algorithm)). *Given $a, b \in \mathbb{Z}, b \neq 0$, then $\exists q, r \in \mathbb{Z}$ s.t. $a = qb + r, 0 \leq r \leq |b|$.*

Example 1 (Exercise 1.4.3). *For the pair $(a, b) = (130, 95)$, find $\gcd(a, b)$ using the Euclidean Algorithm and express it in the form $\gcd(a, b) = sa + tb$ for $s, t \in \mathbb{Z}$.*

$$130 = 95 + 35; \quad 95 = 2 \times 35 + 25$$

$$35 = 25 + 10; \quad 25 = 2 \times 10 + 5$$

$$10 = 2 \times 5 + 0$$

$$5 = 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35$$

$$= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130$$

$$\gcd(130, 95) = \gcd(95, 35) = \gcd(35, 25) = \gcd(25, 10) = \gcd(10, 5) = \gcd(5, 0) = 5$$

We can also express it by matrix

	q	r	s	t
-1		130	1	0
0	1	95	0	1
1	2	35	1	-1
2	1	25	-2	3
3	2	10	3	-4
4	2	5	-8	11

Hence $\gcd(130, 95) = 5 = -8 \cdot 130 + 11 \cdot 95$

1.6 Proposition: $\gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$

Theorem 1. $d = \gcd(a, b)$ is of the form $sa + tb$

证明. We may assume $0 \leq a \leq b$

For $a = 0$, $d = b = 0 \cdot a + 1 \cdot b$.

For $a > 0$, let $b = q \cdot a + r$ with $0 \leq r < a \leq b$. Then

$$\begin{aligned} \{sa + tb : s, t \in \mathbb{Z}\} &= \{sa + t(q \cdot a + r) : s, t \in \mathbb{Z}\} = \{tr + ua : t, u \in \mathbb{Z}\} \\ &= \dots \{x \cdot 0 + y \cdot d : x, y \in \mathbb{Z}\} = \{\dots, -2d, -d, 0, d, 2d, \dots\} \end{aligned}$$

□

Proposition 4 (第二种表示, 第二种证明). $\forall a, b \in \mathbb{Z}$, not both 0, $\gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$. i.e. $\exists m_0, n_0 \in \mathbb{Z}$ s.t. $\gcd(a, b) = m_0 a + n_0 b$.

证明. Let c be the smallest positive integer in the set $M = \{ma + nb \mid m, n \in \mathbb{Z}\}$. $c = m_0a + n_0b > 0$. Let $d = ma + nb \in M$, $d = qc + r$ where $0 \leq r < c$ (by Euclidean Algorithm).

$$r = d - qc = (m - qm_0)a + (n - qn_0)b \in M$$

Since c is the smallest integer in M and $r \in [0, c)$, so $r = 0$. $\Rightarrow d = qc$. So $c \mid d$.

$a = 1a + 0b \in M \Rightarrow c \mid a$, $b = 0a + 1b \in M \Rightarrow c \mid b$.

If $t \mid a, t \mid b$ then $t \mid m_0a + n_0b$ i.e. $t \mid c$. $\Rightarrow c = \gcd(a, b)$. □

1.7 Well-Ordering Principle (Least Integer Axiom)

There is a smallest integer in every nonempty subset S of the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$

1.8 Proposition 1.4.10: $\gcd(b, c), b \mid ac \Rightarrow b \mid a$

Proposition 5 (Proposition 1.4.10). Suppose $a, b, c \in \mathbb{Z}$. If b, c are relatively prime i.e. $\gcd(b, c) = 1$ and $b \mid ac$, then $b \mid a$.

证明. $\gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$ s.t. $1 = mb + nc \Rightarrow a = amb + anc$. Since $b \mid nac, b \mid amb \Rightarrow b \mid a$. □

1.8.1 Corollary: $p \mid ab \Rightarrow p \mid a$ or $p \mid b$

Corollary 1 (Corollary of Prop 1.4.10). $a, b, p \in \mathbb{Z}, p > 1$ prime. If $p \mid ab$, then $p \mid a$ or $p \mid b$.

证明. If $p \mid b$, done. Otherwise, $\gcd(p, b) = 1$. By Prop 1.4.10, $p \mid a$. □

1.9 Fundamental Theorem of Arithmetic: Any integer $a \geq 2$ has a unique prime factorization

1.9.1 Existence

Lemma 1. Any integer $a \geq 2$ is either a prime or a product of primes.

证明. Set $S \subset \mathbb{N}$ be the set of all n without the given property.

Assume that S is nonempty and m is the least element in S .

Since m is not a prime, it can be written as $m = ab$ with $1 < a, b < m$. Since m is the least element in S , $a, b \notin S$. Then m is a product of primes. Contradiction. Thus, $S = \emptyset$. □

1.9.2 Uniqueness

Theorem 2 (Fundamental Theorem of Arithmetic).

Any integer $a > 1$ has a unique prime factorization: $a = p_1^{k_1} \cdot p_2^{k_2} \cdot \dots \cdot p_n^{k_n}$ where $p_i > 1$ is prime, $k_i \in \mathbb{Z}_+, \forall i = 1, \dots, n, p_i \neq p_j, \forall i \neq j$.

证明.

a) Existence: (Previous Lemma)

b) Uniqueness:

1) Method 1:

Suppose $a = p_1^{n_1} \cdot p_2^{n_2} \cdot \dots \cdot p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot \dots \cdot q_j^{r_j}$. Where $p_1 > p_2 > \dots > p_k, q_1 > q_2 > \dots > q_j, n_i, r_i \geq 1$.

$p_1 | a \Rightarrow \exists q_i \text{ s.t. } p_1 | q_i$. Similarly, $\exists q_i \text{ s.t. } q_1 | p_{i'}$.

$q_1 \leq p_{i'} \leq p_1 \leq q_i \Rightarrow q_1 = p_{i'} = p_1 = q_i$

We can also know $n_1 = r_1$, otherwise we would have two prime factorization of the quotient where the largest primes are different by dividing $p_1^{\min\{n_1, r_1\}}$.

Then we can get $b = p_2^{n_2} \cdot \dots \cdot p_k^{n_k} = q_2^{r_2} \cdot \dots \cdot q_j^{r_j}$. Then prove it by induction.

2) Method 2:

Suppose $a = p_1 \cdot p_2 \cdot \dots \cdot p_k = q_1 \cdot q_2 \cdot \dots \cdot q_t$. For a p_i , there must exist a q_j s.t. $p_i = q_j$:

Assume that $p_i \neq q_t$, $\gcd(p_i, q_t) = 1$. Then $\exists a, b$ such that $1 = ap_i + bq_t$. Multiplying both sides by $q_1 \cdot q_2 \cdot \dots \cdot q_{t-1}$:

$$q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} = ap_i q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} + bq_1 \cdot q_2 \cdot \dots \cdot q_t$$

Since $p_i | q_1 \cdot q_2 \cdot \dots \cdot q_t$, we can conclude that $p_i | (ap_i q_1 \cdot q_2 \cdot \dots \cdot q_{t-1} + bq_1 \cdot q_2 \cdot \dots \cdot q_t)$

i.e. $p_i | q_1 \cdot q_2 \cdot \dots \cdot q_{t-1}$ if $p_i \neq q_t$

Then prove by induction.

□

2 Modular arithmetic

2.1 Congruences

2.1.1 Congruent modulo m : $a \equiv b \pmod{m}$

Given $m \in \mathbb{Z}_+$, define a relation on \mathbb{Z} : **congruence modulo m**

$$a \equiv b \pmod{m}, \text{ if } m | (a - b)$$

Read as "a is congruent to b mod n"; Notation: $a \equiv b \pmod{m}$.

Equivalent to: a, b have the same remainder after division by m .

2.1.2 Proposition: For fixed $m \geq 2$, the relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " is an equivalence relation

Proposition 6 (Proposition 1.5.1). For fixed $m \geq 2$, the relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " is an equivalence relation

证明.

- 1) Reflexive: $\forall a \in \mathbb{Z}, m|0 = (a - a)$, so $a \equiv a \pmod{m}$ i.e. $a \sim a$.
- 2) Symmetric: $\forall a, b \in \mathbb{Z}, a \equiv b \pmod{m}$, then $m|(a - b) \Rightarrow m|(b - a) \Rightarrow b \equiv a \pmod{m}$. i.e. $a \sim b \Rightarrow b \sim a$.
- 3) Transitive: $\forall a, b, c \in \mathbb{Z}, a \equiv b \pmod{m}, b \equiv c \pmod{m}$. Then $m|(a - b), m|(b - c) \Rightarrow m|(a - b) + (b - c) = (a - c) \Rightarrow a \equiv c \pmod{m}$.

□

2.1.3 Theorem: the equivalence relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " partitions the integers into m disjoint sets $\Omega_i = \{a | a \sim i\}, i = 0, 1, \dots, m - 1$

Theorem 3. the equivalence relation " $a \sim b \Leftrightarrow a \equiv b \pmod{m}$ " partitions the integers into m disjoint sets $\Omega_i = \{a | a \sim i\}, i = 0, 1, \dots, m - 1$

证明. Prove any $a \in \mathbb{Z}$ belongs to a unique Ω_i .

a) Existence: Division Algorithm $\Rightarrow a = qm + r, 0 \leq r < m. a \in \Omega_r$.

b) Uniqueness: Assume a in two sets, $a \in \Omega_r \cap \Omega_{r^1}, 0 \leq r^1 < r < m$.

Then $m|a - r$ and $m|a - r^1 \Rightarrow m|r - r^1$, which is impossible because $0 < r - r^1 < m$. Contradiction.

□

2.1.4 Proposition: Addition and Mutiplication of Congruences

Proposition 7. Fix integer $m \geq 2$. If $a \equiv r \pmod{m}$ and $b \equiv s \pmod{m}$, then $a + b \equiv r + s \pmod{m}$ and $ab \equiv rs \pmod{m}$

证明.

a) Addition: $m|(a - r), m|(b - s) \Rightarrow m|(a - r) + (b - s) \Rightarrow m|(a + b) - (r + s)$.

b) Mutiplication: $m|(a - r)b + r(b - s) \Rightarrow m|ab - rs$.

□

2.2 Solving Linear Equations on Modular m

2.2.1 Theorem: unique solution of $aX \equiv b \pmod{m}$ if $\gcd(a, m) = 1$

Theorem 4. If $\gcd(a, m) = 1$, then $\forall b \in \mathbb{Z}$ the congruence $aX \equiv b \pmod{m}$ has a unique solution.

证明.

1) Existence: Since $\gcd(a, m) = 1$, $\exists s, t$ such that

$$1 = sa + tm$$

(Version 1)

(Multiplying X)

$$X = saX + tmX$$

$$aX \equiv b \pmod{m} \Leftrightarrow aX = km + b$$

$$\Leftrightarrow X = s(km + b) + b$$

$$\Leftrightarrow X \equiv sb \pmod{m}$$

(Version 2)

(Multiplying s)

$$saX \equiv sb \pmod{m}$$

$$(1 - tm)X \equiv sb \pmod{m}$$

$$X \equiv sb \pmod{m}$$

$X \equiv sb \pmod{m}$ is the solution to $aX \equiv b \pmod{m}$.

2) Uniqueness: Assume x, y are two solutions,

$$ax \equiv b \pmod{m}, ay \equiv b \pmod{m} \Rightarrow a(x - y) \equiv 0 \pmod{m}$$

Since $\gcd(a, m) = 1$, $m | (x - y) \Rightarrow x = y$, $(x, y \in \{0, 1, \dots, m - 1\})$

Example 2. Solve $3X \equiv 5 \pmod{11}$.

$$\gcd(3, 11) = 1, 1 = 4 * 3 - 1 * 11,$$

$$X \equiv 4 * 5$$

$$X \equiv 9$$

□

2.3 Chinese Remaindar Theorem (CRT): unique solution for x modulo mn

Theorem 5 (Chinese Remaindar Theorem (CRT)).

If $\gcd(m, n) = 1$. Then $\begin{cases} x \equiv r \pmod{m} & (1) \\ x \equiv s \pmod{n} & (2) \end{cases}$ have a unique solution for x modulo mn .

证明.

(1) $\Rightarrow x = km + r$ for some $k \in \mathbb{Z}$.

$$\begin{aligned} \text{substitute (2)} \Rightarrow km + r &\equiv s \pmod{n} \\ \Leftrightarrow mk &\equiv s - r \pmod{n} \quad (3) \end{aligned}$$

According to previous theorem, $\gcd(m, n) = 1$, (3) has a **unique** solution.

We say $k \equiv t \pmod{n}$, $k = ln + t$ for some $l \in \mathbb{Z}$

$\Rightarrow x = (ln + t)m + r = lnm + tm + r$, where $tm + r$ is the unique solution to x modulo mn . \square

Example 3. (Similar to CRT) Find the smallest integer x such that

$$x \equiv 1 \pmod{11} \text{ and } x \equiv 9 \pmod{13}$$

$$\gcd(11, 13) = 1 \text{ and } 1 = 6 * 11 - 5 * 13$$

Write $x = 11k + 1$. Substitute in $x \equiv 9 \pmod{13}$:

$$\begin{aligned} 11k &\equiv 8 \pmod{13} \\ 6 * 11k &\equiv 6 * 8 \equiv 9 \pmod{13} \\ (1 + 5 * 13)k &\equiv 9 \pmod{13} \\ k &\equiv 9 \pmod{13} \end{aligned}$$

Then $x = 11k + 1 = 100$.

2.4 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

将给定 n , 相同余数的数分为一组

Fix $n \in \mathbb{Z}_+$, we call $[a]_n = [a]$ the congruence class of a modulo n .

$$[a] = \{b \in \mathbb{Z} | b \equiv a \pmod{n}\} = \{a + kn | k \in \mathbb{Z}\}$$

2.4.1 Set of congruence classes of mod n : $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], \dots, [n-1]\}$

The set of *congruence classes* of mod n is denoted $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

Proposition 8 (Proposition 1.5.2.). For any $n \geq 1$ there are exactly n congruence classes modulo n , which we may write as

$$\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$$

证明.

For any $a \in \mathbb{Z}$. By Euclidean algorithm, $a = qn + r$, $q, r \in \mathbb{Z}$, $0 \leq r < n \Rightarrow a \in [r]$. So, $\mathbb{Z}_n = \{[0], [1], \dots, [n-1]\}$.

When $0 \leq a < b \leq n-1$, $n \nmid (b-a)$, so $[a] \neq [b]$ the n congruence classes listed are all distinct. Hence, there are exactly n congruence classes. \square

2.4.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

Fix $n \in \mathbb{Z}$, we define addition $+$ and multiplication \cdot on \mathbb{Z}_n :

$$\begin{aligned}[a] + [b] &= [a + b] = \{a + b + (k + j)n \mid k, j \in \mathbb{Z}\} \\ [a] \cdot [b] &= [ab] = \{ab + (aj + bk + kjn)n \mid k, j \in \mathbb{Z}\}\end{aligned}$$

This is well defined, follows Lemma 1.5.3.

Proposition 9 (Proposition 1.5.5.). *Let $a, b, c, d, n \in \mathbb{Z}, n \geq 1$, then*

(i) *Addition and multiplication are commutative and associative operations in \mathbb{Z}_n .*

(ii) $[a] + [0] = [a]$.

(iii) $[-a] + [a] = [0]$.

(iv) $[1][a] = [a]$.

(v) $[a]([b] + [c]) = [a][b] + [a][c]$.

证明.

□

2.4.3 Units(i.e. invertible) in Congruence Classes

将与 n 互质的数分为一组

Say $[a] \in \mathbb{Z}_n$ is a **unit** or is **invertible** if $\exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [1]$.

2.4.4 Proposition 1.5.6: Set of units in congruence classes: $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n \mid [a] \text{ is a unit}\} = \{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$

The set of **invertible** elements in \mathbb{Z}_n will be denoted $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n \mid [a] \text{ is a unit}\}$.

Proposition 10 (Proposition 1.5.6.). *For all $n \geq 1$, we have $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$.*

证明.

By Proposition 1.4.8, we know there exists b, c s.t. $ab + cn = 1$. So, $ab \equiv 1 \pmod{n}$, $[1] = [ab] = [a][b]$.

So, $\{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\} \subset \mathbb{Z}_n^\times$

$[a] \text{ is a unit} \Rightarrow \exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow \gcd(a, n) = 1$. So, $\mathbb{Z}_n^\times \subset \{[a] \in \mathbb{Z}_n \mid \gcd(a, n) = 1\}$. □

Note 1. *Inverse of $[a]$ is unique, i.e. $[b] = [a]^{-1}$ is unique.*

$$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$$

2.4.5 Corollary 1.5.7: if p is prime, $\varphi(p) = \mathbb{Z}_p^\times = \{[1], [2], \dots, [p-1]\}$

Corollary 2 (Corollary 1.5.7). *If $p \geq 2$ is prime, $\mathbb{Z}_p^\times = \{[1], [2], \dots, [p-1]\}$.*

2.5 Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$

Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$.

p prime, $\varphi(p) = p - 1$.

2.5.1 $m|n$, $\pi_{m,n}([a]_n) = [a]_m$

Example 4 (Exercise 1.5.4). If $m|n$, we can define $\pi_{m,n} : \mathbb{Z}_n \rightarrow \mathbb{Z}_m$ by $\pi_{m,n}([a]_n) = [a]_m$. Prove it is well-defined.

证明.

We write $[a]_n = [c]_n$, verify that $[a]_m = [c]_m$.

Since $m|n$, there exists $k \in \mathbb{Z}$ s.t. $n = km$.

$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z}$ s.t. $c = a + jn$.

$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m$ □

2.6 Theorem 1.5.8(Chinese Remainder Theorem): $n = mk, \gcd(m, k) = 1, F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

Theorem 6 (Theorem 1.5.8(Chinese Remainder Theorem)). If $m, n, k > 0, n = mk, \gcd(m, k) = 1$, then $F : \mathbb{Z}_n \rightarrow \mathbb{Z}_m \times \mathbb{Z}_k$ which is given by $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$, then F is a bijection.

证明.

(1)Injective: $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$ i.e. $a \equiv b \pmod{m}, a \equiv b \pmod{n}$. $\exists i, j \in \mathbb{Z}$ s.t. $b = a + im = a + jk \Rightarrow k|im$. Since $\gcd(m, k) = 1, k|i \Rightarrow n = mk|im$. Then $[b]_n = [a]_n + [im]_n = [a]_n$.

(2)Surjective: prove $\forall u, v \in \mathbb{Z}, \exists a \in \mathbb{Z}$ s.t. $[a]_m = [u]_m, [a]_k = [v]_k$.

Since $\gcd(m, k) = 1, \exists s, t \in \mathbb{Z}$ so that $1 = sm + tk$.

Let $a = (1 - tk)u + (1 - sm)v, [a]_m = [(u - v)sm + v]_m = [v]_m, [a]_k = [(v - u)tk + u]_k = [u]_k$. □

Note 2. $F([a]_n[b]_n) = F([ab]_n) = ([ab]_m, [ab]_k) = ([a]_m[b]_m, [a]_k[b]_k)$

Since F is a bijection, $[ab]_n = [1]_n$ iff $([a]_m[b]_m, [a]_k[b]_k) = ([1]_m, [1]_k)$.

2.6.1 Proposition 1.5.9+Corollary 1.5.10: $m, n, k > 0, n = mk, \gcd(m, k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$

Proposition 11 (Proposition 1.5.9+Corollary 1.5.10). If $m, n, k > 0, n = mk, \gcd(m, k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$.

2.7 prime factorization: $n = p_1^{r_1} \dots p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$

Proposition 12. *If $n \in \mathbb{Z}$ is positive integer with prime factorization $n = p_1^{r_1} \dots p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$*

证明.

$\mathbb{Z}_{p^r} = \{[0], [1], \dots, [p^r - 1]\}$, the number of multiples of p is $\frac{p^r}{p} = p^{r-1}$. Then $\varphi(p^r) = |\mathbb{Z}_{p^r}^\times| = p^r - p^{r-1} = (p - 1)p^{r-1}$. So,

$$\varphi(n) = \varphi(p_1^{r_1}) \dots \varphi(p_k^{r_k}) = (p_1 - 1)p_1^{r_1-1} \dots (p_k - 1)p_k^{r_k-1}$$

□

3 Complex numbers

$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$, $\mathbb{R} = \{a + 0i | a \in \mathbb{R}\} \subset \mathbb{C}$

Addition & multiplication

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi)(c + di) = ac + bci + adi + bdi^2$$

$$= (ac - bd) + (bc + ad)i$$

Complex conjugation: $z = a + bi$, $\bar{z} = a - bi$, $\overline{z\bar{w}} = \bar{z}\bar{w}$

Absolute value: $|z| = \sqrt{a^2 + b^2}$, $|z|^2 = z\bar{z}$

Additive inverse: $-z = -a - bi$

Multiplicative inverse: $z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

$$z \in \mathbb{C}, \overline{z + \bar{z}} = \bar{z} + \bar{\bar{z}} = z + \bar{z}$$

$$\text{Real part: } \operatorname{Re}(z) = \frac{z + \bar{z}}{2}$$

$$\text{Imaginary part: } \operatorname{Im}(z) = \frac{z - \bar{z}}{2i}$$

3.1 Geometric Meaning of Addition and Multiplication

Addition: parallelogram law

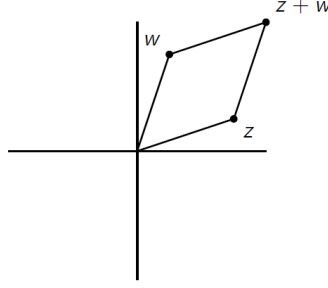
Multiplication:

$$z = a + bi \neq 0$$

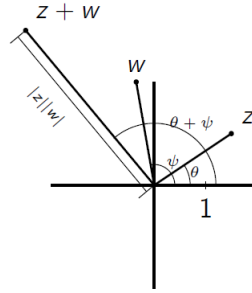
$$= r \cos \theta + r \sin \theta i$$

$$= r(\cos \theta + i \sin \theta)$$

$$|z|^2 = a^2 + b^2 = r^2$$



$$\begin{aligned}
 z &= r(\cos \theta + i \sin \theta) \\
 w &= s(\cos \phi + i \sin \phi) \\
 zw &= rs[\cos \theta \cos \phi - \sin \theta \sin \phi + i(\cos \theta \sin \phi + \cos \phi \sin \theta)] \\
 &= rs[\cos(\theta + \phi) + i \sin(\theta + \phi)] \\
 &= |z||w|[\cos(\theta + \phi) + i \sin(\theta + \phi)]
 \end{aligned}$$



We will write,

$$\begin{aligned}
 \cos \theta + i \sin \theta &= e^{i\theta} \\
 e^{i\theta} e^{i\phi} &= e^{i(\theta+\phi)} \\
 z &= |z|e^{i\theta}
 \end{aligned}$$

3.2 Theorem 2.1.1: $f(x) = a_0 + a_1x + \dots + a_nx^n$ **with coefficients** $a_0, a_1, \dots, a_n \in \mathbb{C}$.

Then f has a root in \mathbb{C} : $\exists \alpha \in \mathbb{C}$ **s.t.** $f(\alpha) = 0$

Theorem 7 (Theorem 2.1.1). *Suppose a nonconstant polynomial $f(x) = a_0 + a_1x + \dots + a_nx^n$ with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$. Then f has a root in \mathbb{C} : $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$.*

3.2.1 Corollary 2.1.2: $f(x) = a_n \prod_{i=1}^n (x - k_i) = a_n(x - k_1)(x - k_2)\dots(x - k_n)$, **where** k_1, k_2, \dots, k_n **are roots of** $f(x)$

Corollary 3 (Corollary 2.1.2). *Every nonconstant polynomial with coefficients $a_0, a_1, \dots, a_n \in \mathbb{C}$ can be factored as $f(x) = a_n \prod_{i=1}^n (x - k_i) = a_n(x - k_1)(x - k_2)\dots(x - k_n)$, where k_1, k_2, \dots, k_n are roots of $f(x)$.*

3.2.2 Corollary 2.1.3: $a_i \in \mathbb{R}$, f can be expressed as a product of linear and quadratic polynomials

Corollary 4 (Corollary 2.1.3). *If $f(x) = a_0 + a_1x + \dots + a_nx^n$ is a nonconstant polynomial $a_0, a_1, \dots, a_n \in \mathbb{R}, a_n \neq 0$. Then f can be expressed as a product of linear and quadratic polynomials.*

这里 a_0, a_1, \dots, a_n 是实数!

证明.

(1) Obviously, the corollary holds at $n = 1$ and $n = 2$.

(2) Suppose the corollary holds for all situations that $n < k$.

When $n = k$, $f(x) = a_0 + a_1x + \dots + a_kx^k, a_k \neq 0$.

By F.T.A., f has a root α in \mathbb{C} .

If $\alpha \in \mathbb{R}$, long division $f(x) = q(x)(x - \alpha)$. q has real coefficients, *degree of $q = k - 1$* . Since the corollary holds at $n = k - 1$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

If $\alpha \notin \mathbb{R}$

$$\begin{aligned} 0 &= f(\alpha) = a_0 + a_1\alpha + \dots + a_k\alpha^k \\ 0 &= \overline{f(\alpha)} = a_0 + a_1\bar{\alpha} + \dots + a_n\bar{\alpha}^n = f(\bar{\alpha}) \end{aligned}$$

Since $\bar{\alpha} \neq \alpha$, $(x - \alpha)(x - \bar{\alpha}) | f$.

$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2$ is a polynomial with coefficients in \mathbb{R} . So $f(x) = q(x)(x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2)$, q has real coefficients with degree $k - 2$. The corollary also holds at $n = k - 2$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

□

4 Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))

Definition: A field is a nonempty set \mathbb{F} with two operations:

1. addition, written $a + b, \forall a, b \in \mathbb{F}$;
2. multiplication, written $a \cdot b = ab, \forall a, b \in \mathbb{F}$.

such that:

- (i) *addition* and *multiplication* are associative and commutative
- (ii) *multiplication* distributes over *addition*: $a(b + c) = ab + ac, \forall a, b, c \in \mathbb{F}$
- (iii) \exists an additive identity $0 \in \mathbb{F}$ s.t. $0 + a = a, \forall a \in \mathbb{F}$.
- (iv) $\forall a \in \mathbb{F}$, \exists an additive inverse $-a$ s.t. $a + (-a) = 0, \forall a \in \mathbb{F}$.
- (v) \exists a multiplicative identity: $1 \in \mathbb{F}$ s.t. $1a = a, \forall a \in \mathbb{F}, 1 \neq 0$.
- (vi) $\forall a \in \mathbb{F}, a \neq 0$, a has a multiplicative inverse $a^{-1} = \frac{1}{a} \in \mathbb{F} : a \cdot \frac{1}{a} = 1$.

Proposition 13 (Proposition 2.2.2). \mathbb{F} a field, $a, b \in \mathbb{F}$, then

- (i) If $a + b = b$ then $a = 0$
- (ii) If $ab = b$ and $b \neq 0$, then $a = 1$
- (iii) $0a = 0$
- (iv) If $a + b = 0$, then $b = -a$
- (v) If $a \neq 0$ and $ab = 1$, then $b = a^{-1}$

Example 5. \mathbb{Z}_4 is not a field. Because $[2]_4$ doesn't have multiplicative inverse in \mathbb{Z}_4 .

4.1 Subfield $(\mathbb{K}, +, \cdot)$: $\mathbb{K} \subseteq \mathbb{F}$, closed under $+$, \cdot and inverse

Definition: Suppose \mathbb{F} is a field and $\mathbb{K} \subseteq \mathbb{F}$ s.t.

$$\begin{aligned} 0, 1 &\in \mathbb{K} \\ \forall a, b \in \mathbb{K}, a + b, ab, -a, a^{-1} (\text{if } a \neq 0) &\in \mathbb{K} \end{aligned}$$

We call \mathbb{K} a subfield of \mathbb{F} .

Example 6. $\mathbb{Q} \subseteq \mathbb{R}, \mathbb{R} \subseteq \mathbb{C}, \mathbb{Q} \subseteq \mathbb{C}$

Example 7. $\mathbb{K} \subseteq \mathbb{Z}_p$ a subfield $\Rightarrow \mathbb{K} = \mathbb{Z}_p$. Prove by induction.

4.1.1 Proposition 2.2.3: Subfield 继承 operations 自成一 field

Proposition 14 (Proposition 2.2.3). Suppose $\mathbb{K} \subset \mathbb{F}$ is a subfield of a field \mathbb{F} . Then the operations of \mathbb{F} make \mathbb{K} into a field.

\Rightarrow We can prove a set is a field by proving it is a subfield of a known field.

5 Polynomials

Let \mathbb{F} be any field. A polynomial over \mathbb{F} in variable x is a formal sum:

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = \sum_{i=0}^n a_ix^i$$

where $n \geq 0$ is an integer, $a_0, a_1, \dots, a_n \in \mathbb{F}$.

Polynomial is a sequence $\{a_k\}_{k=0}^{\infty}$ with $a_m = 0, \forall m > n$.

5.1 $\mathbb{F}[x]$: Polynomial ring 在一个 field 上形成的所有多项式 (方程) 的集合

Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients in the field \mathbb{F} .

$$\mathbb{F}[x] = \left\{ \sum_{i=0}^n a_ix^i \mid n \geq 0, n \in \mathbb{Z}, a_0, \dots, a_n \in \mathbb{F} \right\}$$

We call the $\mathbb{F}[x]$ *polynomial ring* over the field \mathbb{F} .

$$f = \sum_{i=0}^n a_i x^i, g = \sum_{j=0}^n a_j x^j \in \mathbb{F}[x]$$

$$f + g = \sum_{i=0}^n (a_i + b_i) x^i \in \mathbb{F}[x]$$

$$fg = \left(\sum_{i=0}^n a_i x^i \right) \left(\sum_{j=0}^n a_j x^j \right) = \sum_{i=0}^{2n} \left(\sum_{j=0}^i a_j b_{i-j} \right) x^i$$

5.1.1 Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

Proposition 15 (Proposition 2.3.2). *Suppose \mathbb{F} is any field. Then,*

- (i) Addition and multiplication are commutative & associative operations on $\mathbb{F}[x]$
- (ii) Multiplication distributes over addition
- (iii) $0 \in \mathbb{F}$, is additive identity in $\mathbb{F}[x] : \forall f \in \mathbb{F}[x], f + 0 = 0$
- (iv) $\forall f \in \mathbb{F}[x], f = (-1)f$ is the additive inverse: $f + (-1)f = 0$.
- (v) $1 \in \mathbb{F}$, is the multiplicative identity in $\mathbb{F}[x] : 1f = f, \forall f \in \mathbb{F}[x]$

5.2 Degree of a Polynomial: $\deg(f)$

$f = \sum_{i=0}^n a_i x^i$, $\deg(f)$ = degree of f is,

$$\deg(f) = \begin{cases} 0 & \text{if } f \text{ is constant, } f \neq 0 \\ n & \text{if } a_n \neq 0 \text{ in above } (a_n = \text{leading coefficient}) \\ -\infty & \text{if } f = 0 \end{cases}$$

Define $-\infty + a = a + (-\infty) = -\infty \forall a \in \mathbb{Z} \cup \{-\infty\}$

5.2.1 Lemma 2.3.3: $\deg(fg) = \deg(f) + \deg(g)$, $\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$

Lemma 2 (Lemma 2.3.3). *For any field \mathbb{F} and $f, g \in \mathbb{F}[x]$,*

$$\deg(fg) = \deg(f) + \deg(g)$$

$$\deg(f + g) \leq \max\{\deg(f), \deg(g)\}$$

5.3 Corollary 2.3.5: Unit(invertible) in $\mathbb{F}[x]$: constant $\neq 0$ iff $\deg(f) = 0$

Corollary 5 (Corollary 2.3.5). *For any field \mathbb{F} and $f \in \mathbb{F}[x]$, Then f is a unit(i.e. invertible) in $\mathbb{F}[x]$ iff $\deg(f) = 0$.*

证明.

Obviously, $\deg(f) = 0 \Rightarrow f$ is a unit.

Suppose f is a unit, i.e. $\exists g \in \mathbb{F}[x]$ s.t. $fg = 1$.

$$0 = \deg(fg) = \deg(f) + \deg(g) \Rightarrow \deg(f), \deg(g) \geq 0 \Rightarrow \deg(f) = 0, \deg(g) = 0. \quad \square$$

5.4 Irreducible Polynomials: “无法分解为两个 $\text{degree} \geq 1$ 的多项式积”的多项式: 至少一个是 constant (i.e. $\text{degree} = 0$)

A nonconstant polynomial f is irreducible if $f = uv$, $u, v \in \mathbb{F}[x]$, then either u or v is a unit (i.e., constant $\neq 0$)

5.5 Theorem 2.3.6: nonconstant polynomials 可以被唯一地分解

Theorem 8 (Theorem 2.3.6). Suppose \mathbb{F} is a field and $f \in \mathbb{F}[x]$ is any nonconstant. Then $f = ap_1p_2 \dots p_k$ where $a \in \mathbb{F}$, $p_1, \dots, p_k \in \mathbb{F}[x]$ are irreducible monic polynomials (monic = i.e. leading coeff. 1). If $f = bq_1q_2 \dots q_r$ with $b \in \mathbb{F}$ and $q_1, q_2, \dots, q_r \in \mathbb{F}[x]$ monic irreducible, then $a = b, k = r$, and after reindexing $p_i = q_i, \forall i$

Lemma 3 (Lemma 2.3.7). Suppose \mathbb{F} is a field and $f \in \mathbb{F}[x]$ is nonconstant monic polynomial. Then $f = p_1p_2 \dots p_k$ where each p_i is monic irreducible.

证明.

Prove it by induction. When $\text{deg}(f) = 1$, $f = uv$, $u, v \in \mathbb{F}[x]$, $\text{deg}(f) = \text{deg}(u) + \text{deg}(v) \Rightarrow$ one of these is 0.

Suppose the lemma holds for all $\text{degree} < n$. When $\text{deg}(f) = n$,

Either f is irreducible, done.

Suppose $f = uv$ with $\text{deg}(u), \text{deg}(v) \geq 1$

$\Rightarrow \text{deg}(u), \text{deg}(v) < n \Rightarrow u = p_1p_2 \dots p_k, v = q_1q_2 \dots q_j$ So, $f = p_1p_2 \dots p_kq_1q_2 \dots q_j$. □

Example 8. $x^2 - 1 \in \mathbb{Q}[x]$ reducible

$x - 1, x + 1 \in \mathbb{Q}[x]$ irreducible

$x^2 + 1 \in \mathbb{Q}[x]$ irreducible

$x^2 + 1 \in \mathbb{C}[x]$ reducible

$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$ reducible

5.6 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$, f divides g , $f|g$ means $\exists u \in \mathbb{F}[x]$ s.t. $g = fu$.

Proposition 16 (Proposition 2.3.8). $f, h, g \in \mathbb{F}[x]$, then

- (i) If $f \neq 0, f|0$
- (ii) If $f|1$, f is nonzero constant
- (iii) If $f|g$ and $g|f$, then $f = cg$ for some $c \in \mathbb{F}$
- (iv) If $f|g$ and $g|h$, then $f|h$
- (v) If $f|g$ and $f|h$, then $f|(ug + vh)$ for all $u, v \in \mathbb{F}[x]$.

5.6.1 Greatest common divisor of f and g : is not unique, we denote monic Greatest common divisor as $\gcd(f, g)$

If $f, g \in \mathbb{F}[x]$ are nonzero polynomials, a greatest common divisor of f and g is a polynomial $h \in \mathbb{F}[x]$ such that

- (i) $h|f$ and $h|g$, and
- (ii) if $k \in \mathbb{F}[x]$ and $k|f$ and $k|g$, then $k|h$.

the \gcd is not unique, but the monic \gcd is unique. We call it **the monic greatest common divisor**, denote it $\gcd(f, g)$.

Example 9.

$$\begin{aligned} x^2 - 1, x^2 - 2x + 1 &\in \mathbb{Q}[x] \\ (x - 1)(x + 1), (x - 1)^2 &\in \mathbb{Q}[x] \\ x - 1 &= \gcd(x^2 - 1, x^2 - 2x + 1) \end{aligned}$$

5.6.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

Proposition 17 (Proposition 2.3.9). *Given $f, g \in \mathbb{F}[x]$, $g \neq 0$, then $\exists q, r \in \mathbb{F}[x]$ s.t. $\deg(r) < \deg(g)$ and $f = qg + r$*

Example 10.

$$\begin{aligned} f &= 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x] \\ f &= 3g + x^2 - 3x + 2 \end{aligned}$$

5.6.3 Proposition 2.3.10: $\gcd(f, g)$ 是 degree 最小的 f, g 的线性组合

Proposition 18 (Proposition 2.3.10). *Any 2 nonzero polynomials $f, g \in \mathbb{F}[x]$ have a \gcd in $\mathbb{F}[x]$. In fact among all polynomials in the set $M = \{uf + vg | u, v \in \mathbb{F}[x]\}$ any nonconstant of minimal degree are \gcd s.*

证明.

$h \in M$, $\deg(h) = d$ minimal. Let $k|f$ and $k|g \Rightarrow k|uf + vg$, $\forall u, v \Rightarrow k|h$.

Suppose $h' \in M$ is any nonzero element. $\deg(h') \geq \deg(h) \Rightarrow \exists q, r \in \mathbb{F}[x], \deg(r) < \deg(h)$ $h' = qh + r$. $r = h' - qh \in M$. Since $\deg(h) = d$ is nonconstant minimal degree, $r = 0 \Rightarrow h' = qh$. So $\exists q_1, q_2 \in \mathbb{F}[x]$, $1f + 0g = q_1h, 0f + 1g = q_2h \Rightarrow h|g, h|f$. \square

Example 11.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$

$$f = 3g + x^2 - 3x + 2$$

$$g = (x + 1)(x^2 - 3x + 2) + x - 1$$

$$x^2 - 3x + 2 = (x - 2)(x - 1)$$

$$\Rightarrow \gcd(f, g) = x - 1$$

$$x - 1 = g - (x + 1)(x^2 - 3x + 2) = g - (x + 1)(f - 3g) = (3x + 4)g - (x + 1)f$$

Example 12. Find a greatest common divisor of $f = x^3 - x^2 - x + 1$ and $g = x^2 - 3x + 2$ in $\mathbb{Q}[x]$, and express it in form $uf + vg$, $u, v \in \mathbb{Q}[x]$.

$$f = (x + 2)g + 3x - 3$$

$$g = \frac{1}{3}(x - 2)(3x - 3)$$

$$\gcd(f, g) = 3x - 3$$

$$3x - 3 = f - (x + 2)g$$

5.6.4 Proposition 2.3.12: $\gcd(f, g) = 1, f|gh \Rightarrow f|h$

Proposition 19 (Proposition 2.3.12). If $f, g, h \in \mathbb{F}[x]$, $\gcd(f, g) = 1$, and $f|gh$, then $f|h$.

5.6.5 Corollary 2.3.13: irreducible f , $f|gh \Rightarrow f|g$ or $f|h$

Corollary 6 (Corollary 2.3.13). If $f \in \mathbb{F}[x]$ is irreducible, and $f|gh$, then $f|g$ or $f|h$.

Since f is irreducible, we have two possible situations:

1. $\gcd(f, g) = f$, i.e. $f|g$ done.
2. $\gcd(f, g) = 1$, then according to Prop 2.3.12, we can know $f|h$.

5.7 Roots

Root: $\alpha \in \mathbb{F}$ is a root of f if $f(\alpha) = 0$.

5.7.1 Corollary 2.3.16(of Euclidean Algorithm): f 可被分为 $(x - \alpha)q + f(\alpha)$ i.e. if α is a root, then $(x - \alpha)|f$

Corollary 7 (Corollary 2.3.16(of Euclidean Algorithm)). $\forall f \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$, there exists a polynomial $q \in \mathbb{F}[x]$ s.t. $f = (x - \alpha)q + f(\alpha)$. In particular, if α is a root, then $(x - \alpha)|f$.

5.8 Multiplicity

If α is a root of f , say its *multiplicity* is m , if $x - \alpha$ appears m times in irreducible factorization.

5.8.1 Sum of multiplicity $\leq \deg(f)$

Proposition 20 (Proposition 2.3.17). *Given a nonconstant polynomial $f \in \mathbb{F}[x]$, the number of roots of f , counted with multiplicity, is at most $\deg(f)$.*

5.9 Roots in a field may not in its subfield

Note if $\mathbb{F} \subset \mathbb{K}$, then $\mathbb{F}[x] \subset \mathbb{K}$. $f \in \mathbb{F}[x]$ may have no roots in \mathbb{F} , but could have roots in \mathbb{K}

Example 13. $x^n - 1 \in \mathbb{Q}[x]$ has a root in \mathbb{Q} : 1; has 2 roots if n even: ± 1

roots in \mathbb{C} : $\zeta_n = e^{\frac{2\pi i}{n}}$, then $\zeta_n^n = e^{2\pi i} = 1$; $(\zeta_n^k)^n = e^{2\pi k i} = 1$ So, the roots: $\{e^{\frac{2\pi k i}{n}} | k = 0, \dots, n-1\}$

The roots of $x^n - d$: $\{e^{\frac{2\pi k i}{n}} \sqrt[n]{d} | k = 0, \dots, n-1\}$

6 Linear Algebra

6.1 Vector Space $(V, +, \times)$ (over a field \mathbb{F})

A vector space over a field \mathbb{F} is a set V w/ an operation addition $+: V \times V \rightarrow V$ and an operation scalar multiplication $\mathbb{F} \times V \rightarrow V$

- (1) Addition is associative & commutative
- (2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$
- (3) $1v = v \forall v \in V$ (where $1 \in \mathbb{F}$ is multi. id. in \mathbb{F})
- (4) $\forall \alpha, \beta \in \mathbb{F}, v \in V, \alpha(\beta v) = (\alpha\beta)v$
- (5) $\forall v \in V, (-1)v = -v$ we have $v + (-v) = 0$
- (6) $\forall \alpha \in \mathbb{F}, v, u \in V, \alpha(v + u) = \alpha v + \alpha u$
- (7) $\forall \alpha, \beta \in \mathbb{F}, v \in V, (\alpha + \beta)v = \alpha v + \beta v$

6.1.1 A field is a vector space over its subfield

Example 14. $\mathbb{K} \subset \mathbb{F}$ is a subfield of a field \mathbb{F} . Then \mathbb{F} is a vector space over \mathbb{K} . (Since $\mathbb{F} \subset \mathbb{F}[x]$, then $\mathbb{F}[x]$ is a vector space over \mathbb{F} .)

6.1.2 Vector subspace

Suppose that V is a vector space over \mathbb{F} . A vector subspace or just subspace is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W, av \in W, \forall v, w \in W, a \in \mathbb{F}$.

Example 15. $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$, then \mathbb{L} is a subspace of \mathbb{F} over \mathbb{K} .

6.2 Linear independent, Linear combination

6.3 span V, basis, dimension, Proposition 2.4.10

A set of elements $v_1, \dots, v_n \in V$ is said to **span** V if every vector $v \in V$ can be expressed as a linear combination of v_1, \dots, v_n . If v_1, \dots, v_n spans and is linearly independent, then we call the set a **basis** for V .

Proposition 21 (Proposition 2.4.10.). *Suppose V is a vector space over a field \mathbb{F} having a basis $\{v_1, \dots, v_n\}$ with $n \geq 1$.*

(i) For all $v \in V$, $v = a_1v_1 + \dots + a_nv_n$ for exactly one $(a_1, \dots, a_n) \in \mathbb{F}^n$.

(ii) If w_1, \dots, w_n span V , then they are linearly independent.

(iii) If w_1, \dots, w_n are linearly independent, then they span V .

If a vector space V over \mathbb{F} has a basis with n vectors, then V is said to be n -dimensional (over \mathbb{F}) or is said to have **dimension** n .

6.3.1 Standard basis vectors

$$e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, 0, \dots, 0, 1) \in \mathbb{F}^n$$

are a basis for \mathbb{F}^n called the **standard basis vectors**.

6.4 Linear transformation

Given two vector spaces V and W over \mathbb{F} a **linear transformation** is a function $T : V \rightarrow W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$, we have

$$T(av) = aT(v) \text{ and } T(v + w) = T(v) + T(w)$$

Proposition 22 (Proposition 2.4.15.). *If V and W are vector spaces and v_1, \dots, v_n is a basis for V then any function from $\{v_1, \dots, v_n\} \rightarrow W$ extends uniquely to a linear transformation $V \rightarrow W$.*

Any $v \in V$, $\exists (a_1, \dots, a_n)$ s.t. $v = a_1v_1 + \dots + a_nv_n$. Then $T(v) = T(a_1v_1 + \dots + a_nv_n) = a_1T(v_1) + \dots + a_nT(v_n)$

6.4.1 Corollary 2.4.16: 一个线性变换对应一个矩阵 **bijection** $\mathcal{L}(V, M) \rightarrow M_{m \times n}(\mathbb{F})$

Corollary 8 (Corollary 2.4.16.). *If v_1, \dots, v_n is a basis for a vector space V and w_1, \dots, w_m is a basis for a vector space W (both over \mathbb{F}), then any linear transformation $T : V \rightarrow W$ determines (and is*

determined by) the $m \times n$ matrix:

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & \dots & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \dots & w_m \end{bmatrix}^T = A \begin{bmatrix} v_1 & \dots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from V to W ; $M_{m \times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in \mathbb{F} . $T \rightarrow A(T)$ defines a *bijection* $\mathcal{L}(V, M) \rightarrow M_{m \times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** T .

6.4.2 Proposition 2.4.19: 线性变换矩阵相乘仍为线性变换矩阵

Proposition 23 (Proposition 2.4.19). *Suppose that V , W , and U are vector spaces over \mathbb{F} , with fixed chosen bases. If $T : V \rightarrow W$ and $S : W \rightarrow U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \rightarrow U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

6.5 $GL(V)$: invertible(bijective) linear transformations $V \rightarrow V$

Given a vector space V over F , we let $GL(V) \subset \mathcal{L}(V, V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \text{ is a bijection}\} = \mathcal{L}(V, V) \cap \text{Sym}(V)$$

7 Euclidean geometry basics

7.1 Euclidean distance, inner product

Euclidean distance on \mathbb{R}^n :

$$|x - y| = \sqrt{(x_1 - y_1)^2 + \dots + (x_n - y_n)^2}$$

Euclidean inner product:

$$x \cdot y = x_1 y_1 + \dots + x_n y_n = x^T y$$

7.2 Isometry of \mathbb{R}^n : a bijection $\mathbb{R}^n \rightarrow \mathbb{R}^n$ preserves distance

An **isometry** of \mathbb{R}^n is a bijection $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x - y|, \quad \forall x, y \in \mathbb{R}^n$$

7.2.1 $Isom(\mathbb{R}^n)$: set of all isometries of \mathbb{R}^n

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of \mathbb{R}^n ,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n | |\Phi(x) - \Phi(y)| = |x - y|, \forall x, y \in \mathbb{R}^n\}$$

7.2.2 $Isom(\mathbb{R}^n)$ is closed under \circ and inverse

Proposition 24. $\Phi, \Psi \in Isom(\mathbb{R}^n)$, then $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$

证明.

Since Φ, Ψ are bijections, so is $\Phi \circ \Psi$. Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x - y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x - y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

□

7.3 $A \in GL(n, \mathbb{R})$, $T_A(v) = Av$: $A^t A = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a invertible linear transformations $T_A : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t (Aw) = v^t A^t Aw$$

$$A^t A = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)} T_A \in Isom(\mathbb{R}^n)$$

7.4 Linear isometries i.e. orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\}$

We define the all isometries in invertible linear transformations $\mathbb{R}^n \rightarrow \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^t A = I\} \subset GL(n, \mathbb{R})$$

7.4.1 Special orthogonal group $SO(n) = \{A \in O(n) | \det(A) = 1\}$: orthogonal group with $\det(A) = 1$

$O(n)$ are the matrices representing linear isometries of \mathbb{R}^n . $1 = \det(I) = \det(A^t A) = \det(A^t) \det(A) = \det(A)^2 \Rightarrow \det(A) = 1$ or $\det(A) = -1$. We use **special orthogonal group** represents A with $\det(A) = 1$,

$$SO(n) = \{A \in O(n) | \det(A) = 1\}$$

7.5 translation: $\tau_v(x) = x + v$

Define a translation by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \rightarrow \mathbb{R}^n, \tau_v(x) = x + v$$

7.5.1 translation is an isometry

Note 3 (Exercise 2.5.3). $\forall v \in \mathbb{R}^n, \tau_v$ is an isometry.

证明. $|\tau_v(x) - \tau_v(y)| = |(x + v) - (y + v)| = |x - y|$ □

7.6 The composition of a translation and an orthogonal transformation is an isometry $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since the composition of isometries is an isometry, $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. which could account for all isometries.

7.6.1 Theorem 2.5.3: All isometries can be represented by a composition of a translation and an orthogonal transformation, $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

Theorem 9 (Theorem 2.5.3). $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

8 Group

8.1 Group $(G, *)$: a set with a binary operation (associative, identity, inverse)

8.1.1 Definition

A group is a nonempty set G with a binary operation $*$: $G \times G \rightarrow G$ s.t.

- (1) Binary operation on G , $*$: $G \times G \rightarrow G$
- (2) $*$ is **associative**
- (3) G contains an **identity** element e for $*$: $\exists e \in G$ s.t. $e * g = g * e = g \forall g \in G$
- (4) Each element $a \in G$ has an **inverse** $b \in G$ s.t. $a * b = b * a = e$.

A Group is **abelian** if moreover

- (5) $*$ is **commutative**.

$|G|$ = Order of a group $(G, *)$

$(\mathbb{Z}, +)$ is a group and $+$ is commutative, we call this kind of groups (satisfy commutative) *abelian group*.

Example 16. If \mathbb{F} is a field, then $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \cdot)$ are abelian group.

Example 17. If V is a vector space over \mathbb{F} , then $(V, +)$ abelian group.

As we know a V is a vector space over \mathbb{F} means V is a field whose subfields include \mathbb{F} .

8.1.2 Uniqueness of identity and inverse

Lemma 4. 1. Identity of a group is unique. 2. Inverse of any element in a group is also unique.

证明.

1. Let e, e' be two identities in G , then $e * e' = e = e'$.
2. Suppose b, c are both inverse of a , then

$$b = b * e = b * (a * c) = (b * a) * c = e * c = c$$

□

8.1.3 $(Sym(X), \circ)$ symmetric/permutation group of X

Example 18. If X is any nonempty set, permutation group of $X : \{\sigma : X \rightarrow X | \sigma \text{ is a bijection}\}$, then

1. \circ is associative;
2. $id : X \rightarrow X$, $id(x) = x \forall x \in X$ is the identity;
3. $\sigma \in Sym(X)$, $\sigma^{-1} \in Sym(X)$ is the inverse function.

$(Sym(X), \circ)$ is a group called the symmetric group of X

8.1.4 Cancellation Laws

Theorem 10. Let G be a group. The left and right cancelation laws hold in G :

$$1. a * x = a * y \Rightarrow x = y$$

$$2. x * a = y * a \Rightarrow x = y$$

证明.

Let $a * x = a * y$. $\exists a'$ s.t. $a' * a = e$. $a' * (a * x) = a' * (a * y) \Rightarrow (a' * a) * x = (a' * a) * y \Rightarrow e * x = e * y \Rightarrow x = y$

Similar for the right cancel law. □

8.1.5 Unique Solution of Linear Equation

Theorem 11. The linear equation $a * x = b$ and $y * a = b$ has unique solution.

证明.

1. Existence: Multiply by a' : $a' * (a * x) = a' * b \Rightarrow x = a' * b$ is a solution.
2. Uniqueness: if x' is another, $a * x = a * x' = b \Rightarrow x = x'$

□

8.2 Subgroup: $H \leq G$

Definition 1. A subset $H \subseteq G$ is a subgroup of G if H is itself a group.

write $H \leq G$, $H < G$ if H is a subgroup of $(G, *)$. (If $H = G$, H is an improper subgroup. If $H \subsetneq G$, H is a proper subgroup.)

If $H = \{e\}$, then H is a trivial subgroup.

If $H \neq \{e\}$, then H is a nontrivial subgroup.

Theorem 12. A subset $H \subseteq G$ is a subgroup of G if and only if

1. H is closed under $*$. ($\forall g, h \in H, g * h \in H$)
2. identity $e \in H$.
3. Each $a \in H$, the inverse $a' \in H$

证明.

" \Rightarrow ": if $H \leq G$ be a subgroup.

1. H is a group $\Rightarrow *$ is a binary operation on H , $* : H \times H \rightarrow H$ i.e. H is closed under $*$.
2. Identity of H , e_H is also a identity of G , due to the uniqueness of identity, $e_H = e_G$.
3. $a \in H$, a 's inverse $a'_H \in H$ is also an inverse in G , due to the uniqueness of identity, $a'_H = a'_G$.

" \Leftarrow ":

1. H is closed under $*$ $\Rightarrow *$ is a binary operation on H .
2. 2,3 fulfill the requirement of identity and inverse.
3. $*$ is operation of group $G \Rightarrow *$ is associative.

Hence H is itself a group.

4. H is a subset of G , then H is a subgroup of G .

□

8.2.1 Proposition 2.6.8: $H < G$, $(H, *)$ is a group: A group's operation with its any subgroup is also a group

不同的 definition.

Proposition 25 (Proposition 2.6.8). If $(G, *)$ is a group, $H \subset G$ is a subgroup, then $(H, *)$ is a group.

Example 19. $(G, *)$ is a group, then $e < G$, $G < G$.

Example 20. $\mathbb{K} \subset \mathbb{F}$ is a subfield, then $\mathbb{K} < \mathbb{F}$, $\mathbb{K}^\times < \mathbb{F}^\times$.

Example 21. $W \subset V$ is a vector subspace, $W < V$.

Example 22. $1 \in S^1 \subset \mathbb{C}^\times$, $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$. S^1 is a subgroup.

证明.

$S^1 = \{e^{i\theta} \mid \theta \in \mathbb{R}\}$. For any $e^{i\theta}, e^{i\psi} \in S^1$, $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1$, $e^{-i\theta} \in S^1$. □

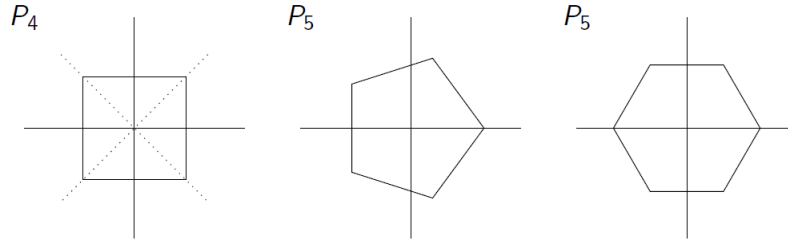
Example 23. $Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$

Example 24. If \mathbb{F} is a field, $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \rightarrow \mathbb{F} \in Sym(\mathbb{F}) \mid \sigma(a+b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$

Example 25. Dihedral Groups:

保留多边形

Let $P_n \subset \mathbb{R}^2$ be a regular n -gon



$D_n < Isom(\mathbb{R}^2)$, $D_n = \{\Phi \in Isom(\mathbb{R}^2) \mid \Phi(P_n) = P_n\}$

8.3 Some Properties of Group Operation

Proposition 26 (Proposition 3.1.1). Let $(G, *)$ be a group with identity $e \in G$, then

- (1) if $g, h \in G$ and either $g * h = h$ or $h * g = h$, then $g = e$
- (2) if $g, h \in G$ and $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

Corollary 9 (Corollary 3.1.2). $e^{-1} = e$, $(g^{-1})^{-1} = g$, $(g * h)^{-1} = h^{-1} * g^{-1}$

8.4 Power of an Element

We define g^n recursively for $n \geq 0$ by setting $g^0 = e$ and for $n \geq 1$, we set $g^n = g^{n-1} * g$. For $n \leq 0$, we define $g^n = (g^{-1})^{-n}$.

Proposition 27 (Proposition 3.1.5). (1) $g^n * g^m = g^{n+m}$; (2) $(g^n)^m = g^{nm}$

8.5 $(G \times H, \otimes)$: Direct Product of G and H

$(G, *)$ a group (H, \star) a group. Define an operation on $G \times H$, \otimes :

$$(h, k) \otimes (h', k') = (h * h', k * k')$$

8.5.1 Proposition 3.1.7: $(G \times H, \otimes)$ is a group

Proposition 28 (Proposition 3.1.7). $(G \times H, \otimes)$ is a group. The identity is (e_G, e_H) , inverse is (g^{-1}, h^{-1})

usually written as

$$(h, k)(h', k') = (hh', kk')$$

8.6 Subgroups and Cyclic Groups

8.6.1 Intersection of Subgroups is a Subgroup

Proposition 29 (Proposition 3.2.2). Let G be a group and suppose \mathcal{H} is any collection of subgroups of G . Then $K = \cap_{H \in \mathcal{H}} H < G$ is a subgroup of G .

8.6.2 Subgroup Generated by A : $\langle A \rangle$

We define **Subgroup Generated by A** :

$$\langle A \rangle = \cap_{H \in \mathcal{H}(A)} H$$

where $\mathcal{H}(A)$ is the set of all subgroups of G containing the set A :

$$\mathcal{H}(A) = \{H < G \mid A \subset H \text{ and } H \text{ is a subgroup of } G\}$$

8.6.3 Cyclic Group: group generated by an element

A group G is cyclic if exists g (an element), $\langle g \rangle = G$.

g is called a generator for G in this case.

Easy to prove

$$G = \langle g \rangle = \{\dots g^{-2}, g^{-1}, e, g^1, g^2 \dots\}$$

8.6.4 Cyclic Subgroup

If A is a subgroup of G , and $A = \langle \{a\} \rangle = \langle a \rangle$. Then A is the cyclic subgroup generated by a :
 $A = \langle a \rangle \leq G$

$$\langle a \rangle = \{\dots a^{-2}, a^{-1}, e, a^1, a^2 \dots\}$$

8.6.5 Subgroups of a Cyclic Group must be Cyclic

Theorem 13. *A subgroup of a cyclic group is cyclic.*

证明.

Let $G = \{a^n : n \in \mathbb{Z}\}$ be a cyclic group. Let $H \leq G$ be a subgroup.

1. If $H = \{e\}$, then H is cyclic.
2. If $H \neq \{e\}$, then $a^n \in H$ for some $n > 0$. Check m be the minimal among all n .

Claim: $H = \langle a^m \rangle$

Proof: Clearly $\langle a^m \rangle \subset H$. $\forall a^n \in H$, $n = qm + r$, $0 \leq r < m$. Then $a^r = a^n(a^m)^{-q}$. Since m is the minimal positive integer s.t. $a^m \in H$, $r = 0$. $\Rightarrow n = qm \Rightarrow a^n \in \langle a^m \rangle$. Hence $H = \langle a^m \rangle$ which is cyclic. □

Example 26 (Subgroups of $(\mathbb{Z}, +)$).

\mathbb{Z} is a cyclic group $\langle 1 \rangle$. Its subgroups are $\langle n \rangle \leq \mathbb{Z}$ for some $n \geq 0$. (which is a multiplier of n . $(n\mathbb{Z})$)
 $n = 0, H = \{0\}; n = 1, H = \mathbb{Z}; n = 2, H = 2\mathbb{Z}$

8.6.6 Theorem: $\langle a^v \rangle < \langle a^n \rangle \Rightarrow \langle a^v \rangle = \langle a^d \rangle, d = \gcd(v, n), |\langle a^v \rangle| = \frac{n}{d}$

Theorem 14. *Let G be a cyclic group of order n . ($G = \{1, a, a^2, \dots, a^{n-1}\}$, where $a^n = 1$). Let $H = \langle a^v \rangle$ be a subgroup of G . Then H is generated by a^d (i.e. $H = \langle a^d \rangle$), $d = \gcd(v, n)$ and $|H| = \frac{n}{d}$.*

证明.

Let $H' = \langle a^d \rangle$, we need to show that $H = H'$. $d = \gcd(v, n) = d|v \Rightarrow a^v \in \langle a^d \rangle \Rightarrow H \subset H'$.

While $d = sv + tn$ for some s, t . $\Rightarrow a^d = (a^v)^s (a^n)^t$. Since $a^n = 1$, $a^d = (a^v)^s \Rightarrow H' \subset H$.

Hence, $H = H' = \langle a^v \rangle$. $H = \{1, a^d, a^{2d}, \dots, a^{n-d}\}, |H| = \frac{n}{d}$ □

8.6.7 Corollary 3.2.4: G is a cyclic group $\Rightarrow G$ is abelian

Corollary 10 (Corollary 3.2.4). *If G is a cyclic group (i.e. exists $g \in G$ s.t. $\langle g \rangle = G$), then G is abelian (i.e. commutative).*

8.6.8 Equivalent properties of order of g : $|g| = |\langle g \rangle| < \infty$

Proposition 30 (Proposition 3.2.6). *Let G be a group for $g \in G$, the following are equivalent:*

- (i) $|g| < \infty$
- (ii) $\exists n \neq m$ in \mathbb{Z} so that $g^n = g^m$
- (iii) $\exists n \in \mathbb{Z}$, $n \neq 0$ so that $g^n = e$
- (iv) $\exists n \in \mathbb{Z}_+$ so that $g^n = e$

If $|g| < \infty$, then $|g| = \text{smallest } n \in \mathbb{Z}_+ \text{ so that } g^n = e$, and $\langle g \rangle = \{e, g, g^2, \dots, g^{n-1}\} = \{g^n \mid n = 0, \dots, n-1\}$

8.6.9 $(\mathbb{Z}, +)$ Theorem 3.2.9: $\langle a \rangle < \langle b \rangle$ if and only if $b|a$

Theorem 15 (Theorem 3.2.9). *If $H < \mathbb{Z}$ is a subgroup, then either $H = \{0\}$, or else $H = \langle d \rangle$, where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently, $a \rightarrow \langle a \rangle$ defines a **bijection** from $N = \{0, 1, 2, \dots\}$ to the set of subgroups of \mathbb{Z} . Furthermore, for $a, b \in \mathbb{Z}_+$, we have $\langle a \rangle < \langle b \rangle$ if and only if $b|a$.*

8.6.10 $(\mathbb{Z}_n, +)$ Theorem 3.2.10: $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$

Theorem 16 (Theorem 3.2.10). *For any $n \geq 2$, if $H < \mathbb{Z}_n$ is a subgroup, then there is a positive divisor d of n so that*

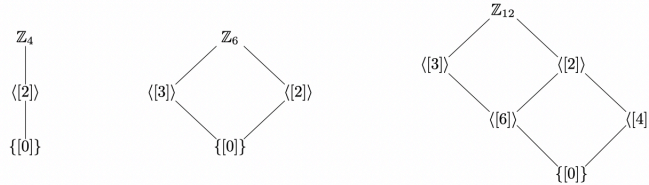
$$H = \langle [d] \rangle$$

Furthermore, this defines a bijection between divisors of H and subgroups of \mathbb{Z}_n . Furthermore, if $d, d' > 0$ are two divisors of n , then $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$.

If $H = \langle [d] \rangle$ is a subgroup of H , then $[n] \in H$, so $d|n$. And $|H| = |\langle [d] \rangle| = \frac{n}{d}$, so $|H||d$

8.6.11 Subgroup Lattice

The set of all subgroups of a group of G , together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup $\{e\}$ at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.



Writing down the subgroup lattice is as easy as writing down the divisibility lattice in which n is placed at the bottom, 1 at the top, and all intermediate divisors in between, connected by edges when there is divisibility. The congruence class of the divisor generates the corresponding subgroup in the subgroup lattice.

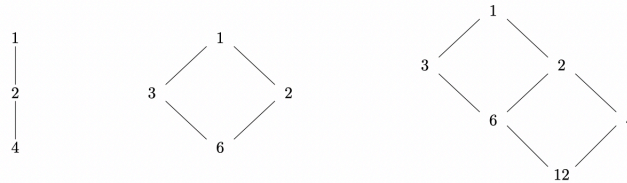


图 1:

8.6.12 Coset

Definition 2. If H is a subgroup of a group G and $a \in G$, then $aH = \{ah|h \in H\} \leq G$ is called left coset of H .

Theorem 17. Let $H \leq G$, $a, b \in G$,

1. $aH = bH$ if and only if $a^{-1}bH$
2. $aH \cap bH = \emptyset$ or $aH = bH$
3. $|aH| = |H| \forall a \in G$

证明.

1. Assume that $aH \cap bH \neq \emptyset$ and let $ah = bk \in aH \cap bH$ with $h, k \in H$.

$$ah = bk \Leftrightarrow h = a^{-1}bk \Leftrightarrow a^{-1}b = hk^{-1} \in H, \text{ thus } a^{-1}b \in H.$$

2. When $aH \cap bH \neq \emptyset \exists k_1, h \in H$ such that $ak_1 = bh \in bH$. Then $\forall k_2 \in H a = bhk_1^{-1} \Rightarrow ak_2 = bhk_1^{-1}k_2$ where $hk_1^{-1}k_2 \in H$ so $ak_2 \in bH, \forall k_2 \in H$.

3. $x \rightarrow ax$ is bijection $\Rightarrow |aH| = |H|$.

□

Claim 1. Coset can generate a partition of group:

$$G = a_1H \cup a_2H \cup \dots \cup a_rH$$

Theorem 18 (Lagrange Theorem). Let $H \leq G$ be a subgroup of finite group G . Then the order $|H|$ divides the order $|G|$.

证明.

Give a partition

$$\begin{aligned} &Ga_1H \cup a_2H \cup \dots \cup a_rH \\ |G| &= |a_1H| + |a_2H| + \dots + |a_rH| \\ &= r|H| \rightarrow |H| \mid |G| \end{aligned}$$

□

Theorem 19 (Order of element/cyclic subgroup). For $a \in G$, the order of a (the smallest m such that $a^m = e$) divides $|G|$. The order of a is the order of cyclic subgroup $\langle a \rangle$ with generator a .

证明.

For $a \in G$, $H = \{a^n, n \in \mathbb{Z}\} \leq G$. $|H|$ is the size of m . With lagrange theorm, $|H| = m \mid |G|$

□

8.7 Homomorphism, Isomorphism

8.7.1 Definition

Definition 3. If $(G, *)$ and (H, \circ) are groups, then a function $f : G \rightarrow H$ is a **homomorphism** if

$$f(x * y) = f(x) \circ f(y), \quad \forall x, y \in G$$

If f is also a bijection, then f is called an **isomorphism**. We say that G and H are **isomorphic** if exists an **isomorphism** f , denoted by $G \cong H$. (since f is bijection, $G \cong H \Leftrightarrow H \cong G$)

Isomorphic means these two groups are the same.

$$\begin{array}{ccc} G \times G & \xrightarrow{*} & G \xrightarrow{f} H \\ G \times G & \xrightarrow{(f,f)} & H \times H \xrightarrow{\circ} H \end{array}$$

Example 27. $(\mathbb{Z}_2, +)$, $(\{-1, 1\}, \times)$ and $\phi : 0 \rightarrow 1; 1 \rightarrow -1$.

$$\begin{aligned} \phi(0 + 0) &= 1 = \phi(0) \times \phi(0) \\ \phi(0 + 1) &= -1 = \phi(0) \times \phi(1) \\ \phi(1 + 1) &= 1 = \phi(1) \times \phi(1) \end{aligned}$$

8.7.2 Theorem: $\left\{ \begin{array}{l} \sigma : G \rightarrow G' \text{ injective} \\ \sigma(xy) = \sigma(x)\sigma(y) \quad \forall x, y \in G \end{array} \right\} \Rightarrow \sigma(G) \leq G', G \text{ is isomorphic to } \sigma(G)$

Theorem 20. Let $\sigma : G \rightarrow G'$ be an injective map s.t.

$$\sigma(xy) = \sigma(x)\sigma(y), \quad \forall x, y \in G$$

Then the image $\sigma(G) = \{\sigma(x) : x \in G\}$ is a subgroup of G' that is isomorphic to G .

证明.

1. Closed: $\forall a = \sigma(x), b = \sigma(y) \in \sigma(G)$, then $ab = \sigma(x)\sigma(y) = \sigma(xy) \in \sigma(G)$.
2. Identity: $\sigma(e) \in \sigma(G)$ is an identity for $\sigma(G)$: $\sigma(e)\sigma(x) = \sigma(ex) = \sigma(x) = \sigma(xe) = \sigma(x)\sigma(e)$
3. Inverse: $\sigma(x^{-1})$ is an inverse in $\sigma(G)$ for $\sigma(x)$: $\sigma(x^{-1})\sigma(x) = \sigma(e) = \sigma(x)\sigma(x^{-1})$

□

8.7.3 Cayley Theorem: G is isomorphic to a subgroup of S_G

Theorem 21 (Cayley Theorem). Let G be a group and S_G is the symmetric group of G (the group of all permutation of G : $S_G = \{\text{Bijection } \sigma : G \rightarrow G\}$) Then G is isomorphic to a subgroup of S_G .

证明.

Set a bijection $\phi : G \rightarrow S_G$ such that $\phi(g) = \lambda_g, \forall g \in G$, where λ_g is a permutation $\lambda_g : x \rightarrow gx$.

Claim: $\lambda_g \in S_G$ (i.e. λ_g is a permutation of G , a bijection $G \rightarrow G$).

1. $\lambda_g : G \rightarrow G$ is injective

$$\lambda_g(x) = \lambda_g(y)$$

$$\Leftrightarrow gx = gy$$

$$\Leftrightarrow x = y$$

2. $\lambda_g : G \rightarrow G$ is surjective. Let $y \in G$

$$\lambda_g(x) = y$$

$$\Leftrightarrow gx = y$$

$$\Leftrightarrow x = g^{-1}y$$

Claim: $\phi(x)\phi(y) = \phi(xy)$

$$\phi(x)\phi(y) = \lambda_x \circ \lambda_y$$

$$(\lambda_x \circ \lambda_y)(z) = \lambda_x(yz) = xyz = \lambda_{xy}(z), \forall z \in G$$

$$\Rightarrow \phi(x)\phi(y) = \phi(xy)$$

According to previous theorem, $\phi(G) \leq S_G$ and G is isomorphic to $\phi(G)$.

□

9 Ring $(R, +, \cdot)$: $+$ is associative, commutative, identity, inverse $\in R$; \cdot is associative, distributes over $+$

Definition 4. A ring is a nonempty set with two operations, called addition and multiplication, $(R, +, \cdot)$ such that

(1): $(R, +)$ is an abelian group: i.e. $+$ is associative and commutative. $0, -a \in R$

(2): \cdot is associative.

(3): \cdot distributes over $+$: $\forall a, b, c \in R, a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

9.1 Commutative ring: ring's \cdot is commutative

If " \cdot " is commutative, we call $(R, +, \cdot)$ a commutative ring.

9.2 Ring with 1: exists multiplication identity $1 \in R$

If there exists an element $1 \in R \setminus \{0\}$ such that $a1 = 1a = a, \forall a \in R$, then we say that R is a ring with 1.

9.3 Field \mathbb{F} is a commutative ring with 1; $\mathbb{F}[x]$ is also a commutative ring with 1

Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))

Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

9.4 $S \subset R$: Subring (closed under $+$ and \cdot ; additive inverse $-a \in S$)

9.4.1 Proposition 2.6.27: $(S, +, \cdot)$ is a ring

Proposition 31 (Proposition 2.6.27). *If $S \subset R$ is a subring, then $+$, \cdot make S into a ring.*

参考文献

- [1] Christopher J Leininger Introduction to Abstract Algebra (Draft) 2017.
- [2] Fraleigh, J. B. (2003). A first course in abstract algebra. Pearson Education India.