Math 417, Sections B13 and X13 Exam 2 (Solutions)
Prof. I.Kapovich April 11, 2016; 7pm-8pm

Problem 1.[10 points]

For each of the following statements indicate whether it is true or false. You DO NOT need to provide justification for your answers in this problem.

(1) If G is an abelian group and $H \le G$ is a subgroup, then $H \lhd G$ and the group G/H is abelian.
(2) If A, B are groups and $\phi : A \to B$ is a function such that $\{a \in A | \phi(a) = e_B\} = \{e_A\}$, then ϕ is one-to-one.
(3) We have $\mathbb{C}^\times/\mathbb{S}^1 \cong (\mathbb{R}_{>0}, \cdot)$.
(4) If G is a finite group and if $n \ge 1$ is an integer such that $n \big| |G|$ then there is an element $g \in G$ with $ord(g) = n$.
(5) If G is a group which is not finitely generated and if $H \lhd G$ is a normal subgroup of G, then the quotient group G/H is not finitely generated.

Answers.

(1) True.

(2) False. (The statement would become true if in addition we assume that the function ϕ is a homomorphism.)

(3) True. Apply the First Isomorphism Theorem to the map $\phi : \mathbb{C}^\times \to \mathbb{R}_{>0}$, $\phi(z) = |z|$.

(4) False. E.g. for $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and for $n = 4$ we have $n \big| |G|$ but G has no elements of order 4.

(5) False. E.g. for any group G (even a non-finitely generated group), taking $H = G$ gives $G/H = \{eH\}$, the trivial group, which is finitely generated.

Problem 2.[10 points]

For each of the following pairs of groups indicate whether or not the groups in this pair are isomorphic.

Give a careful and detailed justification of your answers.

(1) $SL(2, \mathbb{Z})$ and $\mathbb{Z} \times \mathbb{Z}$;
(2) $(\mathbb{R}, +)$ and $(\mathbb{R}^\times, \cdot)$;
(3) $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_{12}$;
(4) $\mathbb{Z}_4 \times \mathbb{Z}_3$ and $\mathbb{Z}_6 \times \mathbb{Z}_2$;
(5) $(\mathbb{Q}, +)$ and $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

Solution.

(1) We have $SL(2, \mathbb{Z}) \not\cong \mathbb{Z} \times \mathbb{Z}$ because $\mathbb{Z} \times \mathbb{Z}$ is abelian but $SL(2, \mathbb{Z})$ is non-abelian.

(2) We have $(\mathbb{R}, +) \not\cong (\mathbb{R}^\times, \cdot)$. Indeed, the group $(\mathbb{R}^\times, \cdot)$ has an element of order 2, namely the element -1. However, in the group $(\mathbb{R}, +)$ every nontrivial element has infinite order, and so $(\mathbb{R}, +)$ has no elements of order 2.

(3) We have $\mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12}$. Indeed, for $a = ([1]_4, [1]_3) \in \mathbb{Z}_4 \times \mathbb{Z}_3$ we have $ord(a) = lcm(4, 3) = 12$. Therefore $|\langle a\rangle| = 12 = |\mathbb{Z}_4 \times \mathbb{Z}_3|$ and hence $\mathbb{Z}_4 \times \mathbb{Z}_3 = \langle a\rangle$. Thus $\mathbb{Z}_4 \times \mathbb{Z}_3$ is a cyclic group of order 12 and therefore $\mathbb{Z}_4 \times \mathbb{Z}_3 \cong \mathbb{Z}_{12}$.

(4) We have $\mathbb{Z}_4 \times \mathbb{Z}_3 \not\cong \mathbb{Z}_6 \times \mathbb{Z}_2$. As we have seen in part (3), the group $\mathbb{Z}_4 \times \mathbb{Z}_3$ has an element of order 12, namely $a = ([1]_4, [1]_3)$. For an arbitrary $b = ([m]_6, [n]_2) \in \mathbb{Z}_6 \times \mathbb{Z}_2$ we have $ord(b) = lcm(r, s)$ where $r|6, s|2$, so that $ord(b) \le 6$. Thus $\mathbb{Z}_6 \times \mathbb{Z}_2$ has no elements of order 12. Therefore $\mathbb{Z}_4 \times \mathbb{Z}_3 \not\cong \mathbb{Z}_6 \times \mathbb{Z}_2$.

(5) We have $(\mathbb{Q},+) \not\cong \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ since, as we proved in class, the group $(\mathbb{Q},+)$ is not finitely generated, but the group $\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}$ is finitely generated (e.g. it is generated by the set $S = \{(1,0,0),(0,1,0),(0,0,1)\}$).

Problem 3.[10 points]

Let $G = GL(2,\mathbb{R})$ and let

$$H := \{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a,b,c \in \mathbb{R}, a \neq 0, c \neq 0\} \leq G.$$

(You can take for granted the fact that H is a subgroup of G and do not need to verify this fact).

Prove that $[G : H] = \infty$.

Solution.

Consider the infinite sequence sequence of matrices

$A_n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix} \in GL(2,\mathbb{R})$, where $n = 0,1,2,3,\dots$

Then $A_n^{-1} = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}$.

Therefore for every $m \neq n$, $m,n \geq 1$ we have

$$A_n^{-1}A_m = \begin{bmatrix} 1 & 0 \\ -n & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ m & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m-n & 1 \end{bmatrix} \not\in H$$

since $m - n \neq 0$.

Hence for every $m \neq n$, $m,n \geq 1$ $A_nH \neq A_mH$, so that the cosets $A_0H, A_1H, A_2H, \dots, A_nH, \dots$ are all distinct. Therefore $[G : H] = \infty$, as required.

Problem 4.[10 points]

Let

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 7 & 4 & 6 & 5 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 2 & 1 & 5 & 4 & 7 \end{pmatrix} \in S_7$$

(1) [5 points] Compute $\sigma_1\sigma_2$, σ_1^{-1} and $sgn(\sigma_1)$.

(2) [5 points] Determine whether or not σ_1 and σ_2 are conjugate in S_7. If they are conjugate, find an element $\mu \in S_7$ such that $\mu\sigma_1\mu^{-1} = \sigma_2$.

Solution.

(1) We have

$$\sigma_1\sigma_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 1 & 2 & 3 & 4 & 7 & 5 \end{pmatrix}, \quad \sigma_1^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 2 & 1 & 5 & 7 & 6 & 4 \end{pmatrix}.$$

We can decompose σ_1 as a product of disjoint cycles as $\sigma_1 = (1\ 3)(4\ 7\ 5)$. Therefore

$$sgn(\sigma_1) = sgn\,((1\ 3))\, sgn\,((4\ 7\ 5)) = (-1)\cdot 1 = -1.$$

(2) We have already found in (1) that σ_1 decomposes as a product of disjoint cycles $\sigma_1 = (1\ 3)(4\ 7\ 5) = (1\ 3)(4\ 7\ 5)(2)(6)$.

Decomposing σ_2 as a product of disjoint cycles we get

$$\sigma_2 = (2\ 3)(1\ 6\ 4)(5)(7)$$

Thus σ_1 and σ_2 have the same cycle structure are therefore they are conjugate in S_7. For a conjugating element $\mu \in S_7$ such that $\mu\sigma_1\mu^{-1} = \sigma_2$ we can take a permutation μ such that $\mu(1) = 2$, $\mu(3) = 3$, $\mu(4) = 1$, $\mu(7) = 6$, $\mu(5) = 4$, $\mu(2) = 5$, $\mu(6) = 7$, that is

$$\mu = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 3 & 1 & 4 & 7 & 6 \end{pmatrix}.$$

Problem 5.[10 points]

For each of the following statements, either prove this statement or give a counter-example.

(1)[5 points] If G is a finitely generated group and if $H \triangleleft G$ is a normal subgroup, then the group G/H is finitely generated.

(2) [5 points] If G is a group, $H \triangleleft G$ is a normal subgroup, G_1 is a group and $\phi : G \to G_1$ is a group homomorphism, then $\phi(H) \triangleleft G_1$.

Solution.

(1) This statement is true. Indeed, suppose that G is finitely generated and let $S \subseteq G$ be a finite subset such that $\langle S \rangle = G$. Put $S_1 = \{sH | s \in S\} \subseteq G/H$. Then S_1 is a finite subset of G/H, since $|S_1| \leq |S| < \infty$. We claim that $\langle S_1 \rangle = G/H$. Indeed, let $g \in G$ be arbitrary. Since $\langle S \rangle = G$, there exists a representation $g = s_1^{\epsilon_1} \dots s_n^{\epsilon_n}$ where $n \geq 0$, $s_i \in S$ and $\epsilon_i = \pm 1$. Then in the quotient group G/H we have

$$gH = s_1^{\epsilon_1} H \dots s_n^{\epsilon_n} H$$

and therefore $gH \in \langle S_1 \rangle$. Since $gH \in G/H$ was arbitrary, it follows that $\langle S_1 \rangle = G/H$. Thus we have found a finite generating set for G/H, and so G/H is finitely generated.

(2) This statement is false. For example, take $G = H = \langle (1\ 2) \rangle \leq S_3$ and $G_1 = S_3$.

Take $\phi : G \to S_3$ to be the inclusion map, $\phi(\sigma) = \sigma$ for every $\sigma \in G$.

Then $H \triangleleft G$, but $\phi(H) = H = \langle (1\,2) \rangle$ is not normal in S_3. Indeed, $(13)(1\,2)(1\,3)^{-1} = (3\ 2) \notin H$.

Other counter-examples of similar kind can be obtained by taking G and G_1 to be any groups such that $G \leq G_1$ but that G is not normal in G_1. Then use $H = G$ and $\phi : G \to G_1$ to be the inclusion map, $\phi(g) = g$ for all $g \in G$. Again we have $H \triangleleft G$, but $\phi(H) = G \not\triangleleft G_1$.