HW6 - 4. The subgroup $H = \{(1), (12)(34), (13)(24), (14)(23)\} \le S_4$ contains the identity, all three permutations with cycle structure $(ab)(cd)$, and no other permutations. Explain how this implies directly that H is a normal subgroup, i.e. that $\sigma H = H\sigma$ or $\sigma H \sigma^{-1} = H$ for all $\sigma \in S_4$.

H is a normal subgroup of G if $gxg^{-1} \in H$ for all $x \in H$ and $g \in G$.
(gxg^{-1} is called a conjugate of x, it is obtained under conjugation of x by g, see Definition 14.15)

From Rotman (midterm1 syllabus): Two permutations a and b of $\{1, 2, \ldots, n\}$ have the same cycle structure if and only if $b = sas^{-1}$. (In the symmetric group S_n, conjugation of an element preserves the cycle structure)

For the given H the three elements $(12)(34), (13)(24), (14)(23)$ are precisely the three elements in S_4 with cycle structure $(ab)(cd)$.

For each $x \in \{(12)(34), (13)(24), (14)(23)\}$ and $g \in S_4$,
$gxg^{-1} \in \{(12)(34), (13)(24), (14)(23)\}$. And thus H is normal.

HW8 - 4. For each of the given polynomials find all zeros in the indicated ring.
a) $x^3 + 7x + 4$ in $\mathbb{Z}_{13}$.
b) $(x + 3)(x + 1)(x - 1)$ in $\mathbb{Z}_{15}$.

a) $x = 2, 3, 8$ (try all $x \in \mathbb{Z}_{13}$; $\mathbb{Z}_{13}$ is a field and there are at most 3 zeros)

b) $x = 1, 2, 4, 6, 7, 9, 11, 12, 14$. A possibility is to try all $x \in \mathbb{Z}_{15}$. With the CRT this can be divided into two searches. Modulo 3 the zeros are $x_1 \equiv -3, -1, +1 \pmod 3$. Modulo 5 the zeros are $x_2 \equiv -3, -1, +1 \pmod 5$. Using the isomorphism $\mathbb{Z}_3 \times \mathbb{Z}_5 \longrightarrow \mathbb{Z}_{15}$, $(x_1, x_2) \mapsto x = 10x_1 + 6x_2$ (so that $x \equiv x_1 \pmod 3$ and $x \equiv x_2 \pmod 5$) we find the zeros in $\mathbb{Z}_{15}$

	$x_2 = 1$	2	4
$x_1 = 0$	6	12	9
$= 1$	1	7	4
$= 2$	11	2	14

HW7 - 1. Let $G = D_{12} = \{1, \rho, \ldots, \rho^5, \sigma, \sigma\rho, \ldots, \sigma\rho^5\}$ be the group of symmetries of a regular hexagon and let $H = [G, G] = \{1, \rho^2, \rho^4\}$ be its commutator subgroup.

a) Give the partition of G into left cosets of H.

b) Determine the structure of the group G/H as abelian group.

a) $G = H \cup \rho H \cup \sigma H \cup \rho\sigma H$.

b) G/H is of size $|G|/|H| = 12/3 = 4$. The two possibilities are $\mathbb{Z}_4$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$.

For any element $g \in G$, $g^2 \in H$. Thus any element in G/H other than the identity is of order 2 and $G/H \simeq \mathbb{Z}_2 \times \mathbb{Z}_2$.

HW7 - 2. Determine the structure as abelian group for the following factor groups.

a) $G = (\mathbb{Z}_8 \times \mathbb{Z}_{12})/\langle(2,2)\rangle$.

b) $G = (\mathbb{Z}_8 \times \mathbb{Z}_{12})/\langle(3,3)\rangle$.

a) The group

$$H = \langle(2,2)\rangle = \{(0,0),(2,2),(4,4),(6,6),(0,8),(2,10),(4,0),(6,2),(0,4),(2,6),(4,8),(6,10)\}.$$

The factor group G/H is of size $|G|/|H| = 8 \cdot 12/12 = 8$. The two possibilities are $\mathbb{Z}_8$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$.

For an element $(a,b) \in G$, $4(a,b) = (4a,4b) \in \{(0,0),(0,4),(0,8),(4,0),(4,4),(4,8)\} \subset H$. Thus any element in G/H has order dividing 4 and $G/H \simeq \mathbb{Z}_4 \times \mathbb{Z}_2$.

b) The group $H = \langle(3,3)\rangle = \{(0,0),(3,3),(6,6),(1,9),(4,0),(7,3),(2,6),(5,9)\}$.

Note that the first coordinate takes on all values of $\mathbb{Z}_8$ once. For each coset $(a,b)+H$ we can therefore assume that it is represented as $(0,b')+H$ for some $b' \in \mathbb{Z}_{12}$. The group $G/H \simeq \mathbb{Z}_{12}$.

Another solution is to look at the order $|G|/|H| = 8 \cdot 12/8 = 12$. The two possibilities are $\mathbb{Z}_{12}$ or $\mathbb{Z}_6 \times \mathbb{Z}_2$. The coset $(0,1)+H$ has order 12 in G/H (the minimal k such that $k(0,1) = (0,k) \in H$ is 12) and thus $G/H \simeq \mathbb{Z}_{12}$.