MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 11 (problem 4 is practice problem for midterm 3, not part of homework)
Due Friday April 22

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1. Let R be the ring of 2×2 matrices over the integers. Let $S=\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\}\subset R$ be the subring of upper triangular matrices.

   a) Is S an R-ideal?

   b) Let $I=\{\begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix}\}\subset R$. Is I an R-ideal? Is I an S-ideal?

   c) Let $J=\{\begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix}\}\subset R$. Is J an R-ideal? Is J an S-ideal?

2. Let $\phi: R\longrightarrow S$ be a ring homomorphism. Let $Q\subset S$ be a prime ideal in S. Show that $\phi^{-1}(Q)=\{x\in R:\phi(x)\in Q\}$ is a prime ideal in R.

3. Fraleigh 27 Exercises 15-17. Let R be the ring $\mathbb{Z}\times\mathbb{Z}$.

   a) Find a maximal ideal of R.

   b) Find a prime ideal of R that is not maximal.

   c) Find a nontrivial proper ideal of R that is not prime.

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4. Let $R=\mathbb{Z}I+\mathbb{Z}A$, for matrices
$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad A=\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

   a) Show that R is a ring and that A is a unit in R.

   b) Let $x_0=0, x_1=1,\ x_{i+1}=x_i+x_{i-1}$ for $i\geq 1$ be the Fibonacci sequence. Show that
$$A^i=\begin{pmatrix} x_{i-1} & x_i \\ x_i & x_{i+1} \end{pmatrix}.$$

   c) Show that $3R\subset R$ is a maximal ideal. Hint: Verify, by considering the Fibonacci sequence modulo 3, that the matrices $I,A,\ldots,A^7$ are distinct modulo 3 and that $I+3R, A+3R,\ldots,A^7+3R$ represent the eight different nonzero elements in the ring $R/3R$.

   d) Show that $5R\subset R$ is not a prime ideal. Hint: Find $x\in\mathbb{Z}$ such that $(I+xA)^2\in 5R$.