

MATH 417 - Introduction to Abstract Algebra
Spring 2022

Homework 4

Due Friday February 11

1. Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ be the set of remainders modulo 5. Label the vertices of a regular pentagon clock-wise with 0, 1, 2, 3, 4.
 - a) Describe the permutation $\rho_3 : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$ with $\rho(x) = x + 3$ geometrically as a transformation of the pentagon and give its disjoint cycle form as a permutation of the set \mathbb{Z}_5 .
 - b) Describe the permutation $\sigma_1 : \mathbb{Z}_5 \rightarrow \mathbb{Z}_5$ with $\sigma(x) = 2 - x$ geometrically as a transformation of the pentagon and give its disjoint cycle form as a permutation of the set \mathbb{Z}_5 .
 - c) Show that the permutations $\{x \mapsto x + b : b \in \mathbb{Z}_5\} \cup \{x \mapsto 2a - x : a \in \mathbb{Z}_5\}$ form a group of size 10.
2. Fraleigh Section 5 Problem 47: Prove that if G is an abelian group, written multiplicatively, with identity element e , then all elements x of G satisfying the equation $x^2 = e$ form a subgroup H of G .
3. Fraleigh Section 5 Problem 51: Let G be a group and let a be one fixed element of G . Show that $H_a = \{x \in G \mid xa = ax\}$ is a subgroup of G .
4. Let $M_n = \{a \in \mathbb{Z}_n : \gcd(a, n) = 1\}$. Then M_n is a group under multiplication (the multiplicative operation is associative, 1 is an identity element, and $1 = sa + tn$ shows that a has an inverse s).
 - a) For a prime p what is the size of the group M_{p^2} ? Is M_9 a cyclic group?
 - b) For distinct primes p and q what is the size of the group M_{pq} ? Is M_{15} a cyclic group?