Preliminaries.

For subgroup $H, K \leq G$. define $HK = \{hk : h \in H, k \in K\}$.

Prop: if H, K are finite $|HK| = \dfrac{|H||K|}{|H \cap K|}$

Proof: HK is a disjoint union of left cosets of K.

$$HK = h_1 K \cup h_2 K \cdots\cdots \cup h_r K.$$

For any two $h, h' \in H$

$$\begin{aligned} hK = h'K &\iff h^{-1}h' \in K \\ &\iff h^{-1}h' \in H \cap K. \\ &\iff h(H \cap K) = h'(H \cap K) \end{aligned}$$

Therefore $H = h_1(H \cap K) \cup \cdots\cdots \cup h_r(H \cap K)$

$$r = \frac{|HK|}{|K|} \qquad r = \frac{|H|}{|H \cap K|}$$

$$\Rightarrow |HK| = \frac{|H||K|}{|H \cap K|}$$

Group action by conjugation.

Let X be the set of all subgroup of a group G

$$(\underset{\in G}{g}, \underset{\in X}{H}) \longmapsto \underset{\in X}{gHg^{-1}}.$$

The stabilizer for this particular action is called the <u>normalizer</u> of H in G.

$$\begin{aligned} N_G(H) &= \{ g \in G : gHg^{-1} = H \}. \\ &= \{ g \in G : gH = Hg \}. \end{aligned}$$

lem: if $K \leq N_G(H)$

Then HK is a subgroup of G

Pf: Let $a = h_1 k_1$, $b = h_2 k_2$

Then $ab = h_1 k_1 h_2 k_2 = h_1 \underbrace{k_1 h_2 k_1^{-1}}_{\in H} k_1 k_2 \in HK.$

$$a^{-1} = k_1^{-1} h_1^{-1} = \underbrace{k_1^{-1} h_1^{-1} k_1}_{\in H} k_1^{-1} \in HK.$$

By the orbit-stabilizer theorem

if $H \lhd N_G(H) \leq G$

then the number of subgroup in G conjugate to H is $[G : N_G(H)]$

<u>Ex:</u> $H = \langle(1,2,3,4)\rangle \lhd \underbrace{D_8 \leq S_4}_{[S_4 : D_8] = 3.}$ $\quad \frac{\cancel{4\times 3\times 2\times 1}}{\cancel{8}} = 3.$

$\langle(1234)\rangle,\ \langle(1324)\rangle,\ \langle(1423)\rangle$

Def: The <u>center</u> of a gp G is

$$Z(G) = \{a \in G : ag = ga \ \forall g \in G\}.$$

A gp G acts on itself by conjugation.

$$(\underset{\in G}{g}, \underset{\in X}{h}) \longmapsto \underset{\in X}{ghg^{-1}}$$

the orbits of $a \in G$ is of the $1 \Longleftrightarrow a \in Z(G)$

<u>Thm:</u> (class Equation).

Let G acts on itself by conjugation and let $O(g_1), \ldots, O(g_r)$ be the orbits of size >1.

Then $|G| = |Z(G)| + \sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|}$

where $|C_G(g_i)|$ is the stabilizer of $g_i \in G$ under conjugation $C_G(g_i) = \{g \in G : g g_i g^{-1} = g_i\}$

Pf: We have $\underset{\substack{\uparrow \\ \text{the unies of} \\ \text{orbits of size } 1.}}{Z(G)} \cup \underbrace{O(g_1) \cup \cdots \cup O(g_r)}_{\text{orbit of size} > 1.}$

$$|G| = |Z(G)| + \sum_{i=1}^{r} |O(g_i)|.$$

$$= |Z(G)| + \sum_{i=1}^{r} [G : C_G(g_i)].$$

$$= |Z(G)| + \sum_{i=1}^{r} \frac{|G|}{|C_G(g_i)|}.$$

Cor. A non-trivial p-group has non-trivial center.

2nd Sylow Thm: Let $|G| = p^\alpha m$, $\gcd(p, m) = 1$.
Let $H \leq G$ be a p-subgroup of maximal order p^β / p^α
Then, all subgroup of size p^β are conjugate to H.
and the number is!

Proof. $S = \{H_1, H_2, \ldots, H_r\}$ be the set of all subgroups of G
$H_1 = H$. that are conjugate on $H = H$.
Let K be any subgroup of G of order p^β. The K acts on S by conjugation.

Two gp H_i, H_j that are conjugate under G.
say $H_j = g H_i g^{-1}$ may not be conjugate under K if no $g \in K$.

The size r of S : $r = [G : \underbrace{N_G(H)}_{\text{Satablizer of H under G}}]$

The orbits of H under conjugate by K is of size
$$[K : N_K(H)] = \frac{|K|}{|N_K(H)|}$$
if $K = H$, $N_K(H) = K$, size of orbit is of size 1.

Claim: In general, $N_K(H) = N_G(H) \cap K = H \cap K$.
Clearly $N_G(H) \cap K \supset H \cap K$.
prove $N_G(H) \cap K \subset H \cap K$.
$N_G(H) H$ is a gp since $H \leq N_G(H)$.
The size $|N_G(H) H| = \dfrac{|N_G(H)|\,|H|}{|N_G(H) \cap H|}$
Since $N_G(H)$ and H are p-groups and $H \leq N_G(H) H$,
H a maximal p-group, we have $H = N_G(H) H$ and $N_G(H) \leq H$.

Then
$$[K : N_K(H)] = \frac{|K|}{|N_K(H)|} = \frac{|K|}{|H \cap K|} > 1.$$

Moreover, the size is divisible by p.

$S = \{H_1, H_2, \ldots, H_r\}$ with $H_1 = H$

will be one orbit with size 1.

orthers of size divisible by p.

$r \equiv 1 \pmod{p}$

If we acts on $K \notin S$, then all sizes of orbits would be divisible by p.