MATH 417

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Contents

1 Function and Set

1.1 Function

$A \times B = \{(a,b)|a \in A, b \in B\}$.
Function is a rule σ that assigns an element B to *every* element of A.

$$\sigma : A \to B$$
$$\forall a \in A, \sigma(a) \in B.$$
$$\sigma(a) = \textit{value of } \sigma \textit{ at } a. \textit{ (the image of } a\textit{)}$$

A set $C \subset B$, we call $\sigma^{-1}(C) = \{a \in A|\sigma(a) \in C\}$ as the *preimage* of a.
An element $b \in B$, we call $\sigma^{-1}(b) = \{a \in A|\sigma(a) = b\}$ as the *fiber* of b.
A is the domain of σ, B is the range of σ.

1.1.1 Composition of functions

$\sigma : A \to B, \tau : B \to C$. The function $\tau \circ \sigma : A \to C$ is $\forall a \in A,\ (\tau \circ \sigma)(a) = \tau(\sigma(a))$

1.1.2 Proposition 1.1.3: Associativity of Functions

Proposition 1 (Proposition 1.1.3)**.** *$\sigma : A \to B, \tau : B \to C, \rho : C \to D$ functions then,*

$$\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$$

1.1.3 Injective, surjective, bijective

A function $\sigma : A \to B$ is called,
1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

1.1.4 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

Lemma 1 (Lemma 1.1.7)**.** *Suppose $\sigma : A \to B, \tau : B \to C$ are functions,*
If σ, τ are injective, then $\tau \circ \sigma$ is injective.
If σ, τ are surjective, then $\tau \circ \sigma$ is surjective.
If σ, τ are bijective, then $\tau \circ \sigma$ is bijective.

1.1.5 Proposition 1.1.8: Inverse of Function

Proposition 2 (Proposition 1.1.8)**.** *A function $\sigma : A \to B$ is a bijection if*
$\exists$ a function $\tau : B \to A$ such that

$$\sigma \circ \tau = id_B = \textit{identity on } B(id_B(x) = x, \forall x \in B)$$
$$\tau \circ \sigma = id_A$$

Such τ is unique, called inverse of σ, $\tau = \sigma^{-1}$.

1.2 Set

1.2.1 Well Defined Set

Definition 1. *A set S is* ***well defined*** *if an object a is either $a \in S$ or $a \notin S$.*

1.2.2 Power Set

Definition 2. *For any set A, we denote by $\mathcal{P}(A)$ the collection of all subsets of A. $\mathcal{P}(A)$ is the* ***power set*** *of A.*

1.2.3 Cardinalities of Sets, Pigeonhole Principle

Definition 3. *If A is a set, $|A| =$ cardinality of $A = \#$ of elements*

$n \in \mathbb{N}, |\{1, \ldots n\}| = n$; $|\emptyset| = 0(\emptyset = \text{ empty set })$.
$|A| = |B|$ if there is a bijection $\sigma : A \to B$.
If there is an *injection* $\sigma : A \to B$, we can write $|A| \leq |B|$;
If there is a *surjection* $\sigma : A \to B$, we can write $|A| \geq |B|$.

Theorem 1 (Pigeonhole Principle)**.** *If A and B are sets and $|A| > |B|$, then there is no injective function $\sigma : A \to B$.*

1.2.4 B^A: Sets of Function

If A, B are sets, then $B^A = \{\sigma : A \to B | \sigma \text{ a function}\}$.

Example 1. *$n \in \mathbb{Z}$, we define a function $f : B^{\{1,\ldots,n\}} \to B^n (= B \times B \times B \times \cdots \times B)$ by the equation $f(\sigma) = \{\sigma(1), ..., \sigma(n)\}$, where $\sigma : \{1, \ldots, n\} \to B$. The f is a bijection.*

Proof.
1. *Injective*:

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), ..., \sigma_1(n)\} = \{\sigma_2(1), ..., \sigma_2(n)\}$$
$$\textit{Since } \sigma : \{1, \ldots, n\} \to B, \textit{ it is sufficient to prove } \sigma_1 = \sigma_2.$$

2. *Surjective*:

$$\forall \{b_1, ..., b_n\}, \textit{ we have } \sigma^*(x) = b_x, x = 1, ..., n. \textit{ s.t. } f(\sigma^*) = \{b_1, ..., b_n\}$$

□

Example 2.

$$C(\mathbb{R}, \mathbb{R}) = \{\textit{continuous functions } \sigma : \mathbb{R} \to \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

1.2.5 Operation definitions

Definition 4. *A binary operation on a set A is a function $* : A \times A \to A$.*
The operation is associative if $a * (b * c) = (a * b) * c, \forall a, b, c \in A$.
The operation is commutative if $a * b = b * a, \forall a, b \in A$.

Example 3. *$+, \circ$ are both associative and commutative operations on $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$; $-$ is a neither associative nor commutative operation on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not $\mathbb{N}$.*

Definition 5. *A subset $H \subset S$ is closed under $*$ if $a * b \in H$ for all $a, b \in H$.*

Definition 6. *$*$ has identity element $e \in S$ if $a * e = e * a = a$ for all $s \in S$.*

2 Equivalence relations and Partition

2.1 Equivalence relations (rational equivalence in micro)

rational equivalence in micro Satisfy: (1)Reflexive, (2)Symmetric, (3)Transitive. Given a set X, a relation on X is a subset of $R \subset X \times X$. We write $a \sim b$.
A relation $\sim$ is said to be
1. *Reflexive* if $\forall x \in X$, we have $x \sim x$.
2. *Symmetric* if $\forall x, y \in X$, $x \sim y \Rightarrow y \sim x$.
3. *Transitive* if $\forall x, y, z \in X$, $x \sim y, y \sim z \Rightarrow x \sim z$.
The *sim* is called **equivalence relation** if it is *reflexive, Symmetric* and *Transitive.*

Example 4. *Set* $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$*, satisfies* $(a, b) \sim (c, d)$ *if* $ad = bc$.

1. *Reflexive*: $(a, b) \sim (a, b), \forall (a, b) \in \mathbb{Z}^2$.
2. *Symmetric*: $\forall (a, b), (c, d) \in \mathbb{Z}^2, (a, b) \sim (c, d) \Rightarrow (c, d) \sim (a, b)$.
3. *Transitive*: $\forall (a, b), (c, d), (u, v) \in \mathbb{Z}^2, (a, b) \sim (c, d), (c, d) \sim (u, v) \Rightarrow ad = bc, cv = du \Rightarrow acv = adu = bcu \Rightarrow av = bu \Rightarrow (a, b) \sim (u, v)$.
So this is an equivalence relation.

Example 5. $f : X \to Y$ *is a function, define* $\sim_f$ *on* X *by* $a \sim_f b$ *if* $f(a) = f(b)$.

1. *Reflexive*: $a \sim a, \forall a \in X$.
2. *Symmetric*: $a, b \in X, a \sim b \Rightarrow b \sim a$.
3. *Transitive*: $\forall a, b, c \in X, a \sim b, b \sim c \Rightarrow f(a) = f(b) = f(c) \Rightarrow a \sim c$.
So $\sim_f$ is an equivalence relation.

2.2 Partition (separate a set into disjoint sets with no element left)

X a set, a partition of X is a collection ω of subsets of X s.t.

1) $\forall A, B \in \omega$ either $A = B$ or $A \cap B = \emptyset$.

2) $\cup_{A \in \omega} A = X$.

The subsets are the **cell**s of partition.

2.3 Equivalence class

2.3.1 $[x]$: equivalence class

Define the **equivalence class** of x to be the subset $[x] \subset X$:

$$[x] = \{y \in X | y \sim x\}$$

Where $\sim$ is an equivalence relation.
$\sim$ is reflexive $\Rightarrow x \in [x]$. We say that any $y \in [x]$ as a **representative** of the equivalence class.

2.3.2 $X/\sim$: set of equivalence classes

Set of equivalence classes is a **set** of division result of an *equivalence relation*
We write the set of equivalence classes

$$X/\sim = \{[x] | x \in X\}$$

2.4 Relationship of *Equivalence relation*, *Set of equivalence classes* and *Partitions*

2.4.1 Theorem 1.2.7: Equivalence relation $\sim$ $\Leftrightarrow$ Set of equivalence classes $X/\sim$; {all Sets of equivalence classes} = {all Partitions}

Theorem 2 (Theorem 1.2.7)**.** *$X/\sim$ is a partition of X. Conversely, given a partition ω of X, there exists a unique equivalence relation $\sim_\omega$ s.t. $X/\sim_\omega = \omega$.*

Proof.
(1)$X/\sim$ is a partition of X:

$$\forall x, y \in X \ s.t. \ [x] \cap [y] \neq \emptyset$$
$$Let \ z \in [x] \cap [y] \Rightarrow z \sim x, z \sim y$$
$$\forall w \in [x] \Rightarrow w \sim x \Rightarrow x \sim w \Rightarrow z \sim w \Rightarrow w \sim z \Rightarrow w \sim y \Rightarrow [x] \subset [y]$$
$$Similarly \ we \ can \ prove \ [y] \subset [x] \Rightarrow [x] = [y]$$

(2)Given a partition ω of X, there exists a unique equivalence relation $\sim_\omega$ s.t. $X/\sim_\omega = \omega$:
(2.1) Prove there exists an equivalence relation s.t. $X/\sim_\omega = \omega$:
We define a relation: $x \sim_\omega y$ if there exists $A \in \omega$ s.t. $x, y \in A \Rightarrow \sim_\omega$ is symmetric and transitive. Since $\cup_{A\in\omega} A = X$, we know $\forall x \in X, \exists A \in \omega \ s.t. \ x \in A \Rightarrow \sim_\omega$ is reflexive. So $\sim_\omega$ is an equivalence relation.
We know $A = [x], \forall A \in \omega, \forall x \in A$ (by $\sim_\omega$), then $X/\sim_\omega = \{[x] | x \in \cup_{A\in\omega} A\} = \{\{A^* | x \in A^*\} | A^* \in \omega\} = \omega$.
(2.2) Prove the equivalence relation is unique:
Set $\sim$ be any equivalence relation that make $X/\sim = \omega$, then we know $\forall A \in \omega, \exists x \in X$ s.t. $[x] = A$. According to the definition of $[x]$, if $x \in A$, $y \sim x$ if and only if $y \in [x] = A$. Which is exactly the $\sim_\omega$. ☐

Example 6 (the same as example 5)**.** *$f : X \to Y$ is a function, define $\sim_f$ on X by $a \sim_f b$ if $f(a) = f(b)$. In this example the **equivalence classes** are precisely the fibers $[x] = f^{-1}(f(x))$. $y \sim_f x \Rightarrow y \in f^{-1}(f(x))$*

Example 7 (the same as example 4)**.** *Set $X = \{(a, b) \in \mathbb{Z}^2 | b \neq 0\}$, satisfies $(a, b) \sim (c, d)$ if $ad = bc$. i.e. we write the equivalence of (a, b) as $\frac{a}{b} = [(a, b)]$. Then $X/\sim = \mathbb{Q}$.*

2.4.2 Proposition 1.2.12: use $X/\sim = \{[x] | x \in X\}$ to infer $\sim_\pi$ equals to $\sim$.

Proposition 3 (Proposition 1.2.12)**.** *If $\sim$ is an equivalence relation on X, define a surjective function $\pi : X \to X/\sim$ by $\pi(x) = [x]$. Then $\sim_\pi = \sim$(the definition of $\sim_f$ in example 6.)*

Proof.
(1)Surjective:
$X/\sim = \{[x] | x \in X\} = \{\pi(x) | x \in X\}$, so $\forall y \in X/\sim$, $y \in \{\pi(x) | x \in X\}$, there exists $x \in X$ s.t. $\pi(x) = y$.
(2)$\sim_\pi = \sim$
$a \sim_\pi b$ if $\pi(a) = \pi(b) \Leftrightarrow [a] = [b]$, which is exactly the definition of $\sim$. ☐

1. Given $\sim$;
2. Get the corresponding $X/\sim = \{[x] | x \in X\}$;
3. $\pi(x) = [x]$;
4. $\sim_\pi$: $a \sim_\pi b$ iff $\pi(a) = \pi(b)$
5. $\sim_\pi = \sim$

Proposition 4 (Proposition 1.2.13). *Given any function $f : X \to Y$ there exists a unique function $\tilde{f} : X/\sim \to Y$ such that $\tilde{f} \circ \pi = f$, where $\pi : X \to X/\sim$ in proposition 3. Furthermore, $\tilde{f}$ is a bijection onto the image $f(X)$.*

Proof.
(1) Existence:
We define $x_1 \sim_f x_2$ if $f(x_1) = f(x_2)$. Set $\tilde{f} : X/\sim_f \to Y$, $\tilde{f}([x]) = f(x)$. Then $\tilde{f}[\pi(x)] = \tilde{f}([x]) = f(x)$. Exactly what we require.
(2) Uniqueness:
Set any $\tilde{f}'$ s.t. $\tilde{f}' \circ \pi = f$, then $\tilde{f}'[\pi(x)] = \tilde{f}'([x]) = f(x)$, i.e. the $\tilde{f}$ is unique.
(3) Bijection:
Surjective, which we proved before $\forall f, \exists \tilde{f}$ s.t.$\tilde{f} \circ \pi = f$;
Injective, we also have proved the uniqueness $f = \tilde{f} \circ \pi = \tilde{f}' \circ \pi \Rightarrow \tilde{f}' = \tilde{f}$. □

3 Permutations

Definition 7. *Let X be a finite set, a permutation is bijection $\sigma : X \to X$.*

Definition 8. *Let S_X ($Sym(X)$) be the set of all bijection $\sigma : X \to X$.*

If $|X| = n$, $|S_X| = n!$.

3.1 $Sym(X) = \{\sigma : X \to X | \sigma \textit{ is a bijection}\}$: permutation group of X; elements in $Sym(X)$: permutations of X

We set $Sym(X) = \{\sigma : X \to X | \sigma \textit{ is a bijection}\} \subset X^X$. We call it **symmetric group of X** or **permutation group of X**. We call the elements in $Sym(X)$ the **permutations of X** or the **symmetries of X**.

3.1.1 Properties of $\circ$ on $Sym(X)$

Proposition 5 (Proposition 1.3.1.). *For any nonempty set X, $\circ$ is an operation on $Sym(X)$ with the following properties:*
(i) $\circ$ is associative.
(ii) $id_X \in Sym(X)$, and for all $\sigma \in Sym(X)$, $id_X \circ \sigma = \sigma \circ id_X = \sigma$, and
(iii) For all $\sigma \in Sym(X)$, $\sigma^{-1} \in Sym(X)$.

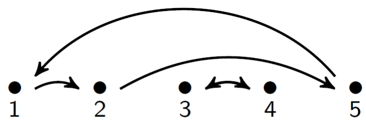
3.1.2 S_n: Permutation group on n elements, σ^i

Note 1. *When $X = \{1, ..., n\}, n \in \mathbb{Z}$, write $S_n = Sym(X)$ **symmetric/permutation group on** n **elements**.*

Note 2. *$\sigma \in Sym(X)$, write $\sigma^n = \sigma \circ \sigma \circ ... \circ \sigma$, $\sigma^0 = id_X$, $\sigma^{-1} = inverse$, $r > 0, \sigma^{-r} = (\sigma^{-1})^r$. So, $r, s \in \mathbb{Z}, \sigma^{r+s} = \sigma^r \circ \sigma^s = \sigma^s \circ \sigma^r$.*

3.1.3 k-cycle, cyclically permute/fix

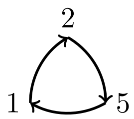
Example 8.



$$1 \overset{\sigma}{\mapsto} 2 \overset{\sigma}{\mapsto} 5 \overset{\sigma}{\mapsto} 1, \quad \tau_1$$

$$3 \overset{\sigma}{\mapsto} 4 \overset{\sigma}{\mapsto} 3, \quad \tau_2$$

Figure 1: Example of Cycle

In the example of *Figure 1*, $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 4 & 3 & 2 \end{pmatrix}$, $\sigma = \tau_1 \circ \tau_2$, where $\tau_1 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 1 & 3 & 4 & 2 \end{pmatrix}$, $\tau_2 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 3 & 5 \end{pmatrix}$. τ_1 is 3-cycle, τ_2 is 2-cycle. We could represent $\tau_1 = (1\ 5\ 2) = (5\ 2\ 1) = (2\ 1\ 5)$, i.e.  Similarly, we can represent $\tau_2 = (3, 4) = (4, 3)$, i.e. $3 \longleftrightarrow 4$
We can find that $\forall x \in \{1, 2, 3, 4, 5\}$, $\tau_1^3(x) = x, \tau_2^2(x) = x$, so we write τ_1 as a **3-cycle** in S_5, τ_2 as a **2-cycle** in S_5.

Given $k \geq 2$, a **k-cycle** in S_n is a permutation σ with the property that $\{1, ..., n\}$ is the union of two disjoint subsets, $\{1, ..., n\} = Y \cup Z$ and $Y \cap Z = \emptyset$, such that
1. $\sigma(x) = x$ for every $x \in Z$, and
2. $|Y| = k$, and for any$x \in Y$,$Y = \{\sigma(x), \sigma^2(x), \sigma^3(x) ... \sigma^k(x) = x\}$.

We say that σ **cyclically permutes** the elements of Y and **fixes** the elements of Z.
$\tau_1 = (1\ 2\ 5)$ **cyclically permutes** the elements of $Y = \{1, 2, 5\}$ and **fixes** the elements of $Z = \{3, 4\}$.
$\tau_2 = (3\ 4)$ **cyclically permutes** the elements of $Y = \{3, 4\}$ and **fixes** the elements of $Z = \{1, 2, 5\}$.

3.2 Disjoint cycles

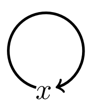
Since the sets are cyclically permuted by τ_1, τ_2 (i.e.Y) are disjoint. We call the **disjoint cycle notation** $\sigma = \tau_1 \circ \tau_2 = (1\ 2\ 5)(3\ 4)$. (Commute the order is irrelevant)

3.2.1 Theorem: Every permutation is a union of disjoint cycles, uniquely.

Given $\sigma \in S_n$, there exists a unique (possibly empty) set of pairwise disjoint cycles.

Theorem 3. *Let X be a finite set, the graph of permutation $\sigma \in S_X$ is a union of disjoint cycle.*

Proof. Prove by induction:

If $|X| = 1$, the graph is circle: 
For $|X| > 1$, let $i_1 \in X$ and let $\mathcal{O}(i_1) = \{\sigma^r(i_1), r \geq 0\} = \{i_1, \sigma(i_1), \sigma^2(i_1), ...\}$. $\mathcal{O}(i_1)$ is finite, and there is a smallest r s.t. $\sigma^r(i_1) \in \{i_1, \sigma(i_1), \sigma^2(i_1), ..., \sigma^{r-1}(i_1)\}$. Then $\sigma^r(i_1) = i_1$ because other elements already have a pre-change under σ.

Then $i_1 \to \sigma(i_1) \to \sigma^2(i_1) \to \cdots \to \sigma^{r-1}(i_1) \to i_1$ is a cycle of length r.
For $j \notin \mathcal{O}(i_1), \sigma(j) \notin \mathcal{O}(i_1), \sigma^{-1}(j) \notin \mathcal{O}(i_1)$. Let $Y = X/\mathcal{O}(i_1)$ then $\sigma : Y \to Y$ is a bijection. Then prove by induction.

□

Example 9. $\sigma_1 = (1\ 2\ 6\ 5)(3)(4)$, *can be written by* $\sigma_1 = (1\ 2\ 6\ 5)$, $\sigma_2 = (2\ 3\ 5\ 4)$

$\sigma_1 \circ \sigma_2 = (1\ 2\ 6\ 5) \circ (2\ 3\ 5\ 4)$

$$1 \xrightarrow{(2\ 3\ 5\ 4)} 1 \xrightarrow{(1\ 2\ 6\ 5)} 2$$
$$2 \xrightarrow{(2\ 3\ 5\ 4)} 3 \xrightarrow{(1\ 2\ 6\ 5)} 3$$
$$3 \xrightarrow{(2\ 3\ 5\ 4)} 5 \xrightarrow{(1\ 2\ 6\ 5)} 1$$
$$4 \xrightarrow{(2\ 3\ 5\ 4)} 2 \xrightarrow{(1\ 2\ 6\ 5)} 6$$
$$5 \xrightarrow{(2\ 3\ 5\ 4)} 4 \xrightarrow{(1\ 2\ 6\ 5)} 4$$
$$6 \xrightarrow{(2\ 3\ 5\ 4)} 6 \xrightarrow{(1\ 2\ 6\ 5)} 5$$

Then $\sigma_1 \circ \sigma_2 = (1\ 2\ 3) \circ (4\ 6\ 5)$
$\sigma_2 \circ \sigma_1 = (2\ 3\ 5\ 4) \circ (1\ 2\ 6\ 5)$

$$1 \xrightarrow{(1\ 2\ 6\ 5)} 2 \xrightarrow{(2\ 3\ 5\ 4)} 3$$
$$2 \xrightarrow{(1\ 2\ 6\ 5)} 6 \xrightarrow{(2\ 3\ 5\ 4)} 6$$
$$3 \xrightarrow{(1\ 2\ 6\ 5)} 3 \xrightarrow{(2\ 3\ 5\ 4)} 5$$
$$4 \xrightarrow{(1\ 2\ 6\ 5)} 4 \xrightarrow{(2\ 3\ 5\ 4)} 2$$
$$5 \xrightarrow{(1\ 2\ 6\ 5)} 1 \xrightarrow{(2\ 3\ 5\ 4)} 1$$
$$6 \xrightarrow{(1\ 2\ 6\ 5)} 5 \xrightarrow{(2\ 3\ 5\ 4)} 4$$

Then $\sigma_2 \circ \sigma_1 = (1\ 3\ 5) \circ (2\ 6\ 4)$
Note: $\sigma_1 \circ \sigma_2 \neq \sigma_2 \circ \sigma_1$

Example 10 (Exercise 1.3.2.)**.** *Consider* $\sigma = (3\ 4\ 8)(5\ 7\ 6\ 9)$ *and* $\tau = (1\ 9\ 3\ 5)(2\ 7\ 4)$ *in* S_9 *expressed in disjoint cycle notation. Compute* $\sigma \circ \tau$ *and* $\tau \circ \sigma$ *expressing both in disjoint cycle notation.*

$$\begin{aligned}
&1 \to \sigma(\tau(1)) = \sigma(9) = 5;\ 2 \to \sigma(\tau(2)) = \sigma(7) = 6;\\
&3 \to \sigma(\tau(3)) = \sigma(5) = 7;\ 4 \to \sigma(\tau(4)) = \sigma(2) = 2;\\
&5 \to \sigma(\tau(5)) = \sigma(1) = 1;\ 6 \to \sigma(\tau(6)) = \sigma(6) = 9;\\
&7 \to \sigma(\tau(7)) = \sigma(4) = 8;\ 8 \to \sigma(\tau(8)) = \sigma(8) = 3;\\
&9 \to \sigma(\tau(9)) = \sigma(3) = 4;\\
\Rightarrow &\sigma \circ \tau = (1\ 5)(2\ 6\ 9\ 4)(3\ 7\ 8)
\end{aligned}$$

$$\begin{aligned}
&1 \to \tau(\sigma(1)) = \tau(1) = 9;\ 2 \to \tau(\sigma(2)) = \tau(2) = 7;\\
&3 \to \tau(\sigma(3)) = \tau(4) = 2;\ 4 \to \tau(\sigma(4)) = \tau(8) = 8;\\
&5 \to \tau(\sigma(5)) = \tau(7) = 4;\ 6 \to \tau(\sigma(6)) = \tau(9) = 3;\\
&7 \to \tau(\sigma(7)) = \tau(6) = 6;\ 8 \to \tau(\sigma(8)) = \tau(3) = 5;\\
&9 \to \tau(\sigma(9)) = \tau(5) = 1;\\
\Rightarrow &\tau \circ \sigma = (1\ 9)(2\ 7\ 6\ 3)(4\ 8\ 5)
\end{aligned}$$

Example 11. *Let $\sigma, \tau \in S_7$, given in disjoint cycle, notation by $\sigma = (1\ 5\ 4)(3\ 7), \tau = (1\ 3\ 2\ 6\ 4)$, Compute $\sigma^2, \sigma^{-1}, \tau \circ \sigma$*

$$\begin{aligned}
\sigma^2 &= (1\ 4\ 5), & \sigma^{-1} &= (4,5,1)(3,7),\\
1 &\to \tau(\sigma(1)) = \tau(5) = 5, & 2 &\to \tau(\sigma(2)) = \tau(2) = 6,\\
3 &\to \tau(\sigma(3)) = \tau(7) = 7, & 4 &\to \tau(\sigma(4)) = \tau(1) = 3,\\
5 &\to \tau(\sigma(5)) = \tau(4) = 1, & 6 &\to \tau(\sigma(6)) = \tau(6) = 4,\\
7 &\to \tau(\sigma(7)) = \tau(3) = 2, & &\\
\Rightarrow &\tau \circ \sigma = (1,5)(2,6,4,3,7) & &
\end{aligned}$$

3.2.2 Cycle Structure

- How many permutation $\sigma \in S_{12}$ has cycle structure $(1\ 2\ 3)(4\ 5\ 6)(7\ 8)(9\ 10)(11\ 12)$?

$$\frac{12!}{3^2 2^3 (2!)(3!)}$$

  12!: Arrange 12 elements in 12 slots.

  3^2: Every cycle with 3 element has 3 forms to represent a same permutation.

  2^3: Every cycle with 2 element has 2 forms to represent a same permutation.

  (2!): Due to the communicative of disjoint permutation, the arrange of cycles with three elements is 2! need to be divided.

  (3!): Due to the communicative of disjoint permutation, the arrange of cycles with two elements is 3! need to be divided.

- $(1\ 2\ 3)(4\ 5)(6) \in S_6$?

$$\frac{6!}{3 \times 2} = 120$$

- General situation: $\sigma \in S_n$, r_i category of length i, $i = 1, 2 ...$

$$\frac{n!}{[1^{r_1} 2^{r_2} 3^{r_3} \cdots][(r_1!)(r_2!)(r_3!) \cdots]}$$

3.3 Transposition

Definition 9. *A **transposition** is a cycle of length 2: $\sigma = (i\ j)$.*

3.3.1 Theorem: Every permutation can be represented by a product of transpositions (not require to be disjoint)

Theorem 4. *Every permutation σ of X is a product of transposition. (the product is not unique)*
Equivalent: *Given $n \geq 2$, any $\sigma \in S_n$ can be expressed as a composition of 2-cycles.(not require disjoint)*

Proof.

Version 1:
$$\begin{aligned}(x_1\ x_k)(x_1\ x_2, ...\ x_{k-1}\ x_k) &= (x_1\ x_2\ ...\ x_{k-1})\\ (x_1\ x_2\ ...\ x_{k-1}\ x_k) &= (x_1\ x_k)(x_1, x_2\ ...\ x_{k-1})\\ &= (\mathbf{x_1\ x_k})(\mathbf{x_1\ x_{k-1}})(\mathbf{x_1\ x_2\ ...\ x_{k-2}})\\ &...\\ &= (\mathbf{x_1\ x_k})(\mathbf{x_1\ x_{k-1}})(\mathbf{x_1\ x_{k-2}})\dots(\mathbf{x_1\ x_2})\end{aligned}$$

Version 2:
$$\begin{aligned}(x_1\ x_2, ...\ x_{k-1}\ x_k)(x_1\ x_k) &= (x_2\ x_3\ ...\ x_k)\\ (x_1\ x_2\ ...\ x_{k-1}\ x_k) &= (x_2\ x_3\ ...\ x_k)(x_1\ x_k)\\ &...\\ &= (\mathbf{x_{k-1}\ x_k})(\mathbf{x_{k-2}\ x_k})\dots(\mathbf{x_2\ x_k})(\mathbf{x_1\ x_k})\end{aligned}$$

□

Claim 1. *Cycle of length k can be written as a product of $k-1$ transpositions.*

3.3.2 Sign of Permutation

Theorem 5. *Although the product of transposition of a permutation is not unique, the parity (odd or even) of the n in a product is unique. We call it the **sign** of permutation.*

$$\begin{aligned}sign(\sigma) &= (-1)^{(\#\ \textit{even-length cycles in}\ \sigma)}\\ &= (-1)^{(\#\ \textit{transpositions in}\ \sigma)}\end{aligned}$$

Example 12.
$\sigma_1 = (1\ 4\ 7\ 9)(2\ 8)(6\ 10)$*:* $N = 3+1+1 = 5$ *is odd.*
$\sigma_2 = (1\ 2\ 3\ 4\ 5)(6\ 7\ 8\ 9\ 10)$*:* $N = 4+4 = 8$ *is even*

What happens to a permutation σ's cycles if $\sigma \to (i\ j)\circ\sigma$?
1. i and j are not contained in σ.
2. i and j appear in the same cycle of σ.
3. i and j appear in disjoint cycles.

$$\begin{aligned}(i\ j)\circ(i--j\sim\sim) &= (i--)\circ(j\sim\sim)\\ (i\ j)\circ(i--)\circ(j\sim\sim) &= (i--j\sim\sim)\end{aligned}$$

Proposition 6. $sign((i\ j)\circ\sigma) = -1\cdot sign(\sigma)$

Proof.

Suppose $\sigma = (a_1\ a_2\ \cdots a_k\ b_1\ b_2\ \cdots b_l)$
Then $(a_1\ b_1)\circ\sigma = (a_1\ a_2\ \cdots a_k)(b_1\ b_2\ \cdots b_l)$

$$sign(\sigma) = \begin{cases}+1 & \text{if } k+l \text{ is odd}\\ -1 & \text{if } k+l \text{ is even}\end{cases}$$

$$sign((a_1\ b_1)\circ\sigma) = \begin{cases}-1 & \text{if } k+l \text{ is odd}\\ +1 & \text{if } k+l \text{ is even}\end{cases}$$

□

4 Integers

4.1 Proposition 1.4.1: Properties of integers $\mathbb{Z}$

Proposition 7 (Proposition 1.4.1.). *The following hold in the integers $\mathbb{Z}$:*

(i) *Addition* and *multiplication* are *commutative* and *associative* operations in $\mathbb{Z}$.
(ii) $0 \in \mathbb{Z}$ is an identity element for addition; that is, $\forall a \in \mathbb{Z}, 0 + a = a$.
(iii) Every $a \in \mathbb{Z}$ has an additive inverse, denoted $-a$ and given by $-a = (-1)a$, satisfying $a+(-a) = 0$.
(iv) $1 \in \mathbb{Z}$ is an identity element for multiplication; that is, for all $a \in \mathbb{Z}$, $1a = a$.
(v) The *distributive* law holds: $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$.
(vi) Both $\mathbb{N} = \{x \in Z | x \geq 0\}$ and $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$ are *closed* under *addition* and *multiplication*. That is, if x and y are in one of these sets, then $x + y$ and xy are also in that set.
(vii) For any two nonzero integers $a, b \in \mathbb{Z}, |ab| \geq \max\{|a|, |b|\}$. Strict inequality holds if $|a| > 1$ and $|b| > 1$.

From this we get cancellation.

$$ab = ac \Rightarrow b = c \ or \ a = 0$$

4.2 Definition: Divide

Suppose $a, b \in \mathbb{Z}, b \neq 0$, b divides a if $\exists m \in \mathbb{Z}$, so that $a = bm, b|a$. Otherwise, write $b \nmid a$.

4.3 Proposition 1.4.2: properties of integer division

Proposition 8 (Proposition 1.4.2). $\forall a, b \in \mathbb{Z}$

(i) if $a \neq 0$, then $a|0$
(ii) if $a|1$, then $a = \pm 1$
(iii) if $a|b$ & $b|a$, then $a = \pm b$
(iv) if $a|b$ & $b|c$, then $a|c$
(v) if $a|b$ & $a|c$, then $a|(mc + nb) \forall m, n \in \mathbb{Z}$

4.4 Definitions: Prime, The Greatest common divisor $gcd(a, b)$

$p > 1, p \in \mathbb{Z}$ is called *prime* if the only divisors are $\pm 1, \pm p$.
Given $a, b \in \mathbb{Z}, a, b \neq 0$, the *greatest common divisor* of a and b is $c \in \mathbb{Z}, c > 0$ s.t.
(1) $c|a$ and $c|b$; (2) if $d|a, d|b$, then $d|c$
The c is unique, we write it $gcd(a, b)$.

4.5 Euclidean Algorithm

Proposition 9 (Proposition 1.4.7(Euclidean Algorithm)). *Given $a, b \in \mathbb{Z}, b \neq 0$, then $\exists q, r \in \mathbb{Z}$ s.t. $a = qb + r, 0 \leq r \leq |b|$.*

Example 13 (Exercise 1.4.3). *For the pair $(a, b) = (130, 95)$, find $gcd(a, b)$ using the Euclidean Algorithm and express it in the form $gcd(a, b) = sa + tb$ for $s, t \in Z$.*

$$\begin{aligned} &130 = 95 + 35; \quad 95 = 2 \times 35 + 25 \\ &35 = 25 + 10; \qquad 25 = 2 \times 10 + 5 \\ &10 = 2 \times 5 + 0 \end{aligned}$$

$$5 = 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35$$
$$= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130$$

$$gcd(130, 95) = gcd(95, 35) = gcd(35, 25) = gcd(25, 10) = gcd(10, 5) = gcd(5, 0) = 5$$

We can also express it by matrix

	q	r	s	t
-1		130	1	0
0	1	95	0	1
1	2	35	1	-1
2	1	25	-2	3
3	2	10	3	-4
4	2	5	-8	11

Hence $gcd(130, 95) = 5 = -8 \cdot 130 + 11 \cdot 95$

4.6 Proposition: $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb|m, n \in \mathbb{Z}\}$

Theorem 6. *$d = gcd(a, b)$ is of the form $sa + tb$*

Proof. We may assume $0 \leq a \leq b$
For $a = 0$, $d = b = 0 \cdot a + 1 \cdot b$.
For $a > 0$, let $b = q \cdot a + r$ with $0 \leq r < a \leq b$. Then

$$\{sa + tb : s, t \in \mathbb{Z}\} = \{sa + t(q \cdot a + r) : s, t \in \mathbb{Z}\} = \{tr + ua : t, u \in \mathbb{Z}\}$$
$$= \dots \{x \cdot 0 + y \cdot d : x, y \in \mathbb{Z}\} = \{..., -2d, -d, 0, d, 2d, ...\}$$

□

Proposition 10 (second form, second proof)**.** *$\forall a, b \in \mathbb{Z}$, not both 0, $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb|m, n \in \mathbb{Z}\}$. i.e. $\exists m_0, n_0 \in \mathbb{Z}$ s.t. $gcd(a, b) = m_0 a + n_0 b$.*

Proof. Let c be the smallest positive integer in the set $M = \{ma+nb|m, n \in \mathbb{Z}\}$. $c = m_0 a+n_0 b > 0$. Let $d = ma + nb \in M$, $d = qc + r$ where $0 \leq r < c$ (by Euclidean Algorithm).

$$r = d - qc = (m - qm_0)a + (n - qn_0)b \in M$$

Since c is the smallest integer in M and $r \in [0, c)$, so $r = 0$. $\Rightarrow d = qc$. So $c|d$.
$a = 1a + 0b \in M \Rightarrow c|a$, $b = 0a + 1b \in M \Rightarrow c|b$.
If $t|a, t|b$ then $t|m_0 a + n_0 b$ i.e. $t|c$. $\Rightarrow c = gcd(a, b)$. □

4.7 Well-Ordering Principle (Least Integer Axiom)

There is a smallest integer in every nonempty subset S of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$

4.8 Proposition 1.4.10: $gcd(b, c)$, $b|ac \Rightarrow b|a$

Proposition 11 (Proposition 1.4.10)**.** *Suppose $a, b, c \in \mathbb{Z}$. If b, c are relatively prime i.e. $gcd(b, c) = 1$ and $b|ac$, then $b|a$.*

Proof. $gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$ s.t. $1 = mb + nc \Rightarrow a = amb + anc$. Since $b|nac, b|amb \Rightarrow b|a$. □

4.8.1 Corollary: $p|ab \Rightarrow p|a$ or $p|b$

Corollary 1 (Corollary of Prop 1.4.10)**.** *$a, b, p \in \mathbb{Z}, p > 1$ prime. If $p|ab$, then $p|a$ or $p|b$.*

Proof. If $p|b$, done. Otherwise, $gcd(p, b) = 1$. By Prop 1.4.10, $p|a$. □

4.9 Fundamental Theorem of Arithmetic: Any integer $a \geq 2$ has a unique prime factorization

4.9.1 Existence

Lemma 2. *Any integer $a \geq 2$ is either a prime or a product of primes.*

Proof. Set $S \subset \mathbb{N}$ be the set of all n without the given property.
Assume that S in nonempty and m is the least element in S.
Since m is not a prime, it can be written as $m = ab$ with $1 < a, b < m$. Since m is the least element in S, $a, b \notin S$. Then m is a product of primes. Contradiction. Thus, $S = \emptyset$. □

4.9.2 Uniqueness

Theorem 7 (Fundamental Theorem of Arithmetic)**.**

Any integer $a > 1$ has a unique prime factorization: $a = p_1^{k_1} \cdot p_2^{k_2} \cdot ...p_n^{k_n}$ where $p_i > 1$ is prime, $k_i \in \mathbb{Z}_+, \forall i = 1, ..., n, p_i \neq p_j, \forall i \neq j$.

Proof.

a) Existence: (Previous Lemma)

b) Uniqueness:

   1) Method 1:
      Suppose $a = p_1^{n_1} \cdot p_2^{n_2} \cdot ...p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot ...q_j^{r_j}$. Where $p_1 > p_2 > ... > p_k, q_1 > q_2 > ... > q_j, n_i, r_i \geq 1$.
      $p_1|a \Rightarrow \exists\ q_i$ *s.t.* $p_1|q_i$. Similarly, $\exists\ q_i$ *s.t.* $q_1|p_{i'}$.
      $q_1 \leq p_{i'} \leq p_1 \leq q_i \Rightarrow q_1 = p_{i'} = p_1 = q_i$
      We can also know $n_1 = r_1$, otherwise we would have two prime factorization of the quotient where the largest primes are different by dividing $p_1^{\min\{n_1, r_1\}}$.
      Then we can get $b = p_2^{n_2} \cdot ...p_k^{n_k} = q_2^{r_2} \cdot ...q_j^{r_j}$. Then prove it by induction.

   2) Method 2:
      Suppose $a = p_1 \cdot p_2 \cdot ...p_k = q_1 \cdot q_2 \cdot ...q_t$. For a p_i, there must exist a q_j s.t. $p_i = q_j$:
      Assume that $p_i \neq q_t$, $gcd(p_i, q_t) = 1$. Then $\exists a, b$ such that $1 = ap_i + bq_t$. Multiplying both sides by $q_1 \cdot q_2 \cdot ...q_{t-1}$:

$$q_1 \cdot q_2 \cdot ...q_{t-1} = ap_iq_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t$$

      Since $p_i|q_1 \cdot q_2 \cdot ...q_t$, we can conclude that $p_i|(ap_iq_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t)$

$$\text{i.e. } p_i|q_1 \cdot q_2 \cdot ...q_{t-1} \text{ if } p_i \neq q_t$$

      Then prove by induction.

□

5 Modular arithmetic

5.1 Congruences

5.1.1 Congruent modulo m: $a \equiv b$ mod m

Given $m \in \mathbb{Z}_+$, define a relation on $\mathbb{Z}$: **congruence modulo m**

$$a \equiv b \text{ mod } m, \text{ if } m|(a-b)$$

Read as "a is congruent to b mod n"; Notation: $a \equiv b$ mod m.
Equivalent to: a, b have the same remainder after division by m.

5.1.2 Proposition: For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" is an equivalence relation

Proposition 12 (Proposition 1.5.1)**.** *For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" is an equivalence relation*

Proof.

1) Reflexive: $\forall a \in \mathbb{Z}, m|0 = (a-a)$, so $a \equiv a$ mod m i.e. $a \sim a$.

2) Symmetric: $\forall a, b \in \mathbb{Z}$, $a \equiv b$ mod m, then $m|(a-b) \Rightarrow m|(b-a) \Rightarrow b \equiv a$ mod m. i.e. $a \sim b \Rightarrow b \sim a$.

3) Transitive: $\forall a, b, c \in \mathbb{Z}$, $a \equiv b$ mod m, $b \equiv c$ mod m. Then $m|(a-b), m|(b-c) \Rightarrow m|(a-b)+(b-c) = (a-c) \Rightarrow a \equiv c$ mod m.

□

5.1.3 Theorem: the equivalence relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$

Theorem 8. *the equivalence relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$*

Proof. Prove any $a \in \mathbb{Z}$ belongs to a unique Ω_i.

a) Existence: Division Algorithm $\Rightarrow a = qm + r$, $0 \leq r < m$. $a \in \Omega_r$.

b) Uniqueness: Assume a in two sets, $a \in \Omega_r \cap \Omega_{r^1}$, $0 \leq r^1 < r < m$.
Then $m|a-r$ and $m|a-r^1 \Rightarrow m|r-r^1$, which is impossible because $0 < r-r^1 < m$. Contradiction.

□

5.1.4 Proposition: Addition and Mutiplication of Congruences

Proposition 13. *Fix integer $m \geq 2$. If $a \equiv r$ mod m and $b \equiv s$ mod m, then $a+b \equiv r+s$ mod m and $ab \equiv rs$ mod m*

Proof.

a) Addition: $m|(a-r), m|(b-s) \Rightarrow m|(a-c)+(b-d) \Rightarrow m|(a+b)-(c-d)$.

b) Mutiplication: $m|(a-r)b + r(b-s) \Rightarrow m|ab - rs$.

□

5.2 Solving Linear Equations on Modular m

5.2.1 Theorm: unique solution of $aX \equiv b$ mod m if $gcd(a,m) = 1$

Theorem 9. *If $gcd(a,m) = 1$, then $\forall b \in \mathbb{Z}$ the congruence $aX \equiv b \mod m$ has a unique solution.*

Proof.

1) Existence: Since $gcd(a,m) = 1$, $\exists s, t$ such that

$$1 = sa + tm$$

$$\begin{aligned}
&\text{(Version 1)} \\
&\qquad \text{(Mutiplying } X\text{)} \\
X &= saX + tmX \\
aX \equiv b \text{ mod } m &\Leftrightarrow aX = km + b \\
&\Leftrightarrow X = s(km + b) + b \\
&\Leftrightarrow X \equiv sb \text{ mod } m \\
&\text{(Version 2)} \\
&\qquad \text{(Mutiplying } s\text{)} \\
saX &\equiv sb \text{ mod } m \\
(1 - tm)X &\equiv sb \text{ mod } m \\
X &\equiv sb \text{ mod } m
\end{aligned}$$

$X \equiv sb$ mod m is the solution to $aX \equiv b$ mod m.

2) Uniqueness: Assume x, y are two solutions,

$$ax \equiv b \text{ mod } , ay \equiv b \text{ mod } m \Rightarrow a(x - y) \equiv 0 \text{ mod } m$$

Since $gcd(a,m) = 1,\ m|(x - y) \Rightarrow x = y,\ (x, y \in \{0, 1, ..., m - 1\})$

Example 14. *Solve* $3X \equiv 5$ *mod* 11.

$gcd(3, 11) = 1$, $1 = 4 * 3 - 1 * 11$,

$$\begin{aligned}
X &\equiv 4 * 5 \\
X &\equiv 9
\end{aligned}$$

□

5.3 Chinese Remaindar Theorem (CRT): unique solution for x modulo mn

Theorem 10 (Chinese Remaindar Theorem (CRT))**.**
If $gcd(m,n) = 1$. Then $\begin{cases} x \equiv r \text{ mod } m & (1) \\ x \equiv s \text{ mod } n & (2) \end{cases}$ have a unique solution for x modulo mn.

Proof.
(1) $\Rightarrow x = km + r$ for some $k \in \mathbb{Z}$.

$$\begin{aligned}
\text{substitute (2) } &\Rightarrow km + r \equiv s \text{ mod } n \\
&\Leftrightarrow mk \equiv s - r \text{ mod } n \quad (3)
\end{aligned}$$

According to previous theorem, $gcd(m,n) = 1$, (3) has a **unique** solution.
We say $k \equiv t$ mod n, $k = ln + t$ for some $l \in \mathbb{Z}$
$\Rightarrow x = (ln + t)m + r = lnm + tm + r$, where $tm + r$ is the unique solution to x modulo mn. □

Example 15. *(Similar to CRT) Find the smallest integer x such that*

$$x \equiv 1 \ mod \ 11 \ and \ x \equiv 9 \ mod \ 13$$

$gcd(11, 13) = 1$ and $1 = 6 * 11 - 5 * 13$
Write $x = 11k + 1$. Substitute in $x \equiv 9$ mod 13:

$$\begin{aligned} 11k &\equiv 8 \text{ mod } 13 \\ 6 * 11k &\equiv 6 * 8 \equiv 9 \text{ mod } 13 \\ (1 + 5 * 13)k &\equiv 9 \text{ mod } 13 \\ k &\equiv 9 \text{ mod } 13 \end{aligned}$$

Then $x = 11k + 1 = 100$.

5.4 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

Fix $n \in \mathbb{Z}_+$, we call $[a]_n = [a]$ the **congruence class** of a modulo n.

$$[a] = \{b \in \mathbb{Z} | b \equiv a \ mod \ n\} = \{a + kn | k \in \mathbb{Z}\}$$

5.4.1 Set of congruence classes of mod n: $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], ..., [n-1]\}$

The set of *congruence classes* of mod n is denoted $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

Proposition 14 (Proposition 1.5.2.). *For any $n \geq 1$ there are exactly n congruences classes modulo n, which we may write as*

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

Proof.
For any $a \in \mathbb{Z}$. By Euclidean algorithm, $a = qn + r$, $q, r \in \mathbb{Z}$, $0 \leq r < n \Rightarrow a \in [r]$. So, $\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$.
When $0 \leq a < b \leq n-1$, $n \nmid (b - a)$, so $[a] \neq [b]$ the n congruence classes listed are all distinct. Hence, there are exactly n congruence classes. □

5.4.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

Fix $n \in \mathbb{Z}$, we define addition$+$ and multiplication$\cdot$ on $\mathbb{Z}_n$:

$$\begin{aligned} [a] + [b] &= [a + b] = \{a + b + (k + j)n | k, j \in \mathbb{Z}\} \\ [a] \cdot [b] &= [ab] = \{ab + (aj + bk + kjn)n | k, j \in \mathbb{Z}\} \end{aligned}$$

This is well defined, follows Lemma 1.5.3.

Proposition 15 (Proposition 1.5.5.). *Let $a, b, c, d, n \in \mathbb{Z}, n \geq 1$, then*
(i) Addition and multiplication are commutative and associative operations in $\mathbb{Z}_n$.
(ii) $[a] + [0] = [a]$.
(iii) $[-a] + [a] = [0]$.
(iv) $[1][a] = [a]$.
(v) $[a]([b] + [c]) = [a][b] + [a][c]$.

Proof.

□

5.4.3 Units(i.e. invertible) in Congruence Classes

Say $[a] \in \mathbb{Z}_n$ is a **unit** or is **invertible** if $\exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [1]$.

5.4.4 Proposition 1.5.6: Set of units in congruence classes: $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \textit{ is a unit}\} = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$

The set of **invertible** elements in $\mathbb{Z}_n$ will be denoted $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \textit{ is a unit}\}$.

Proposition 16 (Proposition 1.5.6.). *For all $n \geq 1$, we have $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$.*

Proof.
By Proposition 1.4.8, we know there exists b, c s.t. $ab + cn = 1$. So, $ab \equiv 1 \ mod \ n$, $[1] = [ab] = [a][b]$. So, $\{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\} \subset \mathbb{Z}_n^\times$
$[a]$ *is a unit* $\Rightarrow \exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow gcd(a, n) = 1$. So, $\mathbb{Z}_n^\times \subset \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$. $\square$

Note 3. *Inverse of $[a]$ is unique, i.e. $[b] = [a]^{-1}$ is unique.*

$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$

5.4.5 Corollary 1.5.7: if p is prime, $\varphi(p) = \mathbb{Z}_p^\times = \{[1], [2], ..., [p-1]\}$

Corollary 2 (Corollary 1.5.7). *If $p \geq 2$ is prime, $\mathbb{Z}_p^\times = \{[1], [2], ..., [p-1]\}$.*

5.5 Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$

Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$.
p prime, $\varphi(p) = p - 1$.

5.5.1 $m|n$, $\pi_{m,n}([a]_n) = [a]_m$

Example 16 (Exercise 1.5.4). *If $m|n$, we can define $\pi_{m,n} : \mathbb{Z}_n \to \mathbb{Z}_m$ by $\pi_{m,n}([a]_n) = [a]_m$. Prove it is well-defined.*

Proof.
We write $[a]_n = [c]_n$, verify that $[a]_m = [c]_m$.
Since $m|n$, there exists $k \in \mathbb{Z}$ s.t. $n = km$.
$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z}$ s.t. $c = a + jn$.
$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m$ $\square$

5.6 Theorem 1.5.8(Chinese Remainder Theorem): $n = mk, gcd(m, k) = 1$, $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

Theorem 11 (Theorem 1.5.8(Chinese Remainder Theorem)). *If $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F : \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_k$ which is given by $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$, then F is a bijection.*

Proof.
(1)Injective: $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$ i.e. $a \equiv b \ mod \ m, a \equiv b \ mod \ n$. $\exists i, j \in \mathbb{Z}$ s.t. $b = a + im = a + jk \Rightarrow k|im$. Since $gcd(m, k) = 1$, $k|i \Rightarrow n = mk|im$. Then $[b]_n = [a]_n + [im]_n = [a]_n$.
(2)Surjective: prove $\forall u, v \in \mathbb{Z}$, $\exists a\mathbb{Z}$ s.t. $[a]_m = [u]_m, [a]_k = [v]_k$.

Since $gcd(m,k) = 1$, $\exists s, t \in \mathbb{Z}$ so that $1 = sm + tk$.
Let $a = (1 - tk)u + (1 - sm)v$, $[a]_m = [(u - v)sm + v]_m = [v]_m$, $[a]_k = [(v - u)tk + u]_k = [u]_k$. $\square$

Note 4. $F([a]_n[b]_n) = F([ab]_n) = ([ab]_m, [ab]_k) = ([a]_m[b]_m, [a]_k[b]_k)$

Since F is a bijection, $[ab]_n = [1]_n$ iff $([a]_m[b]_m, [a]_k[b]_k) = ([1]_m, [1]_k)$.

5.6.1 Proposition 1.5.9+Corollary 1.5.10: $m, n, k > 0, n = mk, gcd(m,k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$

Proposition 17 (Proposition 1.5.9+Corollary 1.5.10)**.** *If $m, n, k > 0, n = mk, gcd(m,k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$.*

5.7 prime factorization: $n = p_1^{r_1}...p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$

Proposition 18. *If $n \in \mathbb{Z}$ is positive integre with prime factorization $n = p_1^{r_1}...p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$*

Proof.
$\mathbb{Z}_{p^r} = \{[0], [1], ..., [p^r - 1]\}$, the number of multiples of p is $\frac{p^r}{p} = p^{r-1}$. Then $\varphi(p^r) = |\mathbb{Z}_{p^r}^\times| = p^r - p^{r-1} = (p-1)p^{r-1}$. So,

$$\varphi(n) = \varphi(p_1^{r_1})...\varphi(p_k^{r_k}) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$$

$\square$

6 Group

6.1 Group $(G, *)$: a set with a binary operation(associative, identity, inverse)

6.1.1 Definition

A *group* is a nonempty set G with a binary operation $* : G \times G \to G$ s.t.

(1) Binary operation on G, $* : G \times G \to G$

(2) $*$ is **associative**

(3) G contains an **identity** element e for $*$: $\exists e \in G$ s.t. $e * g = g * e = g \ \forall g \in G$

(4) Each element $a \in G$ has an **inverse** $b \in G$ s.t. $a * b = b * a = e$.
A Group is **abelian** if moreover

(5) $*$ is **commutative**.

$|G|$ =Order of a group $(G, *)$
$(\mathbb{Z}, +)$ is a group and $+$ is commutative, we call this kind of groups(statify commutative) *abelian group*.

Example 17. *If $\mathbb{F}$ is a field, then $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \cdot)$ are abelian group.*

Example 18. *If V is a vector space over $\mathbb{F}$, then $(V, +)$ abelian group.*

As we know a V is a vector space over $\mathbb{F}$ means V is a field whose subfields include $\mathbb{F}$.

6.1.2 Uniqueness of identity and inverse

Lemma 3. *1. Identity of a group is unique. 2. Inverse of any element in a group is also unique.*

Proof.
1. Let e, e' be two identities in G, then $e * e' = e = e'$.
2. Suppose b, c are both inverse of a, then

$$b = b * e = b * (a * c) = (b * a) * c = e * c = c$$

□

6.1.3 Examples: Permutation group $Sym(X)$, Klein 4-group, alternating group A_n, Dihedral group

Example 19. *If X is any nonempty set, permutation group of X : $\{\sigma : X \to X | \sigma$ is a bijection$\}$, then*

1. $\circ$ is associative;
2. $id : X \to X,\ id(x) = x\ \forall x \in X$ is the idenity;
3. $\sigma \in Sym(X), \sigma^{-1} \in Sym(X)$ is the inverse function.

$(Sym(X), \circ)$ **is a group called the symmetric group of** X

Example 20. *The Klein four-group is a group with four elements, in which each element is self-inverse (composing it with itself produces the identity) and in which composing any two of the three non-identity elements produces the third one. For example, $K \leq S_4$*

$$K = \{(1), (12)(34), (13)(24), (14)(23)\}$$

Example 21. *An alternating group is the group of even permutations of a finite set. An alternating group of degree n, A_n.*

The cycle structure of A_5,
(1) $(abcde)$ - even
(3) (abc) - even
(4) $(ab)(cd)$ - even (odd permutation $\times$ odd permutation)
(6) e - even

Example 22 (Dihedral group)**.**

The dihedral group of order $2n$, denoted D_{2n}, is the group of symmetries of a regular n-gon $A_1A_2 \ldots A_n$, which includes rotations and reflections. It consists of the $2n$ elements

$$\left\{1, \rho, \rho^2, \ldots, \rho^{n-1}, \sigma, \sigma\rho, \sigma\rho^2, \ldots, \sigma\rho^{n-1}\right\}.$$

The element ρ corresponds to rotating the n-gon by $\frac{2\pi}{n}$, while σ corresponds to reflecting it across the line OA_1 (here O is the center of the polygon). So $\rho\sigma$ mean "reflect then rotate" (like with function composition, we read from right to left). In particular, $\rho^n = \sigma^2 = 1$. You can also see that $\rho^k\sigma = \sigma\rho^{-k} = \sigma\rho^{n-k}$.

6.1.4 Cancelation Laws

Theorem 12. *Let G be a group. The left and right cancelation laws hold in G:*

1. $a * x = a * y \Rightarrow x = y$
2. $x * a = y * a \Rightarrow x = y$

Proof.
Let $a{*}x = a{*}y$. $\exists a'$ s.t. $a'{*}a = e$. $a'{*}(a{*}x) = a'{*}(a{*}y) \Rightarrow (a'{*}a){*}x = (a'{*}a){*}y \Rightarrow e{*}x = e{*}y \Rightarrow x = y$
Similar for the right cancel law. □

6.1.5 Unique Solution of Linear Equation

Theorem 13. *The linear equation $a * x = b$ and $y * a = b$ has unique solution.*

Proof.

1. Existence: Multiply by a': $a' * (a * x) = a' * b \Rightarrow x = a' * b$ is a solution.
2. Uniqueness: if x' is another, $a * x = a * x' = b \Rightarrow x = x'$

□

6.2 Subgroup: $H \leq G$

Definition 10. *A subset $H \subseteq G$ is a subgroup of G if H is itself a group.*

write $H \leq G$, $H < G$ if H is a subgroup of $(G, *)$. (If $H = G$, H is an improper subgroup. If $H \subsetneq G$, H is an proper subgroup.)
If $H = \{e\}$, then H is a trivial subgroup.
If $H \neq \{e\}$, then H is a nontrivial subgroup.

Theorem 14. *A subset $H \subseteq G$ is a subgroup of G if and only if*

1. *H is closed under $*$. ($\forall\ g, h \in H$, $g * h \in H$)*
2. *identity $e \in H$.*
3. *Each $a \in H$, the inverse $a' \in H$*

Proof.
"$\Rightarrow$": if $H \leq G$ be a subgroup.

1. H is a group $\Rightarrow *$ is a binary operation on H, $* : H \times H \to H$ i.e. H is closed under $*$.
2. Identity of H, e_H is also a identity of G, due to the uniqueness of identity, $e_H = e_G$.
3. $a \in H$, a's inverse $a'_H \in H$ is also an inverse in G, due to the uniqueness of identity, $a'_H = a'_G$.

"$\Leftarrow$":

1. H is closed under $* \Rightarrow *$ is a binary operation on H.
2. 2,3 fufill the requirement of identity and inverse.
3. $*$ is operation of group $G \Rightarrow *$ is associative.
   Hence H is itself a group.
4. H is a subeset of G, then H is s subgroup of G.

□

6.2.1 Proposition 2.6.8: $H < G$, $(H, *)$ is a group: A group's operation with its any subgroup is also a group

Proposition 19 (Proposition 2.6.8)**.** *If $(G, *)$ is a group, $H \subset G$ is a subgroup, then $(H, *)$ is a group.*

Example 23. *$(G, *)$ is a group, then $e < G$, $G < G$.*

Example 24. *$\mathbb{K} \subset \mathbb{F}$ is a subfield, then $\mathbb{K} < \mathbb{F}$, $\mathbb{K}^{\times} < \mathbb{F}^{\times}$.*

Example 25. *$W \subset V$ is a vector subspace, $W < V$.*

Example 26. *$1 \in S^1 \subset \mathbb{C}^{\times}$, $S^1 = \{z \in \mathbb{C} \| z| = 1\}$. S^1 is a subgroup.*

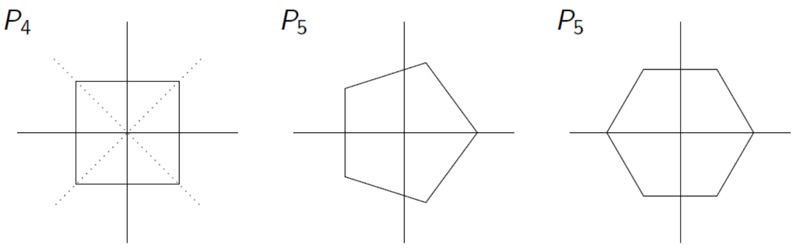
Proof.
$S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. For any $e^{i\theta}, e^{i\psi} \in S^1$, $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1, e^{-i\theta} \in S^1$. □

Example 27. *$Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$*

Example 28. *If $\mathbb{F}$ is a field, $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \to \mathbb{F} \in Sym(\mathbb{F}) | \sigma(a + b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$*

Example 29. *Dihedral Groups:*

Let $P_n \subset \mathbb{R}^2$ be a regular $n - gon$



$D_n < Isom(\mathbb{R}^2)$, $D_n = \{\Phi \in Isom(\mathbb{R}^2) | \Phi(P_n) = P_n\}$

6.3 Some Properties of Group Operation

Proposition 20 (Proposition 3.1.1)**.** *Let $(G, *)$ be a group with identity $e \in G$, then*

(1) if $g, h \in G$ and either $g * h = h$ or $h * g = h$, then $g = e$
(2) if $g, h \in G$ and $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

Corollary 3 (Corollary 3.1.2)**.** *$e^{-1} = e$, $(g^{-1})^{-1} = g$, $(g * h)^{-1} = h^{-1} * g^{-1}$*

6.4 Power of an Element

We define g^n recursively for $n \geq 0$ by setting $g^0 = e$ and for $n \geq 1$, we set $g^n = g^{n-1} * g$. For $n \leq 0$, we define $g^n = (g^{-1})^{-n}$.

Proposition 21 (Proposition 3.1.5)**.** *(1) $g^n * g^m = g^{n+m}$; (2) $(g^n)^m = g^{nm}$*

6.5 $(G \times H, \circledast)$: Direct Product of G and H

$(G, *)$ a group $(H, \star)$ a group. Define an operation on $G \times H$, $\circledast$:

$$(h, k) \circledast (h', k') = (h * h', k * k')$$

6.5.1 Proposition 3.1.7: $(G \times H, \circledast)$ is a group

Proposition 22 (Proposition 3.1.7)**.** *$(G \times H, \circledast)$ is a group. The identity is (e_G, e_H), inverse is (g^{-1}, h^{-1})*

usually written as

$$(h, k)(h', k') = (hh', kk')$$

6.6 Subgroups and Cyclic Groups

6.6.1 Intersection of Subgroups is a Subgroup

Proposition 23 (Proposition 3.2.2)**.** *Let G be a group and suppose $\mathcal{H}$ is any collection of subgroups of G. Then $K = \cap_{H \in \mathcal{H}} H < G$ is a subgroup of G.*

6.6.2 Subgroup Generated by A: $\langle A \rangle$

We define **Subgroup Generated by A:**

$$\langle A \rangle = \cap_{H \in \mathcal{H}(A)} H$$

where $\mathcal{H}(A)$ is the set of all subgroups of G containing the set A:

$$\mathcal{H}(A) = \{H < G | A \subset H \textit{ and } H \textit{ is a subgroup of } G\}$$

6.6.3 Cyclic Group: group generated by an element

A group G is cyclic if exists g (an element), $\langle g \rangle = G$.
g is called a generator for G in this case.
Easy to prove

$$G = \langle g \rangle = \{...g^{-2}, g^{-1}, e, g^1, g^2...\}$$

6.6.4 Cyclic Subgroup

If A is a subgroup of G, and $A = \langle \{a\} \rangle = \langle a \rangle$. Then A is the cyclic subgroup generated by a: $A = \langle a \rangle \leq G$

$$\langle a \rangle = \{...a^{-2}, a^{-1}, e, a^1, a^2...\}$$

6.6.5 Subgroups of a Cyclic Group must be Cyclic

Theorem 15. *A subgroup of a cyclic group is cyclic.*

Proof.
Let $G = \{a^n : n \in \mathbb{Z}\}$ be a cyclic group. Let $H \leq G$ be a subgroup.

1. If $H = \{e\}$, then H is cyclic.

2. If $H \neq \{e\}$, then $a^n \in H$ for some $n > 0$. Check m be the minimal among all n.

   Claim: $H = \langle a^m \rangle$

   Proof: Clearly $\langle a^m \rangle \subset H$. $\forall a^n \in H$, $n = qm + r, 0 \leq r < m$. Then $a^r = a^n (a^m)^{-q}$. Since m is the minimal positive integer s.t. $a^m \in H$, $r = 0$. $\Rightarrow n = qm \Rightarrow a^n \in \langle a^m \rangle$. Hence $H = \langle a^m \rangle$ which is cyclic.

$\square$

Example 30 (Subgroups of $(\mathbb{Z}, +)$)**.**

$\mathbb{Z}$ is a cyclic group $\langle 1 \rangle$. Its subgroups are $\langle n \rangle \leq \mathbb{Z}$ for some $n \geq 0$. (which is a multiplier of n. $(n\mathbb{Z})$)
$n = 0, H = \{0\}; n = 1, H = \mathbb{Z}; n = 2, H = 2\mathbb{Z}$

6.6.6 Theorem: $\langle a^v \rangle < \langle a^n \rangle \Rightarrow \langle a^v \rangle = \left\langle a^d \right\rangle, d = gcd(v, n), |\langle a^v \rangle| = \frac{n}{d}$

Theorem 16. *Let G be a cyclic group of order n. ($G = \{1, a, a^2, ..., a^{n-1}\}$, where $a^n = 1$.). Let $H = \langle a^v \rangle$ be a subgroup of G. Then H is generated by a^d (i.e. $H = \left\langle a^d \right\rangle$), $d = gcd(v, n)$ and $|H| = \frac{n}{d}$.*

Proof.
Let $H' = \left\langle a^d \right\rangle$, we need to show that $H = H'$. $d = gcd(v, n) = d|v \Rightarrow a^v \in \left\langle a^d \right\rangle \Rightarrow H \subset H'$.
While $d = sv + tn$ for some s, t. $\Rightarrow a^d = (a^v)^s (a^n)^t$. Since $a^n = 1$, $a^d = (a^v)^s \Rightarrow H' \subset H$.
Hence, $H = H' = \langle a^v \rangle$. $H = \{1, a^d, a^{2d}, ..., a^{n-d}\}, |H| = \frac{n}{d}$ $\square$

6.6.7 Corollary 3.2.4: G is a cyclic group $\Rightarrow$ G is abelian

Corollary 4 (Corollary 3.2.4)**.** *If G is a cyclic group (i.e. exits $g \in G$ s.t. $\langle g \rangle = G$), then G is abelian (i.e. commutative).*

6.6.8 Equivalent properties of order of g: $|g| = |\langle g \rangle| < \infty$

Proposition 24 (Proposition 3.2.6)**.** *Let G be a group for $g \in G$, the following are equivalent:*

(i) $|g| < \infty$
(ii) $\exists n \neq m$ in $\mathbb{Z}$ so that $g^n = g^m$
(iii) $\exists n \in \mathbb{Z},\ n \neq 0$ so that $g^n = e$
(iv) $\exists n \in \mathbb{Z}_+$so that $g^n = e$
If $|g| < \infty$, then $|g| =$ smallest $n \in \mathbb{Z}_+$so that $g^n = e$, and $\langle g \rangle = \left\{e, g, g^2, \ldots, g^{n-1}\right\} = \{g^n \mid n = 0, \ldots, n-1\}$

6.6.9 $(\mathbb{Z}, +)$ Theorem 3.2.9: $\langle a \rangle < \langle b \rangle$ if and only if $b|a$

Theorem 17 (Theorem 3.2.9)**.** *If $H < \mathbb{Z}$ is a subgroup, then either $H = \{0\}$, or else $H = \langle d \rangle$, where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently, $a \to \langle a \rangle$ defines a **bijection** from $N = \{0, 1, 2, ...\}$ to the set of subgroups of $\mathbb{Z}$. Furthermore, for $a, b \in \mathbb{Z}_+$, we have $\langle a \rangle < \langle b \rangle$ if and only if $b|a$.*

6.6.10 $(\mathbb{Z}_n, +)$ Theorem 3.2.10: $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$

Theorem 18 (Theorem 3.2.10). *For any $n \geq 2$, if $H < \mathbb{Z}_n$ is a subgroup, then there is a positive divisor d of n so that*

$$H = \langle [d] \rangle$$

Furthermore, this defines a bijection between divisors of H and subgroups of $\mathbb{Z}_n$. Furthermore, if $d, d' > 0$ are two divisors of n, then $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$.

If $H = \langle [d] \rangle$ is a subgroup of H, then $[n] \in H$, so $d|n$. And $|H| = |\langle [d] \rangle| = \frac{n}{d}$, so $|H||d$

6.6.11 Subgroup Lattice

The set of all subgroups of a group of G, together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup $\{e\}$ at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.

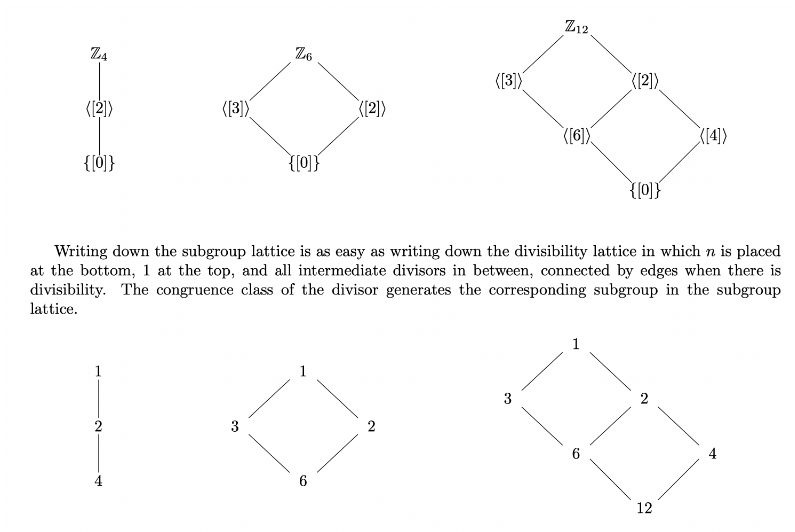


Figure 2:

6.7 Homomorphism

6.7.1 Def: Homomorphism, Image

Definition 11. *If $(G, *)$ and $(H, \circ)$ are groups, then a function $f : G \to H$ is a **homomorphism** if*

$$f(x * y) = f(x) \circ f(y), \ \forall x, y \in G$$

*If f is also a bijection, then f is called an **isomorphism**.*

Example 31. *Let S_n be the symmetric group on n letters, and let $\phi : S_n \to \mathbb{Z}_2$ be defined by*

$$\phi(\sigma) = \begin{cases} 0 & \text{if } \sigma \text{ is an even permutation,} \\ 1 & \text{if } \sigma \text{ is an odd permutation.} \end{cases}$$

Show that ϕ is a homomorphism.

Example 32. *Let $GL(n, \mathbb{R})$ be the multiplicative group of all invertible $n \times n$ matrices. Recall that a matrix A is invertible if and only if its determinant,* $\det(A)$*, is nonzero. Recall also that for matrices $A, B \in GL(n.\mathbb{R})$ we have*

$$\det(AB) = \det(A)$$

Example 33.

1. $\phi : (\mathbb{R}, +) \to (\mathbb{R}^*, x) \quad \phi(x) = 2^x$. Then

$$\phi(x + y) = 2^{x+y} = 2^x 2^y = \phi(x)\phi(y)$$

   ϕ is a homonorphism.

2. $\phi : G \to G \quad \phi(g) = g^{-1}$. Then

$$\phi(gh) = (gh)^{-1} = h^{-1}g^{-1} = \phi(h)\phi(g)$$

   ϕ is not a homonorphism in general; but it is homomorpgism if it is abelian.

Definition 12. *Let ϕ be a mapping of a set X into a set Y, and let $A \subseteq X$ and $B \subseteq Y$. The image $\phi[A]$ of A in Y under ϕ is $\{\phi(a) \mid a \in A\}$. The set $\phi[X]$ is the range of ϕ. The inverse image $\phi^{-1}[B]$ of B i is $\{x \in X \mid \phi(x) \in B\}$*

6.7.2 Properties of Homomorphism

Theorem 19. *Let ϕ be a homomorphism of a group G into a group G', then*

1. if $e \in G$ is an identity in G, then $\phi(e) \in G'$ is the identity in G'.
2. if $a \in G$ has inverse $a' \in G$, then $\phi(a) \in G'$ has inverse $\phi(a') \in G'$.
3. if $H \leq G$ is a subgroup of G, then the image $\phi(H) = \{\phi(h) : h \in G\} \leq G'$ is a subgroup of G'.
4. if $K' \leq G'$ then the inverse image $\phi^{-1}(K') = \{x \in G : \phi(x) \in K'\} \leq G$.

6.7.3 Kernel of Homomorphism

Definition 13. *Let $\phi : G \to G'$ be a homomorphism of groups. The subgroup $\phi^{-1}(e') = \{x \in G : \phi(x) = e'\}$ is the kernel of ϕ, denoted by $Ker(\phi)$.*

$$Ker(\phi) \overset{def}{=} \phi^{-1}(e') = \{x \in G : \phi(x) = e'\}$$

Theorem 20 ($Ker\phi$ is normal)**.** *Let $\phi : G \to G'$ be a homomorphism. $H = Ker\phi$, then for all $a \in G$, $\phi^{-1}[\phi(a)] = \{x \in G : \phi(x) = \phi(a)\}$ is the left coset aH of H, and is also the right coset Ha of H.*

$$aH = Ha = \{x \in G : \phi(x) = \phi(a)\}$$

Proof.

$$\begin{aligned}
& \phi(x) = \phi(a) \\
\Leftrightarrow \quad & \phi(x)\phi(a)^{-1} = e' \\
\Leftrightarrow \quad & \phi(x)\phi(a^{-1}) = e' \\
\Leftrightarrow \quad & \phi(xa^{-1}) = e' \\
\Leftrightarrow \quad & xa^{-1} \in H \\
\Leftrightarrow \quad & x \in Ha
\end{aligned}$$

Similarity, we can prove $x \in aH$. $\square$

Theorem 21. *A homomorphism is injective if and only if $Ker(\phi) = \{e\}$.*

Proof.

$$\begin{aligned}\phi(x) = \phi(y) \Leftrightarrow & \phi(x)\phi^{-1}(y) = e' \\ & \phi(x)\phi(y^{-1}) = e' \\ & \phi(xy^{-1}) = e' \\ \Leftrightarrow & xy^{-1} \in Ker(\phi)\end{aligned}$$

Hence, we can also prove that

$$xy^{-1} \in Ker(\phi) \Leftrightarrow x = y \text{ if and only if } Ker(\phi) = \{e\}$$

□

6.8 Isomorphism

6.8.1 Definition: Isomorphism

Definition 14. *We say that G and H are* ***isomorphic*** *if exists an* ***isomorphism*** *f, denoted by $G \cong H$ or $G \simeq H$. (since f is bijection, $G \cong H \Leftrightarrow H \cong G$)*

Isomophic means these two pathes are the same.

$$\begin{array}{ccc} G \times G \xrightarrow{*} & & G \xrightarrow{f} \ H \\ G \times G \xrightarrow{(f,f)} & H \times H \xrightarrow{\circ} & H \end{array}$$

Example 34. $(\mathbb{Z}_2, +)$, $(\{-1, 1\}, \times)$ *and* $\phi : 0 \to 1;\ 1 \to -1$.

$$\begin{aligned}\phi(0+0) = 1 = \phi(0) \times \phi(0) \\ \phi(0+1) = -1 = \phi(0) \times \phi(1) \\ \phi(1+1) = 1 = \phi(1) \times \phi(1)\end{aligned}$$

6.8.2 Theorem: $\begin{cases} \sigma : G \to G' \textbf{ injective} \\ \sigma(xy) = \sigma(x)\sigma(y)\ \forall x, y \in G \end{cases} \Rightarrow \sigma(G) \leq G'$, G is isomorphic to $\sigma(G)$

Theorem 22. *Let $\sigma : G \to G'$ be an injective map s.t.*

$$\sigma(xy) = \sigma(x)\sigma(y),\ \forall x, y \in G$$

Then the image $\sigma(G) = \{\sigma(x) : x \in G\}$ is a subgroup of G' that is isomorphic to G.

Proof.

1. Closed: $\forall a = \sigma(x), b = \sigma(y) \in \sigma(G)$, then $ab = \sigma(x)\sigma(y) = \sigma(xy) \in \sigma(G)$.
2. Identity: $\sigma(e) \in \sigma(G)$ is an identity for $\sigma(G)$: $\sigma(e)\sigma(x) = \sigma(ex) = \sigma(x) = \sigma(xe) = \sigma(x)\sigma(e)$
3. Inverse: $\sigma(x^{-1})$ is an inverse in $\sigma(G)$ for $\sigma(x)$: $\sigma(x^{-1})\sigma(x) = \sigma(e) = \sigma(x)\sigma(x^{-1})$

□

6.8.3 Cayley Theorem: G is isomorphic to a subgroup of S_G

Theorem 23 (Cayley Theorem)**.** *Let G be a group and S_G is the symmetric group of G (the group of all permutation of G: $S_G = \{\text{Bijection } \sigma : G \to G\}$) Then G is isomorphic to a subgroup of S_G.*

Proof.
Set a bijection $\phi : G \to S_G$ such that $\phi(g) = \lambda_g, \forall g \in G$, where λ_g is a permutation $\lambda_g : x \to gx$.
Claim: $\lambda_g \in S_G$ (i.e. λ_g is a permutation of G, a bijection $G \to G$).

1. $\lambda_g : G \to G$ is injective

$$\begin{aligned} &\lambda_g(x) = \lambda_g(y) \\ &\Leftrightarrow gx = gy \\ &\Leftrightarrow x = y \end{aligned}$$

2. $\lambda_g : G \to G$ is surjective. Let $y \in G$

$$\begin{aligned} &\lambda_g(x) = y \\ &\Leftrightarrow gx = y \\ &\Leftrightarrow x = g^{-1}y \end{aligned}$$

Claim: $\phi(x)\phi(y) = \phi(xy)$

$$\begin{aligned} \phi(x)\phi(y) &= \lambda_x \circ \lambda_y \\ (\lambda_x \circ \lambda_y)(z) = \lambda_x(yz) = xyz &= \lambda_{xy}(z),\ \forall z \in G \\ \Rightarrow \phi(x)\phi(y) &= \phi(xy) \end{aligned}$$

According to previous theorem, $\phi(G) \leq G$ and G is isomorphic to $\phi(G)$.

□

6.9 Coset and Order

Definition 15. *If H is a subgroup of a group G and $a \in G$, then $aH = \{ah | h \in H\} \leq G$ is called left coset of H.*

Theorem 24. *Let $H \leq G$, $a, b \in G$,*

1. *$aH = bH$ if and only if $a^{-1}b \in H$*
2. *$aH \cap bH = \emptyset$ or $aH = bH$*
3. *$|aH| = |H|\ \forall a \in G$*

Proof.

1. Assume that $aH \cap bH \neq \emptyset$ and let $ah = bk \in aH \cap bH$ with $h, k \in H$.
   $ah = bk \Leftrightarrow h = a^{-1}bk \Leftrightarrow a^{-1}b = hk^{-1} \in H$, thus $a^{-1}b \in H$.
2. When $aH \cap bH \neq \emptyset$ $\exists k_1, h \in H$ such that $ak_1 = bh \in bH$. Then $\forall k_2 \in H$ $a = bhk_1^{-1} \Rightarrow ak_2 = bhk_1^{-1}k_2$ wheere $hk_1^{-1}k_2 \in H$ so $ak_2 \in bH$, $\forall k_2 \in H$.
3. $x \to ax$ is bijection $\Rightarrow |aH| = |H|$.

□

Claim 2. *Coset can generate a partition of group:*

$$G = a_1H \cup a_2H \cup \cdots \cup a_rH$$

6.9.1 index of a subgroup

Definition 16. *Let H be a subgroup of a group G. The number of left cosets of H in G is the **index**.*

Note: Since $|aH| = |H| \ \forall a \in G$, the index of a subgroup is the number of subgroups which have order $|H|$.

6.9.2 Lagrange Theorem: Order of subgroup divides the order of group

Theorem 25 (Lagrange Theorem)**.** *Let $H \leq G$ be a subgroup of finite group G. Then the order $|H|$ divides the order $|G|$.*

Proof.
Give a partition

$$\begin{aligned} G &= a_1H \cup a_2H \cup \cdots \cup a_rH \\ |G| &= |a_1H| + |a_2H| + \cdots + |a_rH| \\ &= r|H| \to |H| \Big| |G| \end{aligned}$$

□

6.9.3 Theoerm: Order of element $a \in G = |\langle a \rangle|$ divides $|G|$

Theorem 26 (Order of element/cyclic subgroup)**.** *For $a \in G$, the order of a (the smallest m such that $a^m = e$) divides $|G|$. The order of a is the order of cyclic subgroup $\langle a \rangle$ with generator a.*

Proof.
For $a \in G$, $H = \{a^n, n \in \mathbb{Z}\} \leq G$. H is the size of m. With lagrange theorm, $|H| = m \Big| |G|$ □

Corollary 5. *Every group of prime order is cyclic.*

6.9.4 Theorem: Order n cyclic group is isomorphic to $(\mathbb{Z}_n, +_n)$

Theorem 27. *Let G be a cyclic group with generator a. If the order of G is infinite, then G is isomorphic to $(\mathbb{Z}, +)$. If G has finite order n, then G is isomorphic to $(\mathbb{Z}_n, +_n)$.*

6.10 Direct Products

6.10.1 Cartesian product

Let $G_1, G_2, ..., G_n$ be n groups. Let $G = G_1 \times G_2 \times \cdots \times G_n$ be the Cartesian product.
For $g \in G$, $g = (g_1, ..., g_n)$, $g_i \in G_i$.

Theorem 28. *Then $(G, *)$ becomes a group with operation $*$ defined as*

$$a * b = (a_1, ..., a_n) * (b_1, ..., b_n) = (a_1b_2, ..., a_nb_n) \quad a, b \in G$$

Proof.

(1) Binary operation $* : G \times G \to G$.

(2) $*$ is associative:

$$(a * b) * c = a * (b * c) = (a_1b_1c_1, ..., a_nb_nc_n)$$

(3) Identity: $e = (e_1, ..., e_n) \in G$

$$e * a = a = a * e$$

(4) Inverse: $a^{-1} = (a_1^{-1}, ..., a_n^{-1}) \in G$

$$a * a^{-1} = a^{-1} * a = e$$

□

6.10.2 Theorem: $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn} \Leftrightarrow gcd(m,n) = 1$

Theorem 29. *The group $\mathbb{Z}_m \times \mathbb{Z}_n$ is cyclic and is isomorphic to $\mathbb{Z}_{mn}$ if and only if $gcd(m,n) = 1$.*

Proof.
Claim: $(1,1)$ generate $\mathbb{Z}_m \times \mathbb{Z}_n$
$k(1,1) = (k,k) = (0,0)$ if and only if $m|k$ and $n|k$. The smallest such k is $k = lcm(m,n) = mn$. Hence, $\mathbb{Z}_m \times \mathbb{Z}_n$ is a cyclic group with order mn. Then $\mathbb{Z}_m \times \mathbb{Z}_n$ is isomorphic to $\mathbb{Z}_{mn}$.
We can define an isomorphism

$$\phi : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_{mn}$$

and its inverse

$$\psi : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$$

Since $\mathbb{Z}_{mn} \langle 1 \rangle$, $\mathbb{Z}_m \times \mathbb{Z}_n = \langle (1,1) \rangle$, we can write

$$\psi(x \bmod mn) = (x \bmod m, x \bmod n)$$

ψ is well-defined.
To describe $\phi : \mathbb{Z}_m \times \mathbb{Z}_n \to \mathbb{Z}_{mn}$ at $1 = sm + tn$ and let

$$\phi(a \bmod m, b \bmod n) = (atn + bsm \bmod mn)$$

$$\begin{aligned} \psi(atn + bsm \bmod mn) &= (atn + bsm \bmod m, atn + bsm \bmod n) \\ &= (atn \bmod m, bsm \bmod n) \\ &= (a(1 - sm) \bmod m, b(1 - tn) \bmod n) \\ &= (a \bmod m, b \bmod n) \end{aligned}$$

Hence ψ is the inverse of ϕ. □

Corollary 6. *The group $\prod_{i=1}^{n} \mathbb{Z}_{m_i}$ is cyclic and is isomorphic to $\mathbb{Z}_{m_1 m_2 \cdots m_n}$ if and only if the numbers m_i for $i = 1, ..., n$ are such that the gcd of any two of them is 1.*

Example 35. *If n is written as a product of powers of distinct prime numbers, as it*

$$n = (p_1)^{n_1} (p_2)^{n_2} \cdots (p_r)^{n_r}$$

then $\mathbb{Z}_n$ is isomorphic to

$$\mathbb{Z}_{(p_1)^{n_1}} \times \mathbb{Z}_{(p_2)^{n_2}} \times \cdots \times \mathbb{Z}_{(p_r)^{n_r}}$$

6.10.3 Finitely Generated Abelian Groups

Theorem 30 (Primary Factor Version of the Fundamental Theorem of Finitely Generated Abelian Groups)**.** *Every finitely generated abelian group G is isomorphic to a direct product of cyclic groups in the form*

$$\mathbb{Z}_{(p_1)^{r_1}} \times \mathbb{Z}_{(p_2)^{r_2}} \times \cdots \times \mathbb{Z}_{(p_n)^{r_n}} \times \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$$

where the p_i are primes, not necessarily distinct, and the r_i are positive integers. The number of factors of $\mathbb{Z}$ and the prime powers $(p_i)^{r_i}$ are unique.

- $\mathbb{Z}_{mn} \simeq \mathbb{Z}_m \times \mathbb{Z}_n$ if $gcd(m,n) = 1$.
- Abelian $\Leftrightarrow \mathbb{Z}_m \times \mathbb{Z}_n = \mathbb{Z}_n \times \mathbb{Z}_m$

Example 36. *Find all abelian group of order 16*

5 nonisomorphic abelian group.

$$\begin{cases} \mathbb{Z}_{16} & & & \\ \mathbb{Z}_8 & \times\mathbb{Z}_2 & & \\ \mathbb{Z}_4 & \times\mathbb{Z}_4 & & \\ \mathbb{Z}_4 & \times\mathbb{Z}_2 & \times\mathbb{Z}_2 & \\ \mathbb{Z}_2 & \times\mathbb{Z}_2 & \times\mathbb{Z}_2 & \times\mathbb{Z}_2 \end{cases}$$

Example 37.

$$\begin{aligned} \mathbb{Z}_6 \times \mathbb{Z}_{40} \times \mathbb{Z}_{49} &\simeq \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_{49} \\ \mathbb{Z}_{210} \times \mathbb{Z}_{56} &\simeq \mathbb{Z}_3 \times \mathbb{Z}_7 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \mathbb{Z}_8 \end{aligned}$$

6.11 Def: Normal Subgroup $H \lhd G : aH = Ha, \forall a \in G$

Definition 17. *A subgroup $H \leq G$ is **normal** if its left and right cosets coincide, that is, if*

$$aH = Ha, \quad \forall a \in G$$

Notation: $H \lhd G$

Note that all subgroups of abelian groups are normal.

6.11.1 Thm: Three ways to check if H is normal

Theorem 31. *"$H < G$ is a normal subgroup of G ($H \lhd G$)" is equivalent to*

(1) $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$

(2) $gHg^{-1} = H$ for all $g \in G$

(3) $gH = Hg$ for all $g \in G$

6.11.2 Thm: A subgroup is "Well-defined Left Cosets Multiplication" $\Leftrightarrow$ "Normal"

Theorem 32. *Let H be a subgroup of a group G. Then left coset multiplication is well defined by the equation*

$$(aH)(bH) = (ab)H$$

if and only if $H \lhd G$ (H is a normal subgroup of G).
i.e. 'x $\in aH$ and $y \in bH \Rightarrow xy \in abH$' <u>if and only if</u> '$aH = Ha, \quad \forall a \in G$'

Proof.

- "$\Rightarrow$": $\forall x \in aH$, $a^{-1} \in a^{-1}H \Rightarrow xa^{-1} \in H \Leftrightarrow x \in Ha \Rightarrow aH \subset Ha$;
  Similarly $a^{-1}H \subset Ha^{-1} \Leftrightarrow Ha \subset aH \Rightarrow aH = Ha$
- "$\Leftarrow$": Let $x \in aH$, $y \in bH$. Say $x = ah_1, y = bh_2$

$$\begin{aligned} xy &= (ah_1)(bh_2) \\ &= a(h_1 b)h_2 \\ &= a(bh_3)h_2 \quad \text{(Since } bH = Hb) \\ &= (ab)(h_3 h_2) \in abH \end{aligned}$$

□

6.12 Factor Group $G/H = \{aH : a \in G\}$

Definition 18. *The group $G/H = \{aH : a \in G\}$ with $(aH)(bH) = abH$ is the factor group (or quotient group) of G by H.*

6.12.1 Def: kernel H forms a factor group G/H

Definition 19. *Let $\phi : G \to G'$ be a homomorphism of groups with <u>kernel H</u>. Then the cosets of H form a **factor group**, $G/H = \{aH : a \in G\}$. where $(aH)(bH) = (ab)H$.*

Also, the map $\mu : G/H \to \phi[G]$ defined by $\mu(aH) = \phi(a)$ is an isomorphism. Both coset multiplication and μ are well defined, independent of the choices a and b from the cosets.

6.12.2 Cor: $ker\phi$ is a normal subgroup

Corollary 7. *$ker\phi$ is a normal subgroup: $ker\phi \lhd G$ for all homonorphisms.*

6.12.3 Corollary: normal subgroup H forms a group G/H

By the Thm: A subgroup is "Well-defined Left Cosets Multiplication" $\Leftrightarrow$ "Normal".

Corollary 8. *Let $H \lhd G$ be a **normal subgroup** of G. Then the cosets of H form a group $G/H = \{aH : a \in G\}$ under the binary operation $(aH)(bH) = (ab)H$.*

Proof.

(1) $*$ is associative.

(2) G/H has an identity H.

$$H * aH = aH * H = aH$$

(3) $aH \in G/H$ has inverse $a^{-1}H$

$\square$

Note: This corollary contains the defintion because kernel is normal subgroup(kernel $\Rightarrow$ normal subgroup). (We can then prove they are exactly the same in the next theorem (kernel $\Leftarrow$ normal subgroup))

6.12.4 Thm: normal subgroup is a kernel of a surjective homomorphism $\gamma : G \to G/H$

For any normal subgroup $H \lhd G$, we can define $\gamma(x) = xH$ which is surjective with $ker\gamma = H$

Theorem 33. *Let $H \lhd G$ be a normal subgroup of G. Define $\gamma : G \to G/H$, $\gamma(x) = xH$. Then γ is a surjective homomorphism with $ker\gamma = H$.*

Proof.

1. γ is surjective homomorphism: $\gamma(ab) = abH = (aH)(bH) = \gamma(a)\gamma(b)$
2. $ker\gamma = H$: The identity in G/H is the coset H.

$$\begin{aligned} ker\gamma = \gamma^{-1}(H) &= \{a \in G : \gamma(a) = aH = H\} \\ &= \{a \in G : a \in H\} = H \end{aligned}$$

$\square$

6.12.5 The Fundamental Homomorphism Theorem: Every homomorphism ϕ can be factored to a homomorphism $\gamma : G \to G/H$ and isomorphism $\mu : G/H \to \phi[G]$

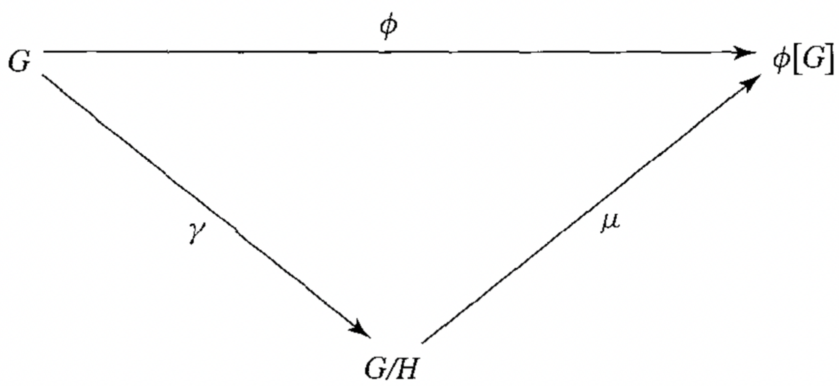


Figure 3: The Fundamental Homomorphism Theorem

Theorem 34 (The Fundamental Homomorphism Theorem)**.**

> *Homomorphism $\phi : G \to G'$ with kernel H can be* ***factored***
>
> $$\phi = \mu\gamma$$
>
> *where $\gamma : G \to G/H$ is a homomorphism, $\mu : G/H \to \phi[G]$ is an isomorphism*
> *where $\gamma(g) = gH$, $\mu(gH) = \phi(g)$*

Let $\phi : G \to G'$ be a group homomorphism with kernel H.
Then $\phi[G]$ is a group isomorphic to G/H, and $\mu : G/H \to \phi[G]$ given by $\mu(gH) = \phi(g)$ is an isomorphism. (If $\gamma : G \to G/H$ is the homomorphism given by $\gamma(g) = gH$, then $\phi(g) = \mu\gamma(g)$ for each $g \in G$.)

Proof. i.e. prove μ is (1) well-deifined, (2) isomorphism.

(1) well-defined: if $aH = bH$, then $a^{-1}b \in H$,

$$\mu(bH) = \mu((a(a^{-1}b))H) = \phi(a(a^{-1}b)) = \phi(a)\phi(a^{-1}b) = \phi(a) = \mu(aH)$$

(2) homomorphism:

$$\mu(aHbH) = \mu(abH) = \phi(ab) = \phi(a)\phi(b) = \mu(aH)\mu(bH)$$

(3) isomorphism i.e. prove $ker(\mu)$ is exactly the identity in G/H:

$$\mu(aH) = e' = \phi(a) \Leftrightarrow a \in ker(\mu), a \in ker(\phi) = H$$
$$\Leftrightarrow aH = H, \quad aH \text{ is the identity in } G/H$$

□

Corollary 9. *Let $\phi : G \to G'$ be a homomorphism for finite group G, G'.*
Then (1).$|\phi(G)| \,\Big|\, |G|$; (2).$|\phi(G)| \,\Big|\, |G'|$

Proof.

(1) According to the Fundamental Homomorphism theorem, $\phi(G)$ is one-to-one corresponse to G/H (H is the kernel of G), then $|\phi(G)| = |G/H| = |\{aH : a \in G\}| \Rightarrow |\phi(G)| = |G|/|H|$

(2) Proved by Lagrange theorem.

□

6.12.6 Thm: $(H \times K)/(H \times e) \simeq K$ and $(H \times K)/(e \times K) \simeq H$

Theorem 35. *Let $G = H \times K$ be the direct product of groups H and K. Then $\bar{H} = \{(h, e) \mid h \in H\}$ is a normal subgroup of G. Also $G/\bar{H}$ is isomorphic to K in a natural way. Similarly, $G/\bar{K} \simeq H$ in a natural way.*

Proof. $\pi : H \times K \to K$ where $\pi(h, k) = k$ has kernal $\bar{H} = \{(h, e) \mid h \in H\}$, then $H \times K/\bar{H}$ is isomorphic to K. Prove $G/\bar{K} \simeq H$ in the same way. □

6.12.7 Thm: factor group of a cyclic group is cyclic [a]/N=[aN]

Theorem 36. *A factor group of a cyclic group is cyclic.* $[a]/N = [aN]$

6.12.8 Ex: 15.11 example $\mathbb{Z}_4 \times \mathbb{Z}_6/(\langle(2, 3)\rangle) \simeq \mathbb{Z}_4 \times \mathbb{Z}_3$ or $\mathbb{Z}_{12}$

6.12.9 Thm: Homomorphism $\phi : G \to G'$ preserves normal subgroups between G and $\phi[G]$.

Theorem 37. *Let $\phi : G \to G'$ be a group homomorphism. If N is a normal subgroup of G, then $\phi[N]$ is a normal subgroup of $\phi[G]$. Also, if N' is a normal subgroup of $\phi[G]$, then $\phi^{-1}[N']$ is a normal subgroup of G.*

Note: $\phi[N]$ is a normal subgroup of $\phi[G]$ not G'. Counterexample: $\phi : \mathbb{Z}_2 \to S_3$, where $\phi(0) = \rho_0$ and $\phi(1) = \mu_1$ is a homomorphism, and $\mathbb{Z}_2$ is a normal subgroup of itself, but $\{\rho_0, \mu_1\}$ is not a normal subgroup of S_3.

6.13 Def: automorphism, inner automorphism

Definition 20.
An isomorphism $\phi : G \to G$ of a group G with itself is an automorphism of G.
The automorphism $\phi_g : G \to G$, where $\phi_g(x) = gxg^{-1}$ for all $x \in G$, is the inner automorphism of G by g. Performing ϕ_g on x is called conjugation of x by g.

6.14 Simple Groups

Definition 21. *A group G is simple if it is nontrivial ($G \neq \{e\}$) and has no proper nontrivial normal subgroups. ($\nexists H \neq \{e\} \lhd G$)*

Theorem 38. *The alternating group A_n is simple for $n \geq 5$*
(alternating group is a group of even permutations on a set of length n)

6.15 The Center and Commutator Subgroups

6.15.1 Def: center and commutator subgroup

Theorem 39. *All finite subgroup G have two normal subgroups,*

(1) The *center* of G, $Z(G) = \{z \in G : za = az, \forall a \in G\} \lhd G$

(2) The *commutator* subgroup of G, $C(G) = [G, G] = \{[a, b] : a, b \in G\}$.

Definition 22. $[a, b] = aba^{-1}b^{-1}$ *is the commutator of a and b. $[a, b] \in G$ is the unique element such that $ab = [a, b]ba$.*

6.15.2 Thm: commutator subgroup is normal

Theorem 40. $[G, G] \lhd G$

Proof. Consider $[a, b] \in [G, G]$, prove that $\forall g \in G, g[a, b]g^{-1} \in [G, G]$

$$\begin{aligned} g[a, b]g^{-1} &= g(aba^{-1}b^{-1})g^{-1} = (gag^{-1})(gbg^{-1})(ga^{-1}g^{-1})(gb^{-1}g^{-1}) \\ &= (gag^{-1})(gbg^{-1})(gag^{-1})^{-1}(gbg^{-1})^{-1} = [gag^{-1}, gbg^{-1}] \in [G, G] \end{aligned}$$

□

Example 38.

(1) For abelian group, $Z(G) = G$, $C(G) = \{e\}$

(2) $G = S_6$, $Z(G) = \{e\}$, $C(G) = \{1, \rho, \rho^2\}$

(3) $G = D_8 = \{1, \rho, \rho^2, \rho^3, \sigma, \sigma\rho, \sigma\rho^2, \sigma\rho^3\}$, $Z(G) = \{1, \rho^2\}$, $C(G) = \{1, \rho^2\}$

(4) $G = D_{12}$, $Z(G) = \{1, \rho^3\}$, $C(G) = \{1, \rho^2, \rho^4\}$

(5) $G = A_4$, $Z(G) = \{(1)\}$, $C(G) = \{(1), (12)(34), (13)(24), (14)(23)\}$

(6) $G = S_4$, $Z(G) = \{(1)\}$, $C(G) = A_4$

Commutator subgroup of S_n is A_n.
Commutator subrgoup of D_{2n} is $\{1, \rho^2, ..., \rho^{n-2}\}$
$\sigma\rho^a = \rho^{n-a}\sigma = \rho^{n-2a}(\rho^a\sigma) \Rightarrow \rho^{n-2a}$ is a commutator $\forall a \in \mathbb{Z} \Rightarrow C(D_{2n}) = \{1, \rho^2, ...\rho^{n-2}\}$ if n is even.

6.15.3 Thm: if $N \lhd G$, "G/N is abelian" $\Leftrightarrow$ "$[G, G] < N$"

Theorem 41. *If N is a normal subgroup of G, then G/N is abelian if and only if $[G, G] < N$.*

Proof.
If N is a normal subgroup of G and G/N is abelian, then $\left(a^{-1}N\right)\left(b^{-1}N\right) = \left(b^{-1}N\right)\left(a^{-1}N\right)$; that is, $aba^{-1}b^{-1}N = N$, so $aba^{-1}b^{-1} \in N$, and $C \leq N$. Finally, if $C \leq N$, then

$$\begin{aligned}(aN)(bN) &= abN = ab\left(b^{-1}a^{-1}ba\right)N \\ &= \left(abb^{-1}a^{-1}\right)baN = baN = (bN)(aN)\end{aligned}$$

□

6.16 Group Action on a Set

6.16.1 Def: action of group G on set X

Definition 23. *Let X be a set and G a group. An **action of G on X** is a map $* : G \times X \to X$ such that*

(1) $ex = x$ for all $x \in X$.

(2) $(g_1g_2)(x) = g_1(g_2x)$ for all $x \in X$ and all $g_1, g_2 \in G$.

*Under these conditions, X **is a G-set**.*

Example: Let X be any set, and let H be a subgroup of the group S_x of all permutations of X. Then X is an H-set.

6.16.2 Thm: If G acts on X, $\phi : G \to S_X$ as $\phi(g) = \sigma_g$ is a homomorphism (where $\sigma_g(x) = gx$)

Theorem 42. *Let group G act on the set X,*

(1) *$\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is well-defined.*
($\sigma_g : X \to X$ defined by $\sigma_g(x) = gx$ for $x \in X$ is a permutation of X)

(2) *$\phi : G \to S_X$ defined by $\phi(g) = \sigma_g$ is a homomorphism with the property that $\phi(g)(x) = gx$.*

Special case: Let G act on itself, we get the **Cayley Theorem**: G is isomorphic to a subgroup of S_G
In general, for a group G act on the set X, the homomorphism $\phi : G \to S_X$ is not injective. We say that G acts faithfully on X if ϕ is injective.

6.16.3 Examples of Group Actions

(Let $H \leq G$ be a subgroup of G)

(1) $G \times G \to G$, $(g_1, g_2) \to g_1g_2$

(2) $G \times G \to G$, $(g_1, g_2) \to g_1g_2g_1^{-1}$ (conjugation)

(3) $G \times G/H \to G/H$, $(g, aH) \to gaH$ (when H is not normal, $X = G/H$ is just a set.)

6.17 Orbits

6.17.1 Thm: Equivalence Relation: X is a G-set, $x_1 \sim x_2 \Leftrightarrow x_2 = gx_1,\ \exists g \in G$

Theorem 43. *For G acting on X, define a relation $\sim$ on X via*

$$x_1 \sim x_2 \Leftrightarrow x_2 = gx_1 \quad \text{for some } g \in G$$

Definition 24. *A group G is **transitive** on a G-set X if for each $x_1, x_2 \in X$, there exists $g \in G$ such that $gx_1 = x_2$.*

6.17.2 Def: $Gx = \{gx|g \in G\}$ is the orbit of x

Definition 25. *For a group action G on X, X partitions into equivalence classes. Denote the class containing x by Gx. $Gx = \{gx|g \in G\}$ is called the orbit of $x \in X$.*

Denote: the partition of X as equivalence classes takes the form

$$X = Gx_1 \cup Gx_2 \cup \cdots Gx_r$$

r disjoint orbits.

6.17.3 Def: $G_x = \{g \in G|gx = x\}$ is the stabilizer of x

Definition 26. *Let G act on X, for $x \in X$, define $G_x = \{g \in G|gx = x\}$, then G_x is a subgroup of G called the **stabilizer** of x. (or the **isotropy subgroup** of x)*

6.17.4 Thm: if X is a G-set, stabilizer $G_x = \{g \in G|gx = x\}$ is subgroup of G, $\forall x \in X$

Let

$$X^g = \{x \in X|gx = x\};\ G_x = \{g \in G|gx = x\}$$

Theorem 44. *Let X be a G-set then G_x is a subgroup of G, $\forall x \in X$.*

Proof.

(1) Closed: $\forall g_1, g_2 \in G_x$, $(g_1g_2)x = g_1(g_2x) = g_1x = x \Rightarrow g_1g_2 \in G_x$.

(2) Identity: $ex = x$.

(3) Inverse: $gx = x$, $x = ex = g^{-1}gx = g^{-1}(gx) = g^{-1}x$.

□

6.17.5 Orbit-Stabilizer Theorem: $|Gx| = \frac{|G|}{|G_x|}$

Theorem 45. *Let G act on X, and let $x \in X$, then $|Gx| = [G : G_x] = |G/G_x| = \frac{|G|}{|G_x|}$*

Proof. Since G_x is the subgroup of G, according to largrange theorem we know $|G_x| \Big| |G|$.

For a $x_1 = g_1x \in Gx$ with $g_1 \notin G_x = \{g \in G|gx = x\}$. $G_{x_1} = \{g \in G|gx_1 = x_1\} = \{g \in G|g_1^{-1}gg_1x = x\}$.
Prove $g \to g_1^{-1}gg_1$ is one to one: assume $g_1^{-1}gg_1 = g_1^{-1}g'g_1$, $\Rightarrow g = g'$.
Hence, $|G_{x_1}| = |G_x| \Rightarrow \frac{|G|}{|G_x|} = |Gx|$ □

6.18 Applications of $G-$sets to Counting

As we showed before, the partition of X as equivalence classes takes the form

$$X = Gx_1 \cup Gx_2 \cup \cdots Gx_r$$

where r is the number of orbits in X.

6.18.1 Burnside's Formula: number of orbits in X: $r = \frac{1}{|G|}\sum_{g\in G}|X^g|$

Theorem 46. *Let G be a finite group and X a finite G-set. If r is the number of orbits in X under G, then*

$$r = \frac{1}{|G|}\sum_{g\in G}|X^g|$$

i.e. r equals to the average $|X^g|$, where $X^g = \{x : gx = x\}$

Proof. Since $G_{x_0} = \{g \in G|gx = x\} = \{(g, x)|gx = x, g \in G, x = x_0\}$,

$$\sum_{x\in X}|G_x| = |\{(g, x)|gx = x, g \in G, x \in X\}|$$

At the same time, $|X^{g_0}| = \{x \in X : gx = x\} = \{(g, x)|gx = x, g = g_0, x \in X\}$, then

$$\sum_{g\in G}|X^g| = |\{(g, x)|gx = x, g \in G, x \in X\}| = \sum_{x\in X}|G_x|$$

As we shoed before, $|G_x| = |G_y|, \forall x, y \in X$

$$\Rightarrow \sum_{x\in X}|G_x| = |G|\sum_{x\in X}\frac{1}{|Gx|} = |G|\sum_{i=1}^{r}\sum_{x\in Gx_i}\frac{1}{|Gx|} = |G|\sum_{i=1}^{r}\frac{|Gx_i|}{|Gx_i|} = |G|r$$

$$\Rightarrow r = \frac{\sum_{x\in X}|G_x|}{|G|} = \frac{\sum_{g\in G}|X^g|}{|G|}$$

□

6.18.2 Example: Counting

Example 39.

7 Ring and Field

7.1 Ring $(R, +, \cdot)$: $+$ is associative, commutative, identity, inverse $\in R$; $\cdot$ is associative, distributes over $+$

7.1.1 Def, Prop

Definition 27. *A ring is a nonempty set with two operations, called addition and multiplication, $(R, +, \cdot)$ such that*

(1): $(R, +)$ is an abelian group: i.e. $+$ is associatve and commucative. $0, -a \in R$
(2): $\cdot$ is associative.
(3): $\cdot$ distributes over $+$: $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

Theorem 47. *If R is a ring with additive identity 0 , then for any $a, b \in R$ we have*
1. $0a = a0 = 0$,
2. $a(-b) = (-a)b = -(ab)$,
3. $(-a)(-b) = ab$.

7.1.2 $S \subset R$: Subring (closed under + and $\cdot$; addictive inverse $-a \in S$)

Proposition 25 (Proposition 2.6.27). *If $S \subset R$ is a subring, then $+, \cdot$ make S into a ring.*

7.1.3 Def: Commutative ring: ring's $\cdot$ is commutative

If "$\cdot$" is commutative, we call $(R, +, \cdot)$ a commutative ring.

7.1.4 Def: A ring with 1: the ring exists multiplication identity $1 \in R$

If there exists an element $1 \in R \backslash \{0\}$ such that $a1 = 1a = a$, $\forall a \in R$,then we say that R is a ring with 1 (a ring with unity).
Note: We usually discuss $1 \neq 0$. If $1 = 0$, $a = 1a = 0 \Rightarrow R = \{0\}$.

7.1.5 Def: In a ring R with 1, u is a unit if $\exists v \in R$ s.t. $uv = vu = 1$

Definition 28. *In a ring R with 1, u is a unit if it has a multiplicative inverse in R i.e. $\exists v \in R$ s.t. $uv = vu = 1$*

Example 40. *units in $\mathbb{Z}$ are $\{-1, +1\}$; in $\mathbb{Z}_n$ are $\{a \in \mathbb{Z}_n : gcd(a, n) = 1\}$*

7.1.6 Def: A ring with 1, R is a division ring if every nonzero element of R is a unit

Definition 29. *A ring with 1, R is a division ring if every nonzero element of R is a unit. This is equalivalent to R has identity and inverse in mutiplication.*

7.1.7 Def: Ring Homomorphism: $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$

Definition 30. *Let R, R' be rings. A map $\phi : R \to R'$ is a ring homomorphism if*

$$\phi(a + b) = \phi(a) + \phi(b)$$

$$\phi(ab) = \phi(a)\phi(b)$$

7.1.8 Def: zero divisor: a $a \neq 0 \in R$ if $\exists b \neq 0 \in R$ s.t. $ba = 0$ or $ab = 0$

Definition 31. *A nonzero element $a \in R$ is called a zero divisor if there exists a nonzero $b \in R$ s.t. $ba = 0$ or $ab = 0$*

Note: Mutiplication cancellation law holds when no zero divisors.

7.1.9 Remark: In $\mathbb{Z}_n$, an element is either 0 or unit or zero divisor

Remark: In $\mathbb{Z}_n$, an element is either (1) 0, (2) a unit, (3) a zero divisor.
$0 \neq a \in \mathbb{Z}_n$ is a $\begin{cases} \text{unit} & \text{if } gcd(a, n) = 1 \\ \text{zero divisor} & \text{if } gcd(a, n) \neq 1 \end{cases}$
In $M_n(R)$ $\begin{cases} \text{unit} & \text{if } rank(A) = n \\ \text{zero divisor} & \text{if } rank(A) < n \end{cases}$
In $R = \mathbb{Z}$, $a \notin \{0, +1, -1\}$ is neither unit nor zero divisor.

7.1.10 Thm: $a \in \mathbb{Z}_n$ is a zero divisor $\Leftrightarrow gcd(a, n) \neq 1$.

Theorem 48. *In the ring $\mathbb{Z}_n$, the zero divisors are precisely those nonzero elements that are not relatively prime to n.*

7.1.11 Cor: $\mathbb{Z}_p$ has no zero divisors if p is prime.

7.1.12 Def: An integral domain is a commutative ring with $1 \neq 0$ that has no zero divisors

Definition 32. *An integral domain is a commutative ring with $1 \neq 0$ that has no zero divisors.*

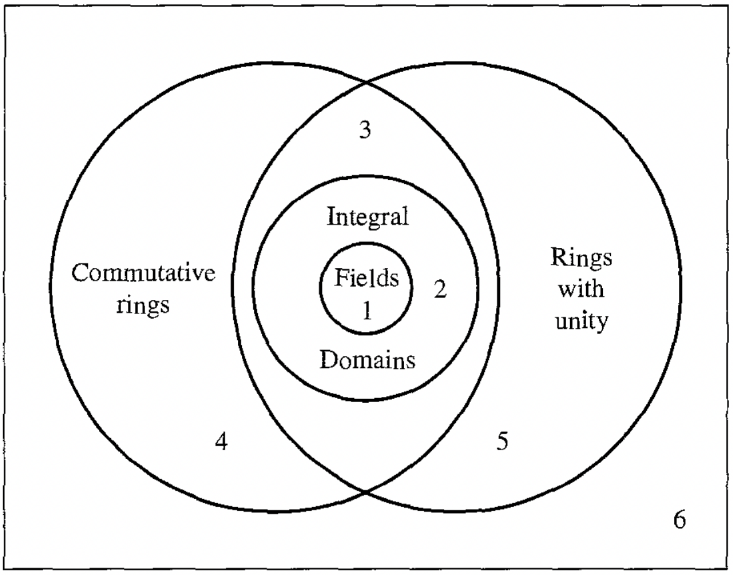
$\mathbb{Z}$ and $\mathbb{Z}_p$ for any prime p are integral domains, but $\mathbb{Z}_p$ is not an integral domain if n is not prime.

7.2 Field $\mathbb{F}$

7.2.1 Def: A field is a commutative division ring.

Definition 33. *A field is a commutative division ring.*

Which is equal to a ring satifies identity, inverse and commutative in multiplication. Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))
Note: nonzero elements of a finite field can form a cyclic (sufficient for abelian) mutiplication group.



19.10 Figure A collection of rings.

Figure 4: example: 1.$\mathbb{Z}_2, \mathbb{Q}$, 2.$\mathbb{Z}$, 3.$\mathbb{Z}_4$, 4.$2\mathbb{Z}$ 5.$M_2(\mathbb{Z})$,$M_2(\mathbb{R})$, 6.upper-triangular matrices with integer entries and all zeros on the main diagonal

7.2.2 Differences between "Field" and "Integral Domain"

Def: An integral domain is a commutative ring with $1 \neq 0$ that has no zero divisors
Def: A field is a commutative ring with $1 \neq 0$ that every nonzero element of R is a unit.

7.2.3 Lemma: A unit is not zero divisor

Proof. $a \in R$ is a unit and $\frac{1}{a}$ is its inverse.

Assume there exists $b \neq 0$ s.t. $ab = 0$, then

$$\begin{aligned} \frac{1}{a}(ab) &= \frac{1}{a}0 = 0 \\ &= (\frac{1}{a}a)b = b \end{aligned}$$

Contradiction!
Assume there exists $b \neq 0$ s.t. $ba = 0$, then

$$\begin{aligned} (ba)\frac{1}{a} &= 0\frac{1}{a} = 0 \\ &= b(a\frac{1}{a}) = b \end{aligned}$$

Contradiction! □

7.2.4 Lemma: A field doesn't has zero divisors

Since a field is a division ring, its nonzero elements are unit which is not zero divisor.

7.2.5 Thm: Every field is an integral domain

Theorem 49. *Every field is an integral domain.*

prove by previous lemma.

7.2.6 Thm: Every finite integral domain is a field

Theorem 50. *Every finite integral domain is a field.*

Proof. The only thing we need to show is that a typical element $a \neq 0$ has a multiplicative inverse. Consider $a, a^2, a^3, \ldots$ Since there are only finitely many elements we must have $a^m = a^n$ for some $m < n$.
Then $0 = a^m - a^n = a^m(1 - a^{n-m})$. Since there are no zero-divisors we must have $a^m \neq 0$ and hence $1 - a^{n-m} = 0$ and so $1 = aa^{n-m-1}$ and we have found a multiplicative inverse for a. □

7.2.7 Note: Finite Integral Domain ⊂ Field ⊂ Integral Domain

$\mathbb{Z}_p$ is a field.
$\mathbb{Z}$ is an integral domain but not a field.

7.3 The Characteristic of a Ring

7.3.1 Def: characteristic n is the least positive integer s.t. $n \cdot a = 0, \forall a \in R$

Definition 34. *If for a ring R a positive integer n exists such that $n \cdot a = 0$ for all $a \in R$, then the least such positive integer is the characteristic of the ring R. If no such positive integer exists, then R is of characteristic 0.*

Example 41. *The ring $\mathbb{Z}_n$ is of characteristic n, while $\mathbb{Z}, \mathbb{Q}, \mathbb{M}$, and $\mathbb{C}$ all have characteristic 0.*

7.3.2 Thm: In a ring with 1, characteristic $n \in \mathbb{Z}^+$ s.t. $n \cdot 1 = 0$

Theorem 51. *Let R be a ring with 1. If $n \cdot 1 \neq 0$ for all $n \in \mathbb{Z}^+$, then R has characteristic 0. If $n \cdot 1 = 0$ for some $n \in \mathbb{Z}^+$, then the smallest such integer n is the characteristic of R.*

8 The Ring $\mathbb{Z}_n$ (Fermat's and Euler's Theorems)

8.1 Fermat's Theorem

8.1.1 Thm: nonzero elements in $\mathbb{Z}_p$ (p is prime) form a group under multiplication

Theorem 52. *The nonzero elements in $\mathbb{Z}_p$ (p is prime) form a group under multiplication.*

Proof. $\mathbb{Z}_p$ is a finite field. ☐

8.1.2 Cor: (Little Theorem of Fermat) $a \in \mathbb{Z}$ and p is prime not dividing a, then $a^{p-1} \equiv 1$ mod p (p divides $a^{p-1} - 1$)

Corollary 10 (Little Theorem of Fermat)**.** *$a \in \mathbb{Z}$ and p is prime not dividing a, then $a^{p-1} \equiv 1 \mod p$ (p divides $a^{p-1} - 1$)*

Proof. Let $G_p = \{a \in \mathbb{Z}_p : a \neq 0\}$, by previous theorem, we know the G_p is a group under multiplication of size $|G_p| = p - 1$.
Then the order of a should divde $|G_p| = p - 1$, then

$$a^{p-1} = 1 \in G_p \Rightarrow a^{p-1} \equiv 1 \text{ mod } p$$

☐

8.1.3 Cor: (Little Theorem of Fermat) If $a \in \mathbb{Z}$, then $a^p \equiv a$ mod p for any prime p

8.2 Euler's Theorem

Euler's Theorem is more general form of Fermat's Theorem.

8.2.1 Thm: $G_n = \{a \in \mathbb{Z}_n : gcd(a, n) = 1\}$ forms a group under multiplication

Theorem 53. *The set G_n of nonzero elements of $\mathbb{Z}_n$ that are not zero divisors ($G_n = \{a \in \mathbb{Z}_n : gcd(a, n) = 1\}$) forms a group under multiplication modulo n.*

8.2.2 Def: Euler phi function $\phi(n) = |G_n|$, where $G_n = \{a \in \mathbb{Z}_n : gcd(a, n) = 1\}$

More generally, any $n \in \mathbb{Z}^+$, $a^{p-1} \equiv 1$ mod p. Then G_n is a group under mutiplication of size $|G_n| = \phi(n)$, we set $\phi(n)$ be the Euler phi function. E.g.

$$\phi(8) = \#\{a \in \mathbb{Z}_8 : gcd(a, 8) = 1\} = 4$$
$$\phi(15) = \#\{1, 2, 4, 7, 8, 11, 13, 14\} = 8$$

8.2.3 Thm: (Euler's Theorem) If $a \in \mathbb{Z}$, $n \geq 2$ s.t. $gcd(a, n) = 1$ then $a^{\phi(n)} \equiv 1$ mod n

Theorem 54. *If a is an integer relatively prime to n, then $a^{\phi(n)-1}$ is divisible by n, that is $a^{\phi(n)} \equiv 1 \mod n$.*

Proof. order of a should divide $|G_n| = \phi(n)$ then $a^{\phi(n)} = 1 \in G_n \Rightarrow a^{\phi(n)} \equiv 1$ mod n ☐

8.3 Application to $ax \equiv b \pmod m$

8.3.1 Thm: find solution of $ax \equiv b \pmod m$, $gcd(a, m) = 1$

Theorem 55. *$a, b \in \mathbb{Z}_m, gcd(a, m) = 1$, then $ax = b$ has a unique solution in $\mathbb{Z}_m$*

Proof. By Euler's Theorem, $a^{\phi(m)} \equiv 1 \bmod m$, which means a is a unit of $\mathbb{Z}_m$, there exists a unique $a^{-1} \in \mathbb{Z}_m$.
Mutiply $a^{-1} \in \mathbb{Z}_m$ on both side, we can get $x = a^{-1}b$ is the solution. □

8.3.2 Thm: $ax \equiv b \pmod m$, $d = gcd(a, m)$ has solutions if $d|b$, the number of solutions is d

Theorem 56. *Let m be a positive integer and let $a, b \in \mathbb{Z}_m$. Let $d = gcd(a, m)$. The equation $ax = b$ has a solution in $\mathbb{Z}_m$ if and only if d divides b. When d divides b, the equation has exactly d solutions in $\mathbb{Z}_m$.*

8.3.3 Cor: $ax \equiv b \pmod m$, $d = gcd(a, m)$, $d|b$, then solutions are $((\frac{a}{d})^{\phi(\frac{m}{d})-1}\frac{b}{d} + k\frac{m}{d}) + (m\mathbb{Z}), \quad k = 0, 1, ..., d-1$

Corollary 11. *Let $d = gcd(a, m)$. The congruence $ax \equiv b$ (mod m) has a solution if and only if d divides b. When this is the case, the solutions are the integers in exactly d distinct residue classes modulo m.*

Steps:

(1) let $a_1 = a/d$, $b_1 = b/d$, $m_1 = m/d$, solve

$$a_1 s \equiv b_1 \bmod m_1 \Rightarrow s = a_1^{-1} b_1$$

where $a_1^{-1} = a_1^{\phi(m_1)-1}$

(2) Solutions are

$$(s + km_1) + (m\mathbb{Z}), \quad k = 0, 1, ..., d-1$$

Example 42. *Find all solutions of* $12x \equiv 27$ *mod* 18

d=gcd(12,18)=6, $d \nmid 27 \Rightarrow$ no solutions.

Example 43. *Find all solutions of* $15x \equiv 27$ *mod* 18

d=gcd(15,18)=3, $a_1 = 5, b_1 = 9, m_1 = 6$. Then $s = a_1^{-1}b_1 = 5 \cdot 9 = 3$, then solutions are $3 + 18\mathbb{Z}$, $9 + 18\mathbb{Z}$, $15 + 18\mathbb{Z}$

9 Ring Homomorphisms and Factor Rings

9.1 Ring Homomorphism

9.1.1 Def: Ring Homomorphism: $\phi(a + b) = \phi(a) + \phi(b)$, $\phi(ab) = \phi(a)\phi(b)$

Definition 35. *Let R, R' be rings. A map $\phi : R \to R'$ is a ring homomorphism if*

$$\phi(a + b) = \phi(a) + \phi(b)$$

$$\phi(ab) = \phi(a)\phi(b)$$

9.1.2 Properties of Ring Homomorphism

1. $\phi(0) = 0'$.
2. $\phi(-a) = -\phi(a)$.
3. $S \subseteq R$ is a subring $\Rightarrow \phi(S) \subseteq R'$ is a subring.
4. $S' \subseteq R'$ is a subring $\Rightarrow \phi^{-1}(S') \subseteq R$ is a subring.
5. If $1 \in R$ is a unity of $R \Rightarrow \phi(1)$ is a unity of $\phi(R)$.

9.1.3 Def: kernel of ring homomorphism (the same as group homomorphism)

$$Ker(\phi) = \phi^{-1}[0'] = \{r \in R : \phi(r) = 0'\}$$

9.1.4 Thm: one-to-one map $\Leftrightarrow Ker(\phi) = \{0\}$

Similiarly, a ring homomorphism is one-to-one map if and only if $Ker(\phi) = \{0\}$.

9.2 Factor(Quotient) Rings

9.2.1 Thm: Homomorphism $\phi : R \to R/H$; Isomorphism: $\mu : R/H \to \phi(R)$

Theorem 57. *Let $\phi : R \to R'$ be a ring homomorphism and let $H = ker\phi$. Then R/H is a ring under the operation.*

$$(a + H) + (b + H) = (a + b) + H$$
$$(a + H)(b + H) = ab + H$$

Also, $\mu : R/H \to \phi[R]$ defined by $\mu(a + H) = \phi(a)$ is an isomorphism.

9.2.2 Def: $N < R$ is ideal $aN \subseteq N$ and $Nb \subseteq N \ \forall a, b \in R$

Definition 36. *An addive subgroup N of a ring R is an **ideal** if $aN \subseteq N$ and $Nb \subseteq N \ \forall a, b \in R$*

9.2.3 Thm: N is ideal $\Rightarrow R/N$ is a ring

Theorem 58. *Let N be an ideal of a ring R. R/N is a ring with operations*

$$(a + H) + (b + H) = (a + b) + H$$
$$(a + H)(b + H) = ab + H$$

*We call this ring R/N is the **factor ring of** R **by** N*

9.2.4 Fundamental Homomorphism Theorem

Theorem 59. *Let $\phi : R \to R'$ be a ring homomorphism with kernel N. Then*

1. *$\phi[R]$ is a ring.*
2. *$\mu : R/N \to \phi[R]$ given by $\mu(x + N) = \phi(x)$ is an isomorphism.*
3. *$\gamma : R \to R/N$ given by $\gamma(x) = x + N$ is a homomorphism.*
4. *$\phi(x) = \mu\gamma(x), \quad \forall x \in R$*

9.2.5 Thm: $I, J \subset R$ be $R-ideals$ and $I + J = R \Rightarrow R/_{I\cap J} \cong R/_I \times R/_J$

Theorem 60. *Let R be a commutative ring with $1 \neq 0$, and $I, J \subset R$ be $R-ideals$ such that $I + J = R$ (I and J are relatively prime). Then,*

$$R/_{I\cap J} \cong R/_I \times R/_J$$

Moreover, $IJ = I \cap J$ and $R/_{IJ} \cong R/_I \times R/_J$

Proof. Using that $I + J = R$ and $1 \in R$, we can write $1 = x + y$, $x \in I, y \in J$.
The natural map (direct product of two projections) $R \to R/I \times R/J$ is a ring homomorphism. $(r \to (r + I, r + J))$.
The ring $R/I \times R/J$ is generated by the element $(1 + I, J), (I, 1 + J)$:

$$(a + I, b + J) = a(1 + I, J) + b(I, 1 + J)$$

Let $x + y = 1, x \in I, y \in J$

$$x \to (x + I, x + J) = (I, 1 - y + J) = (I, 1 + J)$$
$$y \to (y + I, y + J) = (1 - x + I, J) = (1 + I, J)$$

Then $bx + ay = a(1 + I, J) + b(I, 1 + J)$. And $R \to R/I \times R/J$ is surjective.
We can prove that $I \cap J$ is the kernel of the ring $R/I \times R/J$:

$$r \to (r + I, r + J) \text{ maps } r \text{ to } (I, J) = 0 \in R/I \times R/J$$
$$\Leftrightarrow r \in I \text{ and } r \in J.$$
$$\Leftrightarrow r \in I \cap J.$$

Then, according to the FHT $R/_{I\cap J} \cong R/_I \times R/_J$ if $I + J = R$.
Moreover, we can prove $I + J = R \Rightarrow IJ = I \cap J$.

1. $(IJ \subset I \cap J)$: From the definition of ideal $IJ \subset I$ and $IJ \subset J \Rightarrow IJ \subset I \cap J$
2. $(I \cap J \subset IJ)$: Let $1 = x + y, x \in I, y \in J, r \in I \cap J$, then

$$r = r \cdot 1 = r(x + y) = rx + ry = xr + ry \in IJ$$

□

10 Prime and Maximal Ideals

Every nonzero ring R has at least two ideals, the **improper ideal** R and the **trivial ideal** $\{0\}$. For these ideals, the factor rings are R/R, which has only one element, and $R/\{0\}$, which is isomorphic to R. These are uninteresting cases. Let's consider **proper nontrivial ideal** $N \subset R$.

10.1 Thm: N is R-ideal has a unit $\Rightarrow N = R$

Theorem 61. *If R is a ring with 1, and N is an ideal of R containing a unit, then $N = R$.*

Proof. Since N is ideal, $rN \subseteq N, \forall r \in R$. $r^{-1} \in N \Rightarrow 1 \in N \Rightarrow r \cdot 1 \in N, \forall r \in R \Rightarrow N = R$ □

10.1.1 Cor: Ideal of field F is $\{0\}$ or F

Corollary 12. *A field F contains no proper nontrivial ideals, i.e., ideal is $\{0\}$ or F.*

Proof. Every nonzero element of field is unit. □

10.2 Def: Maximal ideal: no other ideal properly contains it

Definition 37. *A proper ideal $M \subsetneq R$ is called **maximal** if*

$$M \subseteq I \subseteq R \Rightarrow M = I \text{ or } I = R \text{ (for } R\text{-ideal } I).$$

i.e, there is no other ideal properly containing M.

10.2.1 Thm: R comm ring with 1, M maximal ideal $\Leftrightarrow$ R/M is a field

Theorem 62. *Let R be a commutative ring with $1 \neq 0$. Then M is a maximal ideal of R if and only if R/M is a field.*

Example 44. *Since $\mathbb{Z}/n\mathbb{Z}$ is isomorphic to $\mathbb{Z}_n$ and $\mathbb{Z}_n$ is a field if and only if n is prime. Then we see that maximal ideals are $p\mathbb{Z}$ where p is any positive prime.*

Example 45. *Let $R = \mathbb{Z}[x]$ has ideals $(2) = 2\mathbb{Z}[x] \subseteq R$, $(x) = x\mathbb{Z}[x] \subseteq R$, $(2, x) = 2\mathbb{Z}[x] + x\mathbb{Z}[x] \subseteq R$*

(1) *$R/(2) \cong \mathbb{Z}_2[x]$, $\mathbb{Z}_2[x]$ is not a field $\Rightarrow$ (2) is not maximal ideal.*

(2) *$R/(x) \cong \mathbb{Z}$, $\mathbb{Z}$ is not a field $\Rightarrow$ (x) is not maximal ideal.*

(3) *$R/(2, x) \cong \mathbb{Z}_2$, $\mathbb{Z}_2$ is a field $\Rightarrow$ $(2, x)$ is maximal ideal.*

10.3 Def: Prime ideal: $ab \in P \Rightarrow a \in P$ or $b \in P$

Definition 38. *An ideal $P \subsetneq R$ in a commutative ring R is a **prime** ideal if $ab \in P \Rightarrow a \in P$ or $b \in P$.*

Note: $\{0\}$ is a prime ideal in $\mathbb{Z}$, and indeed in any integral domain.

Example 46. *$\mathbb{Z} \times \{0\}$ is a prime ideal of $\mathbb{Z} \times \mathbb{Z}$, for if $(a, b)(c, d) \in \mathbb{Z} \times \{0\}$, then we must have $bd = 0$, then either $(a, b) \in \mathbb{Z} \times \{0\}$ or $(c, d) \in \mathbb{Z} \times \{0\}$*

10.3.1 Thm: N prime ideal $\Leftrightarrow$ R/N is an integral domain

Theorem 63. *Let R be a commutative ring with 1, and let $N \subsetneq R$ be an ideal in R. Then R/N is an integral domain if and only if N is a prime ideal in R.*

10.3.2 Cor: maximal ideal $\Rightarrow$ prime ideal

Corollary 13. *Every maximal ideal in a commutative ring R with 1 is a prime ideal.*

10.4 Relation Summary

$$\begin{array}{lcl}
I \text{ is maximal} & \Leftrightarrow & R/I \text{ is a field} \\
\Downarrow & & \Downarrow \\
I \text{ is prime} & \Leftrightarrow & R/I \text{ is an integral domain}
\end{array}$$

10.5 Thm: homomorphism $\phi : \mathbb{Z} \to R$, $\phi(n) = n \cdot 1$

Theorem 64. *If R is a ring with unity 1, then the map $\phi : \mathbb{Z} \to R$ given by*

$$\phi(n) = n \cdot 1$$

for $n \in \mathbb{Z}$ is a homomorphism of $\mathbb{Z}$ into R.

10.5.1 Cor: Ring R 1. characteristic $n > 1 \Rightarrow$ has subring isomorphic to $\mathbb{Z}_n$ 2. characteristic $0 \Rightarrow$ has subring isomorphic to $\mathbb{Z}$

Corollary 14. *If R is a ring with 1 and characteristic $n > 1$, then R contains a subring isomorphic to $\mathbb{Z}_n$. If R has characteristic 0, then R contains a subring isomorphic to $\mathbb{Z}$.*

Review: Characteristic n is the least positive integer s.t. $n \cdot a = 0, \forall a \in R$

10.5.2 Thm: Field F 1. prime characteristic $p \Rightarrow$ has subfield isomorphic to $\mathbb{Z}_p$ 2. characteristic $0 \Rightarrow$ has subfield isomorphic to $\mathbb{Q}$

Theorem 65. *A field F is either of prime characteristic p and contains a subfield isomorphic to $\mathbb{Z}_p$ or of characteristic 0 and contains a subfield isomorphic to $\mathbb{Q}$.*

Definition 39. *We define $\mathbb{Z}_p$ and $\mathbb{Q}$ are prime fields.*

10.6 Def: Pricipal ideal (of comm ring R) generated by a: $\langle a\rangle = \{ra | r \in R\}$

Definition 40. *If R is a commutative ring with 1 and $a \in R$, the ideal $\{ra | r \in R\}$ of all multiples of a is the* ***principal ideal generated by*** *a and is denoted by $\langle a\rangle$. An ideal N of R is a* ***principal ideal*** *if $N = \langle a\rangle$ for some $a \in R$.*

Example 47. *Every ideal of the ring $\mathbb{Z}$ is of the form $n\mathbb{Z}$, which is generated by n, so every ideal of $\mathbb{Z}$ is a principal ideal.*

Example 48. *The ideal $\langle x\rangle$ in $F[x]$ consists of all polynomials in $F[x]$ having zero constant term.*

10.6.1 Thm: field F, every ideal in $F[x]$ is principal

Theorem 66. *If F is a field, every ideal in $F[x]$ is principal.*

11 The Field of Quotients of an Integral Domain

Let D be an integral domain (a ring with 1 has no zero divisors) that we desire to enlarge to a field of quotients F. A coarse outline of the steps we take is as follows:

11.1 Step 1. Define what the elements of F are to be. (Define $S/\sim$)

D is the given integral domain, $S = \{(a, b) | a, b \in D, b \neq 0\} < D \times D$

11.1.1 Def: equivalent relation $(a, b) \sim (c, d) \Leftrightarrow ad = bc$

Definition 41. *Two elements (a, b) and (c, d) in S are equivalent, denoted by $(a, b) \sim (c, d)$, if and only if $ad = bc$.*
Note: we can image it as $\frac{a}{b} = \frac{c}{d}$, but don't use this form.

Lemma 4. *$\sim$ defines an equivalence relation on S.*

Proof. easy to prove (1) reflexive, (2) symmetric, (3) transitive. □

11.2 Step 2. Define the binary operations of addition and multiplication on $S/\sim$.

The relation $\sim$ can define a set of all equvalence classes on $[(a, b)], (a, b) \in S$, $S/\sim = \{[(a, b)] | (a, b) \in S\}$

11.2.1 lemma: well-defined operations $+, \times$

Lemma 5. *For $[(a,b)]$ and $[(c,d)]$ in $S/\sim$, the equations*

$$[(a,b)] + [(c,d)] = [(ad+bc, bd)]$$

and

$$[(a,b)][(c,d)] = [(ac, bd)]$$

give well-defined operations of addition and multiplication on $S/\sim$.

Proof. Assume $(a_1, b_1) \sim (a,b)$, $(c_1, d_1) \sim (c,d)$.
Verify $+ : (ad+bc, bd) \sim (a_1 d_1 + b_1 c_1, b_1 d_1)$

□

11.3 Step 3. Check all the field axioms to show that F is a field under these operations.

11.3.1 Thm: $S/\sim$ is a field with $+, \times$

Theorem 67. *With operation $+, \times$. $S/\sim$ is a field.*

Proof. Check all field axioms:

$$\begin{aligned} Associative :+ : \quad & \checkmark \qquad \times : \checkmark \\ Identity :+ : \quad & [(0,1)] \qquad \times : [(1,1)] \end{aligned}$$

$[(a,b)] + [(0,1)] = [(a,b)]$, $[(a,b)][(1,1)] = [(a,b)]$

$$Inverse :+ : \quad [(-a,b)] \qquad \times : [(b,a)], \forall a \neq 0$$

$[(a,b)] + [(-a,b)] = [(0,b^2)] = [(0,1)]$, where $(0,b^2) \sim (0,1) \Leftrightarrow 0 * 1 = b^2 * 0$;
$[(a,b)][(b,a)] = [(ab,ab)] = [(1,1)]$

$$\begin{aligned} Commucative :+ : \quad & \checkmark \qquad \times : \checkmark \\ \text{Distributive laws} : \quad & \checkmark \end{aligned}$$

□

11.4 Step 4. Show that F can be viewed as containing D as an integral subdomain.

11.4.1 Lem: $\phi(a) = [(a,1)]$ is an isomorphism between D and $\{[(a,1)] | a \in D\}$

Lemma 6. *The map $\phi : D \to F = S/\sim$ given by $\phi(a) = [(a,1)]$ is an* <u>*isomorphism*</u> *of D with a subring of $F (= S/\sim)$.*

Proof.

$$\begin{aligned} \phi(a+b) &= [(a+b, 1)] = [(a,1)] + [(b,1)] \\ \phi(ab) &= [(ab, 1)] = [(a,1)][(b,1)] \end{aligned}$$

Injective: assume $\phi(a) = \phi(b)$, then

$$[(a,1)] = [(b,1)] \Leftrightarrow (a,1) \sim (b,1) \Leftrightarrow a = b$$

Surjective: $\forall [(a,1)]$ is mapped from a
We prove that ϕ is an isomorphism between D and $\{[(a,1)] | a \in D\}$. □

11.4.2 Thm: every element of F can be expressed as a quotient of two elements of D: $[(a,b)] = \frac{\phi(a)}{\phi(b)}$

$\forall [(a,b)] \in F$,

$$[(a,b)] = [(a,1)][(1,b)] = \frac{[(a,1)]}{[(1,b)]^{-1}} = \frac{[(a,1)]}{[(b,1)]} = \frac{\phi(a)}{\phi(b)}$$

Theorem 68. *Any integral domain D can be enlarged to (or embedded in) a field $F = S/\sim$ such that every element of F can be expressed as a quotient of two elements of D. (Such a field F is a* ***field of quotients*** *of D.)*

12 Polynomials

12.1 Def: Polynomials

Let R be any field. A polynomial over R in variable x is a formal sum:

$$a_0 + a_1x + a_2x^2 + ... + a_nx^n = \sum_{i=0}^{n} a_ix^i$$

where $n \geq 0$ is an integer, $a_1, a_1, ..., a_n \in \mathbb{F}$.
Polynomial is a squence $\{a_k\}_{k=0}^{\infty}$ with $a_m = 0, \forall m > n$.
Remark: $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_dx^d$ If $a_d \neq 0$ and $a_i = 0, \forall i > d$, d is the degree of $f(x)$.

12.2 Rings of Polynomials

12.2.1 Thm: $R[x]$ is a ring under addition and multiplication

Theorem 69. *The set $R[x]$ of all polynomials in an indeterminate x with coefficients in a ring R is a ring under polynomial addition and multiplication.*
Note: If R is commutative, then so is $R[x]$, and if R has unity $1 \neq 0$. then 1 is also unity for $R[x]$.

Let $R[x]$ denote the set of all polynomials with coefficients in the ring R.

$$R[x] = \{\sum_{i=0}^{n} a_ix^i | n \geq 0, n \in \mathbb{Z}, a_0, ..., a_n \in R\}$$

We call the $R[x]$ *polynomial ring* over the ring R.

$$f = \sum_{i=0}^{n} a_ix^i, g = \sum_{j=0}^{n} a_jx^j \in R[x]$$

$$f + g = \sum_{i=0}^{n} (a_i + b_i)x^i \in R[x]$$

$$fg = (\sum_{i=0}^{n} a_ix^i)(\sum_{j=0}^{n} a_jx^j) = \sum_{i=0}^{2n} (\sum_{j=0}^{i} a_jb_{i-j})x^i$$

12.2.2 Def: evaluation homomorphism

Definition 42. *Let F be a field, and let $\alpha \in F$. Define an evaluation map. $EV_{x=\alpha} : F[x] \to F$, $\phi_\alpha(\sum_{i=0}^{\infty} a_ix^i) = \sum_{i=0}^{\infty} a_i\alpha^i$. Then,*

$$\phi_\alpha(f(x) + g(x)) = \phi_\alpha(f(x)) + \phi_\alpha(g(x))$$
$$\phi_\alpha(f(x)g(x)) = \phi_\alpha(f(x))\phi_\alpha(g(x))$$

ϕ_α is a ring homomorphism. We call it evaluation homomorphism.

Example 49. *Consider $EV_{x=2} : \mathbb{Q}[x] \to \mathbb{Q}$. $EV_{x=2}$ is a ring homomorphism. In particular it is a group homomorphism for addition.*

$$\phi_2 (a_0 + a_1 x + \cdots + a_n x^n) = a_0 + a_1 2 + \cdots + a_n 2^n$$

Note that

$$\phi_2 \left(x^2 + x - 6\right) = 2^2 + 2 - 6 = 0.$$

Thus $x^2 + x - 6$ is in the kernel N of ϕ_2. Of course,

$$x^2 + x - 6 = (x - 2)(x + 3),$$

and the reason that $\phi_2 \left(x^2 + x - 6\right) = 0$ is that $\phi_2(x - 2) = 2 - 2 = 0$.

Example 50. *Compute $EV_{x=4}(3x^{106} + 5x^{99} + 2x^{53}) \in \mathbb{Z}_7[x]$*

$$EV_{x=4}(3x^{106} + 5x^{99} + 2x^{53}) =$$

According to the little Theorem of Fermat, $x^6 \equiv 1$ mod 7.

$$= 3x^4 + 5x^3 + 2x^5 = 0 \in \mathbb{Z}_7$$

12.2.3 Def: α is zero if $EV_{x=\alpha}(f(x)) = 0$

Definition 43. *We say that α is a zero of $f(x)$ if $EV_{x=\alpha}(f(x)) = 0$.*

Example 51. *Find all zeros of $f(x) = x^3 + 2x + 2$ in $\mathbb{Z}_7$.*
Solve by checking all value $f(x), x = 0, 1, ..., 6 \Rightarrow$ zeros are $x = 2$, $x = 3$.

12.3 Degree of a Polynomial: $deg(f)$

$f = \sum_{i=0}^{n} a_i x^i$, $deg(f)$ = degree of f is,

$$deg(f) = \begin{cases} 0 & \text{if } f \text{ is constant, } f \neq 0 \\ n & \text{if } a_n \neq 0 \text{ in above } (a_n = \text{leading coefficent}) \\ -\infty & \text{if } f = 0 \end{cases}$$

Define $-\infty + a = a + (-\infty) = -\infty \ \forall a \in \mathbb{Z} \cup \{-\infty\}$

12.3.1 Lemma 2.3.3: $deg(fg) = deg(f) + deg(g)$, $deg(f + g) \leq \max\{deg(f), deg(g)\}$

Lemma 7 (Lemma 2.3.3)**.** *For any field $\mathbb{F}$ and f , $g \in \mathbb{F}[x]$,*

$$deg(fg) = deg(f) + deg(g)$$
$$deg(f + g) \leq \max\{deg(f), deg(g)\}$$

12.4 Corollary 2.3.5: Unit(invertible) in $\mathbb{F}[x]$: constant$\neq 0$ iff $deg(f) = 0$

Corollary 15 (Corollary 2.3.5)**.** *For any field $\mathbb{F}$ and $f \in \mathbb{F}[x]$, Then f is a unit(i.e. invertible) in $\mathbb{F}[x]$ iff $deg(f) = 0$.*

Proof.
Obviously, $deg(f) = 0 \Rightarrow f$ is a unit.
Suppose f is a unit, i.e. $\exists g \in \mathbb{F}[x]$ s.t. $fg = 1$.
$0 = deg(fg) = deg(f) + deg(g) \Rightarrow deg(f), deg(g) \geq 0 \Rightarrow deg(f) = 0, deg(g) = 0.$ ☐

12.5 Irreducible Polynomials:

A nonconstant polynomial f is irreducible if $f = uv,\ u, v \in \mathbb{F}[x]$, then either u or v is a unit(i.e., constant$\neq 0$)

12.6 Theorem 2.3.6: nonconstant polynomials can be reduced uniquely

Theorem 70 (Theorem 2.3.6). *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is any nonconstant. Then $f = ap_1p_2 \dots p_k$ where $a \in \mathbb{F},\ p_1, \dots p_k \in \mathbb{F}[x]$ are irreducible monic polynomials (monic = i.e. leading coeff. 1). If $f = bq_1q_2 \dots q_r$ with $b \in \mathbb{F}$ and $q_1, q_2, \dots, q_r \in \mathbb{F}[x]$ monic irreducible, then $a = b, k = r$, and after reindexing $p_i = q_i,\ \forall i$*

Lemma 8 (Lemma 2.3.7). *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is nonconstant monic polynomial. Then $f = p_1p_2 \dots p_k$ where each p_i is monic irreducible.*

Proof.
Prove it by induction. When $deg(f) = 1,\ f = uv,\ u, v \in \mathbb{F}[x]$, $deg(f) = deg(u) + deg(v) \Rightarrow$ one of these is 0.
Suppose the lemma holds for all degree$< n$. When $deg(f) = n$,
Either f is irreducible, done.
Suppose $f = uv$ with/ $deg(u), deg(v) \geq 1$
$\Rightarrow deg(u), deg(v) < n \Rightarrow u = p_1p_2 \dots p_k, v = q_1q_2 \dots q_j$ So, $f = p_1p_2 \dots p_kq_1q_2 \dots q_j$. ☐

Example 52. $x^2 - 1 \in \mathbb{Q}[x]$ *reducible*
$x - 1, x + 1 \in \mathbb{Q}[x]$ *irreducible*
$x^2 + 1 \in \mathbb{Q}[x]$ *irreducible*
$x^2 + 1 \in \mathbb{C}[x]$ *reducible*
$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$ *reducible*

12.7 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$, f divides g, $f|g$ means $\exists u \in \mathbb{F}[x]$ s.t. $g = fu$.

Proposition 26 (Proposition 2.3.8). *$f, h, g \in \mathbb{F}[x]$, then*

(i) If $f \neq 0, f|0$
(ii) If $f|1$, f is nonzero constant
(iii) If $f|g$ and $g|f$, then $f = cg$ for some $c \in \mathbb{F}$
(iv) If $f|g$ and $g|h$, then $f|h$
(v) If $f|g$ and $f|h$, then $f|(ug + vh)$ for all $u, v \in \mathbb{F}[x]$.

12.7.1 Greatest common divisor of f and g: is not unique, we denote monic Greatest common divisor as $gcd(f, g)$

If $f, g \in \mathbb{F}[x]$ are nonzero polynomials, a greatest common divisor of f and g is a polynomial $h \in \mathbb{F}[x]$ such that
(i) $h|f$ and $h|g$, and
(ii) if $k \in \mathbb{F}[x]$ and $k|f$ and $k|g$, then $k|h$.
the *gcd* is not unique, but the monic *gcd* is unique. We call it **the monic greatest common divisor**, denote it $gcd(f, g)$.

Example 53.

$$x^2-1, x^2-2x+1 \in \mathbb{Q}[x]$$
$$(x-1)(x+1), (x-1)^2 \in \mathbb{Q}[x]$$
$$x-1 = gcd(x^2-1, x^2-2x+1)$$

12.7.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

Proposition 27 (Proposition 2.3.9)**.** *Given $f, g \in \mathbb{F}[x]$, $g \neq 0$, then $\exists q, r \in \mathbb{F}[x]$ s.t. $deg(r) < deg(g)$ and $f = qg + r$*

Example 54.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$

12.7.3 Proposition 2.3.10:

Proposition 28 (Proposition 2.3.10)**.** *Any 2 nonzero polynomials $f, g \in \mathbb{F}[x]$ have a gcd in $\mathbb{F}[x]$. In fact among all polynomials in the set $M = \{uf + vg | u, v \in \mathbb{F}[x]\}$ any nonconstant of minimal degree are gcds.*

Proof.
$h \in M$, $deg(h) = d$ minimal. Let $k|f$ and $k|g \Rightarrow k|uf + vg, \ \forall u, v \Rightarrow k|h$.
Suppose $h' \in M$ is any nonzero element. $deg(h') \geq deg(h) \Rightarrow \ \exists q, r \in \mathbb{F}, deg(r) < deg(h) \ h' = qh + r$. $r = h' - qh \in M$. Since $deg(h) = d$ is nonconstant minimal degree, $r = 0 \Rightarrow h' = qh$. So $\exists q_1, q_2 \in \mathbb{F}[x], \ 1f + 0g = q_1 h, 0f + 1g = q_2 h \Rightarrow h|g, h|f$. □

Example 55.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$
$$g = (x+1)(x^2 - 3x + 2) + x - 1$$
$$x^2 - 3x + 2 = (x-2)(x-1)$$
$$\Rightarrow gcd(f, g) = x - 1$$
$$x - 1 = g - (x+1)(x^2 - 3x + 2) = g - (x+1)(f - 3g) = (3x+4)g - (x+1)f$$

Example 56. *Find a greatest common divisor of $f = x^3 - x^2 - x + 1$ and $g = x^2 - 3x + 2$ in $\mathbb{Q}[x]$, and express it in form $uf + vg, \ u, v \in \mathbb{Q}[x]$.*

$$f = (x+2)g + 3x - 3$$
$$g = \frac{1}{3}(x-2)(3x-3)$$
$$gcd(f, g) = 3x - 3$$
$$3x - 3 = f - (x+2)g$$

12.7.4 Proposition 2.3.12: $gcd(f, g) = 1, f|gh \Rightarrow f|h$

Proposition 29 (Proposition 2.3.12)**.** *If $f, g, h \in \mathbb{F}[x]$, $gcd(f, g) = 1$, and $f|gh$, then $f|h$.*

12.7.5 Corollary 2.3.13: irreducible f, $f|gh \Rightarrow f|g$ or $f|h$

Corollary 16 (Corollary 2.3.13)**.** *If $f \in \mathbb{F}[x]$ is irreducible, and $f|gh$, then $f|g$ or $f|h$.*

Since f is irreducible, we have two possible situations:
1. $gcd(f, g) = f$, i.e. $f|g$ done.
2. $gcd(f, g) = 1$, then according to Prop 2.3.12, we can know $f|h$.

12.8 Roots

Root:$\alpha \in \mathbb{F}$ is a root of f if $f(\alpha) = 0$.

12.8.1 Corollary 2.3.16(of Euclidean Algorithm): f can be divided into $(x-\alpha)q+f(\alpha)$i.e. if α is a root, then $(x - \alpha)|f$

Corollary 17 (Corollary 2.3.16(of Euclidean Algorithm))**.** *$\forall f \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$, there exists a polynomial $q \in \mathbb{F}[x]$ s.t. $f = (x - \alpha)q + f(\alpha)$. In particular , if α is a root, then $(x - \alpha)|f$.*

12.9 Multiplicity

If α is a root of f , say its *multiplicity* is m, if $x - \alpha$ appears m times in irreducible factorization.

12.9.1 *Sum of multiplicity* $\leq deg(f)$

Proposition 30 (Proposition 2.3.17)**.** *Given a nonconstant polynomial $f \in \mathbb{F}[x]$, the number of roots of f, counted with multiplicity, is at most $deg(f)$.*

12.10 Roots in a filed may not in its subfield

Note if $\mathbb{F} \subset \mathbb{K}$, then $\mathbb{F}[x] \subset \mathbb{K}$. $f \in \mathbb{F}[x]$ may have no roots in $\mathbb{F}$, but could have roots in $\mathbb{K}$

Example 57. *$x^n - 1 \in \mathbb{Q}[x]$ has a root in $\mathbb{Q}$: 1; has 2 roots if n even: ± 1*
roots in $\mathbb{C}$: $\zeta_n = e^{\frac{2\pi i}{n}}$, then $\zeta_n^n = e^{2\pi i} = 1$; $(\zeta_n^k)^n = e^{2\pi k i} = 1$ So, the roots: $\{e^{\frac{2\pi k i}{n}} | k = 0, ..., n-1\}$
The roots of $x^n - d$: $\{e^{\frac{2\pi k i}{n}} \sqrt{d} | k = 0, ..., n-1\}$

13 Linear Algebra

13.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$)

A vector space over a field $\mathbb{F}$ is a set V w/ an operation addition $+ : V \times V \to V$ and an operation scalar multiplication $\mathbb{F} \times V \to V$
(1) Addition is associative & commutative
(2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$
(3) $1v = v \forall v \in V$(where $1 \in \mathbb{F}$ is multi. id. in $\mathbb{F}$)
(4) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ \alpha(\beta v) = (\alpha\beta)v$
(5) $\forall v \in V,\ (-1)v = -v$ we have $v + (-v) = 0$
(6) $\forall \alpha \in \mathbb{F},\ v, u \in V,\ \alpha(v + u) = \alpha v + \alpha u$
(7) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ (\alpha + \beta)v = \alpha v + \beta v$

13.1.1 A field is a vector space over its subfield

Example 58. $\mathbb{K} \subset \mathbb{F}$ *is a subfield of a field* $\mathbb{F}$*. Then* $\mathbb{F}$ *is a vector space over* $\mathbb{K}$*. (Since* $\mathbb{F} \subset \mathbb{F}[x]$*, then* $\mathbb{F}[x]$ *is a vector space over* $\mathbb{F}$*.)*

13.1.2 Vector subspace

Suppose that V is a vector space over $\mathbb{F}$. A vector subspace or just subspace is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W,\ av \in W,\ \forall v, w \in W,\ a \in \mathbb{F}$.

Example 59. $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$*, then* $\mathbb{L}$ *is a subspace of* $\mathbb{F}$ *over* $\mathbb{K}$*.*

13.2 Linear independent, Linear combination

13.3 span V, basis, dimension, Proposition 2.4.10

A set of elements $v_1, ..., v_n \in V$ is said to **span** V if every vector $v \in V$ can be expressed as a linear combination of $v_1, ..., v_n$. If $v_1, ..., v_n$ spans and is linearly independent, then we call the set a ***basis*** for V.

Proposition 31 (Proposition 2.4.10.)**.** *Suppose* V *is a vector space over a field* $\mathbb{F}$ *having a basis* $\{v_1, ..., v_n\}$ *with* $n \geq 1$*.*

(i) For all $v \in V$, $v = a_1v_1 + ... + a_nv_n$ for exactly one $(a_1, ..., a_n) \in \mathbb{F}^n$.
(ii) If $w_1, ..., w_n$ span V , then they are linearly independent.
(iii)If $w_1, ..., w_n$ are linearly independent, then they span V.
If a vector space V over $\mathbb{F}$ has a basis with n vectors, then V is said to be n-dimensional (over $\mathbb{F}$) or is said to have **dimension** n.

13.3.1 Standard basis vectors

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1) \in \mathbb{F}^n$$

are a basis for $\mathbb{F}^n$ called the **standard basis vectors**.

13.4 Linear transformation

Given two vector spaces V and W over $\mathbb{F}$ a **linear transformation** is a function $T : V \to W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$, we have

$$T(av) = aT(v) \text{ and } T(v + w) = T(v) + T(w)$$

Proposition 32 (Proposition 2.4.15.)**.** *If* V *and* W *are vector spaces and* $v_1, ..., v_n$ *is a basis for* V *then any function from* $\{v_1, ..., v_n\} \to W$ *extends uniquely to a linear transformation* $V \to W$*.*

Any $v \in V$, $\exists(a_1, ..., a_n)$ s.t. $v = a_1v_1 + ... + a_nv_n$. Then $T(v) = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$

13.4.1 Corollary 2.4.16: *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$

Corollary 18 (Corollary 2.4.16.). *If $v_1, ..., v_n$ is a basis for a vector space V and $w_1, ..., w_n$ is a basis for a vector space W (both over $\mathbb{F}$), then any linear transformation $T : V \to W$ determines (and is determined by) the $m \times n$ matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & ... & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}^T = A \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from V to W; $M_{m\times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in $\mathbb{F}$. $T \to A(T)$ defines a *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** T.

13.4.2 Proposition 2.4.19:

Proposition 33 (Proposition 2.4.19). *Suppose that V , W, and U are vector spaces over $\mathbb{F}$, with fixed chosen bases. If $T : V \to W$ and $S : W \to U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \to U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

13.5 GL(V): invertible(bijective) linear transformations $V \to V$

Given a vector space V over F, we let $GL(V) \subset \mathcal{L}(V, V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \textit{ is a bijection}\} = \mathcal{L}(V, V) \cap Sym(V)$$

14 Euclidean geometry basics

14.1 Euclidean distance, inner product

Euclidean distance on $\mathbb{R}^n$:

$$|x - y| = \sqrt{(x_1 - y_1)^2 + ... + (x_n - y_n)^2}$$

Euclidean inner product:

$$x \cdot y = x_1y_1 + \cdots + x_ny_n = x^T y$$

14.2 Isometry of $\mathbb{R}^n$: a bijection $\mathbb{R}^n \to \mathbb{R}^n$ preserves distance

An **isometry** of $\mathbb{R}^n$ is a bijection $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x - y|, \ \forall x, y \in \mathbb{R}^n$$

14.2.1 $Isom(\mathbb{R}^n)$: set of all isometries of $\mathbb{R}^n$

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of $\mathbb{R}^n$,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n | |\Phi(x) - \Phi(y)| = |x - y|, \ \forall x, y \in \mathbb{R}^n\}$$

14.2.2 $Isom(\mathbb{R}^n)$ is closed under $\circ$ and inverse

Proposition 34. *$\Phi, \Psi \in Isom(\mathbb{R}^n)$, then $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$*

Proof.
Since Φ, Ψ are bijections, so is $\Phi \circ \Psi$. Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x - y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x - y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

□

14.3 $A \in GL(n, \mathbb{R})$, $T_A(v) = Av$: $A^tA = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a *invertible linear transofrmations* $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t(Aw) = v^tA^tAw$$

$$A^tA = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)} \; T_A \in Isom(\mathbb{R}^n)$$

14.4 Linear isometries i.e. orthogonal group $O(n) = \{A \in GL(n, \mathbb{R}) | A^tA = I\}$

We define the all isometries in *invertible linear transofrmations* $\mathbb{R}^n \to \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n, \mathbb{R}) | A^tA = I\} \subset GL(n, \mathbb{R})$$

14.4.1 Special orthogonal group $SO(n) = \{A \in O(n) | det(A) = 1\}$: orthogonal group with $det(A) = 1$

$O(n)$ are the matrices representing linear isometries of $\mathbb{R}^n$. $1 = det(I) = det(A^tA) = det(A^t)det(A) = det(A)^2 \Rightarrow det(A) = 1$ or $det(A) = -1$. We use **special orthogonal group** represents A with $det(A) = 1$,

$$SO(n) = \{A \in O(n) | det(A) = 1\}$$

14.5 translation: $\tau_v(x) = x + v$

Define a *translation* by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \to \mathbb{R}^n, \; \tau_v(x) = x + v$$

14.5.1 translation is an isometry

Note 5 (Exercise 2.5.3). *$\forall v \in \mathbb{R}^n, \tau_v$ is an isometry.*

Proof. $|\tau_v(x) - \tau_v(y)| = |(x + v) - (y + v)| = |x - y|$ □

14.6 The composition of a translation and an orthogonal transformation is an isometry $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since *the composition of isometries is an isometry,* $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries**.

14.6.1 Theorem 2.5.3: All isometries can be represented by a composition of *a translation* and *an orthogonal transformation*, $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

Theorem 71 (Theorem 2.5.3)**.** $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

15 Complex numbers

$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$, $\mathbb{R} = \{a + 0i | a \in \mathbb{R}\} \subset \mathbb{C}$
Addition & multiplication

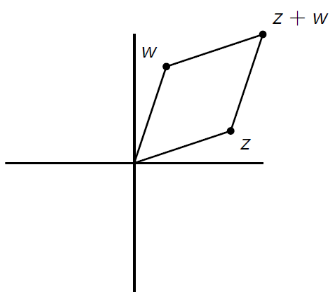
$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi)(c + di) &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

<u>**Complex conjugation**</u>: $z = a + bi, \bar{z} = a - bi, \overline{zw} = \bar{z}\bar{w}$
<u>**Absolute value**</u>: $|z| = \sqrt{a^2 + b^2}$, $|z|^2 = z\bar{z}$
<u>**Additive inverse**</u>: $-z = -a - bi$
<u>**Multiplicative inverse**</u>: $z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

$$z \in \mathbb{C}, \overline{z + \bar{z}} = \bar{z} + \bar{\bar{z}} = z + \bar{z}$$
$$\textit{Real part: } Re(z) = \frac{z + \bar{z}}{2}$$
$$\textit{Imaginary part: } Im(z) = \frac{z - \bar{z}}{2i}$$

15.1 Geometric Meaning of Addition and Multiplication

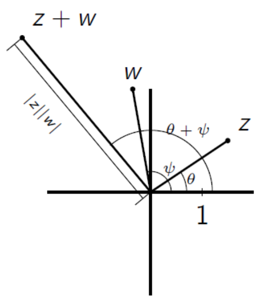
<u>Addition</u>: parallelogram law



<u>Multiplication</u>:

$$\begin{aligned}z &= a + bi \neq 0 \\ &= r\cos\theta + r\sin\theta i \\ &= r(\cos\theta + i\sin\theta) \\ |z|^2 &= a^2 + b^2 = r^2\end{aligned}$$

$$
\begin{aligned}
z =& r(\cos\theta + i\sin\theta)\\
w =& s(\cos\phi + i\sin\phi)\\
zw =& rs[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)]\\
=& rs[cos(\theta+\phi) + i\sin(\theta+\phi)]\\
=& |z||w|[cos(\theta+\phi) + i\sin(\theta+\phi)]
\end{aligned}
$$



We will write,

$$
\begin{aligned}
&\cos\theta + i\sin\theta = e^{i\theta}\\
&e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)}\\
&z = |z|e^{i\theta}
\end{aligned}
$$

15.2 Theorem 2.1.1: $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists\alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$

Theorem 72 (Theorem 2.1.1)**.** *Supose a nonconstant polynomial $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists\alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$.*

15.2.1 Corollary 2.1.2: $f(x) = a_n\prod_{i=1}^{n}(x-k_i) = a_n(x-k_1)(x-k_2)...(x-k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$

Corollary 19 (Corollary 2.1.2)**.** *Every nonconstant polynomial with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$ can be factored as $f(x) = a_n\prod_{i=1}^{n}(x-k_i) = a_n(x-k_1)(x-k_2)...(x-k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$.*

15.2.2 Corollary 2.1.3: $a_i \in \mathbb{R}$, f can be expresses as a product of linear and quadratic polynomials

Corollary 20 (Corollary 2.1.3)**.** *If $f(x) = a_0 + a_1x + ... + a_nx^n$ is a nonconstant polynomial $a_0, a_1, ..., a_n \in \mathbb{R}, a_n \neq 0$. Then f can be expresses as a product of linear and quadratic polynomials.*

$a_0, a_1, ..., a_n$ is real number here!

Proof.
(1)Obviously, the corollary holds at $n = 1$ and $n = 2$.
(2)Suppose the corollary holds for all situations that $n < k$.
When $n = k$, $f(x) = a_0 + a_1x + ... + a_kx^k, a_k \neq 0$.
By F.T.A., f has a root α in $\mathbb{C}$.
If $\alpha \in \mathbb{R}$, long division $f(x) = q(x)(x-\alpha)$. q has real coefficients, *degree of* $q = k-1$. Since the corollary holds at $n = k-1$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds

at $n = k$.
If $\alpha \notin \mathbb{R}$

$$0 = f(\alpha) = a_0 + a_1\alpha + ... + a_k\alpha^k$$
$$0 = \overline{f(\alpha)} = a_0 + a_1\bar{\alpha} + ... + a_n\bar{\alpha}^n = f(\bar{\alpha})$$

Since $\bar{\alpha} \neq \alpha$, $(x-\alpha)(x-\bar{\alpha})|f$.
$(x-\alpha)(x-\bar{\alpha}) = x^2 - (\alpha+\bar{\alpha})x + |\alpha|^2$ is a polynomial with coefficients in $\mathbb{R}$. So $f(x) = q(x)(x^2 - (\alpha+\bar{\alpha})x + |\alpha|^2)$, q has real coefficients with degree $k-2$. The corollary also holds at $n = k-2$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.

$\square$

References

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[2] Fraleigh, J. B. (2003). A first course in abstract algebra. Pearson Education India.