Let $* : S \times S \to S$

$*' : S' \times S' \to S'$

$(S, *)$ and $(S', *')$ are said to be <u>isomophic</u> if there exists a bijection $\phi : S \to S'$ s.t.

$$\phi(x * y) = \phi(x) *' \phi(y). \quad \forall x, y \in S.$$

$$\begin{tikzcd} S \times S \arrow[r, "*"] \arrow[d, "{(\phi,\phi)}"'] & S \arrow[d, "\phi"] \\ S' \times S' \arrow[r, "*'"] & S' \end{tikzcd}$$

$(x, y) \to x * y \to \underline{\phi(x * y)}.$

$=$

$(x, y) \to (\phi(x), \phi(y)) \to \underline{\phi(x) *' \phi(y)}$

Example: $(\mathbb{Z}_2, +)$

	0	1
0	0	1
1	1	0

$(\{-1, 1\}, \times)$

	-1	1
-1	1	-1
1	-1	1

$\phi : 0 \to 1$

$\quad 1 \to -1.$

Let $\phi : G \to G'$ be an injective map s.t.

$$\phi(xy) = \phi(x)\phi(y) \quad \forall x, y \in G.$$

Then the image $\phi(G) = \{\phi(x) : x \in G\}.$

is a subgroup of G' that is isomophiz to G.

Proof: ① closed

② identity

(3) inverse.

(1) Let $a=\phi(x),\ b=\phi(y) \in \phi(G)$

then $ab=\phi(x)\phi(y)=\phi(xy) \in \phi(G)$.

(2) Claim that $\phi(e)\in\phi(G)$ is an identity for $\phi(G)$.

$$\phi(e)\cdot\phi(x)=\phi(e\cdot x)=\phi(x)\in\phi(G)$$

(3). claim $\phi(x^{-1})$ is an inverse in $\phi(G)$ for $\phi(x)$.

$\phi(x^{-1})\phi(x)=\phi(e)$

$\phi(x)\phi(x^{-1})=\phi(e)$.

$\begin{cases}\phi: G\to G' \text{ injective}\\ \phi(xy)=\phi(x)\phi(y)\ \forall x,y\in G\end{cases} \Rightarrow \phi(G)\leq G'$

$\phi: G\to\phi(G)$ is a bijection, it is an isomorphic.

Theorem (Cayley's Theorem).

Let G be a group and let $G'=S_G$ the group of all permutation of G.

$$S_G=\{\text{bijection } \sigma: G\to G\}.$$

Then G is isomorphic to a subgroup of S_G.

Proof: $\phi: G\to S_G$

$g \to ?$

Let $\phi(g)=\lambda_g$ $\quad\quad \lambda_g: x\to gx$.

Observe that indeed.

$\lambda_g \in S_G$ (λ_g is a permutation of G).

$\lambda_g : G \to G$. <u>is injective</u> $\lambda_g(x) = \lambda_g(y)$.

Cancellation $\Leftrightarrow$ $gx = gy$

$\Leftrightarrow$ $x = y$.

λ_g <u>is surjective</u>. Let $y \in G$.

$$\lambda_g(x) = y$$

$$\Leftrightarrow \quad gx = y$$

$$\Leftrightarrow \quad x = g^{-1}y.$$

$\phi(G) \leq S_G$, we apply the previous theorem.

we need $\begin{cases} \phi \text{ is injective} \\ \phi(x)\cdot\phi(y) = \phi(xy) \quad \forall x,y \in G \end{cases}$

$$\phi(g) = \phi(h)$$

$$\Leftrightarrow \lambda_g = \lambda_h$$

$$\Leftrightarrow \lambda_g(x) = \lambda_h(x) \quad \forall x \in G.$$

$$\Leftrightarrow gx = hx$$

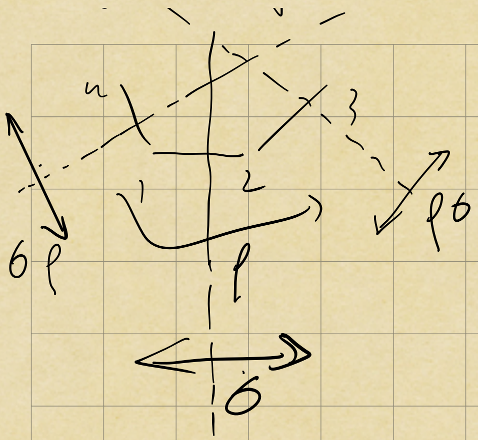
$$\Leftrightarrow g = h. \qquad \Rightarrow \phi \text{ injective}$$

$$\phi(x)\,\phi(y) = \lambda_x \circ \lambda_y$$

$$(\lambda_x \circ \lambda_y)(z) = \lambda_x(yz) = xyz. \quad \forall z \in G.$$

$$= \lambda_{xy}(z) \quad \Rightarrow \phi(x)\phi(y) = \phi(xy).$$

A regular n-gon has $2n$ symmetries group that together form a group called the dih. group D_{2n}.



1	ρ	ρ^2	σ	$\sigma\rho$	$\sigma\rho^2$
ρ	ρ^2	1	$\sigma\rho^2$	σ	$\sigma\rho$
ρ^2	1	ρ	$\sigma\rho$	$\sigma\rho^2$	σ
σ	$\sigma\rho$	$\sigma\rho^2$	1	ρ	ρ^2
$\sigma\rho^2$	σ	$\sigma\rho$	ρ	ρ^2	1
$\sigma\rho$	$\sigma\rho^2$	σ	ρ^2	1	ρ

$D_{2n} \le S_n.$