MATH 417 Lec06-15

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2021

目录

1 Integers

1.1 Proposition 1.4.1: Properties of integers $\mathbb{Z}$

Proposition 1 (Proposition 1.4.1.). *The following hold in the integers $\mathbb{Z}$:*

(i) *Addition* and *multiplication* are *commutative* and *associative* operations in $\mathbb{Z}$.
(ii) $0 \in \mathbb{Z}$ is an identity element for addition; that is, $\forall a \in \mathbb{Z}, 0 + a = a$.
(iii) Every $a \in \mathbb{Z}$ has an additive inverse, denoted $-a$ and given by $-a = (-1)a$, satisfying $a + (-a) = 0$.
(iv) $1 \in \mathbb{Z}$ is an identity element for multiplication; that is, for all $a \in \mathbb{Z}$, $1a = a$.
(v) The *distributive* law holds: $\forall a, b, c \in \mathbb{Z}, a(b + c) = ab + ac$.
(vi) Both $\mathbb{N} = \{x \in Z | x \geq 0\}$ and $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x > 0\}$ are *closed* under *addition* and *multiplication.* That is, if x and y are in one of these sets, then $x + y$ and xy are also in that set.
(vii) For any two nonzero integers $a, b \in \mathbb{Z}$, $|ab| \geq \max\{|a|, |b|\}$. Strict inequality holds if $|a| > 1$ and $|b| > 1$.

From this we get cancellation.

$$ab = ac \Rightarrow b = c \ or \ a = 0$$

1.2 Definition: Divide

Suppose $a, b \in \mathbb{Z}, b \neq 0$, <u>b divides a</u> if $\exists m \in \mathbb{Z}$, so that $a = bm, b|a$. Otherwise, write $b \nmid a$.

1.3 Proposition 1.4.2: properties of integer division

Proposition 2 (Proposition 1.4.2). $\forall a, b \in \mathbb{Z}$

(i) if $a \neq 0$, then $a|0$
(ii) if $a|1$, then $a = \pm 1$
(iii) if $a|b$ & $b|a$, then $a = \pm b$
(iv) if $a|b$ & $b|c$, then $a|c$
(v) if $a|b$ & $a|c$, then $a|(mc + nb) \forall m, n \in \mathbb{Z}$

1.4 Definitions: Prime, The Greatest common divisor $gcd(a, b)$

$p > 1, p \in \mathbb{Z}$ is called <u>*prime*</u> if the only divisors are $\pm 1, \pm p$.
Given $a, b \in \mathbb{Z}, a, b \neq 0$, the <u>*greatest common divisor*</u> of a and b is $c \in \mathbb{Z}, c > 0$ s.t.
(1) $c|a$ and $c|b$; (2) if $d|a, d|b$, then $d|c$
The c is unique, we write it $gcd(a, b)$.

1.5 Euclidean Algorithm

Proposition 3 (Proposition 1.4.7(Euclidean Algorithm)). *Given* $a, b \in \mathbb{Z}, b \neq 0$*, then* $\exists q, r \in \mathbb{Z}$ *s.t.* $a = qb + r, 0 \leq r \leq |b|$.

Example 1 (Exercise 1.4.3). *For the pair* $(a, b) = (130, 95)$*, find* $gcd(a, b)$ *using the Euclidean Algorithm and express it in the form* $gcd(a, b) = sa + tb$ *for* $s, t \in Z$*.*

$$\begin{aligned} &130 = 95 + 35; \quad 95 = 2 \times 35 + 25 \\ &35 = 25 + 10; \qquad 25 = 2 \times 10 + 5 \\ &10 = 2 \times 5 + 0 \end{aligned}$$

$$\begin{aligned} 5 &= 25 - 2 \times 10 = 25 - 2 \times (35 - 25) = 3 \times 25 - 2 \times 35 = 3 \times (95 - 2 \times 35) - 2 \times 35 \\ &= 3 \times 95 - 8 \times 35 = 3 \times 95 - 8 \times (130 - 95) = 11 \times 95 - 8 \times 130 \end{aligned}$$

$$gcd(130, 95) = gcd(95, 35) = gcd(35, 25) = gcd(25, 10) = gcd(10, 5) = gcd(5, 0) = 5$$

We can also express it by matrix

	q	r	s	t
-1		130	1	0
0	1	95	0	1
1	2	35	1	-1
2	1	25	-2	3
3	2	10	3	-4
4	2	5	-8	11

Hence $gcd(130, 95) = 5 = -8 \cdot 130 + 11 \cdot 95$

1.6 Proposition: $gcd(a, b)$ exists and is the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$

Theorem 1. $d = gcd(a, b)$ *is of the form* $sa + tb$

证明. We may assume $0 \leq a \leq b$
For $a = 0$, $d = b = 0 \cdot a + 1 \cdot b$.
For $a > 0$, let $b = q \cdot a + r$ with $0 \leq r < a \leq b$. Then

$$\begin{aligned} \{sa + tb : s, t \in \mathbb{Z}\} &= \{sa + t(q \cdot a + r) : s, t \in \mathbb{Z}\} = \{tr + ua : t, u \in \mathbb{Z}\} \\ &= \ldots \{x \cdot 0 + y \cdot d : x, y \in \mathbb{Z}\} = \{..., -2d, -d, 0, d, 2d, ...\} \end{aligned}$$

□

Proposition 4 (第二种表示，第二种证明). $\forall a, b \in \mathbb{Z}$*, not both 0,* $gcd(a, b)$ *exists and is the smallest positive integer in the set* $M = \{ma + nb | m, n \in \mathbb{Z}\}$*. i.e.* $\exists m_0, n_0 \in \mathbb{Z}$ *s.t.* $gcd(a, b) = m_0 a + n_0 b$*.*

证明. Let c be the smallest positive integer in the set $M = \{ma + nb | m, n \in \mathbb{Z}\}$. $c = m_0 a + n_0 b > 0$. Let $d = ma + nb \in M$, $d = qc + r$ where $0 \leq r < c$ (by Euclidean Algorithm).

$$r = d - qc = (m - qm_0)a + (n - qn_0)b \in M$$

Since c is the smallest integer in M and $r \in [0, c)$, so $r = 0$. $\Rightarrow d = qc$. So $c|d$.
$a = 1a + 0b \in M \Rightarrow c|a$, $b = 0a + 1b \in M \Rightarrow c|b$.
If $t|a, t|b$ then $t|m_0 a + n_0 b$ i.e. $t|c$. $\Rightarrow c = gcd(a, b)$. □

1.7 Well-Ordering Principle (Least Integer Axiom)

There is a smallest integer in every nonempty subset S of the natural numbers $\mathbb{N} = \{0, 1, 2, ...\}$

1.8 Proposition 1.4.10: $gcd(b, c)$, $b|ac \Rightarrow b|a$

Proposition 5 (Proposition 1.4.10)**.** *Suppose $a, b, c \in \mathbb{Z}$. If b, c are relatively prime i.e. $gcd(b, c) = 1$ and $b|ac$, then $b|a$.*

证明. $gcd(b, c) = 1 \Rightarrow \exists m, n \in \mathbb{Z}$ s.t. $1 = mb + nc \Rightarrow a = amb + anc$. Since $b|nac, b|amb \Rightarrow b|a$. □

1.8.1 Corollary: $p|ab \Rightarrow p|a$ or $p|b$

Corollary 1 (Corollary of Prop 1.4.10)**.** *$a, b, p \in \mathbb{Z}, p > 1$ prime. If $p|ab$, then $p|a$ or $p|b$.*

证明. If $p|b$, done. Otherwise, $gcd(p, b) = 1$. By Prop 1.4.10, $p|a$. □

1.9 Fundamental Theorem of Arithmetic: Any integer $a \geq 2$ has a unique prime factorization

1.9.1 Existence

Lemma 1. *Any integer $a \geq 2$ is either a prime or a product of primes.*

证明. Set $S \subset \mathbb{N}$ be the set of all n without the given property.
Assume that S in nonempty and m is the least element in S.
Since m is not a prime, it can be written as $m = ab$ with $1 < a, b < m$. Since m is the least element in S, $a, b \notin S$. Then m is a product of primes. Contradiction. Thus, $S = \emptyset$. □

1.9.2 Uniqueness

Theorem 2 (Fundamental Theorem of Arithmetic)**.**

Any integer $a > 1$ has a unique prime factorization: $a = p_1^{k_1} \cdot p_2^{k_2} \cdot ... p_n^{k_n}$ where $p_i > 1$ is prime, $k_i \in \mathbb{Z}_+, \forall i = 1, ..., n, p_i \neq p_j, \forall i \neq j$.

证明.

a) Existence: (Previous Lemma)

b) Uniqueness:

1) Method 1:

Suppose $a = p_1^{n_1} \cdot p_2^{n_2} \cdot ...p_k^{n_k} = q_1^{r_1} \cdot q_2^{r_2} \cdot ...q_j^{r_j}$. Where $p_1 > p_2 > ... > p_k, q_1 > q_2 > ... > q_j, n_i, r_i \geq 1$.

$p_1|a \Rightarrow \exists\ q_i\ s.t.\ p_1|q_i$. Similarly, $\exists\ q_i\ s.t.\ q_1|p_{i'}$.

$q_1 \leq p_{i'} \leq p_1 \leq q_i \Rightarrow q_1 = p_{i'} = p_1 = q_i$

We can also know $n_1 = r_1$, otherwise we would have two prime factorization of the quotient where the largest primes are different by dividing $p_1^{\min\{n_1, r_1\}}$.

Then we can get $b = p_2^{n_2} \cdot ...p_k^{n_k} = q_2^{r_2} \cdot ...q_j^{r_j}$. Then prove it by induction.

2) Method 2:

Suppose $a = p_1 \cdot p_2 \cdot ...p_k = q_1 \cdot q_2 \cdot ...q_t$. For a p_i, there must exist a q_j s.t. $p_i = q_j$:

Assume that $p_i \neq q_t$, $gcd(p_i, q_t) = 1$. Then $\exists a, b$ such that $1 = ap_i + bq_t$. Multiplying both sides by $q_1 \cdot q_2 \cdot ...q_{t-1}$:

$$q_1 \cdot q_2 \cdot ...q_{t-1} = ap_i q_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t$$

Since $p_i | q_1 \cdot q_2 \cdot ...q_t$, we can conclude that $p_i | (ap_i q_1 \cdot q_2 \cdot ...q_{t-1} + bq_1 \cdot q_2 \cdot ...q_t)$

$$\text{i.e. } p_i | q_1 \cdot q_2 \cdot ...q_{t-1} \text{ if } p_i \neq q_t$$

Then prove by induction.

□

2 Modular arithmetic

2.1 Congruences

2.1.1 Congruent modulo m: $a \equiv b$ mod m

Given $m \in \mathbb{Z}_+$, define a relation on $\mathbb{Z}$: **congruence modulo** m

$$a \equiv b \text{ mod } m, \text{ if } m|(a - b)$$

Read as "a is congruent to b mod n"; Notation: $a \equiv b$ mod m.

Equivalent to: a, b have the same remainder after division by m.

2.1.2 Proposition: For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" is an equivalence relation

Proposition 6 (Proposition 1.5.1)**.** *For fixed $m \geq 2$, the relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" is an equivalence relation*

证明.

1) Reflexive: $\forall a \in \mathbb{Z}, m|0 = (a-a)$, so $a \equiv a$ mod m i.e. $a \sim a$.

2) Symmetric: $\forall a, b \in \mathbb{Z}$, $a \equiv b$ mod m, then $m|(a-b) \Rightarrow m|(b-a) \Rightarrow b \equiv a$ mod m. i.e. $a \sim b \Rightarrow b \sim a$.

3) Transitive: $\forall a, b, c \in \mathbb{Z}$, $a \equiv b$ mod m, $b \equiv c$ mod m. Then $m|(a-b), m|(b-c) \Rightarrow m|(a-b) + (b-c) = (a-c) \Rightarrow a \equiv c$ mod m.

□

2.1.3 Theorem: the equivalence relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$

Theorem 3. *the equivalence relation "$a \sim b \Leftrightarrow a \equiv b$ mod m" partitions the integers into m disjoint sets $\Omega_i = \{a|a \sim i\}, i = 0, 1, ..., m-1$*

证明. Prove any $a \in \mathbb{Z}$ belongs to a unique Ω_i.

a) Existence: Division Algorithm $\Rightarrow a = qm + r$, $0 \leq r < m$. $a \in \Omega_r$.

b) Uniqueness: Assume a in two sets, $a \in \Omega_r \cap \Omega_{r^1}$, $0 \leq r^1 < r < m$.

   Then $m|a-r$ and $m|a-r^1 \Rightarrow m|r-r^1$, which is impossible because $0 < r-r^1 < m$. Contradiction.

□

2.1.4 Proposition: Addition and Mutiplication of Congruences

Proposition 7. *Fix integer $m \geq 2$. If $a \equiv r$ mod m and $b \equiv s$ mod m, then $a + b \equiv r + s$ mod m and $ab \equiv rs$ mod m*

证明.

a) Addition: $m|(a-r), m|(b-s) \Rightarrow m|(a-c) + (b-d) \Rightarrow m|(a+b) - (c-d)$.

b) Mutiplication: $a = r + km, k \in \mathbb{Z}; b = s + jm, j \in \mathbb{Z}$. $ab = rs + rjm + skm + kjm^2 \Rightarrow n|(ab - rs)$.

□

2.2 Congruence Classes: $[a]_n = \{a + kn | k \in \mathbb{Z}\}$

将给定 n，相同余数的数分为一组

Fix $n \in \mathbb{Z}_+$, we call $[a]_n = [a]$ the **congruence class** of a modulo n.

$$[a] = \{b \in \mathbb{Z} | b \equiv a \ mod \ n\} = \{a + kn | k \in \mathbb{Z}\}$$

2.2.1 Set of congruence classes of mod n: $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\} = \{[0], [1], ..., [n-1]\}$

The set of *congruence classes* of mod n is denoted $\mathbb{Z}_n = \{[a]_n | a \in \mathbb{Z}\}$

Proposition 8 (Proposition 1.5.2.). *For any $n \geq 1$ there are exactly n congruences classes modulo n, which we may write as*

$$\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$$

证明.

For any $a \in \mathbb{Z}$. By Euclidean algorithm, $a = qn + r$, $q, r \in \mathbb{Z}$, $0 \leq r < n \Rightarrow a \in [r]$. So, $\mathbb{Z}_n = \{[0], [1], ..., [n-1]\}$.

When $0 \leq a < b \leq n-1$, $n \nmid (b-a)$, so $[a] \neq [b]$ the n congruence classes listed are all distinct. Hence, there are exactly n congruence classes. □

2.2.2 Proposition 1.5.5: Addition and Multiplication on Congruence Classes

Fix $n \in \mathbb{Z}$, we define addition+ and multiplication$\cdot$ on $\mathbb{Z}_n$:

$$[a] + [b] = [a + b] = \{a + b + (k + j)n | k, j \in \mathbb{Z}\}$$
$$[a] \cdot [b] = [ab] = \{ab + (aj + bk + kjn)n | k, j \in \mathbb{Z}\}$$

This is well defined, follows Lemma 1.5.3.

Proposition 9 (Proposition 1.5.5.). *Let $a, b, c, d, n \in \mathbb{Z}, n \geq 1$, then*

(i) Addition and multiplication are commutative and associative operations in $\mathbb{Z}_n$.

(ii) $[a] + [0] = [a]$.

(iii) $[-a] + [a] = [0]$.

(iv) $[1][a] = [a]$.

(v) $[a]([b] + [c]) = [a][b] + [a][c]$.

证明.

□

2.2.3 Units(i.e. invertible) in Congruence Classes

将与 n 互质的数分为一组

Say $[a] \in \mathbb{Z}_n$ is a **unit** or is **invertible** if $\exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [1]$.

2.2.4 Proposition 1.5.6: Set of units in congruence classes: $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \ \textit{is a unit}\} = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$

The set of **invertible** elements in $\mathbb{Z}_n$ will be denoted $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | [a] \ \textit{is a unit}\}$.

Proposition 10 (Proposition 1.5.6.)**.** *For all $n \geq 1$, we have $\mathbb{Z}_n^\times = \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$.*

证明.
By Proposition 1.4.8, we know there exists b, c s.t. $ab + cn = 1$. So, $ab \equiv 1 \ mod \ n$, $[1] = [ab] = [a][b]$. So, $\{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\} \subset \mathbb{Z}_n^\times$
$[a] \ \textit{is a unit} \Rightarrow \exists [b] \in \mathbb{Z}_n$ so that $[a][b] = [ab] = [1] \Rightarrow ab = 1 + kn, k \in \mathbb{Z} \Rightarrow ab - kn = 1, k \in \mathbb{Z} \Rightarrow gcd(a, n) = 1$. So, $\mathbb{Z}_n^\times \subset \{[a] \in \mathbb{Z}_n | gcd(a, n) = 1\}$. □

Note 1. *Inverse of $[a]$ is unique, i.e. $[b] = [a]^{-1}$ is unique.*

$[a][b] = 1, [a][b'] = 1 \Rightarrow [b] = [b][1] = [b][a][b'] = [b']$

2.2.5 Corollary 1.5.7: if p is prime, $\varphi(p) = \mathbb{Z}_p^\times = \{[1], [2], ..., [p - 1]\}$

Corollary 2 (Corollary 1.5.7)**.** *If $p \geq 2$ is prime, $\mathbb{Z}_p^\times = \{[1], [2], ..., [p - 1]\}$.*

2.3 Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$

Euler phi-function: $\varphi(n) = |\mathbb{Z}_n^\times|$.
p prime, $\varphi(p) = p - 1$.

2.3.1 $m|n$, $\pi_{m,n}([a]_n) = [a]_m$

Example 2 (Exercise 1.5.4)**.** *If $m|n$, we can define $\pi_{m,n} : \mathbb{Z}_n \to \mathbb{Z}_m$ by $\pi_{m,n}([a]_n) = [a]_m$. Prove it is well-defined.*

证明.
We write $[a]_n = [c]_n$, verify that $[a]_m = [c]_m$.
Since $m|n$, there exists $k \in \mathbb{Z}$ s.t. $n = km$.
$[a]_n = [c]_n \Rightarrow \exists j \in \mathbb{Z}$ s.t. $c = a + jn$.
$[c]_m = [a + jn]_m = [a + jkm]_m = [a]_m$ □

2.4 Theorem 1.5.8(Chinese Remainder Theorem): $n = mk, gcd(m, k) = 1$, $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$

Theorem 4 (Theorem 1.5.8(Chinese Remainder Theorem))**.** *If $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F : \mathbb{Z}_n \to \mathbb{Z}_m \times \mathbb{Z}_k$ which is given by $F([a]_n) = (\pi_{m,n}([a]_n), \pi_{k,n}([a]_n)) = ([a]_m, [a]_k)$, then F is a bijection.*

证明.
(1)Injective: $F([a]_n) = F([b]_n) \Rightarrow [a]_m = [b]_m, [a]_k = [b]_k$ i.e. $a \equiv b \ mod \ m, a \equiv b \ mod \ n$. $\exists i, j \in \mathbb{Z}$ s.t. $b = a + im = a + jk \Rightarrow k|im$. Since $gcd(m, k) = 1$, $k|i \Rightarrow n = mk|im$. Then $[b]_n = [a]_n + [im]_n = [a]_n$.
(2)Surjective: prove $\forall u, v \in \mathbb{Z}$, $\exists a\mathbb{Z}$ s.t. $[a]_m = [u]_m, [a]_k = [v]_k$.
Since $gcd(m, k) = 1$, $\exists s, t \in \mathbb{Z}$ so that $1 = sm + tk$.
Let $a = (1 - tk)u + (1 - sm)v$, $[a]_m = [(u - v)sm + v]_m = [v]_m$, $[a]_k = [(v - u)tk + u]_k = [u]_k$. □

Note 2. $F([a]_n[b]_n) = F([ab]_n) = ([ab]_m, [ab]_k) = ([a]_m[b]_m, [a]_k[b]_k)$

Since F is a bijection, $[ab]_n = [1]_n$ iff $([a]_m[b]_m, [a]_k[b]_k) = ([1]_m, [1]_k)$.

2.4.1 Proposition 1.5.9+Corollary 1.5.10: $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$

Proposition 11 (Proposition 1.5.9+Corollary 1.5.10)**.** *If $m, n, k > 0, n = mk, gcd(m, k) = 1$, then $F(\mathbb{Z}_n^\times) = \mathbb{Z}_m^\times \times \mathbb{Z}_k^\times$, then $\varphi(n) = \varphi(m)\varphi(k)$.*

2.5 prime factorization: $n = p_1^{r_1}...p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$

Proposition 12. *If $n \in \mathbb{Z}$ is positive integre with prime factorization $n = p_1^{r_1}...p_k^{r_k}$, then $\varphi(n) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$*

证明.
$\mathbb{Z}_{p^r} = \{[0], [1], ..., [p^r - 1]\}$, the number of multiples of p is $\frac{p^r}{p} = p^{r-1}$. Then $\varphi(p^r) = |\mathbb{Z}_{p^r}^\times| = p^r - p^{r-1} = (p - 1)p^{r-1}$. So,

$$\varphi(n) = \varphi(p_1^{r_1})...\varphi(p_k^{r_k}) = (p_1 - 1)p_1^{r_1-1}...(p_k - 1)p_k^{r_k-1}$$

□

3 Complex numbers

$\mathbb{C} = \{a + bi | a, b \in \mathbb{R}\}$, $\mathbb{R} = \{a + 0i | a \in \mathbb{R}\} \subset \mathbb{C}$
Addition & multiplication

$$\begin{aligned}(a + bi) + (c + di) &= (a + c) + (b + d)i \\ (a + bi)(c + di) &= ac + bci + adi + bdi^2 \\ &= (ac - bd) + (bc + ad)i\end{aligned}$$

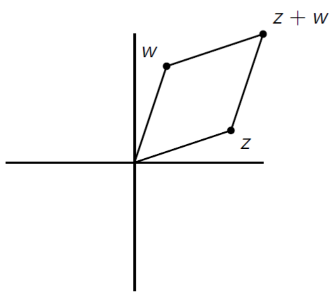
<u>Complex conjugation</u>: $z = a + bi, \bar{z} = a - bi$, $\overline{zw} = \bar{z}\bar{w}$
<u>Absolute value</u>: $|z| = \sqrt{a^2 + b^2}$, $|z|^2 = z\bar{z}$
<u>Additive inverse</u>: $-z = -a - bi$

Multiplicative inverse: $z^{-1} = \frac{1}{z} = \frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{\bar{z}}{|z|^2}$

$$z \in \mathbb{C}, \overline{z+\bar{z}} = \bar{z} + \bar{\bar{z}} = z + \bar{z}$$
$$\textit{Real part: } Re(z) = \frac{z+\bar{z}}{2}$$
$$\textit{Imaginary part: } Im(z) = \frac{z-\bar{z}}{2i}$$

3.1 Geometric Meaning of Addition and Multiplication

Addition: parallelogram law



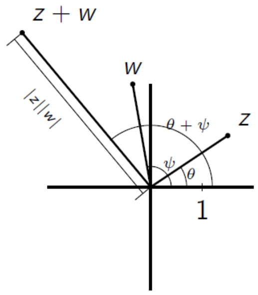
Multiplication:

$$\begin{aligned} z &= a + bi \neq 0 \\ &= r\cos\theta + r\sin\theta i \\ &= r(\cos\theta + i\sin\theta) \\ |z|^2 &= a^2 + b^2 = r^2 \end{aligned}$$

$$\begin{aligned} z =& r(\cos\theta + i\sin\theta) \\ w =& s(\cos\phi + i\sin\phi) \\ zw =& rs[\cos\theta\cos\phi - \sin\theta\sin\phi + i(\cos\theta\sin\phi + \cos\phi\sin\theta)] \\ =& rs[cos(\theta+\phi) + i\sin(\theta+\phi)] \\ =& |z||w|[cos(\theta+\phi) + i\sin(\theta+\phi)] \end{aligned}$$

We will write,

$$\begin{aligned} &\cos\theta + i\sin\theta = e^{i\theta} \\ &e^{i\theta}e^{i\phi} = e^{i(\theta+\phi)} \\ &z = |z|e^{i\theta} \end{aligned}$$



3.2 Theorem 2.1.1: $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$

Theorem 5 (Theorem 2.1.1). *Supose a nonconstant polynomial $f(x) = a_0 + a_1x + ... + a_nx^n$ with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$. Then f has a <u>root</u> in $\mathbb{C}$: $\exists \alpha \in \mathbb{C}$ s.t. $f(\alpha) = 0$.*

3.2.1 Corollary 2.1.2: $f(x) = a_n \prod_{i=1}^{n}(x - k_i) = a_n(x-k_1)(x-k_2)...(x-k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$

Corollary 3 (Corollary 2.1.2). *Every nonconstant polynomial with coefficients $a_0, a_1, ..., a_n \in \mathbb{C}$ can be factored as $f(x) = a_n \prod_{i=1}^{n}(x - k_i) = a_n(x - k_1)(x - k_2)...(x - k_n)$, where $k_1, k_2, ..., k_n$ are roots of $f(x)$.*

3.2.2 Corollary 2.1.3: $a_i \in \mathbb{R}$, f can be expresses as a product of linear and quadratic polynomials

Corollary 4 (Corollary 2.1.3). *If $f(x) = a_0 + a_1x + ... + a_nx^n$ is a nonconstant polynomial $a_0, a_1, ..., a_n \in \mathbb{R}, a_n \neq 0$. Then f can be expresses as a product of linear and quadratic polynomials.*

这里 $a_0, a_1, ..., a_n$ 是实数!

证明.
(1)Obviously, the corollary holds at $n = 1$ and $n = 2$.
(2)Suppose the corollary holds for all situations that $n < k$.
When $n = k$, $f(x) = a_0 + a_1x + ... + a_kx^k, a_k \neq 0$.
By F.T.A., f has a root α in $\mathbb{C}$.
If $\alpha \in \mathbb{R}$, long division $f(x) = q(x)(x - \alpha)$. q has real coefficients, *degree of* $q = k - 1$. Since the corollary holds at $n = k - 1$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n = k$.
If $\alpha \notin \mathbb{R}$

$$0 = f(\alpha) = a_0 + a_1\alpha + ... + a_k\alpha^k$$
$$0 = \overline{f(\alpha)} = a_0 + a_1\bar{\alpha} + ... + a_n\bar{\alpha}^n = f(\bar{\alpha})$$

Since $\bar{\alpha} \neq \alpha$, $(x - \alpha)(x - \bar{\alpha})|f$.
$(x - \alpha)(x - \bar{\alpha}) = x^2 - (\alpha + \bar{\alpha})x + |\alpha|^2$ is a polynomial with coefficients in $\mathbb{R}$. So $f(x) = q(x)(x^2 -$

$(\alpha+\bar{\alpha})x+|\alpha|^2)$, q has real coefficients with degree $k-2$. The corollary also holds at $n=k-2$, $q(x)$ is a product of linear and quadratics. Then, the corollary also holds at $n=k$.

$\square$

4 Field $(\mathbb{F},+,\cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))

Definition: A field is a nonempty set $\mathbb{F}$ with two operations:
1. addition, written $a+b, \forall a,b\in\mathbb{F}$;
2. multiplication, written $a\cdot b=ab, \forall a,b\in\mathbb{F}$.

such that:
(i) *addition* and *multiplication* are associative and commutative
(ii) *multiplication* distributes over *addition*: $a(b+c)=ab+ac, \forall a,b,c\in\mathbb{F}$
(iii) $\exists$ an additive identity $0\in\mathbb{F}$ s.t. $0+a=a, \forall a\in\mathbb{F}$.
(iv)$\forall a\in\mathbb{F}$, $\exists$ an additive inverse $-a$ s.t. $a+(-a)=0, \forall a\in\mathbb{F}$.
(v) $\exists$ a multiplicative identity: $1\in\mathbb{F}$ s.t. $1a=a, \forall a\in\mathbb{F}, 1\neq 0$.
(vi) $\forall a\in\mathbb{F}, a\neq 0$, a has a multiplicative inverse $a^{-1}=\frac{1}{a}\in\mathbb{F}: a\cdot\frac{1}{a}=1$.

Proposition 13 (Proposition 2.2.2)**.** *$\mathbb{F}$ a field, $a,b\in\mathbb{F}$, then*

(i) If $a+b=b$ then $a=0$
(ii) If $ab=b$ and $b\neq 0$, then $a=1$
(iii) $0a=0$
(iv) If $a+b=0$, then $b=-a$
(v) If $a\neq 0$ and $ab=1$, then $b=a^{-1}$

Example 3. *$\mathbb{Z}_4$ is not a field. Because $[2]_4$ doesn't have multiplicative inverse in $\mathbb{Z}_4$.*

4.1 Subfield $(\mathbb{K},+,\cdot)$: $\mathbb{K}\subseteq\mathbb{F}$, closed under $+,\cdot$ and inverse

Definition:Suppose $\mathbb{F}$ is a field and $\mathbb{K}\subseteq\mathbb{F}$ s.t.

$$0,1\in\mathbb{K}$$
$$\forall a,b\in\mathbb{K}, a+b, ab, -a, a^{-1}(\textit{if } a\neq 0)\in\mathbb{K}$$

We call $\mathbb{K}$ a subfield of $\mathbb{F}$.

Example 4. $\mathbb{Q}\subseteq\mathbb{R}, \mathbb{R}\subseteq\mathbb{C}, \mathbb{Q}\subseteq\mathbb{C}$

Example 5. *$\mathbb{K}\subseteq\mathbb{Z}_p$ a subfield $\Rightarrow\mathbb{K}=\mathbb{Z}_p$. Prove by induction.*

4.1.1 Proposition 2.2.3: Subfield 继承 operations 自成一 field

Proposition 14 (Proposition 2.2.3)**.** *Suppose* $\mathbb{K} \subset \mathbb{F}$ *is a subfield of a field* $\mathbb{F}$ *Then the operations of* $\mathbb{F}$ *make* $\mathbb{K}$ *into a field.*

⇒We can prove a set is a field by proving it is a subfield of a known field.

5 Polynomials

Let $\mathbb{F}$ be any field. A polynomial over $\mathbb{F}$ in variable x is a formal sum:

$$a_0 + a_1 x + a_2 x^2 + ... + a_n x^n = \sum_{i=0}^{n} a_i x^i$$

where $n \geq 0$ is an integer, $a_1, a_1, ..., a_n \in \mathbb{F}$.
Polynomial is a squence $\{a_k\}_{k=0}^{\infty}$ with $a_m = 0, \forall m > n$.

5.1 $\mathbb{F}[x]$: Polynomial ring 在一个 field 上形成的所有多项式 (方程) 的集合

Let $\mathbb{F}[x]$ denote the set of all polynomials with coefficients in the field $\mathbb{F}$.

$$\mathbb{F}[x] = \{\sum_{i=0}^{n} a_i x^i | n \geq 0, n \in \mathbb{Z}, a_0, ..., a_n \in \mathbb{F}\}$$

We call the $\mathbb{F}[x]$ *polynomial ring* over the field $\mathbb{F}$.

$$f = \sum_{i=0}^{n} a_i x^i, g = \sum_{j=0}^{n} a_j x^j \in \mathbb{F}[x]$$

$$f + g = \sum_{i=0}^{n} (a_i + b_i) x^i \in \mathbb{F}[x]$$

$$fg(\sum_{i=0}^{n} a_i x^i)(\sum_{j=0}^{n} a_j x^j) = \sum_{i=0}^{2n} (\sum_{j=0}^{i} a_j b_{i-j}) x^i$$

5.1.1 Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

Proposition 15 (Proposition 2.3.2)**.** *Suppose* $\mathbb{F}$ *is any field. Then,*

(i) Addition and multiplication are commutative & associative operations on $\mathbb{F}[x]$
(ii) Multiplication distributes over addition
(iii) $0 \in \mathbb{F}$, is additive identity in $F[x] : \forall f \in \mathbb{F}[x], f + 0 = 0$
(iv) $\forall f \in \mathbb{F}[x], f = (-1)f$ is the additive inverse: $f + (-1)f = 0$.
(v) $1 \in \mathbb{F}$, is the multiplicative identity in $\mathbb{F}[x]$: $1f = f,\ \forall f \in \mathbb{F}[x]$

5.2 Degree of a Polynomial: $deg(f)$

$f = \sum_{i=0}^{n} a_i x^i$, $deg(f)$ = degree of f is,

$$deg(f) = \begin{cases} 0 & \textit{if } f \textit{ is constant, } f \neq 0 \\ n & \textit{if } a_n \neq 0 \textit{ in above } (a_n = \textit{leading coefficent}) \\ -\infty & \textit{if } f = 0 \end{cases}$$

Define $-\infty + a = a + (-\infty) = -\infty \ \forall a \in \mathbb{Z} \cup \{-\infty\}$

5.2.1 Lemma 2.3.3: $deg(fg) = deg(f) + deg(g),\ deg(f+g) \leq \max\{deg(f), deg(g)\}$

Lemma 2 (Lemma 2.3.3)**.** *For any field $\mathbb{F}$ and f , $g \in \mathbb{F}[x]$,*

$$\begin{aligned} &deg(fg) = deg(f) + deg(g) \\ &deg(f+g) \leq \max\{deg(f), deg(g)\} \end{aligned}$$

5.3 Corollary 2.3.5: Unit(invertible) in $\mathbb{F}[x]$: constant$\neq 0$ iff $deg(f) = 0$

Corollary 5 (Corollary 2.3.5)**.** *For any field $\mathbb{F}$ and $f \in \mathbb{F}[x]$, Then f is a* <u>*unit*</u>*(i.e. invertible) in $\mathbb{F}[x]$ iff $deg(f) = 0$.*

证明.
Obviously, $deg(f) = 0 \Rightarrow$ f is a unit.
Suppose f is a unit, i.e. $\exists g \in \mathbb{F}[x]$ s.t. $fg = 1$.
$0 = deg(fg) = deg(f) + deg(g) \Rightarrow deg(f), deg(g) \geq 0 \Rightarrow deg(f) = 0, deg(g) = 0$. □

5.4 <u>Irreducible</u> Polynomials: "无法分解为两个 $degree \geq 1$ 的多项式积"的多项式: 至少一个是 constant (i.e. $degree = 0$)

A nonconstant polynomial f is <u>irreducible</u> if $f = uv,\ u, v \in \mathbb{F}[x]$, then either u or v is a unit(i.e., constant$\neq 0$)

5.5 Theorem 2.3.6：nonconstant polynomials 可以被唯一地分解

Theorem 6 (Theorem 2.3.6)**.** *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is any nonconstant. Then $f = ap_1p_2 \ldots p_k$ where $a \in \mathbb{F}$, $p_1, \ldots p_k \in \mathbb{F}[x]$ are irreducible* <u>*monic*</u> *polynomials (monic = i.e. leading coeff. 1). If $f = bq_1q_2 \ldots q_r$ with $b \in \mathbb{F}$ and $q_1, q_2, \ldots, q_r \in \mathbb{F}[x]$ monic irreducible, then $a = b, k = r$, and after reindexing $p_i = q_i,\ \forall i$*

Lemma 3 (Lemma 2.3.7)**.** *Suppose $\mathbb{F}$ is a field and $f \in \mathbb{F}[x]$ is nonconstant monic polynomial. Then $f = p_1p_2 \ldots p_k$ where each p_i is monic irreducible.*

证明.
Prove it by induction. When $deg(f) = 1,\ f = uv,\ u, v \in \mathbb{F}[x]$, $deg(f) = deg(u) + deg(v) \Rightarrow$ one of

these is 0.
Suppose the lemma holds for all degree$< n$. When $deg(f) = n$,
Either f is irreducible, done.
Suppose $f = uv$ with/ $deg(u), deg(v) \geq 1$
$\Rightarrow deg(u), deg(v) < n \Rightarrow u = p_1p_2 \dots p_k, v = q_1q_2 \dots q_j$ So, $f = p_1p_2 \dots p_kq_1q_2 \dots q_j$. □

Example 6. $x^2 - 1 \in \mathbb{Q}[x]$ *reducible*
$x - 1, x + 1 \in \mathbb{Q}[x]$ *irreducible*
$x^2 + 1 \in \mathbb{Q}[x]$ *irreducible*
$x^2 + 1 \in \mathbb{C}[x]$ *reducible*
$x^2 - 1 = x^2 + 1 = [1]x^2 + [1] \in \mathbb{Z}_2[x]$ *reducible*

5.6 Divisibility of Polynomials

$f, g \in \mathbb{F}[x], f \neq 0$, f divides g, $f|g$ means $\exists u \in \mathbb{F}[x]$ s.t. $g = fu$.

Proposition 16 (Proposition 2.3.8)**.** $f, h, g \in \mathbb{F}[x]$, *then*

(i) If $f \neq 0, f|0$
(ii) If $f|1$, f is nonzero constant
(iii) If $f|g$ and $g|f$, then $f = cg$ for some $c \in \mathbb{F}$
(iv) If $f|g$ and $g|h$, then $f|h$
(v) If $f|g$ and $f|h$, then $f|(ug + vh)$ for all $u, v \in \mathbb{F}[x]$.

5.6.1 Greatest common divisor of f and g: is not unique, we denote monic Greatest common divisor as $gcd(f, g)$

If $f, g \in \mathbb{F}[x]$ are nonzero polynomials, a greatest common divisor of f and g is a polynomial $h \in \mathbb{F}[x]$ such that
(i) $h|f$ and $h|g$, and
(ii) if $k \in \mathbb{F}[x]$ and $k|f$ and $k|g$, then $k|h$.
the gcd is not unique, but the monic gcd is unique. We call it **the monic greatest common divisor**, denote it $gcd(f, g)$.

Example 7.

$$x^2 - 1, x^2 - 2x + 1 \in \mathbb{Q}[x]$$
$$(x - 1)(x + 1), (x - 1)^2 \in \mathbb{Q}[x]$$
$$x - 1 = gcd(x^2 - 1, x^2 - 2x + 1)$$

5.6.2 Proposition 2.3.9: Euclidean Algorithm of polynomials

Proposition 17 (Proposition 2.3.9)**.** *Given* $f, g \in \mathbb{F}[x]$*,* $g \neq 0$*, then* $\exists q, r \in \mathbb{F}[x]$ *s.t.* $deg(r) < deg(g)$ *and* $f = qg + r$

Example 8.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$

5.6.3 Proposition 2.3.10: $gcd(f, g)$ 是 degree 最小的 f,g 的线性组合

Proposition 18 (Proposition 2.3.10)**.** *Any 2 nonzero polynomials* $f, g \in \mathbb{F}[x]$ *have a gcd in* $\mathbb{F}[x]$*. In fact among all polynomials in the set* $M = \{uf + vg | u, v \in \mathbb{F}[x]\}$ *any nonconstant of minimal degree are gcds.*

证明.
$h \in M$, $deg(h) = d$ minimal. Let $k|f$ and $k|g \Rightarrow k|uf + vg,\ \forall u, v \Rightarrow k|h$.
Suppose $h' \in M$ is any nonzero element. $deg(h') \geq deg(h) \Rightarrow\ \exists q, r \in \mathbb{F}, deg(r) < deg(h)\ h' = qh + r$. $r = h' - qh \in M$. Since $deg(h) = d$ is nonconstant minimal degree, $r = 0 \Rightarrow h' = qh$. So $\exists q_1, q_2 \in \mathbb{F}[x],\ 1f + 0g = q_1 h, 0f + 1g = q_2 h \Rightarrow h|g, h|f$. □

Example 9.

$$f = 3x^3 - 5x^2 - 3x + 5, g = x^3 - 2x^2 + 1 \in \mathbb{Q}[x]$$
$$f = 3g + x^2 - 3x + 2$$
$$g = (x + 1)(x^2 - 3x + 2) + x - 1$$
$$x^2 - 3x + 2 = (x - 2)(x - 1)$$
$$\Rightarrow gcd(f, g) = x - 1$$
$$x - 1 = g - (x + 1)(x^2 - 3x + 2) = g - (x + 1)(f - 3g) = (3x + 4)g - (x + 1)f$$

Example 10. *Find a greatest common divisor of* $f = x^3 - x^2 - x + 1$ *and* $g = x^2 - 3x + 2$ *in* $\mathbb{Q}[x]$*, and express it in form* $uf + vg,\ u, v \in \mathbb{Q}[x]$*.*

$$f = (x + 2)g + 3x - 3$$
$$g = \frac{1}{3}(x - 2)(3x - 3)$$
$$gcd(f, g) = 3x - 3$$
$$3x - 3 = f - (x + 2)g$$

5.6.4 Proposition 2.3.12: $gcd(f, g) = 1, f|gh \Rightarrow f|h$

Proposition 19 (Proposition 2.3.12)**.** *If* $f, g, h \in \mathbb{F}[x]$*,* $gcd(f, g) = 1$*, and* $f|gh$*, then* $f|h$*.*

5.6.5 Corollary 2.3.13: irreducible f, $f|gh \Rightarrow f|g$ or $f|h$

Corollary 6 (Corollary 2.3.13)**.** *If $f \in \mathbb{F}[x]$ is irreducible, and $f|gh$, then $f|g$ or $f|h$.*

Since f is irreducible, we have two possible situations:
1. $gcd(f, g) = f$, i.e. $f|g$ done.
2. $gcd(f, g) = 1$, then according to Prop 2.3.12, we can know $f|h$.

5.7 Roots

Root:$\alpha \in \mathbb{F}$ is a root of f if $f(\alpha) = 0$.

5.7.1 Corollary 2.3.16(of Euclidean Algorithm): f 可被分为 $(x - \alpha)q + f(\alpha)$i.e. if α is a root, then $(x - \alpha)|f$

Corollary 7 (Corollary 2.3.16(of Euclidean Algorithm))**.** *$\forall f \in \mathbb{F}[x]$ and $\alpha \in \mathbb{F}$, there exists a polynomial $q \in \mathbb{F}[x]$ s.t. $f = (x - \alpha)q + f(\alpha)$. In particular , if α is a root, then $(x - \alpha)|f$.*

5.8 Multiplicity

If α is a root of f , say its *multiplicity* is m, if $x - \alpha$ appears m times in irreducible factorization.

5.8.1 *Sum of multiplicity* $\leq deg(f)$

Proposition 20 (Proposition 2.3.17)**.** *Given a nonconstant polynomial $f \in \mathbb{F}[x]$, the number of roots of f, counted with multiplicity, is at most $deg(f)$.*

5.9 Roots in a filed may not in its subfield

Note if $\mathbb{F} \subset \mathbb{K}$, then $\mathbb{F}[x] \subset \mathbb{K}$. $f \in \mathbb{F}[x]$ may have no roots in $\mathbb{F}$, but could have roots in $\mathbb{K}$

Example 11. *$x^n - 1 \in \mathbb{Q}[x]$ has a root in $\mathbb{Q}$: 1; has 2 roots if n even: ± 1*
roots in $\mathbb{C}$: $\zeta_n = e^{\frac{2\pi i}{n}}$, then $\zeta_n^n = e^{2\pi i} = 1$; $(\zeta_n^k)^n = e^{2\pi k i} = 1$ So, the roots: $\{e^{\frac{2\pi k i}{n}} | k = 0, ..., n - 1\}$
The roots of $x^n - d$: $\{e^{\frac{2\pi k i}{n}} \sqrt{d} | k = 0, ..., n - 1\}$

6 Linear Algebra

6.1 Vector Space $(V, +, \times)$ (over a field $\mathbb{F}$)

A vector space over a field $\mathbb{F}$ is a set V w/ an operation addition $+ : V \times V \to V$ and an operation scalar multiplication $\mathbb{F} \times V \to V$
(1) Addition is associative & commutative

(2) $\exists 0 \in V$, additive identity: $0 + v = v \forall v \in V$
(3) $1v = v \forall v \in V$(where $1 \in \mathbb{F}$ is multi. id. in $\mathbb{F}$)
(4) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ \alpha(\beta v) = (\alpha\beta)v$
(5) $\forall v \in V,\ (-1)v = -v$ we have $v + (-v) = 0$
(6) $\forall \alpha \in \mathbb{F},\ v, u \in V,\ \alpha(v + u) = \alpha v + \alpha u$
(7) $\forall \alpha, \beta \in \mathbb{F},\ v \in V,\ (\alpha + \beta)v = \alpha v + \beta v$

6.1.1 A field is a vector space over its subfield

Example 12. $\mathbb{K} \subset \mathbb{F}$ *is a subfield of a field* $\mathbb{F}$*. Then* $\mathbb{F}$ *is a vector space over* $\mathbb{K}$*. (Since* $\mathbb{F} \subset \mathbb{F}[x]$*, then* $\mathbb{F}[x]$ *is a vector space over* $\mathbb{F}$*.)*

6.1.2 Vector subspace

Suppose that V is a vector space over $\mathbb{F}$. A vector subspace or just subspace is a nonempty subset $W \subset V$ closed under addition and scalar multiplication. i.e. $v + w \in W,\ av \in W,\ \forall v, w \in W,\ a \in \mathbb{F}$.

Example 13. $\mathbb{K} \subset \mathbb{L} \subset \mathbb{F}$*, then* $\mathbb{L}$ *is a subspace of* $\mathbb{F}$ *over* $\mathbb{K}$*.*

6.2 Linear independent, Linear combination

6.3 span V, basis, dimension, Proposition 2.4.10

A set of elements $v_1, ..., v_n \in V$ is said to **span** V if every vector $v \in V$ can be expressed as a linear combination of $v_1, ..., v_n$. If $v_1, ..., v_n$ spans and is linearly independent, then we call the set a ***basis*** for V.

Proposition 21 (Proposition 2.4.10.)**.** *Suppose* V *is a vector space over a field* $\mathbb{F}$ *having a basis* $\{v_1, ..., v_n\}$ *with* $n \geq 1$*.*

(i) For all $v \in V$, $v = a_1v_1 + ... + a_nv_n$ for exactly one $(a_1, ..., a_n) \in \mathbb{F}^n$.
(ii) If $w_1, ..., w_n$ span V , then they are linearly independent.
(iii)If $w_1, ..., w_n$ are linearly independent, then they span V.
If a vector space V over $\mathbb{F}$ has a basis with n vectors, then V is said to be n-dimensional (over $\mathbb{F}$) or is said to have **dimension** n.

6.3.1 Standard basis vectors

$$e_1 = (1, 0, ..., 0), e_2 = (0, 1, 0, ..., 0), ..., e_n = (0, 0, ..., 0, 1) \in \mathbb{F}^n$$

are a basis for $\mathbb{F}^n$ called the **standard basis vectors**.

6.4 Linear transformation

Given two vector spaces V and W over $\mathbb{F}$ a **linear transformation** is a function $T : V \to W$ such that for all $a \in \mathbb{F}$ and $v, w \in V$, we have

$$T(av) = aT(v) \text{ and } T(v+w) = T(v) + T(w)$$

Proposition 22 (Proposition 2.4.15.). *If V and W are vector spaces and $v_1, ..., v_n$ is a basis for V then any function from $\{v_1, ..., v_n\} \to W$ extends uniquely to a linear transformation $V \to W$.*

Any $v \in V$, $\exists(a_1, ..., a_n)$ s.t. $v = a_1v_1 + ... + a_nv_n$. Then $T(v) = T(a_1v_1 + ... + a_nv_n) = a_1T(v_1) + ... + a_nT(v_n)$

6.4.1 Corollary 2.4.16: 一个线性变换对应一个矩阵 *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$

Corollary 8 (Corollary 2.4.16.). *If $v_1, ..., v_n$ is a basis for a vector space V and $w_1, ..., w_n$ is a basis for a vector space W (both over $\mathbb{F}$), then any linear transformation $T : V \to W$ determines (and is determined by) the $m \times n$ matrix:*

$$A = A(T) = \begin{bmatrix} A_{11} & A_{12} & ... & A_{1n} \\ A_{21} & A_{22} & ... & A_{2n} \\ \vdots & \vdots & ... & \vdots \\ A_{m1} & A_{m2} & ... & A_{mn} \end{bmatrix}$$

$$\begin{bmatrix} w_1 & \cdots & w_m \end{bmatrix}^T = A \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix}^T$$

$\mathcal{L}(V, M)$ denotes the set of all linear transformations from V to W; $M_{m\times n}(\mathbb{F})$ the set of $m \times n$ matrix with entries in $\mathbb{F}$. $T \to A(T)$ defines a *bijection* $\mathcal{L}(V, M) \to M_{m\times n}(\mathbb{F})$. $A(T)$ **represents the linear transformation** T.

6.4.2 Proposition 2.4.19: 线性变换矩阵相乘仍为线性变换矩阵

Proposition 23 (Proposition 2.4.19). *Suppose that V , W, and U are vector spaces over $\mathbb{F}$, with fixed chosen bases. If $T : V \to W$ and $S : W \to U$ are linear transformations represented by matrices $A = A(T)$ and $B = B(S)$, then $ST = S \circ T : V \to U$ is a linear transformation represented by the matrix $BA = B(S)A(T)$.*

6.5 GL(V): invertible(bijective) linear transformations $V \to V$

Given a vector space V over F, we let $GL(V) \subset \mathcal{L}(V, V)$ denote the subset of **invertible linear transformations**.

$$GL(V) = \{T \in \mathcal{L}(V, V) | T \text{ is a bijection}\} = \mathcal{L}(V, V) \cap Sym(V)$$

7 Euclidean geometry basics

7.1 Euclidean distance, inner product

Euclidean distance on $\mathbb{R}^n$:

$$|x-y| = \sqrt{(x_1-y_1)^2 + ... + (x_n-y_n)^2}$$

Euclidean inner product:

$$x \cdot y = x_1y_1 + \cdots + x_ny_n = x^Ty$$

7.2 Isometry of $\mathbb{R}^n$: a bijection $\mathbb{R}^n \to \mathbb{R}^n$ preserves distance

An **isometry** of $\mathbb{R}^n$ is a bijection $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distance, which means,

$$|\Phi(x) - \Phi(y)| = |x-y|, \ \forall x, y \in \mathbb{R}^n$$

7.2.1 $Isom(\mathbb{R}^n)$: set of all isometries of $\mathbb{R}^n$

We use $Isom(\mathbb{R}^n)$ denotes the set of all isometries of $\mathbb{R}^n$,

$$Isom(\mathbb{R}^n) = \{\Phi : \mathbb{R}^n \to \mathbb{R}^n || \Phi(x) - \Phi(y)| = |x-y|, \ \forall x, y \in \mathbb{R}^n\}$$

7.2.2 $Isom(\mathbb{R}^n)$ is closed under $\circ$ and inverse

Proposition 24. *$\Phi, \Psi \in Isom(\mathbb{R}^n)$, then $\Phi \circ \Psi, \Phi^{-1} \in Isom(\mathbb{R}^n)$*

证明.
Since Φ, Ψ are bijections, so is $\Phi \circ \Psi$. Moreover,

$$|\Phi \circ \Psi(x) - \Phi \circ \Psi(y)| = |\Phi(\Psi(x)) - \Phi(\Psi(y))| = |\Psi(x) - \Psi(y)| = |x-y|$$

Since $id \in Isom(\mathbb{R}^n)$,

$$|x-y| = |id(x) - id(y)| = |\Phi \circ \Phi^{-1}(x) - \Phi \circ \Phi^{-1}(y)| = |\Phi^{-1}(x) - \Phi^{-1}(y)|$$

□

7.3 $A \in GL(n, \mathbb{R})$, $T_A(v) = Av$: $A^tA = I \Leftrightarrow T_A \in Isom(\mathbb{R}^n)$

There is a matrix $A \in GL(n, \mathbb{R})$ i.e. a *invertible linear transofrmations* $T_A : \mathbb{R}^n \to \mathbb{R}^n$ is given by $T_A(v) = Av$.

$$T_A(v) \cdot T_A(w) = (Av) \cdot (Aw) = (Av)^t(Aw) = v^tA^tAw$$

$$A^tA = I \Leftrightarrow T_A(v) \cdot T_A(w) = v \cdot w \Leftrightarrow_{(HW4)} \ T_A \in Isom(\mathbb{R}^n)$$

7.4 Linear isometries i.e. orthogonal group $O(n) = \{A \in GL(n,\mathbb{R}) | A^tA = I\}$

We define the all isometries in *invertible linear transofrmations* $\mathbb{R}^n \to \mathbb{R}^n$ as **orthogonal group**

$$O(n) = \{A \in GL(n,\mathbb{R}) | A^tA = I\} \subset GL(n,\mathbb{R})$$

7.4.1 Special orthogonal group $SO(n) = \{A \in O(n) | det(A) = 1\}$: orthogonal group with $det(A) = 1$

$O(n)$ are the matrices representing linear isometries of $\mathbb{R}^n$. $1 = det(I) = det(A^tA) = det(A^t)det(A) = det(A)^2 \Rightarrow det(A) = 1$ or $det(A) = -1$. We use **special orthogonal group** represents A with $det(A) = 1$,

$$SO(n) = \{A \in O(n) | det(A) = 1\}$$

7.5 translation: $\tau_v(x) = x + v$

Define a *translation* by $v \in \mathbb{R}^n$,

$$\tau_v : \mathbb{R}^n \to \mathbb{R}^n,\ \tau_v(x) = x + v$$

7.5.1 translation is an isometry

Note 3 (Exercise 2.5.3)**.** *$\forall v \in \mathbb{R}^n, \tau_v$ is an isometry.*

证明. $|\tau_v(x) - \tau_v(y)| = |(x + v) - (y + v)| = |x - y|$ □

7.6 The composition of a translation and an orthogonal transformation is an isometry $\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$

Since *the composition of isometries is an isometry,* $\forall A \in O(n)$ and $v \in \mathbb{R}^n$, the composition

$$\Phi_{A,v}(x) = \tau_v(T_A(x)) = Ax + v$$

is an isometry. **which could account for all isometries**.

7.6.1 Theorem 2.5.3: All isometries can be represented by a composition of *a translation* and *an orthogonal transformation*, $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

Theorem 7 (Theorem 2.5.3)**.** $Isom(\mathbb{R}^n) = \{\Phi_{A,v} | A \in O(n), v \in \mathbb{R}^n\}$

8 Group $(G, *)$: a set with an operation(associative, identity, inverse)

A *group* is a nonempty set G with an operation $* : G \times G \to G$ s.t. the following holds
(1) $*$ is associative

(2) $\exists e \in G$ an identity element $e * g = g * e = g \ \forall g \in G$
(3) $\forall g \in G, \exists g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$

8.1 $(Sym(X), \circ)$ symmetric/permutation group of X

Example 14. *If X is any nonempty set, permutation group of X : $\{\sigma : X \to X | \sigma$ is a bijection$\}$, then*

1. $\circ$ is associative;
2. $id : X \to X,\ id(x) = x \ \forall x \in X$ is the idenity;
3. $\sigma \in Sym(X), \sigma^{-1} \in Sym(X)$ is the inverse function.

$(Sym(X), \circ)$ **is a group called the symmetric group of** X

8.2 Order of a group $(G, *)$

$|G|$ =Order of a group $(G, *)$

8.3 Abelian group (group that $*$ satisfies commutative)

$(\mathbb{Z}, +)$ is a group and $+$ is commutative, we call this kind of groups(statify commutative) *abelian group.*

Example 15. *If $\mathbb{F}$ is a field, then $(\mathbb{F}, +)$ and $(\mathbb{F}^{\times}, \cdot)$ are abelian group.*

Example 16. *If V is a vector space over $\mathbb{F}$, then $(V, +)$ abelian group.*

As we know a V is a vector space over $\mathbb{F}$ means V is a field whose subfields include $\mathbb{F}$.

8.4 H: Subgroup of $(G, *)$ ($H \neq \emptyset \subset G$ closed under $*$; inverse $g^{-1} \in H$), written as $H < G$

$H \neq \emptyset \subset G$ is a *subgroup* of $(G, *)$ if,
1. $\forall\ g, h \in H,\ g * h \in H.$
2. $\forall\ g \in H,\ g^{-1} \in H.$

write $H < G$ if H is a subgroup of $(G, *)$.

8.4.1 Proposition 2.6.8: $H < G$, $(H, *)$ is a group: A group's operation with its any subgroup is also a group

Proposition 25 (Proposition 2.6.8)**.** *If $(G, *)$ is a group, $H \subset G$ is a subgroup, then $(H, *)$ is a group.*

Example 17. *$(G, *)$ is a group, then $e < G,\ G < G$.*

Example 18. $\mathbb{K} \subset \mathbb{F}$ *is a subfield, then* $\mathbb{K} < \mathbb{F}$, $\mathbb{K}^{\times} < \mathbb{F}^{\times}$.

Example 19. $W \subset V$ *is a vector subspace,* $W < V$.

Example 20. $1 \in S^1 \subset \mathbb{C}^{\times}$, $S^1 = \{z \in \mathbb{C} | |z| = 1\}$. S^1 *is a subgroup.*

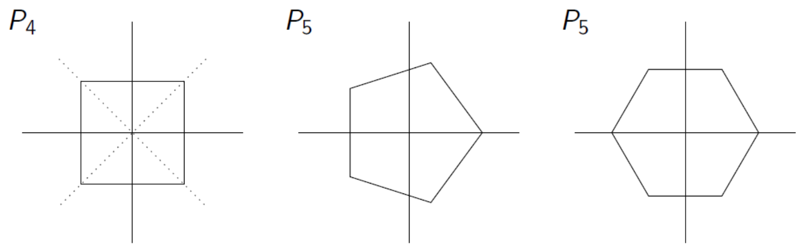
证明.
$S^1 = \{e^{i\theta} | \theta \in \mathbb{R}\}$. For any $e^{i\theta}, e^{i\psi} \in S^1$, $e^{i\theta}e^{i\psi} = e^{i(\theta+\psi)} \in S^1, e^{-i\theta} \in S^1$. □

Example 21. $Isom(\mathbb{R}^n) < Sym(\mathbb{R}^n)$

Example 22. *If* $\mathbb{F}$ *is a field,* $Aut(\mathbb{F}) = \{\sigma : \mathbb{F} \to \mathbb{F} \in Sym(\mathbb{F}) | \sigma(a+b) = \sigma(a) + \sigma(b), \sigma(ab) = \sigma(a)\sigma(b)\} < Sym(\mathbb{F})$

Example 23. *Dihedral Groups:*

保留多边形
Let $P_n \subset \mathbb{R}^2$ be a regular $n - gon$



$D_n < Isom(\mathbb{R}^2)$, $D_n = \{\Phi \in Isom(\mathbb{R}^2) | \Phi(P_n) = P_n\}$

9 Ring $(R, +, \cdot)$: $+$ is associative, commutative, identity, inverse $\in R$; $\cdot$ is associative, distributes over $+$

Definition 1. *A ring is a nonempty set with two operations, called addition and multiplication,* $(R, +, \cdot)$ *such that*

(1): $(R, +)$ is an ablian group: i.e. $+$ is associatve and commucative. $0, -a \in R$
(2): $\cdot$ is associative.
(3): $\cdot$ distributes over $+$: $\forall a, b, c \in R$, $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(b + c) \cdot a = b \cdot a + c \cdot a$

9.1 Commutative ring: ring's $\cdot$ is commutative

If "$\cdot$" is commutative, we call $(R, +, \cdot)$ a commutative ring.

9.2 Ring with 1: exists multiplication identity $1 \in R$

If there exists an element $1 \in R\backslash\{0\}$ such that $a1 = 1a = a, \ \forall a \in R$,then we say that R is a ring with 1.

9.3 Field $\mathbb{F}$ is a commutative ring with 1; $\mathbb{F}[x]$ is also a commutative ring with 1

Field $(\mathbb{F}, +, \cdot)$ (close, associative, commutative, distributive(M over A), identity & inverse(M,A))
Proposition 2.3.2: Polynomial ring (close, associative, commutative, distributive(M over A), identity(M,A), inverse(only A))

9.4 $S \subset R$: Subring (closed under $+$ and $\cdot$; addictive inverse $-a \in S$)

9.4.1 Proposition 2.6.27: $(S, +, \cdot)$ is a ring

Proposition 26 (Proposition 2.6.27)**.** *If $S \subset R$ is a subring, then $+, \cdot$ make S into a ring.*

10 Group theory

10.1 Properties of Group Operation

10.1.1 Proposition 3.1.1: $g * h = h$ or $h * g = h$, then $g = e$; $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

Proposition 27 (Proposition 3.1.1)**.** *Let $(G, *)$ be a group with identity $e \in G$, then*

(1) if $g, h \in G$ and either $g * h = h$ or $h * g = h$, then $g = e$
(2) if $g, h \in G$ and $g * h = e$ then $g = h^{-1}$ and $h = g^{-1}$

10.1.2 Corollary 3.1.: $e^{-1} = e, \ (g^{-1})^{-1} = g, \ (g * h)^{-1} = h^{-1} * g^{-1}$

Corollary 9 (Corollary 3.1.2)**.** $e^{-1} = e, \ (g^{-1})^{-1} = g, \ (g * h)^{-1} = h^{-1} * g^{-1}$

10.1.3 Proposition 3.1.3: $g * h = k * h$ or $h * g = h * k$, then $g = k$

Proposition 28 (Proposition 3.1.3)**.** *If $g * h = k * h$ or $h * g = h * k$, then $g = k$.*

10.1.4 Proposition 3.1.4: $g * x = h$ and $x * g = h$ have unique solutions $x \in G$.

Proposition 29 (Proposition 3.1.4)**.** *$g * x = h$ and $x * g = h$ have unique solutions $x \in G$.*

10.2 Power of an Element

We define g^n recursively for $n \geq 0$ by setting $g^0 = e$ and for $n \geq 1$, we set $g^n = g^{n-1} * g$. For $n \leq 0$, we define $g^n = (g^{-1})^{-n}$.

10.2.1 Proposition 3.1.5: $g^n * g^m = g^{n+m}$, $(g^n)^m = g^{nm}$

Proposition 30 (Proposition 3.1.5). *(1) $g^n * g^m = g^{n+m}$; (2) $(g^n)^m = g^{nm}$*

10.3 $(G \times H, \circledast)$: <u>Direct Product</u> of G and H

$(G, *)$ a group $(H, \star)$ a group. Define an operation on $G \times H$, $\circledast$:

$$(h, k) \circledast (h', k') = (h * h', k * k')$$

10.3.1 Proposition 3.1.7: $(G \times H, \circledast)$ is a group

Proposition 31 (Proposition 3.1.7). *$(G \times H, \circledast)$ is a group. The identity is (e_G, e_H), inverse is (g^{-1}, h^{-1})*

usually written as

$$(h, k)(h', k') = (hh', kk')$$

10.4 Subgroups and cyclic groups

10.4.1 Proposition 3.2.2: Intersection of a Collection of Subgroups is a group

Proposition 32 (Proposition 3.2.2). *Let G be a group and suppose $\mathcal{H}$ is any collection of subgroups of G. Then $K = \cap_{H \in \mathcal{H}} H < G$ is a subgroup of G.*

10.4.2 Subgroup Generated by A: $\langle A \rangle = \cap_{H<G; A \subset H} H$

We define **Subgroup Generated by A:**

$$\langle A \rangle = \cap_{H \in \mathcal{H}(A)} H$$

where $\mathcal{H}(A)$ is the set of all subgroups of G containing the set A:

$$\mathcal{H}(A) = \{H < G | A \subset H \text{ and } H \text{ is a subgroup of } G\}$$

10.4.3 <u>Cyclic Subgroup</u> generated by a: $\langle a \rangle = \cap_{H<G; a \in H} H$ (G is <u>cyclic</u> if exists g, $\langle g \rangle = G$)

If $A = \{a\}$, then $\langle a \rangle \, (= \langle \{a\} \rangle)$ = the <u>cyclic subgroup</u> generated by a
Say G is <u>cyclic</u> if $\exists g \in G$, s.t. $G = \langle g \rangle$; g is called a <u>generator</u> for G in this case.

10.4.4 Proposition 3.2.3: $\langle g \rangle = \{g^n | n \in \mathbb{Z}\}$

Proposition 33 (Proposition 3.2.3). *Let G be a group, $g \in G$. Then*

$$\langle g \rangle = \{g^n | n \in \mathbb{Z}\}$$

10.4.5 Corollary 3.2.4: G is a cyclic group $\Rightarrow$ G is abelian

Corollary 10 (Corollary 3.2.4)**.** *If G is a cyclic group (i.e. exits $g \in G$ s.t. $\langle g \rangle = G$), then G is abelian (i.e. commutative).*

10.4.6 Equivalent properties of order of g: $|g| = |\langle g \rangle| < \infty$

Proposition 34 (Proposition 3.2.6)**.** *Let G be a group for $g \in G$, the following are equivalent:*

(i) $|g| < \infty$
(ii) $\exists n \neq m$ in $\mathbb{Z}$ so that $g^n = g^m$
(iii) $\exists n \in \mathbb{Z},\ n \neq 0$ so that $g^n = e$
(iv) $\exists n \in \mathbb{Z}_+$so that $g^n = e$
If $|g| < \infty$, then $|g| =$ smallest $n \in \mathbb{Z}_+$so that $g^n = e$, and $\langle g \rangle = \left\{e, g, g^2, \ldots, g^{n-1}\right\} = \{g^n \mid n = 0, \ldots, n-1\}$

10.4.7 $(\mathbb{Z}, +)$ Theorem 3.2.9: $H < \mathbb{Z}$ is a subgroup $\Rightarrow$ $H = \{0\}$ or $H = \langle d \rangle$; $\langle a \rangle < \langle b \rangle$ if and only if $b|a$

Theorem 8 (Theorem 3.2.9)**.** *If $H < \mathbb{Z}$ is a subgroup, then either $H = \{0\}$, or else $H = \langle d \rangle$, where*

$$d = \min\{h \in H | h > 0\}$$

*Consequently, $a \to \langle a \rangle$ defines a **bijection** from $N = \{0, 1, 2, ...\}$ to the set of subgroups of $\mathbb{Z}$. Furthermore, for $a, b \in \mathbb{Z}_+$, we have $\langle a \rangle < \langle b \rangle$ if and only if $b|a$.*

10.4.8 $(\mathbb{Z}_n, +)$ Theorem 3.2.10: $H < \mathbb{Z}_n$ is a subgroup $\Rightarrow$ $H = \langle [d] \rangle$; $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$

Theorem 9 (Theorem 3.2.10)**.** *For any $n \geq 2$, if $H < \mathbb{Z}_n$ is a subgroup, then there is a positive divisor d of n so that*

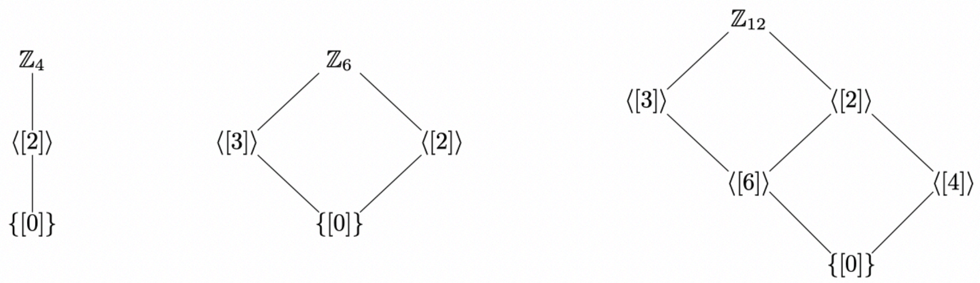
$$H = \langle [d] \rangle$$

Furthermore, this defines a bijection between divisors of H and subgroups of $\mathbb{Z}_n$. Furthermore, if $d, d' > 0$ are two divisors of n, then $\langle [d] \rangle < \langle [d'] \rangle$ if and only if $d'|d$.

If $H = \langle [d] \rangle$ is a subgroup of H, then $[n] \in H$, so $d|n$. And $|H| = |\langle [d] \rangle| = \frac{n}{d}$, so $|H||d$

10.4.9 Subgroup Lattice

The set of all subgroups of a group of G, together with the data of which subgroups contain which others is called the **subgroup lattice**. We often picture the subgroup lattice in a diagram with the entire group at the top, the trivial subgroup $\{e\}$ at the bottom, and the intermediate subgroups in the middle, with lines drawn from subgroups up to larger groups.



Writing down the subgroup lattice is as easy as writing down the divisibility lattice in which n is placed at the bottom, 1 at the top, and all intermediate divisors in between, connected by edges when there is divisibility. The congruence class of the divisor generates the corresponding subgroup in the subgroup lattice.

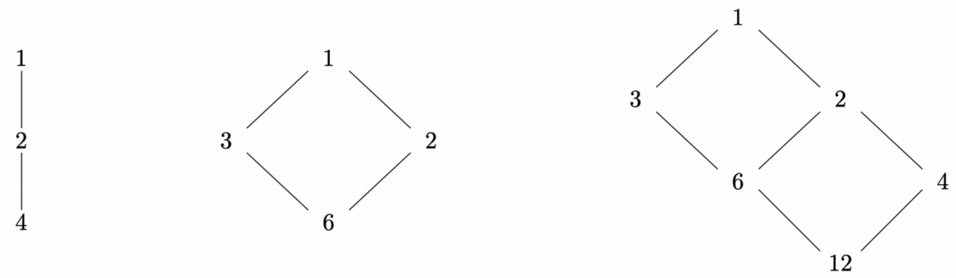


图 1:

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[1] Christopher J Leininger Introduction to Abstract Algebra (Draft) 2017.