Math 482 Linear programming
Fall 2020
Exam 2
10/19/2020
Time Limit: 180 Minutes

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Section: C13

This is the 3 CREDIT EXAM.

This exam contains 7 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books, notes, or any calculator on this exam.

You are required to justify your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	5	
4	5	
5	15	
6	20	
7	15	
Total:	100	

82/100

15/20

1. (20 points) Find the dual program of the linear program

$$\begin{aligned} \underset{x,y,z\in\mathbb{R}}{\text{minimize}} \quad & x - 2y + 2z \\ \text{subject to} \quad & x + y + z \le 3 \\ & y - z \le 1 \\ & x, y, z \ge 0 \end{aligned}$$

$-x-y-z \geqslant -3$

$-y+z \geqslant -1$

$$\begin{aligned} \max_{u_1,u_2,u_3\in R} \quad & -3u_1 - u_2 \\ \text{s.t.} \quad & -u_1 \le 1 \\ & -u_1 - u_2 \le -2 \\ & -u_1 + u_2 \le 2 \\ & u_1\ u_2 \geqslant 0 \end{aligned}$$

✓

in (P) Use complementary slackness to determine whether the primal solution $(x, y, z) = (0, 2, 1)$ is optimal.

$(x, y, z) = (0, 2, 1)$ is feasible

$-x-y-z = -3$

$-y+z = -1$ all tight $\Rightarrow$ satisfy complementary slackness

$\Rightarrow (x, y, z) = (0, 2, 1)$ is optimal.

C.S. is both directions!

You must find a dual sol via C.S and check the other dir of C.S.

Theorem 2 ***Complementary Slackness*** *Assume problem (P) has a solution x^* and problem (D) has a solution y^*.*

1. *If $x_j^* > 0$, then the jth constraint in (D) is binding.*
2. *If the jth constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the ith constraint in (P) is binding.*
4. *If the ith constraint in (P) is not binding, then $y_i^* = 0$.*

in (P) $(x, y, z) = (0, 2, 1)$

$\Rightarrow$ in (D) second and third constraint tight

$$\Rightarrow \begin{cases} u_1 + u_2 = 2 \\ u_1 - u_2 = -2 \end{cases} \Rightarrow \begin{cases} u_1 = 0 \\ u_2 = 2 \end{cases}$$

$(x, y, z) = (0, 2, 1)$, $(u_1, u_2) = (0, 2)$

is primal feasible is dual feasible.

and satisfy CS.

Theorem 1.2. *Let* $\mathbf{x}$ *be a primal feasible solution and let* $\mathbf{u}$ *be a dual feasible solution such that complementary slackness holds between* $\mathbf{x}$ *and* $\mathbf{u}$*. Then* $\mathbf{x}$ *and* $\mathbf{u}$ *are primal optimal and dual optimal, respectively.*

$(x, y, z) = (0, 2, 1)$ is optimal.

2. (20 points) Use the two-phase dual simplex method to solve the linear program

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{minimize}} \quad & 2x - y \\ \text{subject to} \quad & 3x + y \geq 10 \\ & x - y \geq 2 \\ & x, y \geq 0 \end{aligned}$$

12/20

You did simplex method on the dual. This is not the same as dual simplex method.

dual LP: $\max_{u_1,u_2\in\mathbb{R}} \; 10u_1 + 2u_2$

s.t. $3u_1 + u_2 \leq 2$

$u_1 - u_2 \leq -1$

$u_1, u_2 \geq 0$

	u_1	u_2	s_1	s_2	
s_1	3	1	1	0	2
s_2	1	-1	0	1	-1
$-z$	10	2	0	0	0

Pivot on u_2 → remove s_2

	u_1	u_2	s_1	s_2	
s_1	4	0	1	1	1
u_2	-1	1	0	-1	1
$-z$	12	0	0	2	-2

Pivot on u_1 → remove s_1

	u_1	u_2	s_1	s_2	
u_1	1	0	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
u_2	0	1	$\frac{1}{4}$	$-\frac{3}{4}$	$\frac{5}{4}$
$-z$	0	0	-3	-1	-5

→ optimal solution of primal LP is $(x, y) = (3, 1)$

optimal value is 5

$$\min \ 2x - y$$

$$\text{s.t.} \quad -3x - y + S_1 = -10$$

$$-x + y + S_2 = -2$$

$$x, y, S_1, S_2 \geq 0$$

	x	y	S_1	S_2	
S_1	-3	-1	1	0	-10
S_2	-1	1	0	1	-2
$-z$	2	-1	0	0	0

$\downarrow$

	x	y	S_1	S_2	
y	3	1	-1	0	10
S_2	-4	0	1	1	-12
$-z$	5	0	-1	0	10

$\downarrow$

	x	y	S_1	S_2	
y	0	1	$-\frac{1}{4}$	$\frac{3}{4}$	1
x	1	0	$-\frac{1}{4}$	$-\frac{1}{4}$	3
$-z$	0	0	$\frac{1}{4}$	$\frac{5}{4}$	-5

3. (5 points) In the problem below, A is an $m \times n$ matrix, $\mathbf{b} \in \mathbb{R}^m$, and $\mathbf{c} \in \mathbb{R}^n$. (**P**) and (**D**) refer to the two linear programs below:

3/5

Optimal 都考虑了

$$(\mathbf{P}) \begin{cases} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{cases} \qquad (\mathbf{D}) \begin{cases} \underset{\mathbf{u}\in\mathbb{R}^m}{\text{minimize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{u}^\mathsf{T}A \ge \mathbf{c}^\mathsf{T} \\ & \mathbf{u} \ge \mathbf{0} \end{cases}$$

feasible 不用考虑 c   feasible 不用考虑 b.

Check the box next to each statement if it **must** be true.

- X ☐ ~~At least one of (**P**) or (**D**) is feasible.~~
- ☐ At most one of (**P**) or (**D**) is unbounded.
- X ☐ ~~If~~ (**D**) has a feasible solution, (**P**) has a feasible solution.
- ☐ If (**P**) has an optimal solution, (**D**) has an optimal solution.
- ☐ If (**P**) has a unique optimal solution, (**D**) has a unique optimal solution.

$$u^\mathsf{T} b \ge u^\mathsf{T} A x \ge c^\mathsf{T} x$$

4. (5 points) In the problem below, (**P**) and (**D**) refer to the same linear programs as in Problem 3. Moreover, $\mathbf{x} \in \mathbb{R}^n$ is a feasible solution for (**P**), and $\mathbf{u} \in \mathbb{R}^m$ is a feasible solution for (**D**).

Check the box next to each statement if it **must** be true.

5/5

- ☐ $\mathbf{c}^\mathsf{T}\mathbf{x} \ge \mathbf{u}^\mathsf{T}\mathbf{b}$.
- ☐ If $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$, then $\mathbf{x}$ is primal optimal and $\mathbf{u}$ is dual optimal.
- ☐ If $\mathbf{x}$ is primal optimal and $\mathbf{u}$ is dual optimal, then $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$.
- ☐ If $\mathbf{x}$ is primal optimal, $\mathbf{u}$ is dual optimal, and $x_i = 0$, then the i^{th} constraint of (**D**) is slack.
- ☐ If $\mathbf{x}$ is primal optimal, $\mathbf{u}$ is dual optimal, and $u_i > 0$, then the i^{th} constraint of (**P**) is tight.

15/15

5. (15 points) Consider a **minimization** problem, given in standard form with $Ax \le b, x \ge 0$ whose optimal tableau is given below:

	x_1	x_2	s_1	s_2	s_3	
x_1	1	0	0	2	3	4
x_2	0	1	0	1	-2	2
s_1	0	0	1	-1	1	3
$-z$	0	0	0	3	1	-2

(a) Suppose that the right-hand side of the first constraint (with slack variable s_1) is changed by some small amount δ (either positive or negative). Determine the corresponding change in optimal objective value, **and the range of values of** δ for which your prediction holds.

	x_1	x_2	s_1	s_2	s_3	
x_1	1	0	0	2	3	4
x_2	0	1	0	1	-2	2
s_1	0	0	1	-1	1	$3+\delta$
$-z$	0	0	0	3	1	-2

$3+\delta \geq 0$

$\delta \geq -3$ the optimal objective value doesn't change.

(b) Suppose that the coefficient of x_2 in the objective function is changed by some small amount δ (either positive or negative). Determine the corresponding change in optimal objective value, **and the range of values of** δ for which your prediction holds.

	x_1	x_2	s_1	s_2	s_3	
x_1	1	0	0	2	3	4
x_2	0	1	0	1	-2	2
s_1	0	0	1	-1	1	3
$-z$	0	$0+\delta$	0	3	1	-2

$\longrightarrow$

	x_1	x_2	s_1	s_2	s_3	
x_1	1	0	0	2	3	4
x_2	0	1	0	1	-2	2
s_1	0	0	1	-1	1	3
$-z$	0	0	0	$3-\delta$	$1+2\delta$	$-2-2\delta$

the optimal objective value change into $2+2\delta$

$3-\delta \geq 0$

$1+2\delta \geq 0$

$\Rightarrow -\frac{1}{2} \leq \delta \leq 3$

A \ B	B:1	B:2	B:3
A:1	−2	3	−4
A:4	5	−6	7

20/20

6. (20 points) In a variant of the odd-even game discussed in class, Alice holds up either 1 or 4 fingers while Bob simultaneously holds up 1, 2, or 3 fingers. Letting N be the total number of fingers, if N is even then Bob wins \$$N$ from Alice and if N is odd then Alice wins \$$N$ from Bob.

(a) Write down a linear program for Alice's optimal strategy and a linear program for Bob's optimal strategy.

$u = \min\{-2x_1+5x_4,\ 3x_1-6x_4,\ -4x_1+7x_4\}$

$v = \max\{-2y_1+3y_2-4y_3,\ 5y_1-6y_2+7y_3\}$

A's LP:

$$\max_{u,\,x_1,\,x_4\in\mathbb{R}} u$$
$$\text{s.t. } u \le -2x_1+5x_4$$
$$u \le 3x_1-6x_4$$
$$u \le -4x_1+7x_4$$
$$x_1+x_4=1$$
$$x_1, x_4, u \ge 0$$

✓

B's LP:

$$\min_{v,\,y_1,\,y_2,\,y_3\in\mathbb{R}} v$$
$$\text{s.t. } v \ge -2y_1+3y_2-4y_3$$
$$v \ge 5y_1-6y_2+7y_3$$
$$y_1+y_2+y_3=1$$
$$y_1, y_2, y_3, v \ge 0$$

✓

(b) Bob's optimal strategy for this game is to hold up 2 fingers 55% of the time and hold up 3 fingers 45% of the time. Use complementary slackness to determine Alice's optimal strategy.

Bob's opt $(y_1, y_2, y_3) = (0, 0.55, 0.45)$

$$\left.\begin{aligned} -2y_1+3y_2-4y_3 &= -0.15 \\ 5y_1-6y_2+7y_3 &= -0.15 \end{aligned}\right\} \Rightarrow v = -0.15$$

$\Rightarrow$ satisfy complementary $v = u = -0.15$

$$y_2, y_3 \ne 0 \Rightarrow \begin{cases} 3x_1-6x_4 = -0.15 \\ -4x_1+7x_4 = -0.15 \end{cases} \Rightarrow \begin{cases} x_1 = 0.65 \\ x_4 = 0.35 \end{cases}$$

✓

$\Rightarrow$ Alice's opt strategy $= (x_1, x_4) = (0.65, 0.35)$ ✓

65% hold up 1, 35% hold up 4.

12/15

7. (15 points) Use Fourier-Motzkin elimination to give a linear combination of the following inequalities that proves that there is no (x, y, z) satisfying all four inequalities.

$$\begin{array}{lrl} (a) & -x-y-2z & \le -1 \\ (b) & x-y-z & \le -2 \\ (c) & -x+y-z & \le -1 \\ (d) & y+3z & \le 0 \end{array}$$

$$\Rightarrow \begin{cases} (a) & x \geq -y-2z+1 \\ (b) & x \leq -2+y+z \\ (c) & x \geq y-z+1 \\ (d) & y+3z \leq 0 \end{cases}$$

$$\begin{cases} (a)+(b) & -y-2z+1 \leq -2+y+z \\ (b)+(c) & y-z+1 \leq -2+y+z \\ (d) & y+3z \leq 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}[(a)+(b)] & y \geq -\frac{3}{2}z+\frac{3}{2} \\ (d) & y \leq -3z \\ \frac{1}{2}[(b)+(c)] & z \geq \frac{3}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{2}[(a)+(b)]+(d) & -\frac{3}{2}z+\frac{3}{2} \leq -3z \\ \frac{1}{2}[(b)+(c)] & z \geq \frac{3}{2} \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{3}[(a)+(b)]+\frac{2}{3}(d) & z \leq -\cancel{\frac{3}{2}}\ -1 \\ \frac{1}{2}[(b)+(c)] & z \geq \frac{3}{2} \end{cases}$$

$\Rightarrow$ there is no feasible z

$\Rightarrow$ there is no (x, y, z) satisfying all four inequalities.

Read directions.

Asks for a linear combination.

$$\frac{1}{3}(a)+\frac{5}{6}(b)+\frac{1}{2}(c)+\frac{2}{3}(d):\ \frac{5}{2} \leq 0$$