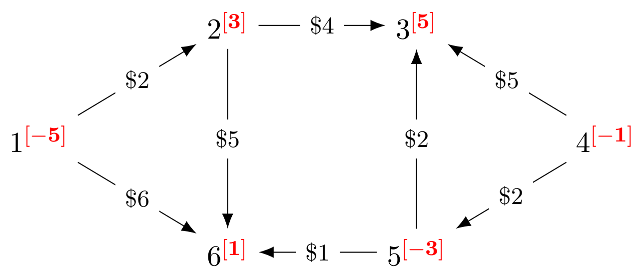
Math 482: Linear Programming[1]	Mikhail Lavrov

A Worked Example of Minimum-Cost Flow

April 13, 2020	*University of Illinois at Urbana-Champaign*

1 The problem

We will use the min-cost flow simplex method to find a minimum-cost flow in the following network:

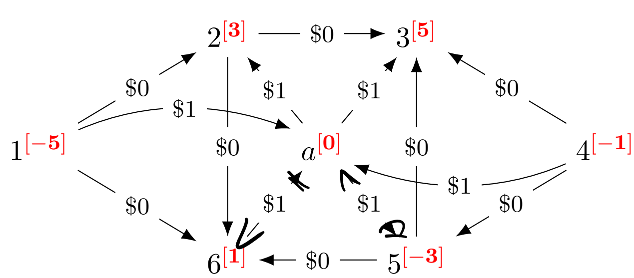


Some comments on notation: first, since we are always going to be dealing with feasible flows, I will usually not write down the demands on the nodes: all we have to do to make sure those are satisfied is to avoid changing the net flow into a node.

Second, when writing down spanning tree solutions, I will only draw the arcs in the spanning tree, and I will label them with the flows along those arcs, not the costs.

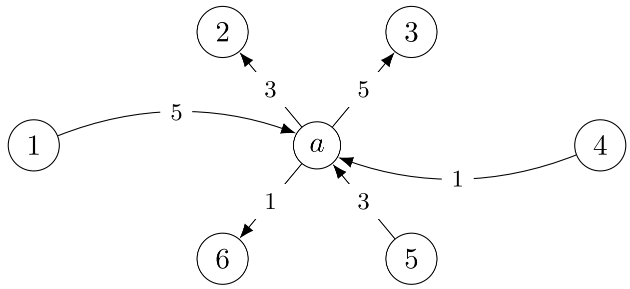
2 The phase-one problem

In the first phase, we modify the network by adding an artificial node a with demand $d_a = 0$. For each node with positive demand, we add an arc to a; for each node with negative demand, we add an arc from a. The costs of the artificial arcs are all 1 for this phase; the costs of the original arcs are all 0.



The reason we set things up like that is so that we can start with the spanning tree solution below:

[1]This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html



We can check that this satisfies all the demands (in red in the previous diagram). Note that these arcs are labeled with the flows along them, not the costs: all six arcs being used have cost 1, so the total cost is $5+3+5+1+3+1=18$.

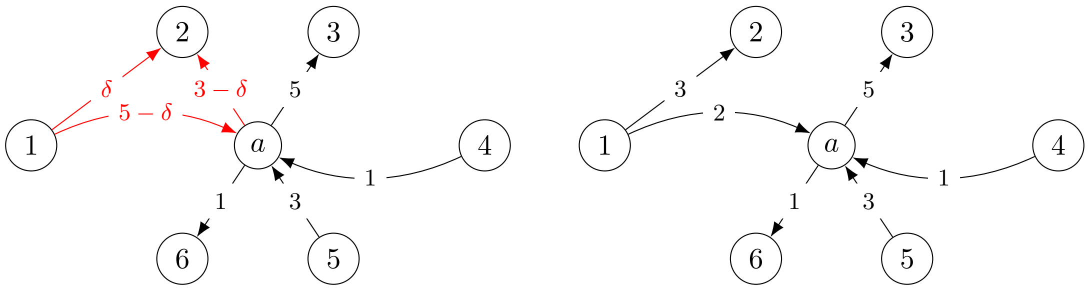
There are 8 different arcs we could bring into the basis. For this step, I will compute all 8 reduced costs as a demonstration:

- Arc $(1,2)$ forms a cycle with arcs $(1,a)$ and $(a,2)$ in the spanning tree. Both of these arcs go in the opposite direction around the cycle, so the reduced cost is $c_{12}-c_{1a}-c_{2a}=0-1-1=-2$.
- Arc $(1,6)$ forms a cycle with arcs $(1,a)$ and $(a,6)$ in the spanning tree. oth of these arcs go in the opposite direction around the cycle, so the reduced cost is $c_{16}-c_{1a}-c_{a6}=0-1-1=-2$.
- Arc $(2,3)$ forms a cycle with arcs $(a,2)$ and $(a,3)$ in the spanning tree. Arc $(a,2)$ has the same direction but arc $(a,3)$ has the opposite direction around the cycle, so the reduced cost is $c_{23}+c_{a2}-c_{a3}=0+1-1=0$.
- Arc $(2,6)$ forms a cycle with arcs $(a,2)$ and $(a,6)$ in the spanning tree. Arc $(a,2)$ has the same direction but arc $(a,6)$ has the opposite direction around the cycle, so the reduced cost is $c_{26}+c_{a2}-c_{a6}=0+1-1=0$.
- Arc $(4,3)$ forms a cycle with arcs $(4,a)$ and $(a,3)$ in the spanning tree. Both of these arcs go in the opposite direction around the cycle, so the reduced cost is $c_{43}-c_{4a}-c_{a3}=0-1-1=-2$.
- Arc $(4,5)$ forms a cycle with arcs $(4,a)$ and $(5,a)$ in the spanning tree. Arc $(4,a)$ has the opposite direction but arc $(5,a)$ has the same direction around the cycle, so the reduced cost is $c_{45}-c_{4a}+c_{5a}=0-1+1=0$.
- Arc $(5,3)$ forms a cycle with arcs $(5,a)$ and $(a,3)$ in the spanning tree. Both of these arcs go in the opposite direction around the cycle, so the reduced cost is $c_{53}-c_{5a}-c_{a3}=0-1-1=-2$.
- Arc $(5,6)$ forms a cycle with arcs $(5,a)$ and $(a,6)$ in the spanning tree. Both of these arcs go in the opposite direction around the cycle, so the reduced cost is $c_{56}-c_{5a}-c_{a6}=0-1-1=-2$.

We see that arcs $(1,2)$, $(1,6)$, $(4,3)$, $(5,3)$, and $(5,6)$ are valid arcs to pivot on; let's just pick the first of these, which is $(1,2)$.

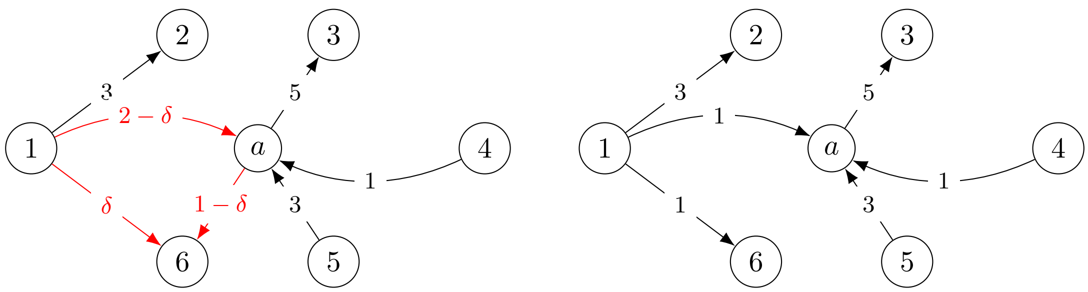
When we increase x_{12} to δ, we must decrease x_{1a} and x_{a2} by δ to preserve feasibility, as in the first diagram below.

Since we want $x_{1a}=5-\delta\geq 0$ and $x_{a2}=3-\delta\geq 0$, we must have $\delta\leq 5$ and $\delta\leq 3$, so we set $\delta=3$. When we do this, x_{a2} becomes 0, so arc $(a,2)$ leaves the spanning tree. We get the updated spanning tree in the second diagram below.



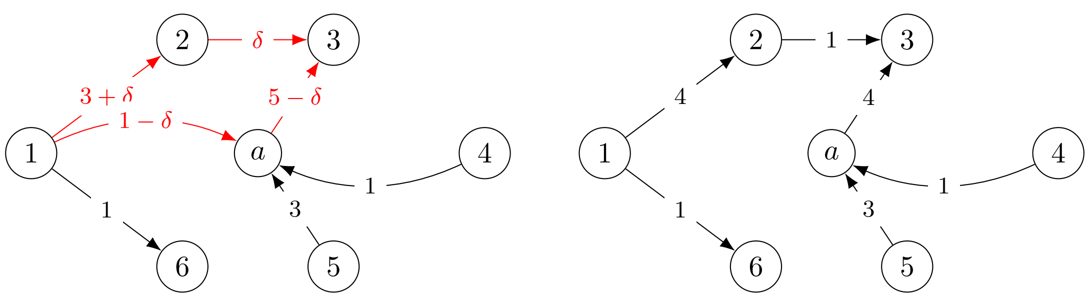
For the next step, note that only a small part of the spanning tree has changed. If the cycle for an arc didn't include arc $(a, 2)$, then it will stay the same, and so will the reduced cost. In particular, the reduced cost of x_{16} is still $c_{16} - c_{1a} - c_{a6} = 0 - 1 - 1 = -2$, so we can pivot on x_{16}.

When we increase x_{16} to δ, we must decrease x_{1a} and x_{a6} by δ to preserve feasibility, as in the first diagram below. Since we want $x_{1a} = 2 - \delta \geq 0$ and $x_{a6} = 1 - \delta \geq 0$, we must have $\delta \leq 2$ and $\delta \leq 1$, so we set $\delta = 1$. When we do this, x_{a6} becomes 0, so arc $(a, 6)$ leaves the spanning tree. We get the updated spanning tree in the second diagram below.



Next, we look at arc $(2, 3)$, which also has a positive reduced cost: it's in a cycle with arcs $(1, 2)$, $(1, a)$, and $(a, 3)$, and arcs $(1, a)$ and $(a, 3)$ both go in the opposite direction around the cycle, so the reduced cost of x_{23} is $c_{23} + c_{12} - c_{1a} - c_{a3} = 0 + 0 - 1 - 1 = -2$. So we can pivot on x_{23}.

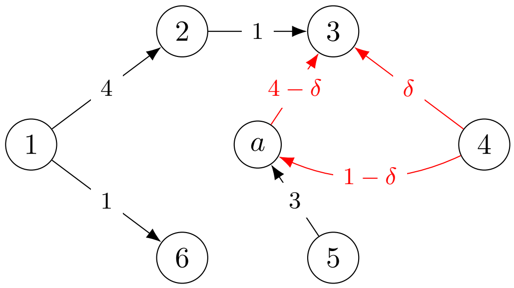
The diagram with the δ-change and the updated spanning tree when we pivot are shown below; x_{1a} leaves the basis.

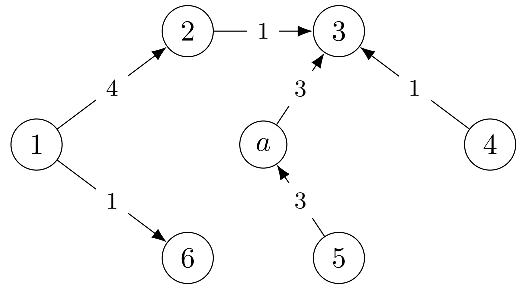


By the way, to verify that the signs on all these δ's are correct, the thing to do is to check that the excess flow at each node around the cycle doesn't depend on δ. For example, at nod a, the flow in is $1 + 3 + (1 - \delta)$, and the flow out is $5 - \delta$, so $\Delta_a(\mathbf{x}) = 1 + 3 + (1 - \delta) - (5 - \delta) = 0$. In general, this quantity should have started at d_k for a node k, and we want to keep it at d_k.

Next, we can pivot on arc $(4, 3)$, whose reduced cost hasn't changed this whole time: it's still in an arc with arcs $(4, a)$ and $(a, 3)$, both of which go in the opposite direction around the cycle.

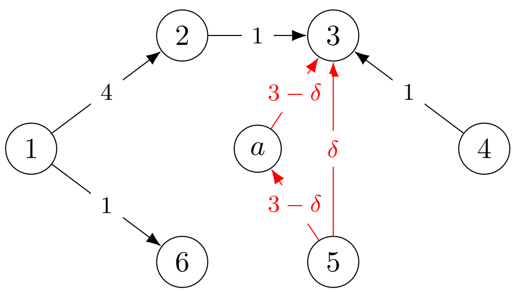
The diagram with the δ-change and the updated spanning tree when we pivot are shown below; x_{4a} leaves the basis.

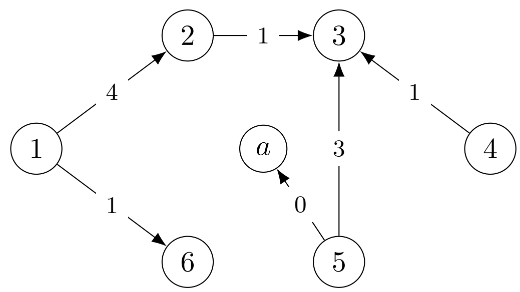




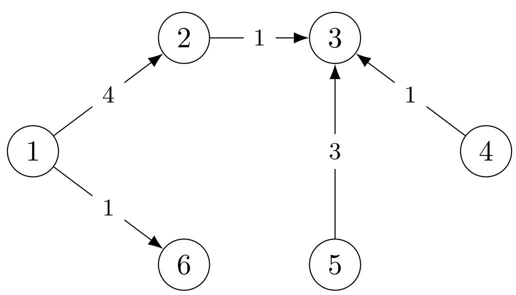
Finally, we can pivot on x_{53}; arc $(5, 3)$ is in a cycle with $(5, a)$ and $(a, 3)$, both of which have opposite directions around the cycle, so the reduced cost is $c_{53} - c_{5a} - c_{a3} = 0 - 1 - 1 = -2$.

When we set $x_{53} = \delta$, we get $x_{5a} = x_{a3} = 3 - \delta$, so at $\delta = 3$, both of them become 0. Normally, this would be a sign of degeneracy, and we'd keep one of them to keep around anyway, even with flow 0. For example, we could keep x_{5a}, as in the second diagram below.





But this particular form of degeneracy is one that we expect at the very end of the first phase of the two-phase method here. In this case, we can just take out both arcs $(5, a)$ and $(a, 3)$, and also take out node a. We are left with a spanning tree solution to the original problem:



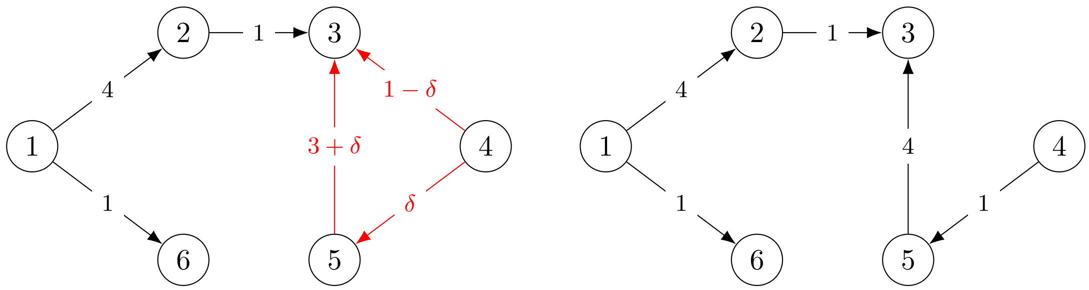
3 The second phase

In the second phase, we have only three arcs to price: the arcs that aren't in the spanning tree are arcs $(2, 6)$, $(4, 5)$, and $(5, 6)$.

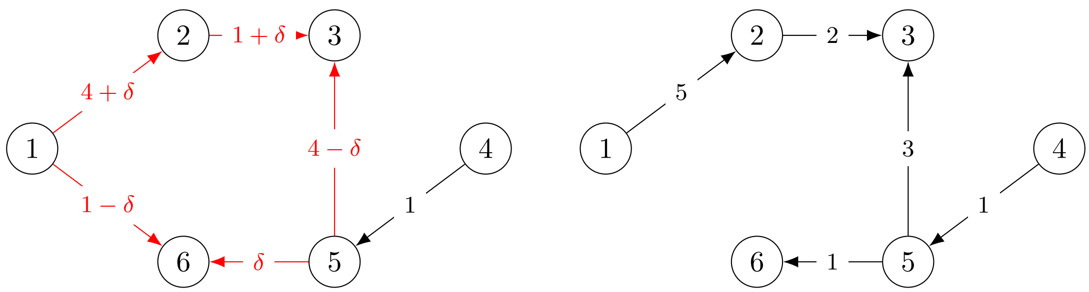
- Arc $(2, 6)$ is in a cycle with arcs $(1, 2)$ and $(1, 6)$. Arc $(1, 2)$ has the same direction around the cycle, and arc $(1, 6)$ has the opposite direction. So the reduced cost of x_{26} is $c_{26} + c_{12} - c_{16} = 5 + 2 - 6 = 1$.

- Arc $(4,5)$ is in a cycle with arcs $(4,3)$ and $(5,3)$. Arc $(4,3)$ has the opposite direction around the cycle, and arc $(5,3)$ has the same direction. So the reduced cost of x_{45} is $c_{45} - c_{43} + c_{53} = 2 - 5 + 2 = -1$.
- Arc $(5,6)$ is in a cycle with arcs $(1,6)$, $(1,2)$, $(2,3)$, and $(5,3)$. Arcs $(1,2)$ and $(2,3)$ have the same direction around the cycle, and arcs $(1,6)$ and $(5,3)$ have the opposite direction. So the reduced cost of x_{56} is $c_{56} - c_{16} + c_{12} + c_{23} - c_{53} = 1 - 6 + 2 + 4 - 2 = -1$.

Let's pivot on arc $(4,5)$ first. This is done in the same way as our pivoting steps in phase one:



Note that we still have the same reduced cost on arcs $(2,6)$ and $(5,6)$, because their cycles haven't changed. Also, arc $(4,3)$ has a positive reduced cost, because pivoting on it would undo our pivoting step just now. So we can continue to pivot on x_{56} without any further pricing calculations:



Next, we have nonbasic variables x_{43}, x_{16}, x_{26} to choose from. The reduced cost of x_{43} hasn't changed, so it's still positive. The reduced cost of x_{16} is positive, because it left the basis just now. But we should recompute the reduced cost of x_{26}.

Arc $(2,6)$ is in a cycle with $(2,3)$, $(5,3)$, and $(5,6)$. Of these, arc $(5,3)$ goes in the same direction, and arcs $(2,3)$ and $(5,6)$ go in the opposite direction. So the reduced cost of x_{26} is $c_{26} - c_{23} + c_{53} - c_{56} = 5 - 4 + 2 - 1 = 2$.

This is positive, so we don't pivot on x_{26}; since all other reduced costs were positive as well, we've found the optimal solution.