notions of corner points: $\begin{cases} \text{feasible basic solutions} \\ \text{vertex.} \\ \text{extreme point.} \end{cases}$

$$\min_{x \in \mathbb{R}^n} \; C^T x$$

$$\text{s.t. } Ax = b$$

$$x \geq 0$$

(rows of A lin. indep.)

Collection of $m$ vars $\underline{\underline{B}}$ to be basic

$n-m$ vars $N$ is nonbasic vars.

$X = (X_1, X_2, X_3, X_4, X_5)$, $B = (2,4)$. $N = (1,3,5)$.

$X_B = (X_2, X_4)$, $X_N = (X_1, X_2, X_3)$

$$\begin{cases} C^T X = C_B^T X_B + C_N^T \cdot X_N \\ AX = A_B X_B + A_N X_N = b. \end{cases}$$

Get basic solution: choose $B$ so that $A_B$ invertible.

$$\underbrace{X_N = 0, \; X_B = A_B^{-1} b \; \Rightarrow AX = b}_{\text{Satisfy}}$$

$\rightarrow$ basic feasible solution.

Since $X_N \geq 0$ if $X_B \geq 0 \Rightarrow X = X_A + X_B \geq 0$.

$\longrightarrow$ $X$ has Basic Feasible Solution.

Other notions of corner points:

(1) A vertex of a set $S \subseteq \mathbb{R}^n$ is a point $x \in S$, s.t. $\underset{\text{exist}}{\text{some}}$ liner function $\alpha^T x$ is strictly minimized at $x$;

$$\{x \mid \alpha^T x < \alpha^T y \quad \forall y \in S, y \neq x\}.$$

(2) An extreme point of $S \subseteq \mathbb{R}^n$, if whenever $x \in [y, y']$ with $y, y' \in S$, either $x = y$ or $x = y'$.

If out set is $F = \underbrace{\{x \in \mathbb{R}^n \mid Ax = b,\ x \geq 0\}}_{\text{feasible region of LP.}}$

all three notions are the same.

Prop 2.1: Any basic feasible solution is a vertex. of feasible region.

Proof: Take any choice of $(B, N)$. $x_N = 0$.

$\Longrightarrow$ basic feasible solution.

Define $\alpha_i = \begin{cases} 1, & i \in N \\ 0, & i \in B \end{cases}$.

$\Longrightarrow x_N = 0$ minimize $\alpha^T x$.

$\Longrightarrow x$ is a vertex.

**Prop 2.2:** Any vertex of $S \subseteq \mathbb{R}^n$ is an extreme point of $S$.

**Proof:** $x \in S$ is a vertex, $\alpha$ is vector s.t. $\alpha^T x < \alpha^T y$ for any $y \in S$, $y \neq x$.

Suppose, for contradiction, $x$ lies on the segment $[y, y']$ with $y, y' \in S$ with $y, y' \neq x$. (not an extreme point.)

Then $x = ty + (1-t)y'$, $t \in (0,1)$

$$\alpha^T x = t(\alpha^T y) + (1-t)(\alpha^T y')$$

$$> t(\alpha^T x) + (1-t)(\alpha^T x) = \alpha^T x$$

$\Longrightarrow$ contradiction.

Hence $x$ must be an extreme point.

**Prop 2.3:** Any extreme point of the feasible region is a basic feasible solution.

**Proof:** Let $x$ be an extreme point

$X = X_W + X_Z$, $W = \{ i \mid x_i \neq 0 \}$, $Z = \{ i \mid x_i = 0 \}$

we can't ask is $A_W$ invertible, $A_W$ may not be square.

instead: Does there exist nonzero $u \in \mathbb{R}^n$?

s.t. (*) $\begin{cases} A_W u_W = 0 \\ u_Z = 0. \end{cases}$

(1) Suppose there is no such $u \in \mathbb{R}^n$.

So, whenever $A_W u_W = 0 \Rightarrow u_W = 0 \Rightarrow u = 0$.

$\Rightarrow$ Colums indexed by $W$ are $|W|$ lin. indep. columns, $|W| \leq m$.

$A$ is lin. indep (full rank) $\Rightarrow$ extend $W$ to $B$ with $|B| = m$, coluns indexed by $B$ are lin. indep.,

Let $A_B$ be square.

$\left[\ \underbrace{\boxed{\phantom{A}}}_{\text{lin. indep}}\ \begin{matrix}0\\0\\0\end{matrix}\ \right] \rightarrow$ still lin. indep.

$A_B$ is invertible.

$N$ be the complement of $B$. $N \subseteq Z$.

$X_Z = 0 \Rightarrow X_N = 0$

$$Ax = A_B x_B + A_N x_N = b$$

$$\Longrightarrow A_B x_B = b \Rightarrow x_B = A_B^{-1} b.$$

$x_B$ is basic feasible solution.

(2) Suppose such a $u$ exists.

Then $Au = A_w u_w + A_z u_z = 0 + 0 = 0.$

$Ax = b \Longrightarrow A(x + tu) = b$, $\forall x \in$ Feasible region

Note: when $|t|$ is large $x + tu < 0$

$\Longrightarrow x + tu \notin$ Feasible region.

but when $|t|$ is small enough

$x + tu > 0$ and $x - tu > 0$

Since $x \in [x - tu, x + tu]$, so $x$ is not a extreme point

$\Longrightarrow$ Contradiction $\times$.