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Math 482: Linear Programming, Fall 2020

Due Friday, October 9, 6PM CST

Homework 5

1. Use the dual simplex method with an artificial objective function to find a solution to the system of equations
$$\begin{cases} x_1 - x_2 + 4x_3 \phantom{{}+x_4} = 4 \\ x_1 + x_2 \phantom{{}+4x_3+x_4} = 2 \\ x_1 + 2x_2 - 2x_3 + x_4 = 3 \end{cases}$$
in which $x_1, x_2, x_3, x_4 \geq 0$.

For problems 2 and 3, suppose that you have solved the linear program below on the left, and gotten the simplex tableau below on the right.

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{minimize}} \quad & x + 2y \\ \text{subject to} \quad & x + y \geq 3 \\ & x + 4y \geq 10 \\ & x \phantom{{}+4y} \geq 2 \\ & x, y \geq 0 \end{aligned}$$

	x	y	s_1	s_2	s_3	
x	1	0	0	0	-1	2
y	0	1	0	$-1/4$	$1/4$	2
s_1	0	0	1	$-1/4$	$-3/4$	1
$-z$	0	0	0	$1/2$	$1/2$	-6

2. Describe how the objective value will change, for sufficiently small values of δ, in each of the following cases. State whether your prediction will be a lower bound or an upper bound in general (when δ is large).
   (a) The objective function changes from $x + 2y$ to $x + (2 + \delta)y$.
   (b) The constraint $x \geq 2$ changes to $x \geq 2 + \delta$.
   (Be careful! In equational form, $x \geq 2 + \delta$ is represented as $-x + s_3 = -2 - \delta$.)
   (c) The constraint $x + y \geq 3$ changes to $x + y \geq 3 + \delta$.
3. Use the dual simplex method to add the constraint $x + 5y \leq 11$ to the linear program and find the new optimal solution.
4. Illinois Instruments (II) is a company that makes calculators. Their three models are:
   - The II-91, which can do basic arithmetic operations.
     (5 ounces of plastic, 3 hours to produce, sells for \$65)
   - The II-92, which can solve linear equations.
     (8 ounces of plastic, 5 hours to produce, sells for \$100)
   - The II-93, which can perform the simplex method.
     (12 ounces of plastic, 8 hours to produce, sells for \$160)

   Their factory in Champaign receives a shipment of 320 ounces of plastic every week. They have 5 employees making calculators, each of which works for 40 hours every week.
   (a) How much of each calculator model should II produce each week to maximize profit?
   (b) The factory manager is considering purchasing more plastic, at a price of \$$D$ per ounce. For which range of D is this profitable?

(c) At most how many extra ounces of plastic per week could be purchased before your answer to (b) might stop being valid?

5. *(Only 4-credit students need to do this problem.)*

All linear programs are either unbounded, infeasible, or have an optimal solution.

Is it possible to have a linear program with constraints $A\mathbf{x} \le \mathbf{b}$ and $\mathbf{x} \ge \mathbf{0}$ such that, just by changing the value of $\mathbf{b}$, we can get a linear program of all three types?

1. $$\begin{cases} x_1 - x_2 + 4x_3 = 4 \\ x_1 + x_2 = 2 \\ x_1 + 2x_2 - 2x_3 + x_4 = 3 \end{cases} \longrightarrow \begin{cases} x_1 - x_2 + 4x_3 = 4 \\ 2x_2 - 4x_3 = -2 \\ 3x_2 - 6x_3 + x_4 = -1 \end{cases}$$

$$\longrightarrow \begin{cases} x_1 + 2x_3 = 3 \\ x_2 - 2x_3 = -1 \\ x_4 = 2 \end{cases}$$

basic vars (x_1, x_2, x_4)

artificial function: x_3

	x_1	x_2	x_3	x_4	
x_1	1	0	2	0	3
x_2	0	1	-2	0	-1
x_4	0	0	0	1	2
$-z$	0	0	1	0	0

$\xrightarrow[\text{remove } x_2]{\text{Pivot on } x_3}$

	x_1	x_2	x_3	x_4	
x_1	1	1	0	0	2
x_3	0	$-\frac{1}{2}$	1	0	$\frac{1}{2}$
x_4	0	0	0	1	2
$-z$	0	$\frac{1}{2}$	0	0	$-\frac{1}{2}$

$\Longrightarrow$ primal feasible solution $(x_1, x_2, x_3, x_4) = (2, 0, \frac{1}{2}, 2)$

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{minimize}} \quad & x + 2y \\ \text{subject to} \quad & x + y \geq 3 \\ & x + 4y \geq 10 \\ & x \geq 2 \\ & x, y \geq 0 \end{aligned}$$

	x	y	s_1	s_2	s_3	
x	1	0	0	0	-1	2
y	0	1	0	$-1/4$	$1/4$	2
s_1	0	0	1	$-1/4$	$-3/4$	1
$-z$	0	0	0	$1/2$	$1/2$	-6

opt sol is $(x, y, S_1, S_2, S_3) = (2, 2, 1, 0, 0)$

opt val is 6.

2. (a) opt val will be $6+2\delta$ (δ is small)

$6+2\delta$ is upper bound (δ is large).

(b)

	x	y	s_1	s_2	s_3	
x	1	0	0	0	-1	$2+\delta$
y	0	1	0	$-1/4$	$1/4$	$2-\frac{\delta}{4}$
s_1	0	0	1	$-1/4$	$-3/4$	$1+\frac{3}{4}\delta$
$-z$	0	0	0	$1/2$	$1/2$	$-6-\frac{\delta}{2}$

opt val will be $6+\frac{\delta}{2}$ (δ is small)

$6+\frac{\delta}{2}$ is upper bound (δ is large)

lower

(c)

	x	y	s_1	s_2	s_3	
x	1	0	0	0	-1	2
y	0	1	0	$-1/4$	$1/4$	2
s_1	0	0	1	$-1/4$	$-3/4$	$1-\delta$
$-z$	0	0	0	$1/2$	$1/2$	-6

opt vall won't change, still be 6.

(δ is small).

lower

6 is upper bound (δ is large).

3.

	x	y	S_1	S_2	S_3	S_4	
x	1	0	0	0	-1	0	2
y	0	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	2
S_1	0	0	1	$-\frac{1}{4}$	$-\frac{3}{4}$	0	1
S_4	1	5	0	0	0	1	11
$-z$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-6

$\longrightarrow$

	x	y	S_1	S_2	S_3	S_4	
x	1	0	0	0	-1	0	2
y	0	1	0	$-\frac{1}{4}$	$\frac{1}{4}$	0	2
S_1	0	0	1	$-\frac{1}{4}$	$-\frac{3}{4}$	0	1
S_4	0	0	0	$\frac{5}{4}$	$-\frac{1}{4}$	1	-1
$-z$	0	0	0	$\frac{1}{2}$	$\frac{1}{2}$	0	-6

pivot on S_3 remove S_4 $\longrightarrow$

	x	y	S_1	S_2	S_3	S_4	
x	1	0	0	-5	0	-4	6
y	0	1	0	1	0	0	1
S_1	0	0	1	-4	0	-3	4
S_3	0	0	0	-5	1	-4	4
$-z$	0	0	0	3	0	2	-8

new opt solution : $(x, y, s_1, s_2, s_3, s_4)$

$= (6, 1, 4, 0, 0, 4)$

opt value is 8.

4. (a)

Max $65x + 100y + 160z$

s.t. $5x + 8y + 12z \leq 320$

$3x + 5y + 8z \leq 200$

$x, y, z \geq 0.$

	x	y	z	S_1	S_2	
S_1	5	8	12	1	0	320
S_2	3	5	8	0	1	200
$-\pi$	65	100	160	0	0	0

pivot on z, remove S_2

	x	y	z	S_1	S_2	
S_1	$\frac{1}{2}$	$\frac{1}{2}$	0	1	$-\frac{3}{2}$	20
z	$\frac{3}{8}$	$\frac{5}{8}$	1	0	$\frac{1}{8}$	25
$-\pi$	5	0	0	0	-20	-4000

Pivot on x remove S_1

	x	y	z	S_1	S_2	
x	1	1	0	2	-3	40
z	0	$\frac{1}{4}$	1	$-\frac{3}{4}$	$\frac{5}{4}$	10
$-\pi$	0	-5	0	-10	-5	-4200

opt sol is $(x, y, z, S_1, S_2) = (40, 0, 10, 0, 0)$

opt value is 4200.

II-91 40, II-92 0, II-93 10.

(b)

$$\text{Max } 65x + 100y + 160z - Dk$$

$$\text{s.t. } 5x + 8y + 12z \leq 320 + k$$

$$3x + 5y + 8z \leq 200$$

$$x, y, z, k \geq 0$$

The initial tableau

	x	y	z	S_1	S_2	k	
S_1	5	8	12	1	0	-1	320
S_2	3	5	8	0	1	0	200
$-\pi$	65	100	160	0	0	$-D$	0

The step 3 tableau.

	x	y	z	S_1	S_2	k	
x	1	1	0	2	-3	-2	40
z	0	$\frac{1}{4}$	1	$-\frac{3}{4}$	$\frac{5}{4}$	$\frac{3}{4}$	10
$-\pi$	0	-5	0	-10	-5	$10-D$	-4200

when $10-D>0$ we can keep going

i.e. $D<10$

Pivot on k remove z

	x	y	z	S_1	S_2	k	
x	1	$\frac{5}{3}$	$\frac{8}{3}$	0	$\frac{1}{3}$	0	$\frac{200}{3}$
k	0	$\frac{1}{3}$	$\frac{4}{3}$	-1	$\frac{5}{3}$	1	$\frac{40}{3}$
$-\pi$	0	$-5-\frac{1}{3}(10-D)$	$-\frac{4}{3}(10-D)$	$-20+D$	$-5-\frac{5}{3}(10-D)$	0	$-4200-\frac{40}{3}(10-D)$

$$4200+\frac{40}{3}(10-D) > 4200 \Rightarrow \text{we need } D<10.$$

(c) max $65x + 100y + 160z$

s.t. $3x + 5y + 8z \leq 200.$

$x, y, z \geq 0.$

Opt sol $(x, y, z) = (\frac{200}{3}, 0, 0).$

extra plastic : $\frac{200}{3} \times 5 - 320 = \frac{40}{3}$