Wenxiao Yang.

Math 482: Linear Programming, Fall 2020

Due Friday, October 16, 6PM CST

Homework 6

1. You are given a set of points on a line at locations $a_1, a_2, \ldots, a_n$. Write down linear programs to find a point x on that line that minimizes:

   (a) The maximum distance from x to any of the points.

   (b) The sum of the distances from x to all of the points.

   Remember that your constraints and objective function must be linear functions.
   (Do not solve either of the linear programs.)

2. Alice and Bob each have two coins: a nickel (5 cents) and a dime (10 cents). They simultaneously put a coin down on the table. If the coins are equal in value, Alice wins Bob's coin; if Alice's coin is more valuable, Bob wins Alice's coin; if Bob's coin is more valuable, nothing happens.

   Determine optimal strategies for Alice and Bob.

   Hint: Verify that pure strategies are not dominant for either player. This tells you, since you only have two variables in both linear programs, that both variables must be greater than 0. Use complementary slackness to find solutions to the primal and dual linear programs.

3. Use Fourier–Motzkin elimination to find a point (x, y, z) satisfying

$$\begin{aligned} x \qquad - \ z &\le 1 \\ x + \ y + \ z &\le 1 \\ -x + \ y + 3z &\le 1 \\ -3x - 5y - \ z &\le -1 \end{aligned}$$

4. *(Only 4-credit students need to do this problem.)*

   Use Farkas's lemma to prove LP duality in the following form: if the linear program $(\mathbf{P})$ below *cannot achieve* an objective value of at least z^*, and the dual program $(\mathbf{D})$ is feasible, then the dual linear program $(\mathbf{D})$ has a feasible solution $\mathbf{u}$ with objective value less than z^*.

$$(\mathbf{P}) \begin{cases} \underset{\mathbf{x} \in \mathbb{R}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \end{cases} \qquad (\mathbf{D}) \begin{cases} \underset{\mathbf{u} \in \mathbb{R}^m}{\text{minimize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{u}^\mathsf{T}A = \mathbf{c}^\mathsf{T} \\ & \mathbf{u} \ge \mathbf{0} \end{cases}$$

1. (a) set $u = \max\{|x-a_1|, |x-a_2|, \cdots, |x-a_n|\}$

$u \geq |x-a_i| \quad u \geq a_i - x \quad u \geq x - a_i$, for all i

LP: $\min\limits_{u\in\mathbb{R},\, x\in\mathbb{R}} u.$

s.t. $u + x \geq a_i$ for all $i \in \{1, 2, \cdots, n\}$.

$u - x \geq -a_i$

$u \geq 0$

(b) set $v_i = |x - a_i| \quad v_i = \max\{x - a_i, a_i - x\}$.

$v_i \geq x - a_i, \quad v_i \geq a_i - x.$

LP: $\min\limits_{x,\, v_i \in \mathbb{R}} \sum_{i=1}^{n} v_i$

s.t. $v_i - x \geq -a_i$ for all $i \in \{1, 2, \cdots, n\}$.

$v_i + x \geq a_i$

$v_i \geq 0$

2.

A \ B	5	10
5	(+5, -5)	(0, 0)
10	(-10, 10)	(10, -10)

$$A = \begin{bmatrix} 5 & 0 \\ -10 & 10 \end{bmatrix}$$

$u = \min\{5x_1 - 10x_2,\ 10x_2\}$

$\max\limits_{u, x_1, x_2 \in \mathbb{R}} u$

s.t. $u \leq 5x_1 - 10x_2$

$u \leq 10x_2$

$x_1 + x_2 = 1$

$x_1, x_2 \geq 0$

	u	x_1	x_2	s_1	s_2	
s_1	1	0	15	1	0	5
s_2	1	0	-10	0	1	0
x_1	0	1	1	0	0	1
$-z$	1	0	0	0	0	0

$\downarrow$ pivot on u remove S_1

	u	x_1	x_2	S_1	S_2	
u	1	0	15	1	0	5
S_2	0	0	-25	-1	1	-5
x_1	0	1	1	0	0	1
$-z$	0	0	-15	-1	0	-5

$\downarrow$

	u	x_1	x_2	S_1	S_2	
u	1	0	0	$\frac{2}{5}$	$\frac{3}{5}$	2
x_2	0	0	1	$\frac{1}{25}$	$-\frac{1}{25}$	$\frac{1}{5}$
x_1	0	1	0	$-\frac{1}{25}$	$\frac{1}{25}$	$\frac{4}{5}$
$-z$	0	0	0	$-\frac{2}{5}$	$-\frac{3}{5}$	-2

Alice's opt solution $(x_1, x_2) = (\frac{4}{5}, \frac{1}{5})$

Bob's opt solution $(y_1, y_2) = (\frac{2}{5}, \frac{3}{5})$

Alice : $\frac{4}{5}$ choose 5 cents $\frac{1}{5}$ choose 10 cents

Bob : $\frac{2}{5}$ choose 5 cents $\frac{3}{5}$ choose 10 cents.

3.
$$\begin{aligned} x \quad\quad - z &\le 1 \\ x + y + z &\le 1 \\ -x + y + 3z &\le 1 \\ -3x - 5y - z &\le -1 \end{aligned} \longrightarrow \begin{cases} y + 2z \le 2 \\ -\frac{5}{3}y - \frac{4}{3}z \le \frac{2}{3} \\ 2y + 4z \le 2 \\ -\frac{2}{3}y + \frac{2}{3}z \le \frac{2}{3} \end{cases}$$

$$-x - \frac{5}{3}y - \frac{1}{3}z \le -\frac{1}{3}$$

$$\longrightarrow \begin{cases} y + 2z \le 2 \\ -y - \frac{4}{5}z \le \frac{2}{5} \\ y + 2z \le 1 \\ -y + z \le 1 \end{cases}$$

$$\longrightarrow \begin{cases} \frac{6}{5}z \leq \frac{12}{5} \\ 3z \leq 3 \\ \frac{6}{5}z \leq \frac{7}{5} \\ 3z \leq 2 \end{cases} \longrightarrow z \leq \frac{2}{3}$$

pick $z = 0$ $\longrightarrow \begin{cases} y \leq 2 \\ -y \leq \frac{2}{5} \\ y \leq 1 \\ -y \leq 1 \end{cases} \longrightarrow -\frac{2}{5} \leq y \leq 1$

Pick $y = 0$ $\longrightarrow \begin{cases} x \leq 1 \\ x \leq 1 \\ -x \leq 1 \\ -3x \leq -1 \end{cases} \longrightarrow \frac{1}{3} \leq x \leq 1$

Pick $x = \frac{1}{3}$ $\longrightarrow$ $(x, y, z) = (\frac{1}{3}, 0, 0)$

is feasible.