Feasibility problem

Given A and b, is $\{x \in \mathbb{R}^n ; Ax \le b\}$ nonempty ?

1. Put into equational form.
2. Find a basic solution (not necessarily feasible).
3. invent an objective function that gives a dual feasible tableau.
4. Use dual simplex method to make it primal feasible.

feasibility problem is as hard as optimization.

$$\begin{cases} \text{optimization} & \max\{c^T x : Ax \le b, x \ge 0\} \\ \text{feasible} & \{(x,u) \in \mathbb{R}^{n+m} : c^T x = u^T b, Ax \le b, u^T A \ge c^T, \\ & x \ge 0, u \ge 0\} \end{cases}$$

Fourier-Motzkin elimination.

Simpler but slow to prove infeasibe

idea: Reduce on n-var problem to an equivalent n-1 var problem. Repeat left with a 1 var feasibility problem.

$x_1, \ldots, x_n$, want to eliminate x_n

①. solve each inequality for x_n

$x_n \leq \dfrac{b - a_1 x_1 - a_2 x_2 \cdots - a_{n-1} x_{n-1}}{a_n}$ or $x_n \geq \dfrac{b - a_1 x_1 - a_2 x_2 \cdots - a_{n-1} x_{n-1}}{a_n}$

end up with a collection of lower and upper bounds $x_n \geq L_1, x_n \geq L_2, \ldots, x_n \geq L_k, x_n \leq U_1, \ldots, x_n \leq U_m$.

it is possible to find x_n if and only if $\max\{L_1, \ldots, L_k\} \leq \min\{U_1, \ldots, U_m\}$.

$L_i \leq x_n \leq U_j$ for all i, j.

We get new system

$$\begin{array}{llll} L_1 \leq U_1 & L_1 \leq U_2 & \cdots & L_1 \leq U_n \\ L_2 \leq U_1 & \cdots & \cdots & \cdots \\ \cdots & & & \\ L_k \leq U_1 & \cdots & \cdots & L_k \leq U_n \end{array}$$

the system only have $n-1$ vars.

Example:

$$\begin{cases} x - y \geq 3 \\ x + 2y \geq 4 \\ x + y \leq 7 \\ x, y \geq 0 \end{cases} \xrightarrow{\text{Eliminate } y} \begin{cases} y \leq x - 3 \\ y \geq \frac{1}{2}(4 - x) \\ y \leq 7 - x \\ y \geq 0, \ x \geq 0 \end{cases}$$

$$\left\{\begin{array}{l} 0 \leq 7-x \\ 0 \leq x-3 \\ \frac{1}{2}(4-x) \leq 7-x \\ \frac{1}{2}(4-x) \leq x-3 \\ x \geq 0 \end{array}\right. \longrightarrow \left\{\begin{array}{l} x \leq 7 \\ 3 \leq x \\ x \leq 10 \\ \frac{10}{3} \leq x \\ x \geq 0 \end{array}\right. \longrightarrow \frac{10}{3} \leq x \leq 7$$

this system is feasible. to find a y pick $x=4$

$$\Longrightarrow \left\{\begin{array}{l} y \leq 1 \\ y \geq 0 \\ y \leq 3 \\ y \geq 0 \end{array}\right. \Longrightarrow 0 \leq y \leq 1. \qquad (1,4) \text{ is feasible.}$$

Complexity.

How many inequalities?

Worst behavior case: Lower bound and upper bound split exactly in half each time.

Starting with 8 inequalities in n vars:

$2^{a_0} = 2^3$	$8 = 4 + 4$	n
$\left(\frac{2^3}{2}\right)^2 = 2^4 = 2^{a_1}$	$4 \times 4 = 16 = 8 + 8$	$n-1$
$\left(\frac{2^4}{2}\right)^2 = 2^6 = 2^{a_2}$	$8 \times 8 = 64 = 32 + 32$	$n-2$
$2^{10} = 2^{a_3}$		

$2^{18} = 2^{a_4}$ $\qquad$ 2^{2^k+2} inequalities $n-k$ vars.

$3, 4, 6, 10, 18$ $\qquad$ $\Uparrow$

$a_{n+1} = 2(a_n - 1)$.

$a_{n+1} - 2 = 2(a_n - 2) \Rightarrow a_n = 2^k + 2$

we can change all inequalities into "$\leq$"

and the coefficient of x_n is "1" or "-1"

add the "$+x_n$" inequality and "$-x_n$" inequality.

$$\begin{cases} x - y \geq 3 \\ x + 2y \geq 4 \\ x + y \leq 7 \\ x, y \geq 0 \end{cases} \longrightarrow \begin{cases} -x + y \leq -3 \\ -x - 2y \leq -4 \\ x + y \leq 7 \\ -x \leq 0 \\ -y \leq 0 \end{cases} \longrightarrow \begin{cases} 2y \leq 4 \\ -y \leq 3 \\ y \leq 7 \end{cases} \longrightarrow -3 \leq y \leq 2$$

pick $y = 1$ $\begin{cases} -x \leq -4 \\ -x \leq -2 \\ x \leq 6 \\ -x \leq 0 \end{cases} \longrightarrow 4 \leq x \leq 6$ pick $x = 4$.

$(1, 4)$ is feasible.

Farkas's Lemma.

For any system of inequalities $Ax \leq b$, either there exists a feasible solution x, or we can take a linear comb of inequalities to derive a contradiction: there exists vector $u \geq 0$ such that $u^T A = 0$, but $u^T b < 0$