Simplex method:

start at a solution go from one solution to another, improving objective value each time, stop at optimal.

$$\max\ C^T X \quad , X \in \mathbb{R}^n$$

$$\text{s.t. } Ax = b$$
$$x \geq 0$$

Rule #1: At least one optimal solution is a vertex of feasible region.

Rule #2: All vertices of feasible region are basic solutions of $Ax = b$.

Example:

$$\begin{cases} x_1 + \cdots + x_3 + 3x_4 + x_5 = 4 \end{cases}$$

$$\begin{cases} x_2 + x_4 + x_5 = 3 \\ x_3 + x_4 - x_5 = 1 \end{cases}$$

$$\left[\begin{array}{ccccc|c} 1 & 0 & 1 & 3 & 1 & 4 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$x_4 = x_5 = 0$

$x_1 = 3$, $x_2 = 3$, $x_1 = 1$ one solution.

basis

$B = (1, 2, 3)$ indicates of desired basic variables

$$A_B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ submatrix of A only take colums from B.

$$A x = b$$

$$A_B^{-1} A x = A_B^{-1} b$$

$= (x_1, x_2, x_3)$

$$x_B = A_B^{-1} b \rightarrow$$ gets x_1, x_2, x_3

$B' = (2, 3, 4)$ different basis.

$$A_{B'} = \begin{bmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad X_{B'} = (X_2, X_3, X_4)$$

$$= A_{B'}^{-1} b = \left(\frac{3}{2}, -\frac{1}{2}, \frac{3}{2}\right)$$

$B=(1,2,3)$ $(x_1, x_2, x_3, x_4, x_5) = (3,3,1,0,0)$ $x \geq 0$

$B'=(2,3,4)$ $(x_1, x_2, x_3, x_4, x_5) = (0, \frac{3}{2}, -\frac{1}{2}, \frac{3}{2}, 0)$ $x \not\geq 0.$

we want only <u>feasible</u> basic solutions.

Pivoting

1. start with a basic feasible solution.
2. add one, remove one.
3. choose which variable is removed to avoid negative signs and arrive at a new feasible solution. Compare $\frac{P_j}{Q_{i,j}}$.

example: $B=(1,2,3)$, pick X_5 to enter basis.

X_4 remains 0.

$$\left[\begin{array}{ccccc:c} 1 & 0 & 0 & 2 & 2 & 3 \\ 0 & 1 & 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \Rightarrow \begin{aligned} x_1 + 2x_5 &= 3 \\ x_2 + x_5 &= 3 \\ x_3 - x_5 &= 1 \end{aligned}$$

$x_5 \uparrow \Longrightarrow x_1$ becomes negative first.

So we remove x_1.

(Two factors: 1. Rate at which old variables change as x_s increases, given by x_s's column $(2, 1, -1)$

2. the starting value of the variables, given by RHS of equation $(3, 3, 1)$).