Math 482: Linear Programming[1] | Mikhail Lavrov

Lecture 13: Complementary Slackness

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0 Rows and columns notation

Previously, if A is an $m \times n$ matrix, we denoted the columns of A as $A_1, A_2, \dots, A_n$.

Today, we'll also need to refer to the rows of A. So let's call those $\mathbf{a}_1^\mathsf{T}, \mathbf{a}_2^\mathsf{T}, \dots, \mathbf{a}_m^\mathsf{T}$.

1 The statement of complementary slackness

Let's consider once again the primal-dual pair

$$(\mathbf{P}) \quad \begin{cases} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{cases} \qquad (\mathbf{D}) \quad \begin{cases} \underset{\mathbf{u}\in\mathbb{R}^m}{\text{minimize}} & \mathbf{u}^\mathsf{T}\mathbf{b} \\ \text{subject to} & \mathbf{u}^\mathsf{T}A \ge \mathbf{c}^\mathsf{T} \\ & \mathbf{u} \ge \mathbf{0} \end{cases}$$

We can prove weak duality for this pair in a quick line of algebra: if $\mathbf{x}$ is primal feasible and $\mathbf{u}$ is dual feasible, then

$$A\mathbf{x} \le \mathbf{b}, \mathbf{u}^\mathsf{T} \ge \mathbf{0}^\mathsf{T} \implies \mathbf{u}^\mathsf{T}A\mathbf{x} \le \mathbf{u}^\mathsf{T}\mathbf{b}$$
$$\mathbf{u}^\mathsf{T}A \ge \mathbf{c}^\mathsf{T}, \mathbf{x} \ge \mathbf{0} \implies \mathbf{u}^\mathsf{T}A\mathbf{x} \ge \mathbf{c}^\mathsf{T}\mathbf{x}.$$

From $\mathbf{c}^\mathsf{T}\mathbf{x} \le \mathbf{u}^\mathsf{T}A\mathbf{x}$ and $\mathbf{u}^\mathsf{T}A\mathbf{x} \le \mathbf{u}^\mathsf{T}\mathbf{b}$, we deduce $\mathbf{c}^\mathsf{T}\mathbf{x} \le \mathbf{u}^\mathsf{T}\mathbf{b}$.

Now suppose that $\mathbf{x}$ and $\mathbf{u}$ are primal and dual optimal, respectively. Strong duality, which we haven't proved yet, assures us that in this case, $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$. And based on the proof above, we know a bit more: that

$$\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}A\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$$

for such a pair.

This lets us get a bit more information out. Let's first focus on the second equation: $\mathbf{u}^\mathsf{T}A\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$. We can rearrange this to say that

$$\mathbf{u}^\mathsf{T}(\mathbf{b} - A\mathbf{x}) = 0 \iff \sum_{i=1}^m u_i(b_i - \mathbf{a}_i^\mathsf{T}\mathbf{x}) = 0.$$

Here, the quantity $b_i - \mathbf{a}_i^\mathsf{T}\mathbf{x}$ has an interpretation: since the i^{th} constraint in $(\mathbf{P})$ is that $\mathbf{a}_i^\mathsf{T}\mathbf{x} \le b_i$, it's the amount of slack in that constraint. We know that both u_i and $b_i - \mathbf{a}_i^\mathsf{T}\mathbf{x}$ are nonnegative: the first, because $\mathbf{u} \ge \mathbf{0}$, and the second, because $\mathbf{a}_i^\mathsf{T}\mathbf{x} \le b_i$. This means that their product $u_i(b_i - \mathbf{a}_i^\mathsf{T}\mathbf{x})$ is nonnegative.

[1] This document comes from the Math 482 course webpage: https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html

The sum of a bunch of nonnegative things can only be 0 in one way: if all of them are 0. So we know that for each i, either $b_i - \mathbf{a}_i^\mathsf{T}\mathbf{x} = 0$ or $u_i = 0$.

A similar thing happens for the first equation, $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}A\mathbf{x}$, which we can rearrange to

$$(\mathbf{u}^\mathsf{T}A - \mathbf{c}^\mathsf{T})\mathbf{x} = 0 \iff \sum_{i=1}^{n}(\mathbf{u}^\mathsf{T}A_i - c_i)x_i.$$

Here, $\mathbf{u}^\mathsf{T}A_i - c_i$ is the amount of slack in the i^{th} constraint of $(\mathbf{D})$: $\mathbf{u}^\mathsf{T}A_i \ge c_i$. So that factor is also nonnegative. By the same reasoning as earlier, the sum can only be 0 if for each i, either $\mathbf{u}^\mathsf{T}A_i - c_i = 0$ or $x_i = 0$.

In summary, we have a result called complementary slackness:

Theorem 1.1 (Complementary slackness)**.** *Let $\mathbf{x}$ be a primal optimal solution and let $\mathbf{u}$ be a dual optimal solution. Then:*

- *For $i = 1, 2, \ldots, m$, either $\mathbf{x}$ satisfies the i^{th} constraint of $(\mathbf{P})$ with equality, or $u_i = 0$.*
- *For $i = 1, 2, \ldots, n$, either $x_i = 0$, or $\mathbf{u}$ satisfies the i^{th} constraint of $(\mathbf{D})$ with equality.*

We say that a $\le$ or $\ge$ constraint is *tight* if equality holds, and *slack* otherwise. Hence the name "complementary slackness". If we pair the constraint $\mathbf{a}_i^\mathsf{T}\mathbf{x} \le b_i$ with the nonnegativity constraint $u_i \ge 0$, or the nonnegativity constraint $x_i \ge 0$ with the constraint $\mathbf{u}^\mathsf{T}A_i \ge c_i$, then **at most one of the constraints in each pair is slack**.

We proved complementary slackness for one specific form of duality: linear programs in the form that $(\mathbf{P})$ and $(\mathbf{D})$ above have. But we can do the same thing with other types of constraints. Complementary slackness holds for all of them, even if it's not always useful: for a $=$ constraint in the primal or dual, the constraint is always tight and we learn nothing about the corresponding variable in the other linear program.

We can actually say slightly more.

Theorem 1.2. *Let $\mathbf{x}$ be a primal feasible solution and let $\mathbf{u}$ be a dual feasible solution such that complementary slackness holds between $\mathbf{x}$ and $\mathbf{u}$. Then $\mathbf{x}$ and $\mathbf{u}$ are primal optimal and dual optimal, respectively.*

Proof. The first form of complementary slackness is equivalent to saying that $\mathbf{u}^\mathsf{T}(A\mathbf{x} - \mathbf{b}) = 0$, which we can rewrite as $\mathbf{u}^\mathsf{T}A\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$. The second form of complementary slackness is equivalent to saying that $(\mathbf{c}^\mathsf{T} - \mathbf{u}^\mathsf{T}A)\mathbf{x} = 0$, which we can rewrite as $\mathbf{u}^\mathsf{T}A\mathbf{x} = \mathbf{c}^\mathsf{T}\mathbf{x}$. Therefore by transitivity $\mathbf{c}^\mathsf{T}\mathbf{x} = \mathbf{u}^\mathsf{T}\mathbf{b}$.

This proves optimality of both $\mathbf{x}$ and $\mathbf{u}$. The value $\mathbf{u}^\mathsf{T}\mathbf{b}$ is an upper bound for all primal objective values, so because $\mathbf{c}^\mathsf{T}\mathbf{x}$ reaches that bound, $\mathbf{x}$ is optimal. Similarly, the value $\mathbf{c}^\mathsf{T}\mathbf{x}$ is a lower bound for all dual objective values, so because $\mathbf{u}^\mathsf{T}\mathbf{b}$ reaches that bound, $\mathbf{u}$ is optimal. $\square$

The intuition behind complementary slackness is that the dual variable u_i measures how useful the constraint $\mathbf{a}_i^\mathsf{T}\mathbf{x} \le b_i$ is in restricting the objective function of $(\mathbf{P})$ from growing. (After all, multiply that constraint by weight u_i before adding it to other things to get an upper bound on the objective function.) If the i^{th} constraint is slack at the optimal solution (that is, if $\mathbf{a}_i^\mathsf{T}\mathbf{x} < b_i$ when $\mathbf{x}$ is optimal), then that constraint is not at all useful, and so we must have $u_i = 0$.

A similar interpretation works with the roles of the primal and dual reversed.

2 Applications and an example

We can use complementary slackness to do two things:

- Go from the optimal primal solution to the optimal dual solution, and vice versa. This will become more and more useful as we learn new uses for duality.
- Verify that a solution is optimal, by checking if there's a dual solution that goes with it.

For example, suppose that we are given the linear program

$$\begin{array}{ll} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{maximize}} & 3x_1 \qquad\quad + 2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 \le 6 \\ & 2x_1 - x_2 + x_3 \le 3 \\ & 3x_1 + x_2 - x_3 \le 3 \\ & x_1, x_2, x_3 \ge 0 \end{array}$$

Your friend used Microsoft Excel to determine that $(x_1, x_2, x_3) = (0, 1.5, 4.5)$ is optimal, but you don't trust Microsoft products, so you'd like to verify this. You also think that there could be other optimal solutions, and you want to find all of them.

A quick check to begin with is that $\mathbf{x} = (0, 1.5, 4.5)$ is feasible. Indeed it is: all three components are nonnegative, and all three equations are satisfied. (If this weren't true, nothing else we did after would be meaningful; also, if $\mathbf{x}$ weren't feasible, it would definitely not be optimal either.)

After that, our first step is to find the dual:

$$\begin{array}{ll} \underset{u_1,u_2,u_3\in\mathbb{R}}{\text{minimize}} & 6u_1 + 3u_2 + 3u_3 \\ \text{subject to} & u_1 + 2u_2 + 3u_3 \ge 3 \\ & u_1 - u_2 + u_3 \ge 0 \\ & u_1 + u_2 - u_3 \ge 2 \\ & u_1, u_2, u_3 \ge 0 \end{array}$$

Now we check what complementary slackness tells us.

The primal solution $(0, 1.5, 4.5)$ has $x_1+x_2+x_3 = 6$ and $2x_1-x_2+x_3 = 3$, but $3x_1+x_2-x_3 = -3 < 3$, so the first two constraints are tight, and the third is slack. This tells us that $u_3 = 0$, while u_1 and u_2 could be zero or nonzero.

Since $x_2 > 0$ and $x_3 > 0$, complementary slackness demands that the second and third dual constraints should be tight: $u_1 - u_2 + u_3 = 0$ and $u_1 + u_2 - u_3 = 2$. It does not say anything about the first constraint.

Putting this together, we get

$$\begin{cases} u_1 - u_2 = 0 \\ u_1 + u_2 = 2 \\ u_3 = 0 \end{cases}$$

which has the unique solution $(u_1, u_2, u_3) = (1, 1, 0)$.

A final important check is that this satisfies the dual feasibility conditions. All three variables are nonnegative, so that's fine. Checking the second and third dual constraints was baked into our method, but we haven't used the first constraint yet, so we check that $u_1 + 2u_2 + 3u_3 = 1 + 2 + 0 = 3 \geq 3$. It is, so $(1, 1, 0)$ is feasible.

Now, because we succeeded in satisfying complementary slackness, we know that $(0, 1.5, 4.5)$ is primal optimal and that $(1, 1, 0)$ is dual optimal.

To see if there are any other primal optimal solutions, we use complementary slackness in the other direction. At the point $(1, 1, 0)$, all three dual constraints are tight, so none of the primal variables are required to be 0. Since $u_1 = u_2 > 0$ and $u_3 = 0$, we know that a feasible primal solution (x_1, x_2, x_3) is optimal if it satisfies $x_1 + x_2 + x_3 = 6$ and $2x_1 - x_2 + x_3 = 3$.

We can parametrize the solutions to these two equations in terms of x_1: points that satisfy $x_1 + x_2 + x_3 = 6$ and $2x_1 - x_2 + x_3 = 3$ are points of the form $(x_1, 1.5 + 0.5x_1, 4.5 - 1.5x_1)$. (If we set $x_1 = 0$, we get back the optimal solution we already knew.) But to ensure feasibility, we also need to have

$$\begin{cases} 3x_1 + x_2 - x_3 \leq 3 \\ x_1 \geq 0 \\ x_2 \geq 0 \\ x_3 \geq 0 \end{cases} \implies \begin{cases} 3x_1 + (1.5 + 0.5x_1) - (4.5 - 1.5x_1) \leq 3 \\ x_1 \geq 0 \\ 1.5 + 0.5x_1 \geq 0 \\ 4.5 - 1.5x_1 \geq 0 \end{cases} \implies \begin{cases} x_1 \leq 1.2 \\ x_1 \geq 0 \\ x_1 \geq -3 \\ x_1 \leq 3 \end{cases}$$

which means that the optimal primal solutions are all the points $(t, 1.5 + 0.5t, 4.5 - 1.5t)$ for $0 \leq t \leq 1.2$. If we want to avoid fractions, for example, we could set $t = 1$ and get $(1, 2, 3)$.