Math 482 Linear programming
Fall 2020
Exam 1
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Time Limit: 180 Minutes

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Section: C13

This is the 3 CREDIT EXAM.

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books, notes, or any calculator on this exam.

You are required to justify your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

Do not write in the table to the right.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

$y \to y^+ - y^-$

1. (20 points) Solve the linear program below using the simplex method.

$$\begin{array}{ll} \text{maximize} & 3x + y \\ \text{subject to} & x + y \le 7 \\ & -x + y \le 1 \\ & -x - 2y \le 1 \\ & x \ge 0, y \text{ unrestricted} \end{array}$$

max $3x + y^+ - y^-$

s.t. $x + y^+ - y^- + S_1 = 7$

$-x + y^+ - y^- + S_2 = 1$

$-x - 2y^+ + 2y^- + S_3 = 1$

$x, y^+, y^-, S_1, S_2, S_3 \ge 0$

	x	y^+	y^-	S_1	S_2	S_3	
S_1	1	1	-1	1	0	0	7
S_2	-1	1	-1	0	1	0	1
S_3	-1	-2	2	0	0	1	1
$-z$	3	1	-1	0	0	0	0

pivot on x, remove S_1 $\longrightarrow$

	x	y	y^-	S_1	S_2	S_3	
x	1	1	-1	1	0	0	7
S_2	0	2	-2	1	1	0	8
S_3	0	-1	1	1	0	1	8
$-z$	0	-2	2	-3	0	0	-21

pivot on y^- remove S_3 $\longrightarrow$

	x	y	y^-	S_1	S_2	S_3	
x	1	0	0	2	0	0	15
S_2	0	0	0	3	1	2	24
y^-	0	-1	1	1	0	1	8
$-z$	0	0	0	-5	0	-2	-37

$\Rightarrow$ when $(x, y, S_1, S_2, S_3) = (15, -8, 0, 24, 0)$

$\max 3x + y = 37.$

2. (20 points) Suppose you are doing the simplex method with lexicographic pivoting and arrive at the tableau:

	x_1	x_2	x_3	x_4	x_5	
x_3	0	-5	1	2	3	$6 + 2\epsilon_1 + 3\epsilon_2$
x_1	1	-7	0	-1	2	$4 - \epsilon_1 + 4\epsilon_2$
$-z$	0	-1	0	0	2	$-10 + 2\epsilon_2$

Suppose your goal is to **maximize** z. Describe your next step starting at the tableau above. What are your entering and exiting variables? Perform one iteration of the simplex method if possible, if it is not possible say what you can conclude about the linear program.

entering x_5. exiting x_1

	x_1	x_2	x_3	x_4	x_5	
x_3	$-\frac{3}{2}$	$\frac{11}{2}$	1	$\frac{7}{2}$	0	$\frac{7}{2}\epsilon_1 - 3\epsilon_2$
x_5	$\frac{1}{2}$	$-\frac{7}{2}$	0	$-\frac{1}{2}$	1	$2 - \frac{\epsilon_1}{2} + 2\epsilon_2$
$-z$	-1	6	0	1	0	$-14 + \epsilon_1 - 2\epsilon_2$

Suppose your goal is to **minimize** z. Describe your next step starting at the tableau above. What are your entering and exiting variables? Perform one iteration of the simplex method if possible, if it is not possible say what you can conclude about the linear program.

entering x_2, but there is no negative entry in x_2 column to pivot on $\Rightarrow$ not possible to perform iteration.

$\Rightarrow$ the linear program is unbounded.

3. (20 points) Consider the following linear program.

$$\begin{array}{lrl} \text{maximize} & -x_1 \qquad\quad - x_3 + 2x_4 & = Z \\ \text{subject to} & x_1 + 2x_2 \qquad\quad + x_4 & = 4 \\ & - x_2 + x_3 - x_4 & = -1 \\ & x_1, x_2, x_3, x_4 \geq 0 & \end{array}$$

Use the two-phase simplex method to find an initial basic feasible solution or to show that this linear program is infeasible. You **do not** need fully solve this linear program.

$$\min \; x_1^a + x_2^a = z^a$$

$$x_1 + 2x_2 \quad + x_4 + x_1^a = 4$$

$$x_2 - x_3 + x_4 + x_2^a = 1$$

$$x_1, x_2, x_3, x_4, x_1^a, x_2^a \geq 0.$$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	2	0	1	1	0	4
x_2^a	0	1	-1	1	0	1	1
$-z^a$	0	0	0	0	1	1	0

$\Rightarrow$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	2	0	1	1	0	4
x_2^a	0	1	-1	1	0	1	1
$-z^a$	-1	-3	1	-2	0	0	-5

Pivot on x_2 remove x_2^a

$\Rightarrow$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_1^a	1	0	2	-1	1	-2	2
x_2	0	1	-1	1	0	1	1
$-z^a$	-1	0	-2	1	0	3	-2

Pivot on x_3 remove x_1^a

$\Rightarrow$

	x_1	x_2	x_3	x_4	x_1^a	x_2^a	
x_3	$\frac{1}{2}$	0	1	$-\frac{1}{2}$	$\frac{1}{2}$	-1	1
x_2	$\frac{1}{2}$	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	2
$-z^a$	0	0	0	0	1	1	0

$\Rightarrow \min z^a = 0$

$\Rightarrow$ basic feasible solution

$$(x_1, x_2, x_3, x_4) = (0, 2, 1, 0)$$

4. (20 points) Solve the linear program below using the **revised** simplex method.

$$\max\ 2x_1+x_2+5x_3-x_4$$
$$\text{s.t. } 2x_1+3x_2+5x_3+x_4+x_5=9$$
$$x_1+x_2+2x_3+x_6=4$$
$$x_1,x_2,x_3,x_4,x_5,x_6\geq 0$$

$$\begin{array}{ll} \text{maximize} & 2x_1 + x_2 + 5x_3 - x_4 \\ \text{subject to} & 2x_1 + 3x_2 + 5x_3 + x_4 \leq 9 \\ & x_1 + x_2 + 2x_3 \leq 4 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{array}$$

Formulas that may be useful:

$$\mathbf{p} = A_{\mathcal{B}}^{-1}\mathbf{b} \qquad Q_j = A_{\mathcal{B}}^{-1}A_j \qquad r_j = c_j - \mathbf{u}^\top A_j \qquad \mathbf{u}^\top = \mathbf{c}_{\mathcal{B}}{}^\top A_{\mathcal{B}}^{-1}$$

$$N=\{1,2,3,4\} \quad B=\{5,6\}$$

$$C^T=[2,1,5,-1,0,0] \quad A=\begin{bmatrix} 2 & 3 & 5 & 1 & 1 & 0 \\ 1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix},\ b=\begin{bmatrix} 9 \\ 4 \end{bmatrix}$$

$$P=A_B^{-1}b=b=\begin{bmatrix} 9 \\ 4 \end{bmatrix} \quad r^T=C_N^T-C_B^T(A_B^{-1}A_N)$$
$$=[2,1,5,-1]$$

choose $j=3$ $\quad Q_j=\begin{bmatrix} 5 \\ 2 \end{bmatrix}$ $\quad i=5$ in B

$$B'=[3,6] \quad N=[1,2,4,5]$$

$$[A_B^{-1} \mid Q_3 \mid P]=\left[\begin{array}{cc|c|c} 1 & 0 & 5 & 9 \\ 0 & 1 & 2 & 4 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c|c} \frac{1}{5} & 0 & 1 & \frac{9}{5} \\ -\frac{2}{5} & 1 & 0 & \frac{2}{5} \end{array}\right]$$
$$=\left[A_B^{-1} \;\middle|\; \begin{matrix} 1 \\ 0 \end{matrix} \;\middle|\; P'\right]$$

$$r'^T=\overset{C_{N'}^T}{[2,1,-1,0]}-\overset{C_{B'}^T}{[5,0]}\overset{A_{B'}^{-1}}{\begin{bmatrix} \frac{1}{5} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix}}\overset{A_{N'}}{\begin{bmatrix} 2 & 3 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}}$$
$$=[0,-2,-2,-1]$$

Hence $\max Z = Z_0' = C_{B'}^T P' = [5,0]\begin{bmatrix} \frac{9}{5} \\ \frac{2}{5} \end{bmatrix} = 9$

when $(x_1,x_2,x_3,x_4)=(0,0,\frac{9}{5},0)$

5. (20 points) Write a linear program for the problem below. (Do not solve.)

   Four roommates want to split up the daily chores in an apartment: washing the dishes, cooking, and taking out the trash. Each chore needs to be done 7 days every week. No roommate should be assigned more than 7 days' worth of chores.

   Different roommates have different preferences about the chores, and have assigned them "unpleasantness points" representing how much they would suffer doing them for 1 day. These are given in the table below.

	Washing the dishes	Cooking	Taking out the trash
Roommate #1	5 points	2 points	3 points
Roommate #2	3 points	3 points	4 points
Roommate #3	2 points	2 points	6 points
Roommate #4	2 points	4 points	4 points

   Your goal is to assign the chores in a way that minimizes suffering. The linear program should calculate the number of days in the week each roommate should do each chore. It does not need to schedule the times when they do those chores.

X_{wi} represents the number of days Roommate i do washing the dishes

X_{ci} represents the number of days Roommate i do Cooking

X_{ti} represents the number of days Roommate i do taking out the trash.

$$\underset{\substack{X_{wi},\, X_{ci},\, X_{ti} \in \mathbb{R}^+ \\ i=1,2,3,4}}{\text{minimize}} \quad 5X_{w1}+2X_{c1}+3X_{t1}+3X_{w2}+3X_{c2}+4X_{t2}+2X_{w3}+2X_{c3}+6X_{t3}+2X_{w4}+4X_{c4}+4X_{t4}$$

Subject to

$$X_{w1}+X_{c1}+X_{t1} \le 7$$
$$X_{w2}+X_{c2}+X_{t2} \le 7$$
$$X_{w3}+X_{c3}+X_{t3} \le 7$$
$$X_{w4}+X_{c4}+X_{t4} \le 7$$
$$\sum_{i=1}^{4} X_{wi} = 7$$
$$\sum_{i=1}^{4} X_{ci} = 7$$
$$\sum_{i=1}^{4} X_{ti} = 7$$

$$X_{wi},\, X_{ci},\, X_{ti} \ge 0 \quad i=1,2,3,4.$$