A is $m \times n$ matrix : columns $A_1, A_2, \dots, A_n$

rows $a_1^T, a_2^T, \dots\dots, a_m^T$.

Complementary slackness (CS).

$$(P): \quad \max\ c^T x \quad \text{s.t. } Ax \le b,\ x \ge 0 \qquad (D): \quad \min\ u^T b \quad \text{s.t. } u^T A \ge c^T,\ u \ge 0.$$

Proof of Weak duality.

If x is primal feasible, u is dual feasible.

$Ax \le b$, $u^T \ge 0^T \Rightarrow u^T A x \le u^T b$.

$u^T A \ge c^T$, $x \ge 0 \Rightarrow u^T A x \ge c^T x$.

$\Rightarrow c^T x \le u^T b$.

(If x, u are opt, $c^T x = u^T b \Rightarrow c^T x = u^T A x = u^T b$)

$u^T (b - Ax) = 0 \iff \sum_{i=1}^{m} u_i (b_i - a_i^T x) = 0$

(from (P): $a_i^T x \le b_i \Rightarrow b_i - a_i^T x \ge 0$)

$\Rightarrow$ for each i, either $\underline{b_i - a_i^T x}$ or $\underline{u_i}$ is 0.

$(U^TA - C^T)X = 0 \iff \sum_{i=1}^{n} (U^TA_i - C_i)X_i = 0$

(from (D): $U^TA \geq C^T \Rightarrow U^TA - C^T \geq 0$)

$\Rightarrow$ for each i, either $U^TA_i - C_i$ or X_i is 0.

Theorem (CS) Let X be primal opt, u be dual opt, Then

- for $i = 1, \ldots, m$, either X satisfies i^{th} constraint of (P) with equality ($a_i^T x = b_i$) or $U_i = 0$
- for $i = 1, \ldots, n$, either U satisfies i^{th} constraint of (D) with equality ($U^TA_i = C_i$) or $X_i = 0$.

(i.e. X, U satisfy Complementary slackness).

Theorem: Suppose x is primal feasible and u is dual feasible, if x, u satisfy CS, then x, u are opt.

Proof: CS $\Rightarrow U^T(Ax - b) = 0 \Rightarrow U^TAx = U^Tb$

$(U^TA - C^T)X = 0 \Rightarrow U^TAX = C^TX.$

$\Rightarrow C^T x = u^T b$

$u^T b$ is an upper bound on all primal objective values. Since $C^T x$ reaches this bound, $C^T x$ is opt.
$C^T x$ is an lower bound on all dual objective values. Since $u^T b$ reaches this bound. $u^T b$ is opt.
u_i measures how useful the constraint $a_i^T x \le b_i$ is in restricting the obj. function of (P).