Zero-Sum Game

Cooperation is impossible. A's payoff is always the negative of B's payoff

mixed strategy X, $\left\{\begin{array}{l}\text{Alice's payoff} \\ \text{Bob's pay}\end{array}\right.$ matrix $A_{a \times b}$

$X^T A$ is the row vector of expected payoff with mixed strategy X, for each opponent's choices.

$\Longrightarrow$ Expected payoff: $X^T A y$.

"Best-worst-case" Strategy - maximin

Alice want to choose the best mixed strategy X. but she can't evaluate how good the strategy is by formula $X^T A y$ since he doesn't know y.

Instead, Alice might try to choose the strategy x with best performance against the worst possible
for Alice.

counter to X. $\Rightarrow$ Alice's maximin

(reasonable in Zero-sum game).

Lemma 1. If Alice's mixed strategy is a fixed $x \in \mathbb{R}^a$, (Bob knows the strategy). Bob's best response is a pure strategy which always picks same option.

Proof: $X^T A$ is the vector of possible payoffs depending on Bob's choices. Bob's choice is $i \in \min_i (X^T A)_i$.

the worst case payoff for Alice's mixed strategy is $\min\{(X^TA)_1, (X^TA)_2, \ldots (X^TA)_b\}$.

$\Rightarrow$ Alice's maximin Strategy is given by the optimal solution to the LP.

$$\max_{x \in \mathbb{R}^a} \min\{(X^TA)_1, \ldots (X^TA)_b\}$$

$$\text{s.t.} \quad x_1 + \cdots + x_a = 1$$

$$x \geq 0$$

Set auxiliary variable $u = \min\{(x^TA)_1, \ldots\ldots, (x^TA)_b\}$

$u \leq (x^TA)_i \qquad i \in [1, b].$

$$\Rightarrow \max_{x \in \mathbb{R}^a,\, u \in \mathbb{R}} u.$$

$$\text{s.t.} \quad u\mathbb{1}^T \leq x^TA$$

$$x^T\mathbb{1} = 1 \qquad \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}$$

$$x \geq 0$$

$$\min \; U^Tb$$

$$\text{s.t.} \quad U^TA \geq C^T$$

$$U \geq 0$$

$$U^T = \frac{x^T}{u} \qquad U^Tb = \frac{1}{u}$$

We can talk about Bob's min max strategy "worst-best-case" for Alice.

Set $v = \max\{(Ay)_1, \ldots\ldots, (Ay)_a\}$

the LP is
$$\min_{y \in \mathbb{R}^b,\, v \in \mathbb{R}} v$$

$$\text{s.t.} \quad \mathbb{1}v \geq Ay$$

$$\mathbb{1}^Ty = 1$$

$$y \geq 0$$

$$\max \; C^TX$$

$$\text{s.t.} \quad AX \leq b$$

$$X \geq 0.$$

$$X = \frac{y}{v} \qquad b = \mathbb{1}$$

$$C^T = \mathbb{1}^T \qquad C^TX = \frac{1}{v}$$

$$\max \ \mathbb{1}^T \cdot \frac{y}{v} \quad \overset{\text{dual}}{\longleftrightarrow} \quad \min \ \frac{x^T}{u} \cdot \mathbb{1}$$

$$\text{s.t. } A \frac{y}{v} \leq \mathbb{1} \qquad\qquad \text{s.t. } \frac{x^T}{u} A \geq \mathbb{1}^T$$

$\Rightarrow$ Alice and Bob's programs are duals of each other

Strong duality $\Rightarrow$ two LPs have same obj value z^*

Alice's mixed strategy x^* $\Rightarrow$ payoff z^*

Bob's mixed strategy y^* $\Rightarrow$ pay z^*