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Math 482: Linear Programming, Fall 2020

Due Wednesday, September 30, 6PM CST

Homework 4

1. Write down the dual of the linear program below. (Do not solve).

$$\begin{aligned} \underset{x,y,z\in\mathbb{R}}{\text{maximize}} \quad & x + y + z \\ \text{subject to} \quad & 2x + y + 2z \le 14 \\ & x + z \le 8 \\ & 2x + 2y - z \le 18 \\ & x, y, z \ge 0. \end{aligned}$$

2. Determine whether $(x, y, z) = (5, 4, 0)$ is the optimal solution to the linear program from problem 1, using complementary slackness.

3. Consider the problem below:

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^n}{\text{maximize}} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} \quad & a_1x_1 + a_2x_2 + \cdots + a_nx_n \le 1, \\ & x_1, x_2, \ldots, x_n \ge 0. \end{aligned}$$

Assume that $a_1, \ldots, a_n, c_1, \ldots, c_n > 0$.

(a) Write down the dual program.

(b) Determine the optimal dual solution. (This will of course depend on $a_1, \ldots, a_n$ and $c_1, \ldots, c_n$, but you should describe how.)

(c) Find a primal solution with the same objective value.

4. Use the simplex method to solve the linear program below. Then, use your final simplex tableau to find the optimal dual solution.

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{maximize}} \quad & x - y + z \\ \text{subject to} \quad & x + 2y + z \le 5 \\ & 2x + y + z \le 6 \\ & x, y, z \ge 0. \end{aligned}$$

5. *(Only 4-credit students need to do this problem.)*

Consider the following linear program discussed in class:

$$
\begin{aligned}
\underset{\mathbf{x}\in\mathbb{R}^d}{\text{maximize}} \quad & x_d \\
\text{subject to} \quad & 0.1 \le x_1 \le 1 - 0.1, \\
& 0.1x_1 \le x_2 \le 1 - 0.1x_1, \\
& \dots \\
& 0.1x_{d-1} \le x_d \le 1 - 0.1x_{d-1}, \\
& x_1, x_2, \dots, x_d \ge 0.
\end{aligned}
$$

Let $\mathcal{P}_d$ be the "terrible trajectory"—the path between adjacent basic feasible solutions defined recursively as follows:

- $\mathcal{P}_1$ starts at $(0, 0, 0)$ and increases x_1 from its lower bound to its upper bound;
- $\mathcal{P}_k$ follows $\mathcal{P}_{k-1}$, then increases x_k from its lower bound to its upper bound, then undoes the steps of $\mathcal{P}_{k-1}$ in reverse order.

Show that the objective value increases with every step along $\mathcal{P}_d$. (Induct on d.)

1. $$\min_{u \in \mathbb{R}^3} \; 14u_1 + 8u_2 + 18u_3$$

$$2u_1 + u_2 + 2u_3 \geq 1$$

$$u_1 \quad + 2u_3 \geq 1$$

$$2u_1 + u_2 - u_3 \geq 1$$

$$u_1, u_2, u_3 \geqslant 0$$

2. $(x, y, z) = (5, 4, 0)$,

$$\begin{cases} 2x + y + 2z = 14 & \text{tight} \\ x \quad + z \leqslant 8 & \text{slack} \Rightarrow u_2 = 0 \\ 2x + 2y - z = 18 & \text{tight} \end{cases}$$

$x, y > 0 \quad \Rightarrow \begin{cases} 2u_1 + u_2 + 2u_3 = 1 \\ u_1 \quad + 2u_3 = 1 \end{cases}$

$\Rightarrow u_1 = u_2 = 0 \quad u_3 = \frac{1}{2}$

$(u_1, u_2, u_3) = (0, 0, \frac{1}{2})$

$x + y + z = 9$, $14u_1 + 8u_2 + 18u_3 = 9$

$\Rightarrow (x, y, z) = (5, 4, 0)$

$(u_1, u_2, u_3) = (0, 0, \frac{1}{2})$ satisfy CS

$\Longrightarrow (x, y, z) = (5, 4, 0)$ is optimal.

3. (a) $\boxed{\begin{aligned} &\min_{u \in \mathbb{R}} \quad u \\ &\text{s.t.} \quad a_i u \geq c_i \quad i = 1, 2, \dots, n \\ &\qquad\quad u \geq 0 \end{aligned}}$

(b) opt solution

$\Longrightarrow$ in (D) $\quad a_i u = c_i$ or $x_i = 0$

for all $i = 1, 2, \dots, n$.

i.e $u = \frac{c_i}{a_i}$ or $x_i = 0$

for all $i = 1, 2, \dots, n$.

in (P) $\quad \sum_{i=1}^{n} a_i x_i = 1$ or $u = 0$

Obviously $x_i = 0$ for $i = 1, 2, \dots, n$ is not a upper bound of primal objective function.

$\Longrightarrow \exists$ set $\mathbb{P}$ and constant $k > 0$. $\forall i \in \mathbb{P}$ $\quad u = \frac{c_i}{a_i} = k \neq 0$

$\forall i \notin \mathbb{P} \quad x_i = 0$.

Since $u \geq \frac{c_i}{a_i}$ for all $i = 1, \dots, n$.

So $k = \max\{ \frac{c_i}{a_i} \mid i = 1, 2, \dots, n \}$

$\Longrightarrow$ The optimal dual solution is $\text{Max}\{\frac{c_i}{a_i} \mid i=1,2\ldots n\}$.

(C) Since $u = k \neq 0$ and $\forall i \notin P$, $x_i = 0$.

we know $\sum_{i=1}^{n} a_i x_i = \sum_{i \in P} a_i x_i = 1$.

for $i \in P$ at least $x_i \neq 0$.

$X = \{ X \in \mathbb{R}^n \mid \sum_{i \in P} a_i x_i = 1,\ x_j = 0, \forall j \notin P \}$

$\Longrightarrow$ one solution : $k \in P$ $X = \{ x_i = 0, i \neq k, x_k = \frac{1}{a_k} \}$.

4.

	x	y	z	S_1	S_2	
S_1	1	2	1	1	0	5
S_2	2	1	1	0	1	6
$-K$	1	-1	1	0	0	0

pivot on x remove S_2

	x	y	z	S_1	S_2	
S_1	0	$\frac{3}{2}$	$\frac{1}{2}$	1	$-\frac{1}{2}$	2
x	1	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{1}{2}$	3
$-K$	0	$-\frac{3}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	-3

Pivot on z remove S_1

	x	y	z	S_1	S_2	
z	0	3	1	2	-1	4
x	1	-1	0	-1	1	1
$-K$	0	-3	0	-1	0	-5

$\Rightarrow$ when $(x, y, z) = (1, 0, 4)$ $\text{Max}(x - y + z) = 5$

Since in the objective function line in the final simplex tableau S_1's reduced cost is -1, S_2's reduced cost is 0.

$\Rightarrow$ $u_1 = -1$ $u_2 = 0$ is the optimal dual sol.