

## Lecture 33: The Branch-and-Bound Method

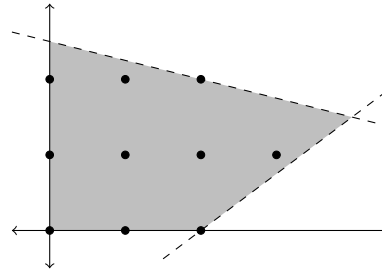
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University of Illinois at Urbana-Champaign

## 1 Solving an example integer program

Consider the following integer program.

$$\begin{array}{ll} \text{maximize} & 4x + 5y \\ \text{subject to} & x + 4y \leq 10 \\ & 3x - 4y \leq 6 \\ & x, y \geq 0 \end{array}$$



The only feasible points of the integer program are the ten points marked as black dots in the diagram above, but we don't know how to find those yet. Right now, the only thing we know how to do is forget about the  $x, y \in \mathbb{Z}$  constraint, and solve the linear program we get by taking  $x, y \in \mathbb{R}$ . This is called solving the *linear programming relaxation* of the integer program.

This is not a complicated linear program, and we can find the optimal solution in just two pivot steps:

$x$	$y$	$s_1$	$s_2$			$x$	$y$	$s_1$	$s_2$			$x$	$y$	$s_1$	$s_2$				
$s_1$	1	4	1	0	10	$\rightsquigarrow$	$y$	$1/4$	1	$1/4$	0	$5/2$	$\rightsquigarrow$	$y$	0	1	$3/16$	$-1/16$	$3/2$
$s_2$	3	-4	0	1	6		$s_2$	4	0	1	1	16		$x$	1	0	$1/4$	$1/4$	4
$-z$	4	5	0	0	0		$-z$	$11/4$	0	$-5/4$	0	$-25/2$		$-z$	0	0	$-31/16$	$-11/16$	$-47/2$



We obtain the point  $(x, y) = (4, 1.5)$  as our optimal solution, with objective value 23.5. But it does not solve the integer program, because  $y$  is not an integer. At the very least we can say the following, however:

**Fact.** The optimal objective value of the linear programming relaxation is an upper bound for the objective value of the integer program.

Whatever the optimal integer solution is, its objective value is at most 23.5. (It could be equal, or it could be smaller. The integer program might even be infeasible, which we'll often think of as an objective value of  $-\infty$  for a maximization problem.)

The "branch" part of the branch-and-bound algorithm is named after the idea that whenever we get such a fractional optimal solution, we should consider two "branches" that both exclude

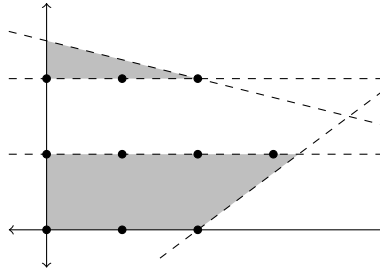
<sup>1</sup>This document comes from the Math 482 course webpage: <https://faculty.math.illinois.edu/~mlavrov/courses/482-spring-2020.html>

it. If the optimal solution gives a fractional value  $f \notin \mathbb{Z}$  to some variable  $x_i$ , we consider two possibilities:

$$x_i \leq \lfloor f \rfloor \quad \text{or} \quad x_i \geq \lceil f \rceil.$$

(Remember that in practical applications, our integer variables are often  $\{0, 1\}$ -valued variables. In that case, the inequalities we get when  $x_i$  is fractional will correspond to  $x_i = 0$  and  $x_i = 1$ .)

In the example above, we have  $y = 1.5 \notin \mathbb{Z}$ , so we should try two possibilities: either  $y \geq 2$ , or  $y \leq 1$ . These correspond to the top and bottom shaded regions of the diagram below, respectively. (Notice that we've eliminated the optimal solution  $(4, 1.5)$  from the shaded region, but we haven't lost any integer points.)



We can add either  $y \leq 1$  or  $y \geq 2$  to our constraints, and we'll probably have to try both. Each one of those gives us a new linear program to solve. Let's try  $y \geq 2$  first.

When we add a constraint, we don't have to solve the linear program from scratch. We can insert a new row for that constraint, getting a tableau that's still dual feasible. Here, we add a row corresponding to the inequality  $y \geq 2$ , or  $-y + s_3 = -2$ :

$$\begin{array}{cc|cc|c} x & y & s_1 & s_2 & \\ \hline y & 0 & 1 & 3/16 & -1/16 & 3/2 \\ x & 1 & 0 & 1/4 & 1/4 & 4 \\ \hline -z & 0 & 0 & -31/16 & -11/16 & -47/2 \end{array} \rightsquigarrow \begin{array}{cc|cc|c} x & y & s_1 & s_2 & s_3 & \\ \hline y & 0 & 1 & 3/16 & -1/16 & 0 & 3/2 \\ x & 1 & 0 & 1/4 & 1/4 & 0 & 4 \\ s_3 & 0 & -1 & 0 & 0 & 1 & -2 \\ \hline -z & 0 & 0 & -31/16 & -11/16 & 0 & -47/2 \end{array}$$

Before we do anything, we should row-reduce the resulting tableau to eliminate the improper  $-1$  in the  $y$  column. This gives us the first tableau below. Then, we use the dual simplex method to find a new optimal solution. (Here, we're done after  $s_3$  replaces  $s_2$  in the basis, giving us the second tableau below.)

$$\begin{array}{c|ccccc|c}
x & y & s_1 & s_2 & s_3 & \\
\hline
y & 0 & 1 & 3/16 & -1/16 & 0 & 3/2 \\
x & 1 & 0 & 1/4 & 1/4 & 0 & 4 \\
s_3 & 0 & 0 & 3/16 & -1/16 & 1 & -1/2 \\
\hline
-z & 0 & 0 & -31/16 & -11/16 & 0 & -47/2
\end{array}
\rightsquigarrow
\begin{array}{c|ccccc|c}
x & y & s_1 & s_2 & s_3 & \\
\hline
y & 0 & 1 & 0 & 0 & -1 & 2 \\
x & 1 & 0 & 1 & 0 & 4 & 2 \\
s_2 & 0 & 0 & -3 & 1 & -16 & 8 \\
\hline
-z & 0 & 0 & -4 & 0 & -11 & -18
\end{array}$$

We've obtained the optimal solution  $(x, y) = (2, 2)$ , with objective value 18. This doesn't mean we're done, though! There was still another branch to consider, the  $y \leq 1$  branch, and that one might lead to a larger objective value.

What we *can* say, however, is that the optimal objective value of the original integer program is at least 18, for the simple reason that we've found a feasible solution with that objective value. In general:

**Fact.** An integer solution to a subproblem in the branch-and-bound method leads to a lower bound on the optimal objective value of the original problem.

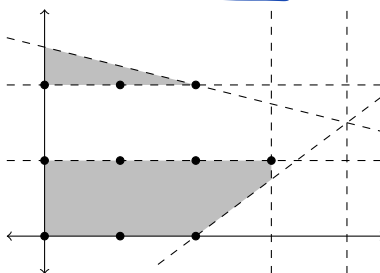
Now let's look at the  $y \leq 1$  case. To solve this, we'd go back to the optimal tableau for the point  $(4, 1.5)$ , and add a row corresponding to the inequality  $y \leq 1$ , or  $y + s_3 = 1$ . Then, proceed as we did in the previous example.

I will skip the details, because the dual simplex method is not today's focus: we know how to do it already. In the  $y \leq 1$  branch, we will get a fractional optimal solution  $(x, y) = (\frac{10}{3}, 1)$ , with objective value  $\frac{55}{3}$ .

In theory, since  $\frac{55}{3} > 18$ , this branch is more promising. So we keep exploring it. Since  $x = \frac{10}{3} \approx 3.333$  is a fractional value, we branch on two more possibilities: either  $x \leq 3$ , or  $x \geq 4$ . These are both added to the linear program *in addition* to the existing condition  $y \leq 1$  we're considering. (Their tableaux will now have four rows, and slack variable  $s_1, s_2, s_3, s_4$ .)

When we solve the linear program with  $y \leq 1$  and  $x \leq 3$ , we get the optimal solution  $(3, 1)$ . It has objective value 17, which is worse than our previous result of 18. So, even though this solution is an integer solution, we don't consider it any further. (And if this point were a fractional point with objective value 17, we still wouldn't branch from it: it couldn't lead us to any integer solutions better than what we've already found.)

When we solve the linear program with  $y \leq 1$  and  $x \geq 4$ , we are unable to obtain a primal feasible tableau. Indeed, we can see in the diagram below that no part of the feasible region remains to the right of the  $x = 4$  line. This branch is already infeasible, and so we don't keep going.



Now, we have considered all the possibilities there were. The best point found so far is the point  $(2, 2)$  with objective value 18, so that is our final answer.

## 2 A general description of the method

In general, the branch-and-bound method can be summarized as follows. (I will assume that we're solving a maximization problem; if we're solving a minimization problem, the upper and lower bounds are reversed, and we use  $+\infty$  instead of  $-\infty$  as a "worst possible" value.)

We maintain two pieces of data:

- A list  $\mathcal{L}$  of “nodes”: linear programs describing all or part of our feasible region. (Initially,  $\mathcal{L}$  consists only of one linear program: the linear programming relaxation of our integer program.)
- A candidate optimal solution  $\mathbf{x}^*$  to the integer program, and its objective value  $z^*$ . (Initially, there is no candidate  $\mathbf{x}^*$ , and we set  $z^* = -\infty$ .)

A step in the algorithm examines a single node in  $\mathcal{L}$ : we solve the linear program associated with that node, and remove that node from  $\mathcal{L}$ . Let  $\mathbf{x}$  be the optimal solution we get, and let  $z$  be its objective value. There are several possibilities:

- If  $z \leq z^*$ , then this node can't lead to any better solutions than what we've already got. We don't explore it further. We call the node pruned by bound.

As a kind of special case, if the linear program is infeasible, then of course we can't continue exploring that node, and we call it pruned by infeasibility. We can think of this as getting the objective value  $z = -\infty$ .

- If  $z > z^*$ , and all components of  $\mathbf{x}$  are integers, we save this as our best solution so far: set  $\mathbf{x}^* = \mathbf{x}$  and  $z^* = z$ . We don't explore the node further; there's no point. We call it pruned by integrality.
- If  $z > z^*$ , but some component  $x_i$  has a fractional value  $f$  in  $\mathbf{x}$ , we branch. We add two new nodes to  $\mathcal{L}$ : one which adds the constraint  $x_i \leq \lfloor f \rfloor$  to the current node, and one which adds the constraint  $x_i \geq \lceil f \rceil$ .

We repeat this procedure for as long as we can. Once  $\mathcal{L}$  is empty (there are no more unexplored nodes), the current value of  $\mathbf{x}^*$  is our optimal solution. It's possible that we still have  $z^* = -\infty$ , and no  $\mathbf{x}^*$ ; if so, there are no feasible solutions to the integer program.

It's convenient to represent this procedure by a tree of the branches we took. For example, here is a tree representation of our example:

