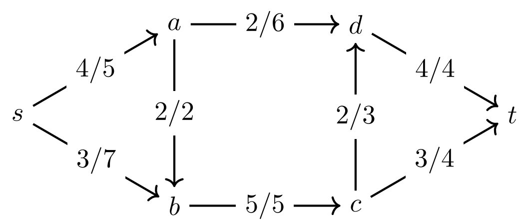
Math 482: Linear Programming, Fall 2020

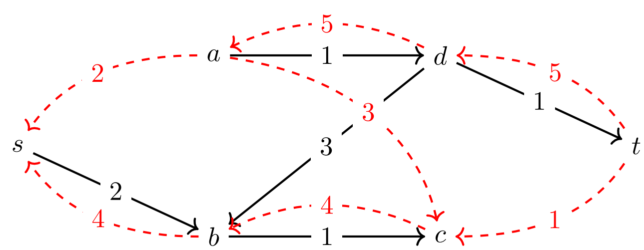
Due Wednesday, November 11, 6PM CST

Homework 8

1. Given the network below, with label x/y denoting a flow of x and a total capacity of y along an edge, draw the residual graph, and use it to list all possible augmenting paths.



2. The diagram below gives a residual graph for a network. (Black edges are "forward" edges, red dashed edges are "backward" edges.)

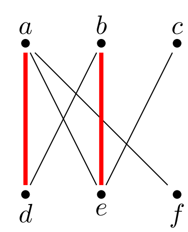


   (a) Determine the edges of the original network, and their capacities.

   (b) Find the flow which produces this residual graph.

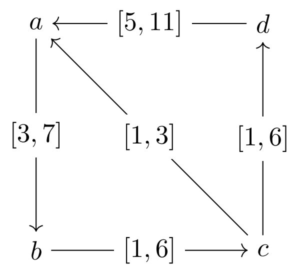
   (c) Find a cut with the same capacity as the value of this flow.

3. Consider the following matching (that is, $M = \{(a, d), (b, e)\}$) in a bipartite graph:



   First, convert this matching into a feasible flow in a network. Then, find an augmenting path in that network, and use it to improve the matching to a larger one.

4. Suppose that we want to find a feasible circulation in the network below with flows on each edge in the specified lower and upper bounds.



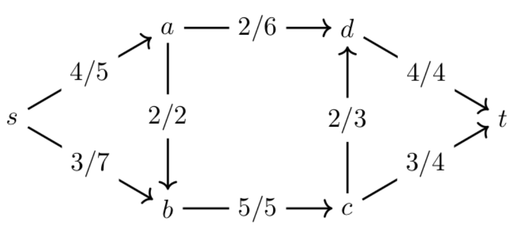
Find a feasible circulation in this network by writing down an equivalent maximum-flow problem, finding the maximum flow in that network (using Ford-Fulkerson), and converting it back to an equivalent feasible circulation.

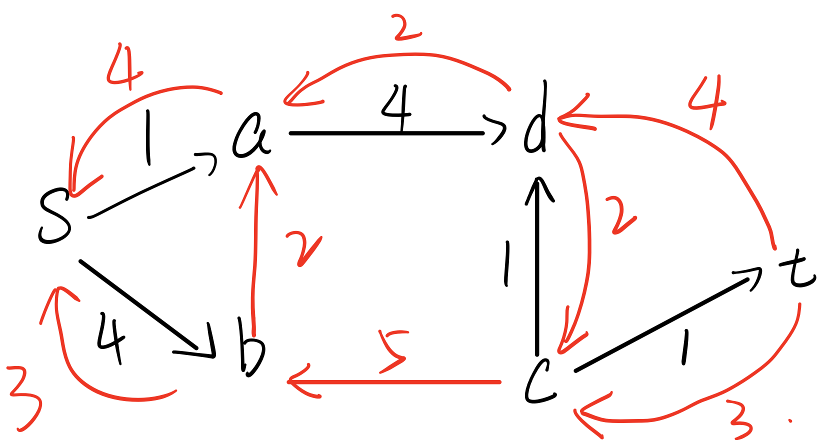
5. *(Only 4-credit students need to do this problem.)*

   Write down a linear program for a general feasible circulation problem. (There is no objective function, so make the objective function just "maximize 0".)

   Then, take the dual of this linear program.

1.



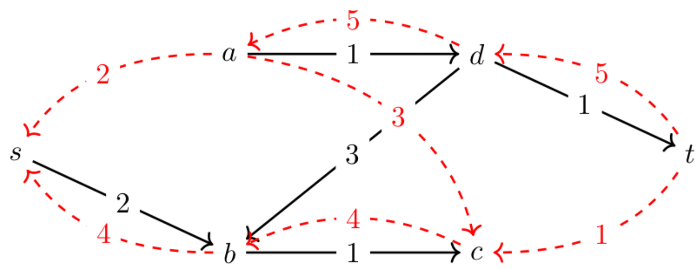


augmenting paths:

$s \to b \leftarrow a \to d \leftarrow c \to t$

$s \to a \to d \leftarrow c \to t$

2.



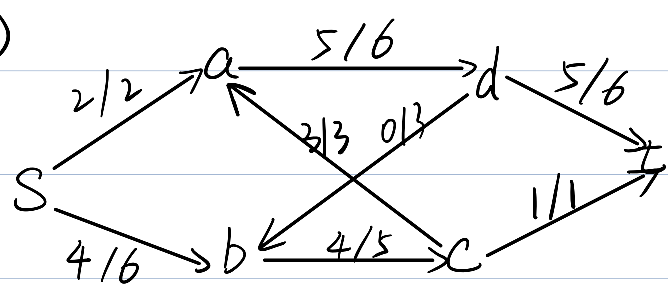
(a) $x_{sb} = 4$, $C_{sb} = 6$, $x_{sa} = 2$, $C_{sa} = 2$

$x_{bc} = 4$, $C_{bc} = 5$, $x_{ad} = 5$, $C_{ad} = 6$

$x_{db} = 0$, $C_{db} = 3$, $x_{ca} = 3$, $C_{ca} = 3$.

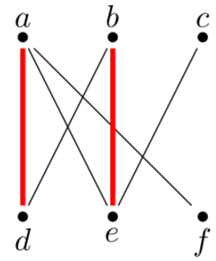
$x_{dt} = 5$, $C_{dt} = 6$, $x_{ct} = 1$, $C_{ct} = 1$.

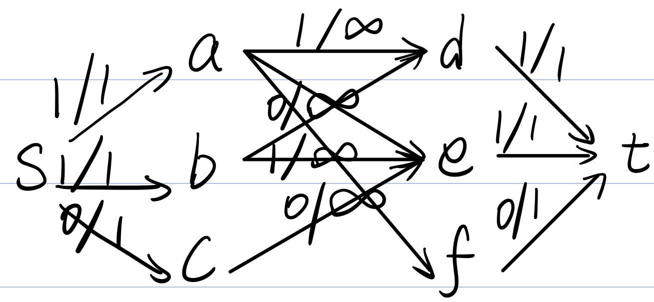
(b)



(c) $(\{s, b, c\}, \{a, d, t\})$.

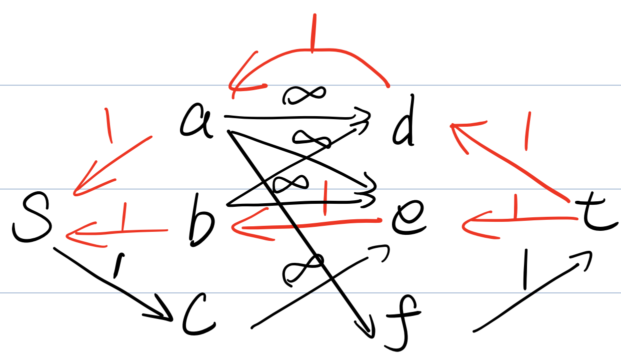
3.

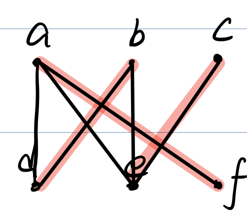




$$\{\, x_{sa} = x_{sb} = x_{ad} = x_{be} = x_{dt} = x_{et} = 1$$

$$x_{sc} = x_{ae} = x_{af} = x_{bd} = x_{ce} = x_{ft} = 0 \,\}.$$



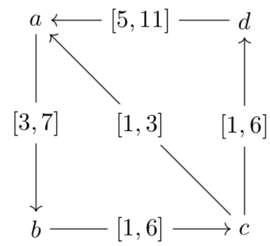


augmenting pathway: $s \to c \to e \leftarrow b \to d \leftarrow a \to f \to t$.

$\Rightarrow$ improved matching:

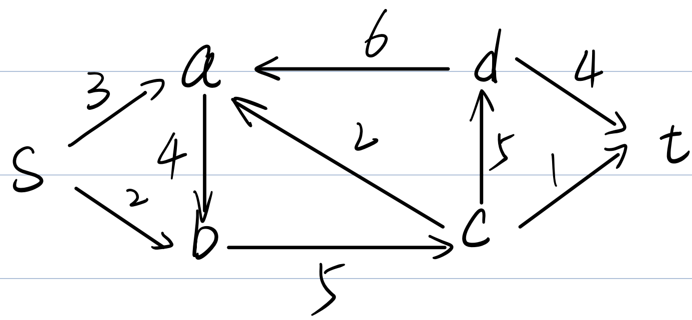
$\{(a,f),\ (b,d),\ (c,e)\}$

4.

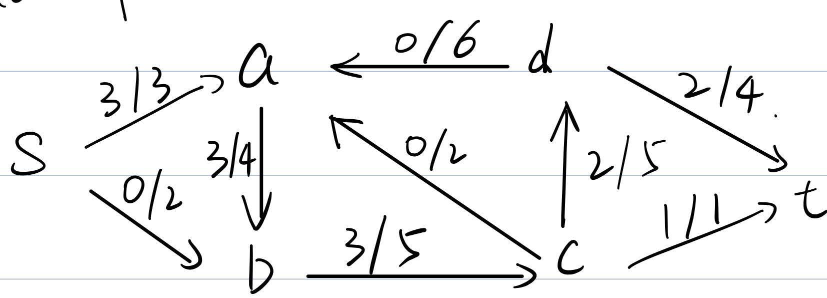


$d_a = 3 - 5 - 1 = -3$, $d_b = 1 - 3 = -2$

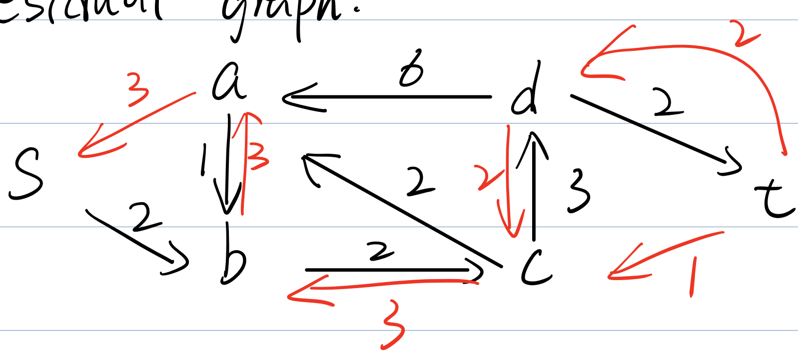
$d_c = 1 + 1 - 1 = 1$, $d_d = 5 - 1 = 4$



Start feasible flow:

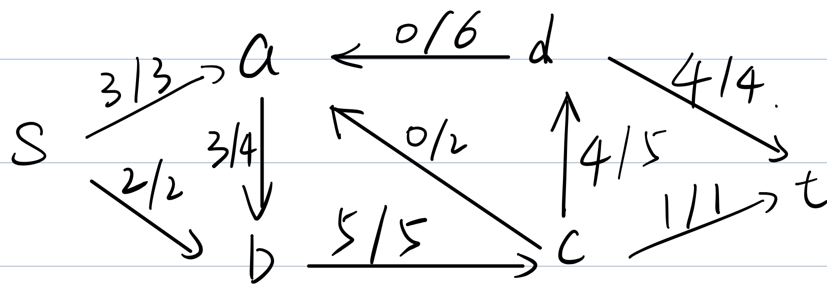


residual graph:



augmenting pathway $S \xrightarrow{2} b \xrightarrow{2} c \xrightarrow{2} d \xrightarrow{2} t$

flow :



$\{ X_{sa} = 3, \ X_{sb} = 2, \ X_{ab} = 3, \ X_{bc} = 5, \ X_{cd} = 4$

$X_{dt} = 4, \ X_{ct} = 1, \ X_{others} = 0 \}.$

which is maximum flow.

equivalent feasible circulation:

$\{ X_{ab} = 6, \ X_{bc} = 6, \ X_{cd} = 5, \ X_{ca} = 1, \ X_{da} = 5 \}.$

5. *(Only 4-credit students need to do this problem.)*

Write down a linear program for a general feasible circulation problem. (There is no objective function, so make the objective function just "maximize 0".)

Then, take the dual of this linear program.

$$\max_{x \in \mathbb{R}^{|A|}} 0$$

$$\text{s.t.} \sum_{i:(i,k)\in A} x_{ik} - \sum_{j:(k,j)\in A} x_{kj} = 0 \quad (k \in N)$$

$$a_{ij} \leq x_{ij} \leq b_{ij} \quad (i,j) \in A$$

$$\begin{cases} -x_{ij} \leq -a_{ij} \\ x_{ij} \leq b_{ij} \end{cases}$$

$$\min \sum_{(i,j)\in A} -a_{ij} V_{ij} + b_{ij} W_{ij}$$

$$V_{ij} + W_{ij} + U_j - U_i = 0 \quad ((i,j)\in A).$$

$V_{ij}, W_{ij} \geq 0$ $\quad U_j, U_i$ unrestricted.