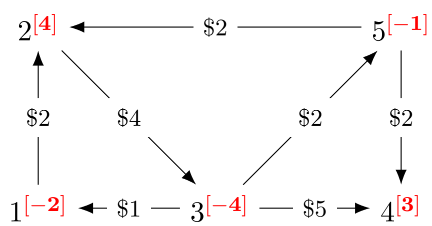
Wenxiao Yang.

Math 482: Linear Programming, Fall 2020

Due Wednesday, November 18, 6PM CST

Homework 9

1. Consider the minimum-cost flow problem given in the diagram below. Solve this by adding an artificial node, and performing diagramatic two-step simplex method as in Lecture 29. Make sure you draw the diagrams at each step.



2. Suppose that we are using the primal-dual method to solve the linear program

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}} \quad & 3x_1 + x_2 + x_4 \\
\text{subject to} \quad & x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& x_1 + x_2 - 2x_4 = 3 \\
& x_1 - 2x_2 - x_3 + 2x_4 = 4 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

and that we are currently at the dual solution $\mathbf{u} = (1, 1, 1)$.

(a) Write down and solve (**RP**): the restricted primal program.

(b) Find the optimal solution $\mathbf{v}$ to (**DRP**): the dual of the restricted primal.

(c) Use $\mathbf{v}$ to find an improved dual solution to the original linear program.

(d) Show that the new dual solution is optimal by finding a corresponding optimal primal solution.

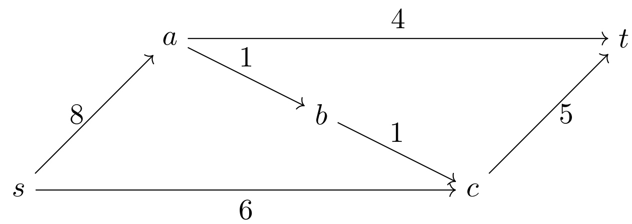
3. Consider the linear program

$$
\begin{aligned}
\underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}} \quad & x_1 + 3x_2 - 2x_3 - x_4 \\
\text{subject to} \quad & x_1 + 2x_2 - 2x_3 - x_4 = 2 \\
& x_1 + x_2 - 2x_4 = 3 \\
& x_1, x_2, x_3, x_4 \geq 0
\end{aligned}
$$

(a) Use the assumption that $x_1 + x_2 + x_3 + x_4 \leq 100$ in all feasible solutions to the linear program above, to write down an equivalent linear program and a feasible dual solution for it.

(b) Perform two iterations of the primal-dual method, starting from the feasible dual solution you found.

4. Use the primal-dual version of Dijkstra's algorithm from Lecture 33 to find the shortest path from s to t in the network below. Make sure you draw the diagram associated to the dual and the dual of the restricted primal at each step.

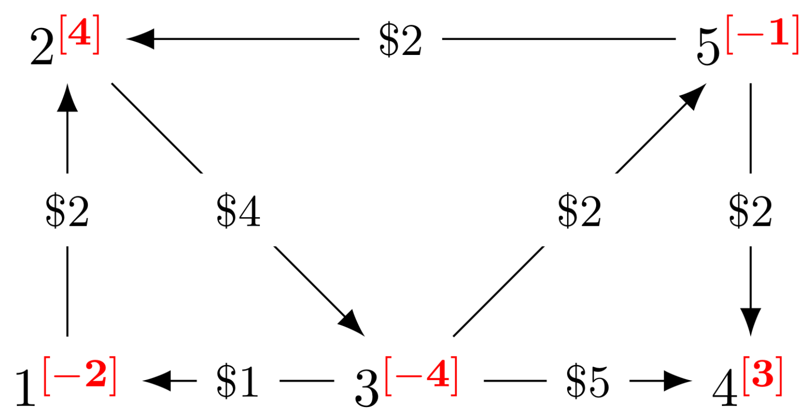


5. *(Only 4-credit students need to do this problem.)*

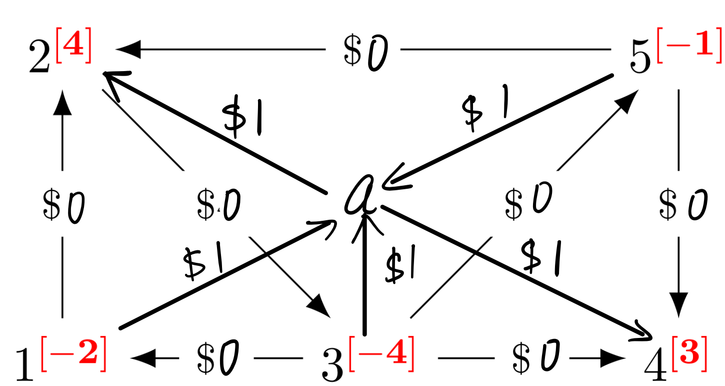
Suppose that you have a network in which, instead of every arc having a capacity, there is a capacity associated through every node (other than the source s or the sink t). The flow along an arc can be arbitrary, but the total flow going into a node (equivalently, the total flow going out of a node) can be at most the capacity of that node.

Explain how to convert a maximum-flow problem for such a network into a standard maximum-flow problem.

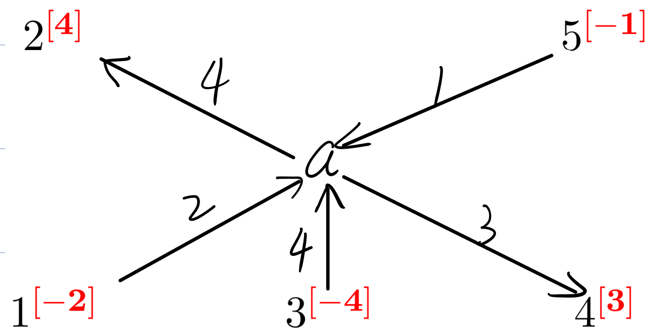
1.



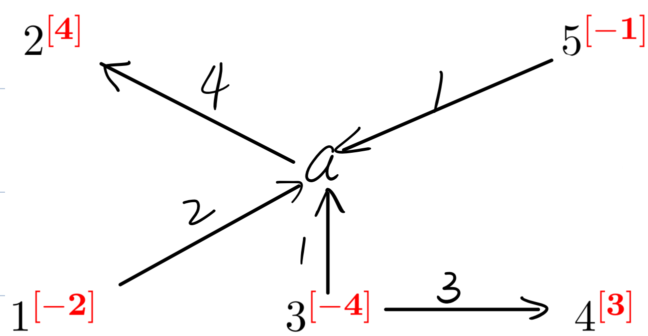
Phase 1 network:



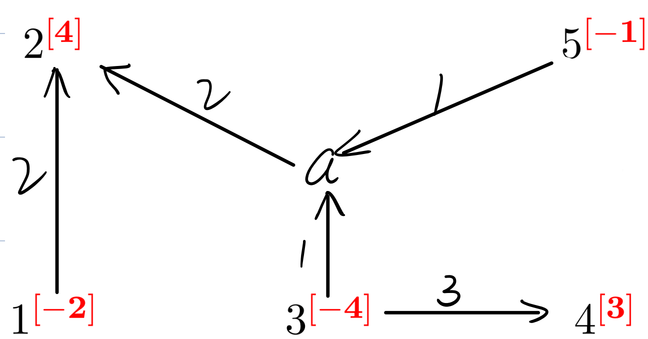
spanning tree:



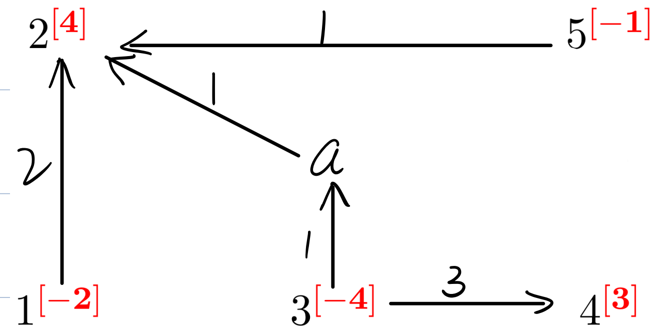
add x_{34}, delete x_{a4}



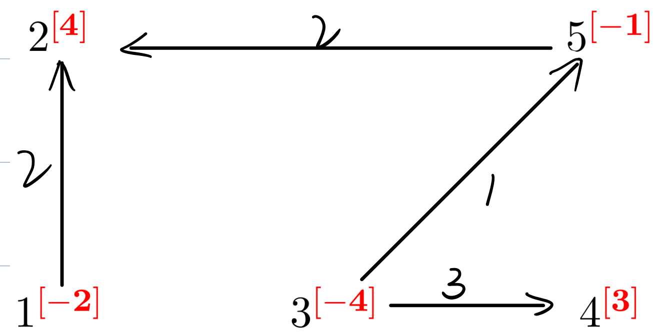
add x_{12}, delete x_{1a}



add x_{52}, delete x_{5a}



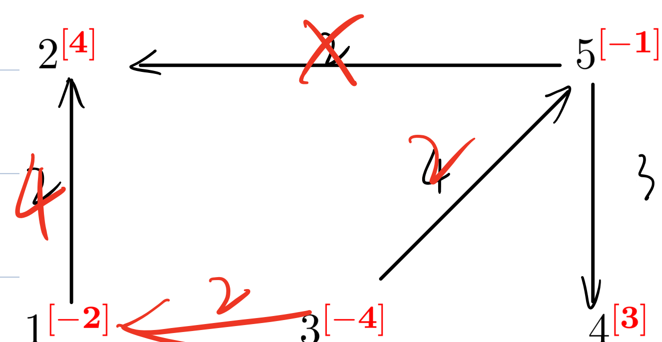
add x_{35}, delete x_{a2}, x_{3a}



this is a feasible basic solution.

phase 2: x_{54}'s reduced cost is -1 $\delta \le 1$

add x_{54} delete x



x_{31}'s reduced cost is -1

this is the optimal solution: $\{x_{12}=4,\ x_{31}=2,\ x_{35}=2,$
$x_{54}=3\}$.

2. (P)

$$\begin{aligned}\underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}}\quad & 3x_1 + x_2 + x_4\\ \text{subject to}\quad & x_1 + 2x_2 - 2x_3 - x_4 = 2\\ & x_1 + x_2 - 2x_4 = 3\\ & x_1 - 2x_2 - x_3 + 2x_4 = 4\\ & x_1, x_2, x_3, x_4 \ge 0\end{aligned}$$

(D)

$$\begin{aligned}\max\quad & 2u_1 + 3u_2 + 4u_3\\ \text{s.t.}\quad & u_1 + u_2 + u_3 \le 3\\ & 2u_1 + u_2 - 2u_3 \le 1\\ & -2u_1 - u_3 \le 0\\ & -u_1 - 2u_2 + 2u_3 \le 1\end{aligned}$$

$u = (1, 1, 1) \Rightarrow$ (DRP):

$$\begin{aligned}\max\quad & 2v_1 + 3v_2 + 4v_3\\ \text{s.t.}\quad & v_1 + v_2 + v_3 \le 0\\ & 2v_1 + v_2 - 2v_3 \le 0\\ & v_1 \le 1\\ & v_2 \le 1\\ & v_3 \le 1\end{aligned}$$

(a) (RP):

$$\begin{aligned}\min\quad & y_1 + y_2 + y_3\\ \text{s.t.}\quad & x_1 + 2x_2 + y_1 = 2\\ & x_1 + x_2 + y_2 = 3\\ & x_1 - 2x_2 + y_3 = 4\end{aligned}$$

	x_1	x_2	y_1	y_2	y_3	
y_1	1	2	1	0	0	2
y_2	1	1	0	1	0	3
y_3	1	-2	0	0	1	4
$-z$	-3	-1	0	0	0	-9

$\rightsquigarrow$

	x_1	x_2	y_1	y_2	y_3	
x_1	1	2	1	0	0	2
y_2	0	-1	-1	1	0	1
y_3	0	-4	-1	0	1	2
$-z$	0	5	3	0	0	-3

opt sol of (RP) is $(x_1, x_2, y_1, y_2, y_3) = (2, 0, 0, 1, 2)$

(b) $v^T = 1^T - r_y^T = (-2, 1, 1)$.

$v = (-2, 1, 1)^T$

(c) $(1-2t,\ 1+t,\ 1+t)$

$-1+2t-1-t \le 0 \Rightarrow t \le 2$

$-1+2t-2-2t+2+2t \le 1 \Rightarrow t \le 1$.

$\Rightarrow t = 1. \Rightarrow$ improved dual solucion:

$u = (-1, 2, 2)$.

$$u_1+u_2+u_3 = 3$$

$$2u_1+u_2-2u_3 = -4 \le 1 \Rightarrow x_2=0.$$

$$-2u_1 \quad -u_3 = 0$$

$$-u_1-2u_2+2u_3=1$$

$$\Rightarrow \begin{cases} x_1-2x_3-x_4 = 2 \\ x_1 \qquad\quad -2x_4 = 3 \\ x_1-x_3+2x_4 = 4 \end{cases} \Rightarrow \begin{cases} x_1 = \frac{27}{7} \\ x_3 = \frac{5}{7} \\ x_4 = \frac{3}{7} \end{cases}$$

$$x=\left(\frac{27}{7}, 0, \frac{5}{7}, \frac{3}{7}\right), \quad u=(-1, 2, 2)$$

$$3x_1+x_2+x_4 = \underline{12}, \qquad 2u_1+3u_2+4u_3=\underline{12}$$

the new dual solution u is optimal.

3.

(a)

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4\in\mathbb{R}}{\text{minimize}} & x_1+3x_2-2x_3-x_4 \\ \text{subject to} & x_1+2x_2-2x_3-\ \ x_4=2 \\ & x_1+\ \ x_2 \qquad\quad -2x_4=3 \\ & x_1,x_2,x_3,x_4\ge 0 \end{array}$$

$$\begin{array}{ll} \underset{x_1,x_2,x_3,x_4,x_5}{\min} & x_1+3x_2-2x_3-x_4 \\ \text{s.t.} & x_1+2x_2-2x_3-x_4 \qquad = 2 \\ & x_1+x_2 \qquad -2x_4 \qquad = 3 \\ & x_1+x_2+x_3+x_4+x_5=100 \\ & x_1,x_2,x_3,x_4,x_5 \ge 0. \end{array} \qquad (P)$$

$$\begin{aligned}
\max \quad & 2u_1 + 3u_2 + 100u_3 \\
\text{s.t.} \quad & u_1 + u_2 + u_3 \le 1 \\
& 2u_1 + u_2 + u_3 \le 3 \\
& -2u_1 \quad + u_3 \le -2 \\
& -u_1 - 2u_2 + u_3 \le -1 \\
& u_3 \le 0
\end{aligned} \qquad (D)$$

a feasible solution : $u = (0, 0, -2)$

(b).

$$(DRP): \begin{aligned}
\max \quad & 2v_1 + 3v_2 + 100v_3 \\
& -v_1 - 2v_2 + v_3 \le 0 \\
& v_1 \le 1 \\
& v_2 \le 1 \\
& v_3 \le 1
\end{aligned}
\qquad
(RP): \begin{aligned}
\min \quad & y_1 + y_2 + y_3 \\
\text{s.t.} \quad & -x_4 + y_1 = 2 \\
& -2x_4 + y_2 = 3 \\
& x_4 + y_3 = 100 \\
& x, y \ge 0
\end{aligned}$$

	x_1 *	x_2 *	x_3 *	x_4	x_5 *	y_1	y_2	y_3	
y_1	1	2	-2	-1	0	1	0	0	2
y_2	1	1	0	-2	0	0	1	0	3
y_3	1	1	1	1	1	0	0	1	100
$-z$	-3	-4	1	2	-1	0	0	0	-105

$$V^T = 1^T - r_y^T = (1, 1, 1).$$

$$\left.\begin{aligned}
t + t - 2 + t \le 1. &\Rightarrow t \le 1 \\
2t + t - 2 + t \le 3 &\Rightarrow t \le \tfrac{5}{4} \\
-2t \quad -2 + t \le -2 &\Rightarrow t \ge 0. \\
-t - 2t - 2 + t \le -1 &\Rightarrow t \ge -\tfrac{1}{2} \\
-2 + t \le 0 &\Rightarrow t \le 2.
\end{aligned}\right\} \Rightarrow t = 1.$$

new $u=(1,1,-1)$.

$\Rightarrow$ (DRP): $\max\ 2v_1+3v_2+100v_3$

s.t. $v_1+v_2+v_3 \le 0$

$v_1 \le 1$

$v_2 \le 1$

$v_3 \le 1$

(RP): $\min\ y_1+y_2+y_3$

$x_1 + y_1 = 2$

$x_1 + y_2 = 3$

$x_1 + y_3 = 100$

	x_1	x_2^*	x_3^*	x_4^*	x_5^*	y_1	y_2	y_3	
y_1	1	2	-2	-1	0	1	0	0	2
y_2	1	1	0	-2	0	0	1	0	3
y_3	1	1	1	1	1	0	0	1	100
$-z$	-3	-4	1	2	-1	0	0	0	-105

$\Downarrow$

	x_1	x_2^*	x_3^*	x_4^*	x_5^*	y_1	y_2	y_3	
y_1	1	2	-2	-1	0	1	0	0	2
y_2	0	-1	2	-1	0	-1	1	0	1
y_3	0	-1	3	2	1	-1	0	1	98
$-z$	0	2	-5	-1	-1	3	0	0	-99

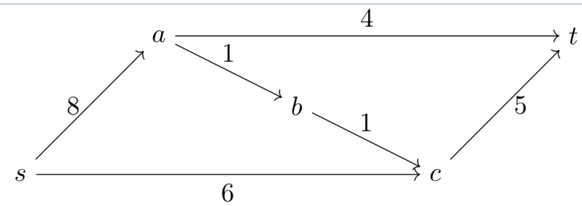
new $v^T = 1^T - \Gamma_y^T = (-2, 1, 1)$.

$$(1-2t)+(1+t)+(-1+t) \le 1$$

$$\left.\begin{array}{ll} 2(1-2t)+(1+t)+(-1+t) \le 3 & \Rightarrow t \ge -\frac{1}{2} \\ -2(1-2t) \quad + (-1+t) \le -2 & \Rightarrow t \le \frac{1}{5} \\ -(1-2t)-2(1+t)+(-1+t) \le -1 & \Rightarrow t \le 3 \\ \quad -1+t \le 0 & \Rightarrow t \le 1 \end{array}\right\} \Rightarrow t = \frac{1}{5}$$

now $u = \left(\frac{3}{5}, \frac{6}{5}, -\frac{4}{5}\right)$

4.



initial $u = (\underset{s}{0}, \underset{a}{0}, \underset{b}{0}, \underset{c}{0}, \underset{t}{0}) \Rightarrow J = \phi$

$\Rightarrow$

0 0

0

0 0

$V = (1, 1, 1, 1, 0)$

$1 \dashrightarrow 0$

1

1 1

$t = \min\{5-0, 4-0\} = 4$

arc (a, t)

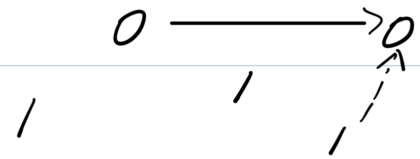
$\Rightarrow u = (4, 4, 4, 4, 0)$, $J = \{(a,t)\}$

$4 \longrightarrow 0$

4

4 4

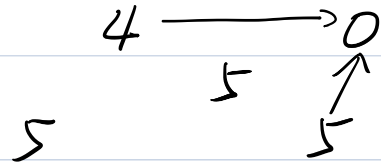
$V = (1, 0, 1, 1, 0)$



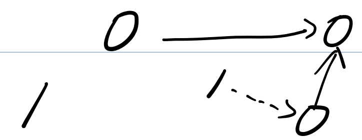
$t = \min\{5-4\} = 1.$

arc (c, t)

$\Rightarrow u = (5, 4, 5, 5, 0) \quad J = \{(a,t), (c,t)\}.$



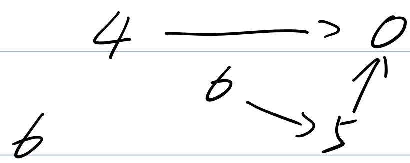
$V = (1, 0, 1, 0, 0)$



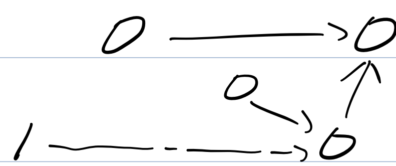
$t = \min\{6,\ 8-1,\ 1-0\} = 1$

arc: (b, c)

$\Rightarrow u = (6, 4, 6, 5, 0), \ J = \{(a,t), (c,t), (b,c)\}.$



$V = (1, 0, 0, 0, 0).$



$t = \min\{8-2,\ 6-1\} = 5.$

arc: (s, c)

$\Rightarrow u = (11, 4, 6, 5, 0), \ J = \{(a,t), (c,t), (b,c)$
$(s,c)\}.$

$V = (0, 0, 0, 0, 0)$

$\Rightarrow$ done!

opt value $= u_s - u_t = 11$

path: $s \to c \to t.$