Given bipartite graph (X, Y, E)

$$\max_{x \in \mathbb{R}^{|E|}} \sum_{i,j:(i,j)\in E} x_{ij}$$

$$\text{s.t.} \sum_{j:(i,j)\in E} x_{ij} \leq 1 \quad \forall\, i \in X.$$

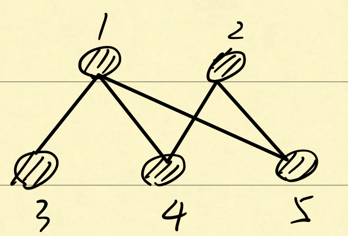
$$\sum_{i:(i,j)\in E} x_{ij} \leq 1 \quad \forall\, j \in Y.$$

$Ax \leq 1$ incidence matrix $(|X|+|Y|) \times |E|$

$A_{ve} = 1$ if v is an endpoint of e.

$A_{ve} = 0$ others.

Example



$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} (1,3) & (1,4) & (1,5) & (2,4) & (2,5) \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

if A is <u>totally unimodular</u> $\xrightarrow{\text{by theorem previous}}$ solution is integral.

Theorem: incidence matrix A is <u>totally unimodular</u>.
det of submatrix $= 0/1/-1$.

Proof: $k=1$: $\det = 0/1$.

$k>1$: $\det \begin{bmatrix} 0 & \\ 0 & \boxed{} \\ \vdots & \\ 0 & \end{bmatrix} = 0$

one col with all 0

one col with one 1. $\det \begin{bmatrix} 0 & \\ 0 & \\ \vdots & \boxed{} \\ 1 & \\ \vdots & \\ 0 & \end{bmatrix} = (-1)^i \det \boxed{}_{(k-1)\times(k-1)}$

all col with two 1. $\det\left[\begin{array}{c|c} 1 & \\ 0 & \\ \vdots & \\ 1 & \\ \vdots & \\ 0 & \end{array}\right]$

one is in X, one is in Y
add all X's rows into first row
add all Y's rows into last row.
first row = last row
$\Rightarrow \det = 0$.

more than two is impossible.

Dual $\min\limits_{u \in \mathbb{R}^{n+m}} \sum\limits_{i \in X \cup Y} u_i$

s.t. $u_i + u_j \geq 1$, $(i, j) \in E$

$u \geq 0$

u describing a set of S: $i \in S$ when $u_i = 1$, $i \notin S$, when $u_i = 0$.

Vertex cover: Set of vertices that cover every edge (include at least one edpoint of each edge).

Looking for smallest S that covers every edge.

$\underbrace{S \text{ is a vertex cover, } M}_{\text{1-to-1.}}$ is a matching.

edge in M $\leq$ # vertices in S. (Weak duality).

Theorem: # edge in Maximum matching

= # vertices in min vertex cover.

Proof: Strong duality, total unimodularity.

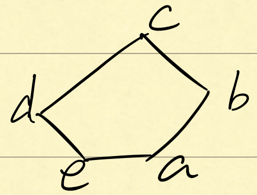
General Form: (V, E).

Matching: share no endpoints.

Vetex cover: set of vertices covering all edges.

there is no bipartiteness ⇒ may not totally unimodular.

Example:



Max matching is of size 2.
(a, b), (c, d)

Min vertex cover is 3 vertices.
{a, c, d}.

Max matching LP ⇒ opt value = 2.5

Min vertex cover ⇒ opt value = 2.5