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Math 482: Linear Programming, Fall 2020

Due Monday, September 21, 6PM CST

Homework 3

1. Let $P \subseteq \mathbb{R}^3$ be the convex polyhedron with only the following four extreme points: $(0,0,0)$, $(0,1,1)$, $(1,0,1)$, and $(1,1,0)$.
   (a) Write down a set of four linear inequalities describing P.
   (b) Show directly from the definition that the point $(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is *not* an extreme point of P.

2. Use the two-phase simplex method to solve the following linear program:

$$\begin{aligned} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{maximize}} \quad & x_1 + x_2 + 3x_3 \\ \text{subject to} \quad & 2x_1 \qquad + \ x_3 = 2 \\ & \qquad x_2 + \ x_3 = 3 \\ & 4x_1 + x_2 + 3x_3 = 7 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

3. Use lexicographic pivoting to solve the following linear program:

$$\begin{aligned} \underset{x,y\in\mathbb{R}}{\text{maximize}} \quad & x - y \\ \text{subject to} \quad & x - 2y \leq 0 \\ & x - 3y \leq 0 \\ & y \leq 3 \\ & x, y \geq 0 \end{aligned}$$

4. Consider the following linear program:

$$\begin{aligned} \underset{\mathbf{x}\in\mathbb{R}^{10}}{\text{maximize}} \quad & x_1 + 2x_2 + 3x_3 + 4x_4 + 2x_5 - x_6 + 4x_7 + 4x_8 + 2x_9 - x_{10} \\ \text{subject to} \quad & x_1 + x_2 - x_3 + 2x_4 - x_5 + 3x_6 + 2x_7 - x_8 + x_9 + 2x_{10} = 3 \\ & 3x_1 + 4x_2 + 2x_3 + 7x_4 + 5x_5 + 6x_6 - 2x_7 + 9x_8 + 8x_9 + 9x_{10} = 10 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \geq 0. \end{aligned}$$

   (a) Starting with basic variables $\mathcal{B} = (x_1, x_2)$, compute the inverse matrix $A_{\mathcal{B}}^{-1}$ and the basic feasible solution corresponding to $\mathcal{B}$.
   (b) Perform one iteration of the revised simplex method from the basic feasible solution you found in part (a). Use Bland's rule for pivoting.

   Your answer should give the new basis $\mathcal{B}$, the new inverse matrix $A_{\mathcal{B}}^{-1}$, and the new basic feasible solution.

5. *(Only 4-credit students need to do this problem.)*

   Your friend was solving a linear program with two inequality constraints on the variables x and y, as well as the nonnegativity constraints $x, y \geq 0$. After adding slack variables s_1, s_2 to deal with the constraints, your friend used the simplex method to arrive at the following tableau:

	x	y	s_1	s_2	
x	1	2	0	0	3
s_2	0	1	-1	1	1
$-z$	0	-2	-1	0	-4

Show that your friend must have made a mistake: there is no linear program of the form described which can result in this final tableau.

(Hint: what would the starting tableau have been?)

四面 $x + ay + bz = 0.$

$(1, a, b) \perp (1,1,0), (0,1,1)$

$1 + a = a + b = 0 \Rightarrow b = 1.$ $a = -1.$

$\Downarrow$

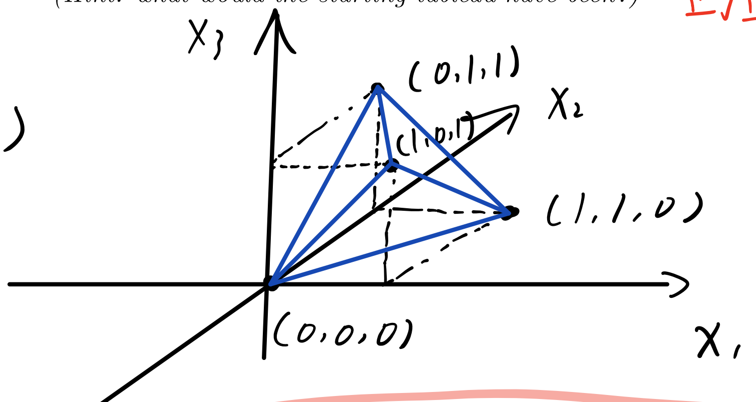
$x + y - z \leq 0.$

$x - y + z \geq 0.$

$-x + y + z \geq 0$

$x + y + z \leq 2.$

1. (a)



$$0 \leq x_1 + x_2 + x_3 \leq 2.$$
$$0 \leq x_1 \leq 1$$
$$0 \leq x_2 \leq 1$$
$$0 \leq x_3 \leq 1.$$

2. exists $u = (\frac{1}{3}, -\frac{1}{3}, 0)$

$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) + u = (1, \frac{1}{3}, \frac{2}{3}) \in$ feasible region

$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) - u = (\frac{1}{3}, 1, \frac{2}{3}) \in$ feasible region.

$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}) \in [(\frac{1}{3}, 1, \frac{2}{3}), (1, \frac{1}{3}, \frac{2}{3})]$

$\Rightarrow (\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ is not extreme point of P.

$\Rightarrow X$

$(\frac{2}{3}, \frac{2}{3}, \frac{2}{3})$ lies on $(1,1,0)$ to $(\frac{1}{2}, \frac{1}{2}, 1) \in P$

2. $\underset{x_1,x_2,x_3\in\mathbb{R}}{\text{maximize}} \quad x_1 + x_2 + 3x_3 = z$

subject to

$$2x_1 + x_3 = 2$$
$$x_2 + x_3 = 3$$
$$4x_1 + x_2 + 3x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$

$\underset{x_i,\, x_i^a \in \mathbb{R}}{\min} \quad x_1^a + x_2^a + x_3^a = z^a$

s.t.

$$2x_1 + x_3 + x_1^a = 2$$
$$x_2 + x_3 + x_2^a = 3$$
$$4x_1 + x_2 + 3x_3 + x_3^a = 7.$$
$$x_1, x_2, x_3, x_1^a, x_2^a, x_3^a \geq 0$$

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1^a	2	0	1	1	0	0	2
x_2^a	0	1	1	0	1	0	3
x_3^a	4	1	3	0	0	1	7
$-z$	1	1	3	0	0	0	0
$-z^a$	0	0	0	1	1	1	0

$\Rightarrow$

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1^a	2	0	1	1	0	0	2
x_2^a	0	1	1	0	1	0	3
x_3^a	4	1	3	0	0	1	7
$-z$	1	1	3	0	0	0	0
$-z^a$	-6	-2	-5	0	0	0	-12

Pivot on x_1, remove x_1^a

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
x_2^a	0	1	1	0	1	0	3
x_3^a	0	1	1	-2	0	1	3
$-z$	0	1	$\frac{5}{2}$	$-\frac{1}{2}$	0	0	-1
$-z^a$	0	-2	-2	3	0	0	-6

$\Rightarrow$

Pivot on x_2 remove x_2^a

	x_1	x_2	x_3	x_1^a	x_2^a	x_3^a	
x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	1
x_2	0	1	1	0	1	0	3
x_3^a	0	0	0	-2	-1	1	0
$-z$	0	0	$\frac{3}{2}$	$-\frac{1}{2}$	-1	0	-4
$-z^a$	0	0	0	3	2	0	0

$\Rightarrow$ basic feesible solution: $(x_1, x_2, x_3, x_1^a, x_2^a, x_3^a) = (1, 3, 0, 0, 0, 0)$

	x_1	x_2	x_3	
x_1	1	0	$\frac{1}{2}$	1
x_2	0	1	1	3
$-z$	0	0	$\frac{3}{2}$	-4

Pivot on x_3 remove x_1

$\Rightarrow$

	x_1	x_2	x_3	
x_3	2	0	1	2
x_2	-2	1	0	1
$-z$	-3	0	0	-7

$\Rightarrow \max z = 7$ when $(x_1, x_2, x_3) = (0, 1, 2)$

3. Let $1 \gg \epsilon_1 \gg \epsilon_2 \gg \epsilon_3$

$$\max_{x,y,s_1,s_2,s_3 \in \mathbb{R}} \; x - y$$

$$\text{s.t.} \quad \begin{aligned} x - 2y + s_1 &= \epsilon_1 \\ x - 3y + s_2 &= \epsilon_2 \\ y + s_3 &= \epsilon_3 + 3 \\ x, y, s_1, s_2, s_3 &\geq 0. \end{aligned}$$

	x	y	s_1	s_2	s_3	
s_1	1	−2	1	0	0	ϵ_1
s_2	1	−3	0	1	0	ϵ_2
s_3	0	1	0	0	1	ϵ_3+3
$-z$	1	−1	0	0	0	0

pivot on x remove s_2 $\Longrightarrow$

	x	y	s_1	s_2	s_3	
s_1	0	1	1	−1	0	$\epsilon_1-\epsilon_2$
x	1	−3	0	1	0	ϵ_2
s_3	0	1	0	0	1	(ϵ_3+3)
$-z$	0	2	0	−1	0	$-\epsilon_2$

Pivot on y remove s_1 $\Longrightarrow$

	x	y	s_1	s_2	s_3	
y	0	1	1	−1	0	$\epsilon_1-\epsilon_2$
x	1	0	3	−2	0	$3\epsilon_1-2\epsilon_2$
s_3	0	0	−1	1	1	$-\epsilon_1+\epsilon_2+\epsilon_3+3$
$-z$	0	0	−2	1	0	$-2\epsilon_1+\epsilon_2$

Pivot on s_2 remove s_3 $\Longrightarrow$

	x	y	s_1	s_2	s_3	
y	0	1	0	0	1	ϵ_3+3
x	1	0	1	0	2	$\epsilon_1+2\epsilon_3+6$
s_2	0	0	−1	1	1	$-\epsilon_1+\epsilon_2+\epsilon_3+3$
$-z$	0	0	−1	0	−1	$-3-\epsilon_1-\epsilon_2$

$$\max z = 3 + \epsilon_1 + \epsilon_2 \approx 3.$$

$$(x, y) = (6, 3).$$

4.

$$\begin{aligned}\underset{\mathbf{x}\in\mathbb{R}^{10}}{\text{maximize}} \quad & x_1+2x_2+3x_3+4x_4+2x_5-x_6+4x_7+4x_8+2x_9-x_{10}\\ \text{subject to} \quad & x_1+x_2-x_3+2x_4-x_5+3x_6+2x_7-x_8+x_9+2x_{10}=3\\ & 3x_1+4x_2+2x_3+7x_4+5x_5+6x_6-2x_7+9x_8+8x_9+9x_{10}=10\\ & x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8,x_9,x_{10}\geq 0.\end{aligned}$$

$C^T = [1\ 2\ 3\ 4\ 2\ -1\ 4\ 4\ 2\ -1]$

$A = \begin{bmatrix} 1 & 1 & -1 & 2 & -1 & 3 & 2 & -1 & 1 & 2 \\ 3 & 4 & 2 & 7 & 5 & 6 & -2 & 9 & 8 & 9 \end{bmatrix}$

$b = [3\ 10]^T$

(a). $\mathcal{B} = (X_1, X_2)$ $\quad A_{\mathcal{B}} = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix}$

$[A_{\mathcal{B}} \mid I] = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|cc} 1 & 0 & 4 & -1 \\ 0 & 1 & -3 & 1 \end{array}\right] = [I \mid A_{\mathcal{B}}^{-1}]$

$A_{\mathcal{B}}^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix}$ $\quad p = A_{\mathcal{B}}^{-1} b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

basic feasible solution: $\underline{\begin{matrix} X_1 = 2 \\ X_2 = 1 \end{matrix} \quad X_i = 0,\ i = 3, 4, \ldots, 10.}$

(b) $C_N^T = [3\ 4\ 2\ -1\ 4\ 4\ 2\ -1]$

$A_N = \begin{bmatrix} -1 & 2 & -1 & 3 & 2 & -1 & 1 & 2 \\ 2 & 7 & 5 & 6 & -2 & 9 & 8 & 9 \end{bmatrix}$

$r^T = C_N^T - C_B^T (A_B^{-1} A_N)$

$= [3\ 4\ 2\ -1\ 4\ 4\ 2\ -1] - [1\ 2] \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 & 3 & 2 & -1 & 1 & 2 \\ 2 & 7 & 5 & 6 & -2 & 9 & 8 & 9 \end{bmatrix}$

$= [-1\ \ \underset{\sim}{1}\ \ -5\ \ -1\ \ 10\ \ -7\ \ -4\ \ -6]$

$j = 4$

$Q = A_B^{-1} A_N = \begin{bmatrix} -6 & 1 & -9 & 6 & 10 & -13 & -4 & -1 \\ 5 & 1 & 8 & -3 & -8 & 12 & 5 & 3 \end{bmatrix}$

$i = 2$

$B' = (x_1, x_4)$

$$\left[A_B^{-1} \mid Q_4 \mid P\right] = \left[\begin{array}{cc:c:c} 4 & -1 & 1 & 2 \\ -3 & 1 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{cc:c:c} 7 & -2 & 0 & 1 \\ -3 & 1 & 1 & 1 \end{array}\right] = \left[A_{B'}^{-1} \mid Q_4' \mid P'\right]$$

$$A_{B'}^{-1} = \begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} \qquad P' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

new basic feasible solution: $x_1 = 1$, $x_4 = 1$, $x_i = 0$, $i = 2, 3, 5, 6, 7, 8, 9, 10$