Duality in the tableau.

Example:

(P) $\max\limits_{x,y\in\mathbb{R}} \; 2x+3y$

$$\begin{aligned} \text{s.t.}\quad -x+y &\le 3 \\ x-2y &\le 2 \\ x+y &\le 7 \\ x,y &\ge 0 \end{aligned}$$

	x	y	s_1	s_2	s_3	
s_1	-1	1	1	0	0	3
s_2	1	-2	0	1	0	2
s_3	1	1	0	0	1	7
$-z$	2	3	0	0	0	0

$\downarrow$

	x	y	s_1	s_2	s_3	
y	0	1	$\frac{1}{2}$	0	$\frac{1}{2}$	5
s_2	0	0	$\frac{3}{2}$	1	$\frac{1}{2}$	10
x	1	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	2
$-z$	0	0	$-\frac{1}{2}$	0	$-\frac{5}{2}$	-19

Opt sol to (P) is $(x,y)=(2,5)$ with opt value 19.

(D) $\min\limits_{u,v,w\in\mathbb{R}} \; 3u+2v+7w$

$$\begin{aligned} \text{s.t.}\quad -u+v+w &\ge 2 \\ u-2v+w &\ge 3 \\ u,v,w &\ge 0 \end{aligned}$$

(P) $\max \; C^T X$, $\quad Ax \le b$, $\quad X \ge 0$

(D) $\min \; U^T b$, $\quad U^T A \ge C^T$, $\quad U \ge 0$

$$r_j = C_j - U^T A_j \,, \quad U^T = \overset{1\times k}{C_B^T}\,\overset{k\times n}{A_B^{-1}} \,, \quad B \text{ is opt basis.}$$

opt solu for dual

Lemma 14.1: If B is an opt basis of (P), then $u^T = C_B^T A_B^{-1}$ is the opt sol to (D)

Proof: B is an opt basis $\Rightarrow$ all reduced costs should be nonpositive: $r_j \leq 0 \ \forall j \Rightarrow C_j - u^T A_j \leq 0$

$\Rightarrow u^T A_j \geq C_j \ \forall j$ u satisfies every dual constraint

in (P) $\quad Z_0 = C_B^T \underbrace{A_B^{-1} b}_{P = A_B^{-1} b} \qquad Z_0 = C_B^T P$

opt $\Rightarrow X_B = P$, $X_N = 0$

$Z_0 = C_B^T X_B = C^T X$ (primal obj function).

in (D) $\quad Z_0 = u^T b$ (dual obj function)

$u^T = C_B^T A_B^{-1} \Rightarrow u^T b = C^T X$ (by the theorem in previous lecture)

: feasible sol for (P) has same obj value as feasible sol to (D)

$\Rightarrow$ both opt.

$u^T = C_B^T A_B^{-1}$ is the opt sol to (D)

Assume started with LP in $Ax \leq b$ form and

if $\geq$ just multiply -1.

add slack vars. Matrix of LP with slack vars: $[A \quad I]$

$A_s = I$.

Reduced costs of slack vars: $r_s^T = C_s^T - u^T A_s$

slack vars don't appear in object fun $\Rightarrow C_s^T = 0^T$

$\Rightarrow r_s^T = -u^T$

Hence we can compute the dual solution $u^T = -r_s^T$

$$\begin{array}{c|ccc|c}
 & \overbrace{x_B \quad x_N}^{X} & & \overbrace{s}^{\text{slack vars}} & \\
\hline
s & \boxed{\overset{A}{A_B \quad A_N}} & & A_s^T = I & b \\
\hline
-z & C_B^T & C_N^T & C_s^T = 0^T & 0
\end{array}$$

$$\Rightarrow \begin{array}{c|ccc|c}
 & x_B & x_N & s & \\
\hline
x_B & I & A_B^{-1}A_N & A_B^{-1} & A_B^{-1}b \\
\hline
-z & 0^T & \boxed{C_N^T - C_B^T A_B^{-1} A_N} & \boxed{-C_B^T A_B^{-1}} & -C_B^T A_B^{-1} b
\end{array}$$

$r_N^T = C_N^T - u^T A_N \qquad r_s^T = -u^T$