Math 482 Linear programming
Fall 2020
Exam 3
11/20/2020
Time Limit: 210 Minutes

Name (Print): Wenxiao Yang

Section: C13

This is the 3 CREDIT EXAM.

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page.

You may use your books, notes, or any calculator on this exam.

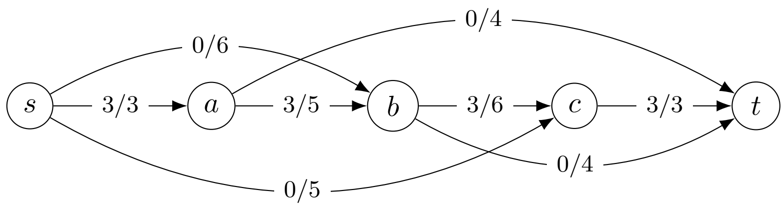
You are required to justify your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. A correct answer, unsupported by calculations, explanation, or algebraic work will receive no credit; an incorrect answer supported by substantially correct calculations and explanations might still receive partial credit.

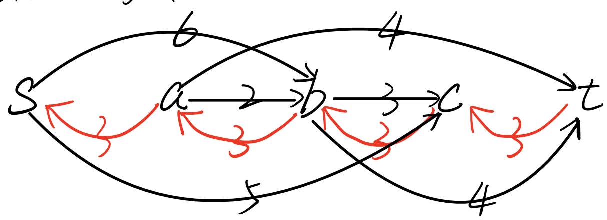
Do not write in the table to the right.

Problem	Points	Score
1	20	
2	8	
3	20	
4	22	
5	15	
6	15	
Total:	100	

1. (20 points) In the network below, someone has already started the process of finding the maximum s,t-flow. Complete their work using the Ford–Fulkerson method (be sure draw the residual graphs and state the augmenting paths you use.)



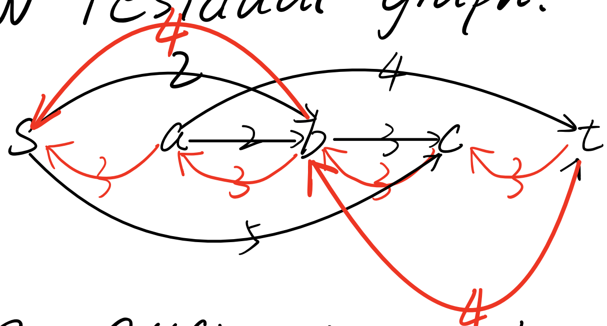
residual graph:

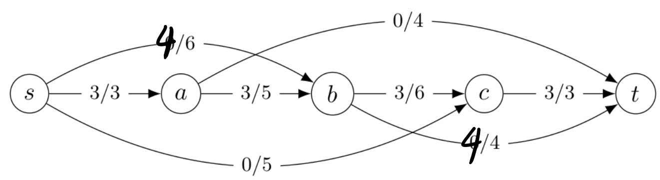


use augmenting path:

$s \to b \to t$ (increases flow by 4.)

$\Rightarrow$ new residual graph:

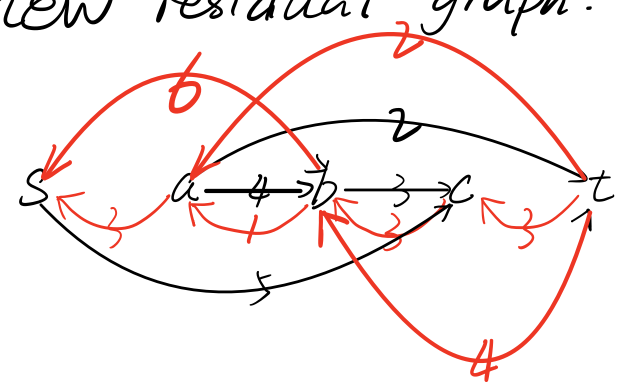


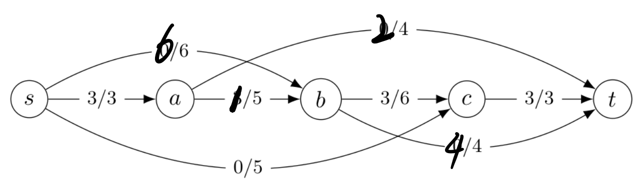


use augmenting path:

$s \to b \leftarrow a \to t$ (increases flow by 2)

$\Rightarrow$ new residual graph:

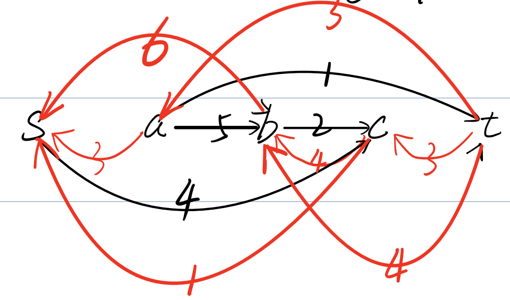


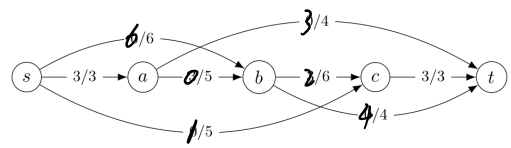


Use augmenting path:

$s \to c \leftarrow b \leftarrow a \to t$ (increases flow by 1.)

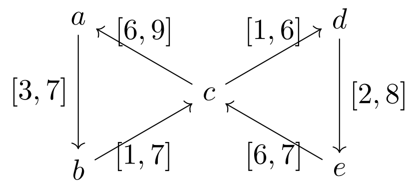
$\Rightarrow$ new residual graph:





no path exists $\Rightarrow$ max flow $= 3+3+4 = 10$.

2. (8 points) Consider the problem of finding a feasible circulation in the network below, where an interval of $[a, b]$ on an edge means the flow on that edge must be between a and b. Draw the network for a maximum flow problem that you could use to find this feasible circulation.



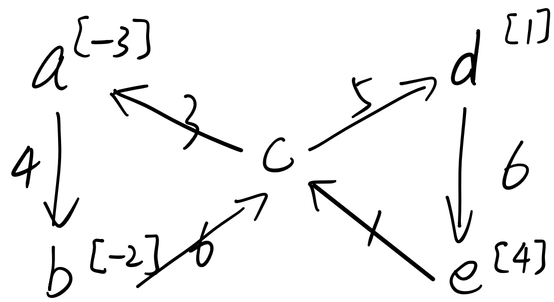
$d_a = 3 - 6 = -3$

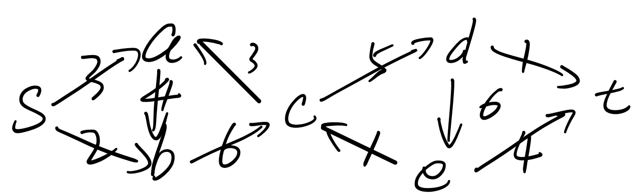
$d_b = 1 - 3 = -2$

$d_c = 6 + 1 - 1 - 6 = 0$

$d_d = 2 - 1 = 1$

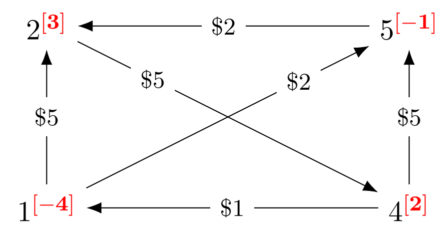
$d_e = 6 - 2 = 4$



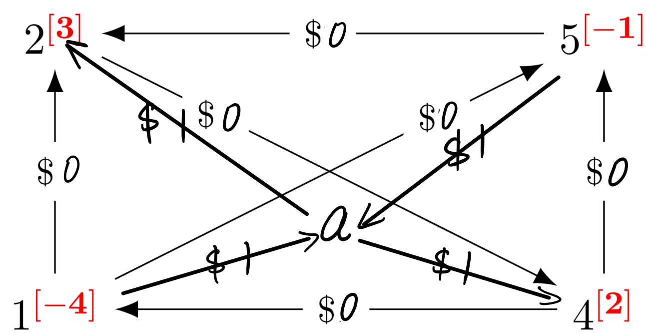


this is the network of a max flow problem.

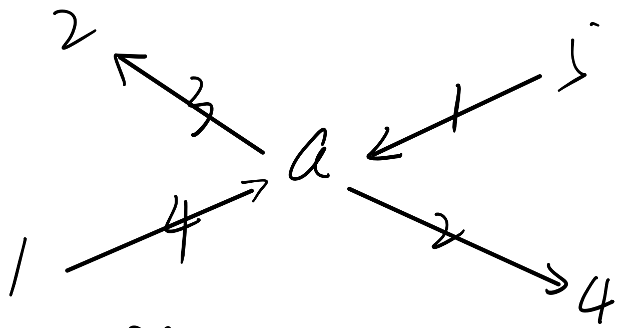
3. (20 points) Consider the minimum-cost flow problem given in the network below. Solve this by adding an artificial node, and performing diagramatic two-step simplex method as in Lecture 29. Make sure you draw the diagrams at each step.



phase network:

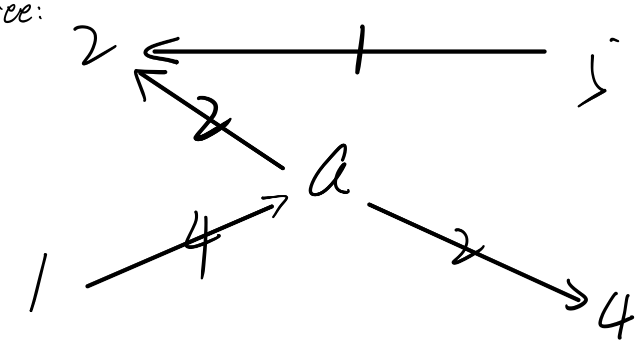


Spanning tree:



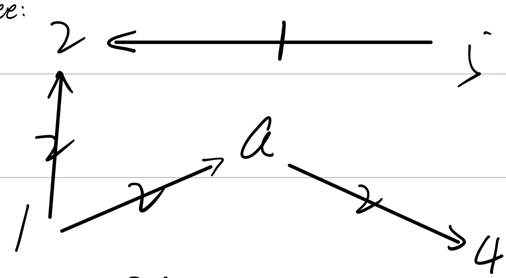
add x_{52} delete x_{5a}

$\Rightarrow$ new spanning tree:



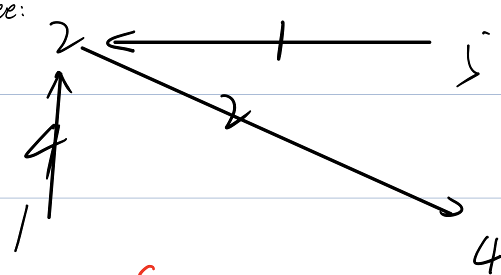
add x_{12} delete x_{a2}

$\Rightarrow$ new spanning tree:

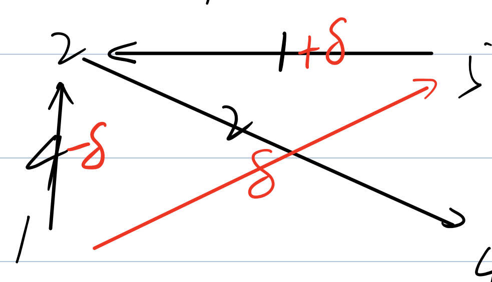


add X_{24} delete X_{a4}, X_{1a}

$\Rightarrow$ new spanning tree:

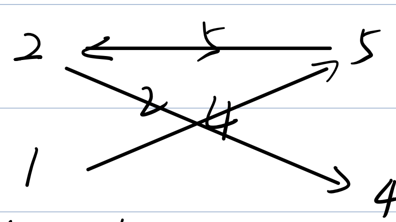


phase2:



X_{15}'s reduced cost is -1. $\Rightarrow$ let $\delta = 4$

add X_{15} delete X_{12}



all reduced costs are positive

$\Rightarrow$ opt solution: $\{ X_{15} = 4,\ X_{52} = 5,\ X_{24} = 2 \}$

4. Consider the linear program

$$\begin{aligned} \underset{x_1,x_2,x_3\in\mathbb{R}}{\text{minimize}} \quad & 3x_1 - x_2 - x_3 \\ \text{subject to} \quad & 5x_1 - 7x_2 + x_3 = 5 \\ & -x_1 + 2x_2 + x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

(a) (4 points) Use the assumption that $x_1+x_2+x_3 \leq 100$ in all feasible solutions to the linear program above, to write down an equivalent linear program and a feasible dual solution.

(b) (14 points) Use the primal-dual algorithm to solve this linear program, starting from the feasible dual solution you found.

(c) (4 points) Check that the assumption that $x_1 + x_2 + x_3 \leq 100$ was valid.

(a)

(P) $$\begin{aligned} \min_{x\in\mathbb{R}^4} \quad & 3x_1 - x_2 - x_3 \\ \text{s.t.} \quad & 5x_1 - 7x_2 + x_3 = 5 \\ & -x_1 + 2x_2 + x_3 = 2 \\ & x_1 + x_2 + x_3 + x_4 = 100 \\ & x_1, x_2, x_3, x_4 \geq 0. \end{aligned}$$

(D) $$\begin{aligned} \max_{u\in\mathbb{R}^3} \quad & 5u_1 + 2u_2 + 100u_3 \\ \text{s.t.} \quad & 5u_1 - u_2 + u_3 \leq 3 \\ & -7u_1 + 2u_2 + u_3 \leq -1 \\ & u_1 + u_2 + u_3 \leq -1 \\ & u_3 \leq 0 \end{aligned}$$

feasible dual solution $(u_1, u_2, u_3) = (0, 0, -1)$

(b)

(DRP) $$\begin{aligned} \max_{v\in\mathbb{R}^3} \quad & 5v_1 + 2v_2 + 100v_3 \\ \text{s.t.} \quad & -7v_1 + 2v_2 + v_3 \leq 0 \\ & v_1 + v_2 + v_3 \leq 0 \\ & v_1 \leq 1 \\ & v_2 \leq 1 \\ & v_3 \leq 1 \end{aligned}$$

(RP) $$\begin{aligned} \min_{x\in\mathbb{R}^2,\, y\in\mathbb{R}^3} \quad & y_1 + y_2 + y_3 \\ \text{s.t.} \quad & -7x_2 + x_3 + y_1 = 5 \\ & 2x_2 + x_3 + y_2 = 2 \\ & x_2 + x_3 + y_3 = 100 \\ & x_2, x_3, y_1, y_2, y_3 \geq 0. \end{aligned}$$

	x_1 *	x_2	x_3	x_4 *	y_1	y_2	y_3	
y_1	5	-7	1	0	1	0	0	5
y_2	-1	2	1	0	0	1	0	2
y_3	1	1	1	1	0	0	1	100
$-z$	0	0	0	0	1	1	1	0

$\Downarrow$

	x_1 *	x_2	x_3	x_4 *	y_1	y_2	y_3	
y_1	5	-7	1	0	1	0	0	5
y_2	-1	2	1	0	0	1	0	2
y_3	1	1	1	1	0	0	1	100
$-z$	-5	4	-3	-1	0	0	0	-107

Pivot on x_3 remove y_2 $\Downarrow$

	x_1 *	x_2	x_3	x_4 *	y_1	y_2	y_3	
y_1	6	-9	0	0	1	-1	0	3
x_3 ~~x_2~~	-1	2	1	0	0	1	0	2
y_3	2	-1	0	1	0	-1	1	98
$-z$	-8	10	0	-1	0	3	0	-101

$$v^T = 1^T - r_y^T = (1, -2, 1)$$

$$u + t v^T = (t, -2t, -1+t) \Rightarrow t \le \frac{1}{2} \Rightarrow t = \frac{1}{2}$$

new $u = (\frac{1}{2}, -1, -\frac{1}{2})$

$$(DRP)\quad \max_{V\in\mathbb{R}^3}\ 5V_1+2V_2+100V_3$$
$$\begin{aligned} \text{s.t.}\quad 5V_1 - V_2 + V_3 &\le 0 \\ V_1 + V_2 + V_3 &\le 0 \\ V_1 &\le 1 \\ V_2 &\le 1 \\ V_3 &\le 1 \end{aligned}$$

$$(RP)\quad \min_{x\in\mathbb{R}^2,\ y\in\mathbb{R}^3}\ y_1+y_2+y_3$$
$$\begin{aligned} \text{s.t.}\quad 5x_1 + x_3 + y_1 &= 5 \\ -x_1 + x_3 + y_2 &= 2 \\ x_1 + x_3 + y_3 &= 100 \\ x_1, x_3, y_1, y_2, y_3 &\ge 0. \end{aligned}$$

	x_1	x_2 *	x_3	x_4 *	y_1	y_2	y_3	
y_1	6	-9	0	0	1	-1	0	3
x_3 $\cancel{x_2}$	-1	2	1	0	0	1	0	2
y_3	2	-1	0	1	0	-1	1	98
$-z$	-8	10	0	-1	0	3	0	-101

pivot on x_1 remove y_1

	x_1	x_2 *	x_3	x_4 *	y_1	y_2	y_3	
x_1	1	$-\frac{3}{2}$	0	0	$\frac{1}{6}$	$-\frac{1}{6}$	0	$\frac{1}{2}$
x_3 $\cancel{x_2}$	0	$\frac{1}{2}$	1	0	$\frac{1}{6}$	$\frac{5}{6}$	0	$\frac{5}{2}$
y_3	0	2	0	1	$-\frac{1}{3}$	$-\frac{2}{3}$	1	97
$-z$	0	-2	0	-1	$\frac{4}{3}$	$\frac{5}{3}$	0	-97

new $V^T = 1^T - r_y^T = (-\frac{1}{3}, -\frac{2}{3}, 1)$

$U + tV^T = (\frac{1}{2} - \frac{1}{3}t, -1 - \frac{2}{3}t, -\frac{1}{2} + t) \Rightarrow t \le \frac{1}{2} \Rightarrow t = \frac{1}{2}$

new $U = (\frac{1}{3}, -\frac{4}{3}, 0)$

new (DRP)

$$\max_{v \in \mathbb{R}^3} \ 5v_1 + 2v_2 + 100v_3$$
$$\begin{aligned} \text{s.t.} \quad 5v_1 - v_2 + v_3 &\le 0 \\ v_1 + v_2 + v_3 &\le 0 \\ v_1 &\le 1 \\ v_2 &\le 1 \\ v_3 &\le 1 \end{aligned}$$

is same as before.

Hence the u can't be improved anymore.

$$(u_1, u_2, u_3) = \left(\tfrac{1}{3}, -\tfrac{4}{3}, 0\right)$$

and $(x_1, x_2, x_3, x_4) = \left(\tfrac{1}{2}, \cancel{\tfrac{5}{2}}\ 0, \cancel{0}\ \tfrac{5}{2}, 0\right)$.

are the opt solutions.

(c) $x_1 + x_2 + x_3 = 3 \Rightarrow x_1 + x_2 + x_3 \le 100$ is valid.

test $u_3 = 0$

5. (15 points) Use the primal-dual version of Dijkstra's algorithm from Lecture 33 to find the shortest path from s to t in the network below. Make sure you draw the diagram associated to the dual and the dual of the restricted primal at each step.

dual: $\max\ u_s - u_t$

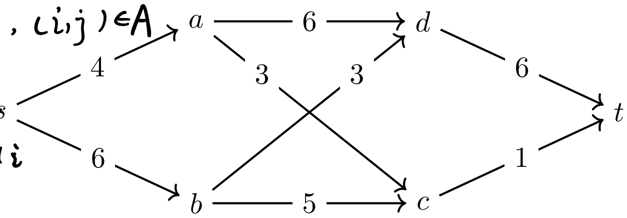
s.t. $u_i - u_j \le w_{ij},\ (i,j) \in A$

DRP: $\max\ v_s - v_t$

s.t. $v_i - v_j \le 0 \quad (i,j) \in J$

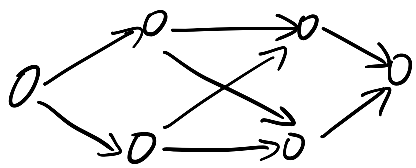
$v_i \le 1$ for all i

$v_t = 0$



initial $u = (0, 0, 0, 0, 0, 0)$

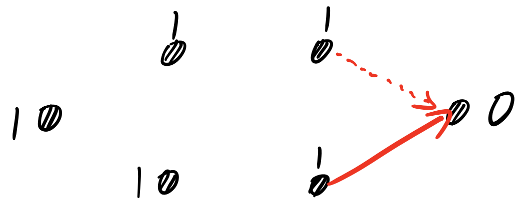
s a b c d t.



$\Rightarrow J = \phi$

$v = (1, 1, 1, 1, 1, 0)$

$\Rightarrow$

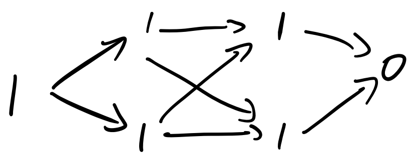


$t = \min\{6-0, 1-0\} = 1.$

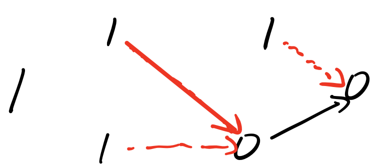
for arc (c, t)

new $u = (1, 1, 1, 1, 1, 0)$ $J = \{(c,t)\}$

$v = (1, 1, 1, 0, 1, 0)$



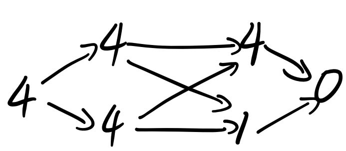
$\Rightarrow$



$t = \min\{5-0, 3-0, 6-1\} = 3$

for arc (a, c)

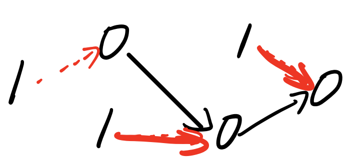
new $u = (4, 4, 4, 1, 4, 0)$



$J = \{(a,c), (c,t)\}$

$v = (1, 0, 1, 0, 1, 0)$

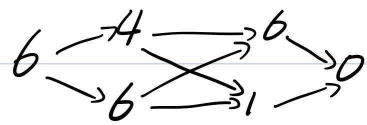
$\Rightarrow$



$t = \min\{4-0, 5-3, 6-4\} = 2$

for arc (b,c) (d,t)

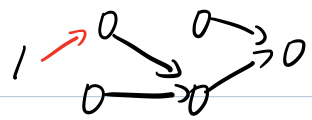
new $u = (6, 4, 6, 1, 6, 0)$



$J = \{(a,c), (c,t), (b,c), (d,t)\}$.

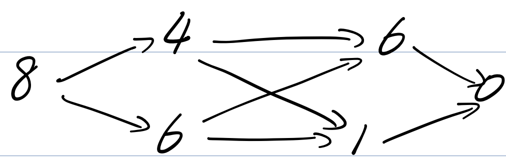
$v = (1, 0, 0, 0, 0, 0)$

$\Rightarrow$



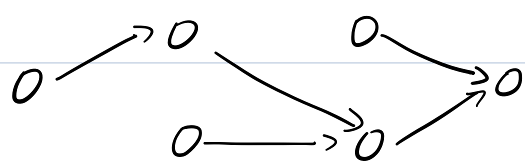
$t = 4 - 2 = 2$ for arc (s, a)

new $u = (8, 4, 6, 1, 6, 0)$.



$J = \{(s,a), (a,c), (c,t), (b,c), (d,t)\}$.

$\Rightarrow$



$v = (0, 0, 0, 0, 0, 0)$.

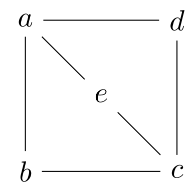
done!

opt value $= 8$.

Shortest path: $s \to a \to c \to t$.

6. A *dominating set* in a graph is a set D of vertices such that for every vertex v in the graph, either $v \in D$ or else there is a vertex $w \in D$ with an edge to v.

(a) (5 points) Write down a linear program for finding a dominating set in the graph below which contains the fewest vertices.



set $A = \{(a,e), (a,d), (a,b), (e,c), (d,c), (b,c)\}$

(i,j) is same as (j,i)

$X = \{a, b, c, d, e\}$

if $i \in D$, $U_i = 1$

$i \notin D$, $U_i = 0$

$$\begin{aligned} \min_{U \in \mathbb{Z}^5} \quad & U_a + U_b + U_c + U_d + U_e \\ \text{s.t.} \quad & U_a + U_b \geq 1 \\ & U_a + U_d \geq 1 \\ & U_a + U_e \geq 1 \\ & U_b + U_c \geq 1 \\ & U_c + U_d \geq 1 \\ & U_c + U_e \geq 1 \\ & U_i \geq 0 \quad \forall i \in X. \end{aligned}$$

(b) (5 points) Find the dual of the linear program you found.

$$\begin{aligned} \text{Dual:} \quad \max_{V_i \in \mathbb{Z}^6} \quad & \sum_{i=1}^{6} V_i \\ \text{s.t.} \quad & V_1 + V_2 + V_3 \leq 1 \\ & V_1 + V_4 \leq 1 \\ & V_4 + V_5 + V_6 \leq 1 \\ & V_2 + V_5 \leq 1 \\ & V_3 + V_6 \leq 1 \end{aligned}$$

(c) (5 points) Give a combinatorial interpretation of your dual. (What kind of objects in the graph do integer feasible solutions of the dual represent?)

find the max number of edges when one vertex can only have one edge at most.