Network : (N, A)

N is a set of nodes

A is a set of arcs: (i, j), $i, j \in N$.

Max-flow: for every arc (i, j) there is a max capacity c_{ij} (non-negative real)

Two special nodes: s (source), t (sink).

Max transport from s to t.

$$x_{ij} \le c_{ij}$$

$$x_{ij} \geq 0$$

$$\sum_{i:(i,k)\in A} x_{ik} = \sum_{j:(k,j)\in A} x_{kj}$$

input in k = output in k.

$$N \xrightarrow{\text{a cut}} S, T.$$

$x_{ij} = 0$ when $i \in T, j \in S$.

$(S \cap T = \phi, S \cup T = N)$

capacity of a cut (S, T) is $\sum_{i\in S} \sum_{j\in T} c_{ij}$.

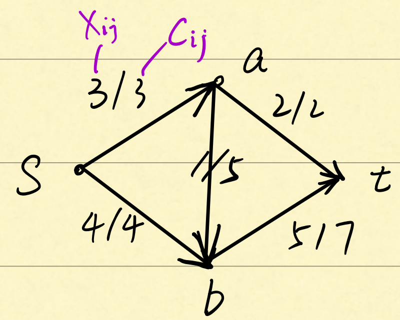
Theorem: if a cut (S,T) has capacity $C(S,T)$ then the value of the flow can't be more than $C(S,T)$.

Proof: $V(X) = \sum_{k\in S}\left(\sum_{j:(k,j)\in A} x_{kj} - \sum_{i:(i,k)\in A} x_{ik}\right)$

$$= \left(\cancel{\sum_{k\in S}\sum_{j\in S} x_{kj}} + \sum_{k\in S}\sum_{j\in T} x_{kj}\right) - \left(\cancel{\sum_{k\in S}\sum_{i\in S} x_{ik}} + \sum_{k\in S}\sum_{i\in T} x_{ik}\right)$$

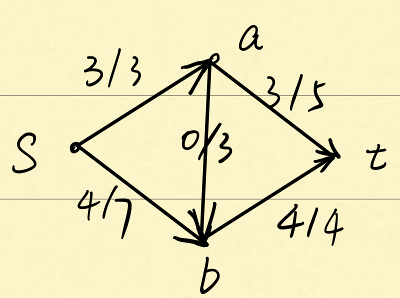
$$= \sum_{k\in S}\sum_{j\in T} \underbrace{x_{kj}}_{\le C_{kj}} - \sum_{k\in S}\sum_{i\in T} \underbrace{x_{ik}}_{\ge 0}$$

$$\le \sum_{k\in S}\sum_{j\in T} C_{kj} = C(S,T)$$



This flow is 7, is optimal.

Total capacity of edges leaving S is 7.



flow is 7, split nodes into $\{s,b\}$ and $\{a,t\}$

Flow from $\{s,b\}$ to $\{a,t\}$ is 7.

can't be increased. $\begin{cases} \{s,b\} \text{ to } \{a,t\} \text{ max} \\ \{a,t\} \text{ to } \{s,b\} = 0. \end{cases}$