A rideshare company has $\underline{m}$ cars at work;

$\underline{n}$ people request a car.

each car can only be used for <u>some requests</u> (not all requests)

due to proximity.

Assign a car to pick up each person.

two types of objects => two types of vertices

(car, people) ($\bullet_1$, $\bullet_2$).

Relationship between vertices of first type and second type => edge connecting vertices.

($\bullet_1$ ——— $\bullet_2$)

Bipartite graph: (X, Y, E)

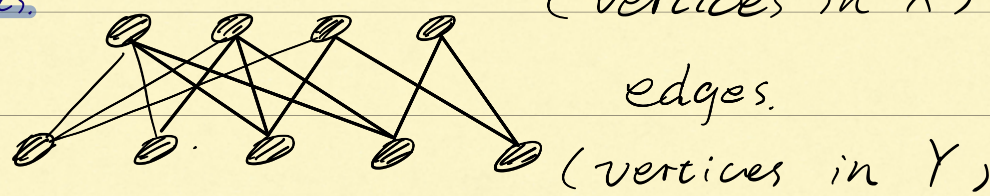
vertices (X, Y) edges (E).

E is ordered pairs (v, w) with $v \in X$, $w \in Y$.

if $(v, w) \in E$, we say v and w are neighbors (adjacent)

v, w are endpoints of edge (v, w).

Graphs.



A <u>matching</u> M in a bipartite graph is a subsets of edges that doesn't have any vertex in X or Y as an endpoint more than once.

Goal: Find a matching that is as large as possible.

Bipartite matching LP.

Label X as $\{1, 2, \ldots, n\}$, Y as $\{n+1, \ldots, n+m\}$.

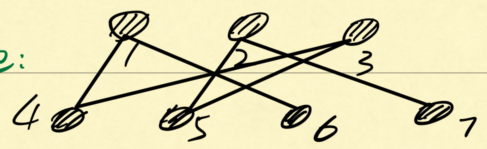
non-negative x_{ij}: for each $(i, j) \in E$.

$$x_{ij} = \begin{cases} 1, & (i,j) \text{ is part of } M. \\ 0, & (i,j) \text{ isn't part of } M. \end{cases}$$

constriant: for each vertex, at most one edge with the vertex as an endpoint can be in M.

i.e. $\sum_i x_{i,j} \le 1$, $\sum_j x_{i,j} \le 1$

Example:



$$\max \quad x_{14} + x_{16} + x_{25} + x_{27} + x_{34} + x_{35}$$

$$\text{s.t.} \quad \begin{aligned} & x_{14} + x_{16} \le 1 \\ & x_{25} + x_{27} \le 1 \\ & x_{34} + x_{35} \le 1 \\ & x_{14} + x_{34} \le 1 \\ & x_{25} + x_{35} \le 1 \\ & x_{16} \le 1, \ x_{27} \le 1 \end{aligned}$$

Basic solution to system $Ax = b$.

for basis B, is given by $x_B = A_B^{-1} b$, $x_N = 0$.

x_B will be integral if both A_B^{-1} and b are composed only of integers. but A_B is composed of integers.

Lemma: M has integer entries, then

M^{-1} has integer entries if and only if $\det(M) = \pm 1$.

Proof: $\Rightarrow$: $\det(M)\det(M^{-1}) = 1$ $\quad\Leftarrow$: $M^{-1} = \frac{1}{\det(M)} \operatorname{adj}(M)$

adjugate matrix.

Theorem: Given an LP with feasible region $\{x \in \mathbb{R}^n : Ax \le b, x \ge 0\}$. All basic feasible sols will be integers if:

①. b is all integers.

②. every square submatrix of A of size $k \times k$ has det $0, 1, -1$ (for all k)

(such an A is called $\boxed{\text{totally unimodular}}$)

Proof: add slack vars: $Ax \le b \to Ax + Is = b$.

Given basis B. basic sol: $x_B = A_B^{-1} b$

A_B is a submatrix of $[A \ I]$

assume A_B has $\underline{k}$ columns from A

$\underline{n-k}$ columns from I.

$\det(A_B)$ can be simplified using the columns from I to the det of a $k \times k$ matrix.

example: $\det \begin{bmatrix} 1 & 5 & 0 & 0 \\ 2 & 6 & 1 & 0 \\ 3 & 7 & 0 & 0 \\ 4 & 8 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 5 & 0 & 0 \\ 3 & 7 & 0 & 0 \\ 2 & 6 & 1 & 0 \\ 4 & 8 & 0 & 1 \end{bmatrix} = -\det \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$

$\Rightarrow \det(A_B)$ equals to a $k \times k$ submatrix of A.

$\Rightarrow \det(A_B) = 0, 1,$ or -1.

if $\det(A_B) = 0$, A_B is not invertible. $\Rightarrow B$ isn't a valid basis.

if $\det(A_B) = 1$ or -1. by lemma, A_B^{-1} has integer entries. $\Longrightarrow x_B = A_B^{-1} b$ has integer entries.