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Math 482: Linear Programming, Fall 2020

Due Friday, October 30, 6PM CST

Homework 7

1. Show that any $n \times n$ matrix following the pattern

$$\begin{bmatrix} 1 & 0 & 1 & \cdots & 0 & 1 \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 1 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 1 \end{bmatrix}$$

is totally unimodular: any submatrix obtained by taking any k rows and any k columns has determinant -1, 0, or 1.

(It may help to consider the cases $k \leq 2$ and $k \geq 3$ separately.)

2. Consider the bipartite graph with vertices $\{a_1, a_2, \ldots, a_{10}\}$ on one side, vertices $\{b_1, b_2, \ldots, b_{10}\}$ on the other side, and an edge between a_i and b_j if the product ij is a multiple of 6.

Find a largest matching in this graph, and show that it cannot be any larger by finding a vertex cover of the same size.

3. A bipartite graph (X, Y, E) has $|X| = |Y| = n$ and is *r-regular*: every vertex (in X or in Y) is the endpoint of exactly r edges.

   (a) Determine $|E|$, the number of edges in the graph.

   (b) Show that any vertex cover must contain at least n vertices.

   (This implies that there is a matching of size n, which matches every vertex in X to a vertex in Y.)

4. Find examples of networks with the following properties:

   (a) A network with a unique maximum flow, but multiple minimum cuts.

   (b) A network with multiple maximum flows, but a unique minimum cut.

   (c) A network with multiple maximum flows and multiple minimum cuts.

   For each example, describe the maximum flow(s) and the minimum cut(s).

5. *(Only 4-credit students need to do this problem.)*

Consider a bipartite graph (X, Y, E) with $X = \{x_1, x_2, \ldots, x_m\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. The *biadjacency matrix* of this graph is the $m \times n$ matrix A where $A_{ij} = 1$ if there is an edge $(x_i, y_j) \in E$, and $A_{ij} = 0$ otherwise.

If $m = n$ (so that the matrix A is square) and $\det(A) = -3$, show that the graph contains a matching of size n.

1. $k=1$ det $=0$ or 1.

$k=2$ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ det $=1$ or -1.

$k \geqslant 3$, obviously we at least have two same rows $\Rightarrow$ det $=0$.

Hence det of submatrix is $0/1/-1$

$\Rightarrow$ totally unimodular.

2. $M = \{(a_1, b_6), (a_2, b_3), (a_3, b_2), (a_6, b_1), (a_8, b_9), (a_9, b_4)\}$

$S = \{a_3, a_6, a_9, b_3, b_6, b_9\}$.

3. (a) $|E| = n \cdot r$

(b) Since every vertex is the endpoint of exactly r edges, we can rearrange X and Y such that

Given $i = 1, 2, \dots, n-r$ $(X'_{(i)}, Y'_{(j)}) \in E$, for all $j = i, i+1, \dots i+r$.

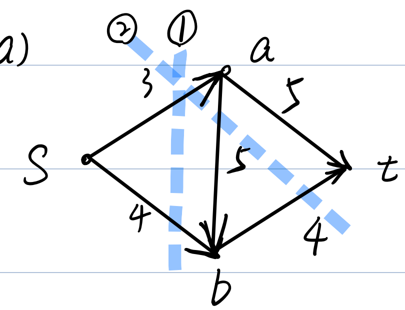
$i = n-r+1, \dots n$ $(X'_{(i)}, Y'_{(j)}) \in E$, for all $j = i, \dots n, 1, 2 \dots i-n+r$

(rearrange by order: $X_{(1)}$ $Y_{(1)}$ $X_{(2)}$ $\dots\dots$ $Y_{(n-1)}$ $X_{(n)}$ $Y_{(n)}$)

Hence exist matching $\{(X_{(i)}, Y_{(i)})\ i=1,2,\dots n\}$

Size is n. $\Rightarrow$ Size of vertex cover $\geq n$.

4. (a)



$C_{sa}=3$, $C_{at}=5$, $C_{sb}=4$

$C_{bt}=4$, $C_{ab}=5$

Max flow is 7 : $\{X_{sa}=3,\ X_{sb}=4=X_{bt}$

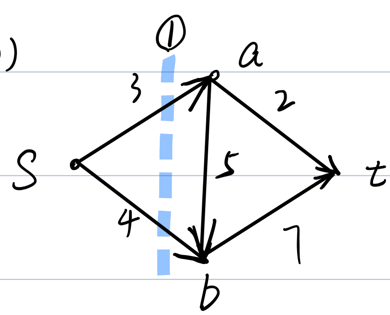
$X_{ab}=0,\ X_{at}=3\}$

which is unique.

Min cut : $(\{S\},\{a,b,t\})$ or

$(\{S,b\},\{a,t\})$

(b)



$C_{sa}=3$, $C_{at}=5$, $C_{sb}=4$

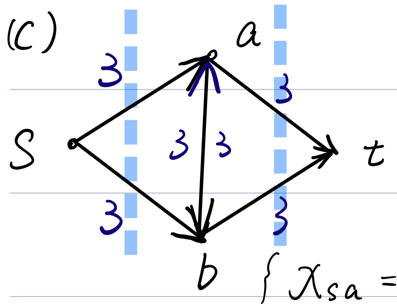
$C_{bt}=7$, $C_{ab}=5$

Max flow is 7 :

$\{X_{sa}=3,\ X_{sb}=4,\ X_{bt}=k,\ X_{at}=7-k,\ X_{ab}=k-4 \mid k=4,5,6,7\}$

Min cut : $(\{S\},\{a,b,t\})$

(c)



$C_{sa} = C_{at} = C_{sb} = C_{ba} = C_{bt} = 3.$

Max flow is 6

$\{x_{sa} = x_{sb} = 3 = x_{at} = x_{bt},\ x_{ab} = x_{ba} = k \mid k = 0, 3\}.$

Min cut : $(\{s\}, \{a, b, t\})$ or

$(\{s, a, b\}, \{t\})$