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Math 482: Linear Programming, Fall 2020

Due Friday, December 11, 6PM CST

Homework 10

1. In chess, a queen is a piece that can move any number of spaces horizontally, vertically, or diagonally.

   The "eight queens" puzzle is to place eight queens on an 8×8 chessboard so that they do not attack each other. That is, no two queens can be placed in the same rank, the same file, or the same diagonal. (To be clear, diagonals are lines with slope 1 or -1.)

   Write an integer linear program that generalizes this puzzle. For an arbitrary n, your program should find the maximum number of queens that can be placed on an $n \times n$ chessboard so that they do not attack each other.

   Since the number of variables and constraints will depend on n, it's fine if you just include one constraint of each type, and explain how it generalizes.

   (Hint: for each $1 \le i \le n$ and $1 \le j \le n$, have a variable x_{ij} telling you if there's a queen on square (i, j).)

2. Solve the integer linear program below using the branch-and-bound method.

$$\begin{array}{ll} \underset{x,y\in\mathbb{Z}}{\text{maximize}} & 2x + y \\ \text{subject to} & -x + y \le 0 \\ & 6x + 2y \le 21 \\ & x, y \ge 0 \end{array}$$

   (Note: in some subproblems, you may be able to solve an LP by looking at it, without using the simplex algorithm. If you do, that's fine. But I do want you to write down, at each step of the branch-and-bound method, which LP you solve, what the optimal solution is, and what its objective value is. If the solution is not obvious, then you should use the simplex method.)

3. Solve the integer linear program from problem 2 again, this time using fractional cuts.

4. Suppose that, in the setting of the traveling salesman problem, we have $n = 3k$ cities, and instead of trying to find a single tour of all of them, we want to find k "triangle tours", each visiting 3 different cities, with the minimum total cost.

   Starting from the incomplete formulation of TSP, add constraints ensuring that the solution $\mathbf{x}$ represents k triangle tours.

5. *(Only 4-credit students need to do this problem.)*

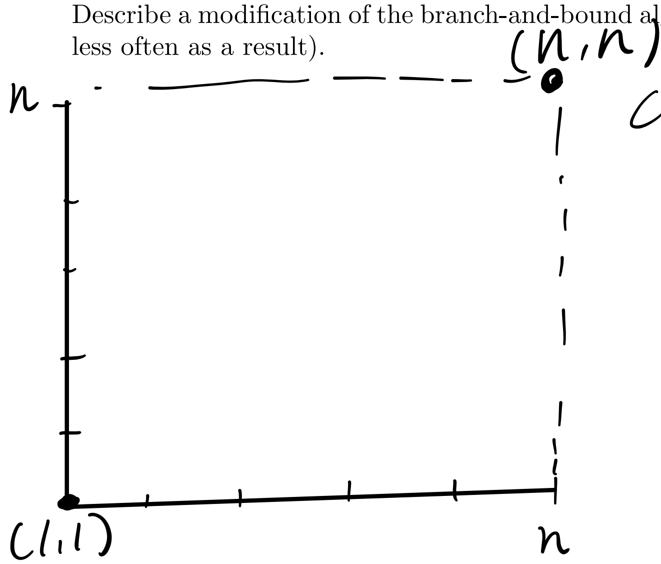
   Suppose we are solving an integer linear program

$$\begin{array}{ll} \underset{\mathbf{x}\in\mathbb{Z}^n}{\text{maximize}} & \mathbf{c}^\mathsf{T}\mathbf{x} \\ \text{subject to} & A\mathbf{x} \le \mathbf{b} \\ & \mathbf{x} \ge \mathbf{0} \end{array}$$

   for which the branch-and-bound algorithm is going to take too long to find the optimal answer (call this unknown optimal answer $\mathbf{x}^*$). However, we're willing to settle for an approximate solution. Specifically, we want a 2-approximation: a feasible (integer) solution $\mathbf{x} \in \mathbb{Z}^n$ such that $\mathbf{c}^\mathsf{T}\mathbf{x} \ge \frac{1}{2}\mathbf{c}^\mathsf{T}\mathbf{x}^*$. (Assume that $\mathbf{c} \ge \mathbf{0}$, so that none of these objective values are negative.)

Describe a modification of the branch-and-bound algorithm that finds a 2-approximation (and branches less often as a result).

1.



Consider a $n \times n$ chessboard.

Let $x_{ij} = \begin{cases} 1 & \text{if queen can place at } (i,j) \\ 0 & \text{if queen can't place at } (i,j) \end{cases}$

$$\max_{x \in \mathbb{Z}} \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij}$$

$$\text{s.t.} \sum_{j=1}^{n} x_{ij} \leq 1$$

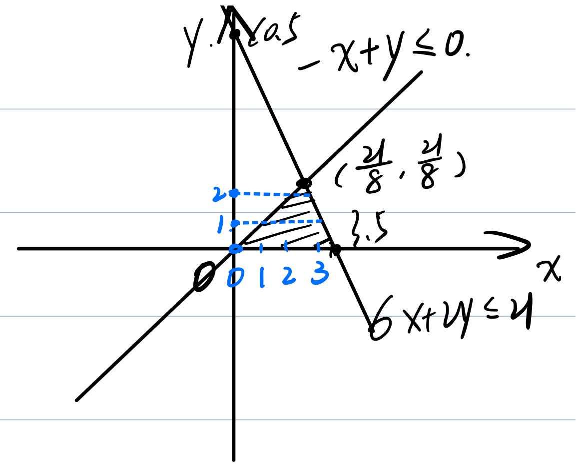
$$\sum_{i=1}^{n} x_{ij} \leq 1$$

$$\sum_{i=1}^{n} x_{i\,i+k} \leq 1 \qquad k = 0, 1, 2 \ldots\ldots, n-1.$$

$$\sum_{i=1}^{n} x_{i\,-i+k} \leq 1 \qquad k = 2, 3, \ldots, n+1$$

2.

$$\begin{aligned} \underset{x,y\in\mathbb{Z}}{\text{maximize}} \quad & 2x + y \\ \text{subject to} \quad & -x + y \le 0 \\ & 6x + 2y \le 21 \\ & x, y \ge 0 \end{aligned}$$



max $2x + y$

s.t. $-x + y + s_1 = 0$

$6x + 2y + s_2 = 21$

$x, y \geq 0$

	x	y	s_1	s_2	
s_1	-1	1	1	0	0
s_2	6	2	0	1	21
$-z$	2	1	0	0	0

pivot x $\longrightarrow$ remove s_2

	x	y	s_1	s_2	
s_1	0	$\frac{4}{3}$	1	$\frac{1}{6}$	$\frac{7}{2}$
x	1	$\frac{1}{3}$	0	$\frac{1}{6}$	$\frac{7}{2}$
$-z$	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	-7

pivot y $\longrightarrow$ remove s_1

	x	y	s_1	s_2	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	$-\frac{63}{8}$

opt: $(x^*, y^*) = (\frac{21}{8}, \frac{21}{8})$

value $z^* = \frac{63}{8}$

$\lceil \frac{21}{8} \rceil = 3$ $\qquad$ $\lfloor \frac{21}{8} \rfloor = 2$

① $y \leq 2$

	x	y	s_1	s_2	s_3	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
s_3	0	1	0	0	1	2
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	0	$-\frac{63}{8}$

$\longrightarrow$

	x	y	s_1	s_2	s_3	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
s_3	0	0	$-\frac{3}{4}$	$-\frac{1}{8}$	1	$-\frac{5}{8}$
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	0	$-\frac{63}{8}$

$\xrightarrow[\text{remove } S_3]{\text{pivot } S_1}$

	x	y	S_1	S_2	S_3	
y	0	1	0	0	1	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	$\frac{5}{6}$
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{23}{3}$

new $(x^*, y^*) = (\frac{17}{6}, 2)$

$z^* = \frac{23}{3}$

$\lceil \frac{17}{6} \rceil = 3 \quad \lfloor \frac{17}{6} \rfloor = 2$

(1.1) $y \leq 2 \quad x \geq 3$

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	0	1	0	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	0	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	0	$\frac{5}{6}$
S_4	-1	0	0	0	0	1	-3
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	0	$-\frac{23}{3}$

$\longrightarrow$

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	0	1	0	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	0	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	0	$\frac{5}{6}$
S_4	0	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	1	$-\frac{1}{6}$
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	0	$-\frac{23}{3}$

$\xrightarrow[\text{remove } S_4]{\text{pivot } S_3}$

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	$\frac{1}{2}$	0	3	1.5
x	1	0	0	0	0	-1	3
S_1	0	0	1	$-\frac{1}{2}$	0	-4	$\frac{3}{2}$
S_3	0	0	0	$-\frac{1}{2}$	1	-3	$\frac{1}{2}$
$-z$	0	0	0	$-\frac{5}{2}$	0	-1	-7.5

new $(x^*, y^*) = (3, 1.5) \quad z^* = 7.5$

$\lceil 1.5 \rceil = 2 \quad \lfloor 1.5 \rfloor = 1$

1.1.1 $\quad y=2 \quad x \geq 3 \Rightarrow (x^*, y^*) = (\frac{17}{6}, 2)$

$z^* = \frac{23}{3}$

$\lceil \frac{17}{6} \rceil = 3 \quad \lfloor \frac{17}{6} \rfloor = 2$

1.1.1.1 $\quad y=2 \quad x=3 \Rightarrow$ infeasible

1.1.1.2 $\quad y=2 \quad x \geq 3$ and $x \leq 2 \Rightarrow$ infeasible

1.1.2 $\quad y \leq 1 \quad x \geq 3$

obviously $(x^*, y^*) = (\frac{19}{6}, 1) \quad z^* = \frac{21}{3}$

$\lfloor \frac{19}{6} \rfloor = 3 \quad \lceil \frac{19}{6} \rceil = 4$

1.1.2.1 $\quad y \leq 1 \quad x=3$

obviously $(x^*, y^*) = (3, 1) \quad z^* = 7$

1.1.2.2 $\quad y \leq 1 \quad x \geq 4 \Rightarrow$ infeasible.

1.2 $\quad y \leq 2 \quad x \leq 2$

obviously $(x^*, y^*) = (2, 2) \quad z^* = 6$

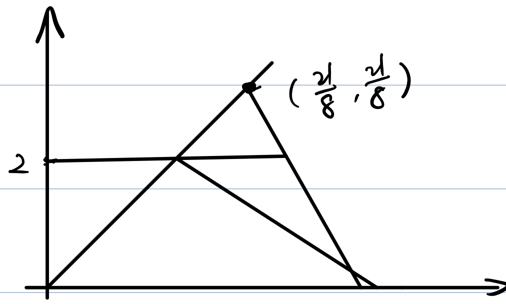
2. $y \geq 3 \Rightarrow$ infeasible

Hence: opt solution is $(x, y) = (3, 1)$

opt value is $z = 7$.

3.

$$\begin{aligned} \underset{x,y\in\mathbb{Z}}{\text{maximize}} \quad & 2x + y \\ \text{subject to} \quad & -x + y \le 0 \\ & 6x + 2y \le 21 \\ & x, y \ge 0 \end{aligned}$$



	x	y	S_1	S_2	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	$-\frac{63}{8}$

opt : $(x^*, y^*) = (\frac{21}{8}, \frac{21}{8})$

value $z^* = \frac{63}{8}$

$$x - \frac{1}{4} S_1 + \frac{1}{8} S_2 = \frac{21}{8}$$

$$x - S_1 \le 2 \qquad \frac{3}{4} S_1 + \frac{1}{8} S_2 \ge \frac{5}{8}$$

$$x - (x - y) \le 2 \qquad -\frac{3}{4} S_1 - \frac{1}{8} S_2 + S_3 = -\frac{5}{8}$$

$$\Rightarrow y \le 2$$

	x	y	S_1	S_2	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	$\frac{21}{8}$
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	$-\frac{63}{8}$

$\longrightarrow$

	x	y	S_1	S_2	S_3	
y	0	1	$\frac{3}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
x	1	0	$-\frac{1}{4}$	$\frac{1}{8}$	0	$\frac{21}{8}$
S_3	0	0	$-\frac{3}{4}$	$-\frac{1}{8}$	1	$-\frac{5}{8}$
$-z$	0	0	$-\frac{1}{4}$	$-\frac{7}{24}$	0	$-\frac{63}{8}$

$\xrightarrow[\text{remove } S_3]{\text{pivot } S_1}$

	x	y	S_1	S_2	S_3	
y	0	1	0	0	1	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	$\frac{5}{6}$
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{23}{3}$

$$x + \frac{1}{6} S_2 - \frac{1}{3} S_3 = \frac{17}{6}$$

$X - S_3 \le 2$ $\qquad \frac{1}{6}S_2 + \frac{2}{3}S_3 \geq \frac{5}{6}$

$x - (2-y) \le 2$

$x + y \le 4$ $\qquad -\frac{1}{6}S_2 - \frac{2}{3}S_3 + S_4 = -\frac{5}{6}$

	x	y	S_1	S_2	S_3	
y	0	1	0	0	1	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	$\frac{5}{6}$
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	$-\frac{23}{3}$

$\longrightarrow$

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	0	1	0	2
x	1	0	0	$\frac{1}{6}$	$-\frac{1}{3}$	0	$\frac{17}{6}$
S_1	0	0	1	$\frac{1}{6}$	$-\frac{4}{3}$	0	$\frac{5}{6}$
S_4	0	0	0	$-\frac{1}{6}$	$-\frac{2}{3}$	1	$-\frac{5}{6}$
$-z$	0	0	0	$-\frac{1}{4}$	$-\frac{1}{3}$	0	$-\frac{23}{3}$

Pivot S_3 $\longrightarrow$ remove S_4

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	$\frac{3}{4}$
x	1	0	0	$\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{13}{4}$
S_1	0	0	1	$\frac{1}{2}$	0	-2	$\frac{5}{2}$
S_3	0	0	0	$\frac{1}{4}$	1	$-\frac{3}{2}$	$\frac{5}{4}$
$-z$	0	0	0	$-\frac{1}{6}$	0	$-\frac{1}{2}$	$-\frac{29}{4}$

$$y - \frac{1}{4}S_2 + \frac{3}{2}S_4 = \frac{3}{4}$$

$y - S_2 + S_4 \le 0$ $\qquad \frac{3}{4}S_2 + \frac{1}{2}S_4 \geq \frac{3}{4}$

$$-\frac{3}{4}S_2 - \frac{1}{2}S_4 + S_5 = -\frac{3}{4}$$

	x	y	S_1	S_2	S_3	S_4	
y	0	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	$\frac{3}{4}$
x	1	0	0	$\frac{1}{4}$	0	$-\frac{1}{2}$	$\frac{13}{4}$
S_1	0	0	1	$\frac{1}{2}$	0	-2	$\frac{5}{2}$
S_3	0	0	0	$\frac{1}{4}$	1	$-\frac{3}{2}$	$\frac{5}{4}$
$-z$	0	0	0	$-\frac{1}{6}$	0	$-\frac{1}{2}$	$-\frac{29}{4}$

$\longrightarrow$

	x	y	S_1	S_2	S_3	S_4	S_5	
y	0	1	0	$-\frac{1}{4}$	0	$\frac{3}{2}$	0	$\frac{3}{4}$
x	1	0	0	$\frac{1}{4}$	0	$-\frac{1}{2}$	0	$\frac{13}{4}$
S_1	0	0	1	$\frac{1}{2}$	0	-2	0	$\frac{5}{2}$
S_3	0	0	0	$\frac{1}{4}$	1	$-\frac{3}{2}$	0	$\frac{5}{4}$
S_5	0	0	0	$-\frac{3}{4}$	0	$-\frac{1}{2}$	1	$-\frac{3}{4}$
$-z$	0	0	0	$-\frac{1}{6}$	0	$-\frac{1}{2}$	0	$-\frac{29}{4}$

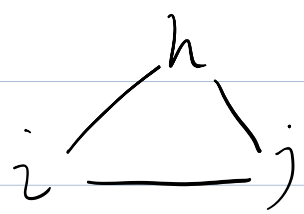
	x	y	S_1	S_2	S_3	S_4	S_5	
y	0	1	0	0	0	$\frac{5}{3}$	$-\frac{1}{3}$	1
x	1	0	0	0	0	$-\frac{2}{3}$	$\frac{1}{3}$	3
S_1	0	0	1	0	0	$-\frac{7}{3}$	$\frac{2}{3}$	2
S_3	0	0	0	0	1	$-\frac{5}{3}$	$\frac{1}{3}$	1
S_5	0	0	0	1	0	$\frac{2}{3}$	$-\frac{4}{3}$	1
$-z$	0	0	0	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	-7

Pivot S_2 $\longrightarrow$ remove S_5

Opt $(x^*, y^*) = (3, 1)$

$z^* = 7$.

4.



$$X_{ij} + X_{hi} = 0 \xRightarrow[\Delta=0 \text{ or } 1]{} X_{hj} = 0 \text{ or } 1.$$

$$X_{ij} + X_{hi} = 1 \xRightarrow[\Delta=1]{} X_{hj} = 0$$

$$X_{ij} + X_{hi} = 2 \xRightarrow[\Delta=1]{} X_{hj} = 1$$

$$(X_{ij} + X_{hi})(X_{ij} + X_{hi} - X_{hj} - 1) = 0$$

$$\Rightarrow \quad \min \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij}$$

$$\text{s.t.} \sum_{\substack{1 \le i \le n \\ i \ne j}} x_{ij} = 1 \qquad j = 1, 2, \cdots, n.$$

$$\sum_{\substack{1 \le k \le n \\ k \ne j}} x_{jk} = 1 \qquad j = 1, 2, \cdots, n.$$

$$(X_{ij} + X_{hi})(X_{ij} + X_{hi} - X_{hj} - 1) = 0 \qquad i \ne h \quad i \ne j \quad h \ne j \quad \forall i, h, j.$$