$Ax = \lambda x \longrightarrow$ eigenvectors.

$\downarrow$

eigenvalues / eigenpairs.

$\Rightarrow$ normalized eigenvectors.

$\|x\|_p = 1$. (numpy.linalg.eig using $p=2$ norm).

$\det(A - \lambda I) = 0$. / $A - \lambda I$ is singular.

We want to approximate the eigenvalues numerically.

a $n \times n$ matrix A with n linearly independent eigenvectors u is said to be diagonalizable.

$$\begin{cases} Au_1 = \lambda_1 u_1 \\ Au_2 = \lambda_2 u_2 \\ \quad \cdots \\ Au_n = \lambda_n u_n \end{cases} \qquad \begin{bmatrix} A \end{bmatrix} \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{U} = \begin{bmatrix} | & | & & | \\ \lambda_1 u_1 & \lambda_2 u_2 & \cdots & \lambda_n u_n \\ | & | & & | \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix}}_{D}$$

$$AU = UD \Rightarrow A = UDU^{-1}$$ i.e. A is similar to D

defective (not diagonalizable).

if $n \times n$ symmetric matrix A has n distinct eigenvalues than A is diagonalizable.

Propose a vector x is lin comb of eigenvectors.

$x = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$.

$Ax = A\alpha_1 u_1 + A\alpha_2 u_2 + \cdots + A\alpha_n u_n$

$= \lambda_1 \alpha_1 u_1 + \lambda_2 \alpha_2 u_2 + \cdots + \lambda_n \alpha_n u_n$.

(Assume $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_n|$).

Goal is to find an eigenvector u_i of A

$x_0 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_n u_n$.

$x_1 = A x_0 = \lambda_1 \alpha_1 u_1 + \lambda_2 \alpha_2 u_2 + \cdots + \lambda_n \alpha_n u_n$.

$x_2 = A x_1 = \lambda_1^2 \alpha_1 u_1 + \lambda_2^2 \alpha_2 u_2 + \cdots + \lambda_n^2 \alpha_n u_n$.

$\cdots$

$x_k = A x_{k-1} = \alpha_1 \lambda_1^k u_1 + \alpha_2 \lambda_2^k u_2 + \cdots + \alpha_n \lambda_n^k u_n$.

$\Downarrow$

Power Iteration:

$$x_k = (\lambda_1)^k \left[\alpha_1 u_1 + \alpha_2 \left(\frac{\lambda_2}{\lambda_1}\right)^k u_2 + \cdots\cdots + \alpha_n \left(\frac{\lambda_n}{\lambda_1}\right)^k u_n \right]$$

Since $|\lambda_1| > |\lambda_2|$, we have $\left(\frac{\lambda_2}{\lambda_1}\right)^k \ll 1$ when k is large.

$\Rightarrow$ Hence, as k increases, x_k converges to a multiple of the first eigenvector u_1, i.e.

$$k \to +\infty \Rightarrow x_k \cong \lambda_1^k \alpha_1 u_1$$