$$\underset{m\times n}{A} = \underbrace{\begin{bmatrix} | & | & & | & & | \\ u_1 & u_2 & \cdots & u_n & \cdots & u_m \\ | & | & & | & & | \end{bmatrix}}_{m\times m} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \\ & 0 & \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} - & v_1^T & - \\ & \vdots & \\ - & v_n^T & - \end{bmatrix}}_{n\times n}$$

$$= \begin{bmatrix} | & | & & | \\ u_1 & u_2 & \cdots & u_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & \sigma_1 v_1^T & - \\ - & \sigma_2 v_2^T & - \\ & \vdots & \\ - & \sigma_n v_n^T & - \end{bmatrix}$$

$$= \sum_{i=1}^{n} \sigma_i \underset{m\times 1}{u_i} \underset{1\times n}{v_i^T} \implies$$ Swaping order will not change A.

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T \qquad k = \min(m, n).$$

if $\sigma_i \neq 0 \ \forall i \implies \operatorname{rank}(A) = k$.

Full rank matrix.

$\operatorname{rank}(A) = r$, $r < k$

matrix rank deficient

Pseudo-inverse. (伪逆)

if A is rank-deficient, Σ is not be invertible.

Pseudo-inverse: $$(\Sigma^+)_i = \begin{cases} \frac{1}{\sigma_i}, & \text{if } \sigma_i \neq 0. \\ 0, & \text{if } \sigma_i = 0. \end{cases}$$

伪 A^{-1}. $A^+ = V\Sigma^+ U^T$

Matrix norm

① The euclidean norm of an orthogonal matrix is equal to 1.

Proof:

$$\|U\|_2 = \max_{\|x\|_2=1} \|Ux\|_2 = \max_{\|x\|_2=1} \sqrt{(Ux)^T(Ux)}$$

$$= \max_{\|x\|_2=1} \sqrt{x^T x} = \max_{\|x\|_2=1} \|x\|_2 = 1.$$

② The euclidean norm of a matrix is given by the largest singular value.

$$\|A\|_2 = \max_{\|x\|_2=1} \|Ax\|_2 = \max_{\|x\|_2=1} \|U\Sigma V^T x\|_2$$

$$\left(\frac{\|U\Sigma V^T x\|_2}{\underbrace{\|U\|_2}_{=1}} \le \|\Sigma V^T x\|_2 \le \underbrace{\|U^{-1}\|_2}_{=1} \|U\Sigma V^T x\|_2 \right)$$

$$\Longrightarrow \|A\|_2 = \max_{\|x\|_2=1} \|U\Sigma V^T x\|_2 = \max_{\|x\|_2=1} \|\Sigma V^T x\|_2$$

Let $y = V^T x$

Similarly: $\|V^T x\|_2 = \|x\|_2 = 1$.

$$= \max_{\|y\|_2=1} \|\underbrace{\Sigma}_{\text{diagonal}} y\|_2.$$

$=$ largest diagonal entry values of Σ.

$$\Longrightarrow \|A\|_2 = \max_i \sigma_i$$

Norm of the inverse of a matrix.

A is full rank, A^{-1} exists.

$$\|A^{-1}\|_2 = \max_{\|x\|_2=1} \|V\Sigma^{-1}U^T x\|_2$$

$$\Rightarrow \|A^{-1}\|_2 = \frac{1}{\sigma_{min}}$$

Norm of the pseudo-inverse matrix.

$\|A^{+}\|_2 = \frac{1}{\sigma_r}$, σ_r is the smallest non-zero singular value.

$= 0$ if A is zero matrix.

Condition number of a matrix.

$$cond_2(A) = \|A\|_2 \|A^{+}\|_2$$

$= \frac{\sigma_{max}}{\sigma_{min}}$ if A is full rank.

$= \infty$ (set) if A is rank deficient

Low-Rank Approximation.

$$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_n u_n v_n^T$$

set $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$, $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$

$k < n$

best rank-k approximation for a $m \times n$ matrix A.

$k \le \min(m, n)$

$$\underbrace{\min_{k} \|A - A_k\|}_{\text{minimizing error.}} = \sigma_{k+1}$$

s.t. $\text{rank}(A_k) \le k$

minimizing error.

Using SVD to solve square system of linear equations.

$$\underset{n \times n}{A} x = b.$$

① $A = U \Sigma V^T \longrightarrow O(n^3)$

$$\Sigma \underbrace{V^T x}_{y} = U^T b \quad O(n^2)$$

② $\Sigma y = U^T b \longrightarrow$ easy! solve for y.

$y = \Sigma^+ U^T b \longrightarrow O(n)$

③ $V^T x = y \longrightarrow x = Vy \longrightarrow O(n^2)$

Value storing.

Storing A : $m \times n$.

Storing rank k approximation:

$$\underset{m\times n}{A} = \underset{m\times m}{\begin{bmatrix} | & | & & | & & | \\ u_1 & u_2 & \cdots & u_n & \cdots & u_m \\ | & | & & | & & | \end{bmatrix}} \underset{m\times n}{\begin{bmatrix} \sigma_1 & & & \\ & \ddots & & \\ & & \sigma_k & \\ & & & \ddots \\ & & & & \sigma_n \\ & & 0 & \end{bmatrix}} \underset{n\times n}{\begin{bmatrix} \cdots & v_1^T & \cdots \\ & \vdots & \\ \cdots & v_n^T & \cdots \end{bmatrix}}$$

$m\times k$ $\quad$ $k\times k$ $\quad$ $k\times n$

number of values need to store

$$= m\times k + k + k\times n = k(m+n+1).$$

Example: $A_{31\times 81}$: 2511 values $\Rightarrow$ 1 value/number

rank-3 approximation: A_3 : $3\times(31+81+1) = 339$ values.