n-dimensions optimization.

$$f(X) = f(x_1, \ldots, x_n)$$

$$\nabla f(X) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \frac{\partial f}{\partial x_2} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix}_{n\times 1}$$

gradient of f

Hessian Matrix

$$H(x) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \cdots & & \ddots & \cdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}_{n\times n}$$

Hessian Matrix is symmetric.

$$\boxed{f(x^*) = \min_x f(x)}$$

First order: $\nabla f(X^*) = 0$ $\overset{\not\Rightarrow}{\Leftarrow}$ X^* is minimizer

(necessary condition)

Second order: $H(X^*)$ is positive definite

$\overset{\Rightarrow}{\not\Leftarrow}$ X^* is minimizer.

not necessary

(sufficient condition).

From linear algebra:

A symmetric $n \times n$ matrix $\boldsymbol{H}$ is **positive definite** if $\boldsymbol{y}^T \boldsymbol{H}\, \boldsymbol{y} > \boldsymbol{0}$ for any $\boldsymbol{y} \neq \boldsymbol{0}$

A symmetric $n \times n$ matrix $\boldsymbol{H}$ is **positive semi-definite** if $\boldsymbol{y}^T \boldsymbol{H}\, \boldsymbol{y} \geq \boldsymbol{0}$ for any $\boldsymbol{y} \neq \boldsymbol{0}$

A symmetric $n \times n$ matrix $\boldsymbol{H}$ is **negative definite** if $\boldsymbol{y}^T \boldsymbol{H}\, \boldsymbol{y} < \boldsymbol{0}$ for any $\boldsymbol{y} \neq \boldsymbol{0}$

A symmetric $n \times n$ matrix $\boldsymbol{H}$ is **negative semi-definite** if $\boldsymbol{y}^T \boldsymbol{H}\, \boldsymbol{y} \leq \boldsymbol{0}$ for any $\boldsymbol{y} \neq \boldsymbol{0}$

A symmetric $n \times n$ matrix $\boldsymbol{H}$ that is not negative semi-definite and not positive semi-definite is called **indefinite** find out if Hessian is positive definite

(λ, y) are eigenpairs of H.

$$Hy = \lambda y \Rightarrow y^T H y = \lambda y^T y = \lambda \|y\|_2^2$$

$$\Rightarrow \lambda = \frac{y^T H y}{\|y\|_2^2}$$

if $\lambda_i > 0 \ \forall i \Rightarrow H$ is positive definite $\Rightarrow X^*$ is minimizer

if $\lambda_i < 0 \ \forall i \Rightarrow H$ is negative definite $\Rightarrow X^*$ is maximizer.

if $\begin{cases} \lambda_i < 0 \\ \lambda_i > 0 \end{cases} \Rightarrow H$ is indefinite $\Rightarrow X^*$ is saddle point.