Nonlinear system of equations

$f(x)=0$. $\quad f: \mathbb{R}^n \to \mathbb{R}^n$

$$f(x) = \begin{bmatrix} f_1(x_1, x_2, \ldots, x_n) \\ f_2(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_n(x_1, x_2, \ldots, x_n) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Newton's method (ND).

Taylor expansion:

$$f(x+s) \approx f(x) + \underbrace{J(x)}_{\text{Jacobian matrix.}}\, s.$$

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad J_{ij} = \frac{\partial f_i}{\partial x_j}$$

$$f(x) + J(x)s = 0 \Rightarrow \underline{\underset{\text{matrix}}{J(x)}\ \underset{\text{vector}}{s} = \underset{\text{vector}}{-f(x)}}.$$

$\hookrightarrow$ solve s.

Algorithm: x_0: initial guess $\quad n\times 1$ vector.

for $i = 1, 2, \ldots$

evaluate $J(x_i) = J$

evaluate $f(x_k) = f$

solve $J s = -f \longrightarrow$ Find s.

update $x_{k+1} = x_k + s$.

Main cost : solve $\underbrace{J}_{n^2}$ and $\underbrace{\text{Newton step.}}_{n^3}$