Finite Difference Method.

$$f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$$

Taylor:

$$f(x+h) = f(x) + f'(x)h + f''(x) \cdot \frac{h^2}{2} + f'''(x) \cdot \frac{h^3}{6} + \cdots$$

$$f(x+h) = f(x) + f'(x)h + O(h^2)$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

Define as $\underline{\underline{df(x)}}$ **Forward Finite Difference.**

truncation error $\left| f'(x) - \frac{f(x+h)-f(x)}{h} \right| \le M.$

Similar way:

$$f(x-h) = f(x) - f'(x)h + O(h^2)$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

Define as $\underline{\underline{df(x)}}$ **Backward Finite Difference.**

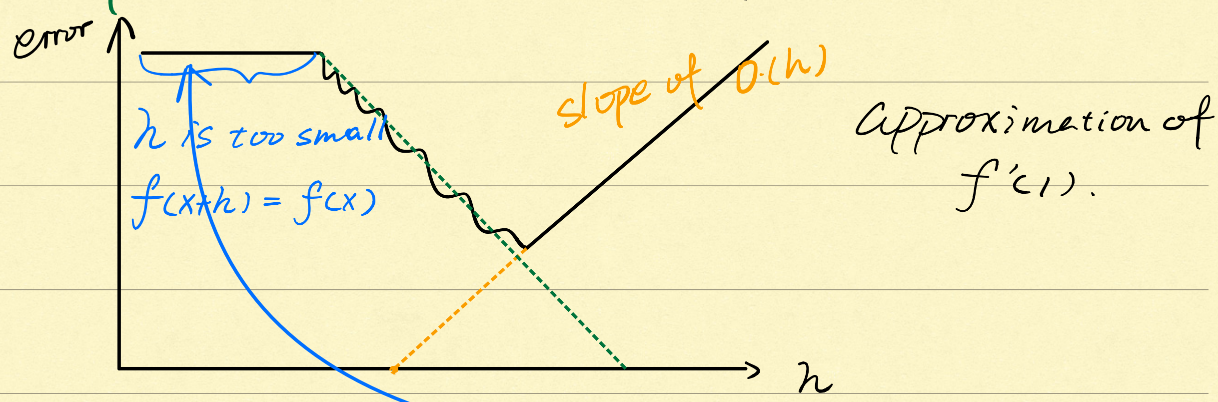
$$f(x+h) - f(x-h) = 2f'(x)h + 2f'''(x)\frac{h^3}{6} + O(h^5)$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

Define as $\underline{\underline{df(x)}}$ **Central Finite Difference.**

Central Finite Difference has better accuracy with (possible) increased computational cost.

Example: $f(x) = e^x - 2$, $f'(x) = e^x \rightarrow f'(1) \approx 2.7$



Forward Finite Difference: $df(1) = \dfrac{f(1+h) - f(1)}{h}$

$\longrightarrow$ due to Cancelation!

when computing the finite difference approximation we have two computing source of errors: Truncation errors and Rounding errors.

$f(x+h) = f(x) + \epsilon_m \cdot 2^n$

when h is really small: $df(x) = \dfrac{|f(x+h) - f(x)|}{h} \leq \dfrac{\epsilon_m |f(x)|}{h}$

(by $\dfrac{|f(x+h) - f(x)|}{|f(x)|} \leq \epsilon_m$)

$h \uparrow$ $\uparrow$

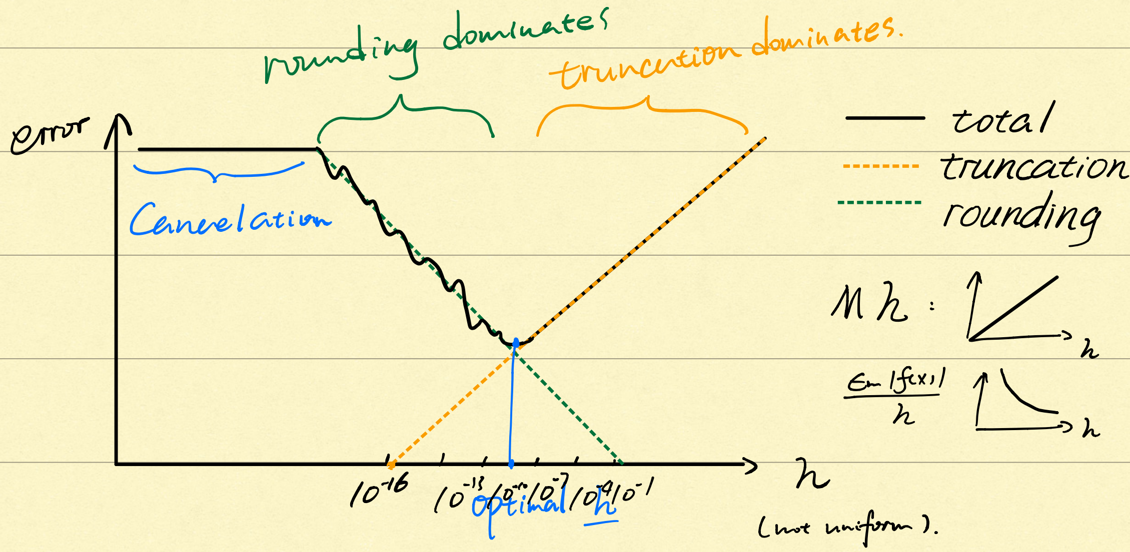
Truncation error: error $\sim Mh$

$h \uparrow$ $\downarrow$

Rounding error: error $\sim \dfrac{\epsilon_m |f(x)|}{h}$

$\Rightarrow$ Total error : $\text{error} \sim Mh + \dfrac{\epsilon_m |f(x)|}{h}$

Minimize total error : $h = \sqrt{\dfrac{\epsilon_m |f(x)|}{M}}$



approx : $h = \sqrt{\epsilon_m}$ 32 bits : $\epsilon_m \sim 10^{-16}$

$h \approx 10^{-8}$