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Homework 12

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Poisson Process Properties
- Inhomogenous Poisson Process

Exercises

1. **Poisson Process**(1 + 2 = 3 points)

   Claims against an insurance company follow a Poisson process with rate $\lambda > 0$. A total of N claims were received over two periods of combined length $t = t_1 + t_2$ with t_1 and t_2 being the lengths of the separate periods.

   (a) Given this information, derive the (conditional) probability distribution of N_1, the number of claims made in the 1st period, given N.

   (b) The amount paid for the ith claim is X_i, with $X_1, X_2, \ldots$ i.i.d. and independent of the claims process. Let $E(X_i) = \mu$ and $Var(X_i) = 2$ for $i = 1, \ldots, N$. Given N, find the mean and variance of the total claims paid in period 1. That is, find these two conditional moments of the quantity

   $$W = \sum_{i=1}^{N_1} X_i$$

   where by convention $W_1 = 0$ if $N_1 = 0$.

2. **Poisson Process**(1 + 2 = 3 points)

   On a certain question and answer website, $N \sim Poi(\lambda_1)$ questions will be posted tomorrow, with λ_1 measured in questions/day. Given N, the post times are i.i.d. and uniformly distributed over the day (a day begins and ends at midnight). When a question is posted, it takes an $Exp(\lambda_2)$ amount of time (in days) for an answer to be posted, independently of what happens with other questions.

(a) Find the probability that a question posted at a uniformly random time tomorrow will not yet have been answered by the end of that day.

(b) Find the joint distribution of how many answered and unanswered questions posted tomorrow there will be at the end of that day.

3. **Inhomogenous Poisson Process**(3 + 2 = 5 points)

An inhomogeneous Poisson process in one dimension is a Poisson process whose rate, instead of being constant, is a nonnegative function $\lambda(t)$ of time. Formally, we require that the number of arrivals in any interval $[t_1, t_2)$ be Poisson distributed with mean $\int_{t_1}^{t_2} \lambda(t)$ and that disjoint intervals be independent. When $\lambda(t)$ is constant, this reduces to to the definition of the ordinary or homogeneous Poisson process.

(a) Show that we can generate arrivals from an inhomogeneous Poisson process in the interval $[t_1, t_2)$ using the following procedure.

1. Let λ_{max} be the maximum value of $\lambda(t)$ in the interval $[t_1, t_2)$. Create a 2D rectangle $[t_1, t_2) \times [0, \lambda_{max}]$ and plot the function $\lambda(t)$ in the rectangle.

2. Generate $N \sim Pois(\lambda_{max}(t_2 - t_1))$ and place N points uniformly at random in the rectangle.

3. For each of the N points: if the point falls below the curve $\lambda(t)$, accept it as an arrival in the process, and take its horizontal coordinate to be its arrival time. If the point falls above the curve $\lambda(t)$, reject it. Hint: Verify that the two conditions in the definition are satisfied.

(b) Suppose we have an inhomogeneous Poisson process with rate function $\lambda(t)$. Let $N(t)$ be the number of arrivals up to time t and T_j the time of the jth arrival. Explain why the hybrid joint PDF of $N(t)$ and $T_1, \ldots, T_{N(t)}$ which constitute all the data observed up to time t is given by

$$f(n, t1, ..., tn) = \exp(-\lambda_{total}) \frac{\lambda_{total}^n}{n!} n! \frac{\lambda(t_1) \ldots \lambda(t_n)}{\lambda_{total}^n}$$

for $0 < t_1 < t_2 < \; < t_n$ and nonnegative integer n, where $\lambda_{total} = \int_0^t \lambda(u) du$.

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

You are encouraged to complete ICES course evaluations. This is a valuable source of course feedback.

1. (a) $N_1 \mid N = n \sim Bin(n, \frac{t_1}{t})$

p.d.f: $P(N_1 = x \mid N = n) = \binom{n}{x} (\frac{t_1}{t_1+t_2})^x (\frac{t_2}{t_1+t_2})^{n-x}$

(b): $E(\sum_{i=1}^{N_1} x_i \mid N=n, N_1=n_1) = n_1 \mu$

$E(\sum_{i=1}^{N_1} x_i \mid N=n) = \sum_{i=0}^{n} E(\sum_{i=1}^{N_1} x_i \mid N=n, N_1=i) \cdot P(N_1 = x \mid N=n).$

$= \mu \cdot n \cdot \frac{t_1}{t_1+t_2}$

$Var(\sum_{i=1}^{N_1} x_i \mid N=n, N_1=n_1) = 2n_1$

$Var(\sum_{i=1}^{N_1} x_i \mid N=n) = Var(E(\sum_{i=1}^{N_1} x_i \mid N=n, N_1=n_1))$

$+ E(Var(\sum_{i=1}^{N_1} x_i \mid N=n, N_1=n_1))$

$= \mu^2 n \cdot \frac{t_1}{t_1+t_2} \cdot \frac{t_2}{t_1+t_2} + 2 \cdot n \cdot \frac{t_1}{t_1+t_2}$

2. (a) $T_1 \sim U(0,1)$. $T_2 \sim Expo(\lambda_2)$

$P(T_1 + T_2 \geq 1)$

$= \int_0^1 P(T_2 \geq 1 - T_1 \mid T_1 = t_1) f_{T_1}(t_1)\, dt_1$

$= \int_0^1 (1 - (1 - e^{-\lambda_2(1-t_1)}))\, dt_1$

$= e^{-\lambda_2} \int_0^1 e^{\lambda_2 t_1}\, dt_1$

$= e^{-\lambda_2} \frac{1}{\lambda_2} e^{\lambda_2 t_1} \Big|_0^1 = \frac{1}{\lambda_2}(1 - e^{-\lambda_2})$

(b) $N \sim Poi(\lambda_1)$.

N_1 is # answered questions, N_2 is # unanswered questions

$$N_2 | N = n \sim Bin\left(n, \frac{1-e^{-\lambda_2}}{\lambda_2}\right)$$

$$P(N_1 = x, N_2 = y \mid N = x+y) = P(N_2 = y \mid N = x+y)$$

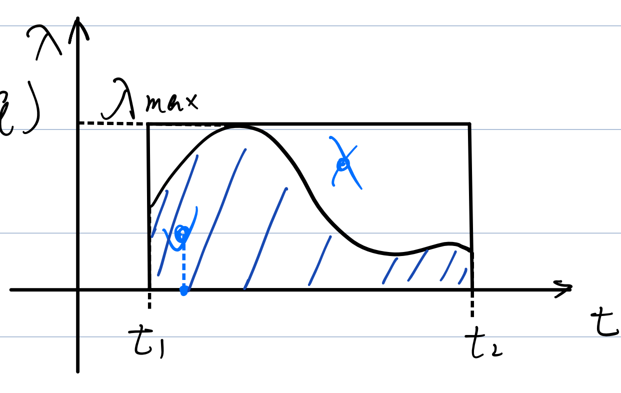
$$= \binom{x+y}{x}\left(\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^y\left(1-\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^x$$

$$P(N_1 = x, N_2 = y) = P(N_1 = x, N_2 = y \mid N = x+y)P(N = x+y)$$

$$= \frac{\lambda_1^{(x+y)}}{(x+y)!}e^{-\lambda_1} \cdot \binom{x+y}{x}\left(\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^y\left(1-\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^x$$

$$= \frac{\lambda_1^{(x+y)}}{x!\,y!}e^{-\lambda_1}\left(\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^y\left(1-\frac{1-e^{-\lambda_2}}{\lambda_2}\right)^x$$

3. (a)



the prob of a point fall below the curve $\lambda(t)$:

$$P = \frac{\int_{t_1}^{t_2} \lambda(t)\,dt}{(t_2-t_1)\lambda_{max}}$$

Hence, any points of N points have prob p be accepted.

Let N_{new} be the number of arrivals from the inhomogeneous Possion process.

$$P(N_{new}=k) = \sum_{i=k}^{\infty} \binom{i}{k} p^k (1-p)^{i-k} P(N=i).$$

$$= \sum_{i=k}^{\infty} \frac{i!}{k!(i-k)!} p^k (1-p)^{i-k} \cdot \frac{(\lambda_{max}(t_2-t_1))^i}{i!} e^{-\lambda_{max}(t_2-t_1)}$$

$$= \sum_{i=k}^{\infty} \frac{1}{k!(i-k)!} \left(\int_{t_1}^{t_2} \lambda(t)dt\right)^k \left((t_2-t_1)\lambda_{max} - \int_{t_1}^{t_2} \lambda(t)dt\right)^{i-k} \cdot e^{-\lambda_{max}(t_2-t_1)}$$

$$= \frac{\left(\int_{t_1}^{t_2} \lambda(t)dt\right)^k}{k!} e^{-\lambda_{max}(t_2-t_1)} \sum_{j=0}^{\infty} \frac{\left((t_2-t_1)\lambda_{max} - \int_{t_1}^{t_2} \lambda(t)dt\right)^j}{j!}$$

$$= \frac{\left(\int_{t_1}^{t_2} \lambda(t)dt\right)^k}{k!} e^{-\lambda_{max}(t_2-t_1)} \frac{1}{e^{-[(t_2-t_1)\lambda_{max} - \int_{t_1}^{t_2} \lambda(t)dt]}}$$

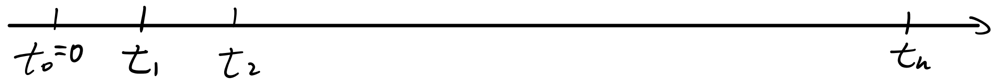
$$= \frac{\left(\int_{t_1}^{t_2} \lambda(t)dt\right)^k}{k!} e^{-\int_{t_1}^{t_2} \lambda(t)dt}$$

$\Rightarrow N_{new} \sim Pois\left(\int_{t_1}^{t_2} \lambda(t)dt\right).$

$\Rightarrow$ We can generate the inhomogeneous Poisson process.

(b)

$N(t) \sim Poiss\left(\int_0^t \lambda(x)dx\right)$



Let the number of arrivals between $(t_{i-1}, t_i]$ be N_i

$(t_n, t]$ be N_n.

From A, we know $N_i \sim \text{Poiss}\left(\int_{t_{i-1}}^{t_i} \lambda(u)\,du\right)$

$N_n \sim \text{Poiss}\left(\int_{t_n}^{t} \lambda(u)\,du\right)$

$$P(T_i > t_i \mid T_{i-1} = t_{i-1}) = P(N_i(t_i) = 0) = e^{-\int_{t_{i-1}}^{t_i} \lambda(u)\,du}$$

$$\Longrightarrow f(t_i \mid t_{i-1}) = \lambda(t_i)\, e^{-\int_{t_{i-1}}^{t_i} \lambda(u)\,du}$$

$$f_T(t_1, \dots, t_n) = f_{T_1}(t_1)\, f_{T_2|T_1}(t_2 \mid t_1) \cdots f_{T_n|T_{n-1}}(t_n \mid t_{n-1})\, P(N_n(t) = 0).$$

$$= \prod_{i=1}^{n} \lambda(t_i)\, e^{-\int_{t_{i-1}}^{t_i} \lambda(u)\,du} \cdot e^{-\int_{t_n}^{t} \lambda(u)\,du}$$

$$= \prod_{i=1}^{n} \lambda(t_i) \cdot e^{-\int_0^t \lambda(u)\,du}$$

$$= \exp(-\lambda_{total}) \cdot \lambda(t_1) \cdots \lambda(t_n)$$