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Section Number: 3UG

STAT 410

Homework #1

(due Friday, September 4, by 5:00 p.m. CDT)

Fall 2020
A. Stepanov

Please include your name (with your last name underlined), your NetID, and your section number at the top of the first page.

No credit will be given without supporting work.

1. Grades on Fall 2020 STAT 410 Exam 1 were not very good*. Graphed, their distribution had a shape similar to the probability density function.

$$f_X(x) = \frac{\sqrt{x+6}}{C}, \qquad 3 \le x \le 75, \qquad \text{zero elsewhere.}$$

a) Find the value of C that makes $f_X(x)$ a valid probability density function.

b) Find the cumulative distribution function of X, $F_X(x) = P(X \le x)$.

"Hint": To double-check your answer: should be $F_X(3) = 0$, $F_X(75) = 1$.

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* The probability distribution is fictional, the exam has not happened yet. Hopefully, the actual grades will be slightly better than these.

a). $\int_3^{75} \frac{\sqrt{x+6}}{c}\,dx = \frac{2}{3c}(x+6)^{\frac{3}{2}}\Big|_3^{75} = \frac{468}{c} = 1.$

$\Rightarrow \quad c = 468$

b). $F_X(x) = \int_3^x \frac{\sqrt{u+6}}{468}\,du = \frac{(x+6)^{\frac{3}{2}} - 27}{702} \quad x \in [3, 75]$

$F_X(x) = 0, \; x < 3, \quad F_X(x) = 1, \; x > 75$

c). $g(x) = 5\sqrt{2x+75}$

$x \in [3, 75] \Rightarrow y = g(x) \in [45, 75].$

d). $F_Y(y) = P(Y \le y) = P(5\sqrt{2x+75} \le y).$

$= P\left(X \le \frac{y^2}{50} - \frac{75}{2}\right) = F_X\left(\frac{y^2}{50} - \frac{75}{2}\right)$

$= \frac{\left(\frac{y^2}{50} - \frac{63}{2}\right)^{\frac{3}{2}} - 27}{702}, \quad y \in [45, 75]$

$F_Y(y) = 0, \; y < 45, \quad F_Y(y) = 1, \; y > 75.$

e) $f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$

$= \frac{\left(\frac{y^2}{50} - \frac{63}{2}\right)^{\frac{1}{2}}}{468} \frac{y}{25} \qquad y \in [45, 75]$

zero elsewhere.

1. (continued)

As a way of "curving" the results, the instructor announced that he would replace each person's grade, X, with a new grade, $\mathrm{Y} = g(\mathrm{X})$, where $g(x) = 5\sqrt{2x+75}$.

c) Find the support (the range of possible values) of the probability distribution of Y.

d) Use part (b) and the c.d.f. approach to find the c.d.f. of Y, $\mathrm{F}_{\mathrm{Y}}(y)$.

"Hint": $\mathrm{F}_{\mathrm{Y}}(y) = \mathrm{P}(\mathrm{Y} \le y) = \mathrm{P}(g(\mathrm{X}) \le y) = \ldots.$

e) Use the change-of-variable technique to find the p.d.f. of Y, $f_{\mathrm{Y}}(y)$.

"Hint": $f_{\mathrm{Y}}(y) = f_{\mathrm{X}}(g^{-1}(y)) \left| \dfrac{dx}{dy} \right|$.

"Hint": To double-check your answer: should be $f_{\mathrm{Y}}(y) = \mathrm{F}_{\mathrm{Y}}'(y)$.

2. Consider a continuous random variable X with the probability density function

$$f_{\mathrm{X}}(x) = \frac{3-x}{8}, \qquad -1 \le x \le 3, \qquad \text{zero elsewhere.}$$

Consider $\mathrm{Y} = g(\mathrm{X}) = \dfrac{9}{\mathrm{X}^2}$. Find the probability distribution of Y.

You are welcome to use a computer and/or calculator on any problem to evaluate any integral. For the supporting work, you should include the full integral (with the function and the bounds) and the answer. For example,

$$\int_0^x u^2\,du = \frac{x^3}{3}, \qquad \int_0^4\left(\int_0^{\sqrt{x}} x^2 y\,dy\right)dx = 32, \qquad \int_1^\infty\left(\int_0^y \frac{1}{(2x+y)^3}\,dx\right)dy = \frac{2}{9}.$$

$$F_Y(y) = P\left(\frac{9}{X^2} \le y\right) = P\left(X^2 \ge \frac{9}{y}\right)$$

$$= \begin{cases} P\left(X \ge \sqrt{\frac{9}{y}}\right) + P\left(X \le -\sqrt{\frac{9}{y}}\right), & y > 9. \\ P\left(X \ge \sqrt{\frac{9}{y}}\right) & y \in [1,9) \end{cases}$$

$$= \begin{cases} \int_{\sqrt{\frac{9}{y}}}^{3} \frac{3-x}{8}\,dx + \int_{-1}^{-\sqrt{\frac{9}{y}}} \frac{3-x}{8}\,dx, & y > 9 \\ \int_{\sqrt{\frac{9}{y}}}^{3} \frac{3-x}{8}\,dx & y \in [1,9] \end{cases}$$

$$= \begin{cases} 1 - \frac{3}{4}\sqrt{\frac{9}{y}}, & y > 9 \\ \frac{9}{16} - \frac{3}{8}\sqrt{\frac{9}{y}} + \frac{9}{16y}, & y \in [1,9] \end{cases}$$

$$F_Y(y) = 0, \quad y < 1$$

$$f_Y(y) = F_Y'(y) = \begin{cases} \frac{9}{8y^{\frac{3}{2}}}, & y > 9 \\ \frac{9}{16y^{\frac{3}{2}}} - \frac{9}{16y^2}, & y \in [1,9] \\ 0, & y < 1. \end{cases}$$