Theorem: (卷积 Convolution Sums and integrals).

If X, Y indep. discrete r.v. the the p.m.f. of $T = X+Y$ is:

$$P(T=t) = \sum_{x} P(Y=t-x)\,P(X=x).$$

$$= \sum_{y} P(X=t-y)\,P(Y=y)$$

Continue

$$f_T(t) = \int_{-\infty}^{\infty} f_Y(t-x)\,f_X(x)\,dx.$$

$$= \int_{-\infty}^{\infty} f_X(t-y)\,f_Y(y)\,dy$$

Special Proof: (Use change of Variable Theorem).

$T = X+Y \qquad W = Y.$

$$g: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} T \\ W \end{pmatrix} = \begin{pmatrix} X+Y \\ Y \end{pmatrix}$$

$$\iint f_{T,W}(t,\omega)\,dt\,dw = \iint f_{X,Y}(x,y)\,dx\,dY.$$

$$\Longrightarrow f_{T,W}(t,\omega) = \left|\frac{\partial(t,\omega)}{\partial(x,y)}\right| f_{X,Y}(t-\omega,\omega)$$

$$= \begin{vmatrix} \frac{\partial X+Y}{\partial X} & \frac{\partial X+Y}{\partial y} \\ \frac{\partial y}{\partial X} & \frac{\partial y}{\partial y} \end{vmatrix} f_X(t-\omega)\,f_Y(\omega)$$

$= |\,1 \quad 1\,|$

$$\begin{vmatrix} 0 & 1 \end{vmatrix} f_X(t-\omega) f_Y(\omega)$$

$$= f_X(t-\omega) f_Y(\omega)$$

$$\Rightarrow f_T(t) = \int_{-\infty}^{+\infty} f_X(t-\omega) f_Y(\omega)\, d\omega.$$

Beta Distribution:

Defn: A r.v. X is said to have the Beta distr with parameters $a, b > 0$, if its p.d.f is:

$$f(x) = \frac{x^{a-1}(1-x)^{b-1}}{\beta(a,b)}, \quad 0 \le x \le 1.$$

$\beta(a,b)$ is a constant. $\int_0^1 f(x) = 1$

$$\beta(a,b) = \int_0^1 u^{a-1}(1-u)^{b-1}\, du$$

Beta $(1,1) \equiv$ Unif $(0,1)$

if $a > 1$, $b > 1$. p.d.f : "U"-shaped.

if $a > 1$, $b > 1$. p.d.f : "∩"-shaped.

if $a = b$ p.d.f is symmetric about $\frac{1}{2}$.

Fact: for any integer k & n with $0 \le k \le n$.

$$\int_0^1 \binom{n}{k} x^k (1-x)^{n-k} \, dx = \frac{1}{n+1}$$

Proof:

$$(x+(1-x))^n = \sum_{k=0}^{n} x^k (1-x)^{n-k} \binom{n}{k}$$

$$\int_0^1 (x+(1-x))^n \, dx = 1 = \sum_{k=0}^{n} \int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx.$$

prove: $\int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx = \int_0^1 x^{k+1} (1-x)^{n-k-1} \binom{n}{k+1} dx.$

k is an interger and $k \in [0.\ n-1]$.

$$\int_0^1 x^{k+1}(1-x)^{n-k-1} \binom{n}{k+1} dx = \int_0^1 \frac{x^{k+1}}{k+1} (n-k)(1-x)^{n-k-1} \binom{n}{k} dx$$

$$= \int_0^1 -\frac{x^{k+1}}{k+1} \binom{n}{k} \, d(1-x)^{n-k}$$

$$= -\frac{x^{k+1}}{k+1}(1-x)^{n-k}\binom{n}{k}\Big|_0^1 - \int_0^1 (1-x)^{n-k} \, d\binom{n}{k}\frac{x^{k+1}}{k+1}$$

$$= \int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx$$

Hence $\int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx =$ a constant.

$\forall$ interger $k \in [0, n]$.

$$\Longrightarrow \int_0^1 (x+(1-x))^n dx = 1 = (n+1) \int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx$$

$$\Longrightarrow \int_0^1 x^k (1-x)^{n-k} \binom{n}{k} dx = \frac{1}{n+1}.$$

$$\Longrightarrow \beta(k+1, n-k+1) = \int_0^1 x^k (1-x)^{n-k} dx$$

$$= \frac{1}{\binom{n}{k}} \cdot \frac{1}{n+1} = \frac{(n-k)!\,k!}{(n+1)!}$$

$$\Longrightarrow \quad a = k+1 \quad b = n-k+1$$

$$k = a-1 \quad n = b+a-1-1 = a+b-2$$

$$\beta(a, b) = \frac{(b-1)!\,(a-1)!}{(a+b-1)!}$$

$X \sim Beta(a, b)$, $EX = \int_0^1 \frac{x^a (1-x)^{b-1}}{\beta(a,b)} dx = \frac{\beta(a+1, b)}{\beta(a,b)} = \frac{a}{a+b}$

Gamma Distribution.

$Beta(1,1) \sim Unif(0,1)$

it generates exponential distribution.

Gamma function: $\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$. $a>0$

Properties: (1) $\Gamma(a+1) = a\Gamma(a) \quad \forall a>0.$

proof: $\Gamma(a+1) = \int_0^\infty x^a e^{-x} dx = -\int_0^\infty x^a d e^{-x}$

$$= -x^a e^{-x}\Big|_0^\infty + \int_0^\infty e^{-x} d x^a$$

$$= 0 + a\int_0^\infty x^{a-1} e^{-x} dx$$

$$= a \cdot \Gamma(a).$$

(2) $\Gamma(n) = (n-1)!$, $n \in \mathbb{N}^+$

Proof: $\Gamma(1) = \int_0^\infty x^0 e^{-x} dx = 1$

$\Gamma(n) = n(n-1)(n-2)\cdots\Gamma(1) = (n-1)!$

(3) $\Gamma(\frac{1}{2}) = \int_0^\infty \frac{1}{\sqrt{x}} e^{-x} dx = \int_0^\infty 2\sqrt{\pi}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{(\sqrt{2x})^2}{2}}\right) d\sqrt{2x} = \sqrt{\pi}$

Def: $X \sim Gamma(a, 1)$ distribution if its p.d.f

$$f_X(x) = \begin{cases} \frac{1}{\Gamma(a)} x^{a-1} e^{-x}, & x > 0. \\ 0, & x < 0 \end{cases}$$

If $X \sim Gamma(a, 1)$ and $\lambda > 0$, then the dist of $Y = \frac{X}{\lambda}$ is called $Gamma(a, \lambda)$. distribution.

$x = \lambda y$ $\quad \frac{dx}{dy} = \lambda.$

$$f_Y(y) = f_X(x) \left|\frac{dx}{dy}\right|$$

$$= \frac{1}{\Gamma(a)} (\lambda y)^{a-1} e^{-\lambda y} \cdot \lambda$$

$$= \frac{1}{\Gamma(a)} \lambda^a y^{a-1} e^{-\lambda y}, \quad y > 0.$$

when $a=1$ $X \sim Gamma(1, \lambda)$

$$f_X(x) = \lambda e^{-\lambda x}, \quad x>0.$$

is same as Exponential (λ)

Consider $X \sim \text{Gamma}(a, 1)$.

$$EX = \int_0^\infty x \frac{1}{\Gamma(a)} x^{a-1} e^{-x} dx = \frac{\Gamma(a+1)}{\Gamma(a)} = a$$

$$EX^2 = \int_0^\infty x^2 \frac{1}{\Gamma(a)} x^{a-1} e^{-x} dx = \frac{\Gamma(a+2)}{\Gamma(a)} = a(a+1)$$

$$\text{Var}X = EX^2 - (EX)^2 = a^2 + a - a^2 = a.$$

$$EX^c = \int_0^\infty x^c \frac{1}{\Gamma(a)} x^{a-1} e^{-x} dx = \frac{\Gamma(a+c)}{\Gamma(a)} = \frac{(a+c)!}{a!}, \quad c > -a$$

Consider $Y \sim \text{Gamma}(a, \lambda)$

$$EY = \frac{EX}{\lambda} = \frac{a}{\lambda}$$

$$\text{Var}Y = \frac{1}{\lambda^2}\text{Var}X = \frac{a}{\lambda^2}$$

Theorem: Let $X_1, \ldots, X_n$ i.i.d. $\text{Exp}(\lambda)$

Then $U = X_1 + X_2 + \cdots + X_n \sim Gamma(n, \lambda)$.

Proof: prove if $X_1 \sim Exp(\lambda)$ (i.e. $Gamma(1, \lambda)$).

$Z_k \sim Gamma(k, \lambda) \quad k \in N^+$

X_1, Z_k indep. then $V = X_1 + Z_k \sim Gamma(k+1, \lambda)$.

$$f_V(v) = \int_0^v f_{Z_k}(v-x) f_{X_1}(x)\,dx.$$

$$= \int_0^v \frac{1}{\Gamma(k)} \cdot \lambda^k (v-x)^{k-1} e^{-\lambda(v-x)} \cdot \lambda e^{-\lambda x} dx$$

$$= \int_0^v \frac{1}{\Gamma(k)} \lambda^{k+1} (v-x)^{k-1} e^{-\lambda v} dx.$$

$$= \frac{-\lambda^{k+1} e^{-\lambda v}}{k\,\Gamma(k)} (v-x)^k \Big|_0^v$$

$$= \frac{1}{\Gamma(k+1)} \lambda^{k+1} v^k e^{-\lambda v}$$

$\Rightarrow V \sim Gamma(k+1, \lambda)$

$\Rightarrow U \sim Gamma(n, \lambda)$.

$X \sim Gamma(a, \lambda)$, $Y \sim Gamma(b, \lambda)$. X, Y idep.

What is the joint distribution of

$$T = X + Y, \quad W = \frac{X}{X+Y}$$

$$g: \begin{pmatrix} X \\ Y \end{pmatrix} \longrightarrow \begin{pmatrix} T \\ W \end{pmatrix} = \begin{pmatrix} X+Y \\ \frac{X}{X+Y} \end{pmatrix}$$

$$\mathbb{R}_+^2 \longrightarrow \mathbb{R}_+^2$$

$$J_{g^{-1}} = \left| \frac{\partial(x, y)}{\partial(t, \omega)} \right| = \left| \frac{\partial(\omega t, t - \omega t)}{\partial(t, \omega)} \right| = \left| \begin{matrix} \omega, & t \\ 1-\omega & -t \end{matrix} \right| = |-t| = t$$

$$f_{T,W}(t, \omega) = f_{X,Y}(x, y) \left| \frac{\partial(x, y)}{\partial(t, w)} \right|$$

$$= t \cdot f_{X,Y}(x, y)$$

$$= t \cdot \frac{1}{\Gamma(a)} \lambda^a x^{a-1} e^{-\lambda x} \cdot \frac{1}{\Gamma(b)} \lambda^b y^{b-1} e^{-\lambda y}$$

$$= \frac{t \lambda^{a+b} (\omega t)^{a-1} (t - \omega t)^{b-1} e^{-\lambda t}}{\Gamma(a) \Gamma(b)}$$

$$= \frac{\lambda^{a+b} t^{a+b-1} \omega^{a-1} (1-\omega)^{b-1} e^{-\lambda t}}{\Gamma(a) \Gamma(b)}$$

$$T = X + Y \sim \text{Gamma}(a+b, \lambda) \Rightarrow f_T(t) = \frac{\lambda^{a+b} t^{a+b-1}}{\Gamma(a+b)} e^{-\lambda t}, \ t > 0$$

$$f_W(\omega) = \int_0^\infty \frac{\lambda^{a+b} t^{a+b-1} \omega^{a-1}}{\Gamma(a) \Gamma(b)} (1-\omega)^{b-1} e^{-\lambda t} \, dt$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1} \int_0^\infty \frac{\lambda^{a+b} t^{a+b-1} e^{-\lambda t}}{\Gamma(a+b)} dt$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} w^{a-1}(1-w)^{b-1}$$

$\Rightarrow f_{T,W}(t,w) = f_T(t) f_W(w) \Rightarrow T, W$ are indep

$$= \frac{(a+b-1)!}{(a-1)!(b-1)!} w^{a-1}(1-w)^{b-1}$$

$$= \frac{w^{a-1}(1-w)^{b-1}}{\beta(a,b)}$$

$\Rightarrow W \sim Beta(a,b).$

Summary.

If $X \sim Gamma(a, \lambda)$, $Y \sim Gamma(b, \lambda)$

X, Y indep. then.

(1) $X+Y$ is indep of $\frac{X}{X+Y}$

(2) $X+Y \sim Gamma(a+b, \lambda)$

(3) $\frac{X}{X+Y} \sim Beta(a,b)$ $\Rightarrow$ $X \sim Expo(1)$ $Y \sim Expo(1)$

$\frac{X}{X+Y} \sim Beta(1,1)$ i.e. $Unif(0,1)$.

Another way to compute mean of Beta distribution

$$\underbrace{E(X)}_{=\frac{a}{\lambda}} = \underbrace{E(X+Y)}_{=\frac{a+b}{\lambda}} E\left(\frac{X}{X+Y}\right)$$

$$\Longrightarrow E\left(\frac{X}{X+Y}\right) = \frac{a}{a+b}$$