Transformation of One Random Variable.

X is a r.v. (we know its distribution).

$Y = g(X)$ is a new r.v.

what is Y's distribution?

①. Discreate Case:

PMF of $g(X)$

$$P(Y=y) = P(g(X)=y)$$
$$= \sum_{x: g(x)=y} P(X=x).$$

if $g(x)$ is $1-1$, There is only one value of x such that $g(x)=y$.

$$\Rightarrow P(Y=y) = P(X=g^{-1}(y))$$

②. Continous Case.

A universal approach start from CDF of $Y=g(X)$ and translate the event

$\{g(X) \le y\}$ to an event about X.

$$F_Y(y) = P\{g(X) \le y\}$$

$$\equiv P\{X \in A\}.$$

if g is continous and <u>strictly increasing</u>.

$$\Rightarrow \{g(X) \le y\} = \{X \le g^{-1}(y)\}.$$

$$F_Y(y) = P\{X \le g^{-1}(y)\}.$$

$$= F_X(g^{-1}(y)).$$

if g is continous and strictly decreasing.

$$\Rightarrow \{g(X) \le y\} = \{X \ge g^{-1}(y)\}.$$

$$F_Y(y) = P\{X \ge g^{-1}(y)\}.$$

$$= 1 - F_X(g^{-1}(y))$$

Change of Variables Theorem:

Let X be a continous random variable with p.d.f. f_X and $Y = g(X)$ where g is differeneciable and strictcly ~~increasing~~.

$$f_Y(y) = f_X(x)\frac{dx}{dy} \Leftarrow$$

decreasing.

The p.d.f of Y is given by $f_Y(y) = -f_X(x)\frac{dx}{dy}$

$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|$$

$$x = g^{-1}(y)$$

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}} = \frac{1}{\frac{dg(x)}{dx}}$$

$$\frac{dx}{dy} = \frac{d\,g^{-1}(y)}{dy}$$

$$f_Y(t) = f_X(g^{-1}(t)) \times \left| g^{-1\prime}(t) \right|$$

Example: $Z \sim N(0,1)$.

The distribution of Z^2 is called χ^2.

Find an expression for the p.d.f. of χ^2?

($g(x) = x^2$ is not strictly increasing or decreasing).

$$P(Z^2 \le t) = P(-\sqrt{t} \le Z \le \sqrt{t}).$$

$$= F_Z(\sqrt{t}) - F_Z(-\sqrt{t}).$$

$$= \Phi(\sqrt{t}) - (1 - \Phi(\sqrt{t})).$$

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$\Longrightarrow$

$$F_{Z^2}(t) = 2\Phi(\sqrt{t}) - 1.$$

$$f_{Z^2}(t) = 2\phi(\sqrt{t}) \cdot \frac{1}{2\sqrt{t}} = \frac{1}{\sqrt{t}}\phi(\sqrt{t})$$

p.d.f. of $N(0,1)$.

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{t}{2}} \times \frac{1}{\sqrt{t}}, \quad t > 0.$$

Simulation.

Theorem: Let F be a CDF which is a continous function and strictly increasing on the support of distribution. This ensures F^{-1} exist $[0,1] \to \mathbb{R}$.

1. Let $U \sim \text{Uniform}(0,1)$, $X = F^{-1}(U)$.
   Then X is a r.v. with c.d.f. F.
2. Let X be a r.v. with c.d.f. F
   Then $F(X) \sim \text{Unif}(0,1)$.

Proof: Let $U \sim \text{Unif}(0,1)$.

$$P(F^{-1}(U) \le t) = P(U \le F(t)) = F(t).$$

$P(F(X) \leq t) = 0 \qquad t < 0.$

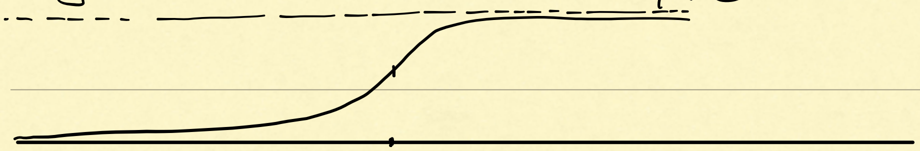
$P(F(X) \leq t) = 1, \quad t > 1.$

$P(F(X) \leq t) = P(X \leq F^{-1}(t)) = F(F^{-1}(t)) = t$

$0 \leq t \leq 1$

Example: Logistic Distribution.

Logistic CDF $F(x) = \dfrac{e^x}{1+e^x} \qquad x \in \mathbb{R}.$



$F^{-1}(U) = \log\left(\dfrac{U}{1-U}\right) \sim$ Logistic

$F(y) = x.$

$\dfrac{e^y}{1+e^y} = x.$

$e^y = \dfrac{x}{1-x} \implies y = \log\left(\dfrac{x}{1-x}\right)$

$\implies F^{-1}(x) = \log\left(\dfrac{x}{1-x}\right)$

Example: How to generate sample from $N(0,1)$?

$$\Phi(t) = \int_{-\infty}^{t} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.$$

It's hard to find inverse of Φ.

Recap: Once we can generate random variables from $(0,1)$, we can generate sample from any discrete as well as continous distribution if we can compute F^{-1}.