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Homework 9

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Definition of Markov Chains
- Irreducibility, Recurrence, Transience
- Gambler's Ruin

Exercises

1. **Gambler's Ruin**(3 points)

   As in the gamblers ruin problem, two gamblers, A and B, make a series of bets, until one of the gamblers goes bankrupt. Let A start out with i dollars and B start out with $N - i$ dollars, and let p be the probability of A winning a bet, with $0 < p < 1$. Each bet is for $\frac{1}{k}$ dollars, with k a positive integer, e.g., $k = 1$ is the original gamblers ruin problem and $k = 20$ means theyre betting nickels. Find the probability that A wins the game, and determine what happens to this as $k \to \infty$.

2. **Card Shuffling**(5 points)

   Suppose you have a deck of 4 cards. Let X_0 denote the original ordering of the cards and be $1, 2, 3, 4$. Suppose I shuffle the cards in the following way. I pick a pair of positions $(i, j) \in \{1, 2, 3, 4\}$ uniformly randomly among the $\binom{4}{2}$ possible pairs of positions and I swap the cards at those positions. For example, if the ordering of the cards is $2, 3, 1, 4$ at any particular stage and I choose the $1, 2$ th pair of positions then the next ordering of the cards would be $3, 2, 1, 4$. Let X_n denote the ordering of the cards after I shuffle the cards n times.

   (a) Show that $X_0, X_1, \ldots$ is a Markov Chain and write down the state space. (1 point)

   (b) Which states are transient if any and which states are recurrent if any? (1 point)

(Hint: Think whether the chain is irreducible and then use a result from the lecture)

(c) What is the period of any given state? (1 point)

(Hint: This will follow from a (rather nice) fact about orderings or permutations. You cannot get back a permutation or ordering from an odd number of swaps. For example, if you start with the ordering 1234 you cannot make an odd number of swaps and get back to this ordering.)

(d) Write down the transition matrix Q. (1 point)

Hint: Just writing one row is sufficient. You can say the other rows follow a similar pattern.

(e) What happens to the n step transition matrix Q^n? Calculate it on a computer and then interpret what this possibly means. (1 point)

Hint: This will require coding. You can do this first when the number of cards is 3 so the state space is small. You can then make an educated guess when the number of cards is 4 and extrapolate it to 52. :)

3. **Not a Markov Chain**(3 points)

   A Markov chain has two states, A and B, with transitions as follows. If the MC is at state A then it goes to B with probability 1/2 and stays at A with probability 1/2. If the Markov Chain is at state B it comes back to state A with probability 1.

   Suppose we do not get to observe this Markov chain, which we will call $X_0, X_1, X_2, \dots$. Instead, whenever the chain transitions from A back to A, we observe a 0, and whenever it changes states, we observe a 1. Let the sequence of 0s and 1s be called $Y_0, Y_1, Y_2, \dots$. For example, if the X chain starts out as $AABABAA$ then the Y chain starts out at $0, 1, 1, 1, 1, 0$.

   (a) Show that the sequence of random variables $Y_0, Y_1, Y_2, \dots$ is not a Markov chain. (1 point)

   (Hint: Think of the conditional distribution of $Y_4 | Y_3 = 1, Y_2 = 0, Y_1 = 0, Y_0 = 0$ and the conditional distribution of $Y_4 | Y_3 = 1, Y_2 = 1, Y_1 = 0, Y_0 = 0$ say. If the markov property were to hold how would these two conditional probabilities be related?

   (b) In the rainy sunny Markov Chain discussed in the lecture we enlarged the state space and created a Markov Chain to incorporate second order dependence. Show that such a thing cannot be done here. That is no matter how large m is,

   $$Z_n = \{the\ (n-m+1)st\ to\ nth\ terms\ of\ the\ Y\ chain\}$$

   is still not a Markov Chain. (2 points)

   Hint: Use a similar reasoning as in the above hint. Show it for $m = 2$ and then observe a similar logic can be given for any $m > 2$.

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

1. the problem is same as A starts out with ki dollars. B starts out with $k(N-i)$ dollars, each bet is 1 dollar.

$P_{ki} = P_{ki+1} \cdot p + P_{ki-1}\, q. \quad P_0 = 0, \quad P_{kN} = 1.$

$$\Rightarrow \quad P_{ki} = \begin{cases} \dfrac{1-(\frac{q}{p})^{ki}}{1-(\frac{q}{p})^{kN}}, & p \neq \frac{1}{2} \\[2ex] \dfrac{i}{N}, & p = \frac{1}{2} \end{cases}$$

prob of A wins is $\begin{cases} \dfrac{1-(\frac{1-p}{p})^{ki}}{1-(\frac{1-p}{p})^{kN}}, & p \neq \frac{1}{2} \\[2ex] \dfrac{i}{N}, & p = \frac{1}{2} \end{cases}$

when $k \to \infty$, prob of A wins is $\begin{cases} 1, & \frac{1}{2} < p < 1 \\[1ex] \dfrac{i}{N}, & p = \frac{1}{2} \\[1ex] 0, & 0 < p < \frac{1}{2} \end{cases}$

2. (1) X_{n+1} is got from X_n. so it only depends on X_n. Hence, $X_0, X_1, \ldots$ is a Markov Chain.

state space $\Omega = \{(i, j, k, l),\ i \in \{1,2,3,4\}.$

$j \in \{1,2,3,4\} \setminus \{i\},\ k \in \{1,2,3,4\} \setminus \{i,j\}$

$,\ l \in \{1,2,3,4\} \setminus \{i,j,k\}\ \}.$

(2) the chain is irreducible. so all states are recurrent.

(3) $\{1,2,3,4\} \longrightarrow \{2,1,3,4\} \longrightarrow \{1,2,3,4\}.$

the gcd is 2. $\Rightarrow$ period is 2 for any state.

(4) $\binom{4}{2} = \frac{4\times 3}{2} = 6.$

first row:

$0,\ \frac{1}{6},\ 0,\ \frac{1}{6}\ \cdots\cdots$

(has six $\frac{1}{6}$, eighteen. 0 in totally)

other rows are similar. $Q_{ij} = \begin{cases} \frac{1}{6}, & \text{if } j \text{ can be transfered from } i \text{ by one step.} \\ 0, & \text{others.} \end{cases}$

Q is also symmetric

(5) $Q^n = \begin{cases} Q & \text{for all odd } n \\ Q' & \text{for all even } n \end{cases}$

$$Q' = \frac{1}{6}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \vdots & 1 & \cdots & \vdots \\ \vdots & & \ddots & \\ 1 & \cdots & \cdots & 1 \end{bmatrix}_{24\times 24} - Q$$

means the transition of the Markov chain has a period of 2.

3. (a) $P(Y_{n+1}=1 \mid Y_n=1, Y_{n-1}=0) = 1.$

$P(Y_{n+1}=1 \mid Y_n=1, Y_{n-1}=1, Y_{n-2}=0) = \frac{1}{2}$

Hence Y_{n+1} not only depends on Y_n

$\Rightarrow Y_0, Y_1, Y_2 \ldots$ is not a Markov chain.

(b) $m=2$: $P(Z_{n+1}=(1,1) \mid Z_n=(1,1)\ Z_{n-1}=(0,1)\} = \frac{1}{2}$

$P(Z_{n+1}=(1,1) \mid Z_n=(1,1)\ Z_{n-1}=(1,1)\ Z_{n-2}=(0,1)\} = 1$

$\Rightarrow m=2$, Z_n is not a Markov Chain.

$m = 2n+1$, $n=1,2,\ldots$

$P(Z_{n+1}=(1,1,\ldots,1) / Z_n=(1,1,\ldots,1)\ Z_{n-1}=(0,1,\ldots 1)\ \} = 1$

$P(Z_{n+1}=(1,1,\ldots 1) / Z_n=(1,1\ldots 1), Z_{n-1}=(1,1\ldots,1), Z_{n-2}=(0,1\ldots 1)\} = \frac{1}{2}$

$\Rightarrow m$ is odd Z_n is not a Markov Chain

$m = 2n$, $n=1,2,\ldots$

$P(Z_{n+1}=(1,1,\ldots,1) / Z_n=(1,1,\ldots,1)\ Z_{n-1}=(0,1,\ldots 1)\ \} = \frac{1}{2}$

$P(Z_{n+1}=(1,1,\dots 1) \mid Z_n=(1,1\dots 1), Z_{n-1}=(1,1\dots 1,1), Z_{n-2}=(0,1\dots 1)\} = 1$

$\Longrightarrow$ m is even Z_n is not a Markov Chain

$\Longrightarrow$ Z_n is not a Markov Chain for all $m \geq 2$.