Definition 13.1.1 (1D Poisson process)**.** A sequence of arrivals in continuous time is a *Poisson process* with rate λ if the following conditions hold:

1. The number of arrivals in an interval of length t is distributed $\text{Pois}(\lambda t)$.
2. The numbers of arrivals in disjoint time intervals are independent.

$E = \lambda t$

$Var = \lambda t$

$N(t) \sim Pois(\lambda t) : P(N(t) = k) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

expected times of arrivals in one unit time.

Let $N(t)$ be number of arrivals in $(0, t]$.

$N(t_2) - N(t_1)$ is # arrivals in $(t_1, t_2]$.

Let T_j be the time of jth arrival

$T_1 > t$ is same as $N(t) = 0$.

$\Rightarrow P(T_1 > t) = P(N(t) = 0) = e^{-\lambda t}$

$\Rightarrow T_1 \sim Expo(\lambda)$

$\Rightarrow T_j - T_{j-1} \sim Expo(\lambda)$

$T_j \sim Gamma(j, \lambda)$

$T \sim Expo(\lambda) : f_T(x) = \lambda e^{-\lambda x}$

$F_T(x) = 1 - e^{-\lambda x}$

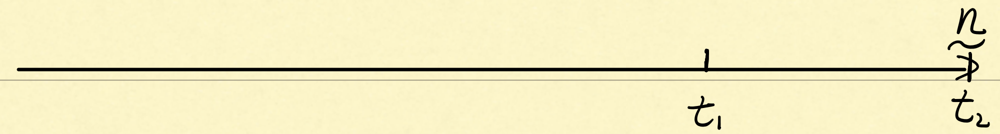
$F_{\lambda T}(x) = P(\lambda T \leq x) = P(T \leq \frac{x}{\lambda})$

$= F_T(\frac{x}{\lambda}) = 1 - e^{-x}$

$\Rightarrow \lambda T \sim Expo(1)$.

Theorem 13.2.1 (Conditional counts)**.** Let $(N(t) : t > 0)$ be a Poisson process with rate λ, and $t_1 < t_2$. The conditional distribution of $N(t_1)$ given $N(t_2) = n$ is

$$N(t_1) \mid N(t_2) = n \sim \text{Bin}\left(n, \frac{t_1}{t_2}\right).$$



n 散落在 $[0, t_2]$ 上的概率每处均等 $= \frac{1}{t_2}$.

散落在 $[0, t_1]$ 上的概率为 $\frac{t_1}{t_2}$

Proof: $P(t_1 = x \mid t_2 = n) = \dfrac{P(N(t_1)=x)\,P(N(t_2-t_1)=n-x)}{P(N(t_2)=n)}$

$$= \frac{\dfrac{(\cancel{\lambda} t_1)^x}{x!} \cancel{e^{-\lambda t_1}} \dfrac{[\cancel{\lambda}(t_2-t_1)]^{(n-x)}}{(n-x)!} \cancel{e^{-\lambda(t_2-t_1)}}}{\dfrac{(\cancel{\lambda} t_2)^n}{n!} \cancel{e^{-\lambda t_2}}}$$

$$= \frac{n!}{x!(n-x)!} \left(\frac{t_1}{t_2}\right)^x \left(\frac{t_2-t_1}{t_2}\right)^{n-x}$$

$$= \binom{n}{x} \left(\frac{t_1}{t_2}\right)^x \left(1-\frac{t_1}{t_2}\right)^{n-x}$$

$$\sim Bin\left(n, \frac{t_1}{t_2}\right)$$

Proposition 13.2.2. In a Poisson process of rate λ, conditional on $N(t) = 1$, the first arrival time T_1 has the $\text{Unif}(0, t)$ distribution.

Proof: $P(T_1 \le s \mid N(t)=1) = \dfrac{P(N(s)=1)P(N(t-s)=0)}{P(N(t)=1)}$

c.d.f.

$$= \frac{\lambda s \cancel{e^{-\lambda s}} \cancel{e^{-\lambda(t-s)}}}{\lambda t \cancel{e^{-\lambda t}}} = \frac{s}{t}$$

Theorem 13.2.3 (Conditional times)**.** In a Poisson process of rate λ, conditional on $N(t) = n$, the joint distribution of the arrival times $T_1, \ldots, T_n$ is the same as the joint distribution of the order statistics of n i.i.d. $\text{Unif}(0, t)$ r.v.s.

order statistics of Unif(0,1) are Betas.

so $\frac{T_j}{t} \sim Beta(j, n-j+1)$.

Theorem 13.2.6 (Superposition)**.** Let $(N_1(t) : t > 0)$ and $(N_2(t) : t > 0)$ be independent Poisson processes with rates λ_1 and λ_2, respectively. Then the combined process $N(t) = N_1(t) + N_2(t)$ is a Poisson process with rate $\lambda_1 + \lambda_2$.

Theorem 13.2.8 (Probability of type-1 event before type-2 event)**.** Consider two independent Poisson processes: a Poisson process of type-1 arrivals, with rate λ_1, and a Poisson process of type-2 arrivals, with rate λ_2. In the superposition of these two processes, the probability of the first arrival being type-1 is $\lambda_1/(\lambda_1 + \lambda_2)$.

Proof: $T \sim \text{Expo}(\lambda_1)$ $V \sim \text{Expo}(\lambda_2)$.

$\tilde{T} = \lambda_1 T \sim \text{Expo}(1)$ $\tilde{V} = \lambda_2 V \sim \text{Expo}(1)$.

$$P(T \leq V) = P\left(\frac{\tilde{T}}{\lambda_1} \leq \frac{\tilde{V}}{\lambda_2}\right)$$

$$= P\left(\frac{\tilde{T}}{\tilde{T}+\tilde{V}} \leq \frac{\tilde{V}}{\tilde{T}+\tilde{V}} \cdot \frac{\lambda_1}{\lambda_2}\right)$$

$\tilde{T}, \tilde{V} \sim \text{Gamma}(0,1) \Rightarrow \frac{\tilde{T}}{\tilde{T}+\tilde{V}} \sim \text{Beta}(1,1)$ i.e. $\text{Unif}(0,1)$.

$$= P\left(U \leq (1-U) \cdot \frac{\lambda_1}{\lambda_2}\right)$$

$$= P\left(U \leq \frac{\lambda_1}{\lambda_1+\lambda_2}\right)$$

$$= \frac{\lambda_1}{\lambda_1+\lambda_2}$$

Theorem 13.2.11 (Projection of superposition into discrete time). Consider the superposition $(N(t) : t > 0)$ of two independent Poisson processes with rates λ_1 and λ_2. For $j = 1, 2, \ldots$, let I_j be the indicator of the jth event being from the Poisson process with rate λ_1. Then the I_j are i.i.d. $\text{Bern}(\lambda_1/(\lambda_1 + \lambda_2))$.

Bernoulli distribution: $f_X(x) = \begin{cases} \frac{\lambda_1}{\lambda_1+\lambda_2}, & \text{if } x=1 \\ \frac{\lambda_2}{\lambda_1+\lambda_2}, & \text{if } x=0 \end{cases}$

Theorem 13.2.12 (Exponential mixture of Poissons is Geometric). Suppose that $X \sim \text{Expo}(\lambda)$ and $Y|X = x \sim \text{Pois}(x)$. Then $Y \sim \text{Geom}(\lambda/(\lambda + 1))$.

Proof. As with the competing risks theorem, we embed X and Y into Poisson processes. Consider two independent Poisson processes, a process of failures arriving at rate 1 and another of successes arriving at rate λ. Let X be the time of the first success; then $X \sim \text{Expo}(\lambda)$. Let Y be the number of failures before the time of the first success. By the definition of a Poisson process with rate 1, $Y|X = x \sim \text{Pois}(x)$. Therefore X and Y satisfy the conditions of the theorem. $\lambda=1$ $t=x$.

To get the marginal distribution of Y, strip out the continuous-time information! In discrete time we have i.i.d. Bernoulli trials with success probability $\lambda/(\lambda + 1)$, and Y is defined as the number of failures before the first success, so by the story of the Geometric distribution, $Y \sim \text{Geom}(\lambda/(\lambda + 1))$. ∎

Theorem 13.2.13 (Gamma mixture of Poissons is Negative Binomial)**.** Suppose that $X \sim \text{Gamma}(r, \lambda)$ and $Y|X = x \sim \text{Pois}(x)$. Then $Y \sim \text{NBin}(r, \lambda/(\lambda + 1))$.

Proof. Consider two independent Poisson processes, a process of failures arriving at rate 1 and another of successes arriving at rate λ. Let X be the time of the rth success, so $X \sim \text{Gamma}(r, \lambda)$. Let Y be the number of failures before the time of the rth success. Then $Y|X = x \sim \text{Pois}(x)$ by definition of Poisson process. We have that Y is the number of failures before the rth success in a sequence of i.i.d. Bernoulli trials with success probability $\lambda/(\lambda + 1)$, so $Y \sim \text{NBin}(r, \lambda/(\lambda + 1))$. ■

$$f(Y) = \binom{y+r-1}{y}\left(\frac{\lambda}{\lambda+1}\right)^r \left(\frac{1}{\lambda+1}\right)^y$$

Negative Binomial 在第 r 次成功时,实验的失败数 x

$$f(k; r, p) = \underbrace{\binom{k+r-1}{r-1}}_{\substack{\text{最后一次前共 } k+r-1 \text{ 次} \\ \text{其中 } r-1 \text{ 次失败.}}} p^r \cdot (1-p)^k$$