Practice Midterm 2

STAT 410 FALL 2020

Thursday, November 19, 2020

Name & UIN ....

Instructions:

1. You should finish this exam in 1 hour and 50 minutes.
2. You can use any result shown in the lectures but you have to clearly state the result here.
3. Show all your supporting work. If you just write the answer you may not get a lot of credit.
4. There are 4 questions each worth 20 points. The maximum you can score is 80. This exam is worth 75 points and the remianing 5 points are for extra credit.

Problem 1: One of two similar looking coins is picked from a hat randomly, where one coin has probability p_1 of Heads and the other has probability p_2 of Heads. I keep on tossing this coin till I get n heads. Let X be the number of tosses till I get n heads. Find the mean and variance of X. (10 + 10 points)

Expection of 1 head $E(X_1 | \text{choose } i) = \frac{1}{p_i}$

n heads are indep with each other

$$\Rightarrow E\left(\sum_{j=1}^{n} X_j \,\middle|\, \text{choose } i\right) = \frac{n}{p_i}$$

$$\Rightarrow E_n\left(\sum_{j=1}^{n} X_j\right) = \frac{1}{2}\left(\frac{n}{p_1} + \frac{n}{p_2}\right)$$

Varianceof 1 head $Var(X_1 | \text{choose } i) = \frac{1-p_i}{p_i^2}$

n heads are indep with each other

$$\Rightarrow Var\left(\sum_{j=1}^{n} X_j \,\middle|\, \text{choose } i\right) = \frac{n(1-p_i)}{p_i^2}$$

$$Var\left(\sum_{j=1}^{n} X_j\right) = E\left(Var\left(\sum_{j=1}^{n} X_j \,\middle|\, \text{choose } i\right)\right) + Var\left(E\left(\sum_{j=1}^{n} X_j \,\middle|\, \text{choose } i\right)\right)$$

$$= \frac{1}{2}\left(\frac{n(1-p_1)}{p_1^2} + \frac{n(1-p_2)}{p_2^2}\right) + \frac{\left[n\left(\frac{1}{p_1} - \frac{1}{p_2}\right)\right]^2}{4}$$

Problem 2:

(a) Let $X_1, \ldots, X_n$ be i.i.d with mean μ and variance σ^2. What is the minimum sample size n (as a specific number) that will ensure that there is at least a 0.9 chance that the sample mean will be within 1 standard deviation of the true mean μ. (10 points)

(b) Let X be a random variable with mean μ. Show that $E|X| \geq |\mu|$. (10 points)

(a) $E(\bar{X}) = \mu \quad Var(\bar{X}) = \frac{\sigma^2}{n}$

$$P\left((\bar{X} - \mu)^2 \cdot \frac{1}{\sigma^2} \leq 1\right)$$

$$= 1 - P(|\bar{X} - \mu| \geq \sigma).$$

$$\geq 1 - \frac{\frac{\sigma^2}{n}}{\sigma^2} = 1 - \frac{1}{n} \geq 0.9$$

minimum $n = 10$.

(b) $f(x) = |x|$ is a convex function.

$$\Rightarrow E(f(x)) \geq f(E(x))$$

$$\Rightarrow E(|x|) \geq |\mu|$$

Problem 3: Let $E_1, E_2, \dots, E_{60}$ be i.i.d $Exp(1)$ random variables and and $X = E_1 + E_2 + \cdots + E_{60}$.

(a) What is the exact distribution of X? Which important distribution is the distribution of X very close to? Specify what the parameters are, and state which theorem justifies your choice. (10 points)

(b) Give a simple but accurate approximation for $P(X > 30)$. You can leave the answer in terms of the CDF of a known distribution. (10 points)

(a) $X \sim \text{Gamma}(60, 1)$.

$EX = 60 \quad \text{Var}X = 60$

close to $N(60, 60)$.

Central Limit Theorem

(b) $P(X > 30) = P\left(\frac{X-60}{\sqrt{60}} > -\frac{30}{\sqrt{60}}\right)$

$= 1 - \Phi\left(-\frac{30}{\sqrt{60}}\right)$

$= \Phi\left(\frac{30}{\sqrt{60}}\right)$

Problem 4: Two lost brothers Nattu and Sattu move independently back and forth between two towns looking for each other. At each time step, Nattu moves from the current town to the other town with probability 0.7. Starting from town 1, Sattu moves to Town 2 with probability 0.4 (and remains otherwise). Starting from town 2, Sattu moves to town 1 with probability 0.5 (and remains otherwise).

(a) Let X_n, Y_n be random variables indicating the current towns the two brothers Nattu and Sattu find themselves in after n steps. Find the stationary distributions of the markov chains X_n and Y_n.(6 points)

(b) Note that there are 4 possible (X_n, Y_n) states: both in town 1, Nattu in town 1 and Sattu in town 2, Nattu in town 2 and Sattu in town 1, and both in town 2. Number these cases $1, 2, 3, 4$ respectively, and let Z_n be the number representing the (X_n, Y_n) state at time n. Is $Z_0, Z_1, Z_2, \ldots$ a Markov chain? If so, what is the state space and transition matrix? (6 points)

(c) Now suppose that the brothers find each other as soon as they are both in the same town. We wish to know the expected time (number of steps taken) until this happens for two initial configurations: when Nattu starts in town 1 and Sattu starts in room 2, and vice versa. Set up a system of two linear equations in two unknowns whose solution is the desired values. (8 points)

$$P_X = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.3 & 0.7 \\ 2 & 0.7 & 0.3 \end{array} \qquad P_Y = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.6 & 0.4 \\ 2 & 0.5 & 0.5 \end{array}$$

(a) $S_X \cdot P_X = S_X$ $\qquad$ $S_Y \cdot P_Y = S_Y$

$\Rightarrow S_X = [\frac{1}{2}, \frac{1}{2}]$ $\qquad$ $\Rightarrow S_Y = [\frac{5}{9}, \frac{4}{9}]$

(b) Yes. Z_n's state only depend on Z_{n-1}

$\Omega = (1, 2, 3, 4)$.

$$P_Z = \begin{array}{c} \\ (1,1) \\ (1,2) \\ (2,2) \\ (2,1) \end{array} \begin{array}{cccc} (1,1) & (1,2) & (2,2) & (2,1) \\ 0.18 & 0.12 & 0.28 & 0.42 \\ 0.15 & 0.15 & 0.35 & 0.35 \\ 0.35 & 0.35 & 0.15 & 0.15 \\ 0.42 & 0.28 & 0.12 & 0.18 \end{array}$$

(c) n_1 : N in town 1, S in town 2.

n_2 : N in town 2, S in town 1.

$$\begin{cases} n_1 = 1 + 0.15n_1 + 0.35n_2 \\ n_2 = 1 + 0.28n_1 + 0.18n_2 \end{cases}$$

$$0.85n_2 - 0.35n_1 = 0.5$$

$$0.85n_1 - 0.35n_2 = 0.5$$