Practice for Midterm 1

STAT 410 FALL 2020

Friday, October 2^{nd} 2019

Name & UIN ....

Instructions:

1. You should finish this exam in 1 hour and 50 minutes.
2. You can use any result shown in the lectures but you have to clearly state the result here.
3. Show all your supporting work. If you just write the answer you may not get a lot of credit.

Problem 1: Fred is going to get himself tested for coronavirus. Given that he has the virus, the probability that the test returns positive is 0.95. Given that he does not have the virus, the probability that the test returns negative is also 0.99. Assume that the unconditional probability of Fred having the virus is 0.1. Fred tests himself twice in two days. Assume that the test results are independent given the real corona virus status of Fred.

(a) Find the probability that Fred tests negative both times. (6 points)

(b) Is whether Fred tested positive in the first test independent of the outcome in the second test?(6 points)

(c) Both the tests returned positive. Given this information, find the probability that Fred has the virus. (8 points)

(a) $0.1 \times 0.05 \times 0.05 + 0.9 \times 0.99 \times 0.99$

(b). $P(X_2 = P \mid X_1 = P) = \dfrac{P(X_1 = P \; X_2 = P)}{P(X_1 = P)} = \dfrac{0.1 \times 0.95 \times 0.95 + 0.9 \times 0.01^2}{0.1 \times 0.95 + 0.9 \times 0.01}$

(c) $P(V \mid X_1 = P \; X_2 = P) = \dfrac{P(X_1 = P, X_2 = P \mid V) P(V)}{P(X_1 = P, X_2 = P)} = \dfrac{0.95 \times 0.95 \times 0.1}{0.1 \times 0.95 \times 0.95 + 0.9 \times 0.01 \times 0.01}$

$= 0.9990038$

Problem 2: (X, Y) is a uniformly random point on the circle with radius 1. This means that the joint pdf should be the same over the whole circle.

(a) Find the joint pdf of X and Y. Be sure to specify the support. (6 points)

(b) Find the marginal pdf of Y. (6 points)

(c) Find the conditional pdf of X given $Y = y$. What is the name of this distribution? (8 points)

(a) $\iint_{x^2+y^2\le 1} dx\,dy = \pi$ $\qquad \iint_{x^2+y^2\le 1} \frac{1}{\pi}\, dx\,dy = 1$

$f_{X,Y}(x,y) = \frac{1}{\pi}$

(b) $\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f_{X,Y}(x,y)\, dx = \frac{2\sqrt{1-y^2}}{\pi}$

(c) $f_{X|Y}(x \mid Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{\frac{1}{\pi}}{\frac{2\sqrt{1-y^2}}{\pi}} = \frac{1}{2\sqrt{1-y^2}}$

Uniform distribution $\quad \text{Unif}(-\sqrt{1-y^2}, \sqrt{1-y^2})$

Problem 3: Let $Z \sim N(0,1)$.

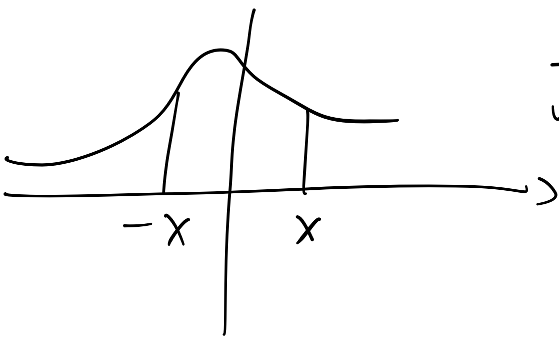
(a) Find the distribution of $|Z|$. (10 points)

(b) Are Z and $|Z|$ uncorrelated? Are Z and $|Z|$ independent? (10 points)

(a) $F_{|Z|}(x) = P(|Z|<x) = P(-x<Z<x)$

$= P(Z<x) - P(Z<-x)$

$= F_Z(x) - F_Z(-x)$



$$f_{|Z|}(x) = f_Z(x) + f_Z(-x) = \frac{2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(b) $f_{Z,|Z|}(x,y) = 0$, when $x \neq -y$ and $x \neq y$

$\neq f_Z(x) f_{|Z|}(y)$ dependent

$Cov(Z, |Z|) = E(Z|Z|) - E(Z)E(|Z|)$.

$= 0$ uncorrelated.

Problem 4:

(a) As a function of U, create a random variable X with CDF $F(x) = 1 - \exp(-x^3)$ for $x > 0$. (10 points)

(b) Let $X \sim Gamma(a, \lambda)$ and $Y \sim Gamma(b, \lambda)$ be independent with a, b positive integers. Show that $X + Y \sim Gamma(a + b, \lambda)$.(10 points)

(a) $X = F^{-1}(U)$

$$y = 1 - e^{-x^3}$$

$$e^{-x^3} = 1 - y.$$

$$-x^3 = \ln(1-y)$$

$$x^3 = \ln \frac{1}{1-y}$$

$$x = \left(\ln \frac{1}{1-y}\right)^{\frac{1}{3}}$$

$$\Rightarrow F^{-1}(U) = \left(\ln \frac{1}{1-U}\right)^{\frac{1}{3}}$$

(b)

$$M_X(t) = E(e^{xt})$$

$$= \int_0^{+\infty} \frac{1}{\Gamma(a)} \lambda^a x^{a-1} e^{(-\lambda+t)x} \, dx.$$

$$= \frac{\lambda^a}{(\lambda - t)^a}$$

$$M_Y(t) = \frac{\lambda^b}{(\lambda - t)^b}$$