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Homework 4

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Convolution
- Beta Distribution
- Gamma Distribution

Exercises

1. **Double Exponential**(3 points)

   Let X and Y be i.i.d. $Exp(\lambda)$. Use the convolution integral formula to show that the PDF of $L = X - Y$ is $f(t) = \frac{\lambda}{2} exp(-\lambda|t|)$ for all real t; this is known as the Laplace distribution or the Double Exponential distribution with parameter λ. (Hint: It is enough to show the above for $\lambda = 1$ and find the p.d.f of $X - Y$ for $t > 0$ only. Why?)

2. **Convolution of independent Normals**(3 points)

   Use the convolution integral formula to show that if $X \sim N(\mu_1, \sigma^2)$ and $Y \sim N(\mu_2, \sigma^2)$ are independent, then $T = X + Y \sim N(\mu_1 + \mu_2, 2\,\sigma^2)$. (Hint: First argue why it is enough to consider the case when $\mu_1 = \mu_2 = 0$ and $\sigma = 1$. Then do the convolution integral. At some point during the integral, you have to complete the square.)

3. **Beta and Gamma Distribution**(3 points)

   Let $B \sim Beta(a, b)$. Find the distribution of $1 - B$ in two ways: (a) using a change of variables (1 point) (b) using the characterization of Beta involving two independent Gamma random variables. That is, we can represent B as $X/(X + Y)$ where X, Y are independent and follow $Gamma(a, \lambda)$ and $Gamma(b, \lambda)$ respectively.(2 points)

4. **Variance of Beta Distribution**(2 points) Let $B \sim Beta(a,b)$. Show that $Var(B) = \frac{ab}{(a+b)^2(a+b+1)}$. (Hint: Use integration and use the property of the gamma function that $\Gamma(a+1) = a\Gamma(a)$ for all $a > 0$.)

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

1. $f_L(t) = \int_{-t}^{\infty} f_X(y+t) f_Y(y)\, dy$

$$= \begin{cases} \int_0^{\infty} \lambda e^{-\lambda(y+t)} \cdot \lambda e^{-\lambda y}\, dy = \frac{\lambda}{2} e^{-\lambda t} \int_0^{\infty} 2\lambda e^{-2\lambda y}\, dy = \frac{\lambda}{2} e^{-\lambda t}, & t > 0 \\ \int_{-t}^{\infty} \lambda e^{-\lambda(y+t)} \lambda e^{-\lambda y} dy = \frac{\lambda}{2} e^{-\lambda t} \cdot \int_{-t}^{\infty} 2\lambda e^{-2\lambda y} dy = \frac{\lambda}{2} e^{-\lambda t} \cdot \left(-e^{-2\lambda y}\Big|_{-t}^{\infty}\right) = \frac{\lambda}{2} e^{\lambda t}, & t < 0. \end{cases}$$

$$\Longrightarrow f_L(t) = \frac{\lambda}{2} e^{-\lambda |t|}$$

2. Let $X' = \frac{X - \mu_1}{\sigma} \sim N(0,1)$, $Y' = \frac{Y - \mu_2}{\sigma} \sim N(0,1)$.

$$T' = X' + Y' = \frac{X + Y - \mu_1 - \mu_2}{\sigma} = \frac{T - \mu_1 - \mu_2}{\sigma}$$

$$f_{T'}(t) = \int_{-\infty}^{+\infty} f_{X'}(t-y) f_{Y'}(y)\, dy = \int_{-\infty}^{+\infty} \frac{1}{2\pi} e^{-\frac{(t-y)^2 + y^2}{2}} dy$$

$$= e^{\frac{-t^2}{4}} \int_{-\infty}^{+\infty} \frac{1}{2\sqrt{2}\pi} e^{-\frac{(\sqrt{2}y - \frac{t}{\sqrt{2}})^2}{2}} d\sqrt{2}y$$

$$= \frac{1}{2\sqrt{\pi}} e^{-\frac{t^2}{4}}$$

Hence $T' \sim N(0, 2)$

$\Rightarrow T \sim N(\mu_1+\mu_2, 2\sigma^2)$.

3. (a) $f_{1-B}(x) = f_B(1-x) \left| \frac{dB}{d(1-B)} \right|$

$$= \frac{(a+b-1)!}{(a-1)!(b-1)!} x^{b-1}(1-x)^{a-1}$$

$\Rightarrow 1-B \sim Beta(b, a)$.

(b) $B = \frac{X}{X+Y}$, $1-B = \frac{Y}{X+Y}$

Let $g: \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} W \\ T \end{pmatrix} = \begin{pmatrix} \frac{Y}{X+Y} \\ Y \end{pmatrix}$

$w = \frac{y}{x+y}$

$t = y$

$(x+t)w = t$

$x = \frac{t}{w} - t$

$$J_{g^{-1}}(w,t) = \left| \frac{\partial(x,y)}{\partial(w,t)} \right| = \begin{vmatrix} -\frac{t}{w^2} & \frac{1}{w} - 1 \\ 0 & 1 \end{vmatrix} = \frac{t}{w^2}$$

$$f_{W,T}(w,t) = f_{X,Y}(x,y) \cdot \left| \frac{\partial(x,y)}{\partial(w,t)} \right|$$

$$= \frac{t}{w^2} \frac{1}{\Gamma(a)\Gamma(b)} \lambda^{a+b} \left(\frac{t}{w} - t\right)^{a-1} t^{b-1} e^{-\lambda \frac{t}{w}}$$

$$f_W(w) = \int_0^{+\infty} f_{W,T}(w,t)\, dt$$

$$= \frac{\lambda^{a+b}\left(\frac{1}{w}-1\right)^{a-1}}{\Gamma(a)\Gamma(b)\, w^2} \int_{-\infty}^{+\infty} t^{a+b-1} e^{-\lambda \frac{t}{w}}\, dt$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \omega^{b-1}(1-\omega)^{a-1} \int_{-\infty}^{+\infty} \frac{1}{\Gamma(a+b)} \left(\frac{\lambda}{\omega}\right)^{a+b} t^{a+b-1} e^{-\frac{\lambda}{\omega}t} dt$$

$$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \omega^{b-1}(1-\omega)^{a-1}$$

$$= \frac{\omega^{b-1}(1-\omega)^{a-1}}{\beta(a+b)} \Rightarrow W = \frac{Y}{X+Y} \sim Beta(b,a)$$

$$\Rightarrow 1-B \sim Beta(b,a).$$

4. $$\operatorname{Var} B = \int_0^1 \left(x - \frac{a}{a+b}\right)^2 \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} dx.$$

$$= \frac{\Gamma(a+2)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+2)} \int_0^1 x^{a+1}(1-x)^{b-1} \frac{\Gamma(a+b+2)}{\Gamma(a+2)\Gamma(b)} dx$$

$$+\left(\frac{a}{a+b}\right)^2 - \frac{2a}{a+b} \frac{\Gamma(a+1)\Gamma(a+b)}{\Gamma(a)\Gamma(a+b+1)} \int_0^1 x^{a}(1-x)^{b-1} \frac{\Gamma(a+b+1)}{\Gamma(a+1)\Gamma(b)} dx$$

$$= \frac{(a+1)a}{(a+b+1)(a+b)} + \left(\frac{a}{a+b}\right)^2 - \frac{2a}{a+b} \cdot \frac{a}{a+b}$$

$$= \frac{ab}{(a+b+1)(a+b)^2}$$