$$p_i = p \cdot p_{i+1} + q \cdot p_{i-1}. \qquad (q = 1-p,\ p \neq 0).$$

Step: ①. guess a sol of the form: $p_i = x^i$

$$\Longrightarrow x^i = p \cdot x^{i+1} + q \cdot x^{i-1}$$

$$\Longrightarrow x = p \cdot x^2 + q$$

i.e. $\underline{p x^2 - x + q = 0}$ characteristic equation

②.

Characteristic equation have $\begin{cases} \text{one root} \\ \text{two roots.} \end{cases}$

if two distinct roots r_1 and r_2, then the solution of the form: $p_i = a r_1^i + b r_2^i$

if only one distinct root r: $p_i = a r^i + b i r^i$

in our case: $x = \dfrac{1 \pm \sqrt{1-4pq}}{2p} = \dfrac{1 \pm \sqrt{1-4p(1-p)}}{2p} = \dfrac{1 \pm |2p-1|}{2p}$

$$= 1, \text{ or } \frac{q}{p}$$

$$\Longrightarrow p_i = \begin{cases} a + b\left(\frac{q}{p}\right)^i, & p \neq q \\ a + bi, & p = q \end{cases}$$

$\Longrightarrow$ solve a, b by two known points.