Joint Distribution

Discrete:

Joint C.D.F of X, Y is $F_{X,Y}: \mathbb{R}^2 \to [0,1]$.

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}.$$

Joint P.m.f $P_{X,Y}(x,y) = P(X=x, Y=y)$.

$$\sum_x \sum_y P_{X,Y}(x,y) = 1.$$

$$\sum\sum_{(x,y)\in A} P_{X,Y}(x,y) = P((X,Y)\in A)$$

Marginal P.m.f For discrete random variables X and Y. the marginal p.m.f of X is

$$P(X=x) = \sum_y P(X=x, Y=y).$$

$$E(g(X,Y)) = \sum_{\text{all } x} \sum_{\text{all } y} g(x,y)\,p(x,y)$$

Continuous

Joint p.d.f. $f(x,y) = \dfrac{\partial^2 F(x,y)}{\partial x \partial y}$, $P((X,Y)\in A) = \iint_A f(x,y)\,dx\,dy$.

Marginal p.d.f $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$.

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\cdot f(x,y)\,dx\,dy.$$

Moment-generating Functions

$M_{X,Y}(t_1, t_2) = E(e^{t_1X+t_2Y})$

$M_{X,Y}(t_1, 0) = M_X(t_1)$

independent

R.V. X and Y are independent if and only if

discrete: $p(x, y) = p_X(x) \cdot p_Y(y)$, $\forall x, y$.

Continuous: $f(x, y) = f_X(x) f_Y(y)$, $\forall x, y$.

c.d.f: $F(x, y) = F_X(x) F_Y(y)$, $\forall x, y$.

Covariance and Correlation Coefficient.

$\sigma_{XY} = Cov(X, Y) = E[(X-\mu_X)(Y-\mu_Y)] = E(XY) - \mu_X\mu_Y$

$Cov(aX+b, Y) = a\, Cov(X, Y)$.

$Cov(X+Y, W) = Cov(X, W) + Cov(Y, W)$

$Cov(aX+bY, cX+dY) = ac\, Var(X) + (ad+bc) Cov(X, Y)$
$+ bd\, Var(Y)$

$$Var(aX+bY)=a^2 Var(X)+2ab\,Cov(X,Y)+b^2 Var(Y)$$

$$\rho_{X,Y}=\frac{\sigma_{X,Y}}{\sigma_X\sigma_Y}=\frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}=E\left[\left(\frac{X-\mu_X}{\sigma_X}\right)\left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]$$

$-1\leq\rho_{X,Y}\leq 1.$

Example: Cauchy distribution.

$(X,Y)\overset{i.i.d}{\sim} N(0,1).$

$T=\frac{X}{Y}$, find c.d.f and p.d.f of T.

$$F_T(t)=P(T\leq t)=P\left(\frac{X}{Y}\leq t\right)$$

distribution of $\frac{X}{Y}$ is same as $\frac{X}{|Y|}$

$$=P\left(\frac{X}{|Y|}\leq t\right).$$

$$=P(X\leq t|Y|)$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{t|y|}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\times\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dx\,dy.$$

$$=\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\left(\int_{-\infty}^{t|y|}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx\right)dy.$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \Phi(t|y|)\, dy.$$

$$f_T(t) = \frac{d}{dt} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \Phi(t|y|)\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \phi(t|y|)\, |y|\, dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi} e^{-\frac{y^2(1+t^2)}{2}} |y|\, dy$$

$$= \frac{1}{\pi} \int_{0}^{\infty} e^{-\frac{y^2(1+t^2)}{2}} y\, dy$$

$$= \frac{1}{\pi(1+t^2)} \int_{0}^{\infty} e^{-\frac{y^2(1+t^2)}{2}} d\,\frac{y^2(1+t^2)}{2}$$

$$= \frac{1}{\pi(1+t^2)}$$

this $\frac{X}{Y}$ called Cauchy distribution.