Wenxiao Yang.

Homework 10

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Stationary Distribution
- Reversibility

Exercises

1. **Stationary Distribution**(2 points)

   Consider the Markov Chain with two states A and B where from one state the probability of moving to the other state is p and the probability of staying put at the same state is $1-p$.

   (a) Find the stationary distribution of the chain. (1 point)

   (b) What is the limit of Q^n as $n \to \infty$ where Q is the transition matrix for the chain. (1 point)

2. **Three Independent Markov Chains**(3 points)

   Alice has 3 dogs: Bishi, Mishi and Rishi. Each dog independently explores the neighbourhood. Let X_n, Y_n, Z_n be the locations at time n of Bishi, Mishi and Rishi respectively, where time is assumed to be discrete and the number of possible locations is a finite number M. Their paths $X_0, X_1, X_2, \ldots; Y_0, Y_1, Y_2, \ldots$ and $Z_0, Z_1, Z_2, \ldots$ are independent Markov chains with the same stationary distribution s. Each dog starts out at a random location generated according to the stationary distribution.

   (a) Let state 0 be home (so s_0 is the stationary probability of the home state). Find the expected number of times that Rishi is at home, up to time 24, i.e., the expected number of how many of $X_0, X_1, \ldots, X_{24}$ are in state 0 (in terms of s_0). (1 point)

(b) If we want to track all 3 dogs simultaneously, we need to consider the vector of positions, (X_n, Y_n, Z_n). There are M^3 possible values for this vector; assume that each is assigned a number from 1 to M^3, e.g., if $M = 2$ we could encode the states $(0,0,0), (0,0,1), (0,1,0), ..., (1,1,1)$ as $1, 2, 3, ..., 8$ respectively. Let W_n be the number between 1 and M^3 representing (X_n, Y_n, Z_n). Determine whether $W_0, W_1, \ldots$ is a Markov chain. (1 point)

(c) Given that all 3 dogs start at home at time 0, find the expected time it will take for all 3 to be at home again at the same time. (1 point)

3. **Reversible Markov Chain**(3 points)

   There are two urns with a total of $2N$ distinguishable balls. Initially, the first urn has N white balls and the second urn has N black balls. At each stage, we pick a ball at random from each urn and interchange them. Let X_n be the number of black balls in the first urn at time n. This is a Markov chain on the state space $\{0, 1, ..., N\}$.

   (a) Give the transition probabilities of the chain. (1 point)

   (b) Show that $(s_0, s_1, \ldots, s_N)$ where

$$s_i = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}}$$

   is the stationary distribution, by verifying the reversibility condition. (2 points)

4. **Random Walk**(3 points)

   In chess, a king can move one square at a time in any direction. Suppose I am on a usual chessboard. At each move, the king chooses uniformly at random where to go, among the legal choices.

   a) Is the Markov chain irreducible and aperiodic? (1 point)

   b) Can you find a probability distribution for which this Markov chain is reversible? (1 point)

   Hint: The king is performing a random walk on a particular graph. Then use the result shown in the lecture about reversibility of random walks on graphs.

   The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

1. $Q = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix}$.

(a) $S = [\, S_1, \ S_2 \,]$

$$\begin{cases} (1-p)S_1 + pS_2 = S_1 \\ pS_1 + (1-p)S_2 = S_2 \\ S_1 + S_2 = 1. \end{cases}$$

$\Rightarrow \ S_1 = S_2 = \frac{1}{2}$

$\Rightarrow$ stationary distribution: $S = [\frac{1}{2}, \frac{1}{2}]$.

(b) if $p \in (0,1)$, $\lim\limits_{n \to \infty} Q^n = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

if $p = 0$ $\quad \lim\limits_{n \to \infty} Q^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

if $p = 1$ $\quad \lim\limits_{n \to \infty} Q^n = \begin{cases} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & n \text{ is odd} \\ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, & n \text{ is even.} \end{cases}$

2. (a) expected time: $25 S_0$

(b) Since $\{X_i\}, \{Y_i\}, \{Z_i\}$ are Markov chains, X_n only depends on X_{n-1}

Y_n only depends on Y_{n-1}

Z_n only depends on Z_{n-1}.

Hence, the distribution of (X_n, Y_n, Z_n)

only depends on $(X_{n-1}, Y_{n-1}, Z_{n-1})$.

Thus, $W_0, W_1, \cdots$ is a Markov Chain.

(c). Let K be the stationary distribution of $\{W_i\}$

k_0 is the stationary prob of $(0,0,0)$.

$$k_0 = S_0^3$$

expected time: $t_0 = \frac{1}{k_0} = \frac{1}{S_0^3}$

3. w: $N - X_n$
b: X_n

w: X_n
b: $N - X_n$

(a) $$P_{ij} = \begin{cases} \left(\frac{N-i}{N}\right)^2, & j = i+1 \\ \left(\frac{i}{N}\right)^2, & j = i-1 \\ 2\,\frac{i(N-i)}{N^2}, & j = i \\ 0, & \text{others} \end{cases}$$

(b) $j = i+1$, $$S_i \cdot P_{ij} = \frac{\binom{N}{i}\binom{N}{N-i}}{\binom{2N}{N}} \cdot \left(\frac{N-i}{N}\right)^2$$

$$S_j \cdot P_{ji} = \frac{\binom{N}{j}\binom{N}{N-j}}{\binom{2N}{N}} \cdot \left(\frac{j}{N}\right)^2$$

$$= \frac{N!}{j!(N-j)!} \cdot \frac{N!}{j!(N-j)!} \quad \frac{N!N!}{2N!} \cdot \frac{j^2}{N^2}$$

$$= \frac{N!}{(j-1)!(N-(j-1))!} \quad \frac{N!}{(j-1)!(N-(j-1))!} \quad \frac{N!N!}{2N!} \left(\frac{N-(j-1)}{N}\right)^2$$

$$= \frac{\binom{N}{j-1}\binom{N}{N-(j-1)}}{\binom{2N}{N}} \cdot \left(\frac{N-(j-1)}{N}\right)^2$$

$$= S_i P_{ij}$$

$j = i-1$. $\quad S_j P_{ji} = \frac{\binom{N}{j}\binom{N}{N-j}}{\binom{2N}{N}} \cdot \left(\frac{N-j}{N}\right)^2$

$$= \frac{\binom{N}{i-1}\binom{N}{N-(i-1)}}{\binom{2N}{N}} \cdot \left(\frac{N-(i-1)}{N}\right)^2$$

$$= S_i P_{ij}.$$

$j = i \qquad S_j = S_j \ . \ P_{ij} = P_{ji} \Rightarrow S_i P_{ij} = S_j P_{ji}$

$\Rightarrow (S_0, S_1 \ldots S_N)$ is stationary distribution.

4. (1) it is irreducible but not aperiodic

its period is 2.

(2) 4 cornor squares

24 edge squares.

36 center squares.

corner square has degree 3.

edge square has degree 5.

center square has degree 8.

$\Rightarrow$ $a = 1 \div (3\times4+24\times5+36\times8) = \frac{1}{420}$

$\Rightarrow$ Stationary probability of corner square is $\frac{1}{140}$

edge square is $\frac{1}{84}$

center square is $\frac{2}{105}$