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Homework 8

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Weak Law of Large Numbers
- Central Limit Theorem
- Chi Square and T distribution

Exercises

1. **Law of Large Numbers**(3 points)

   Let $X_1, X_2, ...$ be i.i.d. positive random variables with mean 1. Let $Y_1, Y_2, ...$ be i.i.d. positive random variables with mean 2. Show that $\frac{X_1+X_2+...X_n}{Y_1+Y_2+...Y_n}$ converges to 1/2 in probability. Does it matter whether the X_i are independent of the Y_j?

2. **Volatile Stock and Law of Large Numbers**(3 points)

   Suppose a very volatile stock rises 70 percent or drops 50 percent in price, with equal probabilities and with different days independent.

   (a) Suppose a hedge fund manager always invests half of her current fortune into the stock each day. Let Y_n be her fortune after n days, starting from an initial fortune of $Y_0 = 100$. What happens to Y_n as $n \to \infty$? (1 point)

   Hint: Think about what happens to $\frac{\log Y_n}{n}$.

   (b) More generally, suppose the hedge fund manager always invests a fraction $0 < \alpha < 1$ of her current fortune into the stock each day (in Part (a), we took $\alpha = 1/2$). With Y_0 and Y_n defined as in Part (a), find the function $g(\alpha)$ such that $\frac{\log Y_n}{n} \to g(\alpha)$ in probability as $n \to \infty$ land find for which α is $g(\alpha)$ maximized. (2 points)

3. **Normal Approximation to the Binomial**(3 points)

   There are n voters in an upcoming election in a certain country, where n is a large, even number. There are two candidates, A and B. Each voter chooses randomly whom to vote for, independently and with equal probabilities.

   (a) Use a Normal approximation to the Binomial (as shown in class) to get an approximation for the probability of a tie, in terms of the normal cdf Φ. (1 point)

   (b) Use a first-order Taylor expansion (linear approximation) to the approximation from Part (a) to show that the probability of a tie is approximately $1/\sqrt{cn}$, where c is a constant (which you should specify). (2 points)

4. **Chi Squared and T Distribution**(2 points)

   Let $V_n \sim \chi^2_n$ and $T_n \sim t_n$ for all positive integers n.

   (a) Find numbers a_n and b_n such that $a_n(V_n - b_n)$ converges in distribution to $N(0,1)$. (1 point)

   (b) Show that $T_n^2/(n+T_n^2)$ has a Beta distribution. Specify the parameters of the Beta distribution. (1 point)

   Hint: You should be able to do part (b) by using the Beta Gamma connection.

   The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

1. By WLLN

$$\frac{X_1 + \cdots + X_n}{n} \to 1 \quad \text{in prob.}$$

$$\frac{Y_1 + \cdots + Y_n}{n} \to 2 \quad \text{in prob.}$$

$n \to \infty$: $\left|\frac{\sum X_i}{n} - 1\right| < \varepsilon_1$, $\left|\frac{\sum Y_i}{n} - 2\right| < \varepsilon_2$.

$$-\varepsilon_1 + 1 < \frac{\sum X_i}{n} < \varepsilon_1 + 1, \quad -\varepsilon_2 + 2 < \frac{\sum Y_i}{n} < \varepsilon_2 + 2$$

$$\frac{-\varepsilon_1 + 1}{\varepsilon_2 + 2} \le \frac{\sum X_i}{\sum Y_i} \le \frac{\varepsilon_1 + 1}{-\varepsilon_2 + 2}$$

$$\frac{-\varepsilon_1 - \frac{1}{2}\varepsilon_2}{\varepsilon_2 + 2} \le \frac{\sum X_i}{\sum Y_i} - \frac{1}{2} \le \frac{\varepsilon_1 + \frac{1}{2}\varepsilon_2}{2 - \varepsilon_2}$$

We can choose $\varepsilon_1, \varepsilon_2$ small enough.

$$-\varepsilon < \frac{-\varepsilon_1 - \frac{1}{2}\varepsilon_2}{\varepsilon_2 + 2} \le \frac{\sum X_i}{\sum Y_i} - \frac{1}{2} \le \frac{\varepsilon_1 + \frac{1}{2}\varepsilon_2}{2 - \varepsilon_2} < \varepsilon$$

$$\Rightarrow P\left(\left|\frac{\sum X_i}{\sum Y_i} - \frac{1}{2}\right| < \varepsilon\right)$$

$$\ge P\left(\frac{-\varepsilon_1 + 1}{\varepsilon_2 + 2} \le \frac{\sum X_i}{\sum Y_i} \le \frac{\varepsilon_1 + 1}{2 - \varepsilon_2}\right)$$

$= 1$ when $n \to +\infty$

Hence $P\left(\left|\frac{\sum X_i}{\sum Y_i} - \frac{1}{2}\right| < \varepsilon\right) = 1$ when $n \to +\infty$.

don't need X_i be indep of Y_i.

1. Let $\bar{X} = \frac{1}{n}(X_1 + \cdots + X_n)$, $\bar{Y} = \frac{1}{n}(Y_1 + \cdots + Y_n)$

$E\bar{X} = 1$, $E\bar{Y} = 2$.

$$E\left(\frac{\bar{X}}{\bar{Y}}\right) = E\left[E\left(\frac{\bar{X}}{\bar{Y}} \mid \bar{Y}\right)\right] = E\left[\frac{1}{\bar{Y}}\right]$$

According to weak law $\bar{Y} \xrightarrow{P} E\bar{Y} = 2$, as $n \to +\infty$.

i.e. $\bar{Y} \xrightarrow{D} 2$

Hence $\frac{1}{\bar{Y}} \xrightarrow{P} \frac{1}{2} = E\left(\frac{1}{\bar{Y}}\right)$ as $n \to +\infty$.

thus. $\frac{\bar{X}}{\bar{Y}} \xrightarrow[\text{as } n \to \infty]{P} E\left[\frac{\bar{X}}{\bar{Y}}\right] = E\left[\frac{1}{\bar{Y}}\right] = \frac{1}{2}$

i.e. $\frac{X_1 + X_2 + \cdots + X_n}{Y_1 + Y_2 + \cdots + Y_n} = \frac{\bar{X}}{\bar{Y}} \xrightarrow[\text{as } n \to \infty.]{P} \frac{1}{2}$

it doesn't matter whether X_i are indep of Y_j.

2. (a) $P(X_i = 1.7) = P(X_i = 0.5) = 0.5$.

$Y_1 = \frac{1}{2}(1 + X_0) Y_0 \quad Y_2 = \frac{1}{2}(1 + X_1) Y_1$

$\cdots\cdots \; Y_n = \frac{1}{2^n}(1 + X_0)(1 + X_1) \cdots (1 + X_{n-1}) Y_0$

$$\lim_{n \to \infty} \frac{\log Y_n}{n} = \lim_{n \to \infty} \frac{1}{n}\left(\sum_{i=0}^{n-1} \log \frac{1 + X_i}{2} + \log Y_0\right)$$

$$= E \log \frac{1 + X_i}{2} + 0 \qquad \text{according to LLN.}$$

$$= 0.5\left(\log\frac{2.7}{2} + \log\frac{1.5}{2}\right) = \frac{1}{2}\log\frac{81}{80}$$

Hence Y_n converges to $\left(\frac{81}{80}\right)^{\frac{n}{2}}$ as $n\to+\infty$.

which means $Y_n \to +\infty$ as $n\to+\infty$.

(b) $Y_1 = \alpha X_0 Y_0 + (1-\alpha)Y_0 = (1+\alpha(X_0-1))Y_0$

$$Y_n = (1+\alpha(X_0-1))(1+\alpha(X_1-1))\cdots(1+\alpha(X_{n-1}-1))Y_0$$

$$\lim_{n\to\infty}\frac{\log Y_n}{n} = \lim_{n\to\infty}\frac{1}{n}\left(\sum_{i=0}^{n-1}\log(1+\alpha(X_i-1)) + \log Y_0\right)$$

$= E\log(1+\alpha(X_i-1))$ according to LLN.

$$= 0.5\left(\log(1+0.7\alpha)+\log(1-0.5\alpha)\right)$$

$$= \frac{1}{2}\log(1+0.7\alpha)(1-0.5\alpha).$$

$$g(\alpha) = \frac{1}{2}\log(1+0.7\alpha)(1-0.5\alpha).$$

when $\alpha = \frac{2}{7}$, $g(\alpha)$ maximized.

3.(a) $X_i = \begin{cases} 1, & \text{vote } A \\ 0, & \text{vote } B \end{cases}$, $Y = \sum_{i=1}^{n} X_i$

$Y \sim N(np, np(1-p))$ for large n.

$$P\left(Y=\frac{n}{2}\right) = P\left(\frac{n}{2}-\frac{1}{2} < Y < \frac{n}{2}+\frac{1}{2}\right)$$

$$= \Phi\left(\frac{\frac{n}{2}+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\frac{\frac{n}{2}-\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$$

Since $p=0.5 \Longrightarrow P(Y=\frac{n}{2}) = \Phi(\frac{1}{\sqrt{n}}) - \Phi(-\frac{1}{\sqrt{n}})$

$$= 2\Phi(\frac{1}{\sqrt{n}}) - 1.$$

(b) $f(x) \approx f(x_0) + f'(x_0)\frac{(x-x_0)}{1!}$

Let $K(x) = 2\Phi(x) - 1 \approx K(0) + K'(0)x$

$$= 0 + 2\phi(0)x$$

$$= \sqrt{\frac{2}{\pi}}\, x$$

Hence $P(Y=\frac{n}{2}) = K(\frac{1}{\sqrt{n}}) = \sqrt{\frac{2}{\pi n}}$

$\Rightarrow c = \frac{\pi}{2}$

4. (a) $EV_n = n$, $Var V_n = 2n$

$b_n = n$, $a_n = \frac{1}{\sqrt{2n}}$

(b) $T_n = \frac{Z_n}{\sqrt{V_n/n}}$, $Z_n \sim N(0,1)$, $V_n \sim \chi^2_{(n)}$

$V_n \sim Gamma(\frac{n}{2}, \frac{1}{2})$.

$T_n^2 = \frac{n Z_n^2}{V_n}$, $Z_n^2 \sim \chi^2_{(1)} \Rightarrow Z_n^2 \sim Gamma(\frac{1}{2}, \frac{1}{2})$

$$\frac{T_n^2}{n+T_n^2} = \frac{\frac{Z_n^2}{V_n}}{1+\frac{Z_n^2}{V_n}} = \frac{Z_n^2}{V_n+Z_n^2} \sim Beta(\frac{1}{2}, \frac{n}{2}).$$