$Z \sim N(0,1)$

p.d.f: $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} \quad \forall z \in \mathbb{R}.$

c.d.f: $\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} e^{-\frac{z^2}{2}} dz$

$X = \mu + \sigma Z \sim N(\mu, \sigma^2).$

c.d.f: $\Phi(t) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{t} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$

p.d.f. $\phi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

Theorem: X, Y indep.

$X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$.

$X+Y \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

Proof: $\varphi_X(t) = E e^{tX} = \int_{-\infty}^{+\infty} e^{tx} \cdot \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x-\mu_1)^2}{2\sigma_1^2}} dx.$

$\varphi_{X+Y}(t) = E e^{t(X+Y)} \qquad = e^{t\mu_1 + \frac{1}{2}t^2\sigma_1^2}$

$= E e^{tX} \cdot E e^{tY} = e^{t(\mu_1+\mu_2) + \frac{1}{2}t^2(\sigma_1^2+\sigma_2^2)}$

$= \varphi_Z(t)$, $Z \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2).$

Multivariate Normal (MVN)

$X = (x_1, \ldots, x_k)$ is said to have a MVN if every linear combination of the X have a Normal Distribution.

$\forall b_i \in \mathbb{R}$ $\quad b_1 x_1 + b_2 x_2 + \cdots + b_k x_k$ has Normal Distribution.

Theorem: $X = (x_1, \ldots x_n) \sim MVN$

$Y = (y_1, \ldots y_m) \sim MVN$

X, Y indep $\Rightarrow W = (x_1, \ldots x_n, y_1, \ldots, y_m) \sim \underline{MVN}$.

Theorem

MVN distribution is completely specified by knowing mean, variance of each comp. and covariance of each pair of components.

$$\left.\begin{array}{l} E(X_1), E(X_2), \ldots E(X_n) \\ Var(X_1), \ldots Var(X_n) \\ Cov(X_1, X_2) \ldots Cov(X_{n-1}, X_n) \end{array}\right\} \Rightarrow \text{Specify a MVN.}$$

Proof:

MGF: $M_X(t) = Ee^{tX} = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ $X \sim N(\mu, \sigma^2)$

$M_X(t_1, t_2, \dots, t_k) = E e^{t_1 X_1 + \dots + t_k X_k}$

$X \sim MVN \Rightarrow W = t_1 X_1 + \dots + t_k X_k$ is normal distribution.

$M_X(t_1, t_2, \dots, t_k) = E e^{W} = e^{E(W) + \frac{1}{2} Var(W)}$

$= \exp\{ \sum_{i=1}^{k} t_k E(X_k) + \frac{1}{2} Var(t_1 X_1 + \dots + t_k X_k) \}$

Hence we can specify any MVN.

BVN

$X \sim N(0,1)$, $Y \sim N(0,1)$, $Cov(X,Y) = \rho$.

$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} e^{\frac{-\frac{1}{2}(x^2+y^2-2\rho xy)}{(1-\rho^2)}}$ $= \rho \cdot \sigma_1 \sigma_2 = \rho$

$=$ Correlation

二元正态 p.d.f.:

$f(x,y) = \frac{1}{2\pi \sigma_1 \sigma_2 \sqrt{1-\rho^2}} \exp\{ -\frac{1}{2(1-\rho^2)} (\frac{(x-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2} + \frac{(y-\mu_2)^2}{\sigma_2^2}) \}$

Generally. Independence is a stronger condition than 0 correlation.

Theorem: If $X \sim MVN$ and $X = (X_1, X_2)$ and every comp in X_1 is uncorrelated with every compent of X_2.

$Cov(X_1, X_2) = 0$.

$\Longrightarrow X_1, X_2$ are independent.

Proof: $X \sim N(\mu_1, \sigma_1^2)$, $Y \sim N(\mu_2, \sigma_2^2)$

$$M_{X,Y}(t_1,t_2) = E e^{Xt_1+Yt_2} = e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}(t_1^2\sigma_1^2+t_2^2\sigma_2^2+\underbrace{2\rho\sigma_1 t_1\sigma_2 t_2}_{=0})}$$

$$= e^{\mu_1 t_1+\mu_2 t_2+\frac{1}{2}(t_1^2\sigma_1^2+t_2^2\sigma_2^2)} = E e^{Xt_1} \cdot E e^{Yt_2}$$

$\Longrightarrow X, Y$ are indep.

Example: $X \sim N(\mu, \sigma^2)$

Prove: $\bar{X}$ is indep of S^2

Solution: $\bar{X} = \frac{1}{n}(X_1+\cdots+X_n) \sim N(\mu, \frac{\sigma^2}{n})$

$$S^2 = \frac{1}{n-1}((X_1-\bar{X})^2+\cdots+(X_n-\bar{X})^2)$$

$$E S^2 = \frac{1}{n-1}(X_1^2+X_2^2+\cdots\cdots+X_n^2-n\bar{X}^2)$$

$$= \frac{1}{n-1}(n\cdot(\sigma^2+\mu^2)-n(\frac{\sigma^2}{n}+\mu^2))$$

$$= \sigma^2$$

$$X_i-\bar{X} = \frac{n-1}{n}X_i - \frac{1}{n}\sum_{j=1, j\neq i}^{n} X_j \sim N(0, \frac{n-1}{n}\sigma^2)$$

$$W = (X_1-\bar{X}, X_2-\bar{X}, \cdots, X_n-\bar{X}) \sim MVN$$

$$Cov(\bar{X}, X_i-\bar{X}) = E(\bar{X}(X_i-\bar{X})) - E(\bar{X})E(X_i-\bar{X})$$

$$= E(\bar{X}(X_i-\bar{X})) = \frac{E(X_i^2)-E[(X_i-\bar{X})^2]-E[(\bar{X})^2]}{2}$$

$$(X_i-\bar{X}+\bar{X})^2 = (X_i-\bar{X})^2+(\bar{X})^2+2\bar{X}(X_i-\bar{X})$$

$$= \frac{\sigma^2+\mu^2-\frac{n-1}{n}\sigma^2-(\frac{\sigma^2}{n}+\mu^2)}{2} = 0.$$

$\Longrightarrow$ $\bar{X}$, W are independent.

$\Longrightarrow$ $\bar{X}$ is independent of any function of $(X_1-\bar{X}, X_2-\bar{X}, \cdots, X_n-\bar{X})$.

So is S^2.

Example: (Bivariable Normal Simulation)

Suppose i.i.d. r.v. $X, Y \sim N(0,1)$

Generate a BVN (Z, W) with $\begin{cases} Cov(Z,W)=\rho \\ Z\sim N(0,1) \\ W\sim N(0,1) \end{cases}$ by X, Y

$Z = aX+bY \qquad W = cX+dY.$

$\Longrightarrow (Z, W) \sim BVN$

$EZ = EW = 0.$

$Var Z = a^2+b^2=1, \; Var W = c^2+d^2=1.$

$Cov(Z,W) = ac\,Cov(X,X) + bd\,Cov(Y,Y)$

$= ac+bd = \rho.$

$$\begin{cases} a^2+b^2=1 \\ c^2+d^2=1 \\ ac+bd=\rho \end{cases}$$

$a=1 \quad b=0$

$c=\rho \quad d=\sqrt{1-\rho^2}$