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Homework 6

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Conditional Expectation given an event.
- Conditional Expectation given a random variable.
- Properties of Conditional Expectation.

Exercises

1. **Conditional Expectation given an event**(3 points)

   There are two envelopes, each of which has a check for a $Unif(0, 1)$ amount of money, measured in thousands of dollars. The amounts in the two envelopes are independent. You get to choose an envelope and open it, and then you can either keep that amount or switch to the other envelope and get whatever amount is in that envelope. Suppose that you use the following strategy: choose an envelope and open it. If you observe U, then stick with that envelope with probability U, and switch to the other envelope with probability $1 - U$.

   (a) Find the probability that you get the larger of the two amounts. (2 points)

   (b) Find the expected value of what you will receive. (1 point)

2. **Expected Waiting Time**(2 points)

   A fair 6 sided die is rolled repeatedly. Find the expected number of rolls needed to get a 1 followed right away by a 2. Hint: Start by conditioning on whether or not the first roll is a 1. (2 points)

   Hint: Start by conditioning on whether or not the first roll is a 1.

3. **Conditional Expectation given a indicator random variable**(2 points)

   Let Y be a discrete random variable, A be an event with $0 < P(A) < 1$, and $\mathbb{I}_A$ be the indicator random variable for A.

   (a) Explain precisely how the random variable $E(Y|\mathbb{I}_A)$ relates to the numbers $E(Y|A)$ and $E(Y|A^c)$. (1 point)

   (b) Show that $E(Y|A) = E(Y\mathbb{I}_A)/P(A)$, directly from the definitions of expectation and conditional expectation. (1 point)

4. **Best Predictions**(4 points)

   Let X, Y be two random variables which follow some joint distribution. Suppose we want to predict Y after observing X and to keep things simple we want to use a linear function of X which can be denoted by $a + bX$. Let a^*, b^* be such that they minimize $\mathbb{E}(Y - a - bX)^2$ over all $a, b \in \mathcal{R}$. The prediction function $a^* + b^*X$ is called the best linear predictor of Y based on X.

   (a) Give an expression for a^*, b^* in terms of means, variances and covariance of X, Y. (1 point)

   (b) Is it true that $\mathbb{E}(Y - \mathbb{E}(Y|X))^2 \leq \mathbb{E}(Y - a - bX)^2$? Why or why not? (1 point)

   Now suppose X, Y are bivariate normal with 0 means, unit variances and correlation ρ.

   (c) Find the conditional expectation $\mathbb{E}(Y|X)$. (1 point)

   Hint: Find the conditional p.d.f of Y given $X = x$ using the joint pdf of a bivariate normal.

   (d) What is $\mathbb{E}(Y - a^* - b^*X)^2 - \mathbb{E}(Y - \mathbb{E}(Y|X))^2$ in this case? (1 point)

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

1. (1) x_1, x_2

$P = x_1$ $\quad$ $P = 1 - x_1$

$P(\text{get the larger}) = P(C = X_1, X_1 > X_2) + P(C = X_2, X_2 > X_1)$

$= \int_0^1 P(C = X_1, X_1 > X_2 \mid X_1 = x_1) f_{X_1}(x_1)\, dx_1$

$+ \int_0^1 P(C = X_2, X_2 > X_1 \mid X_1 = x_1) f_{X_1}(x_1)\, dx_1$

$= \int_0^1 P(C = X_1 \mid X_1)\, P(X_1 > X_2 \mid X_1)$

$\quad + P(C = X_2 \mid X_1)\, P(X_2 > X_1 \mid X_1)\, dx_1$

$= \int_0^1 \left(x_1 \cdot \int_0^{x_1} 1\, dx_2 + (1 - x_1) \int_{x_1}^1 1 \cdot dx_2\right) dx_1$

$= \int_0^1 x_1^2 + (1 - x_1)^2\, dx_1 = \frac{2}{3}$

(2) $E(C) = E\left[E\left(E(C \mid X_1 = x_1, X_2 = x_2)\right)\right]$

$= E\left[\ E(x_1^2 + (1 - x_1)\, x_2)\ \right]$

$= E\left[\int_0^1 x_1^2 + (1 - x_1)\, x_2\, dx_2\right]$

$= E\left[x_1^2 + \frac{1}{2}(1 - x_1)\right]$

$= \int_0^1 x_1^2 - \frac{1}{2} x_1 + \frac{1}{2}\, dx_1$

$= \frac{1}{3} - \frac{1}{4} + \frac{1}{2} = \frac{7}{12}$

2. $E(N) = E(N \mid X_1 = 1)\, P(X_1 = 1) + E(N \mid X_1 \neq 1)\, P(X_1 \neq 1)$

$$= \frac{1}{6}\Big(E(N \mid X_1 = 1, X_2 = 2)\, P(X_2 = 2) + E(N \mid X_1 = 1, X_2 \neq 2)\, P(X_2 \neq 2)\Big) + \frac{5}{6} E(N \mid X_1 \neq 1)$$

$$= \frac{1}{6}\left(1 \times \frac{1}{6} + (2 + E(N)) \times \frac{5}{6}\right) + \frac{5}{6}(E(N) + 1)$$

$$= \frac{41}{36} + \frac{35}{36} E(N)$$

$$\Rightarrow E(N) = 41$$

3. (a) $I_A = \begin{cases} 0, & A^c \\ 1, & A \end{cases}$ $\quad P(I_A = 1) = P(A)$, $\quad P(I_A = 0) = P(A^c)$

$E(Y \mid I_A = 1) = E(Y \mid A)$

$E(Y \mid I_A = 0) = E(Y \mid A^c)$

(b) $$\frac{E(Y I_A)}{P(A)} = \frac{P(I_A=1)\, E(Y I_A \mid I_A = 1) + P(I_A = 0)\, E(Y I_A \mid I_A = 0)}{P(A)}$$

$$= \frac{P(A)\, E(Y \mid I_A = 1) + P(I_A = 0)\, E(0)}{P(A)}$$

$$= E(Y \mid I_A = 1)$$

$$= E(Y \mid A)$$

4. (a) $E(Y-a^*-b^*X)^2$

$= E(Y^2+a^{*2}+(b^*X)^2+2a^*b^*X-2a^*Y-2b^*XY)$

$= E(Y^2)+a^{*2}+b^{*2}E(X^2)+2a^*b^*E(X)-2a^*E(Y)-2b^*E(XY)$

$$\frac{\partial E(Y-a^*-b^*X)^2}{\partial a^*} = 2a^*+2b^*EX-2EY=0$$

$$\Rightarrow a^*+b^*EX-EY=0$$

$$\frac{\partial E(Y-a^*-b^*X)^2}{\partial b^*} = 2b^*EX^2+2a^*EX-2EXY=0$$

$$\Rightarrow a^*EX+b^*EX^2-EXY=0$$

$$\Rightarrow \begin{cases} a^* = EY - EX\cdot\dfrac{Cov(X,Y)}{VarX} \\ b^* = \dfrac{Cov(X,Y)}{Var\,X} \end{cases}$$

(b) True, Let $f(x)=a+bx$.

$$E(Y-a-bx)^2 = E(Y-f(x))^2$$

$$= E(Y-E(Y|X)+E(Y|X)-f(x))^2$$

$$= E(Y-E(Y|X))^2+E(E(Y|X)-f(x))^2$$

$$+2E(Y-E(Y|X))(E(Y|X)-f(x))$$

$E(Y-E(Y|X))(E(Y|X)-f(x))$

$$= E\left(E(Y - E(Y|X))(E(Y|X) - f(x)) \mid X\right)$$

$$= E\left[(E(Y|X) - f(x))(E(Y - E(Y|X)) \mid X\right]$$

Since $E(Y - E(Y|X) \mid X) = E(Y|X) - E(Y|X) = 0$

we know $E(Y - E(Y|X))(E(Y|X) - f(x)) = 0$

Hence $E(Y - a - bX)^2 =$

$$E(Y - E(Y|X))^2 + E(E(Y|X) - f(x))^2$$

$$\Longrightarrow E(Y - a - bX)^2 \geq E(Y - E(Y|X))^2$$

(c) $f(y|x) = \dfrac{f(x,y)}{f(x)} = \dfrac{f(x,y)}{\int_{-\infty}^{+\infty} f(x,y)\,dy} =$

$$\frac{\cancel{\frac{1}{2\pi\sqrt{1-\rho^2}}} \exp\{-\frac{1}{2(1-\rho^2)}(\cancel{x^2} + y^2 - 2\rho xy)\}}{\int_{-\infty}^{+\infty} \cancel{\frac{1}{2\pi\sqrt{1-\rho^2}}} \exp\{-\frac{1}{2(1-\rho^2)}(\cancel{x^2} + y^2 - 2\rho xy)\}\,dy}$$

$$= \frac{\exp\{-\frac{1}{2(1-\rho^2)}(y - \rho x)^2\}}{\int_{-\infty}^{+\infty} \exp\{-\frac{1}{2}\left(\frac{y-\rho x}{\sqrt{1-\rho^2}}\right)^2\}\sqrt{1-\rho^2}\; d\,\frac{y-\rho x}{\sqrt{1-\rho^2}}}$$

$$= \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\{-\frac{1}{2(1-\rho^2)}(y - \rho x)^2\}$$

$$E(Y|X) = \int_{-\infty}^{+\infty} y\, f(y|x)\,dy$$

$$= \int_{-\infty}^{+\infty} y \frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}(y-\rho x)^2\right\} dy$$

$$= \rho x.$$

(d) $$b^* = \frac{Cov(X,Y)}{Var X} = \rho \cdot \frac{\sqrt{Var Y}}{\sqrt{Var X}} = \rho$$

$$a^* = EY - b^* EX = 0$$

Hence

$$E(Y-a^*-b^*X)^2 - E(Y-E(Y|X))^2$$

$$= E(E(Y|X)-a^*-b^*X)^2$$

$$= E(\rho x - 0 - \rho x)^2 = 0.$$