Wenxiao Yang

Homework 11

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Poisson Process
- Poisson Process as the limit of fast coin tossing.

Exercises

1. **Poisson Process**(3 points)

   Passengers arrive at a bus stop according to a Poisson process with rate λ. The arrivals of buses are exactly t minutes apart. Show that on average, the sum of the waiting times of the riders on one of the buses is $\lambda t^2/2$.

2. **Poisson Process**($1 + 2 = 3$ points)

   Earthquakes occur over time according to a Poisson process with rate λ. The jth earthquake has intensity Z_j, where the Z_j are i.i.d. with mean μ and variance σ^2. Find the mean and variance of the cumulative intensity of all the earthquakes up to time t.

3. **Poisson Process**(2 points)

   Emails arrive in an inbox according to a Poisson process with rate 20 emails per hour. Let T be the time at which the 3rd email arrives, measured in hours after a certain fixed starting time. Find $P(T > 0.1)$.

4. **Fast Coin Tossing**(3 points)

   The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities. Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times $0, h, 2h, \ldots$ where h is a small positive number. Let the probability of success of each trial be λh, where λ is a positive constant. Let G be the number of trials till the first success (in discrete time), and T be the time of the first success (in continuous time).

(a) Find a simple equation relating G to T. (1 point)

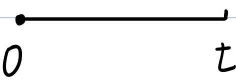
(b) Find the CDF of T. (1 point)

Hint: The distribution of G is Geometric (λh). First find $P(T > t)$.

(c) Show that as $h \to 0$, the CDF of T converges to the $Exp(\lambda)$ CDF, evaluating the CDFs at a fixed $t \geq 0$. (1 point)

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

You are encouraged to complete ICES course evaluations. This is a valuable source of course feedback.

1. 

Waiting time of every buses are same.

so we can only consider $(0, t]$

at each time points the prob of arrival is same: λ.

average sum of the waiting times:

$$\int_0^t \lambda (t-x)\, dx = \frac{\lambda t^2}{2}$$

2. $N(t) \sim Pois(\lambda t)$

$$E(N) = \lambda t \qquad Var(N) = \lambda t$$

mean: $E\left(\sum_{i=1}^{N} Z_i \mid N=n\right) = \mu n$

$$E\left(\sum_{i=1}^{N} Z_i\right) = \sum_{n=0}^{\infty} P(N=n)\, E\left(\sum_{i=1}^{N} Z_i \mid N=n\right)$$

$$= \mu E(N) = \mu \lambda t$$

Variance: $Var\left(\sum_{i=1}^{N} Z_i \mid N=n\right) = \sigma^2 n$

$$Var\left(\sum_{i=1}^{N} Z_i\right) = E\left(Var\left(\sum_{i=1}^{N} Z_i \mid N=n\right)\right) + Var\left(E\left(\sum_{i=1}^{N} Z_i \mid N=n\right)\right)$$

$$= \sigma^2 \lambda t + \mu^2 \lambda t = (\sigma^2 + \mu^2) \lambda t$$

3. $T_3 \sim \text{Gamma}(3, 20)$.

$$P(T_3 > 0.1) = \int_{0.1}^{\infty} \frac{1}{\Gamma(3)} 20^3 x^2 e^{-20x} dx$$

$$= 4000 \int_{0.1}^{\infty} x^2 e^{-20x} dx.$$

$$= 4000 \times \left(-\frac{1}{20}\right) \int_{0.1}^{\infty} x^2 de^{-20x}$$

$$= -200\left(x^2 e^{-20x} \Big|_{0.1}^{\infty} - \int_{0.1}^{\infty} 2x e^{-20x} dx \right).$$

$$= -200\left(-\frac{1}{100} e^{-2} - 2\int_{0.1}^{\infty} x e^{-20x} dx\right)$$

$$= 2e^{-2} + 400 \int_{0.1}^{\infty} x e^{-20x} dx$$

$$= 2e^{-2} + (-20) \int_{0.1}^{\infty} x \, de^{-20x}$$

$$= 2e^{-2} + (-20)\left(e^{-20x} \cdot x \Big|_{0.1}^{\infty} - \int_{0.1}^{\infty} e^{-20x} dx\right).$$

$$= 2e^{-2} + 2e^{-2} + 20\left(-\frac{1}{20} e^{-20x} \Big|_{0.1}^{\infty}\right)$$

$$= 5e^{-2}$$

4. (a) $T = (G-1) \cdot h$

(b) $P(T \le t) = P((G-1) \cdot h \le t)$

$$= P\left(G \le \frac{t}{h} + 1\right).$$

$$= \sum_{i=1}^{[\frac{t}{h}+1]} (1-\lambda h)^{i-1}(\lambda h)$$

$$= \lambda h \cdot \frac{1((1-\lambda h)^{[\frac{t}{h}+1]}-1)}{-\lambda h}$$

$$= 1 - (1-\lambda h)^{[\frac{t}{h}+1]}.$$

(c) $h \to 0$: $P(T \le t) = 1 - e^{\ln(1-\lambda h) \cdot [\frac{t}{h}+1]}.$

$$\lim_{h\to 0} \ln(1-\lambda h) \cdot [\frac{t}{h}+1] = \lim_{h\to 0} \frac{\ln(1-\lambda h)}{h} \cdot t.$$

$$= \lim_{h \to 0} \frac{\frac{-\lambda}{1-\lambda h}}{1} \cdot t.$$

$$= -\lambda t.$$

$$\Longrightarrow \lim_{h\to 0} P(T \le t) = 1 - e^{-\lambda t}.$$

$$\Longrightarrow T \sim Exp(\lambda) \text{ as } h \to 0.$$