Order Statistics

$X_{(1)} \quad X_{(2)} \cdots\cdots X_{(n)}$

$X_1, \cdots, X_n$ i.i.d.

$$F_{X_{(n)}}(x) = P(\max(X_1, \cdots, X_n) \leq x)$$

$$= P(X_1 \leq x, X_2 \leq x, \cdots X_n \leq x)$$

$$= P(X_1 \leq x) \cdot P(X_2 \leq x) \cdots P(X_n \leq x)$$

$$= (P(X_1 \leq x))^n$$

$$= (F(x))^n$$

$$F_{X_{(1)}}(x) = P(\min(X_1, \cdots, X_n) \leq x)$$

$$= 1 - P(\min(X_1, \cdots X_n) > x)$$

$$= 1 - P(X_1 > x) P(X_2 > x) \cdots P(X_n > x)$$

$$= 1 - (1 - F(x))^n$$

$$F_{X_{(j)}}(x) = P(X_{(j)} \leq x)$$

$$= \sum_{i=j}^{n} C_n^i \cdot F^i(x) (1-F(x))^{n-i}$$

$$= \sum_{i=j}^{n} \binom{n}{i} F^i(x) (1-F(x))^{n-i}$$

p.d.f : $P(X_{(j)} = x)$

$\underline{j\text{th var}}$ $\xrightarrow{}$ n possible vars. equals to x.

$\underline{j-1 \text{ vars}}$ lower than x. given jth $\rightarrow$ $\binom{n-1}{j-1}$ possible combs.

$\underline{n-j \text{ vars}}$ higher than x. given j vars ahead only 1 possiblity.

$$f_{X_{(j)}}(x) = f(x) \cdot n \cdot (F(x))^{j-1} \cdot \binom{n-1}{j-1} \cdot (1-F(x))^{n-j} \cdot 1$$

$$= n \cdot \binom{n-1}{j-1} f(x) (F(x))^{j-1} (1-F(x))^{n-j}$$

Example (Unif(0,1))

$$f_{U_{(j)}}(x) = n \binom{n-1}{j-1} x^{j-1} (1-x)^{n-j}$$

$$= \frac{n!}{(j-1)!\,(n-j)!} x^{j-1} (1-x)^{n-j}$$

$\Rightarrow U_{(j)} \sim Beta(j, n-j+1)$

$\Rightarrow E\, U_{(j)} = \frac{j}{n+1}$

Event

$A \to$ Event

$X = 1$ if A happens
$X = 0$ if A^c happens
} Indicator r.v. for the event A.

Bayes Rule

$$P(A \cap B) = P(B)\,P(A|B)$$

$$= P(A)\,P(B|A)$$

$$\Longrightarrow P(A|B) = \frac{P(A)\,P(B|A)}{P(B)}$$

Theorem: For any events $A_1, \ldots, A_n$ with possitive possibilities.

$$P(A_1, \ldots, A_n) = P(A_1) \times P(A_2|A_1) \times \cdots \times P(A_n|A_{n-1} \ldots A_1)$$

Def (odds) $\quad \text{Odds}(A) = \dfrac{P(A)}{P(A^c)}$

Bayes Rule in term of odds:

$$\frac{P(A|B)}{P(A^c|B)} = \frac{P(B|A)}{P(B|A^c)} \times \frac{P(A)}{P(A^c)}$$

Law of Total Prob.

$$P(B) = \sum_{i=1}^{n} P(B|A_i) \times P(A_i)$$

Conditional Bayes Rule

$P(A \cap E) > 0$, $P(B \cap E) > 0$

$$P(A|B, \tilde{E}) = \frac{P(B|A, \tilde{E}) \cdot P(A|\tilde{E})}{P(B|\tilde{E})}$$

$$= \frac{P(A \cap B \cap \tilde{E})}{P(B \cap \tilde{E})}$$

LOTP conditional version.

$$P(B|E) = \sum_{i=1}^{n} P(B \cap A_i | E)$$

$$= \sum P(B|A_i, E) \times P(A_i|E)$$

A, B are indep $\Rightarrow$ $P(A \cap B | E) = P(A|E) P(B|E)$

Continuous Situation: $f_{X|Y}(x|y) = \dfrac{f(x,y)}{f_Y(y)}$

prove by independence:

$$f(x,y) = f(x|y, y) = f_{X|Y}(x|y) f_Y(y)$$