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Homework 2

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Joint Distributions (PMF, CDF)
- Conditional Distributions (PMF, CDF)
- Independence
- Covariance

Exercises

1. **Memoryless Property of Exponentials**(3 points)

   Let $X \sim Exp(\lambda)$ and let c be a positive number.

   (a) (2 points) Derive the conditional cdf of X given $X > c$. Conclude that the conditional distribution of X given $X > c$ is same as the distribution of $X + c$. This is often called the memoryless property of Exponentials. The interpretation is that if your waiting time follows $Exp(\lambda)$, given that you have waited for c units of time, the remaining waiting time still has the same $Exp(\lambda)$ distribution!

   (b) (1 point) Find the conditional cdf of X given $X < c$ and also the conditional pdf of X given $X < c$.

2. **Independence of Sign and Absolute Value of a Normal**(2 points)

   Let $Z \sim N(0, 1)$.

   (a) (1 point) Let $X = |Z|$ and $S_X = sgn(Z)$. Here $sgn(x) = 1$ if $x > 0$ and -1 if $x < 0$. Find the marginal distribution of X and S_X.

(b) (1 point) Show that X and S_X are independent. (Hint: Show that conditional distribution of $X|S_X = 1$ and $X|S_X = -1$ are the same. This is enough to show independence in this case.)

(c) (1 extra credit) Now let X, Y be i.i.d $N(0, 1)$. In the lecture we showed that X/Y has the Cauchy distribution. To show this we used the fact that X/Y has the same distribution as $X/|Y|$. Argue why this is true without explicit computations. (Hint: We can represent $X = S_X|X|$ and $Y = S_Y|Y|$ where $S_X, |X|, S_Y, |Y|$ are jointly independent. Now $X/Y = (S_X/S_Y)|X|/|Y|$ and $X/|Y| = S_X|X|/|Y|$. Show that S_x/S_Y has the same distribution as S_X. Since S_X, S_Y are independent of X, Y you are done.)

Part (c) above carries no points. If you do it correctly, you get one extra credit point.

3. **Continuous Joint PDF**(3 points)

Let X and Y have the joint PDF $f_{X,Y}(x, y) = cxy$ for $0 < x < y < 1$.

(a) (1 point) Find c to make this a valid joint pdf.

(b) (1 point)Are X and Y independent?

(c) (1 point) Find the conditional pdf of Y given $X = x$.

4. **Covariance and Independence**(2 points)

Let X, Y be i.i.d Discrete Uniform on the set $\{1, 2, 3, 4\}$.

(a) (1 point) Compute the covariance of $X + Y$ and $X - Y$.

(b) (1 point) Are $X+Y$ and $X-Y$ independent? (Hint: Think about the distribution of $X - Y|X + Y = 8$)

1. $f_X(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$

(a) $F_{X|X>c}(x) = P\{X \leq x \,/\, X > c\}$

$(x > c)$. $= \dfrac{P\{c < X \leq x\}}{P\{X > c\}}$

$= \dfrac{\int_c^x \lambda e^{-\lambda t}\, dt}{\int_c^{\infty} \lambda e^{-\lambda t}\, dt}$

$= \dfrac{-e^{-\lambda t}\Big|_c^x}{-e^{-\lambda t}\Big|_c^{\infty}} = 1 - e^{-\lambda(x-c)}$

$F_{X+c}(x) = P\{X + c \leq x\}$.

$= P\{X \leq x - c\}$.

$(x > c)$ $= \int_0^{x-c} \lambda e^{-\lambda t}\, dt$

$= -e^{-\lambda t}\Big|_0^{x-c} = 1 - e^{-\lambda(x-c)}$

So. $F_{X|X>c}(x) = F_{X+c}(x)$. $= \begin{cases} 1 - e^{-\lambda(x-c)}, & x > c \\ 0, & x \leq c \end{cases}$

(b) c.d.f :

$F_{X|X<c}(x) = P\{X \leq x \,/\, X < c\}$.

$= \dfrac{P\{X \leq \min(x, c)\}}{P\{X < c\}}$

$= \begin{cases} 1, & x \geq c \\ \dfrac{1 - e^{-\lambda x}}{1 - e^{-\lambda c}}, & 0 \leq x < c \\ 0, & x < 0 \end{cases}$

p.d.f:

$$f_{X|X<c}(x) = \begin{cases} \dfrac{\lambda e^{-\lambda x}}{1-e^{-\lambda c}}, & 0 \le x < c \\ 0, & \text{others.} \end{cases}$$

2. (a)

$$F_X(x) = P(X \le x) = P(|Z| \le x)$$

$$(x \geq 0) \quad = P(-x \le Z \le x).$$

$$= \int_{-x}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$= 2\int_{0}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$f_X(x) = \frac{\partial F_X(x)}{\partial x} = \frac{\partial\left(2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt\right)}{\partial x} = \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}$$

$(x \geq 0)$

$$\therefore f_X(x) = \begin{cases} \sqrt{\frac{2}{\pi}} e^{-\frac{x^2}{2}}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

$$P_{S_X}(1) = \frac{1}{2} \quad P_{S_X}(-1) = \frac{1}{2}.$$

(b). $F_{X|S_X=1}(x) = P(X \le x \mid S_X = 1)$

$$= P(|Z| \le x \mid Z > 0).$$

$$= \begin{cases} 2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt, & x > 0. \\ 0, & x \le 0 \end{cases}$$

$$F_{X|S_x=-1}(x) = P(X \le x \mid S_x = -1)$$

$$= P(|Z| \le x \mid Z < 0)$$

$$= \begin{cases} 2\int_{-x}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt, & x > 0 \\ 0, & x \le 0. \end{cases}$$

Since $\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ is even,

$$2\int_0^x \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt = 2\int_{-x}^0 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

$$\Rightarrow F_{X|S_x=1}(x) = F_{X|S_x=-1}(x)$$

$\Rightarrow$ X and S_x are independent.

(c) $P_{\frac{S_x}{S_Y}}(1) = P_{S_x}(1)P_{S_Y}(1) + P_{S_x}(-1)P_{S_Y}(-1) = \frac{1}{2}$

$P_{\frac{S_x}{S_Y}}(-1) = P_{S_x}(1)P_{S_Y}(-1) + P_{S_x}(-1)P_{S_Y}(1) = \frac{1}{2}$

So $\frac{S_x}{S_Y}$ has the same distribution as S_x

$\Rightarrow \frac{X}{Y} = \frac{S_X |X|}{S_Y |Y|}$ has the same distribution

as $\frac{S_X |X|}{|Y|} = \frac{X}{|Y|}$

3. (a) $\int_0^1 \int_0^y cxy \, dx \, dy$

$= \int_0^1 \frac{1}{2} c y^3 \, dy = \frac{1}{8} c = 1$

$\Rightarrow c = 8$

(b) $f_X(x) = \int_x^1 8xy \, dy = 4X - 4X^3$

$f_Y(y) = \int_0^y 8xy \, dx = 4y^3$

$f_X(x) f_Y(y) = 4y^3(4x - 4x^3) \neq 8xy = f_{X,Y}(x,y)$

$\Rightarrow$ X and Y are not independent.

(c) $f_{Y|X=x}(y) = \frac{f_{X,Y}(x,y)}{f_X(x)} = \frac{8xy}{4X(1-X^2)} = \frac{2y}{1-X^2}$

4. $E(X) = E(Y) = \frac{1}{4} \times (1+2+3+4) = \frac{5}{2}$

$Var(X) = Var(Y) = \frac{1}{4}((\frac{3}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{3}{2})^2) = \frac{5}{4}$

(a) $Cov(X+Y, X-Y) = Var(X) - Var(Y) = 0$

(b) No : $P_{X-Y \mid X+Y=8}(0) = 1.$

$P_{X-Y \mid X+Y=3}(0) = 0$

$\Rightarrow$ $X-Y$ and $X+Y$ are not independent.