Wenxiao Yang

Homework 5

Homework Policy

- You must turn in your own work, do not copy answers from others. This is not helpful to you at all.
- You are welcome and encouraged to work together on homework. If you work with others, write the following on the top of your assignment: I worked with .... on this assignment.
- I am fully aware that it is possible to use online resources and merely copy the derived solutions. This does not help you to learn the material and is going to be detrimental for you in the exams.
- Show all supporting work.

Concepts:

- Order Statistics
- Conditional Probability of Events and Bayes Rule
- Independence and Conditional Independence of Events

Exercises

1. **Conditional Probability**(2 points)

   Alice is trying to communicate with Bob, by sending a message (encoded in binary) across a channel. (a) Suppose for this part that she sends only one bit (a 0 or 1), with equal probabilities. If she sends a 0, there is a 5 percent chance of an error occurring, resulting in Bob receiving a 1; if she sends a 1, there is a 10 percent chance of an error occurring, resulting in Bob receiving a 0. Given that Bob receives a 1, what is the probability that Alice actually sent a 1?(1 point)

   (b) To reduce the chance of miscommunication, Alice and Bob decide to use a repetition code. Again Alice wants to convey a 0 or a 1, but this time she repeats it two more times, so that she sends 000 to convey 0 and 111 to convey 1. Bob will decode the message by going with what the majority of the bits were. Assume that the error probabilities are as in (a), with error events for different bits independent of each other. Given that Bob receives 110, what is the probability that Alice intended to convey a 1? (1 point)

2. **Conditional Independence**(3 points)

   Suppose that in the population of college applicants, being good at baseball is independent of having a good math score on a certain standardized test (with respect to some measure of good). A certain college has a simple admissions procedure: admit an applicant if and only if the applicant is good at baseball or has a good math score on the test.

We can denote the event that an applicant is good at basketball as A, has a good math score as B and $C = A \cup B$ is the event that an applicant gets selected. It is given that A and B are independent. Show that A and B are conditionally dependent given C (as long as $P(A|B) < 1$ and $P(B|A) < 1$), by showing

$$P(A|B, C) < P(A|C).$$

(Give a one line explanation why the above display means that A cannot be independent of B given the event C.)

This phenomenon is known as Berksons paradox, especially in the context of admissions to a school, hospital, etc

(Hint: If you can write $P(A|B, C) = a/b$ for some positive $a < b$ then show that $P(A|C) = (a + c)/(b + c)$ for some $c > 0$. Then show that $(a + c)/(b + c) > a/b$.)

3. **Independence vs Conditional Independence**(3 points)

   There are 3 coins. One is a fair coin, the second one is a biased coin with probability of landing heads 0.8 and the third one is a biased coin with probability of landing heads 0.3. Someone hands you over one of these coins (selected randomly with equal probability).

   (a) What is the probability of getting heads in your first toss?(1 point)

   (b) Given that you got heads in the first toss, what is the probability that you will also get a head in the second toss (assume that, given the coin, the outcomes of the tosses are independent)?(1 point)

   (c) Explain the distinction between assuming that the outcomes of the tosses are independent and assuming that they are conditionally independent given the choice of the coin. Which of these assumptions seems more reasonable, and why?(1 point)

   (Hint: Imagine tossing the coin a 100 times and you get 80 heads. What do you think is the probability of heads in the 101st toss given this information? Now imagine that instead of 80 heads you got 50 heads in the first 100 times. What do you think is the probability of heads in the 101st toss given this information? Does your answer change? If the tosses were actually independent would your answers change? You can replace 100 with any huge number and keep the proportion of heads the same if that helps you.)

4. **Order Statistics**(3 points) Let $X_1, \ldots, X_n$ be i.i.d. continuous r.v.s with PDF f and a strictly increasing CDF F. Suppose that we know that the jth order statistic of n i.i.d. Unif$(0, 1)$ r.v.s is a Beta$(j, n - j + 1)$, but we have forgotten the formula and

derivation for the distribution of the jth order statistic of $X_1, \ldots, X_n$. Show how we can recover the PDF of $X_{(j)}$ using a change of variables.

(Hint: Use the fact that $F^{-1}(U)$ has c.d.f F if $U \sim Unif(0,1)$. This is what we learnt in week 1 under the topic simulation. Also note that since F is strictly increasing on its support so is F^{-1} on $(0,1)$. Therefore, we can represent $X_i = F^{-1}(U_i)$. Now note that because F^{-1} is strictly increasing, $X_{(j)} = F^{-1}(U_{(j)})$. Therefore, we need to find the p.d.f of $F^{-1}(U_{(j)})$ and we know the p.d.f of $U_{(j)}$. We are in good shape now to use the 1d change of variables theorem. Or you can use the CDF approach as well here.)

The maximum you can score in this homework is 11. This homework is worth 10 points in the overall scheme of things as will be every homework. If you score 11 you will earn 1 extra credit point.

$P(B=1 \mid A=0) = 0.05 \qquad P(B=0 \mid A=1) = 0.1$

1.

(a) $$P(A=1 \mid B=1) = \frac{P(A=1, B=1)}{P(B=1)}$$

$$= \frac{P(B=1 \mid A=1)P(A=1)}{P(B=1 \mid A=0)P(A=0) + P(B=1 \mid A=1)P(A=1)}$$

$$= \frac{0.9}{0.05 + 0.9} = 94.7368\%$$

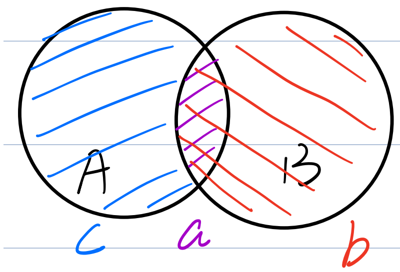
(b) $$P(A=1 \mid B=110) = \frac{P(A=1, B=110)}{P(B=110)}$$

$$= \frac{P(B=110 \mid A=1)P(A=1)}{P(B=110 \mid A=1)P(A=1) + P(B=110 \mid A=0)P(A=0)}$$

$$= \frac{0.9 \times 0.9 \times 0.1}{0.9 \times 0.9 \times 0.1 + 0.05 \times 0.05 \times 0.95} = 97.1514\%$$

2. if A is independent of B given C

Given C, whether B happens doesn't influence the probability of A happens. i.e. $P(A|B,C) = P(A|C)$.



Let $P(A|B,C) = \frac{a}{b} < 1$.

$$P(A|C) = \frac{a+c}{b+c}, \quad c>0.$$

$$\frac{a+c}{b+c} = \frac{a}{b} + \frac{(1-\frac{a}{b})c}{b+c} > \frac{a}{b}$$

Hence $P(A|C) > P(A|B,C)$.

3. (a) $P(A=\text{head}) = \frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times 0.8 + \frac{1}{3}\times 0.3 = \frac{8}{15} \approx 53.33\%$

(b) $P(A_2=\text{head} \mid A_1=\text{head}) = P(A=\text{head}) = 53.33\%$

(c) outcomes are indep means in every toss the coin can be all kinds of coins the prob of head is 53.33% all the time. Previous info is useless.

outcomes are conditionally indep give the choice of the coin means in every toss the coin is a certain choice, the prob of head is depend on the choice, we need to figure out the choice through previous info.

Conditionally independent is more reasonable. In reality, we can guess the kind of coin through the previous info, which can make our prediction about next toss more precisely. For example if we tossing the coin 10^{10000} times and we got head 8×10^{999} times, we can know the prob of the coin is second one is $0.9999\cdots \approx 1$, in this situation if we use the former assumption, next toss gets head's prob is 53.33%, if we use the latter assumption the prob is 80.0%, which seems more reasonable.

4. $f_{U_{(j)}}(x) = f_{B}(x) = \frac{n!}{(j-1)!(n-j)!} x^{j-1}(1-x)^{n-j}$

Let c.d.f function of X is $F_X : \mathbb{R} \to [0,1]$

F_X is strictly increasing. $\Rightarrow F_X^{-1}$ is strictly increasing

$X = F_X^{-1}(U) \qquad X_{(j)} = F_X^{-1}(U_{(j)})$

$$P(X_{(j)} \leq t) = P(F_X^{-1}(U_{(j)}) \leq t)$$

$$= P(U_{(j)} \leq F_X(t)).$$

$$= \int_0^{F(t)} f_{U_{(j)}}(x)\, dx.$$

$$f_{X_{(j)}}(t) = \frac{\partial P(X_{(j)} \leq t)}{\partial t} = f_{U_{(j)}}(F_X(t)) \cdot f_X(t)$$

$$= \frac{n!}{(j-1)!(n-j)!} (F_X(t))^{j-1} (1-F_X(x))^{n-j} f_X(t).$$