1. Intro.

$\{X(t), t \in T\}$

$X(t)$ is the state of the process at time t.

$d_n = X_n - X_{n-1}$

State space Ω

$P_{ij} = P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0)$

probability that the process moves from state i to state j.

$= P(\underline{X_{n+1} = j} \mid \underline{X_n = i})$ for $\forall\, i_0, i_1, \dots, i_{n-1}$

only depend on the state of X_n

Markov Chains.

transition matrix

$$\mathbb{P} = \begin{bmatrix} P_{00} & P_{01} & P_{02} & \cdots \\ P_{10} & P_{11} & P_{12} & \cdots \\ \vdots & \vdots & \vdots & \\ P_{i0} & P_{i1} & P_{i2} & \cdots \\ \vdots & \vdots & \vdots & \end{bmatrix}$$

2. C-K Equations (CHAPMAN-KOLMOGOROV)

$P^{(m)}_{kj}$ = m-step transition probabilities from state k

to State j $= P(X_{n+m} = j \mid X_n = k)$

C-K Equation: $P_{ij}^{(n+m)} = \sum_{k=0}^{\infty} P_{ik}^{(n)} P_{kj}^{(m)}$

$\Rightarrow \mathbb{P}^{(2)} = \mathbb{P} \cdot \mathbb{P} = \mathbb{P}^2$

$\Rightarrow \mathbb{P}^{(n)} = \mathbb{P}^n$

Marginal Distribution: $t = [t_1, \cdots t_m]$ $\quad t_i = P(X_0 = t_i)$ $\quad (\underset{1\times m}{t} \times \mathbb{P}^n_{m\times m})_j = P(X_n = j)$

3. Classification of states

accessible: $\exists\, n$, $P_{ij}^n > 0$

j is accessible from i.

communicate: $i \longleftrightarrow j$.

j is accessible from i and

i is accessible from j.

(1) Reflexivity $\quad i \leftrightarrow i$.

(2) Symmetry. $\quad i \leftrightarrow j \Rightarrow j \leftrightarrow i$.

(3) Transitivity. $\quad i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$.

if $i \leftrightarrow j$, then states i, j are said to be in the same class.

A Markov Chain that has only one class is said to be irreducible

all states are recurrent.

$f_i = P$(ever re-enter state i if started in state i)

State i is recurrent if $f_i = 1$

$\Rightarrow$ the expected number of times it visits state i is infinite.

transient if $f_i < 1$.

$\Rightarrow$ P(visits state i exactly n times)

$$= f_i^{n-1}(1-f_i)$$

$$I_n = \begin{cases} 1, & \text{if } X_n = i \\ 0, & \text{if } X_n \neq i \end{cases}$$

Expected number of times it visits i : $E\left[\sum_{n=0}^{\infty} I_n \mid X_0 = i\right] =$

$$\sum_{n=0}^{\infty} E[I_n \mid X_0 = i] = \sum_{n=0}^{\infty} P\{X_n = i \mid X_0 = i\} = \sum_{n=0}^{\infty} P_{ii}^n$$

expected number: recurrent $\rightarrow \infty$

transient : $\sum_{n=0}^{\infty} f_i^n (1-f_i) n < \infty$.

converge

$$\lim_{n\to\infty} \frac{f_i^{n+1}(1-f_i)(n+1)}{f_i^{n}(1-f_i)\, n} = \lim_{n\to\infty} \frac{(n+1)f_i}{n} = f_i < 1$$

i is recurrent if $\sum_{n=0}^{\infty} P_{ii}^{n} = \infty$

i is transient if $\sum_{n=0}^{\infty} P_{ii}^{n} < \infty$

i can only be visited in finite times.

$\Rightarrow$ there at least has one recurrent state.

i is recurrent, $i \leftrightarrow j \Rightarrow j$ is recurrent.

$f_{ii}^{n} = P\{$first return to state i occurs on the n^{th} transition $|$ starts in state $i\}$

State i is recurrent if and only if $\sum_{n=1}^{\infty} f_{ii}^{n} = f_i = 1$.

Definition:

Period of a state: Greatest common divisor of the possible steps it can take to return to i when starting at i.

A <u>state</u> is <u>aperiodic</u> if period equals 1.
<u>periodic</u> otherwise.

The <u>chain</u> is <u>aperiodic</u> if <u>all</u> its states are aperiodic, <u>periodic</u> otherwise.

Proposition: if an Markov Chain is irreducible, all states have the same period.