Cauchy-Schwarz inequality

: For any r.vs. X and Y with finite variance.

$|EXY| \le \sqrt{EX^2 \cdot EY^2}$

Proof: $E(Y-tX)^2 \ge 0$

$f(t) = EY^2 + t^2 EX^2 - 2t\,EXY \ge 0. \quad \forall t$

$$f'(t) = 2t\,EX^2 - 2EXY = 0$$

$$\Longrightarrow t^* = \frac{EXY}{EX^2}$$

$$\min f(t) = f(t^*) = EY^2 + \frac{E^2XY}{EX^2} - \frac{2E^2XY}{EX^2}$$

$$= EY^2 - \frac{E^2XY}{EX^2} \ge 0$$

$$\Longrightarrow |EXY| \le \sqrt{EY^2 EX^2}$$

Example: Second Moment Method

X is a non-negative r.v.

We want to find an upper bound on $P(X=0)$.

Because X is non-negative, $X = X \cdot I_{(X>0)} = \begin{cases} X, & X>0 \\ 0, & X=0. \end{cases}$

Hence, $EX = EX \cdot I_{(X>0)}$

$$\le \sqrt{EX^2 \cdot E I^2_{(X>0)}}$$

$$= \sqrt{EX^2}\sqrt{P(X>0)}$$

$$\Longrightarrow \quad P(X>0) \geq \frac{(EX)^2}{EX^2}$$

$$\Longrightarrow \quad P(X=0) = 1 - P(X>0) \leq \frac{\mathrm{Var}X}{EX^2}$$

Jensen's Inequality (More general than CS ineq).

: if $g(\cdot)$ is a <u>convex</u> function

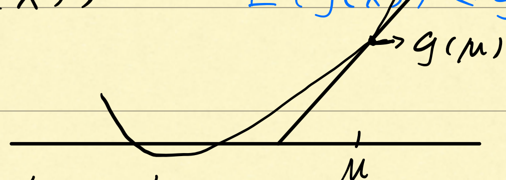
$g: \mathbb{R} \to \mathbb{R}$, $g(\lambda x + (1-\lambda)y) \leq \lambda g(x) + (1-\lambda) g(y)$, $0 \leq \lambda \leq 1$.

$\Longleftrightarrow g''(x) \geq 0$, $\forall x$.

$$E(g(X)) \geq g(E(X))$$

if concave $E(g(X)) \leq g(E(X))$.

Proof: $\mu = EX$



tangent always lie below the curve

$g(x) \geq a + bx$, $\forall x \in \mathbb{R}$

$\Longrightarrow g(X) \geq a + bX$

$$Eg(X) \geq E(a+bX) = a + bEX = a + b\mu$$
$$= g(\mu) = g(EX)$$

if equality holds $Y = g(X) - (a+bX) \geq 0$

$EY = Eg(X) - (a + bEX) = 0$ $\qquad$ $Y \equiv 0$ w.p.1. (with probability 1).

$\Rightarrow g(X) = a + bX$ w.p.1.

i.e. $g(\cdot)$ is a linear function.

Markov's Inequality

: For any r.v. X and a constant $a > 0$.

$$P(|X| \geq a) \leq \frac{E|X|}{a}$$

Proof: $Y = \frac{|X|}{a}$,

$$Y \geq I(Y \geq 1) = \begin{cases} 1, & Y \geq 1 \\ 0, & 0 \leq Y < 1 \end{cases}$$

$$\Rightarrow EY \geq P(Y \geq 1)$$

$$\Rightarrow \frac{E|X|}{a} \geq P(|X| \geq a)$$

Markov's Inequality can also be written as

$$P(X \geq a) \leq \frac{EX}{a}, \quad a > 0, X \text{ is non-negative r.v.}$$

Chebychev's Inequality

: Let X be any r.v. with mean μ, variance $\sigma^2 < \infty$.

then for $a>0$. $P(|X-\mu| \geq a) \leq \frac{\sigma^2}{a^2}$

Proof: $P(|X-\mu| \geq a) = P((X-\mu)^2 \geq a^2)$

$$\leq \frac{E(X-\mu^2)}{a^2} = \frac{\sigma^2}{a^2}$$

Cheroff Inequality.

: For any r.v. X and constant $a>0$, $t>0$.

$$P(X \geq a) \leq \frac{E e^{tx}}{e^{ta}}$$

Proof: $P(X \geq a) = P(e^{tx} \geq e^{ta}) \leq \frac{E e^{tx}}{e^{ta}}$