Linear Algebra Concepts

In this handout, I have summarized all the Linear Algebra terms that we are going to use in this course. A complete list of matrix definitions and properties, including random matrices can be found here: The Matrix Cookbook

Definition of a Matrix

A matrix is a rectangular array arranged in rows and columns. For example

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \\ \alpha_{31} & \alpha_{32} \end{bmatrix}$$

is a $3x2$ matrix. The elements of this matrix are denoted by α_{ij}, where i is the index that corresponds to the row number and j is the index that corresponds to the column number.

Square Matrix

A matrix is said to be square if the number of rows equals the number of columns. For example,

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix}$$

Upper /Lower Triangular Matrix

A square matrix is called upper triangular if all the entries below the main diagonal are zero. Similarly, a square matrix is called lower triangular if all the entries above the main diagonal are zero. For example, a lower triangular matrix is

$$L = \begin{bmatrix} \ell_{11} & 0 & 0 \\ \ell_{21} & \ell_{22} & 0 \\ \ell_{31} & \ell_{32} & \ell_{33} \end{bmatrix}$$

and an upper triangular

$$U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

Diagonal Matrix

A diagonal matrix is a matrix whose off-diagonal elements are all zero. For example,

$$A = \begin{bmatrix} \alpha_{11} & 0 & 0 \\ 0 & \alpha_{22} & 0 \\ 0 & 0 & \alpha_{33} \end{bmatrix}$$

Transpose

The transpose of a matrix A is another matrix, denoted by A^T, that is obtained by interchanging corresponding columns and rows of the matrix A. As a special case, the transpose of a column vector is a row vector and vice versa.

For example,

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & 7 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 6 & 7 \end{bmatrix}$$

Symmetric Matrix

If $A = A^T$ then the matrix A is called symmetric. For example,

$$A = \begin{bmatrix} 1 & 4 & 6 \\ 4 & 2 & 5 \\ 6 & 5 & 3 \end{bmatrix}$$

Identity Matrix

The identity or unit matrix is denoted by **I**. It is a diagonal matrix whose elements on the main diagonal are all 1. For example,

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Linear Dependence

Consider the following matrix

$$A = \begin{bmatrix} 1 & 2 & 5 & 1 \\ 2 & 2 & 10 & 6 \\ 3 & 4 & 15 & 1 \end{bmatrix}$$

Now, think of all the columns of this matrix as vectors. Then, we can think of A as being made up of 4 column vectors. Observe here that the third column is a multiple of the first one. Indeed,

$$\begin{bmatrix} 5 \\ 10 \\ 15 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

If this is the case, we say that A is linearly dependent, since one column can be obtained as a linear combination of the others.

Define a set of c column vectors $C_1, \ldots, C_c$ in an $r \times c$ matrix to be *linearly dependent*, if one vector can be expressed as a linear combination of the others. If not vector in the set can be so expressed, we define the set of vectors to be *linearly independent*.

Rank of a Matrix

The rank of a matrix is defined to be the maximum number of linearly independent columns in the matrix. For example, the rank of matrix A above is 3. The rank of a matrix is unique and can equivalently be defined as the maximum number of linearly independent rows. It follows that the rank of an $r \times c$ matrix cannot exceed $\min(r, c)$. Also, a matrix A is called *full rank,* if $rank(A) = min(r, c)$.

Inverse of a Matrix

The inverse of a matrix is another matrix, denoted by A^{-1}, such that

$$AA^{-1} = A^{-1}A = I$$

Singular Matrix

A square matrix that is not invertible is called singular or degenerate.

Orthogonal Matrix

An orthogonal matrix is a real square matrix whose columns and rows are orthonormal vectors. For an orthogonal matrix

$$A^T A = AA^T = I$$

and as a consequence $A^T = A^{-1}$.

Trace of a Matrix

The trace of a square matrix A, denoted $tr(A)$ is defined to be the sum of the elements on the main diagonal of A.

idempotent

An idempotent matrix is a matrix which, when multiplied by itself, yields itself; that is

$$A^2 = AA = A$$