# One/Two Way ANOVA

*Due:* Monday 11/15 (11.00PM)
*Submission:* On Gradescope

## Part I: Practice Questions

1. A recent study claims that using music in a class enhances the concentration and consequently helps students absorb more information. To test this claim, three different groups of ten randomly selected students (all of the same age) from three different classrooms were selected. Each classroom was provided with a different environment for students to study. *Classroom A* had constant music being played in the background, *Classroom B* had variable music being played and *Classroom C* was a regular class with no music playing. After one month, the students were given a test and their scores were collected (out of 10). The test scores were:

| | | Test Scores $j$ | | | | | | | | | | Mean |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Classroom $i$ | A: constant sound | 7 | 9 | 5 | 8 | 6 | 8 | 6 | 10 | 7 | 4 | 7 |
| | B: variable sound | 4 | 3 | 6 | 2 | 7 | 5 | 5 | 4 | 1 | 3 | 4 |
| | C: no sound | 6 | 1 | 3 | 5 | 3 | 4 | 6 | 5 | 7 | 3 | 4.3 |

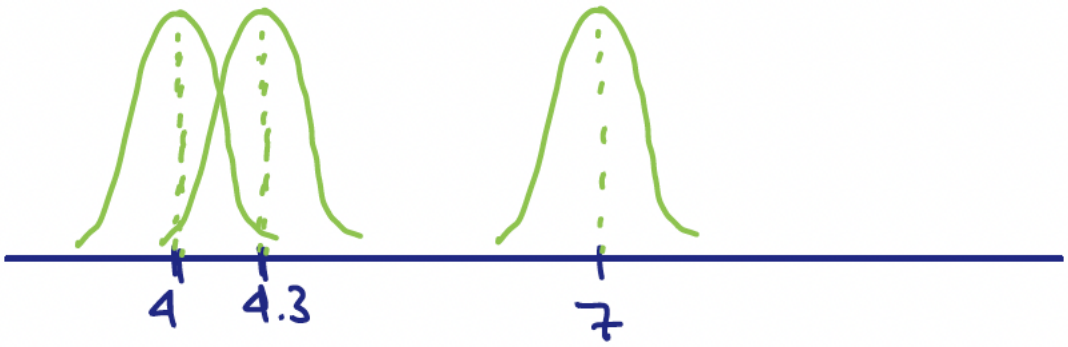
Using the information above, answer the following questions:

(a) Estimate the mean scores of the students in each type of classroom.

In this question we are looking for the point estimates $\hat{\mu}$. Specifically,

$$\hat{\mu}_A = \bar{Y}_{1\cdot} = 7,\ \hat{\mu}_B = \bar{Y}_{2\cdot} = 4,\ \hat{\mu}_C = \bar{Y}_{1\cdot} = 4.3$$

(b) Draw an illustration of the ANOVA model.



(c) Write down the ANOVA model for this problem, including the assumptions. Explain your notation.

All the following models are equivalent:

- Cell Means Model

$$Y_{ij} = \mu_i + \varepsilon_{ij}$$

$Y_{ij}$: test score of student $j$ in classroom $i$
$\mu_i$: mean test score of students in classroom $i$
Note that $\mu_i$ is a parameter, not a statistic, which means that it is an unknown quantity that we want to estimate.
The error terms satisfy the usual assumption $\varepsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$.

- Factor Effects Model

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

$Y_{ij}$: test score of student $j$ in classroom $i$
$\mu$: mean test score of all students (grand mean)
$\alpha_i$: effect of classroom $i$ on test score
The error terms satisfy the usual assumption $\varepsilon_{ij} \sim \mathcal{N}(0, \sigma^2)$.

In order for the factor effects model to be equivalent with the Cell Means Model, we need to impose constraints on the model parameters. Different constraints will lead to different parametrizations. All are equivalent and will lead to the same conclusion, however the estimated values and the interpretation will be different.
My favorite constraint is the **sum** constraint, according to which we impose $\sum_i \alpha_i = 0$.

- Regression Model
In order to write the equivalent regression model, we need to define dummy variables. Our factor has 3 levels, so we will introduce 3-1=2 dummy variables. Those are defined as follows:

$$Z_1 = \begin{cases} 1, \text{ if in Classroom A} \\ 0, \text{otherwise} \end{cases}, Z_2 = \begin{cases} 1, \text{ if in Classroom B} \\ 0, \text{ifotherwise} \end{cases}$$

So, the model is

$$Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \varepsilon$$

Note that this is the model that you see when you run the "summary" function of the lm fitted model.

Depending on the model (as you can see) the parameters are different. However, when we test for the significance of the corresponding factors, then the results we will obtain will be equivalent.

Below, we choose to work with the Cell Means Model notation:

(d) Fill in the degrees of freedom, MS and $F$-value in the ANOVA Table:

| *Source of Variation* | *df* | *SS* | *MS* | *F* | *p*-value |
|---|---|---|---|---|---|
| Between Groups | 3-1=2 | 54.6 | 27.3 | 8.18 | 0.0017 |
| Within Groups | 29-2=27 | 90.1 | 3.34 | | |
| Total | 30-1=29 | 144.7.56 | | | |

(e) Test whether the classroom environment affects students concentration. Use $\alpha = 5\%$. State the null/alternative hypotheses, decision rule and conclusion in the context of the problem.

$$\begin{cases} H_0 : \mu_A = \mu_B = \mu_C \\ H_\alpha : \text{not all } \mu's \text{ are equal} \end{cases}$$

We look at the $p$-value that is equal to 0.0017¡0.05, so we reject the null and we conclude that there is at least on mean that is different than the rest.

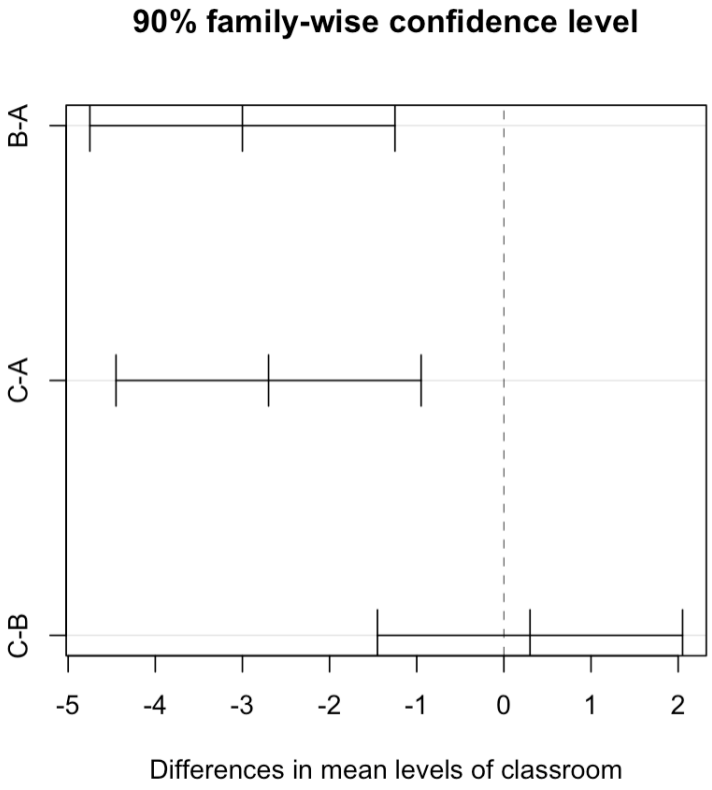
(f) Construct 90% family confidence intervals for all pairwise comparisons of classroom environments. State your conclusions.

We are going to use R to do this part, and we will focus on interpreting the results:

```
> TukeyHSD(aov(score ~ classroom, data=class.data), "classroom", conf.level=0.9)
  Tukey multiple comparisons of means
    90% family-wise confidence level

Fit: aov(formula = score ~ classroom, data = class.data)

$classroom
    diff       lwr        upr     p adj
B-A -3.0 -4.750606 -1.2493942 0.0029108
C-A -2.7 -4.450606 -0.9493942 0.0073313
C-B  0.3 -1.450606  2.0506058 0.9285599
```




Based on the plot and the output, we can conclude that:

- Scores of students in Classrooms B and C do not differ.
- Scores of students in Classrooms C and A differ significantly with the scores of students in classroom A being higher than those in classroom C.
- Scores of students in Classrooms B and A differ significantly with the scores of students in classroom A being higher than those in classroom B.

So, in the context of the problem, we can say that students in a classroom with *constant sound* performed better in the test when compared to students in classrooms B and C, while students in classrooms with variable or no sound performed similarly.

(g) Estimate the following contrast with a 95% confidence interval:

$$L = 2\mu_1 - \mu_2 - \mu_3$$

The ingredients we need to estimate $L$ are computed as follows:

$$\hat{L} = 2\bar{Y}_{1\cdot} - \bar{Y}_{2\cdot} - \bar{Y}_{3\cdot} = 2(7) - 4 - 4.3$$

$$s^2_{\hat{L}} = \frac{MSE}{n}(2^2 + (-1)^2 + (-1)^2)$$

where $MSE = 3.34$ and $n = 10$.

So, the interval will be

$$L \in \left(\hat{L} \pm t_{27}(0.05/2)s_{\hat{L}}\right)$$

2. Consider the ANOVA model and the difference estimator $D = \mu_i - \mu_{i'}$. Show that $E(\hat{D}) = \mu_i - \mu_{i'}$ and that its estimated variance is

$$s^2_{\hat{D}} = MSE\left(\frac{1}{n_i} + \frac{1}{n_{i'}}\right)$$

Construct an $(1-\alpha)100\%$ confidence interval for $D$.

$$\begin{aligned} E(\hat{D}) &= E\left(\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}\right) \\ &= E\left(\bar{Y}_{i\cdot}\right) - E\left(\bar{Y}_{i'\cdot}\right) = \mu_i - \mu_{i'} \end{aligned}$$

$$\begin{aligned} Var(\hat{D}) &= Var\left(\bar{Y}_{i\cdot} - \bar{Y}_{i'\cdot}\right) \\ &= Var\left(\bar{Y}_{i\cdot}\right) + Var\left(\bar{Y}_{i'\cdot}\right) = \frac{\sigma^2}{n_i} + \frac{\sigma^2}{n_i'} \end{aligned}$$

Since $\sigma^2$ is unknown, we estimate it with $MSE$, so we obtain that

$$s^2_{\hat{D}} = MSE\left(\frac{1}{n_i} + \frac{1}{n_{i'}}\right)$$

An $(1-\alpha)100\%$ confidence interval for $D$ is given by

$$D \in \left(\hat{D} \pm t_{n_T - r} s_{\hat{D}}\right)$$

3. An experiment was conducted to determine the effects of four different pesticides on the yield of fruit from three different varieties of a citrus tree. Eight trees of each variety were randomly selected from an orchard. The four pesticides were randomly assigned to two trees of each variety and applications were made according to recommended levels. Yields of fruit (in bushels) were obtained after the test period.

   The data are shown below:

| rep | Pest | Var | Yield |
|---|---|---|---|
| 1 | A | 1 | 49 |
| 2 | A | 1 | 39 |
| 1 | B | 1 | 50 |
| 2 | B | 1 | 55 |
| 1 | C | 1 | 43 |
| 2 | C | 1 | 38 |
| 1 | D | 1 | 53 |
| 2 | D | 1 | 48 |
| 1 | A | 2 | 55 |
| 2 | A | 2 | 41 |
| 1 | B | 2 | 67 |
| 2 | B | 2 | 58 |
| 1 | C | 2 | 53 |
| 2 | C | 2 | 42 |
| 1 | D | 2 | 85 |
| 2 | D | 2 | 73 |
| 1 | A | 3 | 66 |
| 2 | A | 3 | 68 |
| 1 | B | 3 | 85 |
| 2 | B | 3 | 92 |
| 1 | C | 3 | 69 |
| 2 | C | 3 | 62 |
| 1 | D | 3 | 85 |
| 2 | D | 3 | 99 |

   Observe that for every factor level combination we have $n = 2$ data points (this is the "rep" column). Factor A (pesticide) has 4 levels (i.e. a=4), factor B (variety) has 3 levels (i.e. b=3).

   (a) Write down the factor effects model that corresponds to this experiment.

   The factor effects model is as follows:

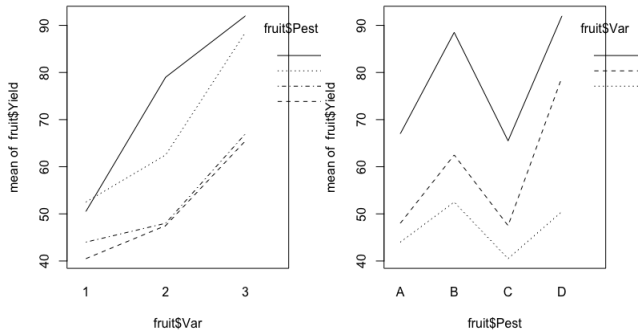
$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij}\varepsilon_{ijk}$$

   $Y_{ijk}$: fruit yield for pesticide $i$, variety $j$, tree $j$
   $\mu$: mean fruit yield of all trees (grand mean)

$\alpha_i$: effect of Pesticide $i$ on fruit yield
$\beta_j$: effect of Variety $j$ on fruit yield
$(\alpha\beta)_{ij}$: interaction term
The error terms satisfy the usual assumption $\varepsilon_{ijk} \sim \mathcal{N}(0, \sigma^2)$.
Sum Constraints: $\sum_i \alpha_i = 0$, $\sum_j \beta_j 0$, $\sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$.

(b) Prepare an estimated interaction plot. What is your conclusion about the presence of interactions.

We do this in R and we obtain:




Some of the lines in the plots intersect, so interactions (probably mild) are present.

(c) Fit an ANOVA model with *yield* as the response, and test whether the interaction term is statistically significant. Use $\alpha = 0.05$. State the alternatives, decision rule and conclusion.

```
> fruit.model1 <- lm(Yield ~ Var*Pest, data=fruit)
> anova(fruit.model1)
Analysis of Variance Table

Response: Yield
          Df Sum Sq Mean Sq F value    Pr(>F)
Var        2 3996.1 1998.04 47.2443 2.048e-06 ***
Pest       3 2227.5  742.49 17.5563 0.0001098 ***
Var:Pest   6  456.9   76.15  1.8007 0.1816844
Residuals 12  507.5   42.29
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
>
```

The hypothesis we test is

$$\begin{cases} H_0 : (\alpha\beta)_{ij} = 0, \forall i, j \\ H_\alpha : \text{not all } (\alpha\beta)_{ij} = 0 \end{cases}$$

Based on the $p$-value above, we conclude that the interaction term is not statistically significant. So, we can remove it from the model.
Note that we can also perform a partial $F$ test with Full/Reduced model. The result will be the same.

(d) Test whether the main effects are statistically significant. Use $\alpha = 0.05$. State the alternatives, decision rule and conclusion.

We will formulate this one as a partial $F$ tests

- Factor A

$$\begin{cases} H_0 : Y_{ijk} = \mu + \beta_j + \varepsilon_{ijk} \\ H_\alpha : Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \end{cases}$$

```
> fruit.model2 <- lm(Yield ~ Var, data=fruit)
> fruit.model3 <- lm(Yield ~ Var + Pest, data=fruit)
> anova(fruit.model2, fruit.model3)
Analysis of Variance Table

Model 1: Yield ~ Var
Model 2: Yield ~ Var + Pest
  Res.Df    RSS Df Sum of Sq      F   Pr(>F)
1     21 3191.9
2     18  964.4  3    2227.5 13.858 6.31e-05 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The p-value is less than 0.05, and therefore we conclude that the reduced model is not adequate and factor A is statistically significant.

- Factor B

$$\begin{cases} H_0 : Y_{ijk} = \mu + \alpha_i + \varepsilon_{ijk} \\ H_\alpha : Y_{ijk} = \mu + \alpha_i + \beta_j + \varepsilon_{ijk} \end{cases}$$

```
> fruit.model4 <- lm(Yield ~ Pest, data=fruit)
> fruit.model3 <- lm(Yield ~ Var + Pest, data=fruit)
> anova(fruit.model4, fruit.model3)
Analysis of Variance Table

Model 1: Yield ~ Pest
Model 2: Yield ~ Var + Pest
  Res.Df    RSS Df Sum of Sq      F    Pr(>F)
1     20 4960.5
2     18  964.4  2    3996.1 37.292 3.969e-07 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The p-value is less than 0.05, and therefore we conclude that the reduced model is not adequate and factor B is statistically significant.

So, we conclude that the additive model is the best one here.

```
> anova(additive.model)
Analysis of Variance Table

Response: Yield
          Df Sum Sq Mean Sq F value    Pr(>F)
Var        2 3996.1 1998.04  37.292 3.969e-07 ***
Pest       3 2227.5  742.49  13.858 6.310e-05 ***
Residuals 18  964.4   53.58
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

(e) Estimate the mean difference in yields of fruit between Variety 1 and Variety 2 with a 95% confidence interval.

What we are looking for here is a 95% CI for

$$D = \mu_{\cdot 1} - \mu_{\cdot 2}$$

(the order in the difference does not matter - if we flip the 1,2 the sign is the only thing that will change).

- Estimator $\hat{D} = \bar{Y}_{\cdot 1\cdot} - \bar{Y}_{\cdot 2\cdot}$.
- Variance: $s^2_{\hat{D}} = \frac{2MSE}{na}$, where MSE = 53.58, $n = 2$ and $a = 4$.

So, the interval is

$$D \in (\hat{D} \pm t_{18}(0.05/2)s_{\hat{D}}$$

(f) Estimate the following contrast with a 95% confidence interval

$$L = \frac{\mu_{A\cdot} + \mu_{B\cdot}}{2} - \frac{\mu_{C,\cdot} + \mu_{D,\cdot}}{2}$$

- Estimator $\hat{L} = \frac{1}{2}\bar{Y}_{1\cdot\cdot} + \frac{1}{2}\bar{Y}_{2\cdot\cdot} - \frac{1}{2}\bar{Y}_{3\cdot\cdot} - \frac{1}{2}\bar{Y}_{4\cdot\cdot}$.
- Variance: $s^2_{\hat{L}} = \frac{MSE}{nb}((1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2)$, where MSE = 53.58, $n = 2$ and $b = 3$.

So, the interval is

$$L \in (\hat{L} \pm t_{18}(0.05/2)s_{\hat{L}}$$

# Part II: Homework Questions – to be submitted

1. A manufacturer of television sets is interested in the effect of tube conductivity of **four** different types of coating color picture tubes. An experiment is conducted and the following conductivity data are obtained:

| Coating Type | Conductivity | | | |
|---|---|---|---|---|
| 1 | 143 | 141 | 150 | 146 |
| 2 | 152 | 149 | 137 | 143 |
| 3 | 134 | 136 | 132 | 127 |
| 4 | 129 | 127 | 132 | 129 |

(a) Is there a difference in conductivity due to coating type? Use $\alpha = 0.05$.

The fitted ANOVA model in R is

```
Call:
lm(formula = conductivity ~ type, data = tube)

Residuals:
   Min     1Q Median     3Q    Max
 -8.25  -2.25  -0.25   3.00   6.75

Coefficients:
            Estimate Std. Error t value Pr(>|t|)
(Intercept)  137.938      1.109 124.350  < 2e-16 ***
type1          7.062      1.921   3.676  0.00317 **
type2          7.313      1.921   3.806  0.00250 **
type3         -5.688      1.921  -2.960  0.01192 *
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 4.437 on 12 degrees of freedom
Multiple R-squared:  0.7814,    Adjusted R-squared:  0.7268
F-statistic:  14.3 on 3 and 12 DF,  p-value: 0.0002881
```

The F-test we perform is the following:

$$\begin{cases} H_0 : Y_{ij} = \mu + \varepsilon_{ij} \\ H_\alpha : Y_{ij} = \mu + \alpha_i + \epsilon_{ij} \end{cases}$$

We can obtain the p-value for this test from the output above. It is equal to 0.0002881¡0.05, which means that we reject the null and conclude that there is a difference in conductivity due to coating type.

Alternatively, we can obtain the anova table:

```
> anova(conductivity.lm)
Analysis of Variance Table

Response: conductivity
          Df Sum Sq Mean Sq F value    Pr(>F)
type       3 844.69 281.562  14.302 0.0002881 ***
Residuals 12 236.25  19.688
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

(b) Compute a 95% confidence interval for the mean of coating type 4.
We can compute this CI using the confint function in R. Here, we compute it by hand:

$$\mu_4 \in (\hat{\mu}_4 \pm T_{12}(0.05/2)\sqrt{MSE}n$$
$$\mu_4 \in (129.25 \pm T_{12}(0.05/2)\sqrt{19.6884}$$

(c) Compute a 99% confidence interval for the for the mean difference between coating types 1 and 4.
The difference we are interested in is

$$D = \mu_1 - \mu_4$$

So, the interval computes as

$$D \in (\hat{D} \pm T_{12}(0.01/2)\sqrt{2MSE}n)$$

$$D \in (145 - 129.25 \pm T_{12}(0.01/2)\sqrt{2(19.688)}4)$$

(d) Test all pairwise differences in means with a family confidence coefficient 90% (you can choose Bonferroni, Scheffe or Tukey). Based on the results, which coating type produces the highest conductivity?

```
> TukeyHSD(aov(conductivity ~ type, data=tube), conf.level=0.9)
  Tukey multiple comparisons of means
    90% family-wise confidence level

Fit: aov(formula = conductivity ~ type, data = tube)

$type
      diff       lwr      upr     p adj
2-1   0.25  -7.78265  8.28265 0.9998078
3-1 -12.75 -20.78265 -4.71735 0.0073964
4-1 -15.75 -23.78265 -7.71735 0.0014707
3-2 -13.00 -21.03265 -4.96735 0.0064441
4-2 -16.00 -24.03265 -7.96735 0.0012913
4-3  -3.00 -11.03265  5.03265 0.7759360
```

Based on the Tukey output, we conclude that the mean conductivity for coating types 1 and 2 is statistically the same (since the corresponding CI contains 0). We also observe that the mean conductivity for coating types 3 and 4 are also the same. Based on the sign of the differences, we can also deduce that the mean conductivity for coating types is lower than that of coating types 1 and 2.

(e) Assuming that coating type 4 is currently in use, what are your recommendations to the manufacturer if they wish to minimize conductivity?
Based on our results above, type 4 is associated with lower conductivities (as well as type 3). Since types 3 and 4 are statistically the same, the manufacturer probably wants to continue using type 4. Now, if there are other factors that make type 3 more desirable (e.g. cost) the manufacturer might want to switch to using type 3.

2. Consider the *butterfat* data set in the *Faraway* library. This data set contains information about the percent of butter fat (more is better) in the milk taken from 100 cows. In the study, there are 5 different breeds of cows and 2 different ages. We are interested in assessing if *Age* and *Breed* affect the butterfat content.

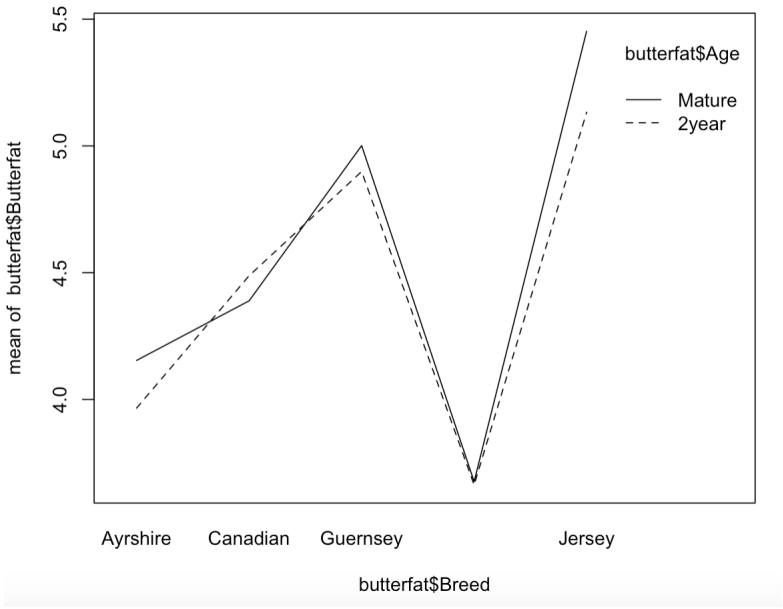
(a) Write down the factor effects model that corresponds to this problem.

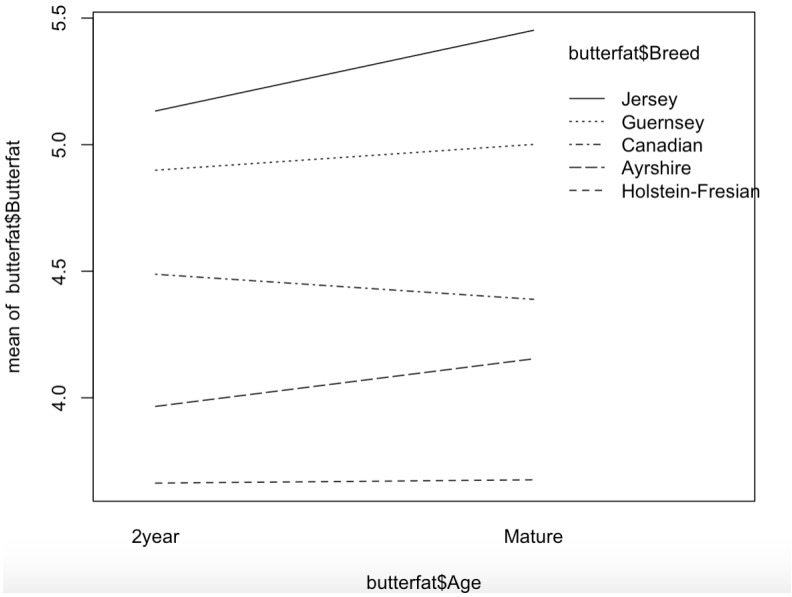
$$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ijk}$$

where $Y_{ijk}$ is the butterfat for cow k of breed i and age j, $\mu$ is the average butterfat for all cows, $\alpha_i$ is the effect of age on butterfat, $\beta_j$ is the effect of breed on butterfat. Also, we have that $i = 1, ..., 5$, $j = 1, 2$ and $k = 1, ...10$. The assumptions for the error terms are: $\epsilon_{ijk} \sim N(0, \sigma^2)$. We also need to impose model constraints. We choose to work with the sum constraints here, but any other choice is also viable.

$$\sum_i \alpha_i = 0, \sum_j \beta_j = 0, \sum_i (\alpha\beta)_{ij} = \sum_j (\alpha\beta)_{ij} = 0$$

(b) Prepare an estimated interaction plot. What is your conclusion about the presence of interactions.






We see that some of the lines intersect, so we conclude that interactions are present.

(c) Fit an ANOVA model with *log(butterfat)* are the response, and test whether the interaction term is statistically significant. Use $\alpha = 0.05$. State the alternatives, decision rule and conclusion.

```
> inter.model = lm(log(Butterfat) ~ Breed*Age, data = butterfat)
> anova(inter.model)
Analysis of Variance Table

Response: log(Butterfat)
          Df  Sum Sq Mean Sq F value Pr(>F)
Breed      4 1.70334 0.42584 56.5179 <2e-16 ***
Age        1 0.01367 0.01367  1.8141 0.1814
Breed:Age  4 0.02232 0.00558  0.7406 0.5668
Residuals 90 0.67811 0.00753
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The hypothesis test we perform is

$$\begin{cases} H_0 : Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \\ H_\alpha : Y_{ij} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \varepsilon_{ij} \end{cases}$$

From the R output, the $p$-value=0.5668¿0.05 which means that we fail to reject the null and conclude that the interactions are not statistically significant.

(d) Test whether the main effects are statistically significant. Use $\alpha = 0.05$. State the alternatives, decision rule and conclusion.

We remove the interactions from the model, and we re-fit testing for the main effects one at a time using the partial F test:

$$\begin{cases} H_0 : Y_{ij} = \mu + \alpha_i + \varepsilon_{ij} \\ H_\alpha : Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \end{cases}$$

```
Analysis of Variance Table

Model 1: log(Butterfat) ~ Breed
Model 2: log(Butterfat) ~ Breed + Age
  Res.Df     RSS Df Sum of Sq      F Pr(>F)
1     95 0.71410
2     94 0.70043  1  0.013668 1.8343 0.1789
```

The p-value is 0.1789¿0.05, which means that we fail to reject the null and conclude that the factor Age is not statistically significant.

$$\begin{cases} H_0 : Y_{ij} = \mu + \beta_j + \varepsilon_{ij} \\ H_\alpha : Y_{ij} = \mu + \alpha_i + \beta_j + \varepsilon_{ij} \end{cases}$$

```
Analysis of Variance Table

Model 1: log(Butterfat) ~ Age
Model 2: log(Butterfat) ~ Breed + Age
  Res.Df     RSS Df Sum of Sq      F    Pr(>F)
1     98 2.40377
2     94 0.70043  4    1.7033 57.149 < 2.2e-16 ***
---
Signif. codes:  0 ‘***’ 0.001 ‘**’ 0.01 ‘*’ 0.05 ‘.’ 0.1 ‘ ’ 1
```

The p-value is $< 10^{-16}$ which means that we reject the null and conclude that the factor Breed is statistically significant.

*** Note that here we can also perform a test of significance of each factor against the null model. The final conclusion will actually be the same.

(e) Estimate the mean difference in butterfat content between Mature and 2year cows with a 95% confidence interval.

The difference we are interested in is the following:

$$\hat{D} = \hat{\mu}_{mature} - \hat{\mu}_{2year} = 0.1046$$

The corresponding estimated variance is

$$s^2 = \frac{2MSE}{nb} = \frac{2(0.00745)}{5(10)}$$

We used the MSE from the additive model here. Also, b=5 the number of levels of the Breed factor and n=10 the number of cows per factor level combination.

So,

$$D \in (0.1946 \pm T_{94}(0.05/2)\sqrt{\frac{2(0.00745)}{5(10)}}$$

$$D \in (0.07, 0.14)$$

(f) Estimate the following contrast with a 95% confidence interval

$$L = \frac{\mu_{Ayrshire,\cdot} + \mu_{Canadian,\cdot}}{2} - \frac{\mu_{Guernsey,\cdot} + \mu_{Jersey,\cdot}}{2}$$

Again, we work with the additive model, so we have

$$\hat{L} = \frac{4.06 + 4.4385}{2} - \frac{4.95 + 5.2925}{2} = -0.872$$

The corresponding estimated variance is

$$s^2_{\hat{L}} = \frac{MSE}{na}(\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4})$$

where MSE=0.00745 (again from the additive model), $n = 10$ and a=2.
Plugging everything in with the same $T_{94}(0.05/2)$ as before gives us

$$(-0.896, -0.848)$$