STAT 426

1.4 Statistical Inference for Categorical Data (Part II)

Outline

Confidence Intervals from Test Inversion

Suppose, for each possible β_0,

$$\text{“Reject if } \; T(\beta_0) \geq c_T \text{ ”}$$

is a (two-sided) level α test of

$$H_0 : \beta = \beta_0 \qquad\qquad H_a : \beta \neq \beta_0$$

Then

$$\left\{ \beta_0 : \; T(\beta_0) < c_T \right\}$$

is a $(1-\alpha)100\%$ **confidence set** for β.

If it is an interval, it is a CI.

For example, the **Wald interval**:

$$\left\{ \beta_0 : \ \frac{\left|\hat{\beta} - \beta_0\right|}{SE} < z_{\alpha/2} \right\} \quad = \quad \hat{\beta} \ \pm \ z_{\alpha/2}\, SE$$

Example (Binomial Probability)

$$Y \sim \text{binomial}(n, \pi)$$

Want to test $H_0 : \pi = \pi_0$ and find a CI for π.

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- Wald

$$z_W = \frac{\hat{\pi} - \pi_0}{\sqrt{\hat{\pi}(1-\hat{\pi})/n}} \qquad (\hat{\pi} = y/n)$$

The $(1-\alpha)$100% Wald interval:

$$\hat{\pi} \pm z_{\alpha/2} \sqrt{\frac{\hat{\pi}(1-\hat{\pi})}{n}}$$

What happens if $y = 0$? $y = n$?

Example (continued)

- Likelihood Ratio

$$
\begin{aligned}
L(\pi_0) &= y \ln \pi_0 + (n-y)\ln(1-\pi_0) \\
L(\hat{\pi}) &= y \ln \hat{\pi} + (n-y)\ln(1-\hat{\pi})
\end{aligned}
$$

Example (continued)

- Likelihood Ratio

$$\begin{aligned} L(\pi_0) &= y \ln \pi_0 + (n-y)\ln(1-\pi_0) \\ L(\hat{\pi}) &= y \ln \hat{\pi} + (n-y)\ln(1-\hat{\pi}) \end{aligned}$$

so the LRT statistic is

$$\begin{aligned} -2\big(L(\pi_0) - L(\hat{\pi})\big) &= 2\left(y \ln \frac{\hat{\pi}}{\pi_0} + (n-y)\ln \frac{1-\hat{\pi}}{1-\pi_0}\right) \\ &= 2\left(y \ln \frac{y}{n\pi_0} + (n-y)\ln \frac{n-y}{n-n\pi_0}\right) \\ &= 2\sum \text{observed} \cdot \ln \frac{\text{observed}}{\text{expected}} \end{aligned}$$

Reject $H_0 : \pi = \pi_0$ when this is $\geq \chi_1^2(\alpha)$.

Example (continued)

The $(1-\alpha)100\%$ likelihood-ratio interval is

$$\left\{\,\pi_0 :\; -2\big(L(\pi_0) - L(\hat{\pi})\big) \;<\; \chi_1^2(\alpha)\right\}$$

For example, when $y = 0$ it is

$$\left\{\,\pi_0 :\; -2n\ln(1-\pi_0) \;<\; \chi_1^2(\alpha)\right\} \;=\; \left(0,\; 1 - e^{-\frac{1}{2n}\chi_1^2(\alpha)}\right)$$

Note: Unlike Wald, this interval is not degenerate.

(For general y, the interval does not have an explicit form.)

Example (continued)

- Score

$$u(\pi_0) = \frac{y - n\pi_0}{\pi_0(1-\pi_0)} \qquad \imath(\pi_0) = \frac{n}{\pi_0(1-\pi_0)}$$

so

$$z_S = \frac{u(\pi_0)}{\sqrt{\imath(\pi_0)}} = \frac{y - n\pi_0}{\sqrt{n\pi_0(1-\pi_0)}} = \frac{\hat{\pi} - \pi_0}{\sqrt{\pi_0(1-\pi_0)/n}}$$

Example (continued)

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The $(1-\alpha)100\%$ score interval is

$$\left\{ \pi_0 : \ |z_S| < z_{\alpha/2} \right\}$$

which has explicit formula in Agresti, eqn. (1.14).
Unlike Wald, it is not degenerate when $y = 0$ or n.

Demonstration of these intervals: Read Agresti, Sec. 1.4.3.

P-Values

If test statistic T tends to be larger under H_a,

$$P\text{-value} \quad = \quad \mathrm{P}_{H_0}(T \geq t_0)$$

where t_0 is the observed value of T.

As usual, can test H_0 by rejecting when P-value is $\leq \alpha$.

The smaller the value, the greater the evidence against H_0.

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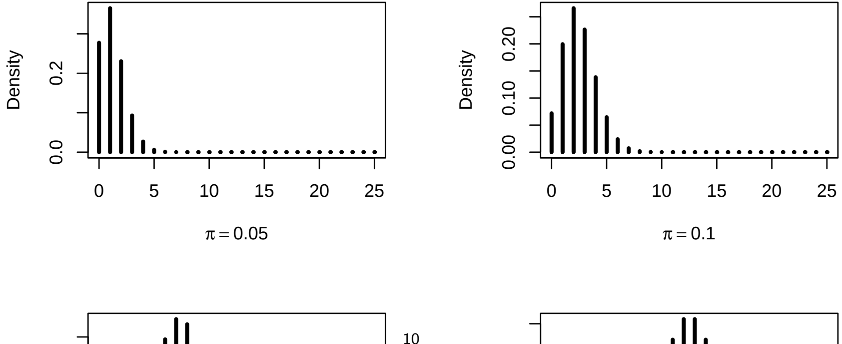
For discrete T, may instead use

$$\text{mid } P\text{-value} \;=\; \frac{1}{2}\mathrm{P}_{H_0}(T = t_0) \,+\, \mathrm{P}_{H_0}(T > t_0)$$

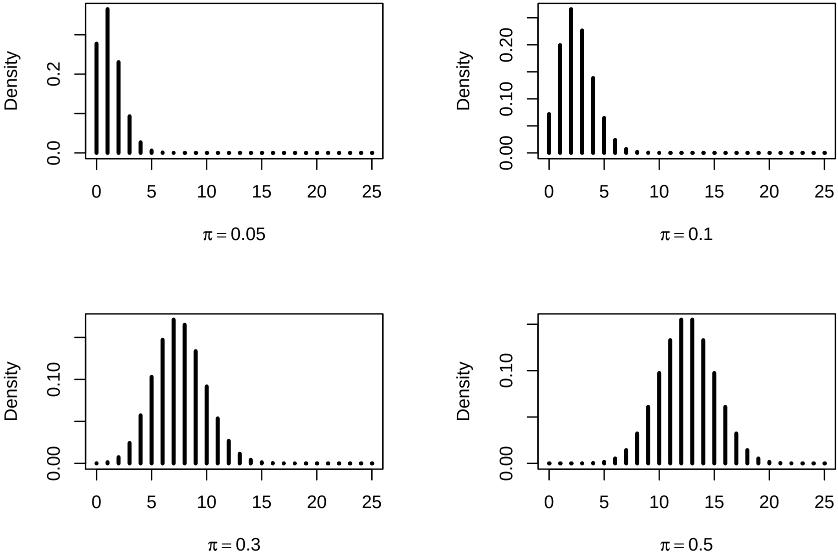
R Examples: Binomial Inference

First view various binomial densities ...

```
n <- 25
pi <- c(0.05, 0.1, 0.3, 0.5)
par(mfrow=c(2,2), mai=c(0.7,1.0,0.5,0.2))
for(i in 1:length(pi))
   plot(0:n, dbinom(0:n,n,pi[i]), type="h",
        xlab=bquote(pi == .(pi[i])), ylab="Density", lwd=3)
```

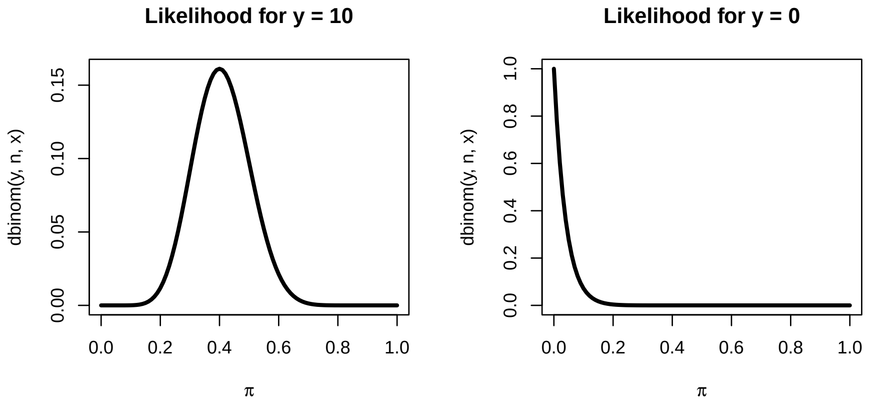


Binomial Densities



Some possible shapes for likelihood $\ell(\pi)$:

```
par(mfrow=c(2,2))
y <- 10
curve(dbinom(y,n,x), xlim=c(0,1), xlab=expression(pi),
      main=paste("Likelihood for y =", y), lwd=3)
y <- 0
curve(dbinom(y,n,x), xlim=c(0,1), xlab=expression(pi),
      main=paste("Likelihood for y =", y), lwd=3)
```



95% Wald CIs for π (c.f. Agresti, Sec. 1.4.3):

```
y <- 10
pihat <- y/n
pihat + c(-1,1) * qnorm(1-0.05/2) * sqrt(pihat*(1-pihat)/n)

## [1] 0.2079635 0.5920365

y <- 0
pihat <- y/n
pihat + c(-1,1) * qnorm(1-0.05/2) * sqrt(pihat*(1-pihat)/n)

## [1] 0 0
```

Remark: Wald interval doesn't work well for y=0.

95% Likelihood-ratio CI for π (c.f. Agresti, Sec. 1.4.3):

```r
y <- 0
c(0, 1 - exp(-qchisq(1-0.05,1)/(2*n)))

## [1] 0.00000000 0.07395197
```

95% score CI for π (c.f. Agresti, Sec. 1.4.3):

```
score1 = prop.test(y, n, correct=FALSE)$conf.int
score1

## [1] 0.0000000 0.1331923
## attr(,"conf.level")
## [1] 0.95

attr(score1,"conf.level")

## [1] 0.95

# ... or with a "continuity correction":

score2 = prop.test(y, n)$conf.int
score2

## [1] 0.000000 0.165773
## attr(,"conf.level")
## [1] 0.95
```

Clopper-Pearson "exact" 95% CI (Agresti, Sec. 16.6.1):

```r
binom.test(y, n)$conf.int

## [1] 0.0000000 0.1371852
## attr(,"conf.level")
## [1] 0.95
```

`prop.test` and `binom.test` can also be used for point-null hypothesis tests — see R help.