ECON 201B

Author: Wenxiao Yang

Institute: Haas School of Business, University of California Berkeley

Date: 2024

Seeking what is true is not seeking what is desirable.

Contents

Chapter 1 Game Theory

Based on

- ”Kreps, D. M., & Sobel, J. (1994). Signalling. *Handbook of game theory with economic applications*, 2, 849-867.”
- Mas-Colell, Whinston, and Green, Microeconomic Theory, Oxford University Press (1995).
- UIUC ECON 530 21Fall, Nolan H. Miller
- UC Berkeley ECON 201A 23Fall, 201B 24Spring
- UC Berkeley MATH 272 23Fall, Alexander Teytelboym
- Jehle, G., Reny, P.: Advanced Microeconomic Theory. Pearson, 3rd ed. (2011). Ch. 6.

1.1 Basic Game Theory

1.1.1 Action and Domination Theorem

Let A be the finite set of possible actions and Ω be the finite set of possible states. A function can map the action and state to a value, $u(a, \omega)$. It can be represented by $\vec{u}(a) = \{u(a, \omega)\}_{\omega \in \Omega}$. It is common in game theory to assume the utility function is given or known.

A **mixed action** is a probability distribution over A, $\sigma \in \Delta(A)$.

A **belief** of the agent is a probability distribution over Ω, $\mu \in \Delta(\Omega)$.

Definition 1.1 (Optimal and Justifiable Mixed Action)

A mixed action $\sigma \in \Delta(A)$ is **optimal** given $\mu \in \Delta(\Omega)$ if

$$\mathbb{E}_\mu u(\sigma, \tilde{\omega}) \geq \mathbb{E}_\mu u(\sigma', \tilde{\omega}),\ \forall \sigma' \in \Delta(A)$$

A mixed action $\sigma \in \Delta(A)$ is **justifiable** if it is optimal for some belief $\mu \in \Delta(\Omega)$. ♣

Definition 1.2 (Dominant and Dominated Action)

A mixed action $\sigma \in \Delta(A)$ is **dominant** if

$$u(\sigma, \omega) > u(\sigma', \omega),\ \forall \omega \in \Omega, \sigma' \in \Delta(A), \sigma \neq \sigma'$$

A mixed action $\sigma \in \Delta(A)$ is **dominated** if

$$u(\sigma, \omega) < u(\sigma', \omega),\ \forall \omega \in \Omega,\ \text{and for some } \sigma' \in \Delta(A)$$

In this case we say σ' dominates σ. ♣

Theorem 1.1 (Domination Theorem: Justifiable = Not Dominated)

A mixed action is justifiable if and only if it is not dominated.

♡

Proof 1.1

$\Rightarrow$ is easily proved by the definition. We focus on proving $\Leftarrow$:

Let $\mathcal{U} = \{\vec{u}(\sigma) : \sigma \in \Delta(A)\}$ and σ^* be an undominated mixed action. Then, we have $\mathcal{U} \cap (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++}) = \emptyset$. Because $\mathcal{U}$ and $\{\vec{u}(\sigma^*)\}+\mathbb{R}^{\Omega}_{++}$ are disjoint, convex, and nonempty, we can use the Separating Hyperplane Theorem ??: $\exists p \in \mathbb{R}^{\Omega}, p \neq 0$ such that $p \cdot a \leq p \cdot b, \forall a \in \mathcal{U}, b \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$.

Claim 1.1

$p \cdot \vec{u}(\sigma) \leq p \cdot \vec{u}(\sigma^*), \forall \sigma \in \Delta(A)$.

♠

Proof 1.2

For any positive number m, $\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m}) \in \{\vec{u}(\sigma')\} + \mathbb{R}^{\Omega}_{++}$. So, for any $\sigma \in \Delta(A)$, $p \cdot \vec{u}(\sigma) \leq p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right)$. By taking limit, $p \cdot \vec{u}(\sigma^*) = \lim_{m\to\infty} p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right) \geq p \cdot \vec{u}(\sigma)$.

Claim 1.2

$p > 0$.

♠

Proof 1.3

Prove by the contradiction. Suppose $p_\omega < 0$ for some $\omega \in \Omega$. Let $z = (\epsilon, ..., \epsilon) + M\mathbb{1}_\omega, M > 0, \epsilon > 0$. So, $\vec{u}(\sigma^*) + z \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$. We have $p \cdot \vec{u}(\sigma^*) \leq p \cdot (\vec{u}(\sigma^*) + z)$ by the result of SHT. There is a contradiction since $p_\omega < 0$. So, we have $p \geq 0$. Because $p \neq 0$, $p > 0$ is proved.

Finally, we normalize p to $\mu = \frac{1}{\sum_\omega p_\omega} p$. Then, σ^* is optimal for the belief μ, which means σ^* is justifiable.

1.1.2 Extensive Game

Definition 1.3 (History)

The sequences of actions are called **histories**. $h' = (\underbrace{a_1, ..., a_n}_{h\text{:prefix of } h'}, a_{n+1}, ...) \in H$. We call h' is the **continuation** of h. h is a **terminal** of H if there is no continuation of h in H. ($\emptyset \in H$.)

♣

Definition 1.4 (Extensive form Perfect Information Game)

Am extensive form game with prefect information is defined as $G = \{N, A, H, Z, P, O, o, \succ_{n\in N}\}$, where N is the set of players, A is the set of actions, H is the set of all histories, Z is the set of all histories that are terminals, $P : H/Z \to N$ is a mapping from a non-terminal histories to a player (who moves after a non-terminal history), O is the set of outcomes, and o is a function from Z to O.

A PIG is finite horizon if there is a bound on the length of its histories. ♣

We denote $A(h)$ as the actions available to player $P(h)$ after a history h.

Let $H_i = \{h \in H/Z : i = P(h)\}$ be the set of histories that player i moves after.

Definition 1.5 (Strategy)

A **strategy** is defined as a function $s_i : H_i \to A$ for which $s_i(h) \in A(h), \forall h \in H_i$. Let S_i be the set of all strategies available to the player i. A **strategy profile** is a collection of strategy $s = (s_i)_{i\in N}$. ♣

Definition 1.6 (Subgame)

A **subgame** of a PIG $G = \{N, A, H, Z, P, O, o, \succ_{n\in N}\}$ is a game (a PIG) that starts after a given finite history $h \in H$. Formally, the subgame $G(h)$ associated with $h = (h_1, ..., h_n) \in H$ is $G(h) = \{N, A, H_h, Z, P_h, O, o_h, \succ_{n\in N}\}$, where

$$H_h = \{(a_1, a_2, ...) : (h_1, ..., h_n, a_1, a_2, ...) \in H\}$$

$$o_h(h') = o(hh'), P_h(h') = P(hh')$$

A strategy s of G defines a strategy s_h of $G(h)$ by $s_h(h') = s(hh')$. ♣

Definition 1.7 (Subgame Perfect Equilibrium (SPNE))

A **subgame perfect equilibrium (SPNE)** of G is a strategy profile s^* such that for every subgame $G(h)$ it holds that $h' \mapsto s_i^*(hh')$ is an optimal strategy in $G(h)$, given beliefs that the rest of the players behave according to s_{-i}^* (or its restriction to $G(h)$). ♣

Definition 1.8 (Profitable Deviation)

Let s be a strategy profile. We say that s_i' is a **profitable deviation** from s for player i at history h if s_i' is a strategy for G such that

$$o_h\left(s_i', s_{-i}\right) \succ_i o_h(s)$$

♣

Note that a strategy profile has no profitable deviations iff it's a SPNE.

Theorem 1.2 (The one-deviation principle)

Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i\in N}\right)$ be a finite horizon, extensive form game with perfect information. Let s be a strategy profile that is not a subgame perfect equilibrium. There exists some history h and a profitable deviation $\bar{s}_i$ for player $i = P(h)$ in $G(h)$ such that $\bar{s}_i(k) = s_i(k)$ for all $k \neq h$. ♡

- Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i\in N}\right)$ be a PIG.
- $A(\emptyset)$ is the set of allowed initial actions for player $i = P(\emptyset)$. For each $b \in A(\emptyset)$, let $s^{G(b)}$ be some strategy

profile for the subgame $G(b)$.

- Given some $a \in A(\emptyset)$, we denote by s^a the strategy profile for G in which player $i = P(\emptyset)$ chooses the initial action a, and for each action $b \in A(\emptyset)$ the subgame $G(b)$ is played according to $s^{G(b)}$.
- So $s_i^a(\emptyset) = a$ and for every player $j, b \in A(\emptyset)$ and $bh \in H \backslash Z$, $s_j^a(bh) = s_j^{G(b)}(h)$.

Lemma 1.1 (Backward Induction)

Let $G = \left(N, A, H, Z, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a finite PIG. Assume that for each $b \in A(\emptyset)$ the subgame $G(b)$ has a subgame perfect equilibrium $s^{G(b)}$. Let $i = P(\emptyset)$ and let a be the $\succ_i$-maximizer over $A(\emptyset)$ of $o_a\left(s^{G(a)}\right)$. Then s^a is a subgame perfect equilibrium of G. ♡

1.1.3 Strategic Form Game

Definition 1.9 (Normal Form Game)

A game in **normal form** is denoted by

$$G = \left(\underbrace{N}_{\text{players}}, \underbrace{\{S_i\}_{i \in N}}_{\text{Strategy Set}}, \underbrace{\{u_i(\cdot)\}_{i \in N}}_{\text{VNM utility}}\right)$$

$u_i : \prod_{i \in I} S_i \to \mathbb{R}$ is the utility function that maps all players' strategies to a player's utilities.

A <u>finite</u> game is a normal-form game in which the set of players N is a finite set, and the set of strategy profiles S is finite. ♣

Definition 1.10 (Mixed/Pure Strategy)

A mixed strategy for player i is a probability distribution $\sigma_i \in \Delta(S_i)$.

Elements of S_i are called pure strategies. ♣

Definition 1.11 (Dominant/Dominated Strategy)

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominant strategy** for i in G, if we have $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \neq \sigma_i, \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i, u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **weakly dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i$, $u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$ and there is a $\sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$, $u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$ ♣

Lemma 1.2

1. *A dominant strategy is always pure.*
2. *A strategy σ_i' dominates σ_i iff $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i})$, for all pure strategy profiles $s_{-i} \in S_{-i}$.* ♡

Definition 1.12 (Belief, Best Response)

A **belief** for player i is a probability distribution $\mu \in \Delta(S_{-i})$.

A strategy $\sigma_i \in \Delta(S_i)$ is the **best response** to beliefs μ if it solves the problem of $\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, s_{-i})$.

Denote the set of all best responses to μ by $\beta_i(\mu)$. ♣

Lemma 1.3 (Mixed Strategy is BR iff its Pure Strategies are Indifferent)

A mixed strategy σ_i is in $\beta_i(\mu)$ iff every pure strategy in the support of σ_i is in $\beta_i(\mu)$. In particular, every strategy in the support of σ_i yields the same payoff to i. ♡

Theorem 1.3 (Domination Theorem rephrased)

In a finite game, a strategy is dominated iff there is no belief to which it is a best response. ♡

Definition 1.13 (Algorithm: Iterated Elimination of Dominated Strategies (IEDS))

Let $(N, (S_i), (u_i))$ be a finite game; $N = [n]$.

- We define (inductively) n sequences of sets of mixed strategies.
- Let $D_i^0 = \Delta(S_i)$.
- Given $D_1^{k-1}, \ldots, D_n^{k-1}$, let

$$D_i^k = \left\{\sigma_i : \nexists \bar{\sigma}_i : u_i(\sigma_i, \sigma_{-i}) < u_i(\bar{\sigma}_i, \sigma_{-i}) \, \forall \sigma_{-i} \in \times_{j \neq i} D_j^{k-1}\right\}.$$

- Note that $\left\{D_i^k\right\}$ is a decreasing sequence of sets.
- Let $D_i = \cap_{k=0}^{\infty} D_i^k$.
- The set $D = \times_{i=1}^n D_i$ be the set of strategies that survive the iterated elimination of dominated strategies.

A game is called **dominance-solvable** if D is a singleton. ♣

Definition 1.14 (Rationalizable Strategies)

- $R_i^0 = \Delta(S_i)$.
- Given $R_1^{k-1}, \ldots, R_n^{k-1}$, Let

$$Z_i^k = \left\{s_i \in S_i : \sigma_i(s_i) > 0 \text{ for some } \sigma_i \in R_i^{k-1}\right\}$$
$$R_i^k = \left\{\sigma_i \in \Delta(S_i) : \exists \mu \in \Delta\left(\times_{j \neq i} Z_j^k\right) \text{ s.t. } \sigma_i \in \beta_i(\mu)\right\}$$

Note: $\left\{R_i^k\right\}_{k=0}^{\infty}$ is a decreasing sequence of sets.

Let $R_i = \cap_{k=0}^{\infty} R_i^k$.

The **rationalizable strategies** are the elements of $R = \times_{i=1}^n R_i$. ♣

Lemma 1.4

In a finite game, R is always non-empty and contains a pure strategy profile.

Proposition 1.1

$\sigma_i \in \Delta(S_i)$ *is* ***rationalizable*** *iff there are sets* $Z_1, ..., Z_n, Z_j \subseteq S_j$ *such that*

1. $\sigma_i \in \beta_i(\mu_i)$ *for some* $\mu_i \in \Delta(\times_{h\neq i} Z_h)$.
2. *for every* $s_j \in Z_j$ *there is* $\mu_j \in \Delta(\times_{h\neq j} Z_h)$ *such that* $s_j \in \beta_j(\mu_j)$.

Corollary 1.1 (Rationalizable = IEDS)

Rationalizable strategies are exactly the strategies survive the iterated elimination of dominated strategies,

$$R = D$$

1.1.4 Nash Equilibrium and Existence

Definition 1.15 (Nash Equilibrium)

A strategy profile $\Sigma = (\sigma_1, ..., \sigma_I)$ is a **Nash** equilibrium of the game G if for every $i \in I$, we have: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Delta(S_i)$ (no profitable deviation). In other words,

1. σ_i is the best response to beliefs $\mu_i \in \Delta(S_{-i})$
2. $\mu_i = \sigma_{-i}$ (correct beliefs).

1. In rationalizable strategies, beliefs can be incorrect.
2. In a Nash equilibrium, beliefs are correct. Any strategy in a Nash equilibrium is rationalizable.

Definition 1.16 (Best Response Correspondence)

In a Nash equilibrium the player i's best response correspondence $\beta_i : \Delta(S_{-i}) \to 2^{\Delta(S_i)}$ is defined as $\beta_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})$. Let $\beta(\sigma) = \times_{i\in I}\beta_i(\sigma_{-i})$. Then σ is a Nash equilibrium iff $\beta(\sigma) = \sigma$. β is called the **best response correspondence** of the game.

Theorem 1.4 (Existence of Nash Equilibrium)

A Nash equilibrium exists in a finite game Γ*, if for all* $i \in I$*,*

(i). S_i *is non-empty, convex, compact, subset of* $\mathbb{R}^m$ *(i.e., for some finite dimensions of real numbers).*

(ii). $u_i(s_i, ..., s_I)$ *is continuous in* $(s_i, ..., s_I)$ *and quasi-concave in any* s_i.

Proof 1.4

We prove a lemma for the best response correspondence $\beta_i(s_{-i}) = \text{argmax}_{s_i\in S_i} u(s_i, s_{-i})$ *firstly.*

Lemma 1.5

Suppose $\{S_i\}_{i\in I}$ are non-empty. Suppose that S_i is compact and convex and u_i is continuous in $(s_i, ..., s_I)$ and quasi-concave in any s_i, then best response correspondence $\beta_i(s_{-i})$ is non-empty, convex-valued and uhc. ♡

Proof 1.5

*This lemma is proved by Berge's Maximum Theorem (Theroem **??**).*

*Consider the best response correspondence of the game β with $\beta(s_i, ..., s_I) = \{\beta_1(s_{-1}), ..., \beta_I(s_{-I})\}$. As we proved β is non-empty, convex-valued and uhc from S to S where S is non-empty, compact, and convex. By the Kakutani's Fixed Point Theorem (Theorem **??**), we have β has a fixed point $s \in S$, which should be the Nash equilibrium.*

1.1.5 Bayesian Game

Definition 1.17 (Bayesian Game)

A **Bayesian game** is defined by

$$\Gamma = (I, \Omega, \{A_i\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}, \{F_i\}_{i\in I})$$

where Ω is the state space, $u_i : A\times\Omega$ is i's payoff function, and $F_i \in \Delta(\Omega \times \Theta_i)$ is the (prior) distribution of the player i's type. ♣

Definition 1.18 (Normal-form Bayesian game)

Assume a finite game. The **normal-form game** can be represented by

$$(I, (S_i, U_i)_{i\in I})$$

defined by letting S_i be the set of strategies based on types $s_i : \Theta_i \to A_i$ and

$$U_i(s) = \sum_{\omega\in\Omega} \sum_{(\theta_i)_{i\in I}\in\Theta} p(\omega, \theta_1, ..., \theta_I) u_i(s_1(\theta_1), ..., s_I(\theta_I), \omega)$$

for all $s \in S$.

A **Bayesian Nash equilibrium** (BNE) of a Bayesian game is a strategy profile $(s_1, ..., s_n)$ that is a Nash equilibrium of the derived normal-form game. ♣

Definition 1.19 (Best Response, Interim Payoff)

s_i is a BR to s_{-i} iff for all θ_i, $s_i(\theta_i)$ maximizes the **interim payoff** of player i. The interim payoff is defined by the expected payoff given the type θ_i of player i by playing action a_i.

$$\mathbb{E}_{\omega\in\Omega,\tilde{\theta}_{-i}\in\Theta_{-i}}[u_i(a_i, s_{-i}(\tilde{\theta}_{-i}), \omega)|\theta_i]$$

♣

1.2 Adverse Selection

Consider a labor market that has many identical firms. In competitive equilibrium, firms' profits are 0. Firms are price (wage) takers, risk-neutral, and CRS. There are continuum of workers with productivity levels $\theta \in [\underline{\theta}, \overline{\theta}]$ (Assume workers work if it is indifferent for them between employment and non-employment).

1. $\theta \sim F$, $F(\cdot)$ is a c.d.f. over $[\underline{\theta}, \overline{\theta}]$.
2. N is the total mass of workers.
3. Type θ worker has a reservation utility $r(\theta)$.

- Suppose the competitive equilibrium wages are $\theta = w^*(\theta)$.
- An allocation is denoted by $I : [\underline{\theta}, \overline{\theta}] \to \{0, 1\}$, where $I(\theta) = 0$ denotes θ is unemployed and $I(\theta) = 1$ denotes θ is employed.
- Aggregate welfare = sum of utilities of all participants

$$= N \int_{\underline{\theta}}^{\overline{\theta}} [I(\theta) \times \theta + [1 - I(\theta)]r(\theta)] \, dF(\theta)$$

  Then we have the optimal allocation satisfies

$$I^*(\theta) = \begin{cases} 1, & \theta > r(\theta) \\ 0, 1 & \theta = r(\theta) \\ 0, & \theta < r(\theta) \end{cases}$$

In the asymmetric information case,

Definition 1.20

w is CE wage if $w = \mathbb{E}[\theta|r(\theta) \leq w]$.

♣

1.2.1 Adverse Selection

Assumption

(A1). r is strictly increasing in θ.

(A2). $F(\cdot)$ has a strictly positive density, $F(\theta) > 0, \forall \theta \in [\underline{\theta}, \overline{\theta}]$.

(A3). $r(\theta) \leq \theta$ (outside option is worse than productivity, i.e., full employment is optimal).

Lemma 1.6

Under A1-A3, $\Phi(w) := \mathbb{E}[\theta|r(\theta) \leq w]$ is well-defined, continuous, and non-decreasing.

♡

Hence, there exists underemployment, which makes 1^{st} welfare theorem fails. There may exist multiple CEs, where the one with the highest wage Pareto dominates others.

Example 1.1 Suppose $\theta \in [0,2]$, $F(\theta) = \frac{\theta}{2}$, $f(\theta) = \frac{1}{2}$, $r(\theta) = \alpha\theta, \alpha \in (0,1)$.

$$\mathbb{E}[\theta | r(\theta) \leq w] = \mathbb{E}\left[\theta | \theta \leq \frac{w}{\alpha}\right] = \begin{cases} 1, & w \geq 2\alpha \\ \frac{1}{F(\frac{w}{\alpha})} \int_0^{\frac{w}{\alpha}} \theta f(\theta) d\theta = \frac{w}{2\alpha}, & w \leq 2\alpha \end{cases}$$

CEs are given by $\mathbb{E}[\theta | r(\theta) \leq w] = w$. $w^* = 0$ is always CE and $w^* = 1$ is CE if $\alpha \leq \frac{1}{2}$.

1.2.2 Game Theoretical Approach

1. Suppose there are two firms setting wages simultaneously.
2. Workers observe the wages in stage 1 and make an employment decision.

Let W^* be the set of CE wages and $w^* := \max W^*$.

Lemma 1.7

$\forall w' \in \left(w^*, \overline{\theta}\right]$: $\mathbb{E}[\theta | r(\theta) \leq w'] < w'$.

Proof 1.6

Suppose by the contradiction that $\exists w' \in \left(w^*, \overline{\theta}\right]$ *s.t.* $\mathbb{E}[\theta | r(\theta) \leq w'] \geq w'$. *Since* $\mathbb{E}[\theta | r(\theta) \leq \overline{\theta}] < \overline{\theta}$, *there must exist a* $w'' \in [w', \overline{\theta})$ *s.t.* $\mathbb{E}[\theta | r(\theta) \leq w''] = w''$ *by intermediate value theorem, which contradicts to the definition of* w^*.

Proposition 1.2

(i). If $w^* > r(\underline{\theta})$ *and* $\exists \epsilon > 0$ *s.t.* $\mathbb{E}[\theta | r(\theta) \leq w'] > w', \forall w' \in (w^* - \epsilon, w^*)$. *Then, there is a unique SPE where both firms set wage* $= w^*$.

(ii). If $w^* = r(\underline{\theta})$ *(complete market shutdown at* w^**), there are multiple SPE that all give the same outcome as complete market shutdown where both firms set wage* $= w^*$.

Proof 1.7

Lemma 1.8

In all SPE, firms make zero profits.

Proof 1.8

Suppose not, i.e., at least one firm makes strictly positive profits. Then, the total profits of firms 1&2,

$$\Pi = M(\bar{w})\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}] - \bar{w}\right] > 0$$

where $\bar{w}$ is the max wage set by the two firms and $M(\bar{w})$ is the mass of workers willing to work at $\bar{w}$. At least one firm, i, makes profit $\leq \frac{\Pi}{2}$. Then, i's profits from setting $\bar{w}+\delta$, with $\delta \to 0^+$, is higher:

$$\begin{aligned} &M(\bar{w}+\delta)\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}+\delta] - \bar{w} - +\delta\right] \\ \geq &M(\bar{w})\left[\mathbb{E}[\theta|r(\theta) \leq \bar{w}+\delta] - \bar{w} - +\delta\right] \to \Pi \text{ as } \delta \to 0 \end{aligned}$$

Hence, the i has incentive to deviate.

Lemma 1.9

In all SPE, firm i sets $w_i \leq w^*, i \in \{1,2\}$. ♡

Proof 1.9

Directly given by Lemma 1.7 and Lemma 1.8.

(i): In SPE, no firm i sets $w_i < w^*$: suppose $w_i < w^*$ and let $j \neq i$, take any w_j' s.t. $w_j' \in (w_i, w^*)$ and $w_j' > w^* - \epsilon$. Then, j gets profit: $M(w_j')\left[\mathbb{E}[\theta|r(\theta) \leq w_j'] - w_j'\right] > 0$ (by Case (i)'s conditions).

(ii): By Lemma 1.9, both firms set $w_i \leq w^* = r(\underline{\theta})$. Check that $\{(w_1, w_2) : w_1, w_2 \leq w^*\}$ is SPE wage profiles.

1.2.3 Planner Intervention

Planner can't observe the true type θ.

The planner's tools:

1. Take over the firms.
2. w_e, employment wage.
3. w_u, unemployment wage.

s.t. budget balanced.

Definition 1.21 (Constrained Efficient)

A CE w is **constrained efficient** if it cannot be Pareto improved upon by an intervention by the planner. ♣

Proposition 1.3 ($w^* := \max W^*$ **is constrained efficient**)

Let W^* be the set of CE wages. $w^* := \max W^*$ is constrained efficient.

Proof 1.10

Note that both firms are making zero profits by the Lemma 1.8. Any CE wage $w \neq w^*$ can be Pareto improved by $\{w_e = w^*, w_u = 0\}$. Then, we prove w^* can't be Pareto improved.

1. *Case 1: if w^* gives full-employment in CE, then w^* is Pareto efficient.*
2. *Case 1: suppose w^* doesn't give full-employment in CE.*

   Consider taking an intervention $w_e \& w_u$. Then, $\{\theta : r(\theta) + w_u \leq w_e\} = [\underline{\theta}, \hat{\theta}]$ for some $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that

   $$r(\hat{\theta}) + w_u = w_e \tag{1.1}$$

   The budget balanced gives

   $$w_e F(\hat{\theta}) + w_u(1 - F(\hat{\theta})) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) \tag{1.2}$$

   Plug (1.1) into (1.2):

   $$\begin{cases} w_u(\hat{\theta}) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) - r(\hat{\theta})F(\hat{\theta}) = F(\hat{\theta}) \left(\mathbb{E}[\theta | \theta \leq \hat{\theta}] - r(\hat{\theta}) \right) \\ w_e(\hat{\theta}) = \int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) + r(\hat{\theta})(1 - F(\hat{\theta})) \end{cases}$$

   Let θ^* be s.t. $r(\theta^*) = w^*$. Because w^* is a CE price, $\mathbb{E}[\theta | \theta \leq \theta^*] = r(\theta^*) = w^*$. So, CE with w^* can be implemented by $w_u(\theta^*) = 0$ and $w_e(\theta^*) = w^*$.

   (a). *If $\hat{\theta} < \theta^*$. $\underline{\theta}$ is worse off under the intervention since $w_e(\hat{\theta}) < w^*$.*

   (b). *If $\hat{\theta} > \theta^*$. $\overline{\theta}$ is worse off under the intervention since $w_u(\hat{\theta}) = F(\hat{\theta}) \left(\mathbb{E}[\theta | \theta \leq \hat{\theta}] - r(\hat{\theta}) \right) < 0$ by the Lemma 1.7*

1.2.4 Signaling

Suppose the worker $\theta \in [\underline{\theta}, \overline{\theta}]$ can properly and costlessly reveal his type to the firms. Then,

Lemma 1.10

All workers revel their types.

Spence's Job Market Signaling Model One worker has productivity $\theta \in \{\theta_L, \theta_H\}$ with $P(\theta_H) = \lambda$. The worker signal through his education with cost $e > 0$. The education doesn't change his productivity. The payoff of the worker is the wage minus the cost:

$$u(w, e|\theta) = w - c(e, \theta)$$

where $c(0,\theta) = 0, c_e(e,\theta) := \frac{\partial c(e,\theta)}{\partial e} > 0, c_\theta(e,\theta) := \frac{\partial c(e,\theta)}{\partial \theta} < 0$, and $c_{e\theta}(e,\theta) := \frac{\partial^2 c(e,\theta)}{\partial e \partial \theta} < 0$ (Single-Crossing Property, the difference $c(e,\theta_L) - c(e,\theta_H)$ is increasing in e (i.e., $c_e(e,\theta_L) - c_e(e,\theta_H) > 0$), which means if $c(e,\theta_L)$ and $c(e,\theta_H)$ intersect as functions of e, they only intersect at one time.)

1. Stage 0: Nature chooses the $\theta \in \{\theta_L, \theta_H\}$ with $P(\theta_H) = \lambda$.
2. Stage 1: The worker learns θ and chooses $e(\theta) \geq 0$.
3. Stage 2: Firms observe $e(\theta)$. Then, they simultaneously make wage offers w_1 and w_2.
4. Stage 3: The worker observes w_1, w_2 and makes employment decision.

Let $r(\theta_L)$ and $r(\theta_H) = 0$. Let $\mu(e) \in [0,1]$ be the probability that in the beginning of stage 2, firms think that the worker is θ_H type with probability $\mu(e)$ when observing e. The corresponding expected productivity (the highest wage) that the firm can pay is

$$w(e) = \mu(e)\theta_H + (1-\mu(e))\theta_L$$

In stage 2, both firm will set $w(e)$ (complete competition).

Definition 1.22 (Perfect Bayesian Equilibrium)

A PBE is a strategy profile ($e^*(\theta_L)$, $e^*(\theta_H)$, $w_1^* : \mathbb{R}_+ \to \mathbb{R}$, $w_2^* : \mathbb{R}_+ \to \mathbb{R}$), and a belief $\mu^* : \mathbb{R} \to [0,1]$ such that

1. $\forall \theta \in \{\theta_L, \theta_H\}$, the worker strategy optimal given firm strategies.
2. The belief $\mu^*(e)$ is derived from $\lambda, e^*(\theta_L), e^*(\theta_H)$ via Bayes' rule whenever possibly (on the equilibrium path). Outside the equilibrium path the belief $\mu^*(e)$ is arbitrarily.
3. Firms offer wages that form a NE of the stage 2 game, where their belief $\mu^*(e)$ about their workers' type. (sequential rationality).

♣

We simplify the game by backward induction:

1. Stage 3: The worker chooses the highest wage off if it is ≥ 0.
2. Stage 2: After observing $e(\theta)$, firms chooses the wage as the expected productivity in NE,

$$w^*(e) = \mu^*(e)\theta_H + (1-\mu^*(e))\theta_L$$

   because it is a Bertrend competition.

Separating Equilibrium In separating equilibrium, $e^*(\theta_L) \neq e^*(\theta_H)$.

Lemma 1.11

In any separating PBE, $w^*(e^*(\theta)) = \theta, \forall \theta \in \{\theta_L, \theta_H\}$.

♡

Proof 1.11

By Bayes' rule, after firm observe $e^*(\theta_L)$, $\mu^*(e^*(\theta_L)) = 0$. Then, $w^*(e^*(\theta_L)) = \theta_L$. ($e^*(\theta_H)$ is similar.)

Lemma 1.12

In separating PBE, low type always chooses zero education, $\theta^*(\theta_L) = 0$.

Proof 1.12

If not, the low type worker always has profitable deviation, $\theta^*(\theta_L) = 0$.

Lemma 1.13

Define $\underline{e}$ and $\overline{e}$ such that

1. $\theta_L = \theta_H - c(\underline{e}, \theta_L)$ *(the lowest effort can prevent the low type from mimicking high type) and*
2. $\theta_L = \theta_H - c(\overline{e}, \theta_H)$ *(the highest effort can prevent the high type from pooling with low type).*

Then, in all separating PBEs, $e \in [\underline{e}, \overline{e}]$.

Conversely, $\forall \hat{e} \in [\underline{e}, \overline{e}]$, there is a separating PBE where $e^*(\theta_H) = \hat{e}$.

These different PBEs are Pareto ranked. High type prefers the PBE with a lower e (the best is the one with $e^*(\theta_H) = \underline{e}$.)

Pooling PBE $e^*(\theta) = e^*, \theta \in \{\theta_L, \theta_H\}$, $\mu^*(e^*) = \lambda$, and $w^*(e^*) = \mathbb{E}[\theta]$.

Lemma 1.14

Define e' by $\theta_L = \mathbb{E}[\theta] - c(e', \theta_L)$ (the highest effort can prevent the low type from choosing $e = 0$ and get $w = \theta_L$.)

Then, for all pooling PBE, $e^*(\theta_L) = e^*(\theta_H) = e^* \in [0, e']$. Conversely, for all $\hat{e} \in [0, e']$, there is a pooling PBE with $e^* = \hat{e}$.

1.2.5 Cho-Kreps Intuitive Criterion

Definition 1.23 (Equilibrium Dominated Message)

A message is **equilibrium dominated** for a type if the type must do strictly worse by sending the message than it does in equilibrium (i.e., payoff in eq. is strictly better than maximum payoff from deviating).

Definition 1.24 (Cho-Kreps Intuitive Criterion)

If an information set is off the eq. path and a message is eq. dominated for a type, then beliefs should assign zero probability to the message coming from that type (if possible).

Fix a PBE $e^*(\theta), \theta \in \{\theta_L, \theta_H\}, \mu^*(\cdot)$ (We know $w_1^*(e) = w_2^*(e) = \mu(e)\theta_H + (1 - \mu(e))\theta_L$). Let $u^*(\theta), \theta \in \{\theta_L, \theta_H\}$ be the PBE utility of the type θ worker.

The criterion requires the (off-path) belief $\mu^*(e) := P(\tilde{\theta} = \theta_H | e) = 1 - P(\tilde{\theta} = \theta_L | e)$ satisfies

$$P(\tilde{\theta} = \theta | e) = 0, \forall e, \theta$$

such that

1. $u^*(\theta) > \max_{w\in[\underline{\theta},\bar{\theta}]}[w - c(e,\theta)]$
2. $\exists\theta'$ s.t. $u^*(\theta') \leq \max_{w\in[\underline{\theta},\bar{\theta}]}[w - c(e,\theta')]$ (make sure the sum of beliefs given e is nonzero.)

In this application, the only PBE that survives Intuitive Criterion is the best separating PBE, $e^*(\theta_H) = \underline{e}$ (the lowest effort).

1.2.6 Screening Model

Workers can undertake a contractible/observable task level $t \geq 0$. The utility of a worker is defined by $u(w,t,\theta) := w - c(t,\theta)$, where $c(\cdot,\cdot)$ satisfies the same assumption as in signaling model 1.2.4.

The Game follows

1. Stage 1: Two firms simultaneously determine sets of contracts, (w,t).
2. Stage 2: The worker observes all offer contracts and makes employment decision. (If indifference, choose lower task contract, favor employment over unemployment. If contracts of firms are indifferent, choose each with probability 1/2.)

The null contract is $(w,t) = (0,0)$. Assume WLOG at stage 1, each firm appears a non-empty set of contracts.

Perfect Information

Proposition 1.4 (Perfect Information)

If firms can observe the worker types, then in SPE firms make zero profit and type θ_i worker signs $(w_i^*, t_i^*) = (\theta_i, 0)$.

Proof 1.13

Claim 1.3

Firms make zero profits from this contract.

Proof 1.14

Suppose not,

- $w_i^* > \theta_i \Rightarrow$ *negative profits, firms benefit from offering null contract.*
- $w_i^* < \theta_i \Rightarrow$ *Let Π be the total profits of the firms. Then one of the firms makes profit $\leq \frac{\Pi}{2}$. Then, this firm can benefit from offering $(w_i^* + \Delta, t_i^*)$, where $\Delta \to 0^+$.*

Then, we prove the firms must choose $(w_i^*, t_i^*) = (\theta_i, 0)$. Suppose by the way of contradiction that $t_i^* > 0$. Then, one firm can profitably deviate by offering $(w_i^*, 0)$.

Asymmetric Information

Lemma 1.15

In any SPE, firms obtain zero profits,

Proof 1.15

Firms must make profits ≥ 0*. Suppose by the way of contradiction that the total profit* $\Pi > 0$*. Let* (w_L, t_L) *be the contract signed by* θ_L *and* (w_H, t_H) *be the contract signed by* θ_H*. One firm can profitably deviate by offering* $(w_L + \Delta, t_L)$ *and* $(w_H + \Delta, t_H)$*, where* $\Delta \in (0, \Pi)$*.*

Lemma 1.16

*There is **no** pooling SPE.*

Proof 1.16

Suppose for a contradiction, $\exists$ *an SPE where both worker types sign* $(w_p = \mathbb{E}[\theta], t_p)$*. Suppose one firm offers* (w_p, t_p)*, then another firm can only employ high type workers by offering* $(\tilde{w}, \tilde{t})$*, where* $\tilde{w} - c(\tilde{t}, \theta_H) > \mathbb{E}[\theta] - c(t_p, \theta_H)$*,* $\tilde{w} - c(\tilde{t}, \theta_L) < \mathbb{E}[\theta] - c(t_p, \theta_L)$*, and* $\tilde{w} < \theta_H$*. (The existence is given by* $\frac{\partial^2 c(t,\theta)}{\partial t \partial \theta} < 0$*.)*

Lemma 1.17

Let (w_L, t_L) *be the contract signed by* θ_L *and* (w_H, t_H) *be the contract signed by* θ_H *in separating SPE. Then,* $w_L = \theta_L$ *and* $w_H = \theta_H$*.*

Proof 1.17

Suppose $w_i > \theta_i, i \in \{L, H\}$*, firms benefit from not offering this contract. So,* $w_L \leq \theta_L$ *and* $w_H \leq \theta_H$*.*

1. $\underline{w_L = \theta_L}$*: Suppose* $w_L < \theta_L$*. Either firm can profitably deviate by setting* (w'_L, t_L) *such that* $w_L < w'_L < \theta_L$*. This offer can win all low-type workers and get a positive profit from hiring them. If* $w'_L - c(t_L, \theta_H) \geq w_H - c(t_H, \theta_H)$*, the offer can also hire high-type workers, which also give positive profit for the firm. Hence, there is a contradiction.*
2. $\underline{w_H = \theta_H}$*: Suppose* $w_H < \theta_H$*, firms get positive profits, which contradicts to the Lemma 1.15.*

Lemma 1.18

θ_L *signs the contract* $(\theta_L, 0)$ *in SPE.*

Proof 1.18

Suppose $t_L > 0$*. One firm can profitably deviate by offering* $(\theta_L - \Delta, 0)$*.*

Proposition 1.5

In any (pure strategy) SPE, θ_L *signs* $(w_L, t_L) = (\theta_L, 0)$ *and* θ_H *signs* $(w_H, t_H) = (\theta_H, t_H)$*, where* t_H *solves*

$$\theta_H - c(t_H, \theta_L) = \theta_L$$

If $\lambda := P(\theta_H)$ is high, the pure SPE may not exist (exist $(\tilde{w}, \tilde{t})$ can attract both types and make positive profit). Cross subsidizing deviation by a firm (prices one product above its market value to fund another product), $(\tilde{w}, \tilde{t})$ (signed by low type) and $(\tilde{\tilde{w}}, \tilde{\tilde{t}})$ (signed by high type), is a profitable deviation if λ is large enough.

Chapter 2 Mechanism Design

2.1 Mechanism Design

Design incentives for agents to reveal their types or achieve particular society outcomes.

Given a “direct” mechanism,

1. the set of agents I with utility function $u_i(x;\theta_i), i \in I$,
2. alternatives (outcomes for the society) X,
3. types (of agents) $\Theta = (\Theta_1, ..., \Theta_I)$ with prior probability ϕ over Θ,
4. and a **social choice function** (SCF) $f : \Theta \to X$.

Definition 2.1 (Mechanism $\Gamma = (S, g)$)

A **mechanism** is represented as

$$\Gamma = (S, g)$$

where $S \triangleq (S_1, ..., S_I)(S_1, ..., S_I)$ represents the set of strategies, S_i represents the strategy set of agent i, and $g : S \triangleq (S_1, ..., S_I) \to X$ is the outcome function that determines the social outcome. ♣

A **Bayesian game induced by** Γ is $(I, S, \Theta, \phi, \tilde{u})$, where the payoffs functions are

$$\tilde{u}_i(s;\theta_i) = u_i(g(s);\theta_i)$$

for all $i \in I, s \in S$, and $\theta_i \in \Theta_i$.

2.1.1 Implement in Dominant Strategies

Definition 2.2 (Γ Implements f)

A mechanism Γ (indirectly) **implements** a social choice function (SCF) f if there exists an “equilibrium” $s^*(\cdot) = (s_1^*(\cdot), ..., s_I^*(\cdot))$ of the Bayesian game induced by Γ such that

$$g(s_1^*(\theta_1), ..., s_I^*(\theta_I)) = f(\theta_1, ..., \theta_I)$$

for all $(\theta_1, ..., \theta_I) \in \Theta$. Here the “equilibrium” is a dominant strategy equilibrium or BNE. ♣

That is, the equilibrium in a game induced by Γ gives the same outcome as the outcome of f given by revealing agents’ true types.

Definition 2.3 (Direct Mechanism)

A mechanism is **direct mechanism** if agents directly report their types (types are observable). $S_i = \Theta_i$ for all $i \in I$ and $g(\theta) = f(\theta)$ for all $\theta = (\theta_1, ..., \theta_I) \in \Theta$. So, a direct mechanism can be represented by $\Gamma = (\Theta, f(\cdot))$. ♣

In indirect mechanism agents don't report their types directly. Types can be observed only indirectly through signals or behavior.

A strategy is weakly dominant if for all $\theta_i \in \Theta_i$ and all $s_{-i}(\cdot) \in S_{-i}$, we have $u_i(g(s_i(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$ for all $s_i' \neq s_i$.

Definition 2.4 (Dominant Strategy Equilibrium)

Strategy profile $s^* = (s_1^*(\cdot), ..., s_I^*(\cdot))$ is a **dominant strategy (D-S) equilibrium** of $\Gamma = (S, g(\cdot))$ if for all $i \in I$ and $\theta_i \in \Theta$, we have, for all $s_i' \in S_i$ and $s_{-i} \in S_{-i}$:

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

equivalently, in the Bayesian game induced by Γ,

$$\tilde{u}_i(s_i^*(\theta_i), s_{-i}, \theta_i) \geq \tilde{u}_i(s_i', s_{-i}, \theta_i)$$

♣

Definition 2.5 (Implement in dominant strategies)

Γ **implements** f in **dominant strategies** if $\exists$ a dominant strategy (D-S) equilibrium s^* of Γ such that $g(s^*(\theta)) = f(\theta)$. ♣

"Γ implements f in dominant strategies" means that the results of the direct mechanism, $(\Theta, f(\cdot))$, are equivalent to the results of a D-S equilibrium of another (indirect) mechanism Γ. That is, Γ can be used as $(\Theta, f(\cdot))$ "equivalently."

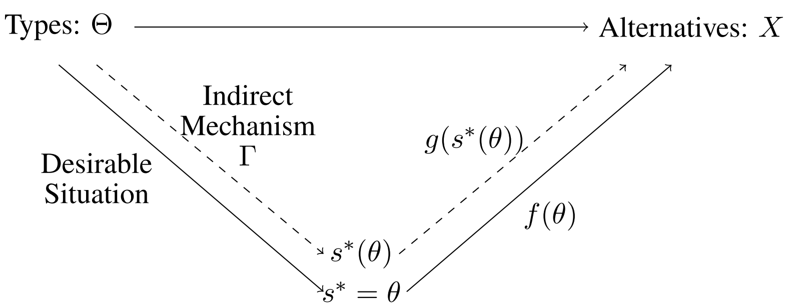

Figure 2.1: How Mechanism Design works

2.1.2 Dominant-Strategy-Incentive-Compatible (DSIC)/Strategy-Proof

Definition 2.6 (Strategy-Proof, DSIC)

f is **strategy-proof** (also called dominant-strategy-incentive-compatible, **DSIC**) if

$$s_i^*(\theta_i) = \theta_i, \quad \forall \theta_i \in \Theta_i, i \in I$$

is a dominant strategy (D-S) equilibrium of the direct mechanism $\Gamma = (\Theta, f(\cdot))$.

Theorem 2.1 (Revelation Principle)

If $\exists$ a mechanism $\Gamma = (S, g(\cdot))$ that implements f in dominant strategies (i.e., $\exists$ a D-S equilibrium s^* of Γ such that $g(s^*(\theta)) = f(\theta)$). Then f is strategy-proof (DSIC).

Note *Based on the Revelation Principle, if a "indirect" mechanism has a D-S equilibrium s^*, then there exists a "direct" DSIC mechanism f with $f(\theta) = g(s^*(\theta))$.*

Proof 2.1

Given Γ implements f in dominant strategies, there is a D-S equilibrium $s^* = (s_1^*(\cdot), ..., s_I^*(\cdot))$ such that $g(s^*(\theta)) = f(\theta)$.

By the definition of D-S equilibrium,

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

By substituting $g(s^*(\theta)) = f(\theta)$, we have

$$u_i(f(\theta_i, \theta_{-i}), \theta_i) \geq u_i(f(\theta_i', \theta_{-i}), \theta_i), \quad \forall \theta_i' \in \Theta_i$$

which gives that f is DSIC.

2.1.3 Bayesian-Incentive-Compatible (BIC)

Definition 2.7 (BIC)

f is Bayesian-incentive-compatible (B.I.C.) if

$$s_i^*(\theta_i) = \theta_i, \quad \forall \theta_i \in \Theta_i, i \in I$$

is a BNE of the Bayesian game induced by the direct mechanism $\Gamma = (\Theta, f(\cdot))$.

BIC is a weaker condition than DSIC, because a BNE must also be a D-S equilibrium.

Theorem 2.2 (Revelation Principle (BIC))

If $\exists$ a mechanism $\Gamma = (S, g(\cdot))$ that implements f in BNE (i.e., $\exists$ a BNE s^* of Γ such that $g(s^*(\theta)) = f(\theta)$). Then f is BIC.

Note *Based on the Revelation Principle, if a "indirect" mechanism has a BNE s^*, then there exists a* *"direct" BIC mechanism f with $f(\theta) = g(s^*(\theta))$.* ♡

2.1.4 Negative Results: dictatorial SCF f

Theorem 2.3 (Gibbard-Satterthwaite Theorem)

*Suppose that $|X| \geq 3$ and a social choice function f is surjective (i.e., $\forall x \in X\ \exists(\theta_1, ..., \theta_I) \in \Theta$ s.t. $f(\theta_1, ..., \theta_I) = x$). Then, f is strategy-proof (DSIC) $\Leftrightarrow$ f is dictatorial (**??**, i.e., $\exists i^* \in \{1, ..., I\}$ such that $f(\theta) \in \operatorname{argmax}_{x\in X} u_{i^*}(x; \theta_{i^*})$ for all $\theta \in \Theta$).* ♡

Note: By the revelation principle, under the conditions of the Theorem, there is no mechanism that implements a non-dictatorial SCF f in dominant strategies.

2.2 Quasi-linear Model

Consider $x = (k, \underbrace{t_1, ..., t_I}_{t}) \in X = K \times \mathbb{R}^I$, in our example, K represents a set of choices for projects and $\mathbb{R}^I$ represents the set of transfers for all agents.

Each agent has a quasi-linear function that represents her utility:

$$u_i(k, t, \theta_i) = v(k, \theta_i) + t_i$$

where $v : K \times \Theta_i \to \mathbb{R}$ represents the utility without transfers.

Let $p(\cdot) = (k(\cdot), t(\cdot))$ represents the "project-choice rule" $k : \Theta \to K$ and the "transfer rule" $t : \Theta \to \mathbb{R}^I$.

Definition 2.8 (ex-post efficient)

$k(\cdot) : \Theta \to K$ is **ex-post efficient** if $\nexists \left(\theta \in \Theta, k' \in K, t = (t_1, ..., t_I) \in \mathbb{R}^I\right)$ such that

(1). $\sum_{i=1}^I t_i = 0$

(2). $v_i(k', \theta_i) + t_i > v_i(k(\theta), \theta_i), \forall i \in I$

i.e., we can't get a higher total social welfare. (Because of the transfers, a higher social welfare can make everyone better off.) ♣

Proposition 2.1 (ex-post efficient $\Leftrightarrow$ maximizing the sum of utilities)

$\forall$ project-choice rule $k(\cdot)$, $k(\cdot)$ is ex-post efficient <u>if and only if</u> $k(\cdot)$ maximizes the sum of utilities, i.e., $\forall \theta \in \Theta$ and $\forall k' \in K$,

$$\sum_{i=1}^I v_i(k(\theta), \theta_i) \geq \sum_{i=1}^I v_i(k', \theta_i)$$

♠

Proof 2.2

"$\Leftarrow$": Suppose by the way of contradiction that there exists (θ, k', t) such that $\sum_{i=1}^{I} t_i = 0$ and $v_i(k', \theta_i) + t_i > v_i(k(\theta), \theta_i)$, $\forall i \in I$. Sum together, there is a contradiction.

"$\Rightarrow$": Suppose by the way of contradiction that there exists (θ, k'), $\sum_{i=1}^{I} v_i(k(\theta), \theta_i) < \sum_{i=1}^{I} v_i(k', \theta_i)$. Then, we can define a t such that satisfies $\sum_{i=1}^{I} t_i = 0$ and $v_i(k', \theta_i) + t_i > v_i(k(\theta), \theta_i)$, $\forall i \in I$. Let $\Delta = \sum_{i=1}^{I} v_i(k', \theta_i) - \sum_{i=1}^{I} v_i(k(\theta), \theta_i)$, then $t_i = v_i(k(\theta), \theta_i) - v_i(k', \theta_i) + \frac{\Delta}{I}, \forall i \in I$ is the transfer-choice we want.

2.2.1 Vickrey-Clarke-Groves Mechanism

Proposition 2.2 (VCG Mechanism)

Suppose $k^*(\cdot)$ is ex-post efficient project choice rule. For each $i \in \{1, ..., I\}$, let $h_i : \Theta_{-i} \to \mathbb{R}$ be an arbitrary function.

Define the transfer rule $t(\cdot)$ as follows

$$t_i(\theta_i, \theta_{-i}) = \sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i})$$

Then the SCF $f(\cdot) = (k^*(\cdot), t(\cdot))$ is DSIC. ♠

Proof 2.3

Take any $i, \theta \in \Theta$ and let $\hat{\theta}_i \in \Theta_i$. Reporting truthfully gives higher profits than misreporting $\hat{\theta}_i$:

$$\begin{aligned} v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) &= v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + \sum_{j \neq i} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}) \\ &= \sum_{j=1}^{I} v_j(k^*(\theta), \theta_j) + h_i(\theta_{-i}) \\ &\geq \sum_{j=1}^{I} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) + h_i(\theta_{-i}) \\ &= v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i}) \end{aligned}$$

Hence, VCG mechanism with SCF $f(\cdot) = (k^*(\cdot), t(\cdot))$ is DSIC.

Definition 2.9 (Pivotal VCG Mechanism (Special Case))

Let $h_i(\theta_{-i}) = \max_{k \in K} \sum_{j \neq i} v_j(k, \theta_j)$.

$$t_i(\theta_i, \theta_{-i}) = \sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) - \max_{k \in K} \sum_{j \neq i} v_j(k, \theta_j) \leq 0$$

1. i is **pivotal** if $k^*(\theta)$ doesn't maximize $\max_{k \in K} \sum_{j \neq i} v_j(k, \theta_j)$.
2. i is **not pivotal** if $k^*(\theta)$ maximizes $\max_{k \in K} \sum_{j \neq i} v_j(k, \theta_j)$.

Note i *is* ***not pivotal*** $\Rightarrow t_i(\theta) = 0$. ♣

Example 2.1 Suppose $k \in \{0,1\}, \theta \in \Theta \subset \mathbb{R}^I$, $v_i(k,\theta_i) = k\theta_i$. Since $\sum_{i=1}^I v_i(k,\theta_i) = k\sum_{i=1}^I \theta_i$, $k^*(\cdot)$ is ex-post efficient: $k^*(\theta) = 1 \Leftrightarrow \sum_{i=1}^I \theta_i \geq 0$. The pivotal VCG transfers:

$$t_i(\theta) = \begin{cases} \sum_{j\neq i}\theta_j - 0 & \text{if } \sum_{j=1}^I \theta_j \geq 0 > \sum_{j\neq i}\theta_j \\ 0 - \sum_{j\neq i}\theta_j & \text{if } \sum_{j=1}^I \theta_j < 0 \leq \sum_{j\neq i}\theta_j \\ 0 & \text{otherwise} \end{cases}$$

Example 2.2 (Second Price Auction) One indivisible object to be allocated to one of $1,...,I$. Social decision is deciding who gets the object, $K = \{1,...,I\}$, $k = i$ means "i receives the object". $\Theta_i \subseteq \mathbb{R}_+$, $\theta_i \in \Theta_i$ denotes i's valuation for the object $v_i(k,\theta_i) = \begin{cases} \theta_i, & \text{if } k = i \\ 0, & \text{if } k \neq i \end{cases}$. Ex-post efficient $k^*(\cdot)$ allocates the object to the individual with the highest valuation. The pivotal VCG transfers:

$$t_i(\cdot) = \begin{cases} 0 - \theta^{(2)}(\text{the second highest}), & \text{if } k^*(\theta) = i \ (i \text{ is pivotal}) \\ 0 = \theta^{(1)} - \theta^{(1)}, & \text{if } k^*(\theta) \neq i \end{cases}$$

Example 2.3 (Uniform-Price Auction) m-identical indivisible objects ($m < I$). Each agent can consume 0 or 1 object. $K = \{M \subset \{1,...,I\} \mid |M| = m\}$, where $k = M$ is the set of agents who receive an object. $v_i(k,\theta_i) = \begin{cases} \theta_i, & i \in k \\ 0, & i \notin k \end{cases}$. The ex-post efficient $k^*(\cdot)$ allocates the objects to top m-valuation agents. The pivotal VCG transfers:

$$t_i(\theta) = \begin{cases} \left(\sum_{j=1}^m \theta_{(j)} - \theta_i\right) - \left(\sum_{j=1}^{m+1} \theta_{(j)} - \theta_i\right) = -\theta^{(m+1)} & , i \in k^*(\theta) \\ 0 & , i \notin k^*(\theta) \end{cases}$$

Example 2.4 (Package Auction) 2 identical indivisible objects to be allocated $I = 3$ agents. Each agent can consume 0, 1, or 2 units. $K = \{(1,1,0),(1,0,1),(0,1,1),(2,0,0),(0,2,0),(0,0,2)\}$. $\Theta_i = \{\theta_i = (v_1,v_2) \in \mathbb{R}_+^2 \mid v_2 \geq v_1 \geq 0\}$. Consider an example,

	θ_1	θ_2	θ_3
v_1	3	4	1
v_2	4	5	6

The ex-post efficient $k^*(\theta) = (1,1,0)$. Then,

$$t_1(\theta) = 4 - 6 = -2, t_2(\theta) = 3 - 6 = -3, t_3(\theta) = 7 - 7 = 0$$

2.2.2 Uniqueness of VCG Mechanism

Assumption *K is a compact subset of a topological space which all singletons are closed (metric spaces $K \subset \mathbb{R}^n$, K compact, or any finite K.)*

Let V_{usc} be the set of upper hemicontinuous functions $v : K \to \mathbb{R}$. (v is upper hemicontinuous if $\forall \alpha \in \mathbb{R}$: $\{k \in K \mid v(k) \geq \alpha\}$ is closed.)

Facts: A upper hemicontinuous function attains the maximum over a compact set. Sum of upper hemicontinuous functions is upper hemicontinuous.

Proposition 2.3 (Green & Laffont 1979)

Suppose that $\forall i : \{v_i(\cdot, \theta_i) : K \to \mathbb{R} \mid \theta_i \in \Theta_i\} = V_{usc}$. *Then, any ex-post efficient and DSIC direct mechanism is a VCG mechanism.*

Proof 2.4

Take any $f(\cdot) = (k^*(\cdot), t_i(\cdot))$ *such that it is ex-post efficient and DSIC. We prove it is VCG mechanism by showing there is a* h_i *satisfies the definition of VCG mechanism.*

Define $\forall i, h_i : \Theta \to \mathbb{R}$ *such that*

$$h_i(\theta) = -\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) + t_i(\theta_i, \theta_{-i})$$

We want to show $h_i(\theta)$ ***is independent of*** θ_i ***and is actually*** $h_i(\theta_{-i})$.

That is, $\forall \theta_i, \hat{\theta}_i, \theta_{-i}$, *we want to show* $h_i(\theta_i, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$.

Lemma 2.1

If $k^*(\theta_i, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$, *then* $h_i(\theta_i, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$.

Proof 2.5

$k^*(\theta_i, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$ *requires*

$$v_i(k^*(\theta_i, \theta_{-i}), \theta_i) + t_i(\theta_i, \theta_{-i}) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i) + t_i(\hat{\theta}_i, \theta_{-i})$$

$$v_i(k^*(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) + t_i(\hat{\theta}_i, \theta_{-i}) \geq v_i(k^*(\theta_i, \theta_{-i}), \hat{\theta}_i) + t_i(\theta_i, \theta_{-i})$$

Since $v_i(k^*(\hat{\theta}_i, \theta_{-i}), \hat{\theta}_i) = v_i(k^*(\theta_i, \theta_{-i}), \hat{\theta}_i)$, *we have* $t_i(\hat{\theta}_i, \theta_{-i}) = t_i(\theta_i, \theta_{-i})$. *Hence,* $h_i(\theta_i, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$.

1. *Case 1:* "$k^*(\theta_i, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$", $h_i(\theta_i, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$ *is given by Lemma 2.1.*
2. *Case 2:* "$k^*(\theta_i, \theta_{-i}) \neq k^*(\hat{\theta}_i, \theta_{-i})$"

   Suppose by the way of contradiction $h_i(\theta_i, \theta_{-i}) \neq h_i(\hat{\theta}_i, \theta_{-i})$, *WLOG, we consider* $h_i(\theta_i, \theta_{-i}) > h_i(\hat{\theta}_i, \theta_{-i})$. *There is an* $\epsilon > 0$ *s.t.* $h_i(\theta_i, \theta_{-i}) > h_i(\hat{\theta}_i, \theta_{-i}) + \epsilon$.

   Define $v : K \to \mathbb{R}$ *such that*

$$v(k) = \begin{cases} -\sum_{j \neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j), & \text{if } k = k^*(\theta_i, \theta_{-i}) \\ -\sum_{j \neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) + \epsilon, & \text{if } k = k^*(\hat{\theta}_i, \theta_{-i}) \\ -C, & \text{otherwise} \end{cases}$$

   where $C > \max_{k \in K} \sum_{j \neq i} v_j(k^*(k, \theta_{-i}), \theta_j)$.

   Hence, v *is upper hemicontinuous,* $v \in V_{usc}$.

By the assumption that $\forall i : \{v_i(\cdot, \theta_i) : K \to \mathbb{R} \mid \theta_i \in \Theta_i\} = V_{usc}$, *we know* $\exists \theta_i^\epsilon \in \Theta_i$ *s.t.* $v_i(\cdot, \theta_i^\epsilon) = v(\cdot)$.

- *Because* $k^*(\cdot)$ *is ex-post efficient,*

$$v_i(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_i^\epsilon) + \sum_{j\neq i} v_j(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_j) \geq v_i(k^*(\hat{\theta}_i, \theta_{-i}), \theta_i^\epsilon) + \sum_{j\neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j)$$
$$= v(k^*(\hat{\theta}_i, \theta_{-i})) + \sum_{j\neq i} v_j(k^*(\hat{\theta}_i, \theta_{-i}), \theta_j) = \epsilon$$

  By the definition of $v(\cdot)$, *we have*

$$v_i(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_i^\epsilon) + \sum_{j\neq i} v_j(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_j) = v(k^*(\theta_i^\epsilon, \theta_{-i})) + \sum_{j\neq i} v_j(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_j) \leq \epsilon$$

  Hence, we can conclude $k^*(\theta_i^\epsilon, \theta_{-i}) = k^*(\hat{\theta}_i, \theta_{-i})$. *Then, by the Lemma 2.1,* $h_i(\theta_i^\epsilon, \theta_{-i}) = h_i(\hat{\theta}_i, \theta_{-i})$.

- *Because* $f(\cdot) = (k^*(\cdot), t_i(\cdot))$ *is DSIC, the agent with* θ_i^ϵ *gets the highest profit from truthfully reporting*

$$v_i(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_i^\epsilon) + t_i(\theta_i^\epsilon, \theta_{-i}) \geq v_i(k^*(\theta_i, \theta_{-i}), \theta_i^\epsilon) + t_i(\theta_i, \theta_{-i})$$
$$\Leftrightarrow -\sum_{j\neq i} v_j(k^*(\theta_i^\epsilon, \theta_{-i}), \theta_j) + \epsilon + t_i(\theta_i^\epsilon, \theta_{-i}) \geq -\sum_{j\neq i} v_j(k^*(\theta_i, \theta_{-i}), \theta_j) + t_i(\theta_i, \theta_{-i})$$
$$\Leftrightarrow h_i(\theta_i^\epsilon, \theta_{-i}) + \epsilon \geq h_i(\theta_i, \theta_{-i})$$
$$\Leftrightarrow h_i(\hat{\theta}_i, \theta_{-i}) + \epsilon \geq h_i(\theta_i, \theta_{-i})$$

  There is a contradiction.

2.2.3 Budget Balancedness of VCG Mechanism

Definition 2.10 (Budget-Balanced VCG Mechanism)

A VCG mechanism is **budget-balanced** if $\sum_{i=1}^{I} t_i(\theta) = 0$. ♣

Based on the Proposition 2.3, we can show the following corollary.

Corollary 2.1

Suppose $I \geq 2$, $|K| \geq 2$, *and* $\forall i : \{v_i(\cdot, \theta_i) : K \to \mathbb{R} \mid \theta_i \in \Theta_i\} = V_{usc}$. *Then, there does not exist a budget- balanced VCG mechanism.* ♡

Example 2.5 $K = \{0, 1\}, \Theta_i = [-1, 1], v_i(k, \theta_i) = k\theta_i$. Take a VCG mechanism $k^*(\cdot)$ ex-post efficient, $h_1 : \Theta_2 \to \mathbb{R}$, $h_2 : \Theta_1 \to \mathbb{R}$.

$$t_1(\theta) + t_2(\theta) = v_2(k^*(\theta), \theta_2) + h_1(\theta_2) + v_1(k^*(\theta), \theta_1) + h_2(\theta_1)$$
$$= \max\{0, \theta_1 + \theta_2\} + h_1(\theta_2) + h_2(\theta_1)$$

Suppose by the contradiction that it is a budget-balanced VCG mechanism.

$$\max\{0, \theta_1 + \theta_2\} + h_1(\theta_2) + h_2(\theta_1) = 0$$

We have

$$h_2(1) - h_2(0) = \max\{0, \theta_1\} - \max\{0, \theta_1 + 1\}$$

The LHS is constant and the RHS is a function of θ_1, which gives a contradiction.

2.2.4 Expected-externality Mechanism (BIC Mechanism)

Definition 2.11 (EE Mechanism)

$(k^*(\cdot), t(\cdot))$ is an Expected-externality (EE/AGV) mechanism if $k^*(\cdot)$ is ex-post efficient and there are functions $h_i : \Theta_i : \mathbb{R}$ for all i s.t.

$$t_i(\theta) = \underbrace{\mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j \neq i} v_j(k^*(\theta_i, \tilde{\theta}_{-i}), \theta_j) \right]}_{\triangleq \xi_i(\theta_i)} + h_i(\theta_{-i}) \tag{2.1}$$

where $\xi_i(\theta_i) \triangleq \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j \neq i} v_j(k^*(\theta_i, \tilde{\theta}_{-i})) \right]$ is the expected externality i imposes on others, from i's interim perspective when her type is θ_i.

♣

Proposition 2.4

EE mechanisms are BIC.

♠

Proof 2.6

Take any i and $\theta_i, \hat{\theta}_i \in \Theta_i$. i's expected payoff from truthfully reproting is

$$\begin{aligned}
& \mathbb{E}_{\tilde{\theta}_{-i}} \left[v_i(k^*(\theta_i, \tilde{\theta}_{-i}), \theta_i) + t_i(\theta_i, \tilde{\theta}_{-i}) \right] \\
\text{(substitute (2.1))} \quad =& \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j=1}^{I} v_j(k^*(\theta_i, \tilde{\theta}_{-i}), \theta_j) \right] + \mathbb{E}_{\tilde{\theta}_{-i}} [h_i(\tilde{\theta}_{-i})] \\
(k^* \text{ is ex-post efficient}) \quad \geq& \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j=1}^{I} v_j(k^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \theta_j) \right] + \mathbb{E}_{\tilde{\theta}_{-i}} [h_i(\tilde{\theta}_{-i})] \\
=& \mathbb{E}_{\tilde{\theta}_{-i}} \left[\sum_{j=1}^{I} v_j(k^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \theta_j) \right] + \mathbb{E}_{\tilde{\theta}_{-i}} [t_i(\hat{\theta}_i, \tilde{\theta}_{-i}) - \xi_i(\hat{\theta}_i)] \\
=& \mathbb{E}_{\tilde{\theta}_{-i}} \left[v_i(k^*(\hat{\theta}_i, \tilde{\theta}_{-i}), \theta_i) + t_i(\hat{\theta}_i, \tilde{\theta}_{-i}) \right]
\end{aligned}$$

2.2.5 Budget-Balanced EE Mechanism

Budget balancedness requires

$$0 = \sum_{i=1}^{I} t_i(\theta) = \sum_{i=1}^{I} [\xi_i(\theta_i) + h_i(\theta_{-i})] \Leftrightarrow \sum_{i=1}^{I} h_i(\theta_{-i}) = -\sum_{i=1}^{I} \xi_i(\theta_i)$$

Suppose $h_i(\theta_{-i})$ is in the form of $h_i(\theta_{-i}) = c\sum_{j\neq i} \xi_j(\theta_j)$. Then,

$$\sum_{i=1}^{I} h_i(\theta_{-i}) = c(I-1)\sum_{i=1}^{I} \xi_i(\theta_i) \Rightarrow c = -\frac{1}{I-1}$$

Proposition 2.5

The EE mechanism where $h_i(\theta_{-i}) = -\frac{1}{I-1}\sum_{j\neq i} \xi_j(\theta_j)$ is budget-balanced,

$$t_i(\theta) = \xi_i(\theta_i) - \frac{1}{I-1}\sum_{j\neq i} \xi_j(\theta_j)$$

Corollary 2.2

$\exists$ a BIC, ex-post efficient and budget-balanced direct mechanism.

Example 2.6 Project choice with $K = \{0, 1\}$, $\theta_i \sim U[-1, 1]$, $v_j(k, \theta_j) = k\theta_j$. Let $k^*(\cdot)$ be:

$$k^*(\theta) = \begin{cases} 1, & \text{if } \theta_1 + \theta_2 \geq 0 \\ 0, & \text{if } \theta_1 + \theta_2 < 0 \end{cases}$$

Then,

$$\begin{aligned} \xi_i(\theta_i) &\triangleq \mathbb{E}_{\tilde{\theta}_{-i}}\left[v_{-i}(k^*(\theta_i, \tilde{\theta}_{-i}))\right] \\ &= \int_{-1}^{-\theta_i} 0 \times \frac{1}{2} d\tilde{\theta}_{-i} + \int_{-\theta_i}^{1} \tilde{\theta}_{-i} \times \frac{1}{2} d\tilde{\theta}_{-i} = \frac{1}{4}(1 - \theta_i^2) \end{aligned}$$

Hence, the budget-balanced EE mechanism is given by

$$t_i(\theta_i) = \xi_i(\theta_i) - \xi_j(\theta_j) = \frac{1}{4}(\theta_j^2 - \theta_i^2)$$

2.2.6 Linear Utility Model

Suppose types are real numbers $\Theta_i = [\underline{\theta}_i, \overline{\theta}_i]$ and $v_i(k, t, \theta_i) = \theta_i \cdot v_i(k) + t_i$, where $v_i : K \to \mathbb{R}_i$.

Given a direct mechanism $(\Theta, k(\cdot), t(\cdot))$, define interim expected values of $v_i(\cdot)$ and $t_i(\cdot)$:

$$\bar{v}_i(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[v_i(\theta_i, \theta_{-i})\right]$$

$$\bar{t}_i(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i, \theta_{-i})\right]$$

The expected utility of agent i when all agents report truthfully,

$$\begin{aligned} U_i(\theta_i) &= \mathbb{E}_{\theta_{-i}}\left[\theta_i v_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})\right] \\ &= \theta_i \bar{v}_i(\theta_i) + \bar{t}_i(\theta_i) \end{aligned}$$

Proposition 2.6

A direct mechanism $(\Theta, k(\cdot), t(\cdot))$ is BIC iff $\forall i \in \{1, ..., I\}$

(1). $\bar{v}_i(\theta_i)$ is non-decreasing in θ_i.

(2). $\forall \theta_i \in \Theta_i$, $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)ds$

Proof 2.7

"$\Rightarrow$": Given the direct mechanism is BIC. Take any i, $\theta_i, \hat{\theta}_i \in \Theta$, agents with $\theta_i, \hat{\theta}_i$ both report truthfully

$$\begin{aligned} \theta_i \bar{v}_i(\theta_i) + \bar{t}_i(\theta_i) &\geq \theta_i \bar{v}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i) \\ \hat{\theta}_i \bar{v}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i) &\geq \hat{\theta}_i \bar{v}_i(\theta_i) + \bar{t}_i(\theta_i) \\ \Rightarrow [\theta_i - \hat{\theta}_i][\bar{v}_i(\theta_i) - \bar{v}_i(\hat{\theta}_i)] &\geq 0 \end{aligned}$$

Hence, $\bar{v}_i(\theta_i)$ is non-decreasing.

$U_i(\theta_i) = \max_{\hat{\theta}_i \in \Theta_i} \left[\theta_i \bar{v}_i(\hat{\theta}_i) + t_i(\hat{\theta}_i)\right]$ *by BIC is maximized at $\hat{\theta}_i = \theta_i$.*

By Envelope Theorem,

$$\begin{aligned} U_i(\theta_i) &= U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} U_i'(s)ds \\ &= U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)ds \end{aligned}$$

"$\Leftarrow$": Take any i, $\theta_i, \hat{\theta}_i \in \Theta$. i's expected interim payoff from reporting $\hat{\theta}_i$ instead of θ_i is

$$\begin{aligned} &\underbrace{\theta_i \bar{v}_i(\theta_i) + \bar{t}_i(\theta_i)}_{U_i(\theta_i)} - \left[\theta_i \bar{v}_i(\hat{\theta}_i) + \bar{t}_i(\hat{\theta}_i)\right] \\ =&U_i(\theta_i) - \left[U_i(\hat{\theta}_i) + (\theta - \hat{\theta}_i)\bar{v}_i(\hat{\theta}_i)\right] \\ =&U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{v}_i(s)ds - [U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\hat{\theta}_i} \bar{v}_i(s)ds] - (\theta - \hat{\theta}_i)\bar{v}_i(\hat{\theta}_i) \\ =&\int_{\theta_i}^{\hat{\theta}_i} (\bar{v}_i(\hat{\theta}_i) - \bar{v}_i(s))ds \geq 0 \end{aligned}$$

So the direct mechanism is BIC.

2.3 Auction

Based on

- Klemperer, P. (1998). Auctions with almost common values: The Wallet Game'and its applications.

European Economic Review, 42(3-5), 757-769.

2.3.1 Examples: Auctions with Common-value

(1). Financial assets;

(2). Oilfields;

(3). A takeover target has a common value if the bidders are financial acquirers (e.g. LBO firms) who will follow similar management strategies if successful;

(4). The Personal Communications Spectrum (PCS) licenses sold by the U.S. Government in the 1995 "Airwaves Auction".

2.3.2 First / Second Price Sealed-bid Auction

- A seller sells an indivisible object.
- There are $N = \{1, ..., n\}$ bidders, $i \in N$.
- Each bidder has a valuation for the object, $X_i \sim F$, $x_i \in [\underline{x}, \overline{x}]$. p.d.f. $f(\cdot)$ is strictly positive and continuous.
- Strategy of i: $b_i : [\underline{x}, \overline{x}] \to \mathbb{R}$, a bid function.

Assumption *1. Independence; 2. Symmetry; 3. Private Values; 4. Risk-neutrality.*

Let $X = (X_1, ..., X_n)$. The k^{th}-order statistic, X^k, is the k^{th} the highest value in $X_1, ..., X_n$.

Definition 2.12 (Second Price Auction)

Highest bidder wins and pays the second-highest bid. (If more than one bidders bid the highest value, they win with equal probability.)

It can be written as the form of Bayesian game: a bidder i's utility function is

$$u_i(b_1, ..., b_n; x_i) = \begin{cases} \frac{1}{|\{j \in N : b_j = b_i\}|} x_i - b^2, & b_i = b^1 \\ 0, & b_i \neq b^1 \end{cases}$$

where b^k is the k-th highest bid. ♣

Theorem 2.4 (Second Price Auction: Bid Truthfully)

In the second-price sealed-bid auction, it is a (weakly) dominant strategy to bid your valuation, i.e., $\forall i \in N, \forall x_i \in [\underline{x}, \overline{x}]$, $b_i(x_i) = x_i$.

That is, $\forall i, \forall b_i' \in \mathbb{R}$,

$$u_i(x_i, b_{-i}; x_i) \geq u_i(b_i', b_{-i}; x_i), \forall b_{-i} \in \mathbb{R}^{n-1}$$

(Moreover, if $\exists b_i' \neq x_i$, then $\exists b_{-i} \in \mathbb{R}^{n-1}$ such that $u_i(x_i, b_{-i}; x_i) > u_i(b_i', b_{-i}; x_i)$.) ♡

Proof 2.8

Player i has value x_i and treats b^1_{-i} as a random variable. The payoff conditional on winning is

$$x_i - b^1_{-i}$$

By bidding $b_i = x_i$, i ensures that i wins if $b_i = x_i > b^1_{-i} \Leftrightarrow x_i - b^1_{-i} > 0$ and i loses if $b_i = x_i < b^1_{-i} \Leftrightarrow x_i - b^1_{-i} < 0$.

Definition 2.13 (First Price Auction)

Highest bidder wins and pays her bid. (If more than one bidder bid the highest value, they win with equal probability.)

It can be written as the form of Bayesian game: a bidder i's utility function is

$$u_i(b_1, ..., b_n; x_i) = \begin{cases} \frac{1}{|\{j \in N : b_j = b_i\}|} x_i - b_i, & b_i = b^1 \\ 0, & b_i \neq b^1 \end{cases}$$

where b^k is the k-th highest bid. ♣

Bayesian Nash Equilibrium Analysis of First Price Auction

Conjecture that $\exists$ a BNE with the following properties:

1. Symmetry: $b_1(\cdot) = b_2(\cdot) = \cdots = b_n(\cdot) := b(\cdot)$.
2. $b(\cdot)$ is differentiable.
3. $b'(\cdot) > 0$.

Take any bidder i with valuation x_i. Assume i knows $b(\cdot)$ and knows that the other bidder use the same $b(\cdot)$. Take any $b_i \in \mathbb{R}$ ($b_i := b(x_i)$). (Not that, by the continuity of X_i, it is impossible to tie in this case.)

Then, i's expected payoff from bidding b_i is

$$P(b(X_j) \leq b_i, \forall j \neq i)(x_i - b_i) = F^{n-1}(b^{-1}(b_i))(x_i - b_i)$$

The necessary F.O.C. gives that optimal b_i satisfies

$$(n-1)f(b^{-1}(b_i))\frac{1}{b'(b^{-1}(b_i))}F^{n-2}(b^{-1}(b_i))(x_i - b_i) - F^{n-1}(b^{-1}(b_i)) = 0$$

Since $b(\cdot)$ is a symmetric BNE, the optimal b_i must be $b(x_i)$, then $b^{-1}(b_i) = x_i$.

$$(n-1)f(x_i)\frac{1}{b'(x_i)}F^{n-2}(x_i)(x_i - b(x_i)) - F^{n-1}(x_i) = 0$$

Hence,

$$\underbrace{(n-1)f(x_i)F^{n-2}(x_i)x_i + F^{n-1}(x_i)}_{\frac{\partial F^{n-1}(x_i)x_i}{\partial x_i}} - F^{n-1}(x_i) = \underbrace{(n-1)f(x_i)F^{n-2}(x_i)b(x_i) + b'(x_i)F^{n-1}(x_i)}_{\frac{\partial F^{n-1}(x_i)b(x_i)}{\partial x_i}}$$

Taking integral at both sides in $[\underline{x}, x]$,

$$F^{n-1}(x)x - \int_{\underline{x}}^{x} F^{n-1}(t)dt = F^{n-1}(x)b(x)$$

That is,

$$b(x) = x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^{x} F^{n-1}(t)dt \tag{2.2}$$

Note $b(\cdot)$ is differentiable and $b'(\cdot) > 0$. We can extend $b(\cdot)$ to $[\underline{x}, \overline{x}]$ by setting $b(\underline{x}) = \lim_{x\to\underline{x}} b(x) = \underline{x}$.

Proposition 2.7 (Symmetric BNE of First Price Auction)

$b(x) = x - \frac{1}{F^{n-1}(x)} \int_{\underline{x}}^{x} F^{n-1}(t)dt$ *is a symmetric BNE of First Price Auction.*

Proof 2.9

Any bid higher than $b(\overline{x})$ is suboptimal, and any bid lower than $b(\underline{x})$ is indifferent to the $b(\underline{x})$.

We prove that, for a bidder with x_i, she prefers to bid $b(x_i)$ than $b(y), \forall y$.

Bidding $b(y)$ gives expected payoff

$$\begin{aligned}
F^{n-1}(y)(x_i - b(y)) &= F^{n-1}(y)(y - b(y)) + F^{n-1}(y)(x_i - y) \\
\text{(by (2.2))} &= \int_{\underline{x}}^{y} F^{n-1}(t)dt + F^{n-1}(y)(x_i - y) \\
&= \int_{\underline{x}}^{x_i} F^{n-1}(t)dt - \underbrace{\int_{y}^{x_i} [F^{n-1}(t) - F^{n-1}(y)]dt}_{\geq 0, \text{ minimized at } y = x_i}
\end{aligned}$$

Theorem 2.5 (Lebrun, 1999)

Consider the bid function $b(\cdot)$ in (2.2), the First Price Auction has essentially unique. (Bidders with types $x > \underline{x}$ bid $b(x)$ and bid for type $x = \underline{x}$ is not pinned down any further than the support $b_i(\underline{x})$ must lie in $(-\infty, b(\underline{x})]$.)

The equilibrium expected payoffs of a bidder in first-price auction and second price auction with valuation x is

$$\int_{\underline{x}}^{x} F^{n-1}(t)dt$$

The equilibrium expected revenue of the seller in first-price auction and second price auction is

$$\overline{x} - \int_{\underline{x}}^{\overline{x}} [nF^{n-1}(t) - (n-1)F^{n}(t)]dt$$

2.4 Revenue Equivalence Theorem and Optimal Auctions

Consider the Optimal Auctions in an Independent Private Values Setting.

Assumption *There is one object and N bidders.*

1. *Bidders are risk-neutral;*

2. *Bidders have private valuations;*
3. *each bidder i's valuation independently drawn from a strictly increasing c.d.f. $F_i(\theta_i)$ (with p.d.f. $f_i(\theta_i), \theta_i \in \Theta_i$) that is continuous and bounded below;*
4. *Seller knows each F_i (use F and f to represent all distributions) and have no value for the object.*

Goal: Find the **optimal auction** that maximizes the seller's expected revenue subject to individual rationality (IR) and Bayesian incentive compatibility for the buyers.

Note *Note that, in symmetric BNE, the bidders' strategies are same $b(\cdot)$, but the bidders' values can draw from different and independent distributions $\{F_i\}_{i=1}^I$*

2.4.1 IR, BIC, Direct Mechanism

Given a direct mechanism $(\Theta, y(\cdot), t(\cdot))$, define

$$\text{Expected Allocation: } \bar{y}_i(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[y_i(\theta_i, \theta_{-i})\right]$$

$$\text{Expected Payment: } \bar{t}_i(\theta_i) = \mathbb{E}_{\theta_{-i}}\left[t_i(\theta_i, \theta_{-i})\right]$$

The expected utility of agent i when all agents report truthfully,

$$\begin{aligned} U_i(\theta_i) &= \mathbb{E}_{\theta_{-i}}\left[\theta_i y_i(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})\right] \\ &= \theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i) \end{aligned}$$

Definition 2.14 (Individual Rationality)

A direct mechanism $(\Theta, y(\cdot), t(\cdot))$ is **individual rationality (IR)** if $\forall i, \theta_i \in \Theta_i$, $U_i(\theta_i) \geq 0$. ♣

Corollary 2.3 (Corollary of Proposition 2.6)

*A direct mechanism $(\Theta, y(\cdot), t(\cdot))$ is **BIC and IR** iff $\forall i \in \{1, ..., I\}$*

(1). $\bar{y}_i(\theta_i)$ is non-decreasing in θ_i.

(2). $\forall \theta_i \in \Theta_i$, $U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s) ds$

(3). $U_i(\underline{\theta}_i) \geq 0$ ♡

For a BIC & IR mechanism, *the expected auction payment* $\bar{t}_i(\theta_i)$ can be represented as

$$U_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s) ds$$

$$\theta_i \bar{y}_i(\theta_i) + \bar{t}_i(\theta_i) = U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s) ds$$

$$\bar{t}_i(\theta_i) = -\theta_i \bar{y}_i(\theta_i) + U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s) ds \tag{2.3}$$

2.4.2 Revenue Equivalence Theorem

For a BIC & IR direct mechanism, the *seller's expected revenues* from bidder i:

$$\begin{aligned}
\mathbb{E}_\theta[-t_i(\theta)] &= -\int_\Theta t_i(\theta)f(\theta)d\theta \\
&= -\int_{\Theta_i} \underbrace{\left(\int_{\Theta_{-i}} t_i(\theta_i,\theta_{-i})f_{-i}(\theta_{-i})d\theta_{-i}\right)}_{\bar{t}_i(\theta_i)} f_i(\theta_i)d\theta_i \\
&= -\int_{\underline{\theta}_i}^{\overline{\theta}_i} \left(-\theta_i\bar{y}_i(\theta_i) + U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)ds\right) f_i(\theta_i)d\theta_i \\
&= -\underbrace{\int_{\underline{\theta}_i}^{\overline{\theta}_i}\int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)ds f_i(\theta_i)d\theta_i}_{\triangleq\star} + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i\bar{y}_i(\theta_i)f_i(\theta_i)d\theta_i - U_i(\underline{\theta}_i)
\end{aligned}$$

applying integration by parts

$$\begin{aligned}
\star &= \int_{\underline{\theta}_i}^{\overline{\theta}_i}\left(\int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)ds\right)dF_i(\theta_i) \\
&= \left(\int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)ds\right)F_i(\theta_i)\Big|_{\underline{\theta}_i}^{\overline{\theta}_i} - \int_{\underline{\theta}_i}^{\overline{\theta}_i} F_i(\theta_i)d\left(\int_{\underline{\theta}_i}^{\theta_i} \bar{y}_i(s)ds\right) \\
&= \int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{y}_i(s)ds - \int_{\underline{\theta}_i}^{\overline{\theta}_i} \bar{y}_i(\theta_i)F_i(\theta_i)d\theta_i \\
&= \int_{\underline{\theta}_i}^{\overline{\theta}_i} (1-F_i(s))\bar{y}_i(s)ds
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathbb{E}_\theta[-t_i(\theta)] &= -\int_{\underline{\theta}_i}^{\overline{\theta}_i}(1-F_i(\theta_i))\bar{y}_i(\theta_i)d\theta_i + \int_{\underline{\theta}_i}^{\overline{\theta}_i} \theta_i\bar{y}_i(\theta_i)f_i(\theta_i)d\theta_i - U_i(\underline{\theta}_i) \\
&= \int_\Theta y_i(\theta)\left[\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right]f(\theta)d\theta - U_i(\underline{\theta}_i)
\end{aligned}$$

The *total expected revenue of the seller* is

$$\int_\Theta \sum_{i=1}^I y_i(\theta)\left[\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}\right]f(\theta)d\theta - \sum_{i=1}^I U_i(\underline{\theta}_i) \tag{2.4}$$

Theorem 2.6 (Revenue Equivalence Theorem)

In the setting of Assumption 2.4, BIC & IR direct mechanisms, with the same allocation rule $y(\cdot)$ and the same interim utilities of the lowest types $(U_i(\underline{\theta}_i))_{i=1,\dots,I}$, generate the same revenues (2.4) and the same expected payments of all types bidders (2.3). ♡

Corollary 2.4 (Revenue Equivalence Theorem (indirect form))

Any two auction formats A and B, fix a BNE of A and a BNE of B such that

1. *For every $\theta \in \Theta$, these two BNEs allocate the object with same probabilities;*

2. *Interim expected payoff of* $\underline{\theta}_i$ *is the same for both BNEs.*

Then, A *and* B *generate the same expected revenues* (2.4) *and the same expected payments of all types bidders* (2.3).

Proof 2.10

Based on the Revelation Principle 2.2, direct auction mechanisms induced by A and B are BIC and IR. By Revenue Equivalence Theorem, the corollary is proved.

2.4.3 Optimal Auctions

The optimal auction design is given by

$$\max_{\text{BIC and IR } y(\cdot)} \int_{\Theta} \sum_{i=1}^{I} y_i(\theta) \left[\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)}\right] f(\theta) d\theta - \sum_{i=1}^{I} U_i(\underline{\theta}_i)$$

Definition 2.15 (Virtual Valuation)

Define bidder i's **virtual valuation** is $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f(v_i)}$.

Assumption *[Regularity Condition] Any bidder* i*'s virtual valuation* $c_i(v_i) = v_i - \frac{1-F_i(v_i)}{f(v_i)}$ *is strictly increasing.*

Corollary 2.5 (Optimal Auction Mechanism)

Assume regularity. Then the expected revenue maximizing direct auction mechanism $(y(\cdot), t(\cdot))$ *can be described as follows*

(1). $y(\cdot) : \Theta \to K$ *is defined as follows. For any* $\theta \in \Theta$, $\max_{i\in\{1,\dots,I\}} c_i(\theta_i) < 0$, *the seller keeps the object* ($y_i(\theta) = 0, \forall i$)*. Otherwise, the object is allocated to the highest virtual valuation bidder.*

(2). Define $t(\cdot) : \Theta \to K$,

$$t_i(\theta) := -\theta_i y_i(\theta_i, \theta_{-i}) + U_i(\underline{\theta}_i) + \int_{\underline{\theta}_i}^{\theta_i} y_i(s, \theta_{-i}) ds$$

which satisfies (2.3).

Example 2.7 Suppose $\Theta_i = [0,1], \theta_i \sim U[0,1]$. $c_i(\theta_i) = \theta_i - \frac{1-\theta_i}{1} = 2\theta_i - 1$, which is strictly increasing in θ_i (regularity satisfied). Then, the optimal auction mechanism is allocating the object to the highest (virtual) valuation bidder (iff his value $\theta_i \geq \frac{1}{2}$).

Definition 2.16 (Bidder-Specific Reserve Price)

Bidder i's bidder-specific reserve price r_i^* is the value for which $c_i(r_i^*) = 0$.

Theorem 2.7 (Myerson (1981))

The optimal (single-good) auction in terms of a direct mechanism: The good is sold to the agent $i = \arg\max_i \phi_i(\hat{v}_i)$, *as long as* $v_i \geq r_i^*$. *If the good is sold, the winning agent* i *is charged the smallest valuation that he could have declared while still remaining the winner:*

$$\inf\{v_i^* : c_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, c_i(v_i^*) \geq c_j(\hat{v}_j)\}$$

♡

2.5 Others

2.5.1 Order Statistics

Definition 2.17 (Order Statistics)

If $X_1, ..., X_n$ is a random sample, then the **characteristics** are the sample values placed in ascending order. Notation:

$$X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$$

♣

Proposition 2.8 (Distribution of $X_n = \max_{i=1,...,n} X_i$)

If $X_1, ..., X_n$ *is a random sample form a distribution with cdf* F *(denoted by "*$X_i \sim$ *i.i.d.* F*"), then*

$$F_{X_{(n)}}(x) = P(X_{(n)} \leq x) = F^n(x)$$

♠

Proposition 2.9 (cdf and pdf)

More generally,

$$F_{X_{(r)}}(x) = \sum_{j=r}^{n} \binom{n}{j} [F_X(x)]^j [1 - F_X(x)]^{n-j}$$

$$f_{X_{(r)}}(x) = \frac{n!}{(r-1)!(n-r)!} f_X(x) [F_X(x)]^{r-1} [1 - F_X(x)]^{n-r}$$

♠

Example 2.8

1. **Order statistics sampled from a uniform distribution on unit interval (Unif$[0, 1]$):** Consider a random sample $U_1, ..., U_n$ from the standard uniform distribution. Then,

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} u^{k-1} (1-u)^{n-k}$$

The k^{th} order statistic of the uniform distribution is a beta-distributed random variable.

$$U_{(k)} \sim \text{Beta}(k, n+1-k)$$

which has mean $\mathbb{E}[U_{(k)}] = \frac{k}{n+1}$.

2. **The joint distribution of the order statistics of the uniform distribution on unit interval (Unif$[0, 1]$):**

Similarly, for $i < j$, the joint probability density function of the two order statistics $U_{(i)} < U_{(j)}$ can be shown to be

$$f_{U_{(i)},U_{(j)}}(u,v) = n!\frac{u^{i-1}}{(i-1)!}\frac{(v-u)^{j-i-1}}{(j-i-1)!}\frac{(1-v)^{n-j}}{(n-j)!}$$

The joint density of the n order statistics turns out to be constant:

$$f_{U_{(1)},U_{(2)},\ldots,U_{(n)}}(u_1,u_2,\ldots,u_n) = n!$$

For $n \geq k > j \geq 1$, $U_{(k)} - U_{(j)}$ also has a beta distribution:

$$U_{(k)} - U_{(j)} \sim \mathrm{Beta}(k-j, n-(k-j)+1)$$

which has mean $\mathbb{E}[U_{(k)} - U_{(j)}] = \frac{k-j}{n+1}$