



Analysis and Something Else

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All models are wrong, but some are useful.

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Chapter 1 Logic

1.1 Main Methods of Proof (@ Lec 01 of ECON 204)

1.1.1 Proof by Induction

1.1.2 Proof by Deduction

1.1.3 Proof by Contradiction

1.1.4 Proof by Contraposition

- $\neg P$ ("not P ") means " P is false".
- $P \wedge Q$ (" P and Q ") means " P is true and Q is true."
- $P \vee Q$ (" P or Q ") means " P is true or Q is true (or possibly both)."
- $\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.
- $P \Rightarrow Q$ (" P implies Q ") means "whenever P is satisfied, Q is also satisfied."

Statement: Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Definition 1.1 (Contrapositive)

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$.



Theorem 1.1 (Prove Contrapositive Instead)

$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.



Chapter 2 Analysis Basis

2.1 Real Number \mathbb{R} (@ Lec 02 of ECON 204)

\mathbb{R} is a field with the usual operations $+$, \cdot , additive identity 0, and multiplicative identity 1.

2.1.1 Order Axiom

Proposition 2.1 (Order Axiom)

There is a complete ordering \leq , i.e. \leq is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that $\forall \alpha, \beta \in \mathbb{R}$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.

The order is compatible with $+$ and \cdot , i.e. $\forall \alpha, \beta, \gamma \in \mathbb{R}$

1. $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.
2. $\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$.



2.1.2 Completeness Axiom

Proposition 2.2 (Completeness Axiom)

Suppose $L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H$ satisfy $\forall l \in L, h \in H, l \leq h$. Then,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall l \in L, h \in H, l \leq \alpha \leq h$$

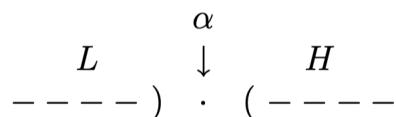


Figure 2.1: Completeness Axiom

Claim 2.1

The Completeness Axiom differentiates \mathbb{R} from \mathbb{Q} :

\mathbb{Q} satisfies all the axioms for \mathbb{R} except the Completeness Axiom.



2.1.3 Supremum $\sup \mathbb{X}$, Infimum $\inf \mathbb{X}$ for $\mathbb{X} \subseteq \mathbb{R}$

Definition 2.1 (Supremum and Infimum)

- (1). Suppose \mathbb{X} is bounded above. The **supremum** of \mathbb{X} , written $\sup \mathbb{X}$, is the least upper bound for \mathbb{X} , i.e. $\sup \mathbb{X}$ satisfies
 - (a). $\sup \mathbb{X} \geq x, \forall x \in \mathbb{X}$ ($\sup \mathbb{X}$ is an upper bound).
 - (b). $\forall y < \sup \mathbb{X}, \exists x \in \mathbb{X}$ s.t. $x > y$ (there is no smaller upper bound).
- (2). Suppose \mathbb{X} is bounded below. The **infimum** of \mathbb{X} , written $\inf \mathbb{X}$, is the greatest lower bound for \mathbb{X} , i.e. $\inf \mathbb{X}$ satisfies
 - (a). $\inf \mathbb{X} \leq x, \forall x \in \mathbb{X}$ ($\inf \mathbb{X}$ is a lower bound).
 - (b). $\forall y > \inf \mathbb{X}, \exists x \in \mathbb{X}$ s.t. $x < y$ (there is no greater lower bound).
- (3). If \mathbb{X} is not bounded above, write $\sup \mathbb{X} = \infty$. If \mathbb{X} is not bounded below, write $\inf \mathbb{X} = -\infty$. By convention, $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.



Proposition 2.3

If $\inf A = x^* \in A$ ($\sup A = x^* \in A$), then $x^* = \min A$ ($x^* = \max A$).



2.1.4 The Supremum Property

Proposition 2.4 (The Supremum Property)

Every nonempty set of real numbers that is bounded above has a supremum, which is a real number.

Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.



Theorem 2.1

The Supremum Property (Prop 2.4) and the Completeness Axiom (Prop 2.2) are equivalent.



2.1.5 Archimedean Property

Theorem 2.2 (Archimedean Property)

$\forall x \in \mathbb{R}, y \in \mathbb{R}^+, \exists n \in \mathbb{N}$ s.t. $ny > x$.



2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)

2.2.1 Metric Space (\mathbb{X}, d) and Metric $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$

Definition 2.2 (Metric Space)

A **metric space** is a pair (\mathbb{X}, d) , where \mathbb{X} is a set and $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ a function satisfying

1. Non-negative: $d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in \mathbb{X}$.
2. Symmetric: $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$.
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathbb{X}$.

A function $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ satisfying 1-3 is called a **metric** on \mathbb{X} . 

A metric gives a notion of distance between elements of \mathbb{X} .

2.2.2 Norm $\|\cdot\|$ and Normed Vector Space $(V, \|\cdot\|)$

Definition 2.3 (Norm)

Let V be a vector space over \mathbb{R} . A **norm** on V is a function $\|\cdot\| : V \rightarrow \mathbb{R}_+$ satisfying

1. Non-negative: $\|x\| \geq 0, \|x\| = 0 \Leftrightarrow x = 0, \forall x \in V$.
2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$.
3. $\|\alpha x\| = |\alpha| \|x\|, \forall \alpha \in \mathbb{R}, x \in V$. 

A norm gives a notion of length of a vector in V .

Definition 2.4 (Normed Vector Space)

A **normed vector space** is a vector space over \mathbb{R} equipped with a norm, $(V, \|\cdot\|)$. 

Example 2.1 Normed Vector Space

- \mathbf{E}^n : n -dimensional Euclidean space.

$$V = \mathbb{R}^n, \|x\|_2 = |x| = \sqrt{\sum_{i=1}^n (x_i)^2}$$

- $V = \mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i|$ (the "taxi cab" norm or L^1 norm)
- $V = \mathbb{R}^n, \|x\|_\infty = \max \{|x_1|, \dots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)
- $C([0, 1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$
- $C([0, 1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$
- $C([0, 1]), \|f\|_1 = \int_0^1 |f(t)| dt$

where $C([0, 1])$ is the space of all continuous real-valued functions on $[0, 1]$.

2.2.3 Theorem: metric can be defined by norm

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 2.3

Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : \mathbb{X} \times \mathbb{X} \rightarrow \mathbb{R}_+$ be defined by $d(v, w) = \|v - w\|$. Then (V, d) is a metric space.



2.2.4 Cauchy-Schwarz Inequality

Theorem 2.4 (Cauchy-Schwarz Inequality)

If $v, w \in \mathbb{R}^n$, then

$$\left(\sum_{i=1}^n v_i w_i \right)^2 \leq \left(\sum_{i=1}^n v_i^2 \right) \left(\sum_{i=1}^n w_i^2 \right)$$

$$\|v \cdot w\|^2 \leq \|v\|^2 \|w\|^2$$

$$\|v \cdot w\| \leq \|v\| \|w\|$$



2.2.5 Lipschitz-equivalent Norm

Definition 2.5 (Lipschitz-equivalent)

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the same vector space V are said to be **Lipschitz-equivalent** (or **equivalent**) if $\exists m, M > 0$ s.t. $\forall x \in V$,

$$m\|x\| \leq \|x\|^* \leq M\|x\|$$

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V, x \neq 0$,

$$m \leq \frac{\|x\|^*}{\|x\|} \leq M$$



Theorem 2.5

All norms on \mathbb{R}^n are equivalent.



However, infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0, 1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \in [0, \frac{1}{n}] \\ 0, & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then $\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} \rightarrow 0$, which means there is no lower bound $m > 0$.

2.2.6 Ball, Radius, Diameter, and Distance

In a metric space (X, d) , define

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\ &= \text{open ball with center } x \text{ and radius } \varepsilon \\ B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\ &= \text{closed ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

We can use the metric d to define a generalization of "radius". In a metric space (X, d) , define the *diameter* of a subset $S \subseteq X$ by

$$\text{diam}(S) = \sup \{d(s, s') : s, s' \in S\}$$

Similarly, we can define the *distance from a point to a set*, and *distance between sets*, as follows:

$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf \{d(a, b) : a \in A, b \in B\} \end{aligned}$$

2.3 Set Theory

2.3.1 Well Defined Set

Definition 2.6

A set S is **well defined** if an object a is either $a \in S$ or $a \notin S$.



2.3.2 Numerically Equivalent (@ Lec 01 of ECON 204)

Definition 2.7

Two sets A, B are **numerically equivalent** (or have the same cardinality) if there is a bijection $f : A \rightarrow B$, that is, 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$).



2.3.3 Finite, Countable Set (@ Lec 01 of ECON 204)

Definition 2.8 (Finite Set)

A set is either **finite** or **infinite**. A set is **finite** if it is numerically equivalent to $\{1, \dots, n\}$ for some n . A set that is not finite is infinite.



We give a more precise definition to classify infinite set:

Definition 2.9 (Countable Set)

An infinite set is **countable** if it is numerically equivalent to \mathbb{N} . An infinite set that is not countable is called **uncountable**.

**Theorem 2.6 (Countable \mathbb{Q})**

The set of rational numbers \mathbb{Q} is countable.

**2.3.4 Power Set (@ Lec 02 of ECON 204)****Definition 2.10 (Power Set: the set of all subsets)**

For any set A , we denote by $\mathcal{P}(A)$ the collection of all subsets of A . $\mathcal{P}(A)$ is the **power set** of A .



We may also use the notation 2^A (in Berkeley ECON 204).

2.3.5 Theorem (Cantor): The power set of \mathbb{N} is uncountable (@ Lec 02 of ECON 204)**Theorem 2.7 (Cantor)**

$\mathcal{P}(\mathbb{N})$ (or denoted by $2^{\mathbb{N}}$), the set of all subsets of \mathbb{N} , is uncountable.

**2.3.6 Cardinalities of Sets (@ Lec 02 of ECON 204)****Definition 2.11 (Cardinality)**

If A is a set, $|A| = \text{cardinality of } A = \# \text{ of elements}$



$n \in \mathbb{N}, |\{1, \dots, n\}| = n; |\emptyset| = 0 (\emptyset = \text{empty set}).$

Proposition 2.5 (Facts about cardinality)

1. If A is numerically equivalent to $\{1, \dots, n\}$ for some $n \in \mathbb{N}$, then $|A| = n$.
2. A and B are numerically equivalent if and only if $|A| = |B|$.
3. If $|A| = n$ (finite) and A is a proper subset of B (that is, $A \subset B$ and $A \neq B$) then $|A| < |B|$.
4. If A is countable and B is uncountable, then $n < |A| < |B|, \forall n \in \mathbb{N}$.
5. If $A \subseteq B$, then $|A| \leq |B|$. (if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable.)
6. If there is an injection $\sigma : A \rightarrow B$, we can write $|A| \leq |B|$;
7. If there is a surjection $\sigma : A \rightarrow B$, we can write $|A| \geq |B|$;
8. If there is a bijection $\sigma : A \rightarrow B$, we can write $|A| = |B|$.



2.3.7 Pigeonhole Principle: $|A| > |B| \Rightarrow$ no injective function $\sigma : A \rightarrow B$

Theorem 2.8 (Pigeonhole Principle)

If A and B are sets and $|A| > |B|$, then there is no injective function $\sigma : A \rightarrow B$.



2.3.8 B^A : Sets of Function

If A, B are sets, then $B^A = \{\sigma : A \rightarrow B | \sigma \text{ a function}\}$.

Example 2.2 $n \in \mathbb{Z}$, we define a function $f : B^{\{1, \dots, n\}} \rightarrow B^n (= B \times B \times B \times \dots \times B)$ by the equation $f(\sigma) = \{\sigma(1), \dots, \sigma(n)\}$, where $\sigma : \{1, \dots, n\} \rightarrow B$. The f is a bijection.

Proof 2.1

1. Injective:

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), \dots, \sigma_1(n)\} = \{\sigma_2(1), \dots, \sigma_2(n)\}$$

Since $\sigma : \{1, \dots, n\} \rightarrow B$, it is sufficient to prove $\sigma_1 = \sigma_2$.

2. Surjective:

$$\forall \{b_1, \dots, b_n\}, \text{ we have } \sigma^*(x) = b_x, x = 1, \dots, n. \text{ s.t. } f(\sigma^*) = \{b_1, \dots, b_n\}$$

Example 2.3

$$C(\mathbb{R}, \mathbb{R}) = \{\text{continuous functions } \sigma : \mathbb{R} \rightarrow \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

2.3.9 Bounded Set (@ Lec 03 of ECON 204)

Definition 2.12 (Bounded Set)

In a metric space (X, d) , a subset $S \subseteq X$ is **bounded** if $\exists x \in X, \beta \in \mathbb{R}$ such that $\forall s \in S, d(s, x) \leq \beta$.



2.3.10 Open, Closed Set (@ Lec 04 of ECON 204)

Definition 2.13 (Open Sets)

Let (X, d) be a metric space. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is **open** if

$\forall x \in \mathbb{X}$ we can draw a ball around x that is contained in \mathbb{X} .

i.e. $\forall x \in \mathbb{X}, \exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{y : d(y, x) < \varepsilon\} \subseteq \mathbb{X}$



Definition 2.14 (Closed Sets)

\mathbb{X} is **closed** if \mathbb{X}^c is open.



Theorem 2.9 (Equivalent definition: Closed Sets)

Equivalent: if A in a metric space (X, d) contains all limit points of all sequences in A , A is closed.

$$\{x_n\} \subset A, x_n \rightarrow x \in X \Rightarrow x \in A$$

**Example 2.4 (Closed and Open Sets on \mathbb{E}_1 i.e., \mathbb{R} with the usual Euclidean metric)**

- 1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open
- 2) \mathbb{R} is both open and closed
- 3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open
- 4) $[1, \infty)$ is closed because its complement open
- 5) $(1, 2]$ is neither open nor closed

Example 2.5 (Closed and Open Sets on other metric space) In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon]$.

Theorem 2.10 (Empty Set and Full Set are both open and closed)

In any metric space (X, d) both \emptyset and X are open, and both \emptyset and X are closed.

**Theorem 2.11 (Union of open sets is open, Intersection of finite open sets is open)**

In any metric space (X, d) ,

1. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
2. *The intersection of a finite collection of open sets is open.*

**2.3.11 Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)**

Given a set $S \subseteq X$, the **point** of X can be classified into three types relative to S :

- **Interior (points)**, denoted $\text{int}(S)$: $\vec{x} \in S$ for which there exists some $B(\vec{x}, r) \subseteq S$, is the largest open set contained in S (the union of all open sets contained in S).
- **Exterior (points)**, denoted $\text{ext}(S)$: $\vec{x} \notin S$ for which there exists some $B(\vec{x}, r)$ containing no points of S , is the largest open set contained in $X \setminus S$.
- **Boundary (points)** denoted $\partial(S)$ or $\text{bd}(S)$: all other points (for which any $B(\vec{x}, r)$ will contain some points of S and some points outside S).
- **Closure of S** , denoted \bar{S} or $\text{cl}(S) = \text{int}(S) \cup \text{bd}(S)$, is the smallest closed set containing S (the intersection of all closed sets containing S).
- Moreover, boundary satisfies $\partial(S) = \overline{(X \setminus S)} \cap \bar{S}$.

(1) A set S is **open** if $S = \text{int}(S)$ - i.e., if S does not contain any of its boundary points.

(2) A set S is **closed** if $S = \bar{S} = \text{int}(S) \cup \text{bd}(S)$ - i.e., if S contains all of its boundary points.

2.3.12 Compact Set

Definition 2.15 (Compact Set)

$\mathcal{L} \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.



Example 2.6 Compact Set $[1, 2] = \{x \in \mathbb{R} : 1 \leq x \leq 2\}; \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 4\}$

2.3.13 Sublevel Set

Definition 2.16 (Sublevel Set)

The sublevel set of a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ (for some level $c \in \mathbb{R}$) is the set

$$\overline{L_c} = \{x \in \mathbb{R}^n : f(x) \leq c\}$$



2.3.14 Set Operations

Definition 2.17

A binary operation on a set A is a function $* : A \times A \rightarrow A$.

The operation is *associative* if $a * (b * c) = (a * b) * c, \forall a, b, c \in A$.

The operation is *commutative* if $a * b = b * a, \forall a, b \in A$.



Example 2.7

$+, \circ$ are both *associative* and *commutative* operations on $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$; $-$ is neither *associative* nor *commutative* operation on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not \mathbb{N} .

Definition 2.18

A subset $H \subset S$ is closed under $*$ if $a * b \in H$ for all $a, b \in H$.



Definition 2.19

$*$ has identity element $e \in S$ if $a * e = e * a = a$ for all $a \in S$.



2.4 Sequences

Sequences $\{x_k\}_{k=1}^{\infty}$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 2.20 (Subsequence)

Suppose $\{x_n\}$ is a sequence and $n_1 < n_2 < \dots$, then $\{x_{n_k}\}$ is called a **subsequence**.



2.4.1 Convergence of Sequences (@ Lec 03 of ECON 204)

Definition 2.21 (Convergence: note $x_k \rightarrow x, \lim_{k \rightarrow \infty} x_k = x$)

Let (X, d) be a metric space. A sequence $\{x_k\}$ converges to x (written $x_k \rightarrow x$ or $\lim_{k \rightarrow \infty} x_k = x$) if

$$\forall \varepsilon > 0, \exists N_\varepsilon \text{ s.t. } d(x_k, x) < \varepsilon, \forall k \geq N_\varepsilon$$



Definition 2.22 (Limit point)

x is a limit point of $\{x_k\}$ if \exists a subsequence of $\{x_k\}$ that converges to x .



Theorem 2.12 (Uniqueness of Limits)

In a metric space (X, d) , if $x_k \rightarrow x$ and $x_k \rightarrow x'$, then $x = x'$.



2.4.2 Cluster Point (@ Lec 03 of ECON 204)

Definition 2.23 (Cluster Point)

An element c is a **cluster point** of a sequence $\{x_n\}$ in a metric space (X, d) if $\forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N}, \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$



Example 2.8 $x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ even} \\ \frac{1}{n}, & \text{if } n \text{ odd} \end{cases}$ has cluster points $\{0, 1\}$.

Theorem 2.13 (Cluster Point \Leftrightarrow exists subsequence converges to it)

Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X . Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k \rightarrow \infty} x_{n_k} = c$.



2.4.3 Sequences in \mathbb{R} and \mathbb{R}^n (@ Lec 03 of ECON 204)

Proposition 2.6

If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it converges.



Theorem 2.14

Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then $\lim_{n \rightarrow \infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ ($\lim_{n \rightarrow \infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$). In particular, the limit exists.



Consider a sequence $\{x_n\}$ of real numbers. Let

$$\alpha_n = \sup\{x_k : k \geq n\}$$

and

$$\beta_n = \inf\{x_k : k \geq n\}$$

Either $\alpha_n = +\infty$ for all n , or $\alpha_n \in \mathbb{R}$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \dots$. Either $\beta_n = -\infty$ for all n , or $\beta_n \in \mathbb{R}$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq \dots$

Definition 2.24 (Lim Sups and Lim Inf)

$$\limsup_{n \rightarrow \infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

$$\liminf_{n \rightarrow \infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$



Theorem 16 Let $\{x_n\}$ be a sequence of real numbers. Then

$$\lim_{n \rightarrow \infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\}$$

$$\Leftrightarrow \limsup_{n \rightarrow \infty} x_n = \liminf_{n \rightarrow \infty} x_n = \gamma$$

Theorem 17 (Theorem 3.2, Rising Sun Lemma) Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.

Proposition 2.7

The largest(smallest) limit point of $\{x_k\}$ is $\limsup_{k \rightarrow \infty} x_k$ ($\liminf_{k \rightarrow \infty} x_k$)



Proposition 2.8

$\{x_k\}$ converges $\Leftrightarrow -\infty < \liminf_{k \rightarrow \infty} x_k = \limsup_{k \rightarrow \infty} x_k < \infty$



2.4.4 Cauchy Sequence

Definition 2.25 (Cauchy Sequence)

$\{x_k\}$ is Cauchy if given $\varepsilon > 0$, $\exists N_\varepsilon$ s.t.

$$\|x_k - x_m\| < \varepsilon, \forall k, m \geq N_\varepsilon.$$



Note:

$\{x_k\}$ converges $\Leftrightarrow \{x_k\}$ is Cauchy

Chapter 3 Functions

3.1 Definitions of Function

Definition 3.1 (Function)

Function is a rule $\sigma : A \rightarrow B$ that assigns an element B to *every* element of A . $\forall a \in A, \sigma(a) \in B$.



3.1.1 Image, Preimage, Fiber

Definition 3.2

1. A is the domain of σ , B is the range of σ .
2. We call $\sigma(a) = \text{value of } \sigma \text{ at } a$ as the image of a .
3. A set $C \subset B$, we call $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$ as the preimage of C .
4. An element $b \in B$, we call $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$ as the fiber of b .



3.1.2 Composition of functions

Definition 3.3 (Function Composition)

The function composition \circ is an operation that takes two functions $\sigma : A \rightarrow B$ and $\tau : B \rightarrow C$, , and produces a function $\tau \circ \sigma : A \rightarrow C$ that fulfills $\forall a \in A, (\tau \circ \sigma)(a) = \tau(\sigma(a))$.



3.1.3 Function Composition is Associative

Proposition 3.1 (Associativity of Functions)

Suppose $\sigma : A \rightarrow B, \tau : B \rightarrow C, \rho : C \rightarrow D$ are functions and \circ is the function composition, then
 $\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$.



3.2 Injection, Surjection, Bijection

3.2.1 Definitions: Injective, surjective, bijective

A function $\sigma : A \rightarrow B$ is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijection* (if injective and surjective)

3.2.2 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

Lemma 3.1 (Lemma 1.1.7)

Suppose $\sigma : A \rightarrow B, \tau : B \rightarrow C$ are functions,

If σ, τ are injective, then $\tau \circ \sigma$ is injective.

If σ, τ are surjective, then $\tau \circ \sigma$ is surjective.

If σ, τ are bijective, then $\tau \circ \sigma$ is bijective.



3.2.3 Proposition 1.1.8: A function is bijection if there exist inverse

Proposition 3.2 (Proposition 1.1.8)

A function $\sigma : A \rightarrow B$ is a bijection if \exists a function $\tau : B \rightarrow A$ such that

$$\sigma \circ \tau = id_B = \text{identity on } B (id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$



Such τ is unique, called inverse of σ , $\tau = \sigma^{-1}$.

3.3 Function Continuity

3.3.1 Continuous Function in \mathbb{R} with Euclidean norm

Definition 3.4 (Continuity at Point)

A real-valued function f is continuous at x if

"For every $\{x_k\}$ converging to x satisfies that $\lim_{k \rightarrow \infty} f(x_k) = f(x)$ ".

Equivalent definition:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } \|y - x\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$$



Continuity at x_0 requires:

1. $f(x_0)$ is defined; and

2. either

- x_0 is an isolated point of X , i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{x\}$; or

- $\lim_{x \rightarrow x_0} f(x)$ exists and equals $f(x_0)$

Definition 3.5 (Continuous Function)

A real-valued function f is continuous if it is continuous at all points in its domain.

**3.3.2 Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)****Definition 3.6 (Continuity in Metric Spaces)**

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \rightarrow Y$ is **continuous** at a point $x_0 \in X$ if

$$\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon.$$

**3.3.3 Theorem: Continuous \Leftrightarrow Preimage of open set is open (@ Lec 04 of ECON 204)****Theorem 3.1 (Continuous \Leftrightarrow Preimage of open set is open)**

Let (X, d) and (Y, ρ) be metric spaces, and $f : X \rightarrow Y$. Then f is **continuous** if and only if

$$f^{-1}(A) = \{x \in X : f(x) \in A\} \text{ is open in } X \quad \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

**3.3.4 Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)****Theorem 3.2 (Continuity is preserved in composition)**

Let (X, d_X) , (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then

$g \circ f : X \rightarrow Z$ is continuous.

**Proof 3.1**

Proved by previous theorem 3.1.

3.3.5 Uniform Continuity (@ Lec 04 of ECON 204)**Definition 3.7 (Uniformly Continuous)**

Suppose $f : (X, d) \rightarrow (Y, \rho)$. f is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

**Claim 3.1**

Uniformly Continuous implies (is stronger than) Continuous.

$(f \text{ is continuous if } \forall x_0 \in X, \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0 \text{ s.t. } d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon)$



Given $\varepsilon > 0$, "uniformly continuous" requires $\delta(\varepsilon)$ that works for all $x_0 \in X$.

3.3.6 Intermediate Value Theorem (@ Lec 02 of ECON 204)

Theorem 3.3 (Intermediate Value Theorem)

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.



3.3.7 Coercive Function

Definition 3.8 (Coercive)

A real-valued function $f : \mathbb{X} \rightarrow \mathbb{R}$ is coercive if for **every** $\{x_k\} \subset \mathbb{X}$ s.t. $\|x_k\| \rightarrow \infty$, $f(x_k) \rightarrow \infty$



Example 3.1 Check coercive

- 1) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2$ - coercive
- 2) $x \in \mathbb{R}$, $f(x) = 1 - e^{-|x|}$ - not coercive
- 3) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (we need $f(x_k) \rightarrow \infty$ for all $\|x_k\| \rightarrow \infty$)

3.3.8 Extreme of Functions

Definition 3.9 (Extreme of Functions)

Let $\mathbb{X} \subseteq \mathbb{R}^n$, $f : \mathbb{X} \rightarrow \mathbb{R}$

$$\inf_{x \in \mathbb{X}} f(x) = \inf\{f(x) : x \in \mathbb{X}\}$$



If $\exists x^* \in \mathbb{X}$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \mathbb{X}} f(x)$

x^* is called a **minimizer** of f , written as $x^* \in \arg \min_{x \in \mathbb{X}} f(x)$. If x^* is unique, we write $x^* = \arg \min_{x \in \mathbb{X}} f(x)$

Similarly, supremum and maximum of f .

3.3.9 Weierstrass' Theorem(Extreme value Theorem)

Theorem 3.4 (Weierstrass' Theorem (Extreme value Theorem))

If f is a **continuous** function on a **compact set**, $\mathbb{X} \subseteq \mathbb{R}^n$, then f attains its min and max on \mathbb{X} i.e.,

$$\exists x_1 \in \mathbb{X} \text{ s.t. } f(x_1) = \inf_{x \in \mathbb{X}} f(x)$$

$$\exists x_2 \in \mathbb{X} \text{ s.t. } f(x_2) = \sup_{x \in \mathbb{X}} f(x)$$



Proof 3.2

(for existence of min; max is similar)

Let $\{\sigma_k\} \subseteq \mathbb{X}$ be s.t.

$$\inf_{x \in \mathbb{X}} f(x) \leq f(\sigma_k) \leq \inf_{x \in \mathbb{X}} f(x) + \frac{1}{k}$$

Then $\lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

\mathcal{L} is bounded $\Rightarrow \{\sigma_k\}$ has at least one limit point x ,

\mathcal{L} is closed $\Rightarrow x_1 \in \mathbb{X}$

f is continuous $\Rightarrow f(x_1) = \lim_{k \rightarrow \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

Corollary 3.1 (Corollary to WT)

Let f be continuous on closed set \mathbb{X} (not necessarily bounded). If f is coercive on \mathbb{X} it attains its min on \mathbb{X} .

**Proof 3.3**

Consider $\{\sigma_k\}$ as in proof of WT.

Since f is closed, $f(x) < \infty, \forall x \in \mathbb{X}$. And f is coercive on \mathbb{X} , which means $f(x) \rightarrow \infty$ if $\|x\| \rightarrow \infty$.

Hence, $\{\sigma_k\} \in \mathbb{X}$ is bounded. Rest of proof same as proof of WT.

Example 3.2 $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) Does f achieve its min and max on $\mathcal{L}_1 = \{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leq 6\}$?

- \mathcal{L}_1 is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over \mathbb{R}^3 ?

- $f \rightarrow \infty$ whenever $\|x\| \rightarrow \infty \Rightarrow f$ is coercive.

- \mathbb{R}^3 is closed.

$\Rightarrow f$ achieves its min. on \mathbb{R}^3 by corollary to WT.

- max does not exist since $f \rightarrow \infty$ as $\|x\| \rightarrow \infty$.

3) Does f achieve its min and max over $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?

- \mathcal{L}_2 is closed, but not bounded.

- Since f is coercive, min achieved.

- max does not exist since setting $x_1 = 0, x_2 = 3 - x_3$ and letting $x_3 \rightarrow \infty$ makes $f \rightarrow \infty$

Chapter 4 Big \mathcal{O} and Small o Notation

4.1 Definition

Complexity:

Definition 4.1

A sequence $f(n)$ is $O(1)$ if $\lim_{n \rightarrow \infty} f(n) < \infty$.



Definition 4.2

A sequence $f(n)$ is $O(g(n))$ if $\frac{f(n)}{g(n)}$ is $O(1)$.



Definition 4.3

A sequence $f(n)$ is $o(1)$ if $\lim_{n \rightarrow \infty} \sup f(n) = 0$.



Definition 4.4

A sequence $f(n)$ is $o(g(n))$ if $\lim_{n \rightarrow \infty} \sup \frac{f(n)}{g(n)} = 0$.



Definition 4.5

A sequence $f(n)$ is asymptotic to $g(n)$ if $\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$. (This is denoted by $f(n) \sim g(n)$ as $a \rightarrow \infty$)



For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x \rightarrow \infty} \frac{f(x)}{g(x)} = 0$.

Example 4.1

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$

$$n^3 + n + 2 = \Theta(n^3)$$

$$n^3 + n + 2 = o(n^4)$$

4.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \rightarrow a$ if $\limsup_{x \rightarrow a} \frac{|f(x)|}{g(x)} < \infty$.

Example 4.2 $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \rightarrow 0$

Chapter 5 Lipschitz Continuous

5.1 Definition (@ Lec 04 of ECON 204)

Definition 5.1 (Lipschitz (Continuous) in Normed Vector Space)

Let X, Y be normed vector spaces, $\mathbb{E} \subseteq X$. A function $f : X \rightarrow Y$ is **Lipschitz** on \mathbb{E} satisfies

$$\exists \gamma > 0, \|f(\mathbf{x}) - f(\mathbf{y})\|_Y \leq \gamma \|\mathbf{x} - \mathbf{y}\|_X, \forall \mathbf{x}, \mathbf{y} \in \mathbb{E}$$

or we call γ -**Lipschitz continuous**;



If f is γ -Lipschitz continuous, then it is also $(\gamma + 1)$ -Lipschitz continuous

The minimal such γ is called a Lipschitz constant of function f

Remark: Here $\|\cdot\|$ can be any given norm of the space \mathbb{R}^n and \mathbb{R}^m , such as Euclidean norm, ℓ_1 -norm, etc.

When not specified, we assume it is Euclidean norm.

5.2 Example

Example 1: $f(x) = 2x$ is 2-Lipschitz continuous;

Example 2: What about $f(\mathbf{x}) = \mathbf{Ax}$, where \mathbf{A} is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).

Example 3: What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞ .

5.3 Contraction Mapping

1. If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.

2. If $\gamma < 1$, then f is called a contraction mapping

Example 1: $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.

Example 2: $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

Chapter 6 Fixed point theorem

1. Fixed point theorem: If f is a contraction mapping that maps \mathbb{R}^n to itself, then the following two results hold:

1) There exists a unique fixed point x^* satisfying

$$\mathbf{x}^* = f(\mathbf{x}^*)$$

2) In addition, the iterated function sequence

$$\mathbf{x}, f(\mathbf{x}), f(f(\mathbf{x})), \dots,$$

converges to this unique fixed point x^* (independent of the initial point x).

2. Remark: This is a special case of "Banach fixed point theorem" (which applies to any complete metric space).