Microeconomic Theory

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All models are wrong, but some are useful.

Contents

Chapter 1 Preference and Utility Function

Based on

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1.1 Preferences

1.1.1 Preference Relation

Definition 1.1 (Weak, Strict, Indifference)

$\succeq$ referred to as the **weak preference relation**: "x is at least as good as y". (ordinal);

"**No better than**": $y \preceq x$ if and only if $x \succeq y$.

"**Strict preference**": $x \succ y$ if and only if $x \succeq y$ and not $y \succeq x$.

"**Indifference**": $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$. ♣

1.1.2 Basic Assumptions

1.1.3 Rational Preference

Definition 1.2 (Rantional Preference)

Rationality: $\succeq$ is **rational** if and only if it is **complete** and **transitive**.

- $\succeq$ is **complete** iff $\forall x, y \in X$, $x \succeq y$ or $y \succeq x$.
- $\succeq$ is **transitive** iff $\forall x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. ♣

The completeness means

- Any two bundles can be compared
- Indifference is allowed

The transitivity

- like transitivity of the real numbers
- extends pairwise preferences to longer chains in the logical way.

1.2 Utility Function

1.2.1 Utility Function $\Leftrightarrow$ Rational Preference

Definition 1.3 (Unitility Function)

We can say a function $u : X \to \mathbb{R}$ represents $\succeq$ if $\forall x, y \in X$,

$$x \succeq y \Leftrightarrow u(x) \geq u(y)$$

Proposition 1.1 (Rational $\succeq \Rightarrow \exists u(\cdot)$)

If $\exists$ a function $u : X \to \mathbb{R}$ represents $\succeq$, then $\succeq$ is rational (i.e., completeness and transitivity)

Note *The reverse may not true.*

1.2.2 Convex Preference

Definition 1.4 (Convex $\succeq$)

$\succeq$ is **convex** if for every $x \in X$ the $\{y \in X : y \succeq x\}$ is convex, i.e., $y_1 \succeq x$ and $y_2 \succeq x \Rightarrow \alpha y_1 + (1-\alpha)y_2 \succeq x$ for all $\alpha \in [0, 1]$.

Convex relations imply *averages are preferred to extremes*.

Definition 1.5 (Strictly Convex)

$\succeq$ is **strictly convex** iff $\forall x, y, z \in X$, if $x \succeq z$ and $y \succeq z$, then $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0, 1)$

1.2.3 Convex Preference $\Leftrightarrow$ Quasiconcave Utility Function

Definition 1.6 (Quasi-Concave Function)

A function u is **quasi-concave** if and only if for all $t \in \mathbb{R}$, $\{x \in X : u(x) \geq t\}$ is convex.

$$\forall x, y \in X, t \in \mathbb{R}, 0 \leq a \leq 1 : u(x) \geq t, u(y) \geq t \Rightarrow u(ax + (1-a)y) \geq t$$

Proposition 1.2 (Concave Function $\Rightarrow$ Quasi-Concave Function)

Any function that is concave is also quasi-concave.

Proposition 1.3 (Convex $\succeq \Leftrightarrow$ quasi-concave $u(\cdot)$)

$\succeq$ is convex, $\Leftrightarrow \exists$ a <u>*quasi-concave*</u> *$u(\cdot)$ that represents $\succeq$.*

1.3 Preferences over Nearby Bundles

1.3.1 Monotone Preference

Definition 1.7 (Monotone $\succeq$)
$\succeq$ is **monotone** if $x, y \in X$ with $x \geq y \Rightarrow x \succeq y$ (and $x > y \Rightarrow x \succ y$).

Proposition 1.4 (Monotone $\succeq \Rightarrow$ monotone $u(\cdot)$)
If $\succeq$ is monotone, then $\exists$ a monotone $u(\cdot)$ that represents $\succeq$.

Note *Complete, transitive, and monotone are three assumptions that made by all theories (either EU or non-EU).*

1.3.2 Strongly monotone

Definition 1.8 (Strongly Monotone $\succeq$)
$\succeq$ is **strongly monotone** if and only if for any $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X$, if $\forall i : x_i \geq y_i$ and $\exists j$ such that $x_j > y_j$, then $x \succ y$.

(When we compare elements that have more than one dimension, strongly monotone holds if at least one relation is not equal.)

$$A = (1, 1), B = (2, 1), C(1, 2), D = (2, 2)$$

Strongly monotone can infer that $D \succ B \succ A, D \succ C \succ A$.

1.3.3 Local Non-Satiation

Even weaker assumptions will ensure that the consumer's choice exhausts their budget.

Definition 1.9 (Local Nonsatiation)
For any bundle x, there is a nearby bundle y in the consumption set such that y is preferred to x. That is, for all $x \in X$ and every $\varepsilon > 0$,

$$\exists y \in |x - y| < \varepsilon, \text{ s.t. } y \succ x$$

We have

$$\text{Strong Monotonicity} \Rightarrow \text{Monotonicity} \Rightarrow \text{Local Nonsatiation}$$

1.4 Common Assumptions of Preference

$\succsim$		u
monotone	$\Longrightarrow$	nondecreasing
strongly monotone	$\Longrightarrow$	strictly increasing
continuous	$\Longrightarrow$	continuous (Debreu's Theorem)
convex	$\Longrightarrow$	quasi-concave (but not concave)
strictly convex	$\Longrightarrow$	strictly concave (and strictly quasi-concave)
homothetic (and continuous)	$\Longrightarrow$	continuous and homogeneous
(so-called) quasi-linear	$\Longrightarrow$	quasi-linear
(so-called) differentiable	$\Longrightarrow$	differentiable
separable	$\Longrightarrow$	separable (form)
strongly separable	$\Longrightarrow$	additively separable (form)

Figure 1.1: Properties of Preference and Utility Function

1.4.1 Independence of Preference

The 'standard' model of decisions under risk is based on von Neumann and Morgenstern Expected Utility (EU), which requires the independence assumption.

Definition 1.10 (Independence of Preference)

Independence: For any $x, y, z \in X$ and $0 < \alpha < 1$, if $x \succeq y$ then $\alpha x + (1-\alpha)z \succeq \alpha y + (1-\alpha)z$. ♣

1.4.2 Continuous Preference

Definition 1.11 (Continuous $\succeq$)

$\succeq$ is **continuous** on X if and only if for any sequence $\{x^n, y^n\}_n = 1^\infty$ with $x^n \succeq y^n$ and we note $x = \lim_{n\to\infty} x^n$, $y = \lim_{n\to\infty} y^n$, we have $x \succeq y$ (i.e., the graph $\{(x,y) \mid x \succeq y \subseteq X \times X\}$ is closed). ♣

Example 1.1 Lexicographic preferences (not continuous) Under Lexicographic preference $\succ$, $x \succ y$ if and only if

- $x_1 > y_1$, or
- $x_1 = y_1$, and $x_2 > y_2$, or
- $x_1 = y_1$ and $x_2 = y_2$ and $x_3 > y_3$, or
- etc.

Under Lexicographic preferences, there is no indifference.

We can find the Lexicographic preference violates continuity: $\left(1+\frac{1}{n},1\right) \succ (1,2)$ and $\lim\left(1+\frac{1}{n},1\right) = (1,1) \prec (1,2)$.

Proposition 1.5 (Debreu's Theorem, Continuous $\succeq \Rightarrow$ continuous $u(\cdot)$)

If $\succeq$ is continuous, then $\exists$ a continuous $u(\cdot)$ that represents $\succeq$. ♠

1.4.3 Homothetic Preference

Definition 1.12 (Homotheticity)

$\succeq$ are homothetic if $x \succeq y \Rightarrow \alpha x \succeq \alpha y$ for all $\alpha > 0$.

Proposition 1.6 (Homothetic preference $\Leftrightarrow$ homogeneous $u(\cdot)$)

A continuous $\succeq$ is homothetic $\Leftrightarrow \exists$ a continuous homogeneous $u(\cdot)$ that represents $\succeq$ such that $u(\alpha x) = \alpha u(x)$ for all $x > 0$.

1.4.4 Quasi-linearity

Definition 1.13 (Quasi-Linearity)

$\succeq$ on X is **quasi-linear** on x_1 if

$$x \succeq y \Rightarrow (x + \epsilon e_1) \succeq (y + \epsilon e_1)$$

where $e_1 = (1, 0, ..., 0)$ and $\epsilon > 0$.

Theorem 1.1 (Quasi-Linearity $\Leftrightarrow u(x) = x_1 + v(x_{-1})$)

A continuous $\succeq$ on $(-\infty, \infty) \times \mathbb{R}_+^{K-1}$ is quasi-linear in $x_1 \Leftrightarrow \exists$ a $u(\cdot)$ that represents $\succeq$ such that

$$u(x) = x_1 + v(x_{-1})$$

where $v(\cdot)$ satisfies $(v(x_{-1}), 0, ..., 0) \sim (0, x_{-1})$.

1.4.5 Separability

Definition 1.14 (Separability)

$\succeq$ satisfies **separability** if for any x_i

$$(x_i, x_{-i}) \succeq (x_i', x_{-i}) \Leftrightarrow (x_i, x_{-i}') \succeq (x_i', x_{-i}')$$

Theorem 1.2 (Separability $\Rightarrow$ Additive $u(\cdot)$)

*$\succeq$ with **separability** admits additive u-representation*

$$u(x) = v_1(x_1) + \cdots v_K(x_K)$$

Note *Strong assumption, usually ignored in practice.*

1.4.6 Differentiable Preference

Consider a vector of values $v(x) \in \mathbb{R}_+^K$ for the K commodities and a feasible the direction $x + \varepsilon d \in X$ from x for small enough $\varepsilon > 0$.

d is considered improvement if and only if

$$d \cdot v(x) > 0$$

Given $v(x) : X \to \mathbb{R}_+^K$, let

$$D_v(x) = \{d : d \cdot v(x) > 0\}$$

be the set of directions that are improvements relative to x.

$d \in \mathbb{R}^k$ is an improvement direction at x if there is $\lambda^* > 0$ such that λd is an improvement

$$x + \lambda d \succ x$$

for any $\lambda \leq \lambda^*$. Let $D_{\succeq}(x)$ be the set of all improvement directions at x.

Any improvement is an improvement direction if

- $\succeq$ are strictly convex.
- $\succeq$ are convex, strongly monotonic, and continuous.

Definition 1.15 (Differentiable Preference)

$\succeq$ is **differentiable** if there exists a function $v(x) : X \to \mathbb{R}_+^K$ such that

$$D_{\succeq}(x) = D_v(x),\ \forall x \in X$$

♣

Example 1.2 $\succeq$ represented by

(1). $\alpha x_1 + \beta x_2$ for $\alpha, \beta > 0$ are differentiable: $v(x) = (\alpha, \beta)$.

(2). $\min\{\alpha x_1, \beta x_2\}$ are differentiable where $\alpha x_1 \neq \beta x_2$: $v(x) = \begin{cases} (1,0) & \text{if } \alpha x_1 < \beta x_2 \\ (0,1) & \text{otherwise} \end{cases}$

Proposition 1.7 (Sufficient condition for differentiable $\succeq$)

Any (monotonic and convex) $\succeq$ can be represented by a (strongly monotonic and quasi-concave) and differentiable u is differentiable.

♠

Chapter 2 Choice Theory

2.1 Choice

2.1.1 Choice

Let $\mathcal{B} = 2^X$ (all subsets of X) and $B \in \mathcal{B}$ be the all potential alternatives that can be chosen.

The choice of an agent can be represented by $C(B) \subseteq B, \forall B \in \mathcal{B}$.

Definition 2.1

A **choice function** C such that $C(A) \subseteq A$ which specifies for each nonempty subset $A \subseteq X$. ♣

A choice function C can be rationalized if there is a preference relation $\succeq$ on X such that $C = C_{\succeq}$, that is

$$C(A) = C_{\succeq}(A) = \{x \in A : x \succeq y, \forall y \in A\}, \forall A \subseteq X$$

Definition 2.2 (Rubinstein's Condition α)

A choice function C satisfies **condition** α if for any two problems A, B, if $A \subseteq B$ and $C(B) \in A$, then $C(A) = C(B)$. ♣

2.1.2 Choice Correspondence

Definition 2.3 (Choice Correspondence (More than one choice))

A choice correspondence C assigns a non-empty <u>subset</u> for every non-empty set A

$$\emptyset \neq C(A) \subseteq A$$

♣

Properties:

(α): If $a \in A \subseteq B$, then $a \in C(B) \Rightarrow a \in C(A)$

(β): If $a, b \in A \subseteq B$, then $a, b \in C(A)$ and $b \in c(B) \Rightarrow a \in C(B)$.

α and β are equivalent to WARP.

2.1.3 Weak Axiom of Revealed Preference (WARP)

Definition 2.4 (Weak Axiom of Revealed Preference)

Given a choice structure $(C, \mathcal{B})$ satisfies **WARP**. If $\exists B \in \mathcal{B}$ with $x, y \in B$, such that $x \in C(B)$. Then, $\forall B' \in \mathcal{B}$ with $x, y \in B', y \in C(B') \Rightarrow x \in C(B')$.

Or we can say, whenever $x, y \in B \cap B'$,

$$x \in C(B) \text{ and } y \in C(B') \Rightarrow x \in C(B')$$

Proposition 2.1 (Rational $\Rightarrow$ WARP)

Given $\succeq$ is rational, then $(C^*_{\succeq}, \mathcal{B})$ satisfies WARP.

($C^*_{\succeq}$ is the choice rule that picks the maximal alternatives by $\succeq$)

2.2 Choice under Uncertainty

We want to model an uncertain prospect corresponding forms of function u.

The literature contains (basically) three sets of answers to these questions, differing in whether uncertainty is objective or subjective.

- Objective uncertainty: von Neumann-Morgenstern (vNM).
- Subjective uncertainty: Savage.
- Horse lottery-roulette wheel theory: Anscombe and Aumann (A-A)

2.2.1 von Neumann-Morgenstern (vNM)

The set of prizes is defined by X and the set of probability measures (or distributions) over X is denoted by P.

A compound lottery: If $p, q \in P$ and $\alpha \in [0, 1]$, then there is an element $\alpha p + (1-\alpha)q \in P$ which is defined by taking the convex combinations of the probabilities of each prize separately, or

$$(\alpha p + (1-\alpha)q)(x) = \alpha p(x) + (1-\alpha)q(x)$$

$(\alpha p + (1-\alpha)q)$ represents a compound lottery.

Definition 2.5 (Three Axioms)

Three Axioms

(A1) $\succ$ is a preference relation (asymmetric and negatively transitive);

(A2) For all $p, q, r \in P$ and $\alpha \in [0, 1]$, $p \succ q \Rightarrow \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$.

(A3) For all $p, q, r \in P$ such that $p \succ q \succ r$, $\exists \alpha, \beta \in (0, 1)$ such that

$$\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$$

Theorem 2.1 (vNM)

$\succ$ on P satisfies axioms (A1)-(A3) if and only if there exists a function $u : X \to \mathbb{R}$ such that

$$p \succ q \Leftrightarrow \sum_x p(x)u(x) > \sum_x q(x)u(x) \qquad (*)$$

Moreover, u is unique up to a positive affine transformation: there is another u' represents $\succ$ in the sense of (*) if and only if there exists $c > 0$ and d such that

$$u'(\cdot) = cu(\cdot) + d$$

♡

Remark

- If u represents $\succ$ then so will $v(\cdot) = f(u(\cdot))$ for any **strictly increasing** f.
- $k(p) = \sum_x p(x)u(x)$ gives an ordinal representation of $\succ$.

Lemma 2.1 (Four Lemmas obtained by the three axioms)

If $\succ$ satisfies (A1) to (A3), then

(L1). If $p \succ q$ and $0 \le \alpha < \beta \le 1$, then

$$\beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$$

(L2). If $p \succeq q \succeq r$ and $p \succ r \Rightarrow$ there exists a unique $\alpha^* \in [0,1]$ such that

$$q \sim \alpha^* p + (1-\alpha^*)r$$

(L3). If $p \sim q$ and $\alpha \in [0,1] \Rightarrow$ for all $r \in P$,

$$\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$$

(L4). For any $x \in X$, let δ_x be the probability distribution degenerate at x, that is $\delta_x(x') = \begin{cases} 1, & \text{if } x' = x \\ 0, & \text{if } x' \neq x \end{cases}$ For all $p \in P$, we have $x_1, x_2 \in X$ such that

$$\delta_{x_1} \succeq p \succeq \delta_{x_2}$$

♡

2.2.2 Savage (1954)

Consider the situation that what the decision maker chooses depends critically on his/her subjectively assesses as the odds of the outcomes.

The basics of the Savage formulation:

- a set of X of prizes/consequences;
- a set S of the nature (states of the world).

Each $s \in S$ is a compilation of all characteristics/factors about which the DM is uncertain and which are relevant to the consequences that will result from her/his choice. The set S is an exhaustive list of mutually exclusive states — only one $s \in S$ will be the realized state.

We denote the choice space by H, as the set of all functions from S to X ($H = X^S$).

Savage seeks to find a subjective taste (the utility function) $u(\cdot)$ and a subjective belief (the probability measure) π such that

$$h \succ h' \Leftrightarrow \sum_{s \in S} \pi(s)u(h(s)) > \sum_{s \in S} \pi(s)u(h'(s))$$

Note that, it contains an assumption that $u(\cdot)$ is a function about x which doesn't depend on the state of the world when it receives x.

2.3 Social Choice

Notations:

1. We consider finite set of alternatives X and finite set of agents I.
2. We use $\mathcal{B}$ to denotes the set of all preference relations.
3. We use $\mathcal{R} \subseteq \mathcal{B}$ to denotes the set of all rational preference relations.
4. We use $\succeq \in \mathcal{R}$ to represents individual rational preference relation.

2.3.1 Social Welfare Function and Properties

Definition 2.6 (Social Welfare Function (SWF))

A **social welfare function** (SWF) is a mapping

$$f : \mathcal{A} \subseteq \mathcal{R}^I \to \mathcal{B}$$

$\trianglerighteq = f(\succeq_1, ..., \succeq_I)$ is interpreted as the **social preference relation**. It doesn't need to be rational (i.e., complete and transitive).

♣

Definition 2.7 (SWF's Properties)

A social welfare function $f : \mathcal{A} \to \mathcal{B}$

- has **unrestricted domain** (UD) if $\mathcal{A} = \mathcal{R}^n$;
- is **transitive** (T) if $f(\succeq_1, ..., \succeq_I)$ is transitive for all $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$;
- is **nondictatorial** (ND) if there is no agent $i \in I$ such that $\forall\{x, y\} \subseteq X$ $x \succeq_i y \Rightarrow x \trianglerighteq y$. (That is there is no distinguished voter who can choose the winner).
- is **weakly Paretian** (PA) if, $\forall\{x, y\} \subseteq X$ and any preference profile $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$, we have $x \succeq_i y, \forall i \in I \Rightarrow x \trianglerighteq y$.

- is **independent of irrelevant alternatives** (IIA) if, $\forall\{x, y\} \subseteq X$, and any $\succeq$ and $\succeq'$ with $\succeq_i|_{x,y} = \succeq'_i|_{x,y}, \forall i \in I$, if $x \trianglerighteq y$ then $x \trianglerighteq' y$.

♣

2.3.2 Arrow's Theorem

Theorem 2.2 (Arrow's impossibility theorem)

Suppose $|X| \geq 3$, $\mathcal{A} = \mathcal{R}^I$ (UD). Then if a SWF f satisfies T, PA, and IIA, then it fails to be ND.

♡

Proof 2.1

Yu, N. N. (2012). A one-shot proof of Arrow's impossibility theorem. *Economic Theory*, 523-525.

Chapter 3 Demand Theory

3.1 Utility Maximization Problem (UMP)

Budget set is given by $B = \{x \in X \subseteq \mathbb{R}_+^K : p \cdot x \leq w\}$, where w is the DM's wealth and p is the vector of prices. Without losing generality, we can assume $w = 1$.

The DM's problem is finding the $\succeq$-optimal bundle $x \in B(p)$. With the corresponding utility function $u(x)$, we can consider a consumer's problem

$$\max_{x \in X} u(x) \qquad s.t.\ p \cdot x \leq w \tag{UMP}$$

The set $\succeq$-optimal bundle is represented by $x(p, w)$.

3.1.1 Marshallian Demand: Existence and Properties

Proposition 3.1 (Continuous Preference⇒ Solution $x(p, w)$ Existence)

If $\succeq$ $(u(\cdot))$ is continuous, then all such problems have a solution $x(p, w)$. ♠

Proof 3.1

By the Weierstrass Extreme Value Theorem.

Proposition 3.2 (Convex Preference⇒ Convex $x(p, w)$)

If $\succeq$ is convex ($u(\cdot)$ is quasi-concave), then $x(p, w)$ is convex. ♠

Proof 3.2

Suppose $x, x' \in X$. The optimal utility $u^* = u(x) = u(x')$. For any $\alpha \in [0, 1]$, let $x'' = \alpha x + (1 - \alpha)x'$. Because $\succeq$ is convex, we have $u(\cdot)$ is quasi-concave, that is $u(x'') \geq u^*$. x'' is also feasible. So, $x'' \in x(p, w)$.

Proposition 3.3 (Strictly Convex Preference⇒ Singleton $x(p, w)$)

If $\succeq$ is strictly convex ($u(\cdot)$ is strictly quasi-concave), then $x(p, w)$ is (at most) a singleton. ♠

Proposition 3.4 (Differentiable Preference⇒ Margianl Utility equals to Price)

If $\succeq$ is differentiable, $x^* \in x(p, w)$, and the vector of marginal values at x^* (as defined above) is denoted by $v(x^*) = (v_1(x^*), ..., v_K(x^*))$, where $v_k(x^*)$ is usually taken by $\frac{\partial u}{\partial x_k}(x^*)$ in "classic" problem. Then, *we have*

$$\frac{v_k(x^*)}{v_j(x^*)} = \frac{p_k}{p_j} \text{ for any } x_k^*, x_j^* > 0$$

and for any k with $x_k^* > 0$ (consumed commodity)

$$\frac{v_k(x^*)}{p_k} \geq \frac{v_j(x^*)}{p_j} \text{ for any } j \neq k \tag{*}$$

Corollary 3.1 (Sufficient Conditions for Optimality)

If $\succeq$ is strongly monotonic, convex, continuous, and differentiable and if $p \cdot x^* = w$ and (*) is satisfied then $x^* \in x(p, w)$

Definition 3.1 (Rationalize)

$\succeq$ **fully rationalize** the demand function x if for any (p, w), the bundle $x(p, w)$ is the unique $\succeq$-maximal bundle within B.

A monotonic $\succeq$ **rationalize** the demand function x if for any (p, w), the bundle $x(p, w)$ is a $\succeq$-maximal bundle within B.

The unique solution is called Marshallian (Uncompensated) Demand.

Proposition 3.5 (Properties of Marshallian Demand)

*(i). **Walras' Law**: If $\succeq$ is local nonsatiation, $\forall x^* \in x(p, w) : p \cdot x^* = w$.*

*(ii). **Homogeneity of degree zero in** (p, w): $x(\alpha p, \alpha w) \equiv x(p, w)$, $\forall \alpha > 0$.*

*(iii). **Continuous in prices and in wealth** if the $\succeq$ is continuous.*

Proposition 3.6 (Weak Axiom of Revealed Preference of Marshallian Demand)

If demand is single valued then WARP(2.4) is equivalent to

$$p \cdot y' \leq w \text{ and } y \neq y' \Rightarrow p' \cdot y > w$$

where $y \equiv x(p, w)$ and $y' \equiv x(p', w')$. (y' is feasible under (p, w) but $y = x(p, w)$, which means y is better and it can't be feasible under (p', w').)

3.1.2 Lagrangian Approach: $\frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i$ and $\lambda^* \left(x_i(p, w) + \sum_{j=1}^K p_j \frac{\partial x_j}{\partial p_i} \right) = 0$

The Lagrangian of the problem is

$$L(x, \lambda) = u(x) - \lambda(p \cdot x - w)$$

By the KKT necessary conditions, we have

$$\frac{\partial L}{\partial x_i}(x^*, \lambda^*) = \frac{\partial u\left(x^*\right)}{\partial x_i} - \lambda^* p_i = 0,\ \forall i = 1, ..., K$$

$$\lambda^* \geq 0 \text{ and } \lambda^*(p \cdot x^* - w) = 0$$

Based on that, we have

Lemma 3.1

(i). $\frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i$;

(ii). $\lambda^* \left(x(p, w) + p \cdot \frac{\partial x(p,w)}{\partial p} \right) = 0$ *i.e.,* $\lambda^* \left(x_i(p, w) + \sum_{j=1}^{K} p_j \frac{\partial x_j}{\partial p_i} \right) = 0.$

♡

3.1.3 Envelope Theorem $\Rightarrow \lambda^* = \frac{\partial u(x(p,w))}{\partial w}$

Theorem 3.1 (Envelope Theorem)

Consider the constrained maximization problem,

$$\max_{x \in \mathbb{R}^n} \quad f(x; \theta)$$

$$\text{s.t. } g(x; \theta) \leq 0$$

where $xin\mathbb{R}^n$ *is the choice variable and* $\theta \in \mathbb{R}^m$ *is some parameter. Let* f, g *be continuously differentiable real-valued functions.*

- *Let the value function of the problem be* $V(\theta) \triangleq f(x^*(\theta), \theta)$.
- *The Lagrangian for this problem is*

$$L(x, \lambda; \theta) = f(x; \theta) - \lambda g(x; \theta)$$

- *Let* x^* *and* λ^* *denote the optimized values of the variables.*

  (By KKT necessary conditions, we have $\frac{\partial f}{\partial x}(x^*; \theta) = \lambda^* \frac{\partial g}{\partial x}(x^*; \theta)$ *and* $\lambda^* g(x^*; \theta) = 0$*)*

Then the following is true for any $\bar{\theta} \in \mathbb{R}^m$

$$\frac{\partial V}{\partial \theta_i}(\bar{\theta}) = \frac{\partial L}{\partial \theta_i}(x^*, \lambda^*; \bar{\theta}) = \frac{\partial f}{\partial \theta_i}(x^*; \bar{\theta}) - \lambda^* \frac{\partial g}{\partial \theta_i}(x^*; \bar{\theta})$$

♡

Proof 3.3

The proof of the envelope theorem is a straightforward calculation.

Firstly, by KKT necessary conditions, we have $\frac{\partial f}{\partial x}(x^*; \bar{\theta}) = \lambda^* \frac{\partial g}{\partial x}(x^*; \bar{\theta})$ *and* $\lambda^* g(x^*; \bar{\theta}) = 0 \Rightarrow$

$\lambda^*\left[\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i}+\frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\right]=0$. *Then we have*

$$\begin{aligned}\frac{\partial V}{\partial\theta_i}(\bar\theta)&=\frac{\partial f}{\partial\theta_i}(x^*;\bar\theta)+\frac{\partial f}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i}\\&\left(\text{by }\frac{\partial f}{\partial x}(x^*;\bar\theta)=\lambda^*\frac{\partial g}{\partial x}(x^*;\bar\theta)\right)\\&=\frac{\partial f}{\partial\theta_i}(x^*;\bar\theta)+\lambda^*\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i}\\&\left(\text{by }\lambda^*\left[\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i}+\frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\right]=0\right)\\&=\frac{\partial f}{\partial\theta_i}(x^*;\bar\theta)-\lambda^*\frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\end{aligned}$$

Corollary 3.2

$\lambda^*=\frac{\partial u(x(p,w))}{\partial w}$.

Proof 3.4

By the envelope theorem, we have $\frac{\partial u(x(p,w))}{\partial w}=\frac{\partial L}{\partial w}\Big|_{x^*,\lambda^*}=\lambda^*$.

3.1.4 Indirect Utility Function $v(p,w)\equiv u(x(p,w))$

Proposition 3.7 (Properties of Indirect Utility Function)

1. $v(p,w)$ *is homogeneous of degree zero in* (p,w)*;*
2. $v(p,w)$ *is strictly increasing in* w *and non-increasing in* p_i*;*
3. $v(p,w)$ *is quasi-convex, that is the set* $\{p: v(p,w)\le u\}$ *is convex for all* $u\in\mathbb{R}$*.*
4. $\lambda^*=\frac{\partial v(p,w)}{\partial w}$ *(Corollary 3.2).*

3.1.5 Roy's Identity $x_i^*=-\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial w}}$: recover $x(p,w)$ from $v(p,w)$

Proposition 3.8 (Roy's Identity)

$x_i^*(p,w)=-\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}}$.

Proof 3.5

By the definition,

$$
\begin{aligned}
v(p,w) &\equiv u(x(p,w)) \\
\frac{\partial v}{\partial p_i} &\equiv \sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial p_i} \\
&= \sum_{j=1}^{K} \lambda^* p_j \frac{\partial x_j}{\partial p_i} && \text{by } \frac{\partial u\left(x^*\right)}{\partial x_i} = \lambda^* p_i \\
&= -\lambda^* x_i^* && \text{by } \lambda^* \left(x_i(p,w) + \sum_{j=1}^{K} p_j \frac{\partial x_j}{\partial p_i} \right) = 0 \\
x_i^* &= -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial w}} && \text{by } \lambda^* = \frac{\partial v(p,w)}{\partial w}
\end{aligned}
$$

3.2 Expenditure Minimization Problem (EMP)

Consider the duality

$$
\begin{aligned}
\min_{x \in X} \quad & p \cdot x \\
s.t. \quad & u(x) \geq u
\end{aligned}
\tag{EMP}
$$

The optimal solutions are represented by $h(p,u)$. With uniqueness, we call it *Hicksian (compensated) demand*.

3.2.1 Hicksian Demand $h(p,u)$: Properties

Proposition 3.9 (Properties of Hicksian Demand)

(i). $h(p,u)$ *is homogeneous of degree zero in* p*:*

$$h(tp,u) = h(p,u), \ \forall t \in \mathbb{R}_+$$

(ii). $u(x)$ *is strictly quasi-concave* $\Rightarrow h(p,u)$ *is unique;*

(iii). For $u > u(0)$ *and* $u(\cdot)$ *is locally non-satiated, constraint is active: for all* $x^* \in h(p,u)$*,*

$$u(x^*) = u$$

Lemma 3.2 ($\sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0$)

If $u(x)$ *is strictly quasi-concave, the Hicksian demand satisfies* $\sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0$

Proof 3.6

$u(h(p,u)) \equiv u \Rightarrow \sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0.$

3.2.2 Expenditure Function $e(p, u) \equiv p \cdot h(p, u)$

Given the Hicksian demand $h(p, u)$, we can define the expenditure function as $e(p, u) \equiv p \cdot h(p, u)$.

Proposition 3.10 (Properties of Expenditure Function)

(i). $e(p, u)$ *is homogeneous of degree* 1 *in* p*:*

$$e(tp, u) = tp \cdot h(tp, u) = tp \cdot h(p, u) = te(p, u)$$

(ii). $e(p, u)$ *is strictly increasing in* u*, non-decreasing in* p_i*;*

(iii). $e(p, u)$ *is* ***concave*** *in* p*;*

(iv). $e(p, u)$ *is continuous in* p *for all* $p >> 0$*;*

(v). For all $x^* \in h(p, u)$*,* $x^* \in x(p, e(p, u))$*;*

(vi). For $w > 0$*,* $e(p, v(p, w)) \equiv w$*;*

(vii). $e(p, u)$*'s derivative property:*

$$\frac{\partial e(p, u)}{\partial p_i} \equiv h_i(p, u)$$

Proof 3.7 (Proof for concavity)

Suppose the price of good 1 increases from p_1^0 *to* p_1^1*:* $p^0 \to p^1$*. Set* $p^a = ap^0 + (1-a)p^1,\ 0 \leq a \leq 1$*.*

So, $p^0 \leq p^a \leq p^1$ *and*

$$\begin{aligned} e(p^a, u) &= p^a \cdot h(p^a, u) \\ &= (ap^0 + (1-a)p^1) \cdot h(p^a, u) \\ &= a[p^0 \cdot h(p^a, u)] + (1-a)[p^1 \cdot h(p^a, u)] \\ &\quad h(p^a, u) \text{ is feasible in both EMP, but not optimal solutions} \\ &\geq a[p^0 \cdot h(p^0, u)] + (1-a)[p^1 \cdot h(p^1, u)] \\ &= ae(p^0, u) + (1-a)e(p^1, u) \end{aligned}$$

Proof 3.8 (Proof for Derivative)

1. *Direct proof:*

$$
\begin{aligned}
e(p,u) &\equiv p \cdot h(p,u) \\
\frac{\partial e}{\partial p_i} &\equiv \sum_{j=1}^{K} p_j \frac{\partial h_j}{\partial p_i} + h_i \\
&\equiv \sum_{j=1}^{K} \frac{1}{\lambda^*} \frac{\partial u\left(x^*\right)}{\partial x_j} \frac{\partial h_j}{\partial p_i} + h_i \qquad \text{by } \frac{\partial u\left(x^*\right)}{\partial x_i} = \lambda^* p_i \\
&= h_i \qquad \text{by } \sum_{j=1}^{K} \frac{\partial u}{\partial x_j} \frac{\partial h_j}{\partial p_i} = 0
\end{aligned}
$$

2. *Envelope Theorem Proof:*

$$
\begin{aligned}
L(x,\lambda;(p,u)) &= p \cdot x - \lambda(u(x) - u) \\
\frac{\partial e(p,u)}{\partial p_i} &= \left.\frac{\partial L(x,\lambda;(p,u))}{\partial p_i}\right|_{x^*=h(p,u)} = x_i|_{x^*=h(p,u)} = h_i(p,u)
\end{aligned}
$$

3.2.3 Law of Compensated Demand: $\frac{\partial h_i}{\partial p_i} \leq 0$

Corollary 3.3 (Law of Compensated Demand)

Hicksian demand is downward sloping in its own price,

$$
\frac{\partial h_i}{\partial p_i} \leq 0
$$

♡

Proof 3.9

By the concavity of $e(p,u)$ (3.10), we can conclude $\nabla^2 e(p,u) \preceq 0$ (negative semi-definite). Then, we know its diagonal elements are non-positive $\frac{\partial e^2}{\partial^2 p_i} = \frac{\partial h_i}{\partial p_i} \leq 0$.

3.2.4 Shifts in Hicksian Demand: $\frac{\partial h_i}{\partial u} \equiv \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial u}$, same direction as $\frac{\partial x_i}{\partial w}$

How does Hicksian demand curve shift when utility changes?

$$
\begin{aligned}
h_i(p,u) &\equiv x_i(p, e(p,u)) \\
\frac{\partial h_i}{\partial u} &\equiv \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial u}
\end{aligned}
$$

We know $\frac{\partial e}{\partial u} > 0$, so the direction of Hicksian demand shift is the same as $\frac{\partial x_i}{\partial w}$.

- Normal good: increasing utility shifts h_i to the right.

- Inferior good: increasing utility shifts h_i to the left.

3.3 UMP and EMP

3.3.1 Slutsky Equation: substitution effect and income effect

Slutsky: how change of p_j (price in good j) affects x_i (the demand of product i).

Proposition 3.11 (Slutsky Equation)

$$\frac{\partial x_i(p,w)}{\partial p_j} = \underbrace{\frac{\partial h_i(p,u)}{\partial p_j}}_{\text{substitution effect}} \underbrace{-\frac{\partial x_i(p,w)}{\partial w}x_j(p,w)}_{\text{income effect}}$$

Proof 3.10

$$\begin{aligned} h_i(p,u) &\equiv x_i(p,e(p,u)) \\ \frac{\partial h_i}{\partial p_j} &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial p_j} \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}h_j(p,u) \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}x_j(p,e(p,u)) \end{aligned}$$

- **Substitution effect**: $\frac{\partial h_i}{\partial p_j}$, the change of relative prices change with constant utility will change the x_i.
- **Income (Wealth) effect**: $-\frac{\partial x_i}{\partial w}x_j(p,w)$, the change of price can be seen as change of wealth, which will also impact the x_i.

3.3.2 Relationship Between UMP and EMP

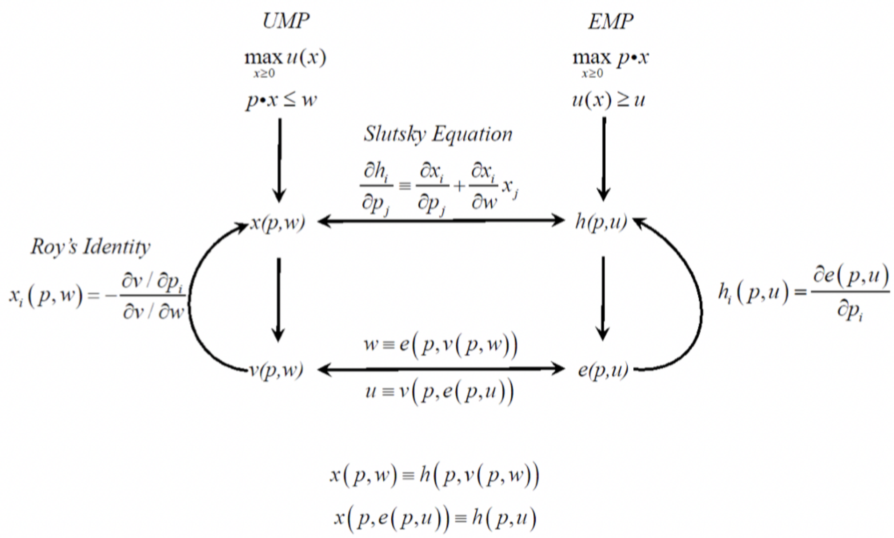

Figure 3.1: Relationship Between UMP and EMP

3.4 Competitive Equilibrium

Definition 3.2 (Competitive Equilibrium)

Given endowment $\{w^i\}_{i\in I}$. A **competitive equilibrium** is a pair $p \in \mathbb{R}^L$ (price vector over L goods) and an allocation $(x^i)_{i\in I}$ such that:

(i). $x^i \in \operatorname{argmax} u^i(x)$ s.t. $p \cdot x^i \leq p \cdot w^i, \forall i \in I$.

(ii). $\sum_{i\in I} x^i_\rho(p, w) = \sum_{i\in I} w^i_\rho, \forall \rho \in L$.

♣

Definition 3.3

An allocation x is **Pareto-effecient** if there doesn't exist an allocation y s.t. $u_i(y) \geq u_i(x)$ for each $i \in I$ and $u_j(y) > u_j(x)$ for some $j \in I$.

♣

(Assume consumers' preferences are strict monotone)

Theorem 3.2 (First-order fundamental welfare theorem)

Suppose (p^*, x^*) is a competitive equilibrium. Then x^* is Pareto-efficient.

♡

If CE exists we can prove a Pareto-effecient allocation is CE.

Theorem 3.3 (Second-order fundamental welfare theorem)

Suppose that x^* is Pareto efficient and consumers receive endowment worth $w^i = p \cdot x^{i^*}$ for all $i \in I$. Then, if a competitive equilibrium exists for such w, then x^* is a competitive equilibrium allocation.

♡

Chapter 4 Game Theory

Based on

- ”Kreps, D. M., & Sobel, J. (1994). Signalling. *Handbook of game theory with economic applications*, 2, 849-867.”
- Mas-Colell, Whinston, and Green, Microeconomic Theory, Oxford University Press (1995).
- UIUC ECON 530 21Fall, Nolan H. Miller
- UC Berkeley ECON 201A 23Fall
- UC Berkeley MATH 272 23Fall, Alexander Teytelboym
- Jehle, G., Reny, P.: Advanced Microeconomic Theory . Pearson, 3rd ed. (2011). Ch. 6.

4.1 Basic Game Theory

4.1.1 Game, Dominant Strategy

A game is denoted by

$$\Gamma = \left(\underbrace{I}_{\text{players}}, \underbrace{\{S_i\}_{i\in I}}_{\text{Strategy Set}}, \underbrace{\{u_i(\cdot)\}_{i\in I}}_{\text{VNM utility}} \right)$$

$u_i : \prod_{i\in I} S_i \to \mathbb{R}$ is the utility function that maps all players' strategies to a player's utilities.

Definition 4.1 (Dominant Strategy)

A strategy $s_i \in S_i$ is **dominant** for i in Γ if for all $s_i' \neq s_i$, we have $u_i(s_i, S_{-i}) \geq u_i(s_i', S_{-i})$. ♣

4.1.2 Nash Equilibrium and Existence

Definition 4.2 (Nash Equilibrium)

A strategy profile $\Sigma = (\sigma_1, ..., \sigma_I)$ is a **Nash** equilibrium of the game Γ if for every $i \in I$, we have: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Delta(S_i)$ (can't benefit from deviating). ♣

Theorem 4.1 (Existence of Nash Equilibrium)

A Nash equilibrium exists in Γ if for all $i \in I$,

(i). S_i is non-empty, convex, compact, subset of $\mathbb{R}^m$ (i.e., for some finite dimensions of real numbers).

(ii). $u_i(s_i, ..., s_I)$ is continuous in $(s_i, ..., s_I)$ and quasi-concave in any s_i. ♡

Proof 4.1

We prove a lemma for the best response correspondence $b_i(s_{-i}) = \text{argmax}_{s_i \in S_i} u(s_i, s_{-i})$ *firstly.*

Lemma 4.1

Suppose $\{S_i\}_{i\in I}$ *are non-empty. Suppose that* S_i *is compact and convex and* u_i *is continuous in* $(s_i, ..., s_I)$ *and quasi-concave in any* s_i*, then* $b_i(s_{-i})$ *is non-empty, convex-valued and uhc.* ♡

Proof 4.2

This lemma is proved by Berge's Maximum Theorem (Theroem **??***).*

Consider the function $b : S \to S$ *with* $b(s_i, ..., s_I) = \{b_1(s_{-1}), ..., b_I(s_{-I})\}$.

As we proved b *is non-empty, convex-valued and uhc from* S *to* S *where* S *is non-empty, compact, and convex. By the Kakutani's Fixed Point Theorem (Theorem* **??***), we have* b *has a fixed point* $s \in S$*, which should be the Nash equilibrium.*

4.1.3 Bayesian Game

Definition 4.3 (Bayesian Game)

A **Bayesian game** is defined by $\Gamma = (I, \{S_i\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}, \{F_i\}_{i\in I})$ where $u_i(s_i, s_{-i}, \theta_i)$ maps the strategies of players and player i's type $\theta_i \in \Theta_i$ to player i's utilities. F_i is the distribution of the player i's type. ♣

4.2 Mechanism Design

Design incentives for agents to reveal their types or achieve particular society outcomes.

Given the set of agents, alternatives (for the society), and types (of agents) are I, X, Θ and a social choice function $f : \Theta = (\Theta_1, ..., \Theta_I) \to X$.

Definition 4.4 (Mechanism)

A **mechanism** is represented as

$$\Gamma = \Big((S_1, ..., S_I), g : S \triangleq (S_1, ..., S_I) \to X\Big)$$

where S_i represents the strategy set of agent i. ♣

Definition 4.5 (Implement)

Γ (indirectly) **implements** a social choice function f if $\exists\, (s_1^*(\cdot), ..., s_I^*(\cdot))$ of a game induced by Γ such that $g(s_1^*(\theta_1), ..., s_I^*(\theta_I)) = f(\theta_1, ..., \theta_I)$ for all $(\theta_1, ..., \theta_I) \in \Theta$ ♣

Definition 4.6 (Direct Mechanism)

A mechanism is **direct mechanism** if $S_i = \Theta_i$ for all $i \in I$ and $g(\theta) = f(\theta)$ for all $\theta = (\theta_1, ..., \theta_I) \in \Theta$. So, a direct mechanism can be represented by $\Gamma = (\Theta, f(\cdot))$. ♣

Definition 4.7 (Weak Dominance)

A strategy is weakly dominant if for all $\theta_i \in \Theta_i$ and all $s_{-i}(\cdot) \in S_{-i}$, we have:

$$u_i(g(s_i(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

for all $s_i' \neq s_i$. ♣

Definition 4.8 (Dominant Strategy Equilibrium)

Strategy profile $s^* = (s_1^*(\cdot), ..., s_I^*(\cdot))$ is a **dominant strategy (D-S) equilibrium** of $\Gamma = (S, g(\cdot))$ if for all $i \in I$ and $\theta_i \in \Theta$, we have:

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

for all $s_i' \in S_i$ and $s_{-i} \in S_{-i}$. ♣

Definition 4.9 (Implement in dominant strategies)

Γ **implements** f in **dominant strategies** if $\exists$ a dominant strategy (D-S) equilibrium s^* of Γ such that $g(s^*(\theta)) = f(\theta)$. ♣

Definition 4.10 (Strategy-Proof, DSIC)

f is **strategy-proof** (also called dominant-strategy-incentive-compatible, DSIC) if "$s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all $i \in I$" is a dominant strategy (D-S) equilibrium of the direct mechanism $\Gamma = (\Theta, f(\cdot))$. ♣

Theorem 4.2 (Revelation Principle)

Suppose that $\exists \Gamma = (S, g(\cdot))$ that (indirectly) implements f in dominant strategies. Then f is strategy-proof (DSIC). ♡

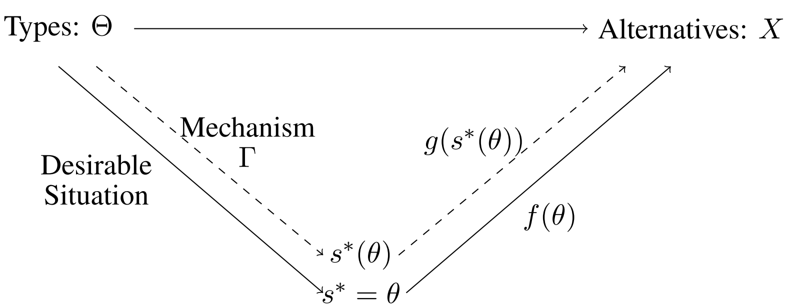

Figure 4.1: How Mechanism Design works

Theorem 4.3 (Gibbard-Satterthwaite theorem)

Suppose that $|X| \geq 3$ and a social choice function f is surjective. Then,

$$f \text{ is strategy-proof (DSIC)} \Leftrightarrow f \text{ is dictatorial (2.7)}$$

Some lemmas can help to prove the theorem.

Lemma 4.2

If f is strategy-proof (DSIC) and $f(\succeq) = x$ and $x \succeq_i y \Rightarrow x \succeq'_i y$ for all $i \in I$ and all $x \neq y \in X$, then $f(\succeq') = x$.

Lemma 4.3 (Pareto Effeciency)

If f is strategy-proof (DSIC) and $x \succ_i y$ for all $i \in I$, then $f(\succeq') \neq y$.

Example 4.1 Define $\succeq = \begin{pmatrix} x & y \\ y & x \\ z & z \end{pmatrix}$ and $\succeq' = \begin{pmatrix} x & y \\ y & z \\ z & x \end{pmatrix}$, each column 1/2 represents player 1/2's preferences.

4.3 Signaling Game

4.3.1 Canonical Game

Definition 4.11 (Canonical Game)

1. There are two players: **S** (sender) and **R** (receiver).
2. **S** holds more information than **R**: the value of some random variable t with support $\mathcal{T}$. (We say that t is the **type** of **S**)
3. Prior belief of **R** concerning t are given by a probability distribution ρ over $\mathcal{T}$ (common knowledge)
4. **S** sends a **signal** $s \in \mathcal{S}$ to **R** drawn from a signal set $\mathcal{S}$.
5. **R** receives this signal, and then takes an **action** $a \in \mathcal{A}$ drawn from a set $\mathcal{A}$ (which could depend on the signal s that is sent).
6. **S**'s payoff is given by a function $u : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ and **R**'s payoff is given by a function $v : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.

4.3.2 Nash Equilibrium

Definition 4.12 (Strategy)

A **behavior strategy** for **S** is given by a function $\sigma : \mathcal{T} \times \mathcal{S} \to [0, 1]$ such that $\sum_s \sigma(t, s)$ for each t.

A **behavior strategy** for **R** is given by a function $\alpha : \mathcal{S} \times \mathcal{A} \to [0, 1]$ such that $\sum_a \alpha(s, a)$ for each t. ♣

Definition 4.13 (Nash Equilibrium)

Behavior strategies α and σ form a **Nash equilibrium** if and only if

1. For all $t \in \mathcal{T}$,

$$\sigma(t, s) > 0 \text{ implies } \textstyle\sum_a \alpha(s, a)u(t, s, a) = \max_{s' \in \mathcal{S}} \left(\sum_a \alpha(s', a)u(t, s', a)\right)$$

2. For each $s \in \mathcal{S}$ such that $\sum_t \sigma(t, s)\rho(t) > 0$,

$$\alpha(s, a) > 0 \text{ implies } \textstyle\sum_t \mu(t; s)v(t, s, a) = \max_{a'} \sum_t \mu(t; s)v(t, s, a')$$

where $\mu(t; s)$ is the $\mathbb{R}$'s posterior belief about t given s, $\mu(t; s) = \frac{\sigma(t,s)\rho(t)}{\sum_{t'} \sigma(t',s)\rho(t')}$ if $\sum_t \sigma(t, s)\rho(t) > 0$ and $\mu(t; s) = 0$ otherwise. ♣

Definition 4.14 (Separating & Pooling Equilibrium)

An equilibrium (σ, α) is called a **separating** equilibrium if each type t sends different signals; i.e., the set $\mathcal{S}$ can be partitioned into (disjoint) sets $\{\mathcal{S}_t; t \in \mathcal{S}\}$ such that $\sigma(t, \mathcal{S}_t) = 1$. An equilibrium (σ, α) is called a **pooling** equilibrium if there is a single signal s^* that is sent by all types; i.e., $\sigma(t, s^*) = 1$ for all $t \in \mathcal{T}$. ♣

4.3.3 Single-crossing

4.3.3.1 Situation over real line

Consider the situation that $\mathcal{T}, \mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$ and $\geq$ is the usual "greater than or equal to" relationship.

1. We let $\Delta\mathcal{A}$ denote the set of probability distributions on $\mathcal{A}$.
2. For each $s \in \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$, we let $\Delta\mathcal{A}(s, T')$ be the set of mixed strategies that are the best responses by **R** to $s \in \mathcal{S}$ for some probability distribution with support $\mathcal{T}'$.
3. For $\alpha \in \Delta\mathcal{A}$, we write $u(t, s, \alpha) \triangleq \sum_{a \in \mathcal{A}} u(t, s, a)\alpha(a)$.

Definition 4.15 (Single-crossing)

The data of the game are said to satisfy the **single-crossing property** if the following holds: If $t \in \mathcal{T}$, $(s, \alpha) \in \mathcal{S} \times \Delta\mathcal{A}$ and $(s', \alpha') \in \mathcal{S} \times \Delta\mathcal{A}$ are such that $\alpha \in \Delta\mathcal{A}(s, \mathcal{T})$, $\alpha' \in \Delta\mathcal{A}(s', \mathcal{T})$, $s > s'$ and $u(t, s, \alpha) \geq u(t, s', \alpha')$, then for all $t' \in T$ such that $t' > t$, $u(t', s, \alpha) \geq u(t', s', \alpha')$. ♣

4.4 Equilibrium Refinement

4.4.1 Cho-Kreps Intuitive Criterion

Definition 4.16 (Equilibrium Dominated Message)

A message is **equilibrium dominated** for a type if the type must do strictly worse by sending the message than it does in equilibrium (i.e., payoff in eq. is strictly better than maximum payoff from deviating). ♣

Definition 4.17 (Cho-Kreps Intuitive Criterion)

If an information set is off the eq. path and a message is eq. dominated for a type, then beliefs should assign zero probability to the message coming from that type (if possible). ♣

Chapter 5 Market Design

Based on

- Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis, Roth, Alvin E.& Sotomayor, Matilda, 1990.
- Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. *Mathematics of Operations research*, 28(1), 103-126.
- Hatfield, J. W., & Kominers, S. D. (2017). Contract design and stability in many-to-many matching. *Games and Economic Behavior*, 101, 78-97.

5.1 Matching One-to-One

Suppose there are doctors (D) and hospitals (H). For a doctor d, define a relation $\succeq_d$ over $H \cup \{d\}$; for a hospital h, define a relation $\succeq_h$ over $D \cup \{h\}$. A matching market is defined by

$$(D, H, \{\succeq_i\}_{i \in D \cup H})$$

Note *Given a matching $\mu : D \cup H \to D \cup H$, we would call $\mu(d)$ be "d's match".*

Definition 5.1 (Involution)

A matching $\mu : D \cup H \to D \cup H$ is **involution** such that

$$\mu(d) \neq d \Rightarrow \mu(d) \in H, \forall d \in D$$

and

$$\mu(h) \neq h \Rightarrow \mu(h) \in D, \forall h \in H$$

♣

Definition 5.2 (Stable)

A matching $\mu : D \cup H \to D \cup H$ is **stable** if it is

- Individually Rational: $\nexists\, i$ for whom $i > \mu(i)$.
- (Pairwise) Unblocked: $\nexists\, (d, h)$ such that $d \succ_h \mu(h)$ and $h \succ_d \mu(d)$.

♣

Theorem 5.1 (Gale-Shapley, 1962)

For any matching market, a stable matching μ exists.

♡

Proof 5.1

We prove it by an algorithm:

Definition 5.3 (Deferred Acceptance Algorithm (DA))

At each round, every doctor applies for his most preferred hospital among those haven't rejected him. Each hospital chooses its most preferred doctors among its applicants and the one on the previous waitlist, and then rejects others. ♣

Observation: DA terminates μ. We want to prove

1. *μ is IR (obviously);*
2. *μ is unblocked.*

Suppose there is a block (d, h) such that $d \succ_h \mu(h)$ and $h \succ_d \mu(d)$. That is impossible, because the $d \neq \mu(h)$, the d must be rejected by h, which means $h \preceq_d \mu(d)$.

Note *We call "h is **achievable** for d" if $\mu(d) = h$ for some stable matching μ.*

5.1.1 Matching Markets: One-to-One

Definition 5.4 (D-Optimal Matching)

A matching $\mu : D \cup H \to D \cup H$ is D-**optimal**, denoted by μ^D, if for any stable μ' we have that $\mu^D \succeq_D \mu'$ (the best stable matching for all doctors). ♣

Theorem 5.2 (Deferred Acceptance Algorithm ⇒ D-Optimal Matching)

Deferred Acceptance Algorithm (with D proposing) terminates in μ^D. ♡

Proof 5.2

...Theorem 2.12 (Gale and Shapley)

Theorem 5.3 (Lone-Wolf Theorem)

The set of matched agent is identical in every stable μ. ♡

Proof 5.3

$|\mu^D(H)| \geq |\mu(H)| \geq |\mu^H(H)|$; *by symmetry,* $|\mu^H(D)| \geq |\mu(D)| \geq |\mu^D(D)|$. *Because* $|\mu^D(H)| = |\mu^D(D)|$ *and* $|\mu^H(H)| = |\mu^H(D)|$ *by one-to-one, so everything is equal.*

5.1.2 Joint and Meet

Definition 5.5 (Joint and Meet)

1. **Join** $\mu \vee_D \mu'$ assign the more preferred match to every d and the less preferred match to every h, that is,

$$\mu \vee_D \mu'(d) = \begin{cases} \mu(d), & \text{if } \mu(d) >_d \mu'(d) \\ \mu'(d), & \text{otherwise} \end{cases}, \forall d \in D$$

$$\mu \vee_D \mu'(h) = \begin{cases} \mu(h), & \text{if } \mu(h) <_h \mu'(h) \\ \mu'(h), & \text{otherwise} \end{cases}, \forall h \in H$$

2. **Meet** $\mu \wedge_D \mu'$ assign the less preferred match to every d and the more preferred match to every h, that is,

$$\mu \wedge_D \mu'(d) = \begin{cases} \mu(d), & \text{if } \mu(d) <_d \mu'(d) \\ \mu'(d), & \text{otherwise} \end{cases}, \forall d \in D$$

$$\mu \wedge_D \mu'(h) = \begin{cases} \mu(h), & \text{if } \mu(h) >_h \mu'(h) \\ \mu'(h), & \text{otherwise} \end{cases}, \forall h \in H$$

♣

Theorem 5.4 (Join and Meet of Stable Matchings are Stable)

If μ and μ' are stable, then $\mu \vee_D \mu'$ and $\mu \wedge_D \mu'$ are stable. ♡

5.1.3 Strategic Incentives

- Type = preference list.
- SCF: $f : \Theta \to \mathcal{M}$, where $\mathcal{M}$ is a set of stable matchings;
- Is f strategy-proof?
- Does there exist a stable and strategy-proof (direct) mechanism?

Definition 5.6

We say a mechanism φ is strategy-proof (SP) if $\varphi(\succ_i, \succ_{-i}) \geq \varphi(\succ_i', \succ_{-i})$ for all $i \in I$ and $\succ_i'$ and $\succ_{-i}$. ♣

Theorem 5.5 (Impossibility theorem (Roth))

There is no stable and strategy-proof (SP) mechanism. ♡

The mechanism that yields the D-optimal stable matching (in terms of the stated preferences) makes it a dominant strategy for each doctor to state his true preferences. (Similarly, the mechanism that yields the H-optimal stable matching makes it a dominant strategy for every hospital to state its true preferences.)

Theorem 5.6 (Dubins and Freedman; Roth)

The doctor(D)-optimal stable mechanism is strategy-proof for doctors.

Proof 5.4

5.2 Matching Many-to-Many

Contracts are denoted by $x \in X, x_D \in D, x_H \in H$. $F \triangleq D \cup H$.

Consider a set of contracts $Y \subseteq X$,

- Y_D = doctors listed in Y;
- Y_d = the contract in Y that list the doctor d;
- $\succ_d$ over set of contracts that name the doctor d;
- The set of contracts $f \in F$ chooses from Y: $C_f(Y) = \max_{\succ_f}\{Z \subseteq X : Z \subseteq Y_f\} \subseteq Y_f$;
- The set of contracts doctors choose from Y: $C_D(Y) = \cup_{d \in D} C_d(Y)$.
- The set of contracts doctors reject from Y: $R_D(Y) = Y \backslash C_D(Y)$.

The outcome of matching is $Y \subseteq X$.

Definition 5.7 (Stable Contracts)

$A \subseteq X$ is **stable** if

- Individually Rational (IR): for all $f \in F$: $C_f(A) = A_f$;
- Unblocked: $\nexists$ non-empty $Z \subseteq X$ such that $Z \cap A = \emptyset$ and for all $f \in F$, $Z_f \subseteq C_f(A \cup Z)$.

Example 5.1 Preferences over doctor d: $\{x, y\} > \{x\} > \emptyset > \{y\}$; Preferences over hospital h: $\{y\} > \{x, y\} > \{x\} > \emptyset$.

$\{x\} \Rightarrow \{x, y\} \Rightarrow \{y\} \Rightarrow \emptyset \Rightarrow \{x\}$.

Definition 5.8 (Substitutability Condition)

Preference of f satisfies the **substitutability condition** if for all $Y \subseteq X$ and $x, z \in X \backslash Y$:

$$z \notin C_f(Y \cup \{z\}) \Rightarrow z \notin C_f(Y \cup \{z\} \cup \{x\})$$

($Y' \subseteq Y \subseteq X \Rightarrow R_f(Y') \subseteq R_f(X)$, where R is the rejection choice.)

If z is rejected given a set, then it should also be rejected given a larger set.

Theorem 5.7

If contracts are substitutes, then $Y \subseteq X$ is stable <u>if and only if</u> pairwise stable.

Proof 5.5

Prove $\Leftarrow$: (If not pairwise stable $\Rightarrow$ not stable)

Suppose that Z is a block. So, $Z \subseteq C_f(A \cup Z)$ for all f listed in Z.

We can pick a $z \in Z$ such that $z \in C_f(A \cup Z)$. By the substitutability condition, $z \in C_f(A \cup \{z\})$. So, it is stable.

Theorem 5.8

If contracts are substitutes, then a stable outcome exists. ♡

Definition 5.9 (Lattice)

On a **lattice**, $L = (X, <, \wedge, \vee)$ (or we just use $L = (X, <)$), $<$ is a partial order on X in such a way that any two elements x and y of X have a unique greatest lower bound (glb) $x \wedge y$ (meet) and a unique lowest upper bound (lub) $x \vee y$ (join). ♣

Definition 5.10 (Complete Lattice)

A lattice $L = (X, <)$ is **complete** if there are both a meet (i.e. a greatest lower bound) and a join (i.e. a least upper bound) for any subset $Y \subseteq X$.

These generalized meet and join operations on Y are denoted by $\wedge Y$ and $\vee Y$. ♣

Definition 5.11 (Monotone Function over Lattice)

A function from one lattice to another lattice $f : (X, <) \to (X', <')$ is **monotone** if $x \leq y \Rightarrow f(x) \leq' f(y)$ for any $x, y \in X$. ♣

Theorem 5.9 (Tarski 1955)

Let $L = (X, <)$ be a complete lattice and $f : L \to L$ be monotone ($\leq$) function on L. Then, the set $\{x \in L : f(x) = x\}$ of fixed points is a non-empty, complete lattice with order $\leq$. ♡

Proof 5.6

Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. Mathematics of Operations research, 28(1), 103-126.

If some contracts are not substitute, there are no stable outcomes exist.

5.3 Matching Many-to-One

Settings

- Doctors, D; Hospitals, H; Contracts $X = D \times H \times$ terms;
- Hospitals preference $\succ_h$ over 2^X;

- Doctors preference $\succ_d$ over X (compare one contract with another one contract, not compare over sets of contracts);
- Outcome is $Y \subseteq X$ s.t. $|Y_d| \leq 1$ for all $d \in D$ (a doctor signs at most one contract).

What restriction do we need to have a stable matching? Not as strong as substitute.

Corollary 5.1

Doctor-proposing DA algorithm produces a doctor-optimal stable matching.

Example 5.2 The preferences of agents are

- $d_1 : h_1 \succ h_2$; $d_2 : h_1 \succ h_2$; $d_3 : h_2 \succ h_1$;
- $h_1 : d_3 \succ d_1, d_2 \succ d_1 \succ d_2$; $h_2 : d_1 \succ d_2 \succ d_3$.

There are two stable outcomes

1. (d_1, h_2), (d_3, h_1);
2. $(d_1, h_1), (d_2, h_1), (d_3, h_2)$.

Remark Lone-Wolf Theorem doesn't hold.

Assume the d_2's true preference is $h_2 \succ h_1$ and he reveals it, there is only one stable matching: (d_1, h_2), (d_3, h_1). So, the d_2 may benefit from lying.

Remark Strategy-proof doesn't hold.

Definition 5.12 (Law of Aggagate Demand/ Cardianlity Monotomicity (CM))

For h, $Y \subseteq Y' \subseteq X \Rightarrow |C_h(Y)| \leq |C_h(Y')|$

Theorem 5.10

Under substitutes and CM, doctor-proposing DA is strategy-proof and LWT holds.

Theorem 5.11 (Rural Hosptial Theorem)

Under substitutes / CM, hospitals have same numbers of contracts in every stable outcome.

Cadets-branch matching

Can be found in:

- Jagadeesan, R. (2019). Cadet-branch matching in a Kelso-Crawford economy. *American Economic Journal: Microeconomics*, 11(3), 191-224.

Remark Contracts are not substitutes.

Definition 5.13 (Unilateral Substitute)

Contracts are **unilateral substitutes** if for all $z, x \in X$ and $Y \subseteq X$ <u>such that $z_D \notin Y_D$</u> if $z \notin C_h(Y \cup \{z\}) \Rightarrow z \notin C_h(Y \cup \{z\} \cup \{x\})$

Remark Preferences of branches satisfying unilateral substitute.

Remark The outcome of doctor-proposing DA algorithm is doctor-optimal and stable.

5.4 Networks

Based on

- Fleiner, T., Jankó, Z., Tamura, A., & Teytelboym, A. (2015). Trading networks with bilateral contracts. arXiv preprint arXiv:1510.01210.
- Fleiner, T., Jankó, Z., Schlotter, I., & Teytelboym, A. (2023). Complexity of stability in trading networks. *International Journal of Game Theory*, 1-20.

Considering a trading network represented by a directed graph, where nodes are firms F and edges X are contracts (income arrow can be understood as buying products and outcome arrow can be understood as selling products).

The choice function of $f \in F$ is represented by C^f, the choice of f over $Y_f \subseteq X_f$ is $C^f(Y_f) \subseteq Y_f$, where X_f is the set of contracts involving f.

The choice sets of buyer side (B) and seller side (S) are defined as

$$C_B^f(Y|Z) \triangleq C^f(Y_f^B \cup Z_f^S) \cap X_f^B$$

$$C_S^f(Z|Y) \triangleq C^f(Z_f^S \cup Y_f^B) \cap X_f^S$$

where Y is the contracts from buyer side and Z is the contratcts from seller side.

Definition 5.14 (Irrelevance of Rejected Contracts)

Irrelevance of Rejected Contracts (IRC): $C(A) \subseteq B \subseteq A \Rightarrow C(A) = C(B)$ ♣

Definition 5.15 (Fully Substitute)

C^f is **fully substitute** if for $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

$$R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$$

$$R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$$

and

$$R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$$

$$R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$$

♣

Define partial order, $(Y, Z) \geq (Y', Z')$ if $Y \subseteq Y'$ and $Z \supseteq Z'$.

Definition 5.16 (Stable Outcome, Hatfield and Kominers (2012))

An outcome $A \subseteq X$ is stable if it is

1. Individual Rational: $\forall f \in F, C^f(A_f) = A_f$;
2. Unblocked: there is no non-empty set $Z \subseteq X$ s.t. $Z \cap A = \emptyset$ and $\forall f \in F(Z), Z_f \subseteq C^f(A \cup Z)$, where $F(Z)$ is the set of the firms are lined to Z.

♣

Definition 5.17 (Trail)

Trail is the set of distinct edges $T = (X^1, X^2, ..., X^M)$ such that the buyer side (the firm who is the buyer in the edge) of X^i is exactly the seller side (the firm who is the seller in the edge) of X^{i+1}, which is denoted by $b(X^i) = s(X^{i+1}), i = 1, ..., M-1$.

♣

Definition 5.18 (Trail-stable Outcome)

An outcome $A \subseteq X$ is **trail-stable** if its is

1. Individual Rational;
2. There is no locally blocking trail $T = (X^1, X^2, ..., X^M)$ such that

   $X^1 \in C^{S(X^1)}(A \cup X^1)$;

   $\{X^i, X^{i+1}\} \in C^{b(X^i)}(A \cup X^i \cup X^{i+1})$;

   $X^M \in C^{b(X^M)}(A \cup X^M)$.

♣

Theorem 5.12 (Fleiner et al. 2016)

If C^f is fully substitute and IRC for all $f \in F$, then a trail-stable outcome exists.

♡

Proof 5.7

$Y \subseteq X$ and $Z \subseteq X$,

$$\Phi(Y, Z) = (X \backslash R_S(Z|Y), X \backslash R_B(Y|Z))$$

where $R_B(Y|Z) = \cup_{f \in F} R_B^f(Y|Z)$.

Claim 5.1

If (Y, Z) is a fixed point of Φ, then $A = Y \cap Z$ is trail-stable outcome.

♠

Lemma 5.1

C^f is fully substitute and IRC, and (Y, Z) such that $Y \cap Z = A$, $C_S(Z|Y) = A$, $C_B(Y|Z) = A$. Then, for a contract $x \in X \backslash A$ and $A \subseteq A' \subseteq X$ if $C_S^{S(x)}(A \cup x|A')$ then $x \in C_S^{S(x)}(Z \cup x|A')$.

♡

Φ will be monotone for the partial order $\geq$. As $(Y, Z) \geq (Y', Z')$, then $\Phi(Y, Z) \geq \Phi(Y', Z')$. Using Tarski fixed-point theorem, there is a (Y, Z) fixed point. .....

Read Fleiner, T., Jankó, Z., Tamura, A., & Teytelboym, A. (2015). Trading networks with bilateral contracts. arXiv preprint arXiv:1510.01210.

Proposition 5.1

A is trail-stable $\Rightarrow \exists\, (Y, Z)$ such that $Y \cap Z = A$ and (Y, Z) is a fixed point of Φ.

5.5 Corporate Game Theory

There is a set of players $N = \{1, ..., n\}$. The subset of players $S \subseteq N$ is called coalition.

There is a value function about coalition $v : 2^N \to \mathbb{R}$, which assumes $v(N) \geq \max_{S\subseteq N} v(S)$.

Definition 5.19 (Cooperative Game)

A cooperative game is described by the pair $\langle N, v\rangle$.

Definition 5.20 (Transferable Utility)

Utility is transferable if one player can losslessly transfer part of its utility to another player.

Assume a TU (transferable utility) Economy. Consider a payoffs vector for all players, $x \in \mathbb{R}^n$. The efficiency requires $\sum_{i\in N} x_i = v(N)$. Individual Rational (IR) requires $x_i \geq v(\{i\})$.

5.5.1 Core of Corporate Game and Farkas' lemma

Definition 5.21 (Core)

The **core** is the set of feasible allocations where no coalition of agents can benefit by breaking away from the grand coalition.

$$C(v, N) = \left\{ x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S), \forall S \subseteq N \right\}$$

Theorem 5.13 (Bondareva-Shapley Theorem)

?? *The core of $\langle N, v\rangle$ is non-empty ($C(v, N) \neq \emptyset$) <u>if and only if</u> for every function $\alpha : 2^N \backslash \{\emptyset\} \to [0, 1]$ where $\forall i \in N : \sum_{S:i\in S} \alpha(S) = 1$, the following condition holds:*

$$\sum_{S\in 2^N\backslash\{\emptyset\}} \alpha(S)v(S) \leq v(N)$$

Consider $B(N)$ be the solutions to: $\begin{cases} \sum_{S:i\in S} y_S = 1, & \forall i \in N \\ y_S \geq 0, & \forall S \subseteq N \end{cases}$

Lemma 5.2 (Farkas' lemma)

Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then, ***exactly one*** *of the following statement is true*

(1). There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

(2). There exists $y \in \mathbb{R}^n$ such that $A^T y \geq 0$ and $b^T y < 0$. ♡

Lemma 5.3

♡

Proof 5.8

Lemma 5.4 ((Alternative) Farkas' lemma)

Let A be $m \times n$ matrix, $b \in \mathbb{R}^m$ and $F = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$. Then, either $Cx = d$ or $\exists z$ such that for $y_S \geq 0$, $C^T z - A^T y_S = 0$ and such that $d^T z - b^T y_S < 0$, but not both. ♡

By using this lemma, we can conclude $\begin{cases} v(N)z - \sum_S v(S)y_S < 0 \\ z - \sum y_S = 0 \\ y_S \geq 0 \end{cases}$ *must hold at the same time, (let $z = 1$, the last two lines are $B(N)$).*

Hence, $\forall y_S \in B(N)$, we have $v(N) \geq \sum_S v(S)y_S$.

5.5.2 Doubly stochastic matrix and Birkhoff-von Neumann Theorem

Consider a matching game between sellers and buyers: $v(\{i,j\}) = v_{ij}$, $v(\{i\}) = 0$ for buyer i and $v(\{j\}) \geq 0$ for seller j.

Core:

$$
\begin{aligned}
\max_{\alpha} \quad & \sum_i \sum_j v_{ij}\alpha_{ij} \\
\text{s.t.} \quad & \sum_i \alpha_{ij} = 1, \forall j \\
& \sum_j \alpha_{ij} = 1, \forall i \\
& \alpha_{ij} \geq 0
\end{aligned}
$$

Definition 5.22 (Doubly Stochastic Matrix)

A **doubly stochastic matrix** is a square matrix $X = (x_{ij})$ of non-negative real numbers, each of whose rows and columns sums to 1.

The class of $n \times n$ doubly stochastic matrices is a convex polytope (convex set in euclidean space) known as the **Birkhoff polytope**. ♣

Theorem 5.14 (Birkhoff-von Neumann Theorem)

A matrix is doubly stochastic if and only if it is a convex combination of permutation matrices.

By this theorem, we can set efficient "integer" assignment.

Can the efficient allocation be competitive equilibrium (CE)?

Theorem 5.15

The core of assignment of game is non-empty.

Proof 5.9

The duality of core can be written as

$$\begin{aligned} \min \quad & \sum_j u_j^S + \sum_i u_i^B \\ \text{s.t.} \quad & u_j^S + u_i^B \geq v_{ij}, \forall i, j \end{aligned}$$

By strong duality, the minimum value should be equal to $V(N)$.

Hence, $\sum_{j \in T} u_j^S + \sum_{i \in T} u_i^B \geq V(T)$ for a subset $T \subseteq N$. That is, the core is non-empty.

Corollary 5.2

For an assignment game, outcome is in the core if and only if the outcome is CE outcome.

5.6 Constrained Demand Theory

5.6.1 Substitutes Valuation

There are buyers $i \in N$ and goods $j \in J$ with quantities $S \in \mathbb{Z}^J$ sold by a seller.

A buyer's utility is $v(x) - p \cdot x$, where $v(0) = 0$, $p \in \mathbb{R}^J$, and $x \in \{0, 1\}^J$. The buyer's demand is represented by $D(p) \operatorname{argmax}_x \{v(x) - p \cdot x\}$.

The competitive equilibrium $\left(p^*, (x^{*i})_{i \in N}\right)$ here are

1. $x^{*i} \in D^i(p^*)$ for every $i \in N$ and
2. $\sum_i x^{*i} \leq S_i$, where the equality holds for $p_i > 0$.

Definition 5.23 (Substitutes Valuation)

A valuation v_i is a **substitutes valuation** if $\forall p : p' = p + \lambda e^j$ $(\lambda > 0)$, where $D^i(p) = \{x\}$ and $D^i(p') = \{x'\}$, we have that $x'_k \geq x_k$ for all $k \neq j$. (The increase of product j's price increases other product's demand).

Theorem 5.16 (Substitutes Valuation ⇒ Competitive Equilibrium Exists)

If agents have substitutes valuations, then a competitive equilibrium exists.

Theorem 5.17

If there exists an agent without substitutes valuation, then we can construct unit-demand preferences for other agents such that no competitive equilibrium exists.

5.6.2 Income Effect

There are buyers $i \in N$ and goods $j \in J$. The endowments (money and goods) of agents are denoted by $w = (w_0, w_I)$.

Outcome: The indivisible (bought) goods is represented by $x_I \in \{0,1\}^J$ and the (left) divisible money is represented by $x_0 \in (\underline{m}, \infty)$.

$$w_0 = x_0 + p_I \cdot x_I$$

must hold, where p_I is the vector of prices of goods.

Utility Function: An agent's utility function is defined by $u^i : (\underline{m}, \infty) \times \{0,1\}^J \to (-\infty, +\infty)$ with assumptions of strictly increasing in x_0, $\lim_{x_0 \to \underline{m}} u^i(x_0, x_I) = -\infty$, and $\lim_{x_0 \to \infty} u^i(x_0, x_I) = +\infty$.

Example 5.3 Examples of feasible utility functions:

1. $u^i(x) = v(x) - p \cdot x$ with $\underline{m} = -\infty$;
2. $u^i(x_0, x_I) = \log(x_0) - \log(-V^i_Q(x_I))$ with $V^i_Q : \{0,1\}^J \to (-\infty, 0)$.

Demand:

- $D^i_{\text{Marshallian}}(p, w) = \{x^* : x^* \in \arg\max_x u^i(x) \text{ s.t. } p \cdot x \le p \cdot w\}$
- $D^i_{\text{Hicksian}}(p, u) = \{x^* : x^* \in \arg\min_x p \cdot x \text{ s.t. } u^i(x) \ge u\}$ which is the dual of $D^i_{\text{Marshallian}}$.

Definition 5.24 (Competitive Equilibrium)

Given $(w^i)_{i \in I}$ s.t. $\sum_{i \in N} w^i_I = y_I$. A **competitive equilibrium** is a price vector $p^*_I \in \mathbb{R}^J$ and $x^{i*}_I \in D_{\text{Marshallian}}(p^*_I, w^i)$ for each $i \in N$ such that $\sum_{i \in N} x^{i*}_I = y_I$.

Based on the idea of duality, we can analyze problem based on the dual demand, Hicksian demand.

Definition 5.25 (Hicksian Valuation)

Hicksian valuation is defined by -1 times "the money that can lead to the utility u with goods x_I":

$$V^i_{\text{Hicksian}}(x_I, u) = -(u^i(\cdot, x_I))^{-1}(u)$$

Proposition 5.2 (Using Hicksian Valuation to Represent Hicksian Demand)

$D^i_{\text{Hicksian}}(p_I, u) = \arg\max_{x_I} \{v^i_{\text{Hicksian}}(x_I, u) - p_I \cdot x_I\}$

Proof 5.10

$D^i_{\text{Hicksian}}(p_I, u) = \arg\min_{x_I} \{(u^i(\cdot, x_I))^{-1}(u) + p_I \cdot x_I\} = \arg\max_{x_I} \{V^i_{\text{Hicksian}}(x_I, u) - p_I \cdot x_I\}$

Definition 5.26 (Hicksian Economy)

Hicksian economy: for a profile $(u^i)_{i\in N}$ is a transferable utility (TU) economy in which each agent's "valuation" is a Hicksian valuation V^i_{Hicksian}. ♣

Hicksian Economy works in finding Competitive Equilibrium

Theorem 5.18 (Equilibrium Existence Duality(EED))

Competitive Equilibrium exists for all feasible endowment profiles if and only if Competitive Equilibrium exists in the Hicksian economies for all profiles of utility levels. ♡

Marshallian	Hicksian
Housing Market	Assignment Game
Utility is not Quasi-linear	Utility is Quasi-linear
Unit Demand	Unit Demand
Existence in Housing Market	Existence in Assignment Game
$\times$	Lattice structure and Convexity of structure of CE prices
Net-substitutes	$\Rightarrow$ Substitutes

Like the Theorem 5.16, we want the Hicksian valuations be "substitutes".

Definition 5.27 (Net-Substitutes)

A agent's utility u^i is net-substitutes if $\forall u$, $D^i_H(p;u) = \{x\}$ and $D^i_H(p'_j, p_{-j}; u) = \{x'\}$, $p'_j > p_j \Rightarrow x'_k \geq x_k$ for all $k \neq j$. ♣

Theorem 5.19

Net-Substitutes Valuation $\Rightarrow$ competitive equilibrium exists. ♡

Proof 5.11

Net-substitutes $\Rightarrow$ substitutes holds in Hicksian economy. Hence, CE exists. By 5.18, CE exists in original economy.

Definition 5.28 (Gross-Substitutes)

A agent's utility u^i is gross-substitutes if $\forall w$, $D^i_M(p;w) = \{x\}$ and $D^i_M(p'_j, p_{-j}; w) = \{x'\}$, $p'_j > p_j \Rightarrow x'_k \geq x_k$ for all $k \neq j$. ♣

Example 5.4 In quasi housing market, we consider an example, of holding a house which price increases, the demand of another bad house doesn't change under Hicksian demand, which makes net-substitutes hold. But, the Marshallian demand decreases, which makes gross-substitutes don't hold.

Example 5.5 Net, but not gross:

Suppose there is a firm f thinking about workers s_1, s_2. f values worker at \$5 each, and the hiring budget is \$6;

- $p_1 = 2, p_2 = 4$;
- $p_1 = 3, p_2 = 4$

Obviously, the gross-substitutes (Marshallian Demand) leads to hiring both under $p_1 = 2, p_2 = 4$ and only hiring s_1 under $p_1 = 3, p_2 = 4$.

Let's consider the net-substitutes (Hicksian Demand): As the utility given under $p_1 = 2, p_2 = 4$ is \$10. We can find hiring two workers is still the optimal strategy.

Example 5.6 Net, but no auction:

Suppose there are two identical firms f_1, f_2 and workers s_1, s_1', s_2. The value of workers is \$5 each, but a firm want at most one of s_1, s_1' and has hiring budget \$6. A worker has reservation wage of \$1.

Equilibrium: \$1 for worker s_1, s_1' and \$5 for s_2; One firm hires one of s_1, s_1' and the other hires s_2.

5.7 Object Allocation

Exchange: $i \in N$ agent; Agents have strict preference $\succ_i$ over objects. (We use $\succ$ denote $\{\succ_i\}_{i \in N}$).

Two settings:

1. Exchange: an agent shows up with exactly one object.
2. Allocation: One planner owns N objects; agents have $\emptyset$.

A **mechanism** $\Phi(\succ)$ gives a outcome μ. We want the final outcome μ be

1. Individual Rationality (IR): for all $i \in N$, $\mu_i \succeq i$ (Exchange) and $\mu_i \succeq \emptyset$ (Allocation).
2. Pareto Efficient (PE): $\nexists \mu'$ such that $\mu_i' \succeq \mu_i$ for all $i \in N$, strict for at least one.
3. Strategy-Proof (SP): Φ induces a game. We want that, in this game, truth-telling is a weakly dominant strategy for all agent $i \in N$.

5.7.1 Allocation

(Random) Serial Dictatorship: Randomly order the agents, ask one by one, and allocate a remaining object. $\Rightarrow$ it satisfies IR, PE, SP, but unfair(?).

5.7.2 Exchange

Definition 5.29 (Core)

The **core** is the set of all allocations μ such that there is no $S \subseteq N$ and μ' for which:

- for $i \in S$, $\mu'_i = j$ for some $j \in S$
- $\mu'_i \succeq \mu_i$ for all $i \in S$, at least one strict.

Core: IR+PE.

Theorem 5.20 (Core is a Singleton)

There is a unique element in the core.

Proof 5.12

Run the algorithm: Top Trading Cycles (TTC).

Definition 5.30 (Top Trading Cycles)

Agent = node.

1. Step 1: every agent point at her favorite object/agent.

   (1A): Find cycles.

   (1B): Allocate object to agent who is pointing at it in cycle.

   (1C): Remove the cycle.

2. Step 2: every (remaining) agent point at her favorite object/agent.

   (2A): Find cycles.

   (2B): Allocate object to agent who is pointing at it in cycle.

   (2C): Remove the cycle.

3. Repeat $\cdots$

Proposition 5.3

TTC produces an allocation that satisfies IR, PE, SP.

Theorem 5.21 (TTC $\Leftrightarrow$ IR, PE, SP (Ma, 1999))

There is at most 1 *IR, PE, SP mechanism (TTC).*

Proof 5.13

Definition 5.31

The **size** of a preference profile $\succ$ is the total number of objects agents find acceptable in $\succ$:

$$S(\succ) = \sum_{i \in N} \#\text{acceptable objects in } \succ_i$$

Consider two Φ *and* Ψ *that disagree for some* $\succ$*, the* $\succ$ *is defined to be bad.*

We define the minimal bad profile as a bad profile of minimal size. Consider the two outcomes given by these mechanisms:

$\Phi(\succ)$	same	$A(\Phi)$
$\Psi(\succ)$	same	$A(\Psi)$

the sum of different parts are $A \triangleq A(\Phi) + A(\Psi)$.

Lemma 5.5

If Φ *and* Ψ *are SP, and* $\succ$ *is a minimal bad profile, then each agent in* A *has exactly two acceptable objects.*

Proof 5.14

Suppose there exists $i \in A$ *such that she has* > 2 *acceptable objects.*

Without losing generality, we consider $\Phi_i(\succ) \succ_i \Psi_i(\succ)$.

Remove all objects from his preference list except $\Phi_i(\succ)$ *and endowment of* i *(call it* $\{i\}$*). The new preference profile is denoted by* $\succ_i'$.

Since Φ *is SP,* $\Phi_i(\succ') = \Phi_i(\succ)$*; since* Ψ *is SP,* $\Psi_i(\succ') \prec_i \Phi_i(\succ)$.

So, we have $\succ'$ *is a bad profile and* $S(\succ') < S(\succ)$*, a contradiction.*

Chapter 6 Auction

Based on

- Klemperer, P. (1998). Auctions with almost common values: TheWallet Game'and its applications. *European Economic Review*, 42(3-5), 757-769.

6.1 Examples

6.1.1 Auctions with Common-value

(1). Financial assets;

(2). Oilfields;

(3). A takeover target has a common value if the bidders are financial acquirers (e.g. LBO firms) who will follow similar management strategies if successful;

(4). The Personal Communications Spectrum (PCS) licenses sold by the U.S. Government in the 1995 "Airwaves Auction".

6.2 Optimal Auctions

Consider the Optimal Auctions in an Independent Private Values Setting

1. Bidders are risk-neutral;
2. Bidders have private valuations;
3. each bidder i's valuation independently drawn from a strictly increasing c.d.f. $F_i(v)$ (with p.d.f. $f_i(v)$) that is continuous and bounded below;
4. Seller knows each F_i and have no value for the object.

Goal: Find the **optimal auction** that maximizes the seller's expected revenue subject to individual rationality (IR) and Bayesian incentive compatibility for the buyers.

Definition 6.1 (Virtual Valuation)

Bidder i's **virtual valuation** is $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f(v_i)}$ ♣

Definition 6.2 (Bidder-Specific Reserve Price)

Bidder i's bidder-specific reserve price r_i^* is the value for which $\phi_i(r_i^*) = 0$. ♣

Theorem 6.1 (Myerson (1981))

The optimal (single-good) auction in terms of a direct mechanism: The good is sold to the agent $i = \arg\max_i \phi_i(\hat{v}_i)$*, as long as* $v_i \geq r_i^*$*. If the good is sold, the winning agent* i *is charged the smallest valuation that he could have declared while still remaining the winner:*

$$\inf\{v_i^* : \phi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \phi_i(v_i^*) \geq \phi_j(\hat{v}_j)\}$$

♡