Analysis and Something Else

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All models are wrong, but some are useful.

Contents

Chapter 1 Logic

1.1 Main Methods of Proof (@ Lec 01 of ECON 204)

1.1.1 Proof by Induction

1.1.2 Proof by Deduction

1.1.3 Proof by Contradiction

1.1.4 Proof by Contraposition

- $\neg P$ ("not P") means "P is false".
- $P \wedge Q$ ("P and Q") means "P is true and Q is true."
- $P \vee Q$ ("P or Q") means "P is true or Q is true (or possibly both)."
- $\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.
- $P \Rightarrow Q$ ("P implies Q") means "whenever P is satisfied, Q is also satisfied."

Statement: Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Definition 1.1 (Contrapositive)

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. ♣

Theorem 1.1 (Prove Contrapositive Insead)

$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true. ♡

Chapter 2 Analysis Basis

2.1 Real Number $\mathbb{R}$ (@ Lec 02 of ECON 204)

$\mathbb{R}$ is a field with the usual operations $+, \cdot$, additive identity 0, and multiplicative identity 1.

2.1.1 Order Axiom

Proposition 2.1 (Order Axiom)

There is a complete ordering $\leq$, i.e. $\leq$ is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that $\forall \alpha, \beta \in \mathbb{R}$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.

The order is compatible with $+$ and $\cdot$, i.e. $\forall \alpha, \beta, \gamma \in \mathbb{R}$

1. $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.
2. $\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$.

2.1.2 Completeness Axiom

Proposition 2.2 (Completeness Axiom)

Suppose $L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H$ satisfy $\forall l \in L, h \in H, l \leq h$. Then,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall l \in L, h \in H,\ l \leq \alpha \leq h$$

$$\begin{array}{ccc} & \alpha & \\ L & \downarrow & H \\ -\,-\,-\,-\,) & \cdot & (\,-\,-\,-\,- \end{array}$$

Figure 2.1: Completeness Axiom

Claim 2.1

The Completeness Axiom differentiates $\mathbb{R}$ from $\mathbb{Q}$:

$\mathbb{Q}$ satisfies all the axioms for $\mathbb{R}$ except the Completeness Axiom.

2.1.3 Supremum $\sup \mathbb{X}$, Infimum $\inf \mathbb{X}$ for $\mathbb{X} \subseteq \mathbb{R}$

Definition 2.1 (Supremum and Infimum)

(1). Suppose $\mathbb{X}$ is bounded above. The **supremum** of $\mathbb{X}$, written $\sup \mathbb{X}$, is the least upper bound for $\mathbb{X}$, i.e. $\sup \mathbb{X}$ satisfies

(a). $\sup \mathbb{X} \geq x, \forall x \in \mathbb{X}$ ($\sup \mathbb{X}$ is an upper bound).

(b). $\forall y < \sup \mathbb{X}$, $\exists x \in \mathbb{X}$ s.t. $x > y$ (there is no smaller upper bound).

(2). Suppose $\mathbb{X}$ is bounded below. The **infimum** of $\mathbb{X}$, written $\inf \mathbb{X}$, is the greatest lower bound for $\mathbb{X}$, i.e. $\inf \mathbb{X}$ satisfies

(a). $\inf \mathbb{X} \leq x, \forall x \in \mathbb{X}$ ($\inf \mathbb{X}$ is a lower bound).

(b). $\forall y > \inf \mathbb{X}$, $\exists x \in \mathbb{X}$ s.t. $x < y$ (there is no greater lower bound).

(3). If $\mathbb{X}$ is not bounded above, write $\sup \mathbb{X} = \infty$. If $\mathbb{X}$ is not bounded below, write $\inf \mathbb{X} = -\infty$. By convention, $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

♣

Proposition 2.3

If $\inf A = x^* \in A$ ($\sup A = x^* \in A$), *then* $x^* = \min A$ ($x^* = \max A$).

♠

2.1.4 The Supremum Property

Proposition 2.4 (The Supremum Property)

Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.

♠

Theorem 2.1

The Supremum Property (Prop 2.4) and the Completeness Axiom (Prop 2.2) are equivalent.

♡

2.1.5 Archimedean Property

Theorem 2.2 (Archimedean Property)

$\forall x \in \mathbb{R}, y \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$ *s.t.* $ny > x$.

♡

2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)

2.2.1 Metric Space $(\mathbb{X}, d)$ and Metric $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$

Definition 2.2 (Metric Space)

A **metric space** is a pair $(\mathbb{X}, d)$, where $\mathbb{X}$ is a set and $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ a function satisfying

1. Non-negative: $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in \mathbb{X}$.
2. Symmetric: $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$.
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathbb{X}$.

A function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ satisfying 1-3 is called a **metric** on $\mathbb{X}$. ♣

A metric gives a notion of distance between elements of $\mathbb{X}$.

2.2.2 Norm $\|\cdot\|$ and Normed Vector Space $(V, \|\cdot\|)$

Definition 2.3 (Norm)

Let V be a vector space over $\mathbb{R}$. A **norm** on V is a function $\|\cdot\| : V \to \mathbb{R}_+$ satisfying

1. Non-negative: $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in V$.
2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$.
3. $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{R}, x \in V$.

♣

A norm gives a notion of length of a vector in V.

Definition 2.4 (Normed Vector Space)

A **normed vector space** is a vector space over $\mathbb{R}$ equipped with a norm, $(V, \|\cdot\|)$. ♣

Example 2.1 Normed Vector Space

- $\mathbf{E}^n$: n-dimensional Euclidean space.

$$V = \mathbb{R}^n, \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

- $V = \mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i|$ (the "taxi cab" norm or L^1 norm)
- $V = \mathbb{R}^n, \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)
- $C([0,1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$
- $C([0,1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$
- $C([0,1]), \|f\|_1 = \int_0^1 |f(t)| dt$

where $C([0,1])$ is the space of all continuous real-valued functions on $[0,1]$.

2.2.3 Theorem: metric can be defined by norm

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 2.3

Let $(V, \|\cdot\|)$ be a normed vector space. Let $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ be defined by $d(v, w) = \|v - w\|$. Then (V, d) is a metric space.

2.2.4 Cauchy-Schwarz Inequality

Theorem 2.4 (Cauchy-Schwarz Inequality)

If $v, w \in \mathbb{R}^n$, then

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \leq \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

$$\|v \cdot w\|^2 \leq \|v\|^2\|w\|^2$$

$$\|v \cdot w\| \leq \|v\|\|w\|$$

2.2.5 Lipschitz-equivalent Norm

Definition 2.5 (Lipschitz-equivalent)

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the same vector space V are said to be **Lipschitz-equivalent** (or **equivalent**) if $\exists m, M > 0$ s.t. $\forall x \in V$,

$$m\|x\| \leq \|x\|^* \leq M\|x\|$$

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V$, $x \neq 0$,

$$m \leq \frac{\|x\|^*}{\|x\|} \leq M$$

Theorem 2.5

All norms on $\mathbb{R}^n$ are equivalent.

However, infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0, 1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \in [0, \frac{1}{n}] \\ 0, & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then $\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} \to 0$, which means there is no lower bound $m > 0$.

2.2.6 Ball, Radius, Diameter, and Distance

In a metric space (X, d), define

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y,x) < \varepsilon\} \\ &= \text{open ball with center } x \text{ and radius } \varepsilon \\ B_\varepsilon[x] &= \{y \in X : d(y,x) \leq \varepsilon\} \\ &= \text{closed ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

We can use the metric d to define a generalization of "radius". In a metric space (X, d), define the *diameter* of a subset $S \subseteq X$ by

$$\operatorname{diam}(S) = \sup \left\{d\left(s, s'\right) : s, s' \in S\right\}$$

Similarly, we can define the *distance from a point to a set*, and *distance between sets*, as follows:

$$\begin{aligned} d(A,x) &= \inf_{a \in A} d(a,x) \\ d(A,B) &= \inf_{a \in A} d(B,a) \\ &= \inf\{d(a,b) : a \in A, b \in B\} \end{aligned}$$

2.3 Set Theory

2.3.1 Well Defined Set

Definition 2.6

*A set S is **well defined** if an object a is either $a \in S$ or $a \notin S$.* ♣

2.3.2 Numerically Equivalent (@ Lec 01 of ECON 204)

Definition 2.7

Two sets A, B are **numerically equivalent** (or have the same cardinality) if there is a bijection $f : A \to B$, that is, 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$). ♣

2.3.3 Finite, Countable Set (@ Lec 01 of ECON 204)

Definition 2.8 (Finite Set)

A set is either **finite** or **infinite**. A set is **finite** if it is numerically equivalent to $\{1, ..., n\}$ for some n. A set that is not finite is infinite. ♣

We give a more precise definition to classify infinite set:

Definition 2.9 (Countable Set)
An infinite set is **countable** if it is numerically equivalent to $\mathbb{N}$. An infinite set that is not countable is called **uncountable**.

Theorem 2.6 (Countable $\mathbb{Q}$)
The set of rational numbers $\mathbb{Q}$ is countable.

2.3.4 Power Set (@ Lec 02 of ECON 204)

Definition 2.10 (Power Set: the set of all subsets)
*For any set A, we denote by $\mathcal{P}(A)$ the collection of all subsets of A. $\mathcal{P}(A)$ is the **power set** of A.*

We may also use the notation 2^A (in Berkeley ECON 204).

2.3.5 Theorem (Cantor): The power set of $\mathbb{N}$ is uncountable (@ Lec 02 of ECON 204)

Theorem 2.7 (Cantor)
$\mathcal{P}(\mathbb{N})$ (or denoted by $2^{\mathbb{N}}$), the set of all subsets of $\mathbb{N}$, is uncountable.

2.3.6 Cardinalities of Sets (@ Lec 02 of ECON 204)

Definition 2.11 (Cardinality)
If A is a set, $|A|$ = cardinality of A = # of elements

$n \in \mathbb{N}, |\{1, \dots n\}| = n$; $|\emptyset| = 0(\emptyset =$ empty set $)$.

Proposition 2.5 (Facts about cardinality)

1. *If A is numerically equivalent to $\{1, ..., n\}$ for some $n \in \mathbb{N}$, then $|A| = n$.*
2. *A and B are numerically equivalent if and only if $|A| = |B|$.*
3. *If $|A| = n$ (finite) and A is a proper subset of B (that is, $A \subset B$ and $A \neq B$) then $|A| < |B|$.*
4. *If A is countable and B is uncountable, then $n < |A| < |B|, \forall n \in \mathbb{N}$.*
5. *If $A \subseteq B$, then $|A| \leq |B|$. (if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable.)*
6. *If there is an injection $\sigma : A \to B$, we can write $|A| \leq |B|$;*
7. *If there is a surjection $\sigma : A \to B$, we can write $|A| \geq |B|$;*
8. *If there is a bijection $\sigma : A \to B$, we can write $|A| = |B|$.*

2.3.7 Pigeonhole Principle: $|A| > |B| \Rightarrow$ no injective function $\sigma : A \to B$

Theorem 2.8 (Pigeonhole Principle)

If A and B are sets and $|A| > |B|$, then there is no injective function $\sigma : A \to B$. ♡

2.3.8 B^A: Sets of Function

If A, B are sets, then $B^A = \{\sigma : A \to B | \sigma \text{ a function}\}$.

Example 2.2 $n \in \mathbb{Z}$, we define a function $f : B^{\{1,\ldots,n\}} \to B^n (= B \times B \times B \times \cdots \times B)$ by the equation $f(\sigma) = \{\sigma(1), ..., \sigma(n)\}$, *where* $\sigma : \{1, \ldots, n\} \to B$. The f is a *bijection*.

Proof 2.1

1. *Injective:*

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), ..., \sigma_1(n)\} = \{\sigma_2(1), ..., \sigma_2(n)\}$$

$$\text{Since } \sigma : \{1, \ldots, n\} \to B, \text{ it is sufficient to prove } \sigma_1 = \sigma_2.$$

2. *Surjective:*

$$\forall \{b_1, ..., b_n\}, \text{ we have } \sigma^*(x) = b_x, x = 1, ..., n. \text{ s.t. } f(\sigma^*) = \{b_1, ..., b_n\}$$

Example 2.3

$$C(\mathbb{R}, \mathbb{R}) = \{\text{continuous functions } \sigma : \mathbb{R} \to \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

2.3.9 Bounded Set (@ Lec 03 of ECON 204)

Definition 2.12 (Bounded Set)

In a metric space (X, d), a subset $S \subseteq X$ is **bounded** if $\exists x \in X, \beta \in \mathbb{R}$ such that $\forall s \in S$, $d(s, x) \leq \beta$. ♣

2.3.10 Open, Closed Set (@ Lec 04 of ECON 204)

Definition 2.13 (Open Sets)

*Let (X, d) be a metric space. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is **open** if*

$\forall x \in \mathbb{X}$ we can draw a ball around x that is contained in $\mathbb{X}$.

i.e. $\forall x \in \mathbb{X}, \exists \varepsilon > 0$ *s.t.* $B_\varepsilon(x) = \{y : d(y, x) < \varepsilon\} \subseteq \mathbb{X}$ ♣

Definition 2.14 (Closed Sets)

*$\mathbb{X}$ is **closed** if $\mathbb{X}^c$ is open.* ♣

Theorem 2.9 (Equivalent definition: Closed Sets)

Equivalent: if A in a metric space (X, d) contains all limit points of all sequences in A, A is closed.

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

♡

Example 2.4 (Closed and Open Sets on $\mathbb{E}_1$ i.e., $\mathbb{R}$ with the usual Euclidean metric)

1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open
2) $\mathbb{R}$ is both open and closed
3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open
4) $[1, \infty)$ is closed because its complement open
5) $(1, 2]$ is neither open nor closed

Example 2.5 (Closed and Open Sets on other metric space) In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon)$.

Theorem 2.10 (Empty Set and Full Set are both open and closed)

In any metric space (X, d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed.

♡

Theorem 2.11 (Union of open sets is open, Intersection of finite open sets is open)

In any metric space (X, d),

1. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
2. *The intersection of a finite collection of open sets is open.*

♡

2.3.11 Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)

Given a set $S \subseteq X$, the **point** of X can be classified into three types relative to S:

- **Interior (points)**, denoted $int(S)$: $\vec{x} \in S$ for which there exists some $B(\vec{x}, r) \subseteq S$, is the largest open set contained in S (the union of all open sets contained in S).
- **Exterior (points)**, denoted $ext(S)$: $\vec{x} \notin S$ for which there exists some $B(\vec{x}, r)$ containing no points of S, is the largest open set contained in $X \backslash S$.
- **Boundary (points)** denoted $\partial(S)$ or $bd(S)$: all other points (for which any $B(\vec{x}, r)$ will contain some points of S and some points outside S).
- **Closure of** S, denoted $\bar{S}$ or $cl(S) = int(S) \cup bd(S)$, is the smallest closed set containing S (the intersection of all closed sets containing S).
- Moreover, boundary satisfies $\partial(S) = \overline{(X \backslash S)} \cap \bar{S}$.

(1) A set S is **open** if $S = int(S)$ - i.e., if S does not contain any of its boundary points.

(2) A set S is **closed** if $S = \bar{S} = int(S) \cup bd(S)$ - i.e., if S contains all of its boundary points.

2.3.12 Compact Set

Definition 2.15 (Compact Set)

$\mathcal{L} \subseteq \mathbb{R}^n$ is compact of it is closed and bounded.

Example 2.6 Compact Set $[1, 2] = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\}$; $\left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 4\right\}$

2.3.13 Sublevel Set

Definition 2.16 (Sublevel Set)

The sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ (for some level $c \in \mathbb{R}$) is the set

$$\overline{L_c} = \{x \in \mathbb{R}^n : f(x) \leq c\}$$

2.3.14 Set Operations

Definition 2.17

A binary operation on a set A is a function $* : A \times A \to A$.

The operation is *associative* if $a * (b * c) = (a * b) * c, \forall a, b, c \in A$.

The operation is *commutative* if $a * b = b * a, \forall a, b \in A$.

Example 2.7

$+, \circ$ are both *associative* and *commutative* operations on $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$; $-$ is a neither *associative* nor *commutative* operation on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not $\mathbb{N}$.

Definition 2.18

A subset $H \subset S$ is closed under $*$ if $a * b \in H$ for all $a, b \in H$.

Definition 2.19

$*$ has identity element $e \in S$ if $a * e = e * a = a$ for all $s \in S$.

2.4 Sequences

Sequences $\{x_k\}_{k=1}, \ldots$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 2.20 (Subsequence)

Suppose $\{x_n\}$ is a sequence and $n_1 < n_2 < \cdots$, then $\{x_{n_k}\}$ is called a **subsequence**.

2.4.1 Convergence of Sequences (@ Lec 03 of ECON 204)

Definition 2.21 (Convergence: note $x_k \to x, \lim_{k\to\infty} x_k = x$)

Let (X, d) be a metric space. A sequence $\{x_k\}$ converges to x (written $x_k \to x$ or $\lim_{k\to\infty} x_k = x$) if

$\forall \varepsilon > 0, \exists N_\varepsilon$ *s.t.* $d(x_k, x) < \varepsilon, \forall k \geqslant N_\varepsilon$ ♣

Definition 2.22 (Limit point)

x is a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to x. ♣

Theorem 2.12 (Uniqueness of Limits)

In a metric space (X, d), if $x_k \to x$ and $x_k \to x'$, then $x = x'$. ♡

2.4.2 Cluster Point (@ Lec 03 of ECON 204)

Definition 2.23 (Cluster Point)

An element c is a **cluster point** of a sequence $\{x_n\}$ in a metric space (X, d) if $\forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N}, \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$ ♣

Example 2.8 $x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ even} \\ \frac{1}{n}, & \text{if } n \text{ odd} \end{cases}$ has cluster points $\{0, 1\}$.

Theorem 2.13 (Cluster Point $\Leftrightarrow$ exists subsequence converges to it)

Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X. Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = c$. ♡

2.4.3 Sequences in $\mathbb{R}$ and $\mathbb{R}^n$ (@ Lec 03 of ECON 204)

Proposition 2.6

*If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it **converges**.* ♠

Theorem 2.14

Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ ($\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$). In particular, the limit exists. ♡

Proposition 2.7

The largest(smallest) limit point of $\{x_k\}$ is $\lim_{k\to\infty} \sup x_k$ ($\lim_{k\to\infty} \inf x_k$) ♠

Proposition 2.8

$\{x_k\}$ *converges* $\iff -\infty < \lim_{k\to\infty} \inf x_k = \lim_{k\to\infty} \sup x_k < \infty$

2.4.4 Cauchy Sequence

Definition 2.24 (Cauchy Sequence)

$\{x_k\}$ *is Cauchy if given* $\varepsilon > 0, \quad \exists N_\varepsilon$ *s.t.*

$$\|x_k - x_m\| < \varepsilon,\ \forall k, m \geqslant N_\varepsilon.$$

Note:

$$\{x_k\} \text{ converges } \iff \{x_k\} \text{ is Cauchy}$$

Chapter 3 Functions

3.1 Definitions of Function

Definition 3.1 (Function)

Function is a rule $\sigma : A \to B$ that assigns an element B to *every* element of A. $\forall a \in A, \sigma(a) \in B$. ♣

3.1.1 Image, Preimage, Fiber

Definition 3.2

1. A is the domain of σ, B is the range of σ.
2. We call $\sigma(a) =$ *value of* σ *at* a as the image of a.
3. A set $C \subset B$, we call $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$ as the *preimage* of C.
4. An element $b \in B$, we call $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$ as the *fiber* of b. ♣

3.1.2 Composition of functions

Definition 3.3 (Function Composition)

The function composition $\circ$ is an operation that takes two functions $\sigma : A \to B$ and $\tau : B \to C$, , and produces a function $\tau \circ \sigma : A \to C$ that fulfills $\forall a \in A,\ (\tau \circ \sigma)(a) = \tau(\sigma(a))$. ♣

3.1.3 Function Composition is Associative

Proposition 3.1 (Associativity of Functions)

Suppose $\sigma : A \to B, \tau : B \to C, \rho : C \to D$ *are functions and* $\circ$ *is the function composition, then* $\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$. ♠

3.2 Injection, Surjection, Bijection

3.2.1 Definitions: Injective, surjective, bijective

A function $\sigma : A \to B$ is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

3.2.2 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

Lemma 3.1 (Lemma 1.1.7)

Suppose $\sigma : A \to B, \tau : B \to C$ are functions,

If σ, τ are injective, then $\tau \circ \sigma$ is injective.

If σ, τ are surjective, then $\tau \circ \sigma$ is surjective.

If σ, τ are bijective, then $\tau \circ \sigma$ is bijective.

3.2.3 Proposition 1.1.8: A function is bijection if there exist inverse

Proposition 3.2 (Proposition 1.1.8)

A function $\sigma : A \to B$ is a bijection if $\exists$ a function $\tau : B \to A$ such that

$$\sigma \circ \tau = id_B = \textit{identity on } B(id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$

Such τ is unique, called inverse of σ, $\tau = \sigma^{-1}$.

3.3 Function Continuity

3.3.1 Continuous Function in $\mathbb{R}$ with Euclidean norm

Definition 3.4 (Continuity at Point)

A real-valued function f is **continuous** at x if

"For every $\{x_k\}$ converging to x satisfies that $\lim_{k\to\infty} f(x_k) = f(x)$".

Equivalent definition:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|y - x\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

Continuity at x_0 requires:

1. $f(x_0)$ is defined; and
2. either

   - x_0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{x\}$; or

   - $\lim_{x\to x_0} f(x)$ exists and equals $f(x_0)$

Definition 3.5 (Continuous Function)

A real-valued function f is continuous if it is continuous at all points in its domain. ♣

3.3.2 Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)

Definition 3.6 (Continuity in Metric Spaces)

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is **continuous** at a point $x_0 \in X$ if $\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0$ s.t. $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$. ♣

3.3.3 Theorem: Continuous ⇔ Preimage of open set is open (@ Lec 04 of ECON 204)

Theorem 3.1 (Continuous ⇔ Preimage of open set is open)

*Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then f is **continuous** if and only if*

$$f^{-1}(A) = \{x \in X : f(x) \in A\} \text{ is open in } X \ \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$

♡

3.3.4 Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)

Theorem 3.2 (Continuity is preserved in composition)

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous. ♡

Proof 3.1

Proved by previous theorem 3.1.

3.3.5 Uniform Continuity (@ Lec 04 of ECON 204)

Definition 3.7 (Uniformly Continuous)

Suppose $f : (X, d) \to (Y, \rho)$. f is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

♣

Claim 3.1

***Uniformly Continuous** implies (is stronger than) **Continuous**.*

(f is continuous if $\forall x_0 \in X, \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0$ s.t. $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$) ♠

Given $\varepsilon > 0$, "uniformly continuous" requires $\delta(\varepsilon)$ that works for all $x_0 \in X$.

3.3.6 Intermediate Value Theorem (@ Lec 02 of ECON 204)

Theorem 3.3 (Intermediate Value Theorem)

Suppose $f : [a, b] \to \mathbb{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.

3.3.7 Coercive Function

Definition 3.8 (Coercive)

*A real-valued function $f : \mathbb{X} \to \mathbb{R}$ is coercive if for **every** $\{x_k\} \subset \mathbb{X}$ s.t. $\|x_k\| \to \infty$, $f(x_k) \to \infty$*

Example 3.1 Check coercive

1) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2$ - coercive

2) $x \in \mathbb{R}$, $f(x) = 1 - e^{-|x|}$ - not coercive

3) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (we need $f(x_k) \to \infty$ for all $\|x_k\| \to \infty$)

3.3.8 Extreme of Functions

Definition 3.9 (Extreme of Functions)

Let $\mathbb{X} \subseteq \mathbb{R}^n$, $f : \mathbb{X} \to \mathbb{R}$

$$\inf_{x \in \mathbb{X}} f(x) = \inf\{f(x) : x \in \mathbb{X}\}$$

If $\exists x^* \in \mathbb{X}$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \mathbb{X}} f(x)$

x^* is called a **minimizer** of f, written as $x^* \in \arg\min_{x \in \mathbb{X}} f(x)$. If x^* is uniqne, we write $x^* = \arg\min_{x \in \mathbb{X}} f(x)$

Similarly, supremum and maximum of f.

3.3.9 Weierstrass' Theorem(Extreme value Theorem)

Theorem 3.4 (Weierstrass' Theorem (Extreme value Theorem))

*If f is a **continuous** function on a **compact set**, $\mathbb{X} \subseteq \mathbb{R}^n$, then f attains its min and max on $\mathbb{X}$ i.e.,*

$$\exists x_1 \in \mathbb{X} \text{ s.t. } f(x_1) = \inf_{x \in \mathbb{X}} f(x)$$

$$\exists x_2 \in \mathbb{X} \text{ s.t. } f(x_2) = \sup_{x \in \mathbb{X}} f(x)$$

Proof 3.2

(for existence of min*;* max *is similar)*

Let $\{\sigma_k\} \subseteq \mathbb{X}$ *be s.t.*

$$\inf_{x \in \mathbb{X}} f(x) \leq f(\sigma_k) \leq \inf_{x \in \mathbb{X}} f(x) + \frac{1}{k}$$

Then $\lim_{k \to \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

$\mathcal{L}$ *is bounded* $\Rightarrow \{\sigma_k\}$ *has it least one limit point* x,

$\mathcal{L}$ *is closed* $\Rightarrow x_1 \in \mathbb{X}$

f *is continuous* $\Rightarrow f(x_1) = \lim_{k \to \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

Corollary 3.1 (Corollary to WT)

Let f *be continuous on closed set* $\mathbb{X}$ *(not necessarily bounded). If* f *is coercive on* $\mathbb{X}$ *it attains its* min *on* $\mathbb{X}$.

♡

Proof 3.3

Consider $\{\sigma_k\}$ *as in proof of* WT.

Since f *is closed,* $f(x) < \infty$, $\forall x \in \mathbb{X}$. *And* f *is coercive on* $\mathbb{X}$, *which means* $f(x) \to \infty$ *if* $\|x\| \to \infty$.

Hence, $\{\sigma_k\} \in \mathbb{X}$ *is bounded. Rest of proof same as proof of* WT.

Example 3.2 $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) *Does* f *achieve its* min *and* max *on* $\mathcal{L}_1 = \left\{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leqslant 6\right\}$*?*

   - $\mathcal{L}_1$ is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over $\mathbb{R}^3$?

   - $f \to \infty$ whenever $\|x\| \to \infty \Rightarrow f$ is coercive.

   - $\mathbb{R}^3$ is closed.

   $\Rightarrow f$ achieves its min. on $\mathbb{R}^3$ by corollary to WT.

   - max does not exist since $f \to \infty$ as $\|x\| \to \infty$.

3) Does f achieve its min and max over $\mathcal{L}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?

   - $\mathcal{L}_2$ is closed, but not bounded.

   - Since f is coercive, min achieved.

   - max does not exist since setting $x_1 = 0$ $x_2 = 3 - x_3$ and letting $x_3 \to \infty$ makes $f \to \infty$

Chapter 4 Big $\mathcal{O}$ and Small o Notation

4.1 Definition

Complexity:

Definition 4.1

A sequence $f(n)$ is $O(1)$ if $\lim_{n\to\infty} f(n) < \infty$. ♣

Definition 4.2

A sequence $f(n)$ is $O(g(n))$ if $\frac{f(n)}{g(n)}$ is $O(1)$. ♣

Definition 4.3

A sequence $f(n)$ is $o(1)$ if $\lim_{n\to\infty} \sup f(n) = 0$. ♣

Definition 4.4

A sequence $f(n)$ is $o(g(n))$ if $\lim_{n\to\infty} \sup \frac{f(n)}{g(n)} = 0$. ♣

Definition 4.5

A sequence $f(n)$ is asymptotic to $g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. (This is denoted by $f(n) \sim g(n)$ as $a \to \infty$) ♣

For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Example 4.1

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$
$$n^3 + n + 2 = \Theta(n^3)$$
$$n^3 + n + 2 = o(n^4)$$

4.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \to a$ if $\limsup_{x\to a} \frac{|f(x)|}{g(x)} < \infty$.

Example 4.2 $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$

Chapter 5 Lipschitz Continuous

5.1 Definition (@ Lec 04 of ECON 204)

Definition 5.1 (Lipschitz (Continuous) in Normed Vector Space)
Let X, Y be normed vector spaces, $\mathbb{E} \subseteq X$. A function $f : X \to Y$ is ***Lipschitz*** *on $\mathbb{E}$ satisfies*

$$\exists \gamma > 0, \|f(\mathbf{x}) - f(\mathbf{y})\|_Y \leq \gamma \|\mathbf{x} - \mathbf{y}\|_X, \forall \mathbf{x}, \mathbf{y} \in \mathbb{E}$$

*or we call γ-**Lipschitz continuous**;* ♣

If f is γ-Lipschitz continuous, then it is also $(\gamma + 1)$-Lipschitz continuous

The minimal such γ is called a Lipschitz constant of function f

Remark: Here $\|\cdot\|$ can be any given norm of the space $\mathbb{R}^n$ and $\mathbb{R}^m$, such as Euclidean norm, ℓ_1-norm, etc.

When not specified, we assume it is Euclidean norm.

5.2 Example

Example 1: $f(x) = 2x$ is 2-Lipschitz continuous;

Example 2: What about $f(\mathbf{x}) = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).

Example 3: What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞.

5.3 Contraction Mapping

1. If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.
2. If $\gamma < 1$, then f is called a contraction mapping

Example 1: $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.

Example 2: $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

Chapter 6 Fixed point theorem

1. Fixed point theorem: If f is a contraction mapping that maps $\mathbb{R}^n$ to itself, then the following two results hold:

1) There exists a unique fixed point x^* satisfying

$$\mathbf{x}^* = f\left(\mathbf{x}^*\right)$$

2) In addition, the iterated function sequence

$$\mathbf{x}, f(\mathbf{x}), f(f(\mathbf{x})), \cdots,$$

converges to this unique fixed point $\mathbf{x}^*$ (independent of the initial point x).

2. Remark: This is a special case of ”Banach fixed point theorem” (which applies to any complete metric space).