Economic Theory and Some Useful Math

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All models are wrong, but some are useful.

Contents

Chapter 1 Stochastic Dominance

Based on

- MIT 14.123 S15 Stochastic Dominance Lecture Notes
- Princeton ECO317 Economics of Uncertainty Fall Term 2007 Notes for lectures 4. Stochastic Dominance
- Jensen, M. K. (2018). Distributional comparative statics. *The Review of Economic Studies*, 85(1), 581-610.

1.1 General Definitions

Definition 1.1 (Jensen (2018), Defnition 1)

Let F and G be two distributions on the same measurable space. Let u be a function for which the following expression is well-defined,

$$\int u(x)dF \geq \int u(x)dG \tag{1.1}$$

Then:

- F **first-order stochastically dominates** G if 1.1 holds for any increasing function u.
- F is a **mean-preserving spread** of G if 1.1 holds for any convex function u.
- F is a **mean-preserving contraction** of G if 1.1 holds for any concave function u.
- F **second-order stochastically dominates** G if 1.1 holds for any concave and increasing function u.
- F **dominates** G **in the convex-increasing order** if 1.1 holds for any convex and increasing function u.

Note *F is a **mean-preserving contraction** of G $\Leftrightarrow$ G is a **mean-preserving spread** of F.*

Definition 1.2 (MPS and MPC)

We define the following notations of sets.

- MPS(f) is the set of all **mean-preserving spread** of f;
- MPC(f) is the set of all **mean-preserving contraction** of f;

1.2 First-order Stochastic Dominance

1.2.1 Two Equivalent Definitions

Definition 1.3 (First-order Stochastic Dominance)

For any lotteries F and G, F **first-order stochastically dominates** G if and only if the decision maker weakly prefers F to G under every weakly increasing utility function u, i.e.,

$$\int u(x)dF \geq \int u(x)dG$$

♣

Definition 1.4 (First-order Stochastic Dominance)

For any lotteries F and G, F **first-order stochastically dominates** G if and only if

$$F(x) \leq G(x), \forall x$$

♣

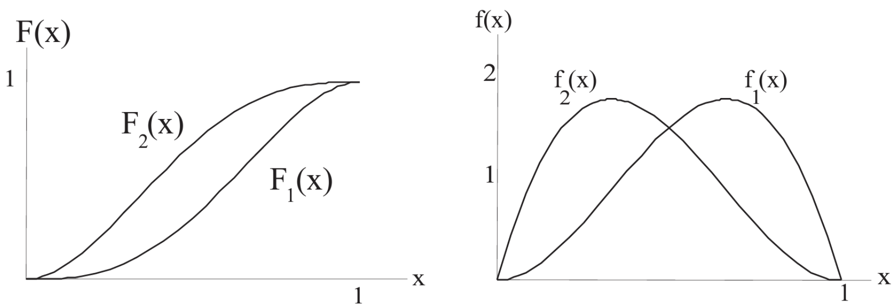

Figure 1.1: F_1 is FOSD over F_2: CDF and density comparison

1.3 Second-order Stochastic Dominance

1.3.1 Definition in terms of final goals

Definition 1.5 (Second-order Stochastic Dominance)

For any lotteries F and G, F **second-order stochastically dominates** G if and only if the decision maker weakly prefers F to G under every weakly **increasing concave** utility function u, i.e.,

$$\int u(x)dF \geq \int u(x)dG$$

♣

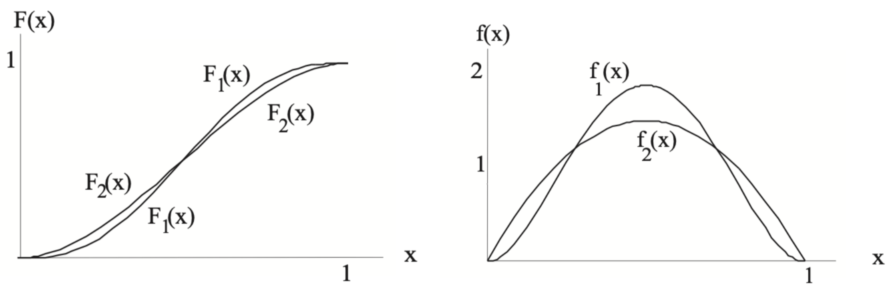

Figure 1.2: F_1 is SOSD over F_2: CDF and density comparison

1.3.2 Mean-Preserving Spread/Contraction

Definition 1.6 (Mean-Preserving Spread)

Let x_F and x_G be the random variables associated with lotteries F and G. Then G is a **mean-preserving spread** of F if and only if

$$x_G \overset{d}{=} x_F + \varepsilon$$

for some random variable ε such that $\mathbb{E}(\varepsilon \mid x_F) = 0, \forall x_F$.

The "$\overset{d}{=}$" means "is equal in distribution to" (that is, "has the same distribution as").

Note *Given G is a **mean-preserving spread** of F, G has larger variance than F.*

Example 1.1 $F(198) = \frac{1}{2}, F(202) = \frac{1}{2}$ and $G(100) = \frac{1}{100}$, $G(200) = \frac{98}{100}$, $G(300) = \frac{1}{100}$. Then

$$x_G \overset{d}{=} x_F + \varepsilon$$

where the distribution of ε can be solved by $\begin{cases} \frac{1}{100} & = \frac{1}{2}P(\varepsilon = 102) + \frac{1}{2}P(\varepsilon = 98) \\ \frac{98}{100} & = \frac{1}{2}P(\varepsilon = 2) + \frac{1}{2}P(\varepsilon = -2) \\ \frac{1}{100} & = \frac{1}{2}P(\varepsilon = -98) + \frac{1}{2}P(\varepsilon = -102) \end{cases}$

1.3.3 For Same Mean Distributions, Second-order Stochastic Dominance is equivalent to Mean-Preserving Spread

Theorem 1.1 (Second-order Stochastic Dominance Equivalence)

Given $\int x dF = \int x dG$ (same mean). The following are equivalent.

1. *F second-order stochastically dominates G: $\int u(x)dF \geq \int u(x)dG$ for every weakly increasing concave utility function u.*
2. *F is a mean-preserving contraction of G (G is a mean-preserving spread of F).*
3. *For every $t \geq 0$, $\int_a^t G(x)dx \geq \int_a^t F(x)dx$.*

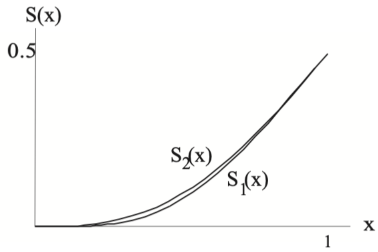

Figure 1.3: F_1 is SOSD over F_2, $S(t) : \int_a^t F_2(x)dx \geq \int_a^t F_1(x)dx$

Chapter 2 Market Design

Based on

- UC Berkeley MATH 272 23Fall, Alexander Teytelboym
- Jehle, G., Reny, P.: Advanced Microeconomic Theory . Pearson, 3rd ed. (2011). Ch. 6.
- Notes on Social Choice and Welfare, Alejandro Saporiti

2.1 Individual Choice to Preferences

Definition 2.1 (Unitility Function)

We can say a function $u : X \to \mathbb{R}$ represents $\succeq$ if $\forall x, y \in X$,

$$x \succeq y \Leftrightarrow u(x) \geq u(y)$$

Proposition 2.1

If $\exists$ a function $u : X \to \mathbb{R}$ represents $\succeq$, then $\succeq$ is rational (i.e., completeness and transitivity)

Note *The reverse may not true.*

Let $\mathcal{B} = 2^X$ (all subsets of X) and $B \in \mathcal{B}$ be the all potential alternatives that can be chosen.

The choice of an agent can be represented by $C(B) \subseteq B, \forall B \in \mathcal{B}$.

2.1.1 Weak Axiom of Revealed Preference (WARP)

Definition 2.2 (Weak Axiom of Revealed Preference)

Given a choice structure $(C, \mathcal{B})$ satisfies **WARP**. If $\exists B \in \mathcal{B}$ with $x, y \in B$, such that $x \in C(B)$. Then, $\forall B' \in \mathcal{B}$ with $x, y \in B'$, $y \in C(B') \Rightarrow x \in C(B')$.

Proposition 2.2 (Rational $\Rightarrow$ WARP)

Given $\succeq$ is rational, then $(C^*_{\succeq}, \mathcal{B})$ satisfies WARP.

($C^*_{\succeq}$ is the choice rule that picks the maximal alternatives by $\succeq$)

2.2 Social Choice

Notations:

1. We consider <u>finite</u> set of alternatives X and <u>finite</u> set of agents I.
2. We use $\mathcal{B}$ to denotes the set of all preference relations.

3. We use $\mathcal{R} \subseteq \mathcal{B}$ to denotes the set of all rational preference relations.
4. We use $\succeq \in \mathcal{R}$ to represents individual rational preference relation.

2.2.1 Social Welfare Function and Properties

Definition 2.3 (Social Welfare Function (SWF))

A **social welfare function** (SWF) is a mapping

$$f : \mathcal{A} \subseteq \mathcal{R}^I \to \mathcal{B}$$

$\trianglerighteq = f(\succeq_1, ..., \succeq_I)$ is interpreted as the **social preference relation**. It doesn't need to be rational (i.e., complete and transitive). ♣

Definition 2.4 (SWF's Properties)

A social welfare function $f : \mathcal{A} \to \mathcal{B}$

- has **unrestricted domain** (UD) if $\mathcal{A} = \mathcal{R}^n$;
- is **transitive** (T) if $f(\succeq_1, ..., \succeq_I)$ is transitive for all $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$;
- is **nondictatorial** (ND) if there is no agent $i \in I$ such that $\forall \{x, y\} \subseteq X\ x \succeq_i y \Rightarrow x \trianglerighteq y$.
- is **weakly Paretian** (PA) if, $\forall \{x, y\} \subseteq X$ and any preference profile $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$, we have $x \succeq_i y, \forall i \in I \Rightarrow x \trianglerighteq y$.
- is **independent of irrelevant alternatives** (IIA) if, $\forall \{x, y\} \subseteq X$, and any $\succeq$ and $\succeq'$ with $\succeq_i|_{x,y} = \succeq'_i|_{x,y}, \forall i \in I$, if $x \trianglerighteq y$ then $x \trianglerighteq' y$. ♣

2.2.2 Arrow's Theorem

Theorem 2.1 (Arrow's impossibility theorem)

Suppose $|X| \geq 3$, $\mathcal{A} = \mathcal{R}^I$ (UD). Then if a SWF f satisfies T, PA, and IIA, then it fails to be ND. ♡

Chapter 3 Signalling Game

Based on

- "Kreps, D. M., & Sobel, J. (1994). Signalling. *Handbook of game theory with economic applications*, 2, 849-867."
-

3.1 Canonical Game

Definition 3.1 (Canonical Game)

1. There are two players: $\mathbf{S}$ (sender) and $\mathbf{R}$ (receiver).
2. $\mathbf{S}$ holds more information than $\mathbf{R}$: the value of some random variable t with support $\mathcal{T}$. (We say that t is the **type** of $\mathbf{S}$)
3. Prior belief of $\mathbf{R}$ concerning t are given by a probability distribution ρ over $\mathcal{T}$ (common knowledge)
4. $\mathbf{S}$ sends a **signal** $s \in \mathcal{S}$ to $\mathbf{R}$ drawn from a signal set $\mathcal{S}$.
5. $\mathbf{R}$ receives this signal, and then takes an **action** $a \in \mathcal{A}$ drawn from a set $\mathcal{A}$ (which could depend on the signal s that is sent).
6. $\mathbf{S}$'s payoff is given by a function $u : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ and $\mathbf{R}$'s payoff is given by a function $v : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.

♣

3.2 Nash Equilibrium

Definition 3.2 (Strategy)

A **behavior strategy** for $\mathbf{S}$ is given by a function $\sigma : \mathcal{T} \times \mathcal{S} \to [0, 1]$ such that $\sum_s \sigma(t, s)$ for each t.

A **behavior strategy** for $\mathbf{R}$ is given by a function $\alpha : \mathcal{S} \times \mathcal{A} \to [0, 1]$ such that $\sum_a \alpha(s, a)$ for each t. ♣

Definition 3.3 (Nash Equilibrium)

Behavior strategies α and σ form a **Nash equilibrium** if and only if

1. For all $t \in \mathcal{T}$,

$$\sigma(t, s) > 0 \text{ implies } \sum_a \alpha(s, a)u(t, s, a) = \max_{s' \in \mathcal{S}} \left(\sum_a \alpha(s', a)u(t, s', a)\right)$$

2. For each $s \in \mathcal{S}$ such that $\sum_t \sigma(t, s)\rho(t) > 0$,

$$\alpha(s, a) > 0 \text{ implies } \sum_t \mu(t; s)v(t, s, a) = \max_{a'} \sum_t \mu(t; s)v(t, s, a')$$

where $\mu(t;s)$ is the $\mathbb{R}$'s posterior belief about t given s, $\mu(t;s) = \frac{\sigma(t,s)\rho(t)}{\sum_{t'}\sigma(t',s)\rho(t')}$ if $\sum_t \sigma(t,s)\rho(t) > 0$ and $\mu(t;s) = 0$ otherwise. ♣

Definition 3.4 (Separating & Pooling Equilibrium)

An equilibrium (σ, α) is called a **separating** equilibrium if each type t sends different signals; i.e., the set $\mathcal{S}$ can be partitioned into (disjoint) sets $\{\mathcal{S}_t; t \in \mathcal{S}\}$ such that $\sigma(t, \mathcal{S}_t) = 1$. An equilibrium (σ, α) is called a **pooling** equilibrium if there is a single signal s^* that is sent by all types; i.e., $\sigma(t, s^*) = 1$ for all $t \in \mathcal{T}$. ♣

3.3 Single-crossing

3.3.1 Situation over real line

Consider the situation that $\mathcal{T}, \mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$ and $\geq$ is the usual "greater than or equal to" relationship.

1. We let $\Delta\mathcal{A}$ denote the set of probability distributions on $\mathcal{A}$.
2. For each $s \in \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$, we let $\Delta\mathcal{A}(s, T')$ be the set of mixed strategies that are the best responses by $\mathbf{R}$ to $s \in \mathcal{S}$ for some probability distribution with support $\mathcal{T}'$.
3. For $\alpha \in \Delta\mathcal{A}$, we write $u(t, s, \alpha) \triangleq \sum_{a \in \mathcal{A}} u(t, s, a)\alpha(a)$.

Definition 3.5 (Single-crossing)

The data of the game are said to satisfy the **single-crossing property** if the following holds: If $t \in \mathcal{T}$, $(s, \alpha) \in \mathcal{S} \times \Delta\mathcal{A}$ and $(s', \alpha') \in \mathcal{S} \times \Delta\mathcal{A}$ are such that $\alpha \in \Delta\mathcal{A}(s, \mathcal{T})$, $\alpha' \in \Delta\mathcal{A}(s', \mathcal{T})$, $s > s'$ and $u(t, s, \alpha) \geq u(t, s', \alpha')$, then for all $t' \in T$ such that $t' > t$, $u(t', s, \alpha) \geq u(t', s', \alpha')$. ♣

Chapter 4 Tools for Comparative Statics

Consider the function $f : (0, 2\pi) \times \mathbb{R} \to \mathbb{R}$ s.t.

$$f(x, a) = \sin x + a$$

Let $X = (0, 2\pi)$ and let $f_a(x) = f(x, a) = \sin x + a$ denote the perturbed function for fixed a.

4.1 Regular and Critical Points and Values

4.1.1 Rank of Derivatives $\text{Rank}df_x = \text{Rank}Df(x)$

Suppose $X \subseteq \mathbb{R}^n$ is open. Suppose $f : X \to \mathbb{R}^m$ is differentiable at $x \in X$, and let $W = \{e_1, ..., e_n\}$ denote the standard basis of $\mathbb{R}^n$. Then $df_x \in L(\mathbb{R}^n, \mathbb{R}^m)$, and

$$\begin{aligned}
\text{Rank}df_x &= \dim \text{Im}(df_x) \\
&= \dim \text{span}\{df_x(e_1), ..., df_x(e_n)\} \\
&= \dim \text{span}\{Df(x)e_1, ..., Df(x)e_n\} \\
&= \dim \text{span}\{\text{column 1 of } Df(x), ..., \text{column n of } Df(x)\} \\
&= \text{Rank}Df(x)
\end{aligned}$$

Thus,

$$\text{Rank}df_x \leq \min\{m, n\}$$

df_x has **full rank** if $\text{Rank}df_x = \min\{m, n\}$, that is, is df_x has the maximum possible rank.

4.1.2 Regular and Critical Points and Values

Definition 4.1 (Regular and Critical Points and Values)

Suppose $X \subseteq \mathbb{R}^n$ is open. Suppose $f : X \to \mathbb{R}^m$ is differentiable at $x \in X$.

1. x is a **regular point** of f if $\text{Rank}df_x = \min\{m, n\}$.
2. x is a **critical point** of f if $\text{Rank}df_x < \min\{m, n\}$.
3. y is a **critical value** of f if there exists $x \in f^{-1}(y)$ such that x is a critical point of f.
4. y is a **regular value** of f if y is not a critical value of f.

♣

Note *Notice that if $y \notin f(X)$, so $f^{-1}(y) = \emptyset$, then y is automatically a regular value of f.*

Example 4.1 Suppose $f(x,y) = (\sin x, \cos y)$, $Df(x,y) = \begin{bmatrix} \cos x & 0 \\ 0 & -\sin y \end{bmatrix}$. Critical point: $\{(\frac{k\pi}{2}, \mathbb{R}) : k \in 2\mathbb{Z}+1\} \cup \{(\mathbb{R}, k\pi) : k \in \mathbb{Z}\}$; Critical values: $\{(x,y) : x = 1$ or $x = -1$ or $y = 1$ or $y = -1\}$

4.2 Inverse and Implicit Function Theorem

4.2.1 Inverse Function Theorem

Using Taylor's theorem to approximate

$$f(x) = f(x_0) + Df(x_0)(x - x_0) + o(x - x_0)$$

The requirement of "regular point" is necessary for the $Df(x_0)$ being invertible.

Theorem 4.1 (Inverse Function Theorem)

Suppose $X \subseteq \mathbb{R}^n$ is open. Suppose $f : X \to \mathbb{R}^n$ is C^1 on X, and $x_0 \in X$. If $\det Df(x_0) \neq 0$ (i.e., x_0 is a regular point of f), then there are open neighborhoods U of x_0 and V of $f(x_0)$ s.t.

$$f : U \to V \text{ is bijective (on-to-on and onto)}$$

$$\exists\, f^{-1} : V \to U \text{ is } C^1$$

$$Df^{-1}(f(x_0)) = [Df(x_0)]^{-1}$$

$$(\text{In } \mathbb{R}, (f^{-1})'(f(x_0)) = (f'(x_0))^{-1})$$

If in addition $f \in C^k$, then $f^{-1} \in C^k$. ♡

4.2.2 Implicit Function Theorem

Using Taylor's theorem to approximate

$$f(x,a) = f(x_0,a_0) + Df(x_0,a_0)(x - x_0) + Df(x_0,a_0)(a - a_0) + \text{remainder}$$

The requirement of "regular point" is necessary for the $Df(x_0, a_0)$ being invertible.

We want to know how the function $x^*(a)$ changes with keeping $f(x^*, a) = 0$.

Theorem 4.2 (Implicit Function Theorem)

Suppose $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^p$ are open and $f : X \times A \to \mathbb{R}^n$ is C^1. Suppose $f(x_0, a_0) = 0$ and $\det(D_x f(x_0, a_0)) \neq 0$, i.e. x_0 is a regular point of $f(\cdot, a_0)$. Then there are open neighborhoods U of x_0 ($U \subseteq X$) and W of a_0 such that

$$\forall a \in W,\ \exists! x \in U \text{ s.t. } f(x,a) = 0$$

For each $a \in W$ let $g(a)$ be that unique x. Then $g : W \to U$ is C^1 and

$$Dg(a_0) = -[D_x f(x_0, a_0)]^{-1}[D_a f(x_0, a_0)]$$

If in addition $f \in C^k$, then $g \in C^k$.

4.2.3 Prove Implicit Function Theorem Given Inverse Function Theorem

Proof 4.1

1. *Firstly, we prove "g is differentiable": The "change of a" incurs the value change:*

$$\begin{aligned} f(x_0, a_0 + h) &= f(x_0, a_0) + D_a f(x_0, a_0)h + o(h) \\ &= D_a f(x_0, a_0)h + o(h) \end{aligned}$$

Find a Δx such that the new x can let the value go back to 0, i.e., $f(x_0 + \Delta x, a_0 + h) = 0$. That is,

$$g(a_0 + h) = x_0 + \Delta x$$

To prove "g is differentiable", we want to prove "$\exists T \in L(A, X)$ s.t. $\Delta x = T(h) + o(h)$"

$$\begin{aligned} 0 &= f(x_0 + \Delta x, a_0 + h) \\ &= f(x_0, a_0) + D_x f(x_0, a_0 + h)\Delta x + D_a f(x_0, a_0)h + o(\Delta x) + o(h) \\ &= D_x f(x_0, a_0 + h)\Delta x + D_a f(x_0, a_0)h + o(\Delta x) + o(h) \end{aligned}$$

$$D_x f(x_0, a_0 + h)\Delta x = -D_a f(x_0, a_0)h + o(\Delta x) + o(h)$$

Because f is C^1 and the determinant is a continuous function of the entries of the matrix, $\det D_x f(x_0, a_0 + h) \neq 0$ for h sufficiently small, so

$$\Delta x = -[D_x f(x_0, a_0 + h)]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h)$$

$$\text{Since } f \in C^1, \Delta x = -[D_x f(x_0, a_0) + o(1)]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h)$$

$$\text{Since } f \in C^1, \Delta x = -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0)h + o(\Delta x) + o(h)$$

Hence, "g is differentiable" is proved and the derivative of g is $Dg(a_0) = -[D_x f(x_0, a_0)]^{-1}[D_a f(x_0, a_0)]$.

2. *Secondly, given the "g is differentiable", we can also compute the derivative by*

$$\begin{aligned} Df(g(a), a)(a_0) &= 0 \\ D_x f(x_0, a_0)Dg(a_0) + D_a f(x_0, a_0) &= 0 \\ Dg(a_0) &= -[D_x f(x_0, a_0)]^{-1} D_a f(x_0, a_0) \end{aligned}$$

Example 4.2 $f : \mathbb{R}^3 \to \mathbb{R}^2$, $f((3,-1,2)) = (0,0)$, $Df(3,-1,2) = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{bmatrix}$. Then, let $(x_0, a_0) = (3,-1,2)$, where $x_0 = 3$ and $a_0 = (-1,2)$. Or, we can let $(x_0, a_0) = (3,-1,2)$, where $x_0 = (3,-1)$ and $a_0 = 2$.

4.2.4 Prove Inverse Function Theorem Given Implicit Function Theorem

Proof 4.2 (Prove Inverse Function Theorem Given Implicit Function Theorem)

Define $F : X \times \mathbb{R}^n$ s.t. $F(x,y) = y - f(x)$. Let $y_0 = f(x_0)$.

$$D_x F(x,y) = -Df(x),\ D_y F(x,y) = I_{n\times n}$$

According to the implicit function theorem, there are open sets $U \subseteq X$ and $V \subseteq \mathbb{R}^n$ such that $x_0 \in U$, $y_0 \in V$ and a function $g : V \to U$ differentiable at y_0 such that $F(g(y), y) = 0$ for all $y \in V$. So, $0 = F(g(y), y) = y - f(g(y))$, we have $f(g(y)) = y$, that is $g = f^{-1}$. $f : U \to V$ is bijective because it has inverse $g : V \to U$.

By the implicit function theorem, $g(y)$ is differentiable and

$$Df^{-1}(y_0) = Dg(y_0) = -[D_x F(x_0, y_0)]^{-1}[D_y F(x_0, y_0)] = [Df(x_0)]^{-1}$$

where $y_0 = f(x_0)$.

By the implicit function theorem, the $g = f^{-1}$ is C^k if f is C^k.

All in all, the inverse function theorem is proved.

4.2.5 Example: Using Implicit Function Theorem in Comparative Statics

Example 4.3 Let us consider a firm that produces a good y; it uses two inputs x_1 and x_2. The firm sells the output and acquires the inputs in competitive markets: The market price of y is p, and the cost of each unit of x_1 and x_2 are w_1 and w_2 respectively. Its technology is given by $f : \mathbb{R}^2_+ \to \mathbb{R}_+$, where $f(x_1, x_2) = x_1^a x_2^b$, $a + b < 1$. Its profits take the form

$$\pi(x_1, x_2; p, w_1, w_2) = p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

The firm selects x_1 and x_2 in order to maximize profits. **We aim to know how its choice of x_1 and x_2 is affected by a change in w_1.**

Assuming an interior solution, the first-order conditions of this optimization problem are

$$\frac{\partial \pi}{\partial x_1}(x_1^*, x_2^*; p, w_1, w_2) = pa(x_1^*)^{a-1}(x_2^*)^b - w_1 = 0$$
$$\frac{\partial \pi}{\partial x_2}(x_1^*, x_2^*; p, w_1, w_2) = pb(x_1^*)^{a}(x_2^*)^{b-1} - w_2 = 0$$

for some $(x_1, x_2) = (x_1^*, x_2^*)$.

Let us define

$$F(x_1^*, x_2^*; p, w_1, w_2) = \begin{bmatrix} pa(x_1^*)^{a-1}(x_2^*)^b - w_1 \\ pb(x_1^*)^a(x_2^*)^{b-1} - w_1 \end{bmatrix}$$

Jacobian matrices are

$$D_{(x_1,x_2)}F(x_1^*, x_2^*; p, w_1, w_2) = \begin{bmatrix} pa(a-1)(x_1^*)^{a-2}(x_2^*)^b & pab(x_1^*)^{a-1}(x_2^*)^{b-1} \\ pab(x_1^*)^{a-1}(x_2^*)^{b-1} & pb(b-1)(x_1^*)^a(x_2^*)^{b-2} \end{bmatrix}$$

$$D_{w_1}F(x_1^*, x_2^*; p, w_1, w_2) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

By the implicit function theorem, we can get

$$\begin{aligned} \begin{bmatrix} \frac{\partial x_1^*}{\partial w_1} \\ \frac{\partial x_2^*}{\partial w_1} \end{bmatrix} &= -[D_{(x_1,x_2)}F(x_1^*, x_2^*; p, w_1, w_2)]^{-1}[D_{w_1}F(x_1^*, x_2^*; p, w_1, w_2)] \\ &= [D_{(x_1,x_2)}F(x_1^*, x_2^*; p, w_1, w_2)]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned}$$

4.2.6 Corollary: $a \to \{x \in X : f(x, a) = 0\}$ is lhc

Corollary 4.1

Suppose $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^p$ are open and $f : X \times A \to \mathbb{R}^n$ is C^1. If 0 is a regular value of $f(\cdot, a_0)$, then the correspondence

$$a \to \{x \in X : f(x, a) = 0\}$$

*is **lower hemicontinuous** at a_0.*

4.3 Transversality and Genericity

4.3.1 Lebesgue Measure Zero

Definition 4.2 (Lebesgue Measure Zero)

Suppose $A \subseteq \mathbb{R}^n$. A has **Lebesgue measure zero** if for every $\varepsilon > 0$ there is a countable collection of rectangles $I_1, I_2, \ldots$ such that

$$\sum_{k=1}^{\infty} \mathrm{Vol}(I_k) < \varepsilon \text{ and } A \subseteq \cup_{k=1}^{\infty} I_k$$

Here by a rectangle we mean $I_k = \times_{j=1}^n (a_j^k, b_j^k) = \{x \in \mathbb{R}^n : x_j \in (a_j^k, b_j^k), \forall j\}$ for some $a_j^k < b_j^k \in \mathbb{R}$, and

$$\text{Vol}(I_k) = \prod_{j=1}^{n} |b_j^k - a_j^k|$$

♣

Example 4.4

1. "Lower-dimensional" sets have Lebesgue measure zero. For example, $A = \{x \in \mathbb{R}^2 : x_2 = 0\}$
2. Any **finite** set has Lebesgue measure zero in $\mathbb{R}^n$.
3. **Finite Union** of sets that have Lebesgue measure zero has Lebesgue measure zero: If A_n has Lebesgue measure zero $\forall n$ then $\cup_{n \in N} A_n$ has Lebesgue measure zero.
4. Every **countable** set (e.g. $\mathbb{Q}$) has Lebesgue measure zero.
5. No open set in $\mathbb{R}^n$ has Lebesgue measure zero.

4.3.2 Sard's Theorem

Theorem 4.3 (Sard's Theorem)

Let $X \subseteq \mathbb{R}^n$ be open, and $f : X \to \mathbb{R}^m$ be C^r with $r \geq 1 + max\{0, n - m\}$. Then the set of all critical values of f has Lebesgue measure zero.

♡

4.3.3 Transversality Theorem

Theorem 4.4 (Transversality Theorem)

Let $X \subseteq \mathbb{R}^n$ and $A \subseteq \mathbb{R}^p$ be open, and $f : X \times A \to \mathbb{R}^m$ be C^r with $r \geq 1 + max\{0, n - m\}$. Suppose that 0 is a regular value of f (that is all (x, a) such that $f(x, a) = 0$ are regular points). Then,

1. *$\exists A_0 \subseteq A$ such that $A \backslash A_0$ has Lebesgue measure zero.*
2. *$\forall a \in A_0$, 0 is a regular value of $f_a = f(\cdot, a)$.*

♡

Example 4.5 $f : \mathbb{R}^4 \to \mathbb{R}^3$ s.t. $f(x, y, z, w) = (g(x) + y, z^3 + 1, w + x + y^2)$

Chapter 5 Fixed Point Theorem

5.1 Contraction Mapping Theorem (@ Lec 05 of ECON 204)

5.1.1 Contraction: Lipschitz continuous with constant < 1

Definition 5.1

Let (X, d) be a nonempty complete metric space. An operator is a function $T : X \to X$. An operator T is a **contraction of modulus** β if $\beta < 1$ and

$$d(T(x), T(y)) \leq \beta d(x, y), \forall x, y \in X$$

♣

A contraction shrinks distances by a *uniform* factor $\beta < 1$.

5.1.2 Theorem: Contraction $\Rightarrow$ Uniformly Continuous

Theorem 5.1 (Contraction $\Rightarrow$ Uniformly Continuous)

Every contraction is uniformly continuous.

♡

Proof 5.1

Let $\delta = \frac{\varepsilon}{\beta}$.

5.1.3 Blackwell's Sufficient Conditions for Contraction

Let X be a set, and let $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$. Then $(B(X), \|\cdot\|_\infty)$ is a normed vector space.

(Notice that below we use shorthand notation that identifies a constant function with its constant value in $\mathbb{R}$, that is, we write interchangeably $a \in \mathbb{R}$ and $a : X \to \mathbb{R}$ to denote the function such that $a(x) = a, \forall x \in X$.)

Theorem 5.2 (Blackwell's Sufficient Conditions)

Consider $B(X)$ *with the sup norm* $\|\cdot\|_\infty$. *Let* $T : B(X) \to B(X)$ *be an operator satisfying*

1. *(monotonicity)* $f(x) \leq g(x), \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x), \forall x \in X$
2. *(discounting)* $\exists \beta \in (0, 1)$ *such that for every* $a \geq 0$ *and* $x \in X$,

$$(T(f + a))(x) \leq (Tf)(x) + \beta a$$

Then T *is a contraction with modulus* β.

♡

Proof 5.2

Fix $f, g \in B(X)$. By the definition of the sup norm,

$$f(x) \leq g(x) + \|f - g\|_\infty \forall x \in X$$

Then

$$(Tf)(x) \leq (T(g + \|f - g\|_\infty))(x) \leq (Tg)(x) + \beta\|f - g\|_\infty \quad \forall x \in X$$

where the first inequality above follows from monotonicity, and the second from discounting. Thus

$$(Tf)(x) - (Tg)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Reversing the roles of f and g above gives

$$(Tg)(x) - (Tf)(x) \leq \beta\|f - g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta\|f - g\|_\infty$$

Thus T is a contraction with modulus β

5.2 Fixed Point Theorem (@ Lec 05 of ECON 204)

5.2.1 Fixed Point

Definition 5.2 (Fixed Point)

A **fixed point** of an operator T is element $x^* \in X$ such that $T(x^*) = x^*$. ♣

Definition 5.3 (Fixed Point of Function)

Let X be a nonempty set and $f : X \to X$. A point $x^* \in X$ is a **fixed point** of f if $f(x^*) = x^*$. ♣

Example 5.1 Let $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$

1. $f(x) = 2x$ has fixed point: $x = 0$.
2. $f(x) = x$ has fixed points: $x \in \mathbb{R}$.
3. $f(x) = x + 1$ doesn't have fixed points.

5.2.2 ★ Contraction Mapping Theorem: contraction ⇒ exist unique fixed point

Theorem 5.3 (Contraction Mapping Theorem)

Let (X, d) be a nonempty complete metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then

1. *T has a unique fixed point x^*.*
2. *For every $x_0 \in X$, the sequence defined by*

$$
\begin{aligned}
x_1 &= T(x_0) \\
x_2 &= T(x_1) = T(T(x_0)) = T^2(x_0) \\
&\vdots \\
x_{n+1} &= T(x_n) = T^{n+1}(x_0)
\end{aligned}
$$

   converges to x^*.

Note that the theorem asserts both the **existence** and **uniqueness** of the fixed point, as well as giving an **algorithm** to find the fixed point of a contraction.

Proof 5.3

Define the sequence $\{x_n\}$ as above. Then,

$$
\begin{aligned}
d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\
&\leq \beta d(x_n, x_{n-1}) \\
&\leq \beta^n d(x_1, x_0)
\end{aligned}
$$

Then for any $n > m$,

$$
\begin{aligned}
d(x_n, x_m) &\leq d(x_1, x_0) \sum_{i=m}^{n-1} \beta^i \\
&< d(x_1, x_0) \sum_{i=m}^{\infty} \beta^i \\
&= \frac{\beta^m}{1-\beta} d(x_1, x_0) \to 0 \text{ as } m \to \infty
\end{aligned}
$$

Fixed $\varepsilon > 0$, we can choose $N(\varepsilon)$ such that $\forall n, m > N(\varepsilon)$,

$$
d(x_n, x_m) < \frac{\beta^m}{1-\beta} d(x_1, x_0) < \varepsilon
$$

Therefore, $\{x_n\}$ is Cauchy. Since (X, d) is complete, $x_n \to x^*$ for some $x^* \in X$.

Next we show that x^* is a fixed point of T.

$$\begin{aligned} T(x^*) &= T\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n\to\infty} x_{n+1} \\ &= x^* \end{aligned}$$

so x^* is a fixed point of T.

Finally, we show that there is at most one fixed point. Suppose x^* and y^* are both fixed points of T, so $T(x^*) = x^*$ and $T(y^*) = y^*$. Then

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\le \beta d(x^*, y^*) \\ \Rightarrow (1-\beta)d(x^*, y^*) &\le 0 \\ \Rightarrow d(x^*, y^*) &\le 0 \end{aligned}$$

So $d(x^*, y^*) = 0$, which implies $x^* = y^*$.

5.2.3 Conditions for Fixed Point's Continuous Dependence on Parameters

Theorem 5.4 (Continuous Dependence on Parameters)

Let (X, d) and (Ω, ρ) be two metric spaces and $T : X \times \Omega \to X$. For each parameter $\omega \in \Omega$ let $T_\omega : X \to X$ be defined by $T_\omega(x) = T(x, \omega)$.

Suppose (1). (X, d) is complete, (2). T is continuous in ω (that is $T(x, \cdot) : \Omega \to X$ is continuous for each $x \in X$), and (3). $\exists \beta < 1$ such that T_ω is a contraction of modulus β $\forall \omega \in \Omega$.

Then the fixed point function (about parameter ω) $x^* : \Omega \to X$ defined by $x^*(\omega) = T_\omega(x^*(\omega))$ is continuous. ♡

5.3 Brouwer's Fixed Point Theorem (@ Lec 13 of ECON 204)

5.3.1 Simple One: One-dimension

Theorem 5.5

Let $X = [a, b]$ for $a, b \in \mathbb{R}$ with $a < b$ and let $f : X \to X$ be continuous. Then f has a fixed point. ♡

Proof 5.4

Easily proved by Intermediate Value Theorem.

5.3.2 ★ Brouwer's Fixed Point Theorem: continuous function has fixed point over compact, convex set

Theorem 5.6 (Brouwer's Fixed Point Theorem)

*Let $X \subseteq \mathbb{R}^n$ be nonempty, **compact**, and **convex**, and let $f : X \to X$ be continuous. Then f has a fixed point.*

♡

Proof 5.5

Consider the case when the set X is the unit ball in $\mathbb{R}^n$.

Using a fact that "Let B be the unit ball in $\mathbb{R}^n$. Then there is no continuous function $h : B \to \partial B$ such that $h(x_0) = x_0$ for every $x_0 \in \partial B$", which is intuitive but hard to prove. (See *J. Franklin, Methods of Mathematical Economics*, for an elementary (but long) proof.)

Then prove by contradiction: suppose f has no fixed points in B. That is, $\forall x \in B$, $x \neq f(x)$. Since x and its image $f(x)$ are distinct points in B for every x, we can carry out the following construction. For each $x \in B$, construct the line segment originating at $f(x)$ and going through x. Let $g(x)$ denote the intersection of this line segment with ∂B. This construction gives a continuous function $g : B \to \partial B$. Furthermore, notice that if $x_0 \in \partial B$, then $x_0 = g(x_0)$. Then, g gives $g(x) = x, \forall x \in \partial B$. Since there are no such functions by the fact above, we have a contradiction.

Chapter 6 Correspondence: $\Psi : X \to 2^Y$ (@ Lec 07 of ECON 204)

Definition 6.1 (Correspondence)

A **correspondence** $\Psi : X \to 2^Y$ from X to Y is a function from X to 2^Y, that is, $\Psi(x) \subseteq Y$ for every $x \in X$. (2^Y is the set of all subsets of Y) ♣

Example 6.1 Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous utility function, $y > 0$ and $p \in \mathbb{R}^n_{++}$, that is, $p_i > 0$ for each i. Define $\Psi : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to 2^{\mathbb{R}^n_+}$ by

$$\Psi(p, y) = \operatorname{argmax} u(x)$$
$$\text{s.t. } x \geq 0$$
$$p \cdot x \leq y$$

Ψ is the demand correspondence associated with the utility function u; typically $\Psi(p, y)$ is multi-valued.

6.1 Continuity of Correspondences

6.1.1 Upper/Lower Hemicontinuous

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$.

Definition 6.2 (Upper Hemicontinuous)

Ψ is **upper hemicontinuous** (uhc) at $x_0 \in X$ if, for every open set V with $\Psi(x_0) \subseteq V$, there is an open set U with $x_0 \in U$ s.t.

$$x \in U \Rightarrow \Psi(x) \subseteq V$$

♣

Upper hemicontinuity reflects the requirement that Ψ doesn't "jump down/implode in the limit" at x_0. *(A set to "jump down" at the limit x_0: It should mean the set suddenly gets smaller – it "implodes in the limit" – that is, there is a sequence $x_n \to x_0$ and points $y_n \in \Psi(x_n)$ that are far from every point of $\Psi(x_0)$ as $n \to \infty$.)*

Definition 6.3 (Lower Hemicontinuous)

Ψ is **lower hemicontinuous** (lhc) at $x_0 \in X$ if, for every open set V with $\Psi(x_0) \cap V \neq \emptyset$, there is an open set U with $x_0 \in U$ s.t.

$$x \in U \Rightarrow \Psi(x) \cap V \neq \emptyset$$

♣

Lower hemicontinuity reflects the requirement that Ψ doesn't "jump up/explode in the limit" at x_0. *(A set to "jump up" at the limit x_0: It should mean that the set suddenly gets bigger – it "explodes in the limit" – that is,* *there is a sequence $x_n \to x_0$ and a point $y_0 \in \Psi(x_0)$ that is far from every point of $\Psi(x_n)$ as $n \to \infty$.)*

Definition 6.4 (Continuous Correspondence)

Ψ is **continuous** at $x_0 \in X$ if it is both **uhc** and **lhc** at x_0. ♣

Proposition 6.1

Ψ is upper hemicontinuous (respectively lower hemicontinuous, continuous) if it is uhc (respectively lhc, continuous) at every $x \in X$. ♠

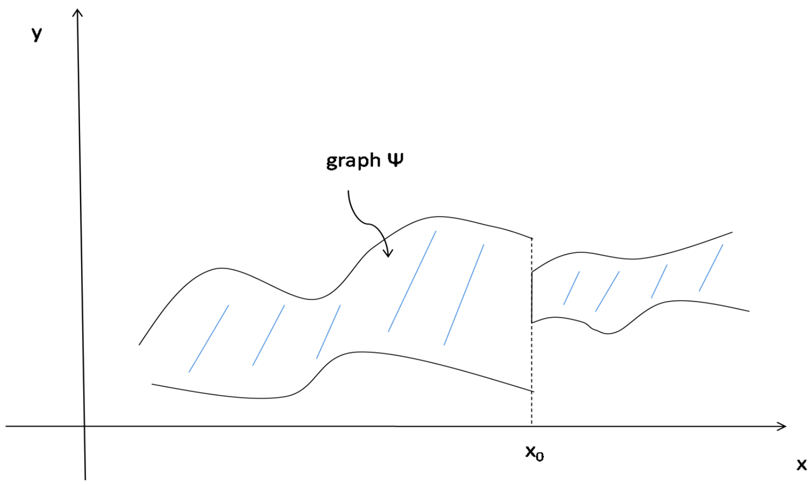

Figure 6.1: The correspondence Ψ "implodes in the limit" at x_0. Ψ is not upper hemicontinuous at x_0.

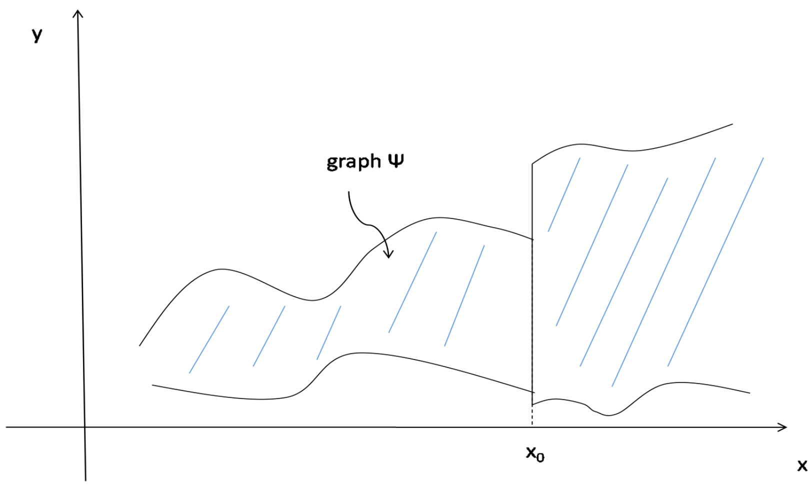

Figure 6.2: The correspondence Ψ "explodes in the limit" at x_0. Ψ is not lower hemicontinuous at x_0.

6.1.2 Theorem: $\Psi(x) = \{f(x)\}$ is uhc $\Leftrightarrow$ f is continuous

Theorem 6.1 ($\Psi(x) = \{f(x)\}$ **is uhc** $\Leftrightarrow$ f **is continuous)**

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$ and $f : X \to Y$. Let $\Psi : X \to 2^Y$ be defined by $\Psi(x) = \{f(x)\}$ for all $x \in X$. Then Ψ is ***uhc*** *if and only if f is* ***continuous****.*

♡

6.1.3 Berge's Maximum Theorem: $v(y) = \max_{x\in\Gamma(y)} f(x, y)$ is continuous; $\{x : f(x, y) = v(y)\}$ is uhc with non-empty compact values

Theorem 6.2 (Berge's Maximum Theorem)

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \to 2^X$. Define $v(y) = \max_{x\in\Gamma(y)} f(x, y)$ and $W(y) = \operatorname{argmax}_{x\in\Gamma(y)} f(x, y) = \{x : f(x, y) = v(y)\}$. Suppose f and Γ are continuous, and that Γ has non-empty compact values. Show that v is continuous and Ω is uhc with non-empty compact values.

♡

6.2 Graph of Correspondence

An alternative notion of continuity looks instead at properties of the graph of the correspondence.

Definition 6.5 (Graph of Correspondence)

The **graph** of a correspondence $\Psi : X \to 2^Y$ is the set

$$\text{graph}\Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$

♣

6.2.1 Closed Graph

By the definition of continuous function $f : \mathbb{R}^n \to \mathbb{R}$, each convergent sequence $\{(x_n, y_n)\}$ in graph f converges to a point (x, y) in graph f, that is, graph f is closed.

Definition 6.6 (Closed Graph)

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$. A correspondence $\Psi : X \to 2^Y$ has closed graph if its graph is a closed subset of $X \times Y$, that is, if for any sequences $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $x_n \to x \in X$, $y_n \to y \in Y$ and $y_n \in \Psi(x_n)$ for each n, then $y \in \Psi(x)$.

♣

Example 6.2 Consider the correspondence $\Psi(x) = \begin{cases} \{\frac{1}{x}\}, & \text{if } x \in (0, 1] \\ \{0\}, & \text{if } x = 0 \end{cases}$ ("implode in the limit")

Let $V = (-0.1, 0.1)$. Then $\Psi(0) = \{0\} \subseteq V$, but no matter how close x is to 0, $\Psi(x) = \{\frac{1}{x}\} \nsubseteq V$, so Ψ is not uhc at 0. However, note that Ψ has closed graph.

6.3 Closed-valued, Compact-valued, and Convex-valued Correspondences

Definition 6.7 (Closed-valued, Compact-valued, and Convex-valued Correspondences)

Given a correspondence $\Psi : X \to 2^Y$,

1. Ψ is **closed-valued** if $\Psi(x)$ is a closed subset of Y for all x;
2. Ψ is **compact-valued** if $\Psi(x)$ is compact for all x.
3. Ψ is **convex-valued** if $\Psi(x)$ is convex for all x.

♣

6.3.1 Closed-valued, uhc and Closed Graph

For closed-valued correspondences these concepts can be more tightly connected. A closed-valued and upper hemicontinuous correspondence must have closed graph. For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Theorem 6.3

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$.

1. *Ψ is **closed-valued** and **uhc** $\Rightarrow$ Ψ has **closed graph**.*
2. *Ψ is **closed-valued** and **uhc** $\Leftarrow$ Ψ has **closed graph**. (If Y is **compact**)*

♡

Theorem 6.4

*Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$. If Ψ has **closed graph** and there is an **open set** W with $x_0 \in W$ and a **compact set** Z such that $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$, then Ψ is **uhc** at x_0.*

♡

6.3.2 Theorem: compact-valued, uhs correspondence of compact set is compact

Theorem 6.5

Let X be a compact set and $\Psi : X \to 2^X$ be a non-empty, compact-valued upper-hemicontinuous correspondence. If $C \subseteq X$ is compact, then $\Psi(C)$ is compact.

♡

Proof 6.1

Given the compact-valued Ψ, we can have an open cover of $\Psi(C)$, $\{U_\lambda : \lambda \in \Lambda\}$. So $\forall x \in C$, there exists $U_{l(x)}, l(x) \in \Lambda$ such that $U_{l(x)}$ is an open cover of $\Psi(x)$.

Consider a $c \in C$. Since Ψ is uhs and $\Psi(c) \subseteq U_{l(c)}$, there exists open set V_c s.t. $c \in V_c$ and $\Psi(x) \subseteq U_{l(c)}, \forall x \in V_c \cap C$.

$\{V_c : c \in C\}$ is an open cover of C. Because C is compact, there is a finite subcover $\{V_{c_i} : i = 1, ..., m\}, m \in \mathbb{N}$, where $\{c_i : i = 1, ..., m\} \subseteq C$.

Because $\Psi(x) \subseteq U_{l(c_i)}, \forall x \in V_{c_i} \cap C$ and $\{V_{c_i} : i = 1, ..., m\}, m \in \mathbb{N}$ is a open cover for C, we can infer $\{U_{l(c_i)} : i = 1, ..., m\}$ is a finite subcover of $\{U_{l(c)} : c \in C\}$ for $\Psi(C)$. Hence, $\Psi(C)$ is compact.

6.4 Fixed Points for Correspondences (@ Lec 13 of ECON 204)

6.4.1 Definition

Definition 6.8 (Fixed Points for Correspondences)

Let X be nonempty and $\psi : X \to 2^X$ be a correspondence. A point $x^* \in X$ is a fixed point of ψ if $x^* \in \psi(x^*)$.

Note *We only need x^* to be in $\psi(x^*)$, not $\{x^*\} = \psi(x^*)$. That is, ψ need not be single-valued at x^*. So x^* can be a fixed point of ψ but there may be other elements of $\psi(x^*)$ different from x^*.*

6.4.2 Kakutani's Fixed Point Theorem: uhs, compact, convex values correspondence has a fixed point over compact convex set

Theorem 6.6 (Kakutani's Fixed Point Theorem)

*Let $X \subseteq \mathbb{R}^n$ be a non-empty, **compact**, **convex** set and $\psi : X \to 2^X$ be an **upper hemi-continuous** correspondence with non-empty, **compact**, **convex** values. Then ψ has a fixed point in X.*

6.4.3 Theorem: $\exists$ compact set $C = \cap_{i=0}^{\infty} \Psi^i(X)$ s.t. $\Psi(C) = C$

Theorem 6.7

Let (X, d) be a compact metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Psi(x)$ is non-empty for every $x \in X$. There exists a compact non-empty subset $C \subseteq X$, such that $\Psi(C) \equiv \cup_{x \in C} \Psi(x) = C$.

Proof 6.2

Let's construct a sequence $\{C_n\}$ such that $C_0 = X$, $C_1 = \Psi(C_0)$, ..., $C_n = \Psi(C_{n-1})$, ... We claim that $C = \cap_{i=0}^{\infty} C_i$ is a non-empty compact set and satisfies $\Psi(C) = C$.

1. *Because we can infer $\Psi(X_1) \subseteq \Psi(X_2)$ if $X_1 \subseteq X_2$, $X = C_0 \supseteq C_1 \Rightarrow C_1 = \Psi(C_0) \supseteq C_2 = \Psi(C_1)$,...., so $C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq \cdots$. Hence, C is not empty.*
2. *Because X is compact, by the theorem 6.5, we can infer C_n is compact for all n. Then, C_n is closed for all n, so C is closed. Because C is a closed set of compact set X, C is compact.*

3. $C \subseteq C_n, \forall n \Rightarrow \Psi(C) \subseteq \Psi(C_n), \forall n \Rightarrow \Psi(C) \subseteq C$
4. *Assume* $C \subseteq \Psi(C)$ *doesn't hold, that is* $\exists y \in C$ *s.t.* $y \notin \Psi(C)$. *Because* $y \in C$ *and* $C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq \cdots$, *there exists* $k \in C_n$ *for all* n *s.t.* $y \in \Psi(k)$. $k \in \cap_{i=1}^{\infty} C_i = C$, *so* $\Psi(k) \subseteq \Psi(C)$, *which contradicts to* $y \notin \Psi(C)$. *Hence,* $C \subseteq \Psi(C)$.

All in all the claim "$C = \cap_{i=0}^{\infty} C_i$ *is a non-empty compact set and satisfies* $\Psi(C) = C$*" is proved.*

Chapter 7 Bayesian Persuasion: Extreme Points and Majorization

Based on

- Kleiner, A., Moldovanu, B., & Strack, P. (2021). Extreme points and majorization: Economic applications. *Econometrica*, 89(4), 1557-1593.
-

7.1 Extreme Points

7.1.1 Extreme Points of Convex Set

Definition 7.1 (Extreme Points)

An **extreme point** of a convex set A is a point $x \in A$ that cannot be represented as a convex combination of points in A.

♣

7.1.2 Krein-Milman Theorem: Existence of Extreme Points

Theorem 7.1 (Krein-Milman Theorem)

*Every non-empty **compact convex** subset of a Hausdorff locally convex topological vector space (for example, a normed space) is the closed, convex hull of its extreme points.*

In particular, this set has extreme points.

♡

7.1.3 Bauer's Maximum Principle: Usefulness of Extreme Points for Optimization

Theorem 7.2 (Bauer's Maximum Principle)

*Any function that is **convex and continuous**, and defined on a set that is **convex and compact**, attains its maximum at some extreme point of that set.*

♡

7.2 Majorization

7.2.1 Majorization and Weak Majorization

Definition 7.2 (Majorization of Non-decreasing Functions)

Consider right-continuous functions that map the unit interval $[0, 1]$ into the real numbers. For two non-decreasing functions $f, g \in L^1$, we say that f **majorizes** g, denoted by $g \prec f$, if the following two conditions hold:

$$\int_x^1 g(s)ds \leq \int_x^1 f(s)ds, \forall x \in [0, 1] \qquad \text{(Condition 1)}$$

$$\int_0^1 g(s)ds = \int_0^1 f(s)ds \qquad \text{(Condition 2)}$$

♣

Definition 7.3 (Weak Majorization)

f **weakly majorizes** g, denoted by $g \prec_w f$, if Condition 1 holds (not necessarily Condition 2). ♣

7.2.2 How to work for non-monotonic functions? – Non-Decreasing Rearrangement

Note How this work with non-monotonic functions?

Suppose f, g are non-monotonic, we compare their non-decreasing rearrangements f^*, g^*.

Definition 7.4 (Rearrangement)

Given a function f, let $m(x)$ denote the Lebesgue measure of the set $\{s \in [0, 1] : f(s) \leq x\}$, that is $m(x) = \int_{s\in\{s\in[0,1]:f(s)\leq x\}} 1ds$ (the "length" of the set). The non-decreasing rearrangement of f, f^*, is defined by

$$f^*(t) = \inf\{x \in \mathbb{R} : m(x) \geq t\},\ t \in [0, 1]$$

♣

7.2.3 Theorem: F majorizes $G \Leftrightarrow G$ is a mean-preserving spread of F

Based on

- Shaked, M., & Shanthikumar, J. G. (2007). *Stochastic orders*. New York, NY: Springer New York.

Definition 7.5 (Generalized Inverse)

Suppose G is defined on the interval $[0, 1]$, we can define the **generalized inverse**

$$G^{-1}(x) = \sup\{s : G(s) \leq x\}, x \in [0, 1]$$

♣

Let X_F and X_G be now random variables with distributions F and G, defined on the interval $[0, 1]$.

Theorem 7.3 (Shaked & Shanthikumar (2007), Section 3.A)

$$G \prec F \Leftrightarrow F^{-1} \prec G^{-1} \Leftrightarrow X_G \leq_{ssd} X_F \text{ and } \mathbb{E}[X_G] = \mathbb{E}[X_F]$$

where $\leq_{ssd}$ denotes the standard second-order stochastic dominance. ♡

Based on Theorem 1.1 and the Condition 2 of Majorization, we can conclude

Corollary 7.1 (Majorization ⇔ Mean-preserving Contraction)

F majorizes $G \Leftrightarrow F$ is a mean-preserving contraction of G (G is a mean-preserving spread of F) ♡

That is, we can construct random variables X_F, X_G, jointly distributed on some probability space, such that $X_F \sim F$, $X_G \sim G$ and such that $X_F = \mathbb{E}[X_G \mid X_F]$.

7.3 Capture Extreme Points in Economic Applications

Let L^1 denote the real-valued and integrable functions defined on $[0, 1]$.

In this section, we focus on **non-decreasing (weakly increasing) functions**, for example, a cumulative distribution function in Bayesian persuasion, or an incentive-compatible allocation in mechanism design.

7.3.1 Definitions of $\mathcal{MPS}(f)$, $\mathcal{MPS}_w(f)$, $\mathcal{MPC}(f)$

Based on Corollary 7.1, we can define following sets

Definition 7.6

1. The set of non-decreasing functions that are majorized by f is denoted by
$$\begin{aligned}\mathcal{MPS}(f) &= \text{MPS}(f) \cap \{g \in L^1 \mid g \text{ is non-decreasing}\} \\ &= \{g \in L^1 \mid g \text{ is non-decreasing and } g \prec f\}\end{aligned}$$
2. The set of non-negative, non-decreasing functions that are weakly majorized by f is denoted by
$$\mathcal{MPS}_w(f) = \{g \in L^1 \mid g \text{ is non-negative, non-decreasing and } g \preceq f\}$$
3. The set of non-decreasing functions that majorize f and satisfy $f(0) \leq g \leq f(1)$ is denoted by
$$\begin{aligned}\mathcal{MPC}(f) &= \text{MPC}(f) \cap \{g \in L^1 \mid g \text{ is non-decreasing and } f(0) \leq g \leq f(1)\} \\ &= \{g \in L^1 \mid g \text{ is non-decreasing and } g \succ f \text{ and } f(0) \leq g \leq f(1)\}\end{aligned}$$
where $f(0) \leq g \leq f(1)$ is used to ensure compactness. ♣

7.3.2 Proposition: $\mathcal{MPS}(f)$, $\mathcal{MPS}_w(f)$, $\mathcal{MPC}(f)$ have extreme points and any element is a combination of extreme points

Following two propositions are the Proposition 1 of the Kleiner et al. (2021).

Proposition 7.1 (Non-decreasing $f \Rightarrow \mathcal{MPS}(f)$, $\mathcal{MPS}_w(f)$, and $\mathcal{MPC}(f)$ have extreme points)

Suppose $f \in L^1$ is non-decreasing. Then $\mathcal{MPS}(f)$, $\mathcal{MPS}_w(f)$, and $\mathcal{MPC}(f)$ are convex and compact in the norm topology $\Rightarrow$ (by Krein-Milman Theorem 7.1) they all have non-empty set of extreme points. ♠

Note *We use* ext$A$ *to denote the set of extreme points of set A.*

Proposition 7.2 (Non-decreasing $f \Rightarrow$ any distribution is a combination of extreme points)

Suppose $f \in L^1$ is non-decreasing. For any $g \in \mathcal{MPS}(f)$, $\exists$ a probability measure λ_g over ext$\mathcal{MPS}(f)$ *such that*

$$g = \int_{\text{ext}\mathcal{MPS}(f)} h \; d\lambda_g(h)$$

(also hold for any $g \in \mathcal{MPS}_w(f)$ and $g \in \mathcal{MPC}(f)$). ♠

7.3.3 Extreme Points in $\mathcal{MPS}(f)$

Theorem 7.4 (Form of Extreme Points in MPS(f): Kleiner et al. (2021), Theorem 1)

Let f be non-decreasing. Then g is an ***extreme point*** *in $\mathcal{MPS}(f)$ <u>if and only if</u> there exists a collection of disjoint intervals $\{[\underline{x}_i, \overline{x}_i)\}_{i\in I}$ such that*

$$g(x) = \begin{cases} f(x), & \text{if } x \notin \cup_{i\in I}[\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} f(s)ds}{\overline{x}_i - \underline{x}_i}, & \text{if } x \in [\underline{x}_i, \overline{x}_i) \end{cases}$$

♡

g is an extreme point of $\mathcal{MPS}(f)$ implies either that $g(x) = f(x)$ or that g is constant at x.

Definition 7.7 (Exposed Element)

An element x of a convex set A is **exposed** if there exists a continuous linear functional that attains its maximum on A uniquely at x. ♣

Note *Every exposed point is extreme, but the converse is not true in general.*

Corollary 7.2 (Kleiner et al. (2021), Corollary 1)

Every extreme point of $\mathcal{MPS}(f)$ is exposed. ♡

7.3.4 Extreme Points in $\mathcal{MPS}_w(f)$

For a set $A \subseteq [0,1]$, we use $\mathbf{1}_A(x)$ denote the indicator function of set A: it equals to 1 if $x \in A$ and 0 otherwise.

Corollary 7.3 (Kleiner et al. (2021), Corollary 2)

Suppose that f is non-decreasing and non-negative. A function g is an extreme point of $\mathcal{MPS}_w(f)$ if and only if there is $\theta \in [0,1]$ such that g is an extreme point of $\mathcal{MPS}(f)$ and $g(x) = 0, \forall x \in [0,\theta)$. ♡

7.3.5 Extreme Points in $\mathcal{MPC}(f)$

Theorem 7.5 (Kleiner et al. (2021), Theorem 2)

Let f be non-decreasing and continuous. Then $g \in \mathcal{MPC}(f)$ is an extreme point of $\mathcal{MPC}(f)$ if and only if there exists a collection of intervals $[\underline{x}_i, \bar{x}_i)$, (potentially empty) sub-intervals $\left[\underline{y}_i, \bar{y}_i\right) \subseteq [\underline{x}_i, \bar{x}_i)$, and numbers v_i indexed by $i \in I$ such that for all $x \in [0,1]$,

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i\in I} [\underline{x}_i, \bar{x}_i) \\ f(\underline{x}_i) & \text{if } x \in \left[\underline{x}_i, \underline{y}_i\right) \\ v_i & \text{if } x \in \left[\underline{y}_i, \bar{y}_i\right) \\ f(\bar{x}_i) & \text{if } x \in [\bar{y}_i, \bar{x}_i) \end{cases} \tag{7.1}$$

Moreover, a function g as defined in (7.1) is in $\mathcal{MPC}(f)$ if the following three conditions are satisfied:

$$\left(\bar{y}_i - \underline{y}_i\right) v_i = \int_{\underline{x}_i}^{\bar{x}_i} f(s)\mathrm{d}s - f(\underline{x}_i)\left(\underline{y}_i - \underline{x}_i\right) - f(\bar{x}_i)(\bar{x}_i - \bar{y}_i) \tag{7.2}$$

$$f(\underline{x}_i)(\bar{y}_i - \underline{x}_i) + f(\bar{x}_i)(\bar{x}_i - \bar{y}_i) \le \int_{\underline{x}_i}^{\bar{x}_i} f(s)\mathrm{d}s \le f(\underline{x}_i)\left(\underline{y}_i - \underline{x}_i\right) + f(\bar{x}_i)\left(\bar{x}_i - \underline{y}_i\right) \tag{7.3}$$

If $v_i \in \left(f\left(\underline{y}_i\right), f(\bar{y}_i)\right)$, then for an arbitrary point m_i satisfying $f(m_i) = v_i$ it must hold that

$$\int_{m_i}^{\bar{x}_i} f(s)\mathrm{d}s \le v_i(\bar{y}_i - m_i) + f(\bar{x}_i)(\bar{x}_i - \bar{y}_i) \tag{7.4}$$

♡

Condition (7.2) in the theorem ensures that g and f have the same integrals for each sub-interval $[\underline{x}_i, \bar{x}_i)$, analogously to the condition imposed in Theorem 7.3.3. Condition (7.3) ensures that $v_i \in (f(\underline{x}_i), f(\bar{x}_i))$, ensuring that g is non-decreasing. If f crosses g in the interval $\left[\underline{y}_i, \bar{y}_i\right]$, then there is $m_i \in \left[\underline{y}_i, \bar{y}_i\right]$ such that $f(m_i) = v_i$. In this case, Condition (7.4) ensures that $\int_s^{\bar{x}_i} f(t)\mathrm{d}t \le \int_s^{\bar{x}_i} g(t)\mathrm{d}t$ for all $s \in [\underline{x}_i, \bar{x}_i)$ and thus that $f \prec g$. If $v_i \notin \left(f\left(\underline{y}_i\right), f(\bar{y}_i)\right)$, Condition (7.3) is enough to ensure that $f \prec g$ and thus Condition (7.4) is not necessary.

Chapter 8 Bayesian Persuasion: Bi-Pooling

Based on

- Arieli, I., Babichenko, Y., Smorodinsky, R., & Yamashita, T. (2023). Optimal persuasion via bi-pooling. *Theoretical Economics*, 18(1), 15-36.