Analysis and Something Else

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All models are wrong, but some are useful.

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- $\neg P$ (”not P”) means ”P is false”.
- $P \wedge Q$ (”P and Q”) means “P is true and Q is true.”
- $P \vee Q$ (”P or Q”) means “P is true or Q is true (or possibly both).”
- $\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.
- $P \Rightarrow Q$ (”P implies Q”) means “whenever P is satisfied, Q is also satisfied.”

Statement: Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

Definition 1.1 (Contrapositive)

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement $\neg Q \Rightarrow \neg P$. ♣

Theorem 1.1 (Prove Contrapositive Insead)

$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true. ♡

Chapter 2 Analysis Basis

2.1 Real Number $\mathbb{R}$ (@ Lec 02 of ECON 204)

$\mathbb{R}$ is a field with the usual operations $+$, $\cdot$, additive identity 0, and multiplicative identity 1.

2.1.1 Order Axiom

Proposition 2.1 (Order Axiom)

There is a complete ordering $\leq$, i.e. $\leq$ is reflexive, transitive, antisymmetric ($\alpha \leq \beta, \beta \leq \alpha \Rightarrow \alpha = \beta$) with the property that $\forall \alpha, \beta \in \mathbb{R}$ either $\alpha \leq \beta$ or $\beta \leq \alpha$.

The order is compatible with $+$ and $\cdot$, i.e. $\forall \alpha, \beta, \gamma \in \mathbb{R}$

1. $\alpha \leq \beta \Rightarrow \alpha + \gamma \leq \beta + \gamma$.
2. $\alpha \leq \beta, 0 \leq \gamma \Rightarrow \alpha\gamma \leq \beta\gamma$.

2.1.2 Completeness Axiom

Proposition 2.2 (Completeness Axiom)

Suppose $L, H \subseteq \mathbb{R}, L \neq \emptyset \neq H$ satisfy $\forall l \in L, h \in H, l \leq h$. Then,

$$\exists \alpha \in \mathbb{R} \text{ s.t. } \forall l \in L, h \in H,\ l \leq \alpha \leq h$$

$$\begin{array}{ccc} & \alpha & \\ L & \downarrow & H \\ ----) & \cdot & (---- \end{array}$$

Figure 2.1: Completeness Axiom

Claim 2.1

The Completeness Axiom differentiates $\mathbb{R}$ from $\mathbb{Q}$:

$\mathbb{Q}$ satisfies all the axioms for $\mathbb{R}$ except the Completeness Axiom.

2.1.3 Supremum $\sup \mathbb{X}$, Infimum $\inf \mathbb{X}$ for $\mathbb{X} \subseteq \mathbb{R}$

Definition 2.1 (Supremum and Infimum)

(1). Suppose $\mathbb{X}$ is bounded above. The **supremum** of $\mathbb{X}$, written $\sup \mathbb{X}$, is the least upper bound for $\mathbb{X}$, i.e. $\sup \mathbb{X}$ satisfies

(a). $\sup \mathbb{X} \geq x, \forall x \in \mathbb{X}$ ($\sup \mathbb{X}$ is an upper bound).

(b). $\forall y < \sup \mathbb{X}$, $\exists x \in \mathbb{X}$ s.t. $x > y$ (there is no smaller upper bound).

(2). Suppose $\mathbb{X}$ is bounded below. The **infimum** of $\mathbb{X}$, written $\inf \mathbb{X}$, is the greatest lower bound for $\mathbb{X}$, i.e. $\inf \mathbb{X}$ satisfies

(a). $\inf \mathbb{X} \leq x, \forall x \in \mathbb{X}$ ($\inf \mathbb{X}$ is a lower bound).

(b). $\forall y > \inf \mathbb{X}$, $\exists x \in \mathbb{X}$ s.t. $x < y$ (there is no greater lower bound).

(3). If $\mathbb{X}$ is not bounded above, write $\sup \mathbb{X} = \infty$. If $\mathbb{X}$ is not bounded below, write $\inf \mathbb{X} = -\infty$. By convention, $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$.

♣

Proposition 2.3

If $\inf A = x^* \in A$ ($\sup A = x^* \in A$), *then* $x^* = \min A$ ($x^* = \max A$).

♠

2.1.4 The Supremum Property

Proposition 2.4 (The Supremum Property)

Every nonempty set of real numbers that is bounded above has a supremum, which is a real number. Every nonempty set of real numbers that is bounded below has an infimum, which is a real number.

♠

Theorem 2.1

The Supremum Property (Prop 2.4) and the Completeness Axiom (Prop 2.2) are equivalent.

♡

2.1.5 Archimedean Property

Theorem 2.2 (Archimedean Property)

$\forall x \in \mathbb{R}, y \in \mathbb{R}^+$, $\exists n \in \mathbb{N}$ *s.t.* $ny > x$.

♡

2.2 Metric Spaces and Normed Spaces (@ Lec 03 of ECON 204)

2.2.1 Metric Space $(\mathbb{X}, d)$ and Metric $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$

Definition 2.2 (Metric Space)

A **metric space** is a pair $(\mathbb{X}, d)$, where $\mathbb{X}$ is a set and $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ a function satisfying

1. Non-negative: $d(x, y) \geq 0$, $d(x, y) = 0 \Leftrightarrow x = y, \forall x, y \in \mathbb{X}$.
2. Symmetric: $d(x, y) = d(y, x), \forall x, y \in \mathbb{X}$.
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathbb{X}$.

A function $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ satisfying 1-3 is called a **metric** on $\mathbb{X}$. ♣

A metric gives a notion of distance between elements of $\mathbb{X}$.

2.2.2 Norm $\|\cdot\|$ and Normed Vector Space $(V, \|\cdot\|)$

Definition 2.3 (Norm)

Let V be a vector space over $\mathbb{R}$. A **norm** on V is a function $\|\cdot\| : V \to \mathbb{R}_+$ satisfying

1. Non-negative: $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in V$.
2. Triangle inequality: $\|x + y\| \leq \|x\| + \|y\|, \forall x, y \in V$.
3. $\|\alpha x\| = |\alpha|\|x\|, \forall \alpha \in \mathbb{R}, x \in V$.

♣

A norm gives a notion of length of a vector in V.

Definition 2.4 (Normed Vector Space)

A **normed vector space** is a vector space over $\mathbb{R}$ equipped with a norm, $(V, \|\cdot\|)$. ♣

Example 2.1 Normed Vector Space

- $\mathbf{E}^n$: n-dimensional Euclidean space.

$$V = \mathbb{R}^n, \|x\|_2 = |x| = \sqrt{\sum_{i=1}^{n} (x_i)^2}$$

- $V = \mathbb{R}^n, \|x\|_1 = \sum_{i=1}^n |x_i|$ (the "taxi cab" norm or L^1 norm)
- $V = \mathbb{R}^n, \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\}$ (the maximum norm, or sup norm, or L^∞ norm)
- $C([0,1]), \|f\|_\infty = \sup\{|f(t)| : t \in [0,1]\}$
- $C([0,1]), \|f\|_2 = \sqrt{\int_0^1 (f(t))^2 dt}$
- $C([0,1]), \|f\|_1 = \int_0^1 |f(t)| dt$

where $C([0,1])$ is the space of all continuous real-valued functions on $[0,1]$.

2.2.3 Theorem: metric can be defined by norm

In an abstract normed vector space, the norm can be used analogously to define a notion of distance.

Theorem 2.3

Let $(V, \|\cdot\|)$ *be a normed vector space. Let* $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$ *be defined by* $d(v, w) = \|v - w\|$*. Then* (V, d) *is a metric space.*

♡

2.2.4 Cauchy-Schwarz Inequality

Theorem 2.4 (Cauchy-Schwarz Inequality)

If $v, w \in \mathbb{R}^n$*, then*

$$\left(\sum_{i=1}^{n} v_i w_i\right)^2 \le \left(\sum_{i=1}^{n} v_i^2\right)\left(\sum_{i=1}^{n} w_i^2\right)$$

$$\|v \cdot w\|^2 \le \|v\|^2 \|w\|^2$$

$$\|v \cdot w\| \le \|v\|\|w\|$$

♡

2.2.5 Lipschitz-equivalent Norm

Definition 2.5 (Lipschitz-equivalent)

Two norms $\|\cdot\|$ and $\|\cdot\|^*$ on the same vector space V are said to be **Lipschitz-equivalent** (or **equivalent**) if $\exists m, M > 0$ s.t. $\forall x \in V$,

$$m\|x\| \le \|x\|^* \le M\|x\|$$

Equivalently, $\exists m, M > 0$ s.t. $\forall x \in V$, $x \neq 0$,

$$m \le \frac{\|x\|^*}{\|x\|} \le M$$

♣

Theorem 2.5

All norms on $\mathbb{R}^n$ *are equivalent.*

♡

However, infinite-dimensional spaces support norms that are not equivalent. For example, on $C([0, 1])$, let f_n be the function

$$f_n(t) = \begin{cases} 1 - nt, & \text{if } t \in [0, \frac{1}{n}] \\ 0, & \text{if } t \in (\frac{1}{n}, 1] \end{cases}$$

Then $\frac{\|f_n\|_1}{\|f_n\|_\infty} = \frac{1}{2n} \to 0$, which means there is no lower bound $m > 0$.

2.2.6 Ball, Radius, Diameter, and Distance

In a metric space (X, d), define

$$\begin{aligned} B_\varepsilon(x) &= \{y \in X : d(y, x) < \varepsilon\} \\ &= \text{open ball with center } x \text{ and radius } \varepsilon \\ B_\varepsilon[x] &= \{y \in X : d(y, x) \leq \varepsilon\} \\ &= \text{closed ball with center } x \text{ and radius } \varepsilon \end{aligned}$$

We can use the metric d to define a generalization of ”radius”. In a metric space (X, d), define the *diameter* of a subset $S \subseteq X$ by

$$\operatorname{diam}(S) = \sup \left\{d\left(s, s'\right) : s, s' \in S\right\}$$

Similarly, we can define the *distance from a point to a set*, and *distance between sets*, as follows:

$$\begin{aligned} d(A, x) &= \inf_{a \in A} d(a, x) \\ d(A, B) &= \inf_{a \in A} d(B, a) \\ &= \inf\{d(a, b) : a \in A, b \in B\} \end{aligned}$$

2.3 Set Theory

2.3.1 Well Defined Set

Definition 2.6

*A set S is **well defined** if an object a is either $a \in S$ or $a \notin S$.* ♣

2.3.2 Numerically Equivalent (@ Lec 01 of ECON 204)

Definition 2.7

Two sets A, B are **numerically equivalent** (or have the same cardinality) if there is a bijection $f : A \to B$, that is, 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and onto ($\forall b \in B, \exists a \in A$ s.t. $f(a) = b$). ♣

2.3.3 Finite, Countable Set (@ Lec 01 of ECON 204)

Definition 2.8 (Finite Set)

A set is either **finite** or **infinite**. A set is **finite** if it is numerically equivalent to $\{1, ..., n\}$ for some n. A set that is not finite is infinite. ♣

We give a more precise definition to classify infinite set:

Definition 2.9 (Countable Set)
An infinite set is **countable** if it is numerically equivalent to $\mathbb{N}$. An infinite set that is not countable is called **uncountable**.

Theorem 2.6 (Countable $\mathbb{Q}$)
The set of rational numbers $\mathbb{Q}$ is countable.

2.3.4 Power Set (@ Lec 02 of ECON 204)

Definition 2.10 (Power Set: the set of all subsets)
For any set A, we denote by $\mathcal{P}(A)$ the collection of all subsets of A. $\mathcal{P}(A)$ is the ***power set*** *of A.*

We may also use the notation 2^A (in Berkeley ECON 204).

2.3.5 Theorem (Cantor): The power set of $\mathbb{N}$ is uncountable (@ Lec 02 of ECON 204)

Theorem 2.7 (Cantor)
$\mathcal{P}(\mathbb{N})$ (or denoted by $2^{\mathbb{N}}$), the set of all subsets of $\mathbb{N}$, is uncountable.

2.3.6 Cardinalities of Sets (@ Lec 02 of ECON 204)

Definition 2.11 (Cardinality)
If A is a set, $|A|$ = cardinality of $A = \#$ of elements

$n \in \mathbb{N}, |\{1, \ldots n\}| = n$; $|\emptyset| = 0$($\emptyset =$ empty set).

Proposition 2.5 (Facts about cardinality)

1. *If A is numerically equivalent to $\{1, ..., n\}$ for some $n \in \mathbb{N}$, then $|A| = n$.*
2. *A and B are numerically equivalent if and only if $|A| = |B|$.*
3. *If $|A| = n$ (finite) and A is a proper subset of B (that is, $A \subset B$ and $A \neq B$) then $|A| < |B|$.*
4. *If A is countable and B is uncountable, then $n < |A| < |B|, \forall n \in \mathbb{N}$.*
5. *If $A \subseteq B$, then $|A| \leq |B|$. (if B is countable and $A \subseteq B$, then A is at most countable, that is, A is either empty, finite, or countable.)*
6. *If there is an injection $\sigma : A \to B$, we can write $|A| \leq |B|$;*
7. *If there is a surjection $\sigma : A \to B$, we can write $|A| \geq |B|$;*
8. *If there is a bijection $\sigma : A \to B$, we can write $|A| = |B|$.*

2.3.7 Pigeonhole Principle: $|A| > |B| \Rightarrow$ no injective function $\sigma : A \to B$

Theorem 2.8 (Pigeonhole Principle)

If A and B are sets and $|A| > |B|$, then there is no injective function $\sigma : A \to B$. ♡

2.3.8 B^A: Sets of Function

If A, B are sets, then $B^A = \{\sigma : A \to B | \sigma \textit{ a function}\}$.

Example 2.2 $n \in \mathbb{Z}$, we define a function $f : B^{\{1,\ldots,n\}} \to B^n (= B \times B \times B \times \cdots \times B)$ by the equation $f(\sigma) = \{\sigma(1), ..., \sigma(n)\}$, *where* $\sigma : \{1, \ldots, n\} \to B$. The f is a *bijection*.

Proof 2.1

1. *Injective:*

$$f(\sigma_1) = f(\sigma_2) \Rightarrow \{\sigma_1(1), ..., \sigma_1(n)\} = \{\sigma_2(1), ..., \sigma_2(n)\}$$

$$\textit{Since } \sigma : \{1, \ldots, n\} \to B, \textit{ it is sufficient to prove } \sigma_1 = \sigma_2.$$

2. *Surjective:*

$$\forall \{b_1, ..., b_n\}, \textit{ we have } \sigma^*(x) = b_x, x = 1, ..., n. \textit{ s.t. } f(\sigma^*) = \{b_1, ..., b_n\}$$

Example 2.3

$$C(\mathbb{R}, \mathbb{R}) = \{\textit{continuous functions } \sigma : \mathbb{R} \to \mathbb{R}\} \subset \mathbb{R}^{\mathbb{R}}$$

2.3.9 Bounded Set (@ Lec 03 of ECON 204)

Definition 2.12 (Bounded Set)

In a metric space (X, d), a subset $S \subseteq X$ is **bounded** if $\exists x \in X, \beta \in \mathbb{R}$ such that $\forall s \in S$, $d(s, x) \leq \beta$. ♣

Equivalent: $\exists \beta > 0$ and $s \in S$ such that $S \subseteq B_\beta(s)$.

2.3.10 Open, Closed Set (@ Lec 04 of ECON 204)

Definition 2.13 (Open Sets)

*Let (X, d) be a metric space. A set $\mathbb{X} \subseteq \mathbb{R}^n$ is **open** if*

$\forall x \in \mathbb{X}$ we can draw a ball around x that is contained in $\mathbb{X}$.

i.e. $\forall x \in \mathbb{X}, \exists \varepsilon > 0$ *s.t.* $B_\varepsilon(x) = \{y : d(y, x) < \varepsilon\} \subseteq \mathbb{X}$ ♣

Definition 2.14 (Closed Sets)

*$\mathbb{X}$ is **closed** if $\mathbb{X}^c$ is open.* ♣

Theorem 2.9 (Equivalent definition: Closed Sets)

Equivalent: if A in a metric space (X, d) contains all limit points of all sequences in A, A is closed.

$$\{x_n\} \subset A, x_n \to x \in X \Rightarrow x \in A$$

♡

Example 2.4 (Closed and Open Sets on $\mathbb{E}_1$ i.e., $\mathbb{R}$ with the usual Euclidean metric)

1) $(1, 2) = \{x \in \mathbb{R} : 1 < x < 2\}$ - open
2) $\mathbb{R}$ is both open and closed
3) $(-\infty, 1) = \{x \in \mathbb{R} : x < 1\}$ - open
4) $[1, \infty)$ is closed because its complement open
5) $(1, 2]$ is neither open nor closed

Example 2.5 (Closed and Open Sets on other metric space) In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon\} = [0, \varepsilon)$.

Theorem 2.10 (Empty Set and Full Set are both open and closed)

In any metric space (X, d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed.

♡

Theorem 2.11 (Union of open sets is open, Intersection of finite open sets is open)

In any metric space (X, d),

1. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
2. *The intersection of a finite collection of open sets is open.*

♡

2.3.11 Interior, Exterior, Boundary, Closure (@ Lec 04 of ECON 204)

Given a set $S \subseteq X$, the **point** of X can be classified into three types relative to S:

- **Interior (points)**, denoted $int(S)$: $\vec{x} \in S$ for which there exists some $B(\vec{x}, r) \subseteq S$, is the largest open set contained in S (the union of all open sets contained in S).
- **Exterior (points)**, denoted $ext(S)$: $\vec{x} \notin S$ for which there exists some $B(\vec{x}, r)$ containing no points of S, is the largest open set contained in $X \backslash S$.
- **Boundary (points)** denoted $\partial(S)$ or $bd(S)$: all other points (for which any $B(\vec{x}, r)$ will contain some points of S and some points outside S).
- **Closure of** S, denoted $\bar{S}$ or $cl(S) = int(S) \cup bd(S)$, is the smallest closed set containing S (the intersection of all closed sets containing S).
- Moreover, boundary satisfies $\partial(S) = \overline{(X \backslash S)} \cap \bar{S}$.

(1) A set S is **open** if $S = int(S)$ - i.e., if S does not contain any of its boundary points.

(2) A set S is **closed** if $S = \bar{S} = int(S) \cup bd(S)$ - i.e., if S contains all of its boundary points.

2.3.12 Compact Set

Definition 2.15 (Compact Set)

$\mathbb{X} \subseteq \mathbb{R}^n$ is compact of it is closed and bounded. ♣

Example 2.6 Compact Set $[1,2] = \{x \in \mathbb{R} : 1 \leqslant x \leqslant 2\}$; $\left\{x \in \mathbb{R}^2 : x_1^2 + x_2^2 \leqslant 4\right\}$

2.3.13 Sublevel Set

Definition 2.16 (Sublevel Set)

The sublevel set of a function $f : \mathbb{R}^n \to \mathbb{R}$ (for some level $c \in \mathbb{R}$) is the set

$$\overline{L_c} = \{x \in \mathbb{R}^n : f(x) \leq c\}$$

♣

2.3.14 Set Operations

Definition 2.17

A binary operation on a set A is a function $* : A \times A \to A$.

The operation is *associative* if $a * (b * c) = (a * b) * c, \forall a, b, c \in A$.

The operation is *commutative* if $a * b = b * a, \forall a, b \in A$. ♣

Example 2.7

$+, \circ$ are both *associative* and *commutative* operations on $\mathbb{Z}, \mathbb{N}, \mathbb{Q}, \mathbb{R}$; $-$ is a neither *associative* nor *commutative* operation on $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, but not $\mathbb{N}$.

Definition 2.18

A subset $H \subset S$ is closed under $*$ if $a * b \in H$ for all $a, b \in H$. ♣

Definition 2.19

$*$ has identity element $e \in S$ if $a * e = e * a = a$ for all $s \in S$. ♣

2.4 Sequences

Sequences $\{x_k\}_{k=1}, \ldots$ or $\{x_k\}, x_k \in \mathbb{R}^n$

Definition 2.20 (Subsequence)

Suppose $\{x_n\}$ is a sequence and $n_1 < n_2 < \cdots$, then $\{x_{n_k}\}$ is called a **subsequence**. ♣

2.4.1 Convergence of Sequences (@ Lec 03 of ECON 204)

Definition 2.21 (Convergence: note $x_k \to x, \lim_{k\to\infty} x_k = x$)

Let (X, d) be a metric space. A sequence $\{x_k\}$ converges to x (written $x_k \to x$ or $\lim_{k\to\infty} x_k = x$) if $\forall \varepsilon > 0, \exists N_\varepsilon$ s.t. $d(x_k, x) < \varepsilon,\ \forall k \geqslant N_\varepsilon$ ♣

Definition 2.22 (Limit point)

x is a limit point of $\{x_k\}$ if $\exists$ a subsequence of $\{x_k\}$ that converges to x. ♣

Theorem 2.12 (Uniqueness of Limits)

In a metric space (X, d), if $x_k \to x$ and $x_k \to x'$, then $x = x'$. ♡

2.4.2 Cluster Point (@ Lec 03 of ECON 204)

Definition 2.23 (Cluster Point)

An element c is a **cluster point** of a sequence $\{x_n\}$ in a metric space (X, d) if $\forall \varepsilon > 0, \{n : x_n \in B_\varepsilon(c)\}$ is an infinite set. Equivalently,

$$\forall \varepsilon > 0, N \in \mathbb{N}, \exists n > N \text{ s.t. } x_n \in B_\varepsilon(c)$$ ♣

Example 2.8 $x_n = \begin{cases} 1 - \frac{1}{n}, & \text{if } n \text{ even} \\ \frac{1}{n}, & \text{if } n \text{ odd} \end{cases}$ has cluster points $\{0, 1\}$.

Theorem 2.13 (Cluster Point $\Leftrightarrow$ exists subsequence converges to it)

Let (X, d) be a metric space, $c \in X$, and $\{x_n\}$ a sequence in X. Then c is a cluster point of $\{x_n\}$ if and only if there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = c$. ♡

2.4.3 Proposition: Sequences in $\mathbb{R}^n$, Bounded above and non-decreasing sequence $\Rightarrow$ Converge

Proposition 2.6

*If $\{x_k\}$ is bounded above(below) and non-decreasing(non-increasing) it **converges**.* ♠

2.4.4 Theorem: Increasing/Decreasing Sequences in $\mathbb{R}^n$, Limit is sup/inf (@ Lec 03 of ECON 204)

Theorem 2.14

Let $\{x_n\}$ be an increasing (decreasing) sequence of real numbers. Then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$ ($\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$). In particular, the limit exists. ♡

2.4.5 Definition: $\limsup$ and $\liminf$ (@ Lec 03 of ECON 204)

Consider a sequence $\{x_n\}$ of real numbers. Let

$$\alpha_n = \sup\{x_k : k \geq n\}$$

and

$$\beta_n = \inf\{x_k : k \geq n\}$$

Either $\alpha_n = +\infty$ for all n, or $\alpha_n \in \mathbb{R}$ and $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq ....$ Either $\beta_n = -\infty$ for all n, or $\beta_n \in \mathbb{R}$ and $\beta_1 \leq \beta_2 \leq \beta_3 \leq ....$

Definition 2.24 (Lim Sups and Lim Infs)

$$\limsup_{n\to\infty} x_n = \begin{cases} +\infty & \text{if } \alpha_n = +\infty \text{ for all } n \\ \lim \alpha_n & \text{otherwise.} \end{cases}$$

$$\liminf_{n\to\infty} x_n = \begin{cases} -\infty & \text{if } \beta_n = -\infty \text{ for all } n \\ \lim \beta_n & \text{otherwise.} \end{cases}$$

♣

2.4.6 Exists Limit $\Leftrightarrow \lim = \limsup = \liminf$ (@ Lec 03 of ECON 204)

Theorem 2.15

Let $\{x_n\}$ be a sequence of real numbers. Then

$$\lim_{n\to\infty} x_n = \gamma \in \mathbf{R} \cup \{-\infty, \infty\}$$

$$\Leftrightarrow \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \gamma$$

♡

2.4.7 Rising Sun Lemma: Sequence in $\mathbb{R}^n$ contains monotone subsequence (@ Lec 03 of ECON 204)

Theorem 2.16 (Rising Sun Lemma)

Every sequence of real numbers contains an increasing subsequence or a decreasing subsequence or both.

♡

2.4.8 Bolzano-Weierstrass: Bounded Sequence in $\mathbb{R}^n$ contains a convergent subsequence (@ Lec 03 of ECON 204)

Theorem 2.17 (Bolzano-Weierstrass)

Every bounded sequence of real numbers contains a convergent subsequence. ♡

2.5 Complete Metric Spaces (@ Lec 05 of ECON 204)

Roughly, a metric space is complete if "every sequence that ought to converge to a limit has a limit to converge to."

2.5.1 Cauchy Sequence

Definition 2.25 (Cauchy Sequence)

*A sequence $\{x_k\}$ in a metric space (X, d) is **Cauchy** if $\forall \varepsilon > 0, \exists N(\varepsilon)$ s.t.*

$$d(x_n, x_m) < \varepsilon, \ \forall n, m > N(\varepsilon)$$

♣

Note: Cauchy property depends only on the sequence and the metric d, not on the ambient metric space.

Note: In $\mathbb{E}^1 = (\mathbb{R}, \|\cdot\|)$, $\{x_k\}$ converges $\iff$ $\{x_k\}$ is Cauchy.

A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to. Any sequence that does converge must be Cauchy, however, by the argument above.

2.5.2 Theorem: Convergent $\Rightarrow$ Cauchy

Theorem 2.18 (Convergent $\Rightarrow$ Cauchy)

Every convergent sequence in a metric space is Cauchy. ♡

2.5.3 Complete Metric Space and Banach Space

Definition 2.26 (Complete Metric Spaces)

A metric space (X, d) is **complete** if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$. ♣

Example 2.9

1. Consider the earlier example of $X = (0, 1]$ with d the usual Euclidean metric. Since $x_n = \frac{1}{n}$ is Cauchy but does not converge in that metric space, $((0, 1], d)$ is not complete.
2. $\mathbb{Q}$ is not complete in the Euclidean metric: $x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n} \to \sqrt{2}$.

Definition 2.27 (Banach Space)

A **Banach space** is *a normed space* that is complete in the metric generated by its norm. ♣

Theorem 2.19 ($\mathbb{E}^1$ **is complete)**

$\mathbb{R}$ is complete with the usual metric (so $\mathbb{E}^1$ is a Banach space). ♡

Theorem 2.20 ($\mathbb{E}^n$ **is complete)**

$\mathbb{E}^n$ is complete for every $n \in \mathbb{N}$. ♡

Theorem 2.21 ($(C(X), \|\cdot\|_\infty)$ **is Complete)**

Given $X \subseteq \mathbb{R}^n$, let $C(X)$ be the set of bounded continuous functions from X to R with

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Then $C(X)$ is a Banach space. ♡

2.5.4 Theorem: Subset of Complete Metric Space is Complete ⇔ Subset is Closed

Theorem 2.22 (Subset of Complete Metric Space is Complete ⇔ Subset is Closed)

Suppose (X, d) is a complete metric space and $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if Y is a closed subset of X. (where $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}_+$; $d|_Y : \mathbb{Y} \times \mathbb{Y} \to \mathbb{R}_+$) ♡

2.6 Compactness over Metric Spaces (@ Lec 06 of ECON 204)

2.6.1 Open Cover

Definition 2.28 (Open Cover)

A collection of sets $\mathcal{U} = \{U_\lambda : \lambda \in \Lambda\}$ in a metric space (X, d) is an **open cover** of $A \subseteq X$ if

1. U_λ is open for all $\lambda \in \Lambda$ and
2. $A \subseteq \cup_{\lambda \in \Lambda} U_\lambda$

♣

2.6.2 Definition of Compact Set

Definition 2.29 (Compact Set)

A set A in a metric space is **compact** if every open cover of A contains a finite subcover of A. In other words, if $\{U_\lambda : \lambda \in \Lambda\}$ is an open cover of A, there exist $n \in N$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that $A \subseteq \cup_{i=1}^n U_{\lambda_i}$ ♣

Example 2.10

1. $(0, 1]$ is not a compact set over $\mathbb{E}^1$: Consider the open cover $\{(\frac{1}{m}, 2) : m \in \mathbb{N}\}$.
2. $[0, \infty)$ is not a compact set over $\mathbb{E}^1$: Consider the open cover $\{(-1, m) : m \in \mathbb{N}\}$.

2.6.3 Theorem: Closed Subset of Compact Set is Compact

Theorem 2.23 (Closed Subset of Compact Set is Compact)

Every closed subset A of a compact metric space (X, d) is compact.

Proof 2.2

Let $\{U_\lambda : \lambda \in \Lambda\}$ be an open cover of A.

Let $U'_\lambda = U_\lambda \cup (X\backslash A)$. Since A is closed, $X\backslash A$ is open; Since U_λ and $X\backslash A$ are both open, U'_λ is open.

By the definition of open set, if $a \in A$, $a \in \cup_{\lambda\in\Lambda} U_\lambda$. If $a \in X\backslash A$, $a \in U'_\lambda$ for sure. So, $X \subseteq \cup_{\lambda\in\Lambda} U'_\lambda$.

Hence, we can conclude $\{U_\lambda \cup (X\backslash A) : \lambda \in \Lambda\}$ is an open cover of X.

Since X is compact, there exists $n \in N$ and $\lambda_1, \cdots, \lambda_n \in \Lambda$ such that $X \subseteq \cup_{i=1}^n U'_{\lambda_i}$.

Then,

$$\begin{aligned} a \in A &\Rightarrow a \in X \\ &\Rightarrow a \in U'_{\lambda_i} \text{ for some } i \\ &\Rightarrow a \in U_{\lambda_i} \text{ for some } i \end{aligned}$$

Hence, we can conclude $A \subseteq \cup_{i=1}^n U_{\lambda_i}$. Thus, A is compact.

2.6.4 Theorem: Compact Set over Metric Space is Closed

Theorem 2.24 (Compact Set over Metric Space is Closed)

If A is a compact subset of the metric space (X, d), then A is closed.

Proof 2.3

Suppose $X\backslash A$ is not open, so we can find a point $x \in X\backslash A$ such that, for every $\varepsilon > 0$, $A \cap B_\varepsilon(x) \neq \emptyset$, and hence $A \cap B_\varepsilon[x] \neq \emptyset$. For $n \in \mathbb{N}$, let

$$U_n = X\backslash B_{1/n}[x]$$

Each U_n is open, and

$$\cup_{n\in\mathbb{N}} U_n = X\backslash\{x\} \supseteq A$$

since $x \notin A$. Therefore, $\{U_n : n \in \mathbb{N}\}$ is an open cover for A. Since A is compact, there is a finite *subcover* $\{U_{n_1}, \ldots, U_{n_k}\}$. *Let* $n = \max\{n_1, \ldots, n_k\}$. *Then*

$$\begin{aligned} U_n &= X \backslash B_{1/n}[x] \\ &\supseteq X \backslash B_{1/n_j}[x] (j = 1, \ldots, k) \\ U_n &\supseteq \cup_{j=1}^{k} U_{n_j} \\ &\supseteq A \end{aligned}$$

But $A \cap B_{1/n}[x] \neq \emptyset$, *so* $A \nsubseteq X \backslash B_{1/n}[x] = U_n$. *This is a contradiction, which proves that* A *is closed.*

2.6.5 Definition of Sequentially Compact

Definition 2.30 (Sequentially Compact)

A set A in a metric space (X, d) is **sequentially compact** if every sequence of elements of A contains a convergent subsequence whose limit lies in A. ♣

2.6.6 Theorem: Compact ⇔ Sequentially Compact

Theorem 2.25 (Compact ⇔ Sequentially Compact)

A set A *in a metric space* (X, d) *is* ***compact*** *if and only if it is* ***sequentially compact****.* ♡

2.6.7 Definition of Totally Bounded

Definition 2.31 (Totally Bounded)

A set A in a metric space (X, d) is **totally bounded** if, for every $\varepsilon > 0$,

$$\exists x_1, ..., x_n \in A \text{ s.t. } A \subseteq \cup_{i=1}^{n} B_\varepsilon(x_i).$$ ♣

Claim 2.2

"Totally bounded" is stronger than "bounded". ♠

Example 2.11

1. $X = [0, 1]$ is totally bounded with Euclidean metric: Given $\varepsilon > 0$, let $x_i = \frac{i}{n}, i = 1, ..., n-1$.
2. $X = [0, 1]$ is not totally bounded with discrete metric $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$: Given $\varepsilon = \frac{1}{2}$, $B_\varepsilon(x_i) = \{x_i\}$. Hence, $\cup_i^n B_\varepsilon(x_i) = \{x_1, ..., x_n\} \nsupseteq [0, 1]$.

   Note: X is bounded but not totally bounded.

2.6.8 Theroem: Compact $\Leftrightarrow$ Complete/Closed and Totally Bounded

Theorem 2.26 (Subset is Compact $\Leftrightarrow$ Complete and Totally Bounded)

*Let A be a subset of a metric space (X, d). Then A is **compact** if and only if A is **complete** and **totally bounded**.*

Proof 2.4

Compact $\Rightarrow$ totally bonded (by definition). Compact $\Leftrightarrow$ sequentially compact $\Rightarrow$ every Cauchy sequence converges to a limit in the subset (i.e., Complete).

Conversely, suppose complete and totally bounded. Because totally bounded, we can extract a Cauchy subsequence. Because complete, subsequence converges to a limit in the subset, which shows sequentially compact and hence compact.

Corollary 2.1 (Subset is Compact $\Leftrightarrow$ Closed and Totally Bounded)

*Let A be a subset of a metric space (X, d). Then A is **compact** if and only if A is **closed** and **totally bounded**.*

Proof 2.5

Directly by theorem 2.22 and theorem 2.26.

2.6.9 Heine-Borel Theorem: With Euclidean Metric, Compact $\Leftrightarrow$ Closed and Bounded

Claim 2.3 (With Euclidean Metric, Compact $\Leftrightarrow$ Closed and Bounded)

*If $A \subseteq \mathbb{E}^n$, then A is **totally bounded** if and only if A is **bounded**.*

Theorem 2.27 (Heine-Borel Theorem: With Euclidean Metric, Compact $\Leftrightarrow$ Closed and Bounded)

*If $A \subseteq \mathbb{E}^n$, then A is **compact** if and only if A is **closed** and **bounded**.*

2.6.10 Theorem: Continuous Images of Compact Set is Compact

Theorem 2.28 (Continuous Images of Compact Set is Compact)

Let (X, d) and (Y, ρ) be metric spaces. If $f : X \to Y$ is continuous and C is a compact subset of (X, d), then $f(C)$ is compact in (Y, ρ).

Chapter 3 Functions

3.1 Definitions of Function

Definition 3.1 (Function)

Function is a rule $\sigma : A \to B$ that assigns an element B to *every* element of A. $\forall a \in A, \sigma(a) \in B$.

3.1.1 Image, Preimage, Fiber

Definition 3.2

1. A is the domain of σ, B is the range of σ.
2. We call $\sigma(a) =$ *value of* σ *at* a as the image of a.
3. A set $C \subset B$, we call $\sigma^{-1}(C) = \{a \in A | \sigma(a) \in C\}$ as the *preimage* of C.
4. An element $b \in B$, we call $\sigma^{-1}(b) = \{a \in A | \sigma(a) = b\}$ as the *fiber* of b.

3.1.2 Composition of functions

Definition 3.3 (Function Composition)

The function composition $\circ$ is an operation that takes two functions $\sigma : A \to B$ and $\tau : B \to C$, , and produces a function $\tau \circ \sigma : A \to C$ that fulfills $\forall a \in A,\ (\tau \circ \sigma)(a) = \tau(\sigma(a))$.

3.1.3 Function Composition is Associative

Proposition 3.1 (Associativity of Functions)

Suppose $\sigma : A \to B, \tau : B \to C, \rho : C \to D$ *are functions and* $\circ$ *is the function composition, then* $\rho \circ (\tau \circ \sigma) = (\rho \circ \tau) \circ \sigma$.

3.1.4 Homeomorphism (@ Lec 04 of ECON 204)

Definition 3.4 (Homeomorphism)

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called a **homeomorphism** if it is one-to-one, onto, continuous, and its inverse function is continuous.

3.2 Injection, Surjection, Bijection

3.2.1 Definitions: Injective, surjective, bijective

A function $\sigma : A \to B$ is called,

1. *Injective (1 to 1)*

$$\forall a_1, a_2 \in A, \sigma(a_1) = \sigma(a_2) \Rightarrow a_1 = a_2$$

2. *Surjective (onto)*

$$\forall b \in B, \exists a \in A, s.t. \sigma(a) = b$$

3. *Bijective* (if injective and surjective)

3.2.2 Lemma 1.1.7: injective/surjective/bijective is preserved in composition

Lemma 3.1 (Lemma 1.1.7)

Suppose $\sigma : A \to B, \tau : B \to C$ are functions,

If σ, τ are injective, then $\tau \circ \sigma$ is injective.

If σ, τ are surjective, then $\tau \circ \sigma$ is surjective.

If σ, τ are bijective, then $\tau \circ \sigma$ is bijective.

♡

3.2.3 Proposition 1.1.8: A function is bijection if there exist inverse

Proposition 3.2 (Proposition 1.1.8)

A function $\sigma : A \to B$ is a bijection if $\exists$ a function $\tau : B \to A$ such that

$$\sigma \circ \tau = id_B = \textit{identity on } B(id_B(x) = x, \forall x \in B)$$

$$\tau \circ \sigma = id_A$$

♠

Such τ is unique, called inverse of σ, $\tau = \sigma^{-1}$.

3.3 Function Continuity

3.3.1 Continuous Function in $\mathbb{R}$ with Euclidean norm

Definition 3.5 (Continuity at Point)

A real-valued function f is **continuous** at x if

"For every $\{x_k\}$ converging to x satisfies that $\lim_{k\to\infty} f(x_k) = f(x)$".

Equivalent definition:

$\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\|y - x\| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$ ♣

Continuity at x_0 requires:

1. $f(x_0)$ is defined; and
2. either

    - x_0 is an isolated point of X, i.e. $\exists \varepsilon > 0$ s.t. $B_\varepsilon(x) = \{x\}$; or

    - $\lim_{x\to x_0} f(x)$ exists and equals $f(x_0)$

Definition 3.6 (Continuous Function)

A real-valued function f is continuous if it is continuous at all points in its domain. ♣

3.3.2 Continuous Function in Metric Spaces (@ Lec 04 of ECON 204)

Definition 3.7 (Continuity in Metric Spaces)

Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is **continuous** at a point $x_0 \in X$ if $\forall \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0$ s.t. $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$. ♣

3.3.3 Theorem: Continuous $\Leftrightarrow$ Preimage of open set is open (@ Lec 04 of ECON 204)

Theorem 3.1 (Continuous $\Leftrightarrow$ Preimage of open set is open)

*Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$. Then f is **continuous** if and only if*

$$f^{-1}(A) = \{x \in X : f(x) \in A\} \text{ is open in } X \ \forall A \subseteq Y \text{ s.t. } A \text{ is open in } Y$$ ♡

3.3.4 Theorem: Continuity is preserved in composition (@ Lec 04 of ECON 204)

Theorem 3.2 (Continuity is preserved in composition)

Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous. ♡

Proof 3.1

Proved by previous theorem 3.1.

3.3.5 Uniform Continuity (@ Lec 04 of ECON 204)

Definition 3.8 (Uniformly Continuous)

Suppose $f : (X, d) \to (Y, \rho)$. f is **uniformly continuous** if

$$\forall \varepsilon > 0, \exists \delta(\varepsilon) > 0 \text{ s.t. } \forall x_0 \in X, d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

♣

Claim 3.1

Uniformly Continuous *implies (is stronger than)* ***Continuous****.*

(f is continuous if $\forall x_0 \in X, \varepsilon > 0, \exists \delta(x_0, \varepsilon) > 0$ s.t. $d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$)

♠

Given $\varepsilon > 0$, "uniformly continuous" requires $\delta(\varepsilon)$ that works for all $x_0 \in X$.

3.3.6 Theorem: Continuous function with Compact domain is Uniformly Continuous (@ Lec 06 of ECON 204)

Theorem 3.3 (Continuous function with Compact domain is Uniformly Continuous)

Let (X, d) and (Y, ρ) be metric spaces, C a ***compact*** *subset of X, and $f : C \to Y$ a* ***continuous*** *function. Then f is* ***uniformly continuous*** *on C.*

♡

Proof 3.2

3.3.7 Lipschitz Continuous (@ Lec 04 of ECON 204)

Definition 3.9 (Lipschitz (Continuous) in Normed Vector Space)

Let X, Y be normed vector spaces, $\mathbb{E} \subseteq X$.

(1). A function $f : X \to Y$ is ***Lipschitz*** *on $\mathbb{E}$ satisfies*

$$\exists \gamma > 0, \|f(\mathbf{x}) - f(\mathbf{y})\|_Y \leq \gamma \|\mathbf{x} - \mathbf{y}\|_X, \forall \mathbf{x}, \mathbf{y} \in \mathbb{E}$$

or we call ***γ-Lipschitz continuous****;*

(2). f is ***locally Lipschitz*** *on $\mathbb{E}$ if*

$$\forall x_0 \in \mathbb{E}, \exists \varepsilon > 0 \text{ s.t. } f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap \mathbb{E}$$

♣

Claim 3.2

1). *If f is γ-Lipschitz continuous, then it is also $(\gamma+1)$-Lipschitz continuous. The minimal such γ is called a Lipschitz constant of function f*

2). *The definition can be extended to general metric spaces by replacing $\|\cdot\|_X$ and $\|\cdot\|_Y$ to general metrics.*

Example 3.1 (Lipschitz)

1. $f(x) = 2x$ is 2-Lipschitz continuous;
2. What about $f(\mathbf{x}) = \mathbf{Ax}$, where $\mathbf{A}$ is a matrix? Spectral norm $\|\mathbf{A}\|_2$ (for Euclidean norm).
3. What about $f(x) = x^2$? Not Lipschitz continuous, or the Lipschitz constant is ∞.

Example 3.2 (Locally Lipschitz) Every C^1 function is locally Lipschitz. (Recall that a function $f : \mathbb{R}^m \to \mathbb{R}^n$ is said to be C^1 if all its first partial derivatives exist and are continuous.)

Claim 3.3 (Lipschitz continuity vs. continuity vs. uniform continuity)

Lipschitz continuity is stronger than either continuity or uniform continuity:

- *locally Lipschitz $\Rightarrow$ continuous*
- *Lipschitz $\Rightarrow$ uniformly continuous*

3.3.8 Contraction (from the view of Lipschitz constant)

a). If the Lipschitz constant $\gamma \leq 1$, then f is called a non-expansive mapping.

b). If the Lipschitz constant $\gamma < 1$, then f is called a contraction mapping.

Example 3.3

1. $f(x) = 2x$ is not a contraction mapping; $f(x) = 0.5x$ is.
2. $f(x) = Ax$ is a contraction mapping (with respect to Euclidean norm) iff $\|A\|_2 < 1$.

3.4 Extreme Value, Intermediate Value of Functions

3.4.1 Intermediate Value Theorem (@ Lec 02 of ECON 204)

Theorem 3.4 (Intermediate Value Theorem)

Suppose $f : [a,b] \to \mathbb{R}$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = d$.

3.4.2 Coercive Function

Definition 3.10 (Coercive)

A real-valued function $f : \mathbb{X} \to \mathbb{R}$ *is* <u>*coercive*</u> *if for* ***every*** $\{x_k\} \subset \mathbb{X}$ *s.t.* $\|x_k\| \to \infty$, $f(x_k) \to \infty$ ♣

Example 3.4 Check coercive

1) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2$ - coercive

2) $x \in \mathbb{R}$, $f(x) = 1 - e^{-|x|}$ - not coercive

3) $x \in \mathbb{R}^2$, $f(x) = x_1^2 + x_2^2 - 2x_1x_2$ - not coercive (we need $f(x_k) \to \infty$ for all $\|x_k\| \to \infty$)

3.4.3 Extreme of Functions

Definition 3.11 (Extreme of Functions)

Let $\mathbb{X} \subseteq \mathbb{R}^n$, $f : \mathbb{X} \to \mathbb{R}$

$$\inf_{x \in \mathbb{X}} f(x) = \inf\{f(x) : x \in \mathbb{X}\}$$

♣

If $\exists x^* \in \mathbb{X}$ s.t. $\inf f(x) = f(x^*)$. Then, f achieves (attains) its minimum and $f(x^*) = \min_{x \in \mathbb{X}} f(x)$

x^* is called a **minimizer** of f, written as $x^* \in \arg\min_{x \in \mathbb{X}} f(x)$. If x^* is uniqne, we write $x^* = \arg\min_{x \in \mathbb{X}} f(x)$

Similarly, supremum and maximum of f.

3.4.4 Weierstrass' Theorem (Extreme value Theorem) (@ Lec 06 of ECON 204)

Theorem 3.5 (Weierstrass' Theorem (Extreme value Theorem))

Given a ***compact set*** C *in a metric space* (X, d) *and a* ***continuous*** *function* $f : C \to \mathbb{R}$*, then* f *is bounded on* C *and attains its min and max on* C *i.e.,*

$$\exists x_1 \in C \text{ s.t. } f(x_1) = \inf_{x \in C} f(x)$$

$$\exists x_2 \in C \text{ s.t. } f(x_2) = \sup_{x \in C} f(x)$$

♡

Proof 3.3

(for existence of min*;* max *is similar)*

Let $\{\sigma_k\} \subseteq \mathbb{X}$ *be s.t.* $\inf_{x \in \mathbb{X}} f(x) \leq f(\sigma_k) \leq \inf_{x \in \mathbb{X}} f(x) + \frac{1}{k}$

Then $\lim_{k \to \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

$\mathbb{X}$ *is bounded* $\Rightarrow$ $\{\sigma_k\}$ *has it least one limit point* x_1*,*

$\mathbb{X}$ *is closed* $\Rightarrow$ $x_1 \in \mathbb{X}$

f *is continuous* $\Rightarrow$ $f(x_1) = \lim_{k \to \infty} f(\sigma_k) = \inf_{x \in \mathbb{X}} f(x)$

Corollary 3.1 (Corollary to WT)

Let f be continuous on closed set $\mathbb{X}$ (not necessarily bounded). If f is coercive on $\mathbb{X}$ it attains its min *on* $\mathbb{X}$.

Proof 3.4

Consider $\{\sigma_k\}$ as in proof of WT.

Since f is closed, $f(x) < \infty$, $\forall x \in \mathbb{X}$. And f is coercive on $\mathbb{X}$, which means $f(x) \to \infty$ if $\|x\| \to \infty$.

Hence, $\{\sigma_k\} \in \mathbb{X}$ is bounded. Rest of proof same as proof of WT.

Example 3.5 $f(x) = f(x_1, x_2, x_3) = x_1^4 + 2x_2^2 + e^{-x_3} + e^{2x_3}$

1) *Does f achieve its* min *and* max *on* $\mathbb{X}_1 = \left\{x \in \mathbb{R}^3 : x_1^2 + 2x_2^2 + 3x_3^2 \leqslant 6\right\}$*?*

   - $\mathbb{X}_1$ is compact and f is continuous. Both min and max are achieved (WT).

2) Does f achieve its min and max over $\mathbb{R}^3$?

   - $f \to \infty$ whenever $\|x\| \to \infty \Rightarrow f$ is coercive.

   - $\mathbb{R}^3$ is closed.

   $\Rightarrow f$ achieves its min. on $\mathbb{R}^3$ by corollary to WT.

   - max does not exist since $f \to \infty$ as $\|x\| \to \infty$.

3) Does f achieve its min and max over $\mathbb{X}_2 = \{x \in \mathbb{R}^3 : x_1 + x_2 + x_3 = 3\}$?

   - $\mathbb{X}_2$ is closed, but not bounded.

   - Since f is coercive, min achieved.

   - max does not exist since setting $x_1 = 0$ $x_2 = 3 - x_3$ and letting $x_3 \to \infty$ makes $f \to \infty$

3.5 Monotonic Function (@ Lec 05 of ECON 204)

Definition 3.12

A function $f : \mathbb{R} \to \mathbb{R}$ is **monotonically increasing** if $y > x \Rightarrow f(y) \geq f(x)$.

3.5.1 Theorem: Monotonically Increasing $\Rightarrow$ One-sided Limits exist

Theorem 3.6

Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then the one-sided limits: $f(t^+) = \lim_{u \to t^+} f(u)$ and $f(t^-) = \lim_{u \to t^-} f(u)$ exist and are real numbers for all $t \in (a, b)$.

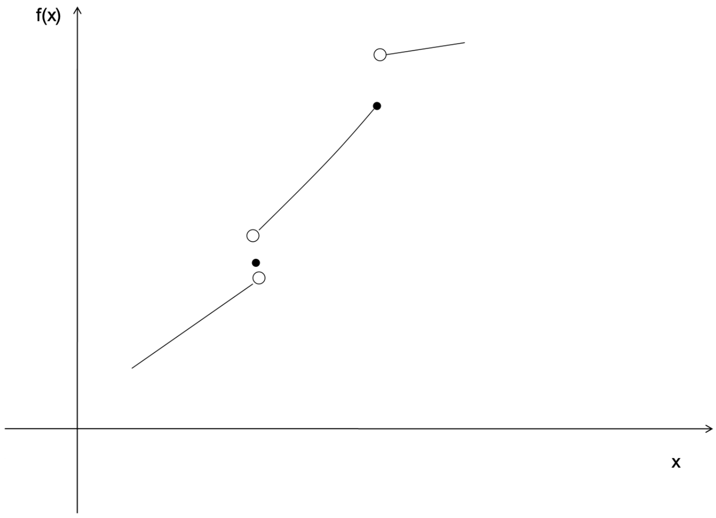

Figure 3.1: A monotonic function has only simple jump discontinuities.

3.5.2 Theorem: Monotonically Increasing ⇒ Set of Points are discontinuous is finite (possibly empty) or countable

Theorem 3.7

Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : (a, b) \to \mathbb{R}$ be monotonically increasing. Then $D = \{t \in (a, b) : f \text{ is discontinuous at } t\}$ is finite (possibly empty) or countable.

3.6 Contraction Mapping Theorem (@ Lec 05 of ECON 204)

3.6.1 Contractions

Definition 3.13

Let (X, d) be a nonempty complete metric space. An operator is a function $T : X \to X$. An operator T is a **contraction of modulus** β if $\beta < 1$ and

$$d(T(x), T(y)) \leq \beta d(x, y), \forall x, y \in X$$

A contraction shrinks distances by a *uniform* factor $\beta < 1$.

3.6.2 Theorem: Contraction ⇒ Uniformly Continuous

Theorem 3.8 (Contraction ⇒ Uniformly Continuous)

Every contraction is uniformly continuous.

Proof 3.5

Let $\delta = \frac{\varepsilon}{\beta}$.

3.6.3 Theorem: Blackwell's Sufficient Conditions for Contraction

Let X be a set, and let $B(X)$ be the set of all bounded functions from X to $\mathbb{R}$. Then $(B(X), \|\cdot\|_\infty)$ is a normed vector space.

(Notice that below we use shorthand notation that identifies a constant function with its constant value in $\mathbb{R}$, that is, we write interchangeably $a \in \mathbb{R}$ and $a : X \to \mathbb{R}$ to denote the function such that $a(x) = a, \forall x \in X$.)

Theorem 3.9 (Blackwell's Sufficient Conditions)

Consider $B(X)$ *with the sup norm* $\|\cdot\|_\infty$. *Let* $T : B(X) \to B(X)$ *be an operator satisfying*

1. *(monotonicity)* $f(x) \leq g(x), \forall x \in X \Rightarrow (Tf)(x) \leq (Tg)(x), \forall x \in X$
2. *(discounting)* $\exists \beta \in (0,1)$ *such that for every* $a \geq 0$ *and* $x \in X$,

$$(T(f+a))(x) \leq (Tf)(x) + \beta a$$

Then T *is a contraction with modulus* β.

Proof 3.6

Fix $f, g \in B(X)$. *By the definition of the sup norm,*

$$f(x) \leq g(x) + \|f - g\|_\infty \forall x \in X$$

Then

$$(Tf)(x) \leq (T(g + \|f-g\|_\infty))(x) \leq (Tg)(x) + \beta\|f-g\|_\infty \quad \forall x \in X$$

where the first inequality above follows from monotonicity, and the second from discounting. Thus

$$(Tf)(x) - (Tg)(x) \leq \beta\|f-g\|_\infty \quad \forall x \in X$$

Reversing the roles of f *and* g *above gives*

$$(Tg)(x) - (Tf)(x) \leq \beta\|f-g\|_\infty \quad \forall x \in X$$

Thus

$$\|T(f) - T(g)\|_\infty \leq \beta\|f-g\|_\infty$$

Thus T *is a contraction with modulus* β

3.7 Fixed Point Theorem (@ Lec 05 of ECON 204)

3.7.1 Fixed Point

Definition 3.14 (Fixed Point)

A **fixed point** of an operator T is element $x^* \in X$ such that $T(x^*) = x^*$.

3.7.2 Contraction Mapping Theorem: Existence and Uniqueness of Fixed Point

Theorem 3.10 (Contraction Mapping Theorem)

Let (X, d) be a nonempty complete metric space and $T : X \to X$ a contraction with modulus $\beta < 1$. Then

1. *T has a unique fixed point x^*.*
2. *For every $x_0 \in X$, the sequence defined by*

$$\begin{aligned} x_1 &= T(x_0) \\ x_2 &= T(x_1) = T(T(x_0)) = T^2(x_0) \\ &\vdots \\ x_{n+1} &= T(x_n) = T^{n+1}(x_0) \end{aligned}$$

   converges to x^*.

Note that the theorem asserts both the **existence** and **uniqueness** of the fixed point, as well as giving an **algorithm** to find the fixed point of a contraction.

Proof 3.7

Define the sequence $\{x_n\}$ as above. Then,

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) \\ &\leq \beta^n d(x_1, x_0) \end{aligned}$$

Then for any $n > m$,

$$\begin{aligned} d(x_n, x_m) &\leq d(x_1, x_0) \sum_{i=m}^{n-1} \beta^i \\ &< d(x_1, x_0) \sum_{i=m}^{\infty} \beta^i \\ &= \frac{\beta^m}{1-\beta} d(x_1, x_0) \to 0 \text{ as } m \to \infty \end{aligned}$$

Fixed $\varepsilon > 0$, *we can choose* $N(\varepsilon)$ *such that* $\forall n, m > N(\varepsilon)$,

$$d(x_n, x_m) < \frac{\beta^m}{1-\beta} d(x_1, x_0) < \varepsilon$$

Therefore, $\{x_n\}$ *is Cauchy. Since* (X, d) *is complete,* $x_n \to x^*$ *for some* $x^* \in X$.

Next we show that x^* *is a fixed point of* T.

$$\begin{aligned} T(x^*) &= T\left(\lim_{n\to\infty} x_n\right) \\ &= \lim_{n\to\infty} T(x_n) \text{ since } T \text{ is continuous} \\ &= \lim_{n\to\infty} x_{n+1} \\ &= x^* \end{aligned}$$

so x^* *is a fixed point of* T.

Finally, we show that there is at most one fixed point. Suppose x^* *and* y^* *are both fixed points of* T, *so* $T(x^*) = x^*$ *and* $T(y^*) = y^*$. *Then*

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \beta d(x^*, y^*) \\ \Rightarrow (1-\beta) d(x^*, y^*) &\leq 0 \\ \Rightarrow d(x^*, y^*) &\leq 0 \end{aligned}$$

So $d(x^*, y^*) = 0$, *which implies* $x^* = y^*$.

3.7.3 Theorem: Fixed Point's Continuous Dependence on Parameters

Theorem 3.11 (Continuous Dependence on Parameters)

Let (X, d) *and* (Ω, ρ) *be two metric spaces and* $T : X \times \Omega \to X$. *For each parameter* $\omega \in \Omega$ *let* $T_\omega : X \to X$ *be defined by* $T_\omega(x) = T(x, \omega)$.

Suppose (1). (X, d) *is complete, (2).* T *is continuous in* ω *(that is* $T(x, \cdot) : \Omega \to X$ *is continuous for each* $x \in X$*), and (3).* $\exists \beta < 1$ *such that* T_ω *is a contraction of modulus* β $\forall \omega \in \Omega$.

Then the fixed point function (about parameter ω*)* $x^* : \Omega \to X$ *defined by* $x^*(\omega) = T_\omega(x^*(\omega))$ *is continuous.*

♡

Chapter 4 Big $\mathcal{O}$ and Small o Notation

4.1 Definition

Complexity:

Definition 4.1

A sequence $f(n)$ is $O(1)$ if $\lim_{n\to\infty} f(n) < \infty$. ♣

Definition 4.2

A sequence $f(n)$ is $O(g(n))$ if $\frac{f(n)}{g(n)}$ is $O(1)$. ♣

Definition 4.3

A sequence $f(n)$ is $o(1)$ if $\lim_{n\to\infty} \sup f(n) = 0$. ♣

Definition 4.4

A sequence $f(n)$ is $o(g(n))$ if $\lim_{n\to\infty} \sup \frac{f(n)}{g(n)} = 0$. ♣

Definition 4.5

A sequence $f(n)$ is asymptotic to $g(n)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1$. (This is denoted by $f(n) \sim g(n)$ as $a \to \infty$) ♣

For two scalar functions $f(x) \in \mathbb{R}, g(x) \in \mathbb{R}_+$, where $x \in \mathbb{R}$, we write:

1. $f(x) = \mathcal{O}(g(x))$ if $\limsup_{x\to\infty} \frac{|f(x)|}{g(x)} < \infty$; we say f is dominated by g asymptotically.
2. $f(x) = \Omega(g(x))$ if $\liminf_{x\to\infty} \frac{|f(x)|}{g(x)} > 0$.
3. $f(x) = \Theta(g(x))$ if $f(x) = \mathcal{O}(g(x))$ and $f(x) = \Omega(g(x))$ both hold.
4. $f(x) = o(g(x))$ if $\liminf_{x\to\infty} \frac{f(x)}{g(x)} = 0$.

Example 4.1

$$n^3 + n + 2 = \Omega(1), n^3 + n + 2 = \Omega(n^2)$$
$$n^3 + n + 2 = \Theta(n^3)$$
$$n^3 + n + 2 = o(n^4)$$

4.1.1 Extension

$f(x) = \mathcal{O}(g(x))$ as $x \to a$ if $\limsup_{x\to a} \frac{|f(x)|}{g(x)} < \infty$.

Example 4.2 $\varepsilon^2 + \varepsilon^3 = \mathcal{O}(\varepsilon^2)$ as $\varepsilon \to 0$