Miguel Class

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All models are wrong, but some are useful.

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Chapter 1 Pricing

1.1 Monopoly

1.1.1 Base Case

The firm decides its price p to maximize $\Pi(p) = p \cdot D(p) - C(D(p))$, where $D(\cdot)$ is the demand function and $C(\cdot)$ is the cost function.

The monopoly problem is maximizing the profit

$$\max_p \Pi(p) = p \cdot D(p) - C(D(p))$$

The F.O.C. (first-order condition) is

$$\frac{\partial \Pi(p)}{\partial p} = D(p) + pD'(p) - C'(D(p))D'(p) = 0$$

and the S.O.C. (second-order condition) is

$$\frac{\partial \Pi^2(p)}{\partial p^2} < 0$$

The F.O.C. gives that

$$\begin{aligned}
(p - C')D' &= -D \\
p - C' &= -\frac{D}{D'} \\
\underbrace{\frac{p - C'}{p}}_{\text{Lerner Index}} &= -\frac{D}{D'p} \\
&= -\frac{1}{\frac{dD}{dp}\frac{p}{D}} = -\frac{1}{\frac{\frac{dD}{D}}{\frac{dp}{p}}} := \frac{1}{E}
\end{aligned}$$

where $\frac{\frac{dD}{D}}{\frac{dp}{p}} < 0$ is the elasticity of demand with respect to price. The absolute value of the elasticity is denoted by E.

E is supposed to be greater than 1, otherwise, the optimal price is negative.

In the demand function $D(p) = kp^{-E}$, where the elasticity is constant. Its elasticity is $-E$.

The monopolist gives the production that is lower than social-optimal to maximize the profit (dead weight loss).

Rent dissipation can give larger dead weight loss.

1.1.2 Multiple Products

$$\max_{p} \sum_{i=1}^{N} p_i D_i(p) - C(D_1(p), ..., D_N(p))$$

Related Demand and Separable Costs: $C(D_1(p), ..., D_N(p)) = C_1(D_1(p)) + ... + C_N(D_N(p))$. The optimal pricing in this case satisfies

$$\frac{p_i - C_i'}{p_i} = \frac{1}{E_{ii}} - \sum_{j \neq i} \frac{(p_j - C_j')D_j E_{ij}}{R_i E_{ii}}$$

where $E_{ij} = \frac{\partial D_i}{\partial p_j} \frac{p_j}{D_i}$ and R_i is the revenue.

Intuition: In the case of substitutes/complements, we want to increase/decrease the price of products compared to the one product case. (Positive/negative externality by increasing price of substitutes).

Similar Intuition: Consider a two-period model that the demand at second period depends on the price at first period (assuming $\frac{\partial D_2}{\partial p_1} < 0$).

1. $q_1 = D_1(p_1)$; $C_1(q_1)$
2. $q_2 = D_2(p_2, p_1)$; $C_2(q_2)$

Then, $\frac{p_1 - C_1'}{p_1} < \frac{1}{E_1}$ (the negative externality of prices).

Independent Demands and Related Costs:

Example 1.1

Different intensity of demand across periods.

1. Period 1: Low demand. $q_1 = D_1(p_1)$.
2. Period 2: High demand. $q_2 = D_2(p_2)$, where $D_1(p) = \lambda D_2(p)$ for some $\lambda < 1$.
3. Marginal cost of Production is c and the Marginal cost of capacity is γ.

Intuition: if $\lambda \to 0$, the marginal cost at period $\to c + \gamma$ and the marginal cost at period $1 = c$. Then, we have

$$\frac{p_2 - (c + \gamma)}{p_2} = \frac{1}{E_2}, \frac{p_1 - c}{p_1} = \frac{1}{E_1}$$

Now, let's consider a not too small λ. The problem is given as

$$\max_{p_1, p_2, k} (p_1 - c)D_1(p_1) + (p_2 - c)D_2(p_2) - \gamma k$$

$$s.t.\ D_1(p_1) \leq k$$

$$D_2(p_2) \leq k$$

The Lagrangian is given by

$$\mathcal{L} = (p_1 - c)D_1(p_1) + (p_2 - c)D_2(p_2) - \gamma k + \lambda_1(k - D_1(p_1)) + \lambda_2(k - D_2(p_2))$$

$$\frac{\partial \mathcal{L}}{\partial k} = -\gamma + \lambda_1 + \lambda_2 = 0 \Leftrightarrow \gamma = \lambda_1 + \lambda_2$$

Skip the process: $\frac{p_1-(c+\lambda_1)}{p_1} = \frac{1}{E_1}, \frac{p_2-(c+\lambda_2)}{p_2} = \frac{1}{E_2}$. Example: If $\lambda_1 = 0, k > D_1(p_1)$, the second period pays all the capacity cost.

Example 1.2 (Learning by Doing)

Suppose there are two periods $t = 1, 2$. The demand is $q_t = D_t(p_t)$. The cost in period one is $c_1(q_1)$ and $c_2(q_2, q_1)$ ($\frac{\partial c_2}{\partial q_1} < 0$, the more you produce in period one, the lower the cost you are facing in period two).

In continuous form, the cost form is

$$C(w(t))$$

where $\dot{w}(t) = \frac{dw}{dt} = q(t)$. We want to maximize

$$\max_{q(t),w(t)} \int_0^{\infty} e^{-\pi t}[q(t)p(q(t)) - C(w(t))q(t)]dt$$

$$\text{s.t. } \dot{w}(t) = q(t)$$

By Hamiltonian (skip), average of future marginal costs is

$$A(t) = \int_t^{\infty} C(w(s))\pi e^{-\pi(s-t)}ds$$

$$\frac{P(t) - A(t)}{P(t)} = \frac{1}{E(t)}$$

1.1.3 Durable Goods

The demand in one period is substitute to demand in other periods.

Example 1.3

Two periods $t = 1, 2$. Three consumers: $v_1 = 1$ per period, $v_2 = 2$ per period, and $v_3 = 3$ per period. The cost of production is zero. The seller chooses p_1, p_2.

(Consumer may forward-looking).

Moorthy (1988), Levinthal, D. A., & Purohit, D. (1989).

$t = 1, 2$ and zero production cost. The values of consumers $v \sim U[0, 1]$ and the discount factor is $\delta < 1$. The selling price p_1, p_2.

Suppose the consumers bought in first period have $v \geq v_1^*$, which must satisfies

$$\delta(v_1^* - p_2) = v_1^* - p_1 + \delta v_1^*$$

$$v_1^* = p_1 - \delta p_2$$

The price in second period should be $p_2 = \frac{v_1^*}{2}$. Then, $v_1^* = p_1 - \delta\frac{v_1^*}{2} \Rightarrow v_1^* = \frac{2p_1}{2+\delta}$. The price in the first period is given by

$$\max_{p_1} \; p_1(1-v_1^*) + \delta \frac{(v_1^*)^2}{4} = p_1 \frac{2+\delta-2p_1}{2+\delta} + \delta \frac{p_1^2}{(2+\delta)^2}$$

Leasing (instead of selling): The leasing price is p in each period, which is given by $\operatorname{argmax}_p p(1-p) = \frac{1}{2}$. Leasing may generate more profits than selling for the seller.

Intuition:

1. Too much flexibility for seller ⇒ losses of capital of first period buyers.
2. Intertemporal price discrimination (first period buyers pay higher price) "price skimming".

Theorem 1.1 (Coase Conjecture)

Suppose the seller can change the price faster and faster. What happens to the profits of the seller? The profit goes to zero.

Why there is selling in the world?

1. Moral hazard of leasing.
2. Leasing is not anonymous. Reveal reservation price ⇒ Price discrimination in leasing. (Even worse than selling.) (Long-term contract + renegotiation = selling).
3. Commit to sequences of prices.
   (a). Deposit to third party
   (b). Reputation
4. Increasing cost
5. "Most-favored Nation" clause.
6. Consumers are not informed about the production costs.
7. New consumers coming into the market.

1.1.4 Learning Demand

Firms may not be able to learn the demand function perfectly.

It is relatively easy to learn a quasi-concave profit function. In the case that the profit function is not quasi-concave, the firm may not be able to learn the profit function. (stay at local maximum because of the loss from learning).

Learning the optimal features: By assuming the distribution of marginal increase $\frac{\partial \pi_j}{\partial x_j}$ is symmetric about 0, we can use Brownian motion to model the continuous profit function.

1.2 Short-run Competition

1.2.1 Bertrand Paradox

Consider two firms with marginal cost c:

$$\Pi^i(p_i, p_j) = (p_i - c)D_i(p_i, p_j)$$

where

$$D_i(p_i, p_j) = \begin{cases} D_i(p_i), & p_i < p_j, \\ \frac{1}{2}D_i(p_i), & p_i = p_j, \\ 0, & p_i > p_j \end{cases}$$

The NE is $p_i = p_j = c$. $\Pi^i = \Pi^j = 0$.

1.2.2 Static Solution to Bertrand Paradox

Capacity Constraints

Edgeworth: there may exist some constraints of the capacity.

1. Firms choose capacity K_i, K_j.
2. Firms choose prices p_i, p_j.

Solving by backward induction: That is, firstly solve $p_i^*(K_i, K_j)$ such that

$$\max_{p_i} (p_i - c)D_i(p_i, p_j)$$

$$\text{s.t. } D_i(p_i, p_j) \leq K_i$$

and then solve

$$\max_{K_i} \ (p_i^*(K_i, K_j) - c)\, D_i(p_i^*(K_i, K_j), p_j^*(K_i, K_j)) - \gamma K_i$$

where γ is the marginal cost of capacity.

Best response in prices: positive correlated, which is called “strategic complements”.

Best response in quantities (Cournot competition): negative correlated, which is called “strategic substitutes”.

(Quantity competition gives higher profits.)

Example 1.4 (Simple Example of Couront Competition)

$P(q_1, q_2) = 1 - q_1 - q_2$ and $\gamma \in (\frac{3}{4}, 1)$.

$$\max_{q_1} \ q_1\,(1 - q_1 - q_2 - \gamma)$$

$$\Rightarrow q_1^*(q_2) = \frac{1 - q_2 - \gamma}{2}$$

Similarly, $q_2^*(q_1) = \frac{1-q_1-\gamma}{2}$. Thus, $q_1^* = q_2^* = \frac{1-\gamma}{3}$.

Similar to the Cournot competition, we can get positive profits with capacity constraints.

Differentiation

Idea: it is easier to change prices than to change products.

Basic case: Spatial Competition: There are consumers in $[0,1]$ (uniform distribution). The position chosen by both firms is $\frac{1}{2}$ (the center of the market).

With price competition: Transportation cost is tx^2, where x is the distance. Suppose the firm A locates at 0 and the firm B locates at 1. The profit of consumer x from purchasing A is $v - p_A - tx^2$ and the profit of consumer x from purchasing B is $v - p_B - t(1-x)^2$. The indifferent consumer is

$$v - p_A - tx^2 = v - p_B - t(1-x)^2$$

$$p_A - p_B = t(1-2x)$$

$$\Rightarrow x = \frac{1}{2} - \frac{p_A - p_B}{2t}$$

Therefore, the demand of A is

$$D_A(p_A, p_B) = \frac{1}{2} - \frac{p_A - p_B}{2t}$$

and the demand of B is

$$D_B(p_A, p_B) = \frac{1}{2} + \frac{p_A - p_B}{2t}$$

Note *If the transportation cost is tx, the demand function is the same as above.*

The p_A^* and p_B^* are given by

$$\left.\begin{aligned} p_A^*(p_B) &= \text{argmax}_{p_A}(p_A - c)\left(\tfrac{1}{2} - \tfrac{p_A-p_B}{2t}\right) = \tfrac{c+t+p_B}{2} \\ p_B^*(p_A) &= \text{argmax}_{p_B}(p_B - c)\left(\tfrac{1}{2} + \tfrac{p_A-p_B}{2t}\right) = \tfrac{c+t+p_A}{2} \end{aligned}\right\} \Rightarrow p_A^* = p_B^* = c + t$$

Then, the profits are $\Pi_A^* = \Pi_B^* = \frac{t}{2}$.

Endogenous Differentiation: Denote the position of firm A as a and the position of firm B as $1-b$. Then, the indifferent consumer is

$$p_A + t(x-a)^2 = p_B + t(1-b-x)^2$$

$$\Rightarrow x = \frac{p_B - p_A + t\left[(1-b)^2 - a^2\right]}{2t(1-b-a)},$$

and the demands are

$$D_A(p_A, p_B) = \frac{p_B - p_A + t\left[(1-b)^2 - a^2\right]}{2t(1-b-a)},\ D_B(p_A, p_B) = 1 - D_A(p_A, p_B)$$

Then, the equilibrium prices given a and b are

$$\begin{aligned} p_A^* &= t(1-a-b)\left(1 + \frac{a-b}{3}\right) \\ p_B^* &= t(1-a-b)\left(1 + \frac{b-a}{3}\right) \end{aligned}$$

where $c := 0$. The corresponding profits are

$$\begin{aligned} \Pi_A(a,b) = p_A^* D_A^* &= \left(1 + \frac{a-b}{3}\right)\frac{t(1-b-a)(1-b+a)}{2} \\ \Pi_B(a,b) = p_B^* D_B^* &= \left(1 + \frac{b-a}{3}\right)\frac{t(1-b-a)(1-a+b)}{2} \\ \frac{\partial \Pi_A(a,b)}{\partial a} &= \underbrace{(p_A^* - c)\frac{\partial D_A^*}{\partial a}}_{\text{direct effect} > 0} + \underbrace{\frac{\partial D_A^*}{\partial p_A^*}}_{=0}\frac{\partial p_A^*}{\partial a} + \underbrace{(p_A^* - c)\frac{\partial D_A^*}{\partial p_B^*}\frac{\partial p_B^*}{\partial a}}_{\text{strategic effect} < 0} \end{aligned}$$

Which effect dominates depends on the model. In this model, the strategic effect dominates the direct effect. That is, $a = b = 0$. (If allowing negative values, $a = b = -\frac{1}{4}$.)

Note *If the transportation cost is tx, the equilibrium may not exist.*

Other models:

1. vertical differentiation (S. Moorthy);
2. defender model (John Hauser);
3. logit/limited defender model, $u_{ij} = v_j - \alpha p_j + \epsilon_{ij}$, where $\epsilon_{ij} \sim EVI$. The outside option is modeled as $u_{i0} = \epsilon_{i0} \sim EVI$. The market share is given as

   $$s_j = \Pr(u_{ij} \geq u_{ij'}, \forall j') = \frac{\exp(v_j - \alpha p_j)}{1 + \sum_{j'} \exp(v_{j'} - \alpha p_{j'})}$$

   The demand is given by

   $$D_j = s_j \cdot \text{Market Size}$$

   Estimated by taking inversion,

   $$\ln s_j - \ln s_0 = v_j - \alpha p_j$$

Proliferation of Products to Deter Entry: Entry with products locating uniformly to deter entry.

Spoke Model.

Chapter 2 Search

2.1 Optimal Search

2.1.1 Individual Choice

1. *Simple example*, where the cost of search for each price is c and a consumer buys one or zero unit.

   Definition 2.1 (Optimal Stopping Rule)

   The optimal rule of searching is an **optimal stopping rule**: Stop and buy once the consumer finds a price less or equal to a reservation price, R.

   The R is constructed as the critical value when the expected return from an extra search equals to the marginal cost:

$$\begin{aligned} c &= \mathbb{E}[R - p \mid p < R]\Pr(p < R) \\ &= \int_0^R (R - p) f(p) dp \end{aligned}$$

   where $f(\cdot)$ is the density distribution of prices.

2. *Consuming multiple units (general case, the one unit case can be modeled by assuming the demand function)*, Remind that the consumer surplus given price p is given as

$$s(p) = \max_q \{U(q) - p \cdot q\}$$

   and its derivatives are

$$s'(p) = -D(p), \; s''(p) = -D'(p) > 0 \text{ (convexity)}$$

   In this case, the optimal stopping rule R that maximizing the consumer surplus is given as

$$c = \int_0^R [s(p) - s(R)] f(p) dp$$

Note *Given a greater variance of the price distribution, the R decreases.*

2.1.2 Homogeneous Markets

Can we find a fixed point of search behavior (i.e., $f(p)$)?

Assumption 2.1

All firms are identical with marginal cost c_f. All consumers are identical with search costs c.

Define the "monopoly price":

$$p^M = \underset{p}{\text{argmax}} \{p \cdot D(p) - c_f D(p)\}$$

Assumption 2.2

The consumers want to visit the first store in equilibrium, $s(p^M) > c$.

Theorem 2.1 (Diamond Theorem)

The unique equilibrium is for all firms to charge monopoly price p^M and consumers do not search.

Proof 2.1 (Sketch)

Firstly, we can prove this proposed equilibrium exists: all monopoly prices $\Rightarrow$ no search; no search $\Rightarrow$ all monopoly prices.

Secondly, we prove the uniqueness: Given $f(p)$, the corresponding reservation price of search is R. All firms charge $p = R$ and the consumers do not search.

1. Firstly, no firms should charge below R: if $p < R$, the consumer purchases once he visits the store. So, there is no firm charging $p < R$.
2. Secondly, we prove no firms charge above R: Suppose by the way of contradiction that there is a firm charging $p = R + \epsilon$. Once a consumer visits the store, the consumer's highest surplus from an extra search is $s(R) - s(R + \epsilon) - c$. There always exists an $\epsilon > 0$ such that purchases at $R + \epsilon$ is profitable for the consumer.

Therefore, the consumers do not search, and then all firms charge p^M.

2.1.3 Solutions for Diamond Paradox

1. Firms are different.
   (a). Different costs, which give different p^M.
   (b). Inflation + menu costs.
2. Consumers are different.
   (a). Different search costs.
      I. If mass zero at zero search cost, then everything is the same.
      II. If no mass zero at zero, then equal breaks down.
   (b). Different preferences [Choi, Dai, and Kim(2018)].
   (c). Lack of common knowledge [Kuksov(2006)].

2.1.4 [Choi, Dai, and Kim(2018)]

Suppose the utility of consumer from consuming product j is

$$U(v_j, z_j, p_j, N) = v_j + z_j - p_j - \underbrace{\sum_{k \in N} s_k}_{\text{search costs based on his search history}}$$

where (v_j, z_j) are the parameters of the consumer that reflects his preference, h v_j is known before search, z_j is learning during search, p_j is the price, N is set of the products searched by the consumer, s_k is the search cost of product (firm) k.

Assumption 2.3

$v_j \sim F_j \in \Delta[\underline{v}_j, \bar{v}_j]$ and $z_j \sim G_j \in \Delta[\underline{z}_j, \bar{z}_j]$.

Definition 2.2 (Gittins index)

Weitzman index (Gittins index) z_j^* is given by solving

$$s_j = \int_{z_j^*}^{\bar{z}_j} [(1 - G_j(z_j))] dz_j$$

which is equivalent to $s_j = \int_{z_j^*}^{\bar{z}_j} (z_j - z_j^*) g(z_j) dz_j$ (integration by part).

Remark Upper Confidence Bound (UCB) Algorithm

Proposition 2.1 (Optimal Search Policy)

The optimal policy: Visit stores in descending order of $v_j + z_j^* - p_j$ and stop if

$$\max\left\{u_0, \max_{j \in N} v_j + z_j - p_j\right\} > \max_{j \notin N} v_j + z_j^* - p_j$$

where u_0 is the utility from the outside option.

Note *Since it is descending order of $v_j + z_j^* - p_j$, stop after visiting i if $z_i \geq z_i^*$ holds.*

Define $w_i := v_i + \min\{z_i, z_i^*\}$. Then, shopping outcome is that consumer buys product i <u>iff</u>

1. $w_i - p_i > u_0$ and
2. $w_i - p_i > w_j - p_j$ for all $j \neq i$.

Note *Latent utility framework that maximizes $w_i - p_i$.*

Proof 2.2

Prove the sufficiency: Given $w_i - p_i > v_0$,

$$w_i - p_i > v_0 \Rightarrow v_i + z_i^* - p_i > u_0$$

By the search policy, we must have $v_i + z_i - p_i > u_0$.

Given $w_i - p_i > w_j - p_j$.

1. If $z_j^* \leq z_j$, we have $w_j = v_j + z_j^* - p_j$. As $v_i + z_i^* - p_i \geq w_i - p_i > w_j - p_j = v_j + z_j^* - p_j$ and the order is descending in $v_j + z_j^* - p_j$, i must be visited before j. The consumer does not want to visit j after visit i because $v_i + z_i - p_i \geq w_i - p_i > v_j + z_j^* - p_j$.
2. If $z_j^* > z_j$, ...

Prove the necessary:

Solving the Equilibrium Let $H_i(\cdot)$ be the CDF of $w_i := v_i + \min\{z_i, z_i^*\}$.

$$H_i(w_i) := \int_{\underline{z}_i}^{z_i^*} F_i(w_i - z_i)dG(z_i) + \int_{z_i^*}^{\bar{z}_i} F_i(w_i - z_i^*)dG(z_i)$$

The best alternative is defined as $x_i = \max\{u_0, \max_{j \neq i} w_j - p_j\}$. Its CDF is given as

$$\tilde{H}(x) = \text{Prob}[x_i \leq x]$$

Note that $\tilde{H}(x)$ depends on p_{-i} but not p_i.

Therefore, the demand for the product i given the price vector p can be given as

$$D_i(p) = \int \underbrace{[1 - H_i(x_i + p_i)]}_{\text{Prob}[w_i - p_i \geq x_i]} d\tilde{H}(x_i)$$

Thus, the optimization problem of pricing is given by

$$\max_{p_i} (p_i - MC_i) D_i(p_i, p_{-i})$$

Main Results

1. Observable prices: as the search costs increase $\Rightarrow$ the benefit from search decreases $\Rightarrow$ attract consumers to search first is more important $\Rightarrow$ Lower prices.
2. Unobservable prices: as the search costs increase $\Rightarrow$ the benefit from search increases $\Rightarrow$ try to exploit consumers $\Rightarrow$ Higher prices.

Given more information pre-search, there are two effects:

1. Less benefit from search $\Rightarrow$ attract consumers to search first is more important $\Rightarrow$ Lower prices.
2. More dispersed consumer preferences $\Rightarrow$ Larger differentiation among products $\Rightarrow$ Higher prices.

2.1.5 [Kuksov(2006)]: Lack of Common Knowledge

Suppose there are identical buyers have valuation v for products (1 unit). Sellers are uncertain about the valuation v and choose their own prices based on private signals.

Suppose a seller j gets a signal

$$x_j = v + \eta_j = \begin{cases} v + 0, & \text{with prob } \frac{1}{2} \\ v - \delta(v), & \text{with prob } \frac{1}{2} \end{cases}$$

The cost of first search is zero and the cost of sequential search is s.

If the signal v I received is the high signal, then others who receive the low signal gets $f(v) := v - \delta(v)$

If the signal v I received is the low signal, then others who receive the high signal gets $g(v) := f^{-1}(v)$

Thus, if a seller receives signal x_j, he believes the possible other sellers' signals are $\{f(x_j), x_j, g(x_j)\}$.

Equilibrium Consumers are facing two kinds of prices set by firms $P(v)$ and $P(f(v))$.

1. If consumers see $P(f(v))$, they stop.
2. If consumers see $P(v)$, they buy the product if

$$\frac{1}{2}(P(v) - P(f(v))) \leq s$$

Therefore, the equilibrium prices should satisfy

$$P(v) = P(f(v)) + 2s$$

Results

1. Prices are below the monopoly price.
2. As s goes to 0, the price $P(v)$ goes to MC.
3. As $\delta(v)$ goes to zero, the price $P(v)$ goes to the monopoly price.

2.1.6 [Kuksov and Villas-Boas(2010)]: Alternative Overload

Consumers can either search with some costs or buy at random.

The value of consumers is defined as

$$V(v, x, I, N) = \max \Bigg\{ \underbrace{0}_{\text{not choose an alternative}}, \underbrace{\max_{i \in I}(v - t|x - z_i|)}_{\text{choose the best checked product}}, \underbrace{\sum_{i \notin I} \frac{1}{N - M} (v - t|x - z_i|)}_{\text{randomly choose from unchecked products}}, \underbrace{-c + \sum_{i \notin I} \frac{1}{N - M} V(v, x, I \cup \{i\}, N)}_{\text{(randomly) search one more}} \Bigg\}$$

where v is the basic value of choosing an alternative, x is the location of the consumer, I is the set of products checked, M is the number of products checked, N is the number of total products, and z_i is the location of the alternative i.

Infinite Alternatives: In this case the expected values from different strategies are given as

1. Random purchase: x gets expected disutility $d = \int_0^1 t|x-z|dz$.
2. Search: the search strategy can be given as
   (a). Stop search if $z \in (x-\delta, x+\delta)$. This gives disutility $d = \int_{x-\delta}^{x+\delta} t|x-z|dz$, which is minimized for consumers located at $x = \frac{1}{2}$ who gets disutility $d = \frac{t}{4}$.
   (b). Search otherwise. This requires a search cost of c.

   In equilibrium, the search costs equal to the disutility from stopping:
$$c = \int_{x-\delta}^{x+\delta} t|x-z|dz \Rightarrow \delta = \sqrt{\frac{c}{t}}$$
   The corresponding expected disutility is
$$d(\infty) = t\delta = \sqrt{ct}$$

Therefore, all consumers choose to search instead of randomly purchase if and only if $\frac{c}{t} \leq \frac{1}{16}$.

Results There can be too many products. To show this, we compare $\hat{N}$ products with infinity. ($\hat{N}$ products uniformly locate on $[0,1]$ in equilibrium).

Let's consider a strategy (not optimal): <u>Search until find the closest alternative</u>: without search costs, it gives the expected disutility $\frac{t}{4\hat{N}}$ for a consumer on average.

1. Without search costs, it gives the expected disutility $\frac{t}{4\hat{N}}$ for a consumer on average.
2. The expected search cost to find this product is
$$c\frac{1}{\hat{N}} + 2c\frac{\hat{N}-1}{\hat{N}}\frac{1}{\hat{N}-1} + \cdots \left[(N-1)c\frac{\hat{N}-1}{\hat{N}}\frac{\hat{N}-2}{N\hat{-}1}\cdots\frac{1}{2}\right] = (\hat{N}-1)\left(\frac{1}{2}+\frac{1}{\hat{N}}\right)c$$

The expected disutility for a consumer on average is
$$\frac{t}{4\hat{N}} + (\hat{N}-1)\left(\frac{1}{2}+\frac{1}{\hat{N}}\right)c$$
its derivate of $\hat{N}$ is $-\frac{t}{4\hat{N}^2} + \left(\frac{1}{2}+\frac{1}{\hat{N}}\right)c - \frac{(\hat{N}-1)}{\hat{N}^2}c \propto -\frac{t}{4} + \left(\frac{\hat{N}^2}{2}+1\right)c$.

For some $\hat{N} \approx \frac{1}{2\sqrt{c/t}}$ such that
$$\frac{t}{4\hat{N}} + (\hat{N}-1)\left(\frac{1}{2}+\frac{1}{\hat{N}}\right)c < \frac{3}{4}\sqrt{ct} + c \leq \sqrt{ct} < d(\infty)$$
Therefore, there is a finite number of products $\hat{N}$ that benefits consumers compared to infinity.

2.2 Search for Information

2.2.1 Multiple Attributes

Example 2.1

Suppose there is one attribute. The search cost is c. The initial value is $v = -1$. The attribute is $z = -10$ or $z = 10$ with equal probability.

The expected payoff from search is $\frac{9}{2} - c$. That is, search if and only if $c < \frac{9}{2}$.

Two Attributes Suppose you get $v + z$ (positive) or $v - z$ (negative) after the first search. Should you search second attribute? (assume $v < 0$)

1. Given $v + z$, the expected payoff from searching the second attribute is
$$\frac{1}{2}(v + 2z) + \frac{1}{2}\max\{v, 0\} - c = \frac{v}{2} + z - c$$
and the expected payoff from not searching is $v + z$. Therefore, search if and only if $c < -\frac{v}{2}$.
2. Given $v - z$, the expected payoff from searching the second attribute is
$$\frac{1}{2}\max\{v, 0\} + \frac{1}{2}\max\{v - z, 0\} - c = -c$$
and the expected payoff from not searching is 0. Therefore, never search in this case.

Therefore, the expected payoff from the first search is
$$\begin{aligned}&\frac{1}{2}0 + \frac{1}{2}\max\{\frac{v}{2} + z - c, v + z\} - c\\ =&\max\{\frac{v}{4} + \frac{z}{2} - \frac{3c}{2}, \frac{v + z}{2} - c\}\end{aligned}$$
Thus, search if and only if $c < \max\{\frac{v+z}{2}, \frac{v+2z}{6}\}$.

More Attributes Every search X_i gives $+z$ or $-z$ $\mathbb{E}[X_i] = 0$. $U = v + \sum_{i=1}^{N} X_i$. After search n attributes, $x_i, i = 1, ..., n$. The expected payoff is given by
$$u = v + \sum_{i=1}^{n} x_i + \sum_{i=n+1}^{N} \mathbb{E}[X_i] = v + \sum_{i=1}^{n} x_i$$
In a continuous form,
$$du = \sigma dW$$
where W is the standard Brownian motion. Let $z = \sigma\sqrt{dt}$.

At each point in search process, the decision maker chooses among: 1. Continue to search (gather information) at cost c per unit of attributes; 2. Stop search and buy the product; 3. Stop search without buying the product.

The expected payoff of keeping on searching is given by

$$V(u,t) = -cdt + \mathbb{E}[V(u+du, t+dt)]$$

where u is the current expected payoff if buying now and r is the mass of attributes searched.

By the Taylor expansion,

$$V(u+du, t+dt) = V(u,t) + V_u du + V_t dt + V_{uu}\frac{(du)^2}{2} + V_{ut} du dt + o((dt)^2)$$

where $\mathbb{E}du = 0$, $\mathbb{E}(du)^2 = \sigma^2 dt$, and $V_t = 0$. Thus,

$$\begin{aligned} V(u,t) &= -cdt + V(u,t) + V_u \mathbb{E}du + V_t dt + \frac{V_{uu}}{2}\mathbb{E}[(du)^2] + V_{ut} dt \mathbb{E}du \\ &= -cdt + V(u,t) + \frac{V_{uu}}{2}\sigma^2 dt \end{aligned}$$

Therefore,

$$V_{uu} = \frac{2c}{\sigma^2}$$

That is the function form of $V(u)$ must be

$$V(u) = A_2 + A_1 u + \frac{c}{\sigma^2}u^2$$

for some constants A_1, A_2. Now, we solve for the A_1 and A_2.

The rule of searching can be given by $\overline{U}$ and $\underline{U}$, where $\overline{U}$ is the utility that the DM buys the product if $u \geq \overline{U}$ and $\underline{U}$ is the utility that the DM stops search without buying if $u \leq \underline{U}$.

1. *Value matching conditions* are

$$V(\overline{U}) = \overline{U}$$

$$V(\underline{U}) = 0$$

2. *Smooth pasting conditions* are

$$V'(\overline{U}) = 1$$

$$V'(\underline{U}) = 0$$

Remark Intuition for $V'(\overline{U}) = 1$: Suppose the DM chooses to search given $u = \overline{U}$, the expected payoff is

$$\frac{1}{2}\left(\overline{U} + \mathbb{E}[du \mid du > 0]\right) + \frac{1}{2}\mathbb{E}\left[V(\overline{U} + du) \mid du < 0\right] - cdt$$

where $V(\overline{U} + du) = \bar{U} + V'(\overline{U})du$, $\mathbb{E}[du \mid du > 0] = \sigma\frac{\sqrt{dt}}{\sqrt{2\pi}}$, and $\mathbb{E}[du \mid du < 0] = -\sigma\frac{\sqrt{dt}}{\sqrt{2\pi}}$. Then, the expected payoff is equal to

$$\frac{1}{2}\left(\overline{U} + \sigma\frac{\sqrt{dt}}{\sqrt{2\pi}}\right) + \frac{1}{2}\left[\bar{U} - V'(\overline{U})\sigma\frac{\sqrt{dt}}{\sqrt{2\pi}}\right] - cdt = \overline{U} + \frac{1}{2}\left(1 - V'(\overline{U})\right)\sigma\frac{\sqrt{dt}}{\sqrt{2\pi}} - cdt$$

By the definition of $\overline{U}$, the expected payoff must be equal to $\overline{U}$. As dt has a higher order than $\sqrt{dt}$, we need to have $V'(\overline{U}) = 1$.

The solution is $\overline{U} = -\underline{U} = \frac{\sigma^2}{4c}$

$$V(u) = \frac{\sigma^2}{16c} + \frac{1}{2}u + \frac{c}{\sigma^2}u^2$$

The **probability of purchase** is

$$\text{Prob}(v) = \frac{v - \underline{U}}{\overline{U} - \underline{U}} = \frac{1}{2} + \frac{2cv}{\sigma^2}$$

The **expected number of attributes searched**:

$$V(v) = \text{Prob}(v)\overline{U} - c\mathbb{E}[\text{number of attributes searched}]$$

$$\Rightarrow \mathbb{E}[\text{number of attributes searched}] = \frac{\text{Prob}(v)\overline{U} - V(u)}{c} = \frac{\sigma^2}{16c^2} - \frac{v^2}{\sigma^2}$$

Optimal Price: Suppose the marginal cost is g. The v becomes $v - p$. That is probability of purchase with $v - p \in [\underline{U}, \overline{U}]$ is

$$\text{Prob}(v) = \frac{1}{2} + \frac{2c(v-p)}{\sigma^2}$$

Since we always make $v - p \in [\underline{U}, \overline{U}]$, the profit is given by

$$\Pi = (p - g)\text{Prob}(v) = (p - g)\left[\frac{1}{2} + \frac{2c(v-p)}{\sigma^2}\right]$$

1. If v is extremely high ($v > g + 3\overline{U}$),

$$p^* = v - \overline{U}$$

$$\Pi^* = v - \overline{U} - g$$

2. If v is not extremely high,

$$p^* = \frac{1}{2}\left(v + \overline{U} + g\right)$$

$$\Pi^* = \frac{(v - g + \overline{U})^2}{8\overline{U}}$$

Intuition about surplus, as c decreases $\Rightarrow$ p can increase $\Rightarrow$ consumer surplus can fall.

2.2.2 Discounting

Consider the discounting, the expected payoff of keeping on searching is given by

$$V(u) = -cdt + e^{-\pi dt}\mathbb{E}[V(u + du)]$$

By Taylor's expansion, we have

$$V(u) = -cdt + e^{-\pi dt}[V(u) + \frac{\sigma^2}{2}V_{uu}(u)dt]$$

$$V(u)\frac{1 - e^{-\pi dt}}{dt} = -c + e^{-\pi dt}\frac{\sigma^2}{2}V_{uu}(u)$$

As $dt \to 0$,

$$V(u)\pi = -c + \frac{\sigma^2}{2}V_{uu}(u)$$

Then, we can solve

$$V(\mu) = A_1 \exp\left(\frac{2\pi}{\sigma^2}\mu\right) + A_2 \exp\left(-\frac{2\pi}{\sigma^2}\mu\right) - \frac{c}{\pi}$$

We also have the same conditions that

1. *Value matching conditions* are

$$V(\overline{U}) = \overline{U}$$

$$V(\underline{U}) = 0$$

2. *Smooth pasting conditions* are

$$V'(\overline{U}) = 1$$

$$V'(\underline{U}) = 0$$

Thus, we can solve that

$$\overline{U} = \sqrt{\frac{\sigma^2}{\pi^2} + \frac{\sigma^2}{2\pi}} - \frac{c}{\pi}$$

Intuition: discounting $\Rightarrow$ search less, $\overline{U}$ decreases and $\underline{U}$ increases.

In previous case, we have $|\overline{U}| = |\underline{U}|$. However, in this case, $|\overline{U}| < |\underline{U}|$ (because the option of not purchasing, the consumer is more likely to search with discounting at low value).

As $c \to 0$, $\underline{U} \to -\infty$. If the outside option is positive (i.e. there exists a cost from receiving outside option later), $\underline{U} \nrightarrow -\infty$.

2.2.3 Signals of Value of Products (heterogeneous attributes)

In this case, we have

$$du = \sigma_t dW,$$

where σ_t changes over time.

Consider the case of normal learning, that is the signal $s_i = U + \epsilon_i$, where $U \sim \mathcal{N}\left(v, e^2\right)$ and $\epsilon_i \sim \mathcal{N}\left(0, s^2\right)$.

In discrete form, the posterior belief of u after receiving t number of signals is a normal distribution with

$$u(t) = \frac{\frac{v}{e^2} + \sum_{j=1}^{t} \frac{s_j}{s^2}}{\frac{1}{e^2} + \frac{t}{s^2}}, \sigma(t)^2 = \frac{1}{\frac{1}{e^2} + \frac{t}{s^2}}$$

σ_t in continuous form is induced by the equations above.

Most induction is similar to previous case, but the V is related to t this time. Then, we have

$$-c + V_t(u,t) + \frac{\sigma_t^2}{2} V_{uu}(u,t) = 0$$

Combing with

1. *Value matching conditions* are

$$V(\overline{U}_t, t) = \overline{U}_t$$

$$V(\underline{U}_t, t) = 0$$

2. *Smooth pasting conditions* are

$$V'(\overline{U}_t, t) = 1$$

$$V'(\underline{U}_t, t) = 0$$

Intuition: as time goes, the marginal information can be obtained is less, so $\overline{U}_t$ and $\underline{U}_t$ converges to zero.

2.2.4 Finite Mass of Attributes: T

In this case, $V(u, T) = \max\{u, 0\}$. We have $\overline{U}(T) = \underline{U}(T) = 0$.

$$-c + V_t(u, t) + \frac{\sigma^2}{2} V_{uu}(u, t) = 0$$

2.2.5 Finite Mass and Heterogeneous Attributes

2.2.6 Choosing Search Intensity

Suppose the consumer can choose the search intensity, σ^2, with search cost $c(\sigma^2)$ such that $c' > 0, c'' > 0$. With discounting, we have

$$\pi V = -c + \frac{\sigma^2}{2} V_{uu}$$

Thus, $\sigma^2 V_{uu}$ is increasing in u.

$$\max_{\sigma^2} V$$

The F.O.C. of this problem is

$$c'(\sigma^2) = \frac{V_{uu}}{2}$$

Therefore, the σ^2 is increasing in u.

2.2.7 Information Overload

Assumption 2.4

1. The firm decides how much information to provide, T.
2. Information is not structured.
3. Given a certain amount of information, the firm provides the more relevant information.

Let σ_i denote the importance of attribute i, which is decreasing in i and $\sigma_{i\to\infty}\sigma_i = 0$. Let the average of σ over T be

$$\bar{\sigma}_T = \frac{1}{T}\int_0^T \sigma_i di,$$

which is decreasing in T. The $T\bar{\sigma}_T$ is the total information perceived, which is increasing in T.

Stationary: Each attribute has a constant hazard rate (with parameter 1) of running out. With information available T: Probability of running out of attribute $T\sigma$ check is $\frac{dt}{T}$. (Consider an attribute I am checking, I do not know whether the next marginal checking is still available. The attribute is "running' out' if the next marginal checking is not available.)

$$\begin{aligned} V(u) &= -cdt + \frac{dt}{T}\max\{u,0\} + \left(1-\frac{dt}{T}\right)\mathbb{E}[u+du] \\ &= -cdt + \frac{dt}{T}\max\{u,0\} + \left(1-\frac{dt}{T}\right)\left(V(u) + \frac{\bar{\sigma}_T^2}{2}V_{uu}dt\right) \end{aligned}$$

Then, we have

$$-cT + \max\{u,0\} - V + \frac{T\bar{\sigma}_T^2}{2}V_{uu} = 0$$

1. *Value matching conditions* are

$$V(\overline{U}) = \overline{U}$$

$$V(\underline{U}) = 0$$

$$V(0^+) = V(0^-)$$

2. *Smooth pasting conditions* are

$$V'(\overline{U}) = 1$$

$$V'(\underline{U}) = 0$$

$$V'(0^+) = V'(0^-)$$

We have $\overline{U} = -\underline{U}$ with $\overline{U}$ increases in T firstly and then decreases.

With initial value $v < 0$. As the T increases, the purchase probability increases firstly and the decreases.

As the initial value ($v < 0$) close to zero, the optimal information T increases.

As the T increases, the payoff of the consumer $V(v)$ increases firstly and then decreases.

The optimal information provided by the firm is more than the optimal information for the consumer.

2.2.8 Search for information of Multiple Products

Suppose there are two alternatives. The consumer can

1. search information for alternative 1,

2. search information for alternative 2,
3. purchase alternative 1,
4. purchase alternative 2,
5. exit without purchases.

The optimal policy is not going to be a (single) index-based policy (what we do in multi-armed bandit, Gittins index policy). Gittins index: $K_i := V(I_i, K_i)$: Value of playing arm i with information I_i with the possibility of exiting and getting K_i.

Claim 2.1

Single index-based policy may not be optimal in search problem.

Example 2.2 (An example that Gittins index policy is not optimal, Bergman)

There are two alternatives, A and B. The initial expected values are $v_A = 10$ and $v_B = 4$.

1. The first search: no learning.
2. The second search: A becomes 20 or 0 with equal probability. B becomes 18 or -10 with equal probability.

The search cost of each search is $c = 1$.

(i). **Gittins index policy**: Gittins indexes are given by

$$K_A = \underbrace{-2}_{\text{search cost}} + \frac{1}{2}20 + \frac{1}{2}K_A \Rightarrow K_A^* = 16$$

$$K_B = \underbrace{-2}_{\text{search cost}} + \frac{1}{2}18 + \frac{1}{2}K_B \Rightarrow K_B^* = 14$$

Since $K_A^* > K_B^*$, we search A first. The expected payoff is

$$-2 + \frac{1}{2}20 + \frac{1}{2}\left(-2 + \frac{1}{2}18 + \frac{1}{2}0\right) = 11.5$$

(ii). **Optimal Policy**: Search B first is better. The expected payoff is

$$-2 + \frac{1}{2}18 + \frac{1}{2}10 = 12$$

Continuous Search Suppose the expected value of product i from searching follows $d\mu_i = \sigma dW_i$. The search phase depends on the value function of keep searching.

$$V(\mu_1, \mu_2) = -cdt + \max\left\{\mathbb{E}_{d\mu_1}\left[V(\mu_1 + d\mu_1, \mu_2)\right] + \mathbb{E}_{d\mu_2}\left[V(\mu_1, \mu_2 + d\mu_2)\right]\right\}$$

By the Taylor expansion, we have $\mathbb{E}_{d\mu_1}\left[V(\mu_1 + d\mu_1, \mu_2)\right] = V(\mu_1, \mu_2) + V_{\mu_1\mu_1}(\mu_1, \mu_2)\frac{\sigma^2}{2}dt$. Then,

$$\frac{2c}{\sigma^2} = \max\left\{V_{\mu_1\mu_1}, V_{\mu_2\mu_2}\right\}$$

The consumer searches A if $V_{\mu_1\mu_1} = \frac{2c}{\sigma^2} > V_{\mu_2\mu_2}$.

The optimal search policy can be defined as $\mu_i = \overline{U}_i(\mu_j)$ (Buy i), $\mu_i = \underline{U}_i(\mu_j)$ (Buy j or zero).

The value matching moment is

$$V(\mu_1, \mu_2)\Big|_{\mu_i = \overline{U}_i(\mu_j)} = \overline{U}_i(\mu_j)$$

$$V(\mu_1, \mu_2)\Big|_{\mu_i = \underline{U}_i(\mu_j)} = \max\{0, \mu_j\}$$

Smooth pasting moment is

$$V_{\mu_1}(\mu_1, \mu_2)\Big|_{\mu_1 = \overline{U}_1(\mu_2)} = 1$$

$$V_{\mu_1}(\mu_1, \mu_2)\Big|_{\mu_2 = \underline{U}_2(\mu_1)} = 0$$

... The solutions are

$$\underline{U}(\mu) = -\frac{\sigma^2}{4c}, \overline{U}(\mu) = ... \geq \frac{\sigma^2}{4c}$$

Observations:

1. Search alternative with the highest μ_i.
2. Endogenous consideration sets.
3. Require more information of an alternative if μ of the other alternative is high.
4. Purchase thresholds $\overline{U}(\mu)$ narrow as μ's increase and converge to $\frac{\sigma^2}{4c}$.
5. The expected utility is increasing in σ^2 and decreasing in c.

<u>Purchase Probability:</u> The probability of purchase alternative i is increasing in μ_i. The probability of purchase one of the alternatives can be decreased by increasing one of the μ_i (which induces more searches).

<u>Number of Alternatives:</u> For $N = 2$, the thresholds converge to $\frac{\sigma^2}{4c}$. For $N = 3$, the thresholds converge to $\frac{4}{3}\frac{\sigma^2}{4c}$. For $N = \infty$, the thresholds converge to $2\frac{\sigma^2}{4c}$.

<u>Heterogeneous Search Cost or σ^2:</u> Search more (larger thresholds) for the alternative with higher σ^2 or lower cost.

Chapter 3 Dynamic Competition

3.1 Competition with Multiple Products

3.1.1 Bertrand Supertraps

Products interact on demand and supply.

$$\frac{P - MC}{P} = \frac{1}{E}$$

If products are complementary, P's are lower. $\frac{P-MC}{P} < \frac{1}{E}$. In competition, the complementary properties decrease the prices, giving lower expected profits. (Bertrand Supertraps)

Consider two products A and B.

No Bundling: Reminds that in Bertrand the price is $p_A = p_B = c+t$, where t is the parameter for transportation cost. The profits are $\Pi_A = \Pi_B = \frac{t}{2}$ in both A and B markets. The total profit of a firm is $\Pi_A + \Pi_B = t$.

Bundling: A and B sold together. In this case, the bundled price is $p_{BD} = 2c + t$ with the total profit being $\Pi_{BD} = \frac{t}{2}$. (Bundling induces more intensive competition, giving lower profits).

3.1.2 Two-Sided Markets

Interaction between two end users: Sellers (Suppliers) (S) and Buyers (B).

Example 3.1

Video Games: In the platform, buyers (gamers) buy games and sellers (game publishers) sell games.

OS: App developers and users.

Advertising: Viewers and Ad companies in media.

There exists membership charges A^S, A^B that gives membership externalities (benefits/hurts the other part) and usage charges a^S, a^B that gives usage externalities.

Monopoly Platform Suppose there is a fixed cost C_i per member on side i of market and a marginal cost c per interaction between 2 members. The average benefit per transaction is b_i and the fixed benefit of membership is B_i.

Assumption 3.1

Transaction involves no payments between end-users.

Let N^S and N^B be the numbers of sellers and buyers, respectively. The number of potential transactions: $N^B \cdot N^S$. The expected payoff of agent i is

$$U^i = (b^i - a^i)N^j + B^i - A^i$$

where $N^j = \text{Prob}\left(U^j \geq 0\right)$.

Define "per-interaction price" as

$$P_i := a^i + \frac{A^i - C^i}{N^j}$$

$$\begin{aligned} U^i \geq 0 &\Leftrightarrow (b^i - a^i)N^j + B^i - A^i \geq 0 \\ &\Leftrightarrow b_i + \frac{B^i - C^i}{N^j} \geq P_i \end{aligned}$$

Thus,

$$N^i = \text{Prob}\left(U^i \geq 0\right) = \text{Prob}\left(b_i + \frac{B^i - C^i}{N^j} \geq P_i\right) := D^i\left(P_i, N^j\right)$$

Similarly, we can have

$$\begin{cases} N^S & = D^S\left(P_S, N^B\right) \\ N^B & = D^B\left(P_B, N^S\right) \end{cases} \Rightarrow \begin{cases} N^S & = n^S\left(P_S, P_B\right) \\ N^B & = n^B\left(P_S, P_B\right) \end{cases}$$

The platform's profit is given by

$$\begin{aligned} \Pi &= \left(A^B - C^B\right)N^B + \left(A^S - C^S\right)N^S + (a^B + a^S - c)N^S N^B \\ &= (P_B + P_S - c)\, n^S\left(P_S, P_B\right) n^B\left(P_S, P_B\right) \end{aligned}$$

3.2 Dynamic Competition

1. Detection lags incurs more competition.
2. Firm asymmetry leads to more competition.
3. Multi-market compact may increase the collusion.
4. Larger number of competitors / Longer horizon increases the competition.

Suppose there are two firms with marginal cost c selling homogeneous products. The demand follows

$$D_i(p_i, p_j) = \begin{cases} D_i(p_i), & p_i < p_j, \\ \frac{1}{2}D_i(p_i), & p_i = p_j, \\ 0, & p_i > p_j \end{cases}$$

Suppose there is a finite time T. The payoff of firm i is

$$\sum_{t=1}^{T} \delta^{t-1}\Pi^i(p_{it}, p_{jt})$$

(δ can be written as $\delta = \frac{1}{1+r}$).

When deciding p_{it}, the firm i knows history until $t-1$.

$T=1$ (only one period): it is exactly the static case. $p_1 = p_2 = c$.

For any finite T (finite periods): Since we must have $p_{1T} = p_{2T} = c$ in the period T, we must have $p_{1t} = p_{2t} = c$ in any period.

3.2.1 Infinite Periods

$p_{1t} = p_{2t} = c$ is still an equilibrium.

Another equilibrium can be achieved by the strategy: Collude at monopoly price $p^M := \text{argmax}_p (p-c)D(p)$ (denote the total profit in each period as $\Pi^M := \max_p (p-c)D(p)$) and move to $p=c$ if the competitor deviates. The existence of this equilibrium requires

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} \geq \Pi^M \Leftrightarrow \delta \geq \frac{1}{2}$$

(Folk Theorem)

N Competitors In the case of N competitors, the condition becomes

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{N} \geq \Pi^M \Leftrightarrow \delta \geq 1 - \frac{1}{N}$$

Detection Lags The deviation is not detected at once, suppose after two periods. Then the condition becomes

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} \geq (1+\delta)\Pi^M \Leftrightarrow \delta \geq \frac{1}{\sqrt{2}}$$

Barriers to Entry

Differentiation Higher differentiation makes collusion less likely, as it decreases the punishment.

Fluctuations in Demand When demand is high, firms have a higher temptation to deviate $\Rightarrow$ Price Wars during booms.

Secrete Price Cuts with Demand Fluctuations Suppose the prices are not observed but the demand fluctuates. Given a decrease in sales, you do not know whether it is incurred by the competitor's deviation or the fluctuation of the demand. Firms give punishment in some degree $\Rightarrow$ Price wars during recessions.

Cost Asymmetries

Multi-Market Contract Suppose there are two markets: Market A meets every period and Market B meet every two periods.

As we know the collusion occurs in A if $\delta \geq \frac{1}{2}$ and in B if $\delta \geq \frac{1}{\sqrt{2}}$, if A and B are separated.

If we consider these two markets together, the condition is

$$\sum_{t=1}^{\infty} \delta^{t-1} \frac{\Pi^M}{2} + \sum_{t=1}^{\infty} \delta^{2t-2} \frac{\Pi^M}{2} \geq 2\Pi^M$$

Therefore, there exists $\delta \in \left[\frac{1}{2}, \frac{1}{\sqrt{2}}\right]$ that can make collusion in both markets.

3.2.2 Empirical Analysis

Suppose the demand in time t at market s is given by

$$P_{ts} = f\left(q_{1ts} + q_{2ts}, Z_{ts}\right)$$

where q's are quantities and Z is IV.

The cost of production is given by

$$C_{its} = F_{its} + C^{VC}\left(q_{its}, w_{ts}\right)$$

where F_{its} is the fixed cost, $C^{VC}(\cdot)$ is the variance cost, and w_{ts} are prices of inputs.

1. In perfect competition, $P_{ts} = MC_{its}$.
2. In perfect collusion, $(q_{1ts}, q_{2ts}) = \operatorname{argmax} P_{ts}Q_{ts} - C_{1ts} - C_{2ts}$, where $Q_{ts} = q_{1ts} + q_{2ts}$. The F.O.C. is
$$P_{ts} + Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}} - MC_{its} = 0$$
$$P_{ts} = MC_{its} - Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$$
3. Cournot competition: $P_{ts} = MC_{its} - \frac{1}{2}Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$.

Then, testing the perfect competition or collusion, can be given by estimating the θ:

$$P_{ts} = MC_{its} - \theta Q_{ts}\frac{\partial P_{ts}}{\partial q_{1ts}}$$

1. Perfect competition: $H_0 : \theta = 0$
2. Perfect collusion: $H_0 : \theta = 1$
3. Cournot competition: $H_0 : \theta = \frac{1}{2}$

3.2.3 Market Continue with Probability x

Suppose the market continues with probability x. The condition becomes

$$\sum_{t=1}^{\infty} (x\delta)^{t-1} \frac{\Pi^M}{2} \geq \Pi^M \Leftrightarrow \delta \geq \frac{1}{2x}$$

3.3 Repeated Game

3.3.1 Reputation

	C	$\tilde{C}$
C	1,1	-1,1.5
$\tilde{C}$	1.5,-1	0,0

There are two periods.

Suppose there is a probability $\frac{1}{2}$ that the player 1 is "irrational" such that it always chooses C in the first period, and it chooses C in the second period if the player 2 chooses C in the first period and chooses $\tilde{C}$ in the second period if the player 2 chooses $\tilde{C}$ in the first period.

"Rational" player may choose either C or $\tilde{C}$ in the first period and always choose $\tilde{C}$ in the second period.

The payoffs of two rational players (each player has an equal probability to meet the other rational player or the irrational player) from the game is

	C	$\tilde{C}$
C	$\frac{7}{4}, \frac{7}{4}$	$\frac{3}{4}, \frac{3}{2}$
$\tilde{C}$	$\frac{3}{2}, \frac{3}{4}$	$\frac{3}{4}, \frac{3}{4}$

There are two Nash equilibrium, (C, C) and $(\tilde{C}, \tilde{C})$.

3.3.2 Multiple Equilibria

	p_H	p_M	p_L
p_H	1,1	-1,1.5	-2,1
p_M	1.5,-1	0,0	-2,0
p_L	1,-2	0,-2	-1,-1

For two periods of game, we have the following equilibrium:

1. In first period, play (p_H, p_H).
2. In second period, play (p_M, p_M) if (p_H, p_H) is played in the first period, and play (p_L, p_L) if other outcome is played in the first period.

3.4 Markov Perfect Equilibrium

1. Switching cost / loyalty effect.

2. Consumer learning.
3. Durable goods.
4. Network effects.
5. Learning by doing.
6. Advertising dynamics.
7. Entry with costs.

A refinement: you can only condition actions on payoff relevant state variables. Thus, if $\Pi(a, h_1) = \Pi(a, h_2), \forall a$, we have h_1 and h_2 in the same state.

Player i's payoff with action a_t at time t is written as $\Pi_t^i(a_t, s_t)$, where s_t is the payoff relevant state at time t.

The value function is

$$V^i(s_t) = \max_{a_t^i} \Pi_t^i((a_t^i, a_t^{-i}), s_t) + \delta V^i(h(s_t, a_t))$$

where $s_{t+1} = h(s_t, a_t)$ is the transition function and the solution $a_t^i(s_t)$ is called policy function.

Application

Uncertainty of evolution process Transportation probability is Prob $(s_{t+1} \mid s_t, a_t)$. Then,

$$V^i(s_t) = \max_{a_t^i} \Pi_t^i((a_t^i, a_t^{-i}), s_t) + \sum_{s'} \delta V^i(s') \text{Prob}\left(s' \mid s_t, a_t\right)$$

Numerical Computation:

1. Value iteration: (usually slow) start at $[V^i]^{(0)}$.

$$[V^i(s_t)]^{(k+1)} = \max_{a_t^i} \Pi_t^i((a_t^i, a_t^{-i}), s_t) + \sum_{s'} \delta [V^i(s')]^{(k)} \text{Prob}\left(s' \mid s_t, a_t\right)$$

until $[v^i]^{(k)}$ converges.

2. Policy iteration: start at $a^{(0)}(s_t)$. There are N states and J players.

$$\underbrace{\begin{pmatrix} V^i(s_t) \end{pmatrix}}_{N\times 1} = \underbrace{\begin{pmatrix} \Pi_t^i(a^{(k)}(s_t), s_t) \end{pmatrix}}_{N\times 1} + \delta \underbrace{\begin{pmatrix} \text{Prob}\left(s' \mid s_t, a^{(k)}(s_t)\right) \end{pmatrix}}_{N\times N} \underbrace{\begin{pmatrix} V^i(s') \end{pmatrix}}_{N\times 1}$$

$$V = \Pi + \delta P V$$

$$V = (I - \delta P)^{-1} \Pi$$

$$[a^i]^{(k+1)} = \operatorname*{argmax}_{a_t^i} \Pi_t^i((a_t^i, a_t^{-i}), s_t) + \sum_{s'} \delta [V^i(s')]^{(k)} \text{Prob}\left(s' \mid s_t, a_t\right)$$

Obtain empirically policy functions: We can regress

(a). a_t on s_t to get policy function.

(b). s_{t+1} on a_t and s_t.

Plug in $V = (I - \delta P)^{-1}\Pi$ to obtain V.

1. s_t is discrete or continuous (interpolation).
2. Uncertainty on profit function $\Pi^i(a_t, s_t, \epsilon_t)$. That is, the profit also depends on a noise ϵ_t:

Known ϵ_t:

$$V^i(s_t, \epsilon_t) = \max_{a_t^i} \Pi_t^i((a_t^i, a_t^{-i}), s_t, \epsilon_t) + \sum_{s'} \delta V^i(s', \epsilon_{t+1}) \text{Prob}\left(s', \epsilon_{t+1} \mid s_t, a_t, \epsilon_t\right)$$

Unknown ϵ_t:

$$\Pi_t^i(a_t, s_t) = \mathbb{E}_{\epsilon_t} \tilde{\Pi}_t^i(a_t, s_t, \epsilon_t)$$

Private information on ϵ_t.

3.4.1 Switching Costs

2-Period Model

Suppose consumers are uniformly distributed on $[0, 1]$. $t \sim$ Unif$[0, 1]$ is the transportation cost, s is he switching cost, f is the fraction of consumers that have the same preferences from period to period, and δ_F, δ_S are the discount factors.

Consider the second period first. Suppose q_1^i consumers bought i in the first period, where $q_1^i = 1 - q_1^{-i}$. Assume the consumers who bought from one firm in the first period and did not change preferences continue to buy from that firm in the second period. Then,

$$q_2^i = yq_1^i + (1-y)q_1^i \frac{t + s + p_2^{-i} - p_2^i}{2t} + (1-y)(1-q_1^i)\frac{t - s + p_2^{-i} - p_2^i}{2t}$$

Thus, the best response to p_2^{-i} is given by

$$\max_{p_2^i} p_2^i q_2^i$$

By FOC, we have

$$p_2^i = t + \frac{2t}{3}\frac{y}{1-y}\left(1 + q_1^i\right) + \frac{s}{3}\left(2q_1^i - 1\right)$$

Then, $p_2^i = \frac{t}{1-y}$ if $q_1^i = \frac{1}{2}$. Moreover, we have

$$\frac{\partial q_2^i}{\partial p_2^{-i}} = \frac{1-y}{2t}, \; \frac{\partial q_2^i}{\partial q_1^i} = y + (1-y)\frac{s}{t}$$

$$\frac{\partial p_2^{-i}}{\partial q_1^i} = -\frac{2}{3}\left(\frac{ty}{1-y} + s\right), \; \frac{\partial \Pi_2^i}{\partial q_1^i} = \frac{2}{3}\left(\frac{yt}{1-y} + s\right)$$

In the first period, the indifferent consumer t should be

$$p_1^i + tq_1^i + y\delta_C\left[p_2^i + tq_1^i\right] + (1-y)\delta_C M^i = p_1^{-i} + t(1-q_1^i) + y\delta_C\left[p_2^{-i} + tq_1^{-i}\right] + (1-y)\delta_C M^{-i}$$

where $M^i := \mathbb{E}_z \min\left\{p_2^i + tz, p_2^j + t(1-z) + s\right\}$ is the expected cost in second period if consumers bought i in the first period and change preferences. ... We can have $M^i + M^{-i} = \frac{s}{t}\left[p_2^i - p_2^{-i}\right]$.

Firm i's problem is $\Pi_1^i = \max_{p_1^i} p_1^i q_1^i + \delta_F \Pi_2^i(q_1^i)$.

As δ_F increases, p_1^i and Π_1^i decrease. As δ_C increases, p_1^i and Π_1^i increase.

Infinite Horizon

Overlapping Generations: Consumers only live for 2 periods.

$$V(q_{t-1}^i) = \max_{p_t^i} p_t^i \cdot (q_t^i + q_{ot}^i) + V(q_t^i)$$

where the q_{t-1}^i is the number of new consumers in period $t-1$ who chooses i and the q_{ot}^i is a function about q_{t-1}^i which represents the number consumers in previous period who chooses i in this period. (q_{ot}^i is the $q_2^i = yq_1^i + (1-y)q_1^i \frac{t+s+p_2^{-i}-p_2^i}{2t} + (1-y)(1-q_1^i)\frac{t-s+p_2^{-i}-p_2^i}{2t}$ in previous case.)

Given the linear demand function. The relationship between the demand and prices is constructed as

$$\frac{dq_t^i}{dp_t^i} = -\frac{dq_t^j}{dp_t^i} = -\frac{1}{\Delta}$$

Δ is some function (omitted). The F.O.C.s' forms are $p_t^i = c + dq_{t-1}^i$. ...

The results are: $\frac{\partial p_t^i}{\partial \delta^F} < 0$; as $\delta_P = \delta_C$ increases, Π decreases.

3.4.2 Experience Goods

3.5 Price Discrimination

The problem of price discrimination: arbitrage

1. Transfer of goods, which can be impeded by transaction costs.
2. Transfer of demands.

Price discrimination can increase the quantity supplied.

3.5.1 First Degree Price Discrimination (Perfect)

Consider a linear demand of homogeneous consumers, the perfect price discrimination can be implemented as:

Tariff, total payment

$$T = A + p \cdot q$$

So that the average price can vary according to the demand of the consumer.

Consider a 1 unit demand for heterogeneous consumers, the perfect price discrimination can be implemented as setting individual prices.

3.5.2 Third Degree Price Discrimination (External Signal)

Given exogenous information, the price is based on that exogenous information. (e.g. student ID, covid or not, age, gender, income).

Pricing based on $\frac{p-MC}{p} = \frac{1}{E_d}$ in each submarket.

Example 3.2 (Spatial Discrimination)

Suppose the firm locates at 0 and consumers located at x. There is a transportation cost $t \cdot x$ incurred by the firm to sell to consumers

The optimal price for consumers at x is charged as

$$p^*(x) = \underset{p}{\operatorname{argmax}}(p - tx - c)D(p) = \frac{a + btx + bc}{2b}$$

where $D(p) = a - bp$. Thus,

$$\frac{\partial p^*(x)}{\partial x} = \frac{t}{2}$$

The discrimination against consumers close-by.

Now, consider $D(p) = ae^{-bp}$. The optimal price for consumers at x is charged as

$$p^*(x) = \underset{p}{\operatorname{argmax}}(p - tx - c)D(p) = tx + c + \frac{1}{b}$$

Thus,

$$\frac{\partial p^*(x)}{\partial x} = t$$

There is no discrimination compared to cost.

Now, consider $D(p) = ap^{-b}$. The optimal price for consumers at x is charged as

$$p^*(x) = \underset{p}{\operatorname{argmax}}(p - tx - c)D(p) = \frac{b}{b-1}(tx + c)$$

Thus,

$$\frac{\partial p^*(x)}{\partial x} = \frac{b}{b-1}t > t$$

There is price discrimination against consumers far-away. (Which may not happen, if there is a competitive transportation industry.)

Example 3.3 (2-Vertical Intergration, PS3 #4)

Suppose there is a manufacturer with two downstream industries C_1 and C_2 such that the elasticity follows $\epsilon_2 > \epsilon_1$. Thus, the monopoly prices are $p_1^* = \frac{c}{1-1/\epsilon_1} > p_2^* = \frac{c}{1-1/\epsilon_2}$. The manufacturer integrates C_2 can achieve third-price discrimination. (He sets the monopoly price that is lower than his competitor's MC in the industry as he sets the monopoly price in C_1.)

3.5.3 Second Degree Price Discrimination (Self-Selection)

Offer a menu for price-quantity (or price-quality) options.

Suppose the utility of consumers is defined as

$$U = \begin{cases} \theta V(q) - T, & \text{if buy} \\ 0, & \text{if not buy} \end{cases}$$

where θ measures how much consumer cares about quantity. θ equals to the marginal utility of income in consumption theory ($U(\text{income} - T) + V(q)$ vs. $U(\text{income})$, approximated by Taylor expansion).

Suppose there are two types of consumers, $\theta_2 > \theta_1$ with the proportion of θ_1 being λ. Let the $V(q) = \frac{1-(1-q)^2}{2}$ such that $V'(q) = 1 - q$.

Let the marginal cost $c : c < \theta_1 < \theta_2$. The consumer problem is

$$\max_q \theta_i V(q) - pq$$

Thus, $q^* = 1 - \frac{p}{\theta_i}$. The surplus of consumer is given as

$$S_i(p) = \theta_i \left[\frac{1-(1-D_i(p))^2}{2}\right] - pD_i(p) = \frac{(\theta_i - p)^2}{2\theta_i}$$

Define the Harmonic mean:

$$\frac{1}{\theta} := \frac{\lambda}{\theta_1} + \frac{1-\lambda}{\theta_2}$$

Then, we can have the demand:

$$D(p) = \lambda D_1(p) + (1-\lambda) D_2(p) = 1 - \frac{p}{\theta}$$

1. Prefect price discrimination: charge price $p = c$ and $A_i = \frac{(\theta_i - c)^2}{2\theta_i}$. $\Pi = \lambda \frac{(\theta_1 - c)^2}{2\theta_1} + (1-\lambda) \frac{(\theta_2 - c)^2}{2\theta_2}$.
2. Monopoly price:

$$p = \text{argmax}(p-c)D(p) = \frac{c+\theta}{2}$$

   The total profit is $\Pi = \frac{(\theta - c)^2}{4\theta}$.
3. Two-part Tariffs (Linear Price): $T(q) = A + p \cdot q$ (quantity discount). $A = \min\{S_1(p), S_2(p)\} = S_1(p)$. The optimal p is given by

$$p^* = \underset{p}{\text{argmax}}(p-c)D(p) + S_1(p) = \frac{c^2}{2c - \theta}$$

4. Non-linear Price: provide a two item menu $\{(q_1^*, T_1^*), (q_2^*, T_2^*)\}$, which satisfies

$$\theta_1 V(q_1^*) - T_1^* \geq \theta_1 V(q_2^*) - T_2^*$$

$$\theta_2 V(q_2^*) - T_2^* \geq \theta_2 V(q_1^*) - T_1^*$$

$$\theta_i V(q_i^*) - T_i^* \geq 0, \ i = 1, 2$$

   We have $\theta_1 V_1(q_1^*) - T_1^* = 0$, $\theta_2 V(q_2^*) - T_2^* = \theta_2 V(q_1^*) - T_1^*$, $q_2^* = \text{argmax}_{q_2}\{\theta_2 V(q_2) - cq_2\}$. ($q_1^*$ is lower than the efficient amount, $q_1^* = \text{argmax}_{q_1}\{\theta_1 V(q_1) - cq_1\}$).

3.5.4 Behavior-Based Price Discrimination

What you bought will affect the price next.

Suppose there is a two periods model. Consumers' values distribute over $[0,1]$. Consumers with $v \geq v^* \in (0,1)$ bought in the first period. In the second period, we can price based whether a consumer bought in the first period.

Forward-looking consumers hurt firms.

3.6 Branding

3.6.1 Umbrella Branding

Whether tell consumers two firms belong to me. One of the reason: Signal quality.

3.6.2 Brand Values

Social interactions: you want to match with some types of people. Brand can signal type.

3.7 Signaling

High quality one wants to signal its quality.

Example 3.4 (Money-back Guarantees)

Marginal costs are $c_H > c_L = 0$. (ϕ is the probability of being high-quality).

Let $v_i, i \in \{L, H\}$ be the reservation value if it works and is i-quality.

Let $f_i, i \in \{L, H\}$ be the probability of falling if it is i-quality. ($f_H < f_L$ and $(1-f_H)v_H > (1-f_L)v_L$).

Firms' decisions are (1). Price p; (2). Decision to offer or not offer money-back guarantees (if fails).

Suppose there are transaction costs of using money-back guarantees: t_B for buyers and t_S for sellers.

With complete information, there is no money-back guarantee in equilibrium.

With incomplete information, there can have a separating equilibrium that H-quality offers MBG while L-quality does not, and may have a pooling equilibrium. The condition to make the separating equilibrium exist: the low-quality does not want to imitate the high-quality,

$$(1 - f_L)p_H - t_S f_L \leq p_L$$

High-quality does not benefit from not offering MBG,

$$(1 - f_H)p_H - t_S f_H - c_H \geq \max\{0, p_L - c_H\}$$

and $p_H > t_B$.

Then, we define consistent on-path beliefs and off-paths beliefs that cannot lead to deviations.

Result: the separating equilibrium exists if the t_S is either not too small or not too large.

3.8 Distribution Channels

3.8.1 Base Case

Suppose there is a manufacturer, a retailer, and consumers. The demand of consumers is $D(p; s)$, where s is the service offered by retailer.

1. Uniform Pricing: $T(q) = p_W \cdot q$.
2. Franchise Fee: $T(q) = A + p_W \cdot q$.
3. Fixed Prices: $p \in [\underline{p}, \overline{p}]$.
4. Fixed Quantities: $q \in [\underline{q}, \overline{q}]$.

Some constraints:

1. Exclusive Territories: Constraints on manufacturer.
2. Exclusive Dealing: Constraints on retailer.
3. Tie-In Sales/Royalties.

3.8.2 Double Marginalization (under uniform price)

Suppose $D(p) = 1 - p$.

Coordinated channel: $p^* = \text{argmax}_p (p - c)(1 - p) = \frac{1+c}{2}$. Total profit is $\frac{(1-c)^2}{4}$.

If not coordinated, the manufacturer chooses a price for retailer and the retailer chooses a price for consumers.

Given the price p_W set by the manufacturer, the retailer can choose a price for consumers:

$$p = \underset{p}{\text{argmax}}\, \Pi^R(p) = \underset{p}{\text{argmax}} (p - p_W)(1 - p) = \frac{1 + p_W}{2}$$

Then, the corresponding profit of the manufacturer from setting p_W is $\Pi^M(p_W) = \frac{(p_W - c)(1 - p_W)}{2}$. The optimal p_W is solved by

$$p_W^* = \underset{p_W}{\text{argmax}}\, \Pi^M(p_W) = \underset{p_W}{\text{argmax}} \frac{(p_W - c)(1 - p_W)}{2} = \frac{1 + c}{2} \Rightarrow p^* = \frac{3 + c}{4}$$

Total profit is $\frac{(1-c)^2}{16} + \frac{(1+c)^2}{8} = \frac{3(1-c)^2}{16}$.

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