Microeconomic Theory

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All models are wrong, but some are useful.

Contents

Chapter 1 Correspondence: $\Psi : X \to 2^Y$ (@ Lec 07 of ECON 204)

Definition 1.1 (Correspondence)

A **correspondence** $\Psi : X \to 2^Y$ from X to Y is a function from X to 2^Y, that is, $\Psi(x) \subseteq Y$ for every $x \in X$. (2^Y is the set of all subsets of Y) ♣

Example 1.1 Let $u : \mathbb{R}^n_+ \to \mathbb{R}$ be a continuous utility function, $y > 0$ and $p \in \mathbb{R}^n_{++}$, that is, $p_i > 0$ for each i. Define $\Psi : \mathbb{R}^n_{++} \times \mathbb{R}_{++} \to 2^{\mathbb{R}^n_+}$ by

$$\Psi(p, y) = \operatorname{argmax} u(x)$$

$$\text{s.t. } x \geq 0$$

$$p \cdot x \leq y$$

Ψ is the demand correspondence associated with the utility function u; typically $\Psi(p, y)$ is multi-valued.

1.1 Continuity of Correspondences

1.1.1 Upper/Lower Hemicontinuous

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$.

Definition 1.2 (Upper Hemicontinuous)

Ψ is **upper hemicontinuous** (uhc) at $x_0 \in X$ if, for every open set V with $\Psi(x_0) \subseteq V$, there is an open set U with $x_0 \in U$ s.t.

$$x \in U \Rightarrow \Psi(x) \subseteq V$$

♣

Upper hemicontinuity reflects the requirement that Ψ doesn't "jump down/implode in the limit" at x_0. *(A set to "jump down" at the limit x_0: It should mean the set suddenly gets smaller – it "implodes in the limit" – that is, there is a sequence $x_n \to x_0$ and points $y_n \in \Psi(x_n)$ that are far from every point of $\Psi(x_0)$ as $n \to \infty$.)*

Definition 1.3 (Lower Hemicontinuous)

Ψ is **lower hemicontinuous** (lhc) at $x_0 \in X$ if, for every open set V with $\Psi(x_0) \cap V \neq \emptyset$, there is an open set U with $x_0 \in U$ s.t.

$$x \in U \Rightarrow \Psi(x) \cap V \neq \emptyset$$

♣

Lower hemicontinuity reflects the requirement that Ψ doesn't "jump up/explode in the limit" at x_0. *(A set to "jump up" at the limit x_0: It should mean that the set suddenly gets bigger – it "explodes in the limit" – that is,* *there is a sequence $x_n \to x_0$ and a point $y_0 \in \Psi(x_0)$ that is far from every point of $\Psi(x_n)$ as $n \to \infty$.)*

Definition 1.4 (Continuous Correspondence)

Ψ is **continuous** at $x_0 \in X$ if it is both **uhc** and **lhc** at x_0. ♣

Proposition 1.1

Ψ is upper hemicontinuous (respectively lower hemicontinuous, continuous) if it is uhc (respectively lhc, continuous) at every $x \in X$. ♠

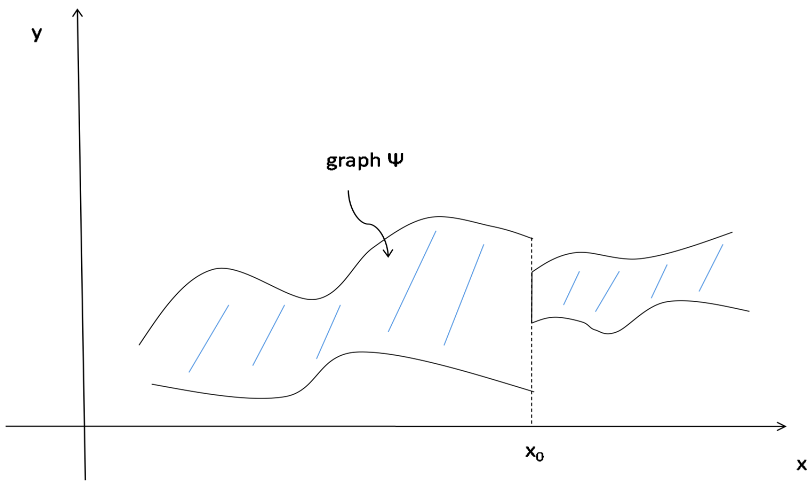

Figure 1.1: The correspondence Ψ "implodes in the limit" at x_0. Ψ is not upper hemicontinuous at x_0.

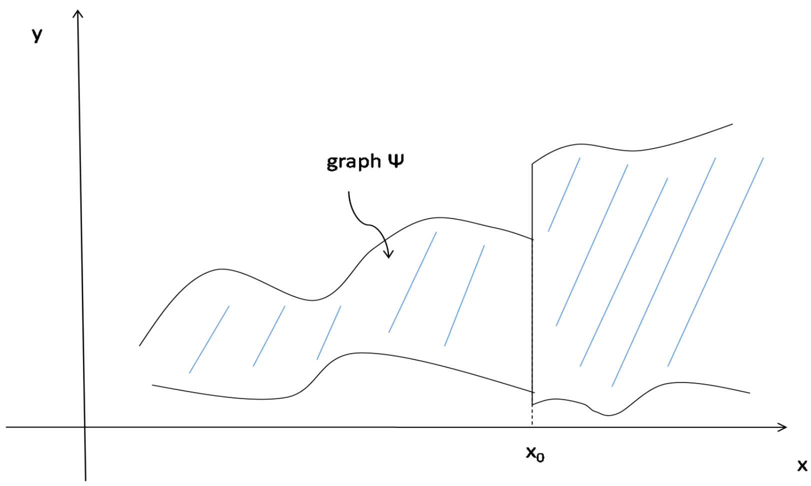

Figure 1.2: The correspondence Ψ "explodes in the limit" at x_0. Ψ is not lower hemicontinuous at x_0.

1.1.2 Theorem: $\Psi(x) = \{f(x)\}$ is uhc $\Leftrightarrow$ f is continuous

Theorem 1.1 ($\Psi(x) = \{f(x)\}$ **is uhc** $\Leftrightarrow$ f **is continuous)**

*Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$ and $f : X \to Y$. Let $\Psi : X \to 2^Y$ be defined by $\Psi(x) = \{f(x)\}$ for all $x \in X$. Then Ψ is **uhc** if and only if f is **continuous**.*

♡

1.1.3 Berge's Maximum Theorem: the set of maximizers is uhc with non-empty compact values

Theorem 1.2 (Berge's Maximum Theorem)

Let $X \subseteq \mathbb{R}^n$ and $Y \subseteq \mathbb{R}^m$. Consider the function $f : X \times Y \to \mathbb{R}$ and the correspondence $\Gamma : Y \to 2^X$. Define $v(y) = \max_{x \in \Gamma(y)} f(x, y)$ and the set of maximizers

$$\Omega(y) = \operatorname*{argmax}_{x \in \Gamma(y)} f(x, y) = \{x : f(x, y) = v(y)\}$$

Suppose f and Γ are continuous, and that Γ has non-empty compact values. Then, v is continuous and Ω is uhc with non-empty compact values.

♡

1.2 Graph of Correspondence

An alternative notion of continuity looks instead at properties of the graph of the correspondence.

Definition 1.5 (Graph of Correspondence)

The **graph** of a correspondence $\Psi : X \to 2^Y$ is the set

$$\text{graph}\Psi = \{(x, y) \in X \times Y : y \in \Psi(x)\}$$

♣

1.2.1 Closed Graph

By the definition of continuous function $f : \mathbb{R}^n \to \mathbb{R}$, each convergent sequence $\{(x_n, y_n)\}$ in graph f converges to a point (x, y) in graph f, that is, graph f is closed.

Definition 1.6 (Closed Graph)

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$. A correspondence $\Psi : X \to 2^Y$ has closed graph if its graph is a closed subset of $X \times Y$, that is, if for any sequences $\{x_n\} \subseteq X$ and $\{y_n\} \subseteq Y$ such that $x_n \to x \in X$, $y_n \to y \in Y$ and $y_n \in \Psi(x_n)$ for each n, then $y \in \Psi(x)$.

♣

Example 1.2 Consider the correspondence $\Psi(x) = \begin{cases} \{\frac{1}{x}\}, & \text{if } x \in (0, 1] \\ \{0\}, & \text{if } x = 0 \end{cases}$ ("implode in the limit")

Let $V = (-0.1, 0.1)$. Then $\Psi(0) = \{0\} \subseteq V$, but no matter how close x is to 0, $\Psi(x) = \{\frac{1}{x}\} \not\subseteq V$, so Ψ is not uhc at 0. However, note that Ψ has closed graph.

1.3 Closed-valued, Compact-valued, and Convex-valued Correspondences

Definition 1.7 (Closed-valued, Compact-valued, and Convex-valued Correspondences)

Given a correspondence $\Psi : X \to 2^Y$,

1. Ψ is **closed-valued** if $\Psi(x)$ is a closed subset of Y for all x;
2. Ψ is **compact-valued** if $\Psi(x)$ is compact for all x.
3. Ψ is **convex-valued** if $\Psi(x)$ is convex for all x.

♣

1.3.1 Closed-valued, uhc and Closed Graph

For closed-valued correspondences these concepts can be more tightly connected. A closed-valued and upper hemicontinuous correspondence must have closed graph. For a closed-valued correspondence with a compact range, upper hemicontinuity is equivalent to closed graph.

Theorem 1.3 (uhc and Closed Graph)

Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$.

1. *Ψ is **closed-valued** and **uhc** $\Rightarrow$ Ψ has **closed graph**.*
2. *Ψ is **closed-valued** and **uhc** $\Leftarrow$ Ψ has **closed graph**. (If Y is **compact**)*

♡

Theorem 1.4

*Let $X \subseteq \mathbb{E}^n$, $Y \subseteq \mathbb{E}^m$, and $\Psi : X \to 2^Y$. If Ψ has **closed graph** and there is an **open set** W with $x_0 \in W$ and a **compact set** Z such that $x \in W \cap X \Rightarrow \Psi(x) \subseteq Z$, then Ψ is **uhc** at x_0.*

♡

1.3.2 Theorem: compact-valued, uhs correspondence of compact set is compact

Theorem 1.5

Let X be a compact set and $\Psi : X \to 2^X$ be a non-empty, compact-valued upper-hemicontinuous correspondence. If $C \subseteq X$ is compact, then $\Psi(C)$ is compact.

♡

Proof 1.1

Given the compact-valued Ψ, we can have an open cover of $\Psi(C)$, $\{U_\lambda : \lambda \in \Lambda\}$. So $\forall x \in C$, there exists $U_{l(x)}, l(x) \in \Lambda$ such that $U_{l(x)}$ is an open cover of $\Psi(x)$.

Consider a $c \in C$. Since Ψ is uhs and $\Psi(c) \subseteq U_{l(c)}$, there exists open set V_c s.t. $c \in V_c$ and $\Psi(x) \subseteq U_{l(c)}, \forall x \in V_c \cap C$.

$\{V_c : c \in C\}$ is an open cover of C. Because C is compact, there is a finite subcover $\{V_{c_i} : i = 1, ..., m\}, m \in \mathbb{N}$, where $\{c_i : i = 1, ..., m\} \subseteq C$.

Because $\Psi(x) \subseteq U_{l(c_i)}, \forall x \in V_{c_i} \cap C$ and $\{V_{c_i} : i = 1, ..., m\}, m \in \mathbb{N}$ is a open cover for C, we can infer $\{U_{l(c_i)} : i = 1, ..., m\}$ is a finite subcover of $\{U_{l(c)} : c \in C\}$ for $\Psi(C)$. Hence, $\Psi(C)$ is compact.

1.4 Fixed Points for Correspondences (@ Lec 13 of ECON 204)

1.4.1 Definition

Definition 1.8 (Fixed Points for Correspondences)

Let X be nonempty and $\psi : X \to 2^X$ be a correspondence. A point $x^* \in X$ is a fixed point of ψ if $x^* \in \psi(x^*)$.

Note *We only need x^* to be in $\psi(x^*)$, not $\{x^*\} = \psi(x^*)$. That is, ψ need not be single-valued at x^*. So x^* can be a fixed point of ψ but there may be other elements of $\psi(x^*)$ different from x^*.*

1.4.2 Kakutani's Fixed Point Theorem: uhs, compact, convex values correspondence has a fixed point over compact convex set

Theorem 1.6 (Kakutani's Fixed Point Theorem)

*Let $X \subseteq \mathbb{R}^n$ be a non-empty, **compact**, **convex** set and $\psi : X \to 2^X$ be an **upper hemi-continuous** correspondence with non-empty and **convex** values. Then ψ has a fixed point in X.*

1.4.3 Theorem: $\exists$ compact set $C = \cap_{i=0}^{\infty} \Psi^i(X)$ s.t. $\Psi(C) = C$

Theorem 1.7

Let (X, d) be a compact metric space and let $\Psi(x) : X \to 2^X$ be a upper-hemicontinuous, compact-valued correspondence, such that $\Psi(x)$ is non-empty for every $x \in X$. There exists a compact non-empty subset $C \subseteq X$, such that $\Psi(C) \equiv \cup_{x \in C} \Psi(x) = C$.

Proof 1.2

Let's construct a sequence $\{C_n\}$ such that $C_0 = X$, $C_1 = \Psi(C_0)$, ..., $C_n = \Psi(C_{n-1})$, ... We claim that $C = \cap_{i=0}^{\infty} C_i$ is a non-empty compact set and satisfies $\Psi(C) = C$.

1. *Because we can infer $\Psi(X_1) \subseteq \Psi(X_2)$ if $X_1 \subseteq X_2$, $X = C_0 \supseteq C_1 \Rightarrow C_1 = \Psi(C_0) \supseteq C_2 = \Psi(C_1)$,...., so $C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq \cdots$. Hence, C is not empty.*

2. *Because X is compact, by the theorem 1.5, we can infer C_n is compact for all n. Then, C_n is closed for all n, so C is closed. Because C is a closed set of compact set X, C is compact.*
3. $C \subseteq C_n, \forall n \Rightarrow \Psi(C) \subseteq \Psi(C_n), \forall n \Rightarrow \Psi(C) \subseteq C$
4. *Assume $C \subseteq \Psi(C)$ doesn't hold, that is $\exists y \in C$ s.t. $y \notin \Psi(C)$. Because $y \in C$ and $C_0 \supseteq C_1 \supseteq \cdots C_n \supseteq \cdots$, there exists $k \in C_n$ for all n s.t. $y \in \Psi(k)$. $k \in \cap_{i=1}^{\infty} C_i = C$, so $\Psi(k) \subseteq \Psi(C)$, which contradicts to $y \notin \Psi(C)$. Hence, $C \subseteq \Psi(C)$.*

All in all the claim "$C = \cap_{i=0}^{\infty} C_i$ is a non-empty compact set and satisfies $\Psi(C) = C$" is proved.

Chapter 2 Preference and Utility Function

Based on

- Mas-Colell, Whinston, and Green, Microeconomic Theory, Oxford University Press (1995).
- UIUC ECON 530 21Fall, Nolan H. Miller
- UC Berkeley ECON 201A 23Fall
- UC Berkeley MATH 272 23Fall, Alexander Teytelboym
- Jehle, G., Reny, P.: Advanced Microeconomic Theory. Pearson, 3rd ed. (2011). Ch. 6.
- Notes on Social Choice and Welfare, Alejandro Saporiti
- Yu, N. N. (2012). A one-shot proof of Arrow's impossibility theorem. *Economic Theory*, 523-525.

2.1 Preferences

2.1.1 Preference Relation

Definition 2.1 (Weak, Strict, Indifference)

$\succeq$ referred to as the **weak preference relation**: "x is at least as good as y". (ordinal);

"**No better than**": $y \preceq x$ if and only if $x \succeq y$.

"**Strict preference**": $x \succ y$ if and only if $x \succeq y$ and not $y \succeq x$.

"**Indifference**": $x \sim y$ if and only if $x \succeq y$ and $y \succeq x$. ♣

2.1.2 Basic Assumptions

2.1.3 Rational Preference

Definition 2.2 (Rantional Relation = Preference)

A binary relation $\succeq$ on X is a **preference relation** if it is a weak order, i.e., **complete** and **transitive**.

Rationality: $\succeq$ is **rational** <u>if and only if</u> it is **complete** and **transitive**.

- $\succeq$ is **complete** iff $\forall x, y \in X$, $x \succeq y$ or $y \succeq x$.
- $\succeq$ is **transitive** iff $\forall x, y, z \in X$, if $x \succeq y$ and $y \succeq z$, then $x \succeq z$. ♣

The completeness means

- Any two bundles can be compared
- Indifference is allowed

The transitivity

- like transitivity of the real numbers
- extends pairwise preferences to longer chains in the logical way.

2.2 Utility Function

2.2.1 Utility Function $\Leftrightarrow$ Rational Preference

Definition 2.3 (Unitility Function)

We can say a function $u : X \to \mathbb{R}$ represents $\succeq$ if $\forall x, y \in X$,

$$x \succeq y \Leftrightarrow u(x) \geq u(y)$$

Proposition 2.1 (Rational $\succeq \Rightarrow \exists u(\cdot)$)

If $\exists$ a function $u : X \to \mathbb{R}$ represents $\succeq$, then $\succeq$ is rational (i.e., completeness and transitivity)

Note *The reverse may not true.*

2.2.2 Convex Preference

Definition 2.4 (Convex $\succeq$)

$\succeq$ is **convex** if for every $x \in X$ the $\{y \in X : y \succeq x\}$ is convex, i.e., $y_1 \succeq x$ and $y_2 \succeq x \Rightarrow \alpha y_1 + (1-\alpha)y_2 \succeq x$ for all $\alpha \in [0,1]$.

Convex relations imply *averages are preferred to extremes*.

Definition 2.5 (Strictly Convex)

$\succeq$ is **strictly convex** iff $\forall x, y, z \in X$, if $x \succeq z$ and $y \succeq z$, then $\alpha x + (1-\alpha)y \succ z$ for all $\alpha \in (0,1)$

2.2.3 Convex Preference $\Leftrightarrow$ Quasiconcave Utility Function

Definition 2.6 (Quasi-Concave Function)

A function u is **quasi-concave** if and only if for all $t \in \mathbb{R}$, $\{x \in X : u(x) \geq t\}$ is convex.

$$\forall x, y \in X, t \in \mathbb{R}, 0 \leq a \leq 1 : u(x) \geq t, u(y) \geq t \Rightarrow u(ax + (1-a)y) \geq t$$

Proposition 2.2 (Concave Function $\Rightarrow$ Quasi-Concave Function)

Any function that is concave is also quasi-concave.

Proposition 2.3 (Convex $\succeq$ $\Leftrightarrow$ quasi-concave $u(\cdot)$)

$\succeq$ is convex, $\Leftrightarrow \exists$ a quasi-concave $u(\cdot)$ that represents $\succeq$.

2.3 Preferences over Nearby Bundles

2.3.1 Monotone Preference

Definition 2.7 (Monotone $\succeq$)

$\succeq$ is **monotone** if $x, y \in X$ with $x \geq y \Rightarrow x \succeq y$ (and $x > y \Rightarrow x \succ y$).

Proposition 2.4 (Monotone $\succeq$ $\Rightarrow$ monotone $u(\cdot)$)

If $\succeq$ is monotone, then $\exists$ a monotone $u(\cdot)$ that represents $\succeq$.

Note *Complete, transitive, and monotone are three assumptions that made by all theories (either EU or non-EU).*

2.3.2 Strongly monotone

Definition 2.8 (Strongly Monotone $\succeq$)

$\succeq$ is **strongly monotone** if and only if for any $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in X$, if $\forall i : x_i \geq y_i$ and $\exists j$ such that $x_j > y_j$, then $x \succ y$.

(When we compare elements that have more than one dimension, strongly monotone holds if at least one relation is not equal.)

$$A = (1, 1), B = (2, 1), C(1, 2), D = (2, 2)$$

Strongly monotone can infer that $D \succ B \succ A, D \succ C \succ A$.

2.3.3 Local Non-Satiation

Even weaker assumptions will ensure that the consumer's choice exhausts their budget.

Definition 2.9 (Local Nonsatiation)

For any bundle x, there is a nearby bundle y in the consumption set such that y is preferred to x. That is, for all $x \in X$ and every $\varepsilon > 0$,

$$\exists y \in |x - y| < \varepsilon, \text{ s.t. } y \succ x$$

We have

$$\text{Strong Monotonicity} \Rightarrow \text{Monotonicity} \Rightarrow \text{Local Nonsatiation}$$

2.4 Common Assumptions of Preference

$\succsim$		u
monotone	$\Longrightarrow$	nondecreasing
strongly monotone	$\Longrightarrow$	strictly increasing
continuous	$\Longrightarrow$	continuous (Debreu's Theorem)
convex	$\Longrightarrow$	quasi-concave (but not concave)
strictly convex	$\Longrightarrow$	strictly concave (and strictly quasi-concave)
homothetic (and continuous)	$\Longrightarrow$	continuous and homogeneous
(so-called) quasi-linear	$\Longrightarrow$	quasi-linear
(so-called) differentiable	$\Longrightarrow$	differentiable
separable	$\Longrightarrow$	separable (form)
strongly separable	$\Longrightarrow$	additively separable (form)

Figure 2.1: Properties of Preference and Utility Function

2.4.1 Independence of Preference

The 'standard' model of decisions under risk is based on von Neumann and Morgenstern Expected Utility (EU), which requires the independence assumption.

Definition 2.10 (Independence of Preference)

Independence: For any $x, y, z \in X$ and $0 < \alpha < 1$, if $x \succeq y$ then $\alpha x + (1-\alpha)z \succeq \alpha y + (1-\alpha)z$. ♣

2.4.2 Continuous Preference

Definition 2.11 (Continuous $\succeq$)

$\succeq$ is **continuous** on X if and only if for any sequence $\{x^n, y^n\}_{n=1}^{\infty}$ with $x^n \succeq y^n$ and we note $x = \lim_{n\to\infty} x^n$, $y = \lim_{n\to\infty} y^n$, we have $x \succeq y$ (i.e., the graph $\{(x, y) \mid x \succeq y \subseteq X \times X\}$ is closed). ♣

Proposition 2.5 (Debreu's Theorem, Continuous $\succeq \Rightarrow$ continuous $u(\cdot)$)

If $\succeq$ is continuous (on X, a convex subset of $\mathbb{R}^k$), then $\exists$ a continuous $u(\cdot)$ that represents $\succeq$. ♠

Example 2.1 (Lexicographic preferences (not continuous)) Under Lexicographic preference $\succ$, $x \succ y$ if and only if

- $x_1 > y_1$, or
- $x_1 = y_1$, and $x_2 > y_2$, or
- $x_1 = y_1$ and $x_2 = y_2$ and $x_3 > y_3$, or
- etc.

Under Lexicographic preferences, there is no indifference.

We can find the Lexicographic preference violates continuity: $\left(1+\frac{1}{n},1\right) \succ (1,2)$ and $\lim\left(1+\frac{1}{n},1\right) = (1,1) \prec (1,2)$.

Example 2.2 (Utility Representation for Lexicographic Preferences) Consider the lexicographic preference $\succeq$ over the restricted domain $X = (\mathbb{Q} \cap [0,1]) \times [0,1]$. Enumerate the rationals in $[0,1]$ as $\mathbb{Q} \cap [0,1] = \{q^1, q^2, q^3, ...\}$ where $q^i \neq q^j$ if $i \neq j$. The utility representation of this preference is

$$u(x_1,x_2) = \sum_{q^i<q^j} \frac{1}{2^i} + \frac{1}{2^j}x_2, \text{ where } q^j = x_1$$

2.4.3 Homothetic Preference

Definition 2.12 (Homotheticity)

$\succeq$ are homothetic if $x \succeq y \Rightarrow \alpha x \succeq \alpha y$ for all $\alpha > 0$. ♣

Proposition 2.6 (Homothetic preference $\Leftrightarrow$ homogeneous $u(\cdot)$)

A continuous $\succeq$ is homothetic $\Leftrightarrow \exists$ a continuous homogeneous $u(\cdot)$ that represents $\succeq$ such that $u(\alpha x) = \alpha u(x)$ for all $x > 0$. ♠

2.4.4 Quasi-linearity

Definition 2.13 (Quasi-Linearity)

$\succeq$ on X is **quasi-linear** on x_1 if

$$x \succeq y \Rightarrow (x + \epsilon e_1) \succeq (y + \epsilon e_1)$$

where $e_1 = (1,0,...,0)$ and $\epsilon > 0$. ♣

Theorem 2.1 (Quasi-Linearity $\Leftrightarrow u(x) = x_1 + v(x_{-1})$)

A continuous $\succeq$ on $(-\infty,\infty) \times \mathbb{R}_+^{K-1}$ is quasi-linear in $x_1 \Leftrightarrow \exists$ a $u(\cdot)$ that represents $\succeq$ such that

$$u(x) = x_1 + v(x_{-1})$$

where $v(\cdot)$ satisfies $(v(x_{-1}),0,...,0) \sim (0,x_{-1})$. ♡

2.4.5 Separability

Definition 2.14 (Separability)

$\succeq$ satisfies **separability** if for any x_i

$$(x_i, x_{-i}) \succeq (x'_i, x_{-i}) \Leftrightarrow (x_i, x'_{-i}) \succeq (x'_i, x'_{-i})$$

♣

Theorem 2.2 (Separability $\Rightarrow$ Additive $u(\cdot)$)

*$\succeq$ with **separability** admits additive u-representation*

$$u(x) = v_1(x_1) + \cdots v_K(x_K)$$

♡

Note *Strong assumption, usually ignored in practice.*

2.4.6 Differentiable Preference

Consider a vector of values $v(x) \in \mathbb{R}^K_+$ for the K commodities and a feasible the direction $x + \varepsilon d \in X$ from x for small enough $\varepsilon > 0$.

d is considered improvement if and only if

$$d \cdot v(x) > 0$$

Given $v(x) : X \to \mathbb{R}^K_+$, let

$$D_v(x) = \{d : d \cdot v(x) > 0\}$$

be the set of directions that are improvements relative to x.

$d \in \mathbb{R}^k$ is an improvement direction at x if there is $\lambda^* > 0$ such that λd is an improvement

$$x + \lambda d \succ x$$

for any $\lambda \leq \lambda^*$. Let $D_{\succeq}(x)$ be the set of all improvement directions at x.

Any improvement is an improvement direction if

- $\succeq$ are strictly convex.
- $\succeq$ are convex, strongly monotonic, and continuous.

Definition 2.15 (Differentiable Preference)

$\succeq$ is **differentiable** if there exists a function $v(x) : X \to \mathbb{R}^K_+$ such that

$$D_{\succeq}(x) = D_v(x),\ \forall x \in X$$

♣

Example 2.3 $\succeq$ represented by

(1). $\alpha x_1 + \beta x_2$ for $\alpha, \beta > 0$ are differentiable: $v(x) = (\alpha, \beta)$.

(2). $\min\{\alpha x_1, \beta x_2\}$ are differentiable where $\alpha x_1 \neq \beta x_2$: $v(x) = \begin{cases} (1,0) & \text{if } \alpha x_1 < \beta x_2 \\ (0,1) & \text{otherwise} \end{cases}$

Proposition 2.7 (Sufficient condition for differentiable $\succeq$)

Any (monotonic and convex) $\succeq$ can be represented by a (strongly monotonic and quasi-concave) and differentiable u is differentiable.

Chapter 3 Choice Theory

3.1 Choice

Let $\mathcal{B} = 2^X$ (all subsets of X) and $B \in \mathcal{B}$ be the all potential alternatives that can be chosen.

The choice of an agent can be represented by $C(B) \subseteq B, \forall B \in \mathcal{B}$.

Definition 3.1 (Choice Correspondence (More than one choice))

A choice correspondence C assigns a non-empty subset for every non-empty set A

$$\emptyset \neq C(A) \subseteq A$$

♣

Definition 3.2 (Induced Choice Rule)

Given a binary relation $\succeq$, the **induced choice rule** $C_{\succeq}$ is defined by $C(A) = C_{\succeq}(A) = \{x \in A : x \succeq y, \forall y \in A\}, \forall A \subseteq X$.

A choice function c can be **rationalizable** if there is a preference relation $\succeq$ on X such that $c = c_{\succeq}$. ♣

Definition 3.3 (Revealed Preference)

Given a choice rule $\succeq$, its **revealed preference relation** $\succeq_C$ is defined by $x \succeq_C y$ if there exists some A such that $x, y \in A$ and $x \in C(A)$.

♣

Proposition 3.1

If C is rationalized by $\succeq$, then $\succeq = \succeq_C$.

♠

Definition 3.4 (Rubinstein's Condition α)

A choice function c satisfies **condition** α if for any two problems A, B, if $A \subseteq B$ and $c(B) \in A$, then $c(A) = c(B)$.

♣

3.1.1 Choice Function

Definition 3.5 (Choice Function)

A **choice function** c such that $c(A) \in A$ which specifies a unique element for each nonempty subset $A \subseteq X$ (no indifferent preferences).

♣

Proposition 3.2 (Rubinstein's Condition $\alpha \Rightarrow$ Rationalizable Choice Function c)

(1). Let c be a choice function defined on a domain containing at least all subsets of X of size of at most 3. If c satisfies condition α, then there is a preference $\succeq$ on X such that $c = c_{\succeq}$.

(2). Let c be a choice function with a domain D satisfying that if $A, B \in D$, then $A \cup B \in D$. If c satisfies condition α, then there is a preference relation $\succeq$ on X such that $c = c_{\succeq}$.

3.1.2 Choice Correspondence

Definition 3.6 (Sen's α or Independence of Irrelevant Alternatives)

If $a \in A \subseteq B$, then $a \in C(B) \Rightarrow a \in C(A)$.

Definition 3.7 (Sen's β)

If $a, b \in A \subseteq B$, then $a, b \in C(A)$ and $b \in C(B) \Rightarrow a \in C(B)$.

α and β are equivalent to WARP.

Definition 3.8 (Weak Axiom of Revealed Preference (WARP))

Given a choice structure $(C, \mathcal{B})$ satisfies **WARP**. If $\exists B \in \mathcal{B}$ with $x, y \in B$, such that $x \in C(B)$. Then, $\forall B' \in \mathcal{B}$ with $x, y \in B'$, $y \in C(B') \Rightarrow x \in C(B')$.

Or we can say,

$$x, y \in B \cap B', x \in C(B), \text{ and } y \in C(B') \Rightarrow x \in C(B')$$

Proposition 3.3 (Rational $\Rightarrow$ WARP)

Given $\succeq$ is rational, then $(C^*_{\succeq}, \mathcal{B})$ satisfies WARP.

($C^*_{\succeq}$ is the choice rule that picks the maximal alternatives by $\succeq$)

Proposition 3.4 (Sen's Condition $\alpha, \beta \Rightarrow$ Rationalizable Choice Correspondence C)

Let C be a choice correspondence defined on a domain containing at least all subsets of X of size of at most 3. If C satisfies condition α and β, then there is a preference $\succeq$ on X such that $C = C_{\succeq}$.

3.2 Revealed Preference

Given choice data (p^t, x^t), we say u-fucntion rationalizes the observed behavior (p^t, x^t) if for all $t = 1, ..., T$, $p^t x^t \geq p^t x \Rightarrow u(x^t) \geq u(x)$, that is, $u(\cdot)$ achieves its maximum value on the budget set at the chosen bundles. If "locally non-satiated" u-function, $p^t x^t > p^t x \Rightarrow u(x^t) > u(x)$.

Definition 3.9 (Revealed Preferred)

We say x^t is

- $x^t R^D x$: **directly** revealed preferred to x, if $p^t x^t \geq p^t x$; (x is available under p^t)
- $x^t P^D x$: **strictly directly** revealed preferred to x, if $p^t x^t > p^t x$;

- $x^t R x$: **indirectly** revealed preferred to x, if $\exists$ a sequence $\{x_k\}_{k=1}^K$ with $x_1 = x^t$ and $x_K = x$ such that $x_k R^D x_{k+1}$ for all $k = 1, ..., K-1$, i.e., $p^t x^t = p^t x_1 \geq p^t x_2 \geq ... \geq p^t x_K = p^t x$. ♣

Definition 3.10 (Generalized Axiom of Revealed Preference (GARP))

Consider two observations (p^t, x^t) and (p^s, x^s), GARP is satisfied if

$$x^t R x^s \Rightarrow \text{ not } x^s P^D x^t$$

$$\text{i.e., } x^t R x^s \Rightarrow p^s x^t \geq p^s x^s$$

♣

GARP is a generalization of various other revealed preference tests

Definition 3.11

Weak Axiom of Revealed Preference (WARP):

$$x^t R^D x^s, x^t \neq x^s \Rightarrow \text{ not } x^s P^D x^t$$

$$\text{i.e., } p^t x^t \geq p^t x^s, x^t \neq x^s \Rightarrow p^s x^t \geq p^s x^s$$

Strong Axiom of Revealed Preference (SARP):

$$x^t R x^s, x^t \neq x^s \Rightarrow \text{ not } x^s R x^t$$

♣

Theorem 3.1 (Afriat's Theorem)

The following conditions are equivalent:

1. *The data satisfies GARP;*
2. *There exists a non-satiated u-function that rationalizes the data;*
3. *There exists a concave, monotonic, continuous, non-satiated u-function that rationalizes the data.*
4. *There exist positive numbers (u^t, λ^t) for $t = 1, ..., T$ that satisfy the so-called Afriat inequalities:*

$$u^s \leq u^t + \lambda^t p^t (x^s - x^t), \ \forall t, s$$

♡

3.3 Choice under Uncertainty

We want to model an uncertain prospect corresponding forms of function u.

The literature contains (basically) three sets of answers to these questions, differing in whether uncertainty is objective or subjective.

- Objective uncertainty: von Neumann-Morgenstern (vNM).
- Subjective uncertainty: Savage.
- Horse lottery-roulette wheel theory: Anscombe and Aumann (A-A)

3.3.1 von Neumann-Morgenstern (vNM)

The set of prizes is defined by X and the set of probability measures (or distributions) over X is denoted by P.

A compound lottery: If $p, q \in P$ and $\alpha \in [0,1]$, then there is an element $\alpha p + (1-\alpha)q \in P$ which is defined by taking the convex combinations of the probabilities of each prize separately, or

$$(\alpha p + (1-\alpha)q)(x) = \alpha p(x) + (1-\alpha)q(x)$$

$(\alpha p + (1-\alpha)q)$ represents a compound lottery.

Definition 3.12 (Three Axioms)

Three Axioms

(A1) $\succ$ is a preference relation (asymmetric and negatively transitive);

(A2) For all $p, q, r \in P$ and $\alpha \in [0,1]$, $p \succ q \Rightarrow \alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$.

(A3) For all $p, q, r \in P$ such that $p \succ q \succ r$, $\exists \alpha, \beta \in (0,1)$ such that

$$\alpha p + (1-\alpha)r \succ q \succ \beta p + (1-\beta)r$$

♣

Theorem 3.2 (vNM)

$\succ$ on P satisfies axioms (A1)-(A3) if and only if there exists a function $u : X \to \mathbb{R}$ such that

$$p \succ q \Leftrightarrow \sum_x p(x)u(x) > \sum_x q(x)u(x) \qquad (*)$$

Moreover, u is unique up to a positive affine transformation: there is another u' represents $\succ$ in the sense of (*) if and only if there exists $c > 0$ and d such that

$$u'(\cdot) = cu(\cdot) + d$$

♡

Remark

- If u represents $\succ$ then so will $v(\cdot) = f(u(\cdot))$ for any **strictly increasing** f.
- $k(p) = \sum_x p(x)u(x)$ gives an ordinal representation of $\succ$.

Lemma 3.1 (Four Lemmas obtained by the three axioms)

If $\succ$ satisfies (A1) to (A3), then

(L1). If $p \succ q$ and $0 \leq \alpha < \beta \leq 1$, then

$$\beta p + (1-\beta)q \succ \alpha p + (1-\alpha)q$$

(L2). If $p \succeq q \succeq r$ and $p \succ r \Rightarrow$ there exists a unique $\alpha^* \in [0,1]$ such that

$$q \sim \alpha^* p + (1-\alpha^*)r$$

(L3). If $p \sim q$ and $\alpha \in [0,1] \Rightarrow$ for all $r \in P$,

$$\alpha p + (1-\alpha)r \succ \alpha q + (1-\alpha)r$$

(L4). For any $x \in X$, let δ_x be the probability distribution degenerate at x, that is $\delta_x(x') = \begin{cases} 1, & \text{if } x' = x \\ 0, & \text{if } x' \neq x \end{cases}$ For all $p \in P$, we have $x_1, x_2 \in X$ such that

$$\delta_{x_1} \succeq p \succeq \delta_{x_2}$$

3.3.2 Savage (1954)

Consider the situation that what the decision maker chooses depends critically on his/her subjectively assesses as the odds of the outcomes.

The basics of the Savage formulation:

- a set of X of prizes/consequences;
- a set S of the nature (states of the world).

Each $s \in S$ is a compilation of all characteristics/factors about which the DM is uncertain and which are relevant to the consequences that will result from her/his choice. The set S is an exhaustive list of mutually exclusive states — only one $s \in S$ will be the realized state.

We denote the choice space by H, as the set of all functions from S to X ($H = X^S$).

Savage seeks to find a subjective taste (the utility function) $u(\cdot)$ and a subjective belief (the probability measure) π such that

$$h \succ h' \Leftrightarrow \sum_{s \in S} \pi(s)u(h(s)) > \sum_{s \in S} \pi(s)u(h'(s))$$

Note that, it contains an assumption that $u(\cdot)$ is a function about x which doesn't depend on the state of the world when it receives x.

3.4 Social Choice

Notations:

1. We consider finite set of alternatives X and finite set of agents I.
2. We use $\mathcal{B}$ to denotes the set of all preference relations.
3. We use $\mathcal{R} \subseteq \mathcal{B}$ to denotes the set of all rational preference relations.
4. We use $\succeq \in \mathcal{R}$ to represents individual rational preference relation.

3.4.1 Social Welfare Function and Properties

Definition 3.13 (Social Welfare Function (SWF))

A **social welfare function** (SWF) is a mapping

$$f : \mathcal{A} \subseteq \mathcal{R}^I \to \mathcal{B}$$

$\trianglerighteq = f(\succeq_1, ..., \succeq_I)$ is interpreted as the **social preference relation**. It doesn't need to be rational (i.e., complete and transitive). ♣

Definition 3.14 (SWF's Properties)

A social welfare function $f : \mathcal{A} \to \mathcal{B}$

- has **unrestricted domain** (UD) if $\mathcal{A} = \mathcal{R}^n$;
- is **transitive** (T) if $f(\succeq_1, ..., \succeq_I)$ is transitive for all $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$;
- is **nondictatorial** (ND) if there is no agent $i \in I$ such that $\forall \{x, y\} \subseteq X$ $x \succeq_i y \Rightarrow x \trianglerighteq y$. (That is there is no distinguished voter who can choose the winner).
- is **weakly Paretian** (PA) if, $\forall \{x, y\} \subseteq X$ and any preference profile $(\succeq_1, ..., \succeq_I) \in \mathcal{A}$, we have $x \succeq_i y, \forall i \in I \Rightarrow x \trianglerighteq y$.
- is **independent of irrelevant alternatives** (IIA) if, $\forall \{x, y\} \subseteq X$, and any $\succeq$ and $\succeq'$ with $\succeq_i|_{x,y} = \succeq'_i|_{x,y}, \forall i \in I$, if $x \trianglerighteq y$ then $x \trianglerighteq' y$. ♣

3.4.2 Arrow's Theorem

Theorem 3.3 (Arrow's impossibility theorem)

Suppose $|X| \geq 3$, $\mathcal{A} = \mathcal{R}^I$ *(UD). Then if a SWF* f *satisfies T, PA, and IIA, then it fails to be ND.* ♡

Proof 3.1

Yu, N. N. (2012). A one-shot proof of Arrow's impossibility theorem. *Economic Theory*, 523-525.

Chapter 4 Demand Theory

4.1 Utility Maximization Problem (UMP)

Budget set is given by $B = \{x \in X \subseteq \mathbb{R}^K_+ : p \cdot x \leq w\}$, where w is the DM's wealth and p is the vector of prices. Without losing generality, we can assume $w = 1$.

The DM's problem is finding the $\succeq$-optimal bundle $x \in B(p)$. With the corresponding utility function $u(x)$, we can consider a consumer's problem

$$\max_{x \in X} u(x) \qquad s.t.\ p \cdot x \leq w \tag{UMP}$$

The set $\succeq$-optimal bundle is represented by $x(p, w)$.

4.1.1 Marshallian Demand: Existence and Properties

Proposition 4.1 (Continuous Preference$\Rightarrow$ Solution $x(p, w)$ Existence)

If $\succeq$ $(u(\cdot))$ is continuous, then all such problems have a solution $x(p, w)$.

Proof 4.1

By the Weierstrass Extreme Value Theorem.

Proposition 4.2 (Convex Preference$\Rightarrow$ Convex $x(p, w)$)

If $\succeq$ is convex ($u(\cdot)$ is quasi-concave), then $x(p, w)$ is convex.

Proof 4.2

Suppose $x, x' \in X$. The optimal utility $u^* = u(x) = u(x')$. For any $\alpha \in [0, 1]$, let $x'' = \alpha x + (1 - \alpha)x'$. Because $\succeq$ is convex, we have $u(\cdot)$ is quasi-concave, that is $u(x'') \geq u^*$. x'' is also feasible. So, $x'' \in x(p, w)$.

Proposition 4.3 (Strictly Convex Preference$\Rightarrow$ Singleton $x(p, w)$)

If $\succeq$ is strictly convex ($u(\cdot)$ is strictly quasi-concave), then $x(p, w)$ is (at most) a singleton.

Proposition 4.4 (Differentiable Preference$\Rightarrow$ Margianl Utility equals to Price)

If $\succeq$ is differentiable, $x^* \in x(p, w)$, and the vector of marginal values at x^* (as defined above) is denoted by $v(x^*) = (v_1(x^*), ..., v_K(x^*))$, where $v_k(x^*)$ is usually taken by $\frac{\partial u}{\partial x_k}(x^*)$ in "classic" problem. Then, *we have*

$$\frac{v_k(x^*)}{v_j(x^*)} = \frac{p_k}{p_j} \text{ for any } x_k^*, x_j^* > 0$$

and for any k with $x_k^* > 0$ (consumed commodity)

$$\frac{v_k(x^*)}{p_k} \geq \frac{v_j(x^*)}{p_j} \text{ for any } j \neq k \tag{*}$$

Corollary 4.1 (Sufficient Conditions for Optimality)

If $\succeq$ is strongly monotonic, convex, continuous, and differentiable and if $p \cdot x^* = w$ and (*) is satisfied then $x^* \in x(p, w)$

Definition 4.1 (Rationalize)

$\succeq$ **fully rationalize** the demand function x if for any (p, w), the bundle $x(p, w)$ is the unique $\succeq$-maximal bundle within B.

A monotonic $\succeq$ **rationalize** the demand function x if for any (p, w), the bundle $x(p, w)$ is a $\succeq$-maximal bundle within B.

The unique solution is called Marshallian (Uncompensated) Demand.

Proposition 4.5 (Properties of Marshallian Demand)

(i). ***Walras' Law****: If $\succeq$ is local nonsatiation, $\forall x^* \in x(p, w) : p \cdot x^* = w$.*

(ii). ***Homogeneity of degree zero in*** (p, w)**:** $x(\alpha p, \alpha w) \equiv x(p, w)$, $\forall \alpha > 0$.

(iii). ***Continuous in prices and in wealth*** *if the $\succeq$ is continuous.*

Proposition 4.6 (Weak Axiom of Revealed Preference of Marshallian Demand)

If demand is single valued then WARP(3.8) is equivalent to

$$p \cdot y' \leq w \text{ and } y \neq y' \Rightarrow p' \cdot y > w$$

where $y \equiv x(p, w)$ and $y' \equiv x(p', w')$. (y' is feasible under (p, w) but $y = x(p, w)$, which means y is better and it can't be feasible under (p', w').)

4.1.2 Lagrangian Approach: $\frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i$ and $\lambda^* \left(x_i(p, w) + \sum_{j=1}^{K} p_j \frac{\partial x_j}{\partial p_i} \right) = 0$

The Lagrangian of the problem is

$$L(x, \lambda) = u(x) - \lambda(p \cdot x - w)$$

By the KKT necessary conditions, we have

$$\frac{\partial L}{\partial x_i}(x^*, \lambda^*) = \frac{\partial u\,(x^*)}{\partial x_i} - \lambda^* p_i = 0,\ \forall i = 1, ..., K$$

$$\lambda^* \geq 0 \text{ and } \lambda^*(p \cdot x^* - w) = 0$$

Based on that, we have

Lemma 4.1

(i). $\frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i$;

(ii). $\lambda^* \left(x(p, w) + p \cdot \frac{\partial x(p,w)}{\partial p} \right) = 0$ *i.e.,* $\lambda^* \left(x_i(p, w) + \sum_{j=1}^{K} p_j \frac{\partial x_j}{\partial p_i} \right) = 0.$ ♡

4.1.3 Envelope Theorem $\Rightarrow \lambda^* = \frac{\partial u(x(p,w))}{\partial w}$

Theorem 4.1 (Envelope Theorem)

Consider the constrained maximization problem,

$$\max_{x \in \mathbb{R}^n} \quad f(x; \theta)$$

$$\text{s.t. } g(x; \theta) \leq 0$$

where $xin\mathbb{R}^n$ *is the choice variable and* $\theta \in \mathbb{R}^m$ *is some parameter. Let* f, g *be continuously differentiable real-valued functions.*

- *Let the value function of the problem be* $V(\theta) \triangleq f(x^*(\theta), \theta)$.
- *The Lagrangian for this problem is*

$$L(x, \lambda; \theta) = f(x; \theta) - \lambda g(x; \theta)$$

- *Let* x^* *and* λ^* *denote the optimized values of the variables.*

  (By KKT necessary conditions, we have $\frac{\partial f}{\partial x}(x^*; \theta) = \lambda^* \frac{\partial g}{\partial x}(x^*; \theta)$ *and* $\lambda^* g(x^*; \theta) = 0$*)*

Then the following is true for any $\bar{\theta} \in \mathbb{R}^m$

$$\frac{\partial V}{\partial \theta_i}(\bar{\theta}) = \frac{\partial L}{\partial \theta_i}(x^*, \lambda^*; \bar{\theta}) = \frac{\partial f}{\partial \theta_i}(x^*; \bar{\theta}) - \lambda^* \frac{\partial g}{\partial \theta_i}(x^*; \bar{\theta})$$ ♡

Proof 4.3

The proof of the envelope theorem is a straightforward calculation.

Firstly, by KKT necessary conditions, we have $\frac{\partial f}{\partial x}(x^*; \bar{\theta}) = \lambda^* \frac{\partial g}{\partial x}(x^*; \bar{\theta})$ *and* $\lambda^* g(x^*; \bar{\theta}) = 0 \Rightarrow$

$\lambda^* \left[\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i} + \frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\right] = 0$. *Then we have*

$$\begin{aligned}\frac{\partial V}{\partial\theta_i}(\bar\theta) &= \frac{\partial f}{\partial\theta_i}(x^*;\bar\theta) + \frac{\partial f}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i} \\ &\left(\text{by } \frac{\partial f}{\partial x}(x^*;\bar\theta) = \lambda^*\frac{\partial g}{\partial x}(x^*;\bar\theta)\right) \\ &= \frac{\partial f}{\partial\theta_i}(x^*;\bar\theta) + \lambda^*\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i} \\ &\left(\text{by } \lambda^*\left[\frac{\partial g}{\partial x}(x^*;\bar\theta)\frac{\partial x^*(\bar\theta)}{\partial\theta_i} + \frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\right] = 0\right) \\ &= \frac{\partial f}{\partial\theta_i}(x^*;\bar\theta) - \lambda^*\frac{\partial g}{\partial\theta_i}(x^*;\bar\theta)\end{aligned}$$

Corollary 4.2

$\lambda^* = \frac{\partial u(x(p,w))}{\partial w}$.

Proof 4.4

By the envelope theorem, we have $\frac{\partial u(x(p,w))}{\partial w} = \frac{\partial L}{\partial w}\Big|_{x^*,\lambda^*} = \lambda^*$.

4.1.4 Indirect Utility Function $v(p,w) \equiv u(x(p,w))$

Proposition 4.7 (Properties of Indirect Utility Function)

1. $v(p,w)$ *is homogeneous of degree zero in* (p,w)*;*
2. $v(p,w)$ *is strictly increasing in* w *and non-increasing in* p_i*;*
3. $v(p,w)$ *is quasi-convex, that is the set* $\{p : v(p,w) \le u\}$ *is convex for all* $u \in \mathbb{R}$*.*
4. $\lambda^* = \frac{\partial v(p,w)}{\partial w}$ *(Corollary 4.2).*

4.1.5 Roy's Identity $x_i^* = -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial w}}$: recover $x(p,w)$ from $v(p,w)$

Proposition 4.8 (Roy's Identity)

$x_i^*(p,w) = -\frac{\frac{\partial v(p,w)}{\partial p_i}}{\frac{\partial v(p,w)}{\partial w}}$.

Proof 4.5

By the definition,

$$
\begin{aligned}
v(p,w) &\equiv u(x(p,w)) \\
\frac{\partial v}{\partial p_i} &\equiv \sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial x_j}{\partial p_i} \\
&= \sum_{j=1}^{K} \lambda^* p_j \frac{\partial x_j}{\partial p_i} && \text{by } \frac{\partial u(x^*)}{\partial x_i} = \lambda^* p_i \\
&= -\lambda^* x_i^* && \text{by } \lambda^* \left(x_i(p,w) + \sum_{j=1}^{K} p_j \frac{\partial x_j}{\partial p_i} \right) = 0 \\
x_i^* &= -\frac{\frac{\partial v}{\partial p_i}}{\frac{\partial v}{\partial w}} && \text{by } \lambda^* = \frac{\partial v(p,w)}{\partial w}
\end{aligned}
$$

4.2 Expenditure Minimization Problem (EMP)

Consider the duality

$$
\begin{aligned}
\min_{x \in X} \quad & p \cdot x \\
s.t. \quad & u(x) \geq u
\end{aligned}
\tag{EMP}
$$

The optimal solutions are represented by $h(p,u)$. With uniqueness, we call it *Hicksian (compensated) demand*.

4.2.1 Hicksian Demand $h(p,u)$: Properties

Proposition 4.9 (Properties of Hicksian Demand)

(i). $h(p,u)$ *is homogeneous of degree zero in* p*:*

$$h(tp,u) = h(p,u), \ \forall t \in \mathbb{R}_+$$

(ii). $u(x)$ *is strictly quasi-concave* $\Rightarrow h(p,u)$ *is unique;*

(iii). For $u > u(0)$ *and* $u(\cdot)$ *is locally non-satiated, constraint is active: for all* $x^* \in h(p,u)$*,*

$$u(x^*) = u$$

Lemma 4.2 ($\sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0$)

If $u(x)$ *is strictly quasi-concave, the Hicksian demand satisfies* $\sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0$

Proof 4.6

$u(h(p,u)) \equiv u \Rightarrow \sum_{j=1}^{K} \frac{\partial u}{\partial x_j}\frac{\partial h_j}{\partial p_i} = 0.$

4.2.2 Expenditure Function $e(p, u) \equiv p \cdot h(p, u)$

Given the Hicksian demand $h(p, u)$, we can define the expenditure function as $e(p, u) \equiv p \cdot h(p, u)$.

Proposition 4.10 (Properties of Expenditure Function)

(i). $e(p, u)$ *is homogeneous of degree* 1 *in* p*:*

$$e(tp, u) = tp \cdot h(tp, u) = tp \cdot h(p, u) = te(p, u)$$

(ii). $e(p, u)$ *is strictly increasing in* u*, non-decreasing in* p_i*;*

(iii). $e(p, u)$ *is* ***concave*** *in* p*;*

(iv). $e(p, u)$ *is continuous in* p *for all* $p >> 0$*;*

(v). For all $x^* \in h(p, u)$*,* $x^* \in x(p, e(p, u))$*;*

(vi). For $w > 0$*,* $e(p, v(p, w)) \equiv w$*;*

(vii). $e(p, u)$*'s derivative property:*

$$\frac{\partial e(p, u)}{\partial p_i} \equiv h_i(p, u)$$

♠

Proof 4.7 (Proof for concavity)

Suppose the price of good 1 increases from p_1^0 *to* p_1^1*:* $p^0 \to p^1$*. Set* $p^a = ap^0 + (1-a)p^1,\ 0 \leq a \leq 1$*.*

So, $p^0 \leq p^a \leq p^1$ *and*

$$\begin{aligned}
e(p^a, u) &= p^a \cdot h(p^a, u) \\
&= (ap^0 + (1-a)p^1) \cdot h(p^a, u) \\
&= a[p^0 \cdot h(p^a, u)] + (1-a)[p^1 \cdot h(p^a, u)] \\
& h(p^a, u) \text{ is feasible in both EMP, but not optimal solutions} \\
&\geq a[p^0 \cdot h(p^0, u)] + (1-a)[p^1 \cdot h(p^1, u)] \\
&= ae(p^0, u) + (1-a)e(p^1, u)
\end{aligned}$$

Proof 4.8 (Proof for Derivative)

1. *Direct proof:*

$$
\begin{aligned}
e(p,u) &\equiv p \cdot h(p,u) \\
\frac{\partial e}{\partial p_i} &\equiv \sum_{j=1}^{K} p_j \frac{\partial h_j}{\partial p_i} + h_i \\
&\equiv \sum_{j=1}^{K} \frac{1}{\lambda^*} \frac{\partial u\left(x^*\right)}{\partial x_j} \frac{\partial h_j}{\partial p_i} + h_i \qquad \text{by } \frac{\partial u\left(x^*\right)}{\partial x_i} = \lambda^* p_i \\
&= h_i \qquad \text{by } \sum_{j=1}^{K} \frac{\partial u}{\partial x_j} \frac{\partial h_j}{\partial p_i} = 0
\end{aligned}
$$

2. *Envelope Theorem Proof:*

$$
\begin{aligned}
L(x,\lambda;(p,u)) &= p \cdot x - \lambda(u(x) - u) \\
\frac{\partial e(p,u)}{\partial p_i} &= \left.\frac{\partial L(x,\lambda;(p,u))}{\partial p_i}\right|_{x^*=h(p,u)} = x_i|_{x^*=h(p,u)} = h_i(p,u)
\end{aligned}
$$

4.2.3 Law of Compensated Demand: $\frac{\partial h_i}{\partial p_i} \leq 0$

Corollary 4.3 (Law of Compensated Demand)

Hicksian demand is downward sloping in its own price,

$$\frac{\partial h_i}{\partial p_i} \leq 0$$

♡

Proof 4.9

By the concavity of $e(p,u)$ (4.10), we can conclude $\nabla^2 e(p,u) \preceq 0$ (negative semi-definite). Then, we know its diagonal elements are non-positive $\frac{\partial e^2}{\partial^2 p_i} = \frac{\partial h_i}{\partial p_i} \leq 0$.

4.2.4 Shifts in Hicksian Demand: $\frac{\partial h_i}{\partial u} \equiv \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial u}$, same direction as $\frac{\partial x_i}{\partial w}$

How does Hicksian demand curve shift when utility changes?

$$
\begin{aligned}
h_i(p,u) &\equiv x_i(p, e(p,u)) \\
\frac{\partial h_i}{\partial u} &\equiv \frac{\partial x_i}{\partial w} \frac{\partial e}{\partial u}
\end{aligned}
$$

We know $\frac{\partial e}{\partial u} > 0$, so the direction of Hicksian demand shift is the same as $\frac{\partial x_i}{\partial w}$.

- Normal good: increasing utility shifts h_i to the right.

- Inferior good: increasing utility shifts h_i to the left.

4.3 UMP and EMP

4.3.1 Slutsky Equation: substitution effect and income effect

Slutsky: how change of p_j (price in good j) affects x_i (the demand of product i).

Proposition 4.11 (Slutsky Equation)

$$\frac{\partial x_i(p,w)}{\partial p_j} = \underbrace{\frac{\partial h_i(p,u)}{\partial p_j}}_{\text{substitution effect}} \underbrace{-\frac{\partial x_i(p,w)}{\partial w} x_j(p,w)}_{\text{income effect}}$$

Proof 4.10

$$\begin{aligned} h_i(p,u) &\equiv x_i(p,e(p,u)) \\ \frac{\partial h_i}{\partial p_j} &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w}\frac{\partial e}{\partial p_j} \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} h_j(p,u) \\ &\equiv \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial w} x_j(p,e(p,u)) \end{aligned}$$

- **Substitution effect**: $\frac{\partial h_i}{\partial p_j}$, the change of relative prices change with constant utility will change the x_i.
- **Income (Wealth) effect**: $-\frac{\partial x_i}{\partial w} x_j(p,w)$, the change of price can be seen as change of wealth, which will also impact the x_i.

4.3.2 Relationship Between UMP and EMP

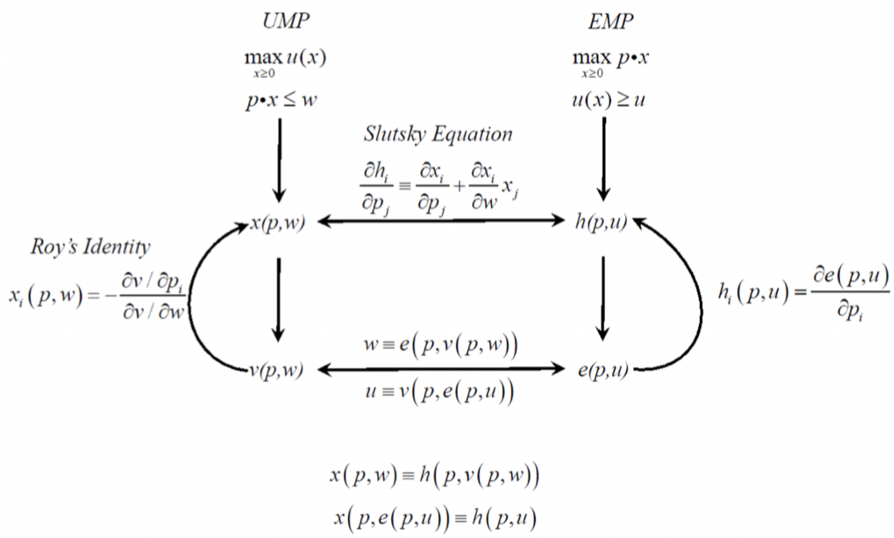

Figure 4.1: Relationship Between UMP and EMP

Chapter 5 General Equilibrium

5.1 Exchange Economy

1. There are L perfectly divisible commodities indexed by $l = 1, ..., L$ over $\mathbb{R}^L$.
2. There are m agents, indexed by $i = 1, ..., m$. $N = \{1, ..., m\}$.
3. Each agent has a preference relation $\succeq_i$ on $\mathbb{R}_+^L$ represented by a utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$.
4. Each agent has a vector of initial endowments $w_i \in \mathbb{R}_+^L$.
5. The aggregate endowment is $w = \sum_{i=1}^m w_i$.

Example 5.1 (Endowment Box Economy) The endowment box economy has 2 goods ($L = 2$) and 2 agents ($m = 2$). The commodity space is $\mathbb{R}^2$.

Each agent's consumption set is $\mathbb{R}_+^2 = \{x \in \mathbb{R}^2 : x \geq 0\}$.

Each agent $i = a, b$ has preference relation $\succ_i$ over $\mathbb{R}_+^2$ represented by a utility function $u_i : \mathbb{R}_+^2 \to \mathbb{R}$.

Each agent has a vector of initial endowments $w_i = (w_{i1}, w_{i2}) \in \mathbb{R}^2$.

Definition 5.1 (Allocation)

An **allocation** in an exchange economy is an assignment of goods to agents $x = (x_1, ..., x_m) \in \mathbb{R}_+^{L\times m}$ such that $\sum_{i=1}^m x_i = w$. ♣

5.1.1 Pareto Optimal/Efficient

Definition 5.2 (Pareto Optimal)

An allocation x is **Pareto optimal/efficient** if there doesn't exist an allocation y s.t. $u_i(y_i) \geq u_i(x_i)$ ($y_i \succeq_i x_i$) for each i and $u_j(y_j) > u_j(x_j)$ ($y_j \succ_j x_j$) for some j. ♣

Consider the following social planner's problem:

- Fix an agent j and $\{\bar{u}_i\}_{i\neq j}$,

$$\begin{aligned} \max_{(x_1,\ldots,x_n)\in\mathbb{R}_+^{L\times m}} \quad & u_j(x_j) \\ \text{s.t.} \quad & u_i(x_i) \geq \bar{u}_i, i \neq j \\ & \sum_{i=1}^m x_i = w \\ & x_i \geq 0, \forall i \end{aligned} \tag{P}$$

Proposition 5.1 (P.O. $\Leftrightarrow$ Solutions of Problem (P))

Suppose each agent's utility function is continuous and strongly monotone. Then, an allocation x^* in an exchange economy is Pareto-Optimal iff it is a solution of Problem (P) for some choice of $\{\bar{u}_i\}_{i\neq j}$. ♠

Proof 5.1

1. "$\Leftarrow$": *Suppose x^* is a solution to Problem (P) for $\{\bar{u}_i\}_{i\neq j}$. Suppose by the way of contradiction that x^* is not Pareto-Optimal. Then there is another allocation $\hat{x}$ such that*

   (i). Either: $u_j(\hat{x}_j) > u_j(x_j^*)$ and $u_i(\hat{x}_i) \geq u_i(x_i^*)$ for all $i \neq j$.

   (ii). Or: $u_j(\hat{x}_j) \geq u_j(x_j^*)$, $u_k(\hat{x}_k) \geq u_k(x_k^*)$ for some $k \neq j$, and $u_i(\hat{x}_i) \geq u_i(x_i^*)$ for all $i \neq j, k$.

   Suppose (i) holds: Since $\hat{x}$ is an allocation, $\sum_{i=1}^m \hat{x}_i = m$ and $\hat{x}_i \geq 0, \forall i$. By assumption and x^* is solution of Problem (P), $u_i(\hat{x}_i) \geq u_i(x_i^*) \geq \bar{u}_i$ for all $i \neq j$. So, $\hat{x}$ satisfies the constraints of Problem (P). Because $u_j(\hat{x}_j) > u_j(x_j^*)$, x^* is not the solution to Problem (P). Contradiction!

   Suppose (ii) holds: Prove by constructing another allocation $\tilde{x}$ as follows: By continuity, $\exists \epsilon > 0$ sufficiently small s.t. $u_k((1-\epsilon)\hat{x}_k) \geq u_k(x_k^*)$. Set $\tilde{x}_k = (1-\epsilon)\hat{x}_k$, $\tilde{x}_j = \hat{x}_j + \epsilon\hat{x}_k$, and $\tilde{x}_i = \hat{x}_i$ for all $i \neq j, k$. Then, $\sum_{i=1}^m \tilde{x}_i = \sum_{i=1}^m \hat{x}_i = w$, $u_i(\tilde{x}_i) \geq u_i(x_i^*) \geq \bar{u}_i$ for all $i \neq j$ and $u_j(\tilde{x}_j) > u_j(x_j^*) \geq \bar{u}_j$ (by strong monotonicity). Hence, x^* is not the solution to Problem (P). Contradiction!

2. "$\Rightarrow$": *Suppose x^* is Pareto-Optimal. Set $\bar{u}_i = u_i(x^*)$ for all $i \neq j$.*

   Claim 5.1

   x^* solves Problem (P) given $\{\bar{u}_i\}_{i\neq j}$. ♠

   Firstly, x^* is feasible for Problem (P). Then, suppose by the way of contradiction that there is another allocation x' such that $\sum_{i=1}^m x_i' = w$, $u_i(x_i') \geq \bar{u}_i = u_i(x_i^*)$ for all $i \neq j$, and $u_j(x_j') > u_j(x_j^*)$. Hence, x^* is not Pareto-Optimal, which is a contradiction!

Proposition 5.2

From the first-order condition (FOC) of Problem (P), a necessary condition for interior Pareto-Optimal allocations when each u_i is also differentiable is

- *$Du_j(x_j^*) = \lambda_i Du_i(x_i^*)$ for some $\lambda_i > 0$ and $\forall i \neq j$ where $x_i^* >> 0, \forall i$.* ♠

5.1.2 Individually Rational, Block, Core

Are all Pareto-Optimal allocations equally likely are reasonable?

How the initial endowment allocation affects the Pareto-Optimal allocation?

One agent should block any proposed trades leading to allocations that generate lower utility.

Definition 5.3 (Individually Rational)

A bundle x_i is **individually rational** (IR) for agent i if $x_i \succeq_i w_i$.

An allocation $x = (x_1, ..., x_m)$ is **individually rational** (IR) if $x_i \succeq_i w_i$ for all $i = 1, ..., m$.

Let $N := \{1, ..., m\}$ be the set of agents. A **coalition** is a nonempty subset $S \subseteq N$.

Example 5.2 With two agents $\{a, b\}$, there are 3 possible coalitions: $\{a\}, \{b\}, \{a, b\}$.

Definition 5.4 (Block)

A coalition S can **block** an allocation $x = (x_1, ..., x_m)$ if there exists bundles $y_i \in \mathbb{R}^L_+$ for all $i \in S$ s.t.

1. $\sum_{i \in S} y_s = \sum_{i \in S} w_i$
2. $y_i \succeq_i x_i$ for all $i \in S$;
3. $y_j \succ_j x_j$ for some $j \in S$.

Definition 5.5 (Core)

The **core** is the set of allocations that cannot be blocked by any coalition.

Note *(Core and P.O.)*

- *Every allocation in the core is Pareto-Optimal (directly by definition).*
- *Not every Pareto-Optimal allocation is in the core.*
- *For the two agent case, the core is the set of individually rational Pareto-Optimal allocations.*

5.1.3 Competitive Equilibrium

Assumption Suppose there are markets for all available goods and all agents are price-takers in these markets.

Given a vector of price $p \in \mathbb{R}^L$, agent i chooses x_i^* to solve the following problem:

$$\max_{x_i \in \mathbb{R}^L_+} u_i(x_i)$$

$$s.t. \; p \cdot x_i \leq p \cdot w_i$$

Definition 5.6 (Competitive Equilibrium)

Given endowment $w = (w_i)_{i \in N}$. A **competitive (Walrasian) equilibrium** in an exchange economy is a pair $p^* \in \mathbb{R}^L$ (price vector over L goods) and an allocation $x^* = (x_i^*)_{i \in N}$ such that:

(i). $x_i^* \in \operatorname{argmax} u_i(x)$ s.t. $p^* \cdot x_i \leq p^* \cdot w_i, \forall i \in N$.

(ii). $\sum_{i \in N} x_i^* = w$.

We call $x^* = (x_i^*)_{i \in N}$ the competitive equilibrium (Walrasian) allocation

and p^* the competitive equilibrium (Walrasian) price vector.

Demand notations:

Definition 5.7 (Excess Demand)

Let $x_i(p) := x_i(p, p \cdot w_i)$ denote agent i's **demand** given the price vector $p \in \mathbb{R}^L$ and income $p \cdot w_i$.

Agent $i's$ **individual excess demand at** p is $x_i(p) - w_i$.

The **aggregate excess demand** at p is $\sum_{i \in N} x_i(p) - w$.

♣

Note *(Excess Demand and Competitive Equilibrium)*

- *p^* is a competitive equilibrium price vector if and only if $0 \in \sum_{i \in N} x_i(p) - w$*
- *(x^*, p^*) is a competitive equilibrium if and only if x^* satisfies $x_i^* \in x_i(p^*), \forall i$ and $\sum_{i \in N} x_i^* - w = 0$.*

5.1.4 First Welfare Theorem: CE $\Rightarrow$ P.O.

Given non-satiated preference, every CE is P.O. (P.E.).

Theorem 5.1 (First-order (fundamental) Welfare Theorem: CE $\Rightarrow$ P.O.)

If each agent's preference relation is locally non-satiated, then every competitive equilibrium allocation is Pareto optimal (Pareto efficient).

♡

Proof 5.2

Let $x^* = (x_1^*, ..., x_n^*)$ be a CE allocation with corresponding CE price vector p^*.

Suppose by way of contradiction that x^* is not P.O. allocation. Then, there is another allocation $\hat{x} = (\hat{x}_1, ...\hat{x}_n)$ such that $\hat{x}_j \succ_j x_j^*$ for some $j \in N$ and $\hat{x}_i \succeq_i x_i^*$ for all $i \neq j$.

By the definition of CE, $\hat{x}_j$ should not be affordable for j, i.e., $p^*\hat{x}_j > p^*w_j$. By the definitions of non-satiated and CE, $p^*\hat{x}_i \geq p^*w_i$ for all $i \neq j$. (If $p^*\hat{x}_i < p^*w_i$, $\exists \tilde{x}_i$ s.t. $p^*\tilde{x}_i \leq p^*w_i$ and $\tilde{x}_i \succ_i \hat{x}_i \succeq_i x_i^*$, which contradicts to the definition of CE.)

Add up all inequalities, we get

$$p^* \cdot \left(\sum_{i=1}^{n} \hat{x}_i \right) > p^* \cdot \left(\sum_{i=1}^{n} w_i \right) = p^* \cdot w$$

which contradicts to the definition of a feasible allocation that $\sum_i^n \hat{x}_i = w$. Hence, x^* is P.O.

Note *This requires **only** local non-satiation of preferences. In particular, does not require convexity of preferences.*

5.1.5 CE $\Rightarrow$ IR; CE $\subseteq$ P.O. $\cap$ Core

Note *[CE $\Rightarrow$ IR] At any prices p, an agent can always afford their initial endowment, so by revealed preference, every CE allocation is individually rational.*

Corollary 5.1 (CE Allocation is in Core)

If each agent's preference relation is locally non-satiated, then every CE allocation is in the core.

Proof 5.3

In exercise.

Note *Not every P.O. allocation is in the core. But every CE allocation is a P.O. allocation in the core.*

$$\text{CE} \subseteq \text{P.O.} \cap \text{Core}$$

5.1.6 Equilibrium with Transfers

What scope does planner have for redistribution using only decentralized market mechanism?

Not every P.O. allocation is "equitable." To implement a more "equitable" allocation. Some possible mechanisms:

- will need taxes or transfers (should be budget-balancing, i.e., no money leaves the economy).
- taxes/transfers should be lump-sum.

Definition 5.8 ("Supportable" as a Price Equilibrium with Transfers)

An allocation x^* is **supportable** as a **price equilibrium with transfers** if there exists a price vector $p^* \in \mathbb{R}^L$ and lump-sum budget-balancing transfers $\{T_i : i = 1, ..., m\}$ so that $\sum_{i=1}^m T_i = 0$, such that $\forall i$:

$$x_i^* \in \arg \max_{x \in \mathbb{R}_+^L \text{ s.t. } p^* \cdot x_i \leq p^* \cdot w_i + T_i} u_i(x_i)$$

5.1.7 Second Welfare Theorem: sufficient condition for P.O. be supported as a price equilibrium with transfers

Remark Is every P.O. allocation supportable as a price equilibrium with transfers? **No.** (e.g. a non-convex indifference curve (preference relation): for a P.O. allocation, there exists an allocation such that gives a bundle with lower cost but equal utility for an agent.)

Formally, a sufficient condition can be given:

Theorem 5.2 (Second Welfare Theorem)

If each consumers' preference relation is convex, continuous, and strongly monotone, then every interior P.O. allocation in an exchange economy can be supported as a price equilibrium with transfers.

To give the proof of the theorem, we need firstly give some definitions and results.

Definition 5.9 (Supported by a price; Supported)

An allocation $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ in an exchange economy is **supported by a non-zero price vector** $p \in \mathbb{R}^L$ if

$$\forall i : x_i \succeq_i \bar{x}_i \Rightarrow p \cdot x_i \geq p \cdot \bar{x}_i$$

If an allocation $\bar{x} = (\bar{x}_1, ..., \bar{x}_m)$ is **supported**, then a common price p supports each agent's "better-than set" at $\bar{x}_i$ ($\{x_i \in \mathbb{R}_L^+ : x_i \succeq_i \bar{x}_i\}$): $\forall x_i \in \{x_i \in \mathbb{R}_L^+ : x_i \succeq_i \bar{x}_i\} : p \cdot x_i \geq p \cdot \bar{x}_i$. ♣

Note *An allocation is supported as a price equilibrium with transfers ⇔ the allocation that is supported (i.e., all agents' bundles are supported by a common price).*

Recall:

Theorem 5.3 (Separating Hyperplane Theorem)

Let $A, B \subseteq \mathbb{R}^n$ be non-empty disjoint, convex sets. Then $\exists p \in \mathbb{R}^n, p \neq 0$, s.t.

$$p \cdot a \leq p \cdot b,\ \forall a \in A, \forall b \in B$$

♡

Proof 5.4 (Second Welfare Theorem 5.2)

Let $x^* = (x_1^*, ..., x_m^*)$ be an interior P.O. allocation, so $x_i^* >> 0, \forall i$. Let

$$P_i := \left\{x_i \in \mathbb{R}_+^L : u_i(x_i) > u_i(x_i^*)\right\},\ \forall i$$

Properties about P_i:

(1). By strong monotonicity, $P_i \neq \emptyset$ (interior allocation) for all i.

(2). By convexity, P_i is convex for all i.

Let

$$\begin{aligned} P :&= P_1 + \cdots + P_m \\ &= \left\{ z \in \mathbb{R}_L^+ : z = \sum_{i=1}^m x_i \text{ for some } (x_1, ..., x_m) \in \mathbb{R}_+^{L\times m} \text{ s.t. } u_i(x_i) > u_i(x_i^*), \forall i \right\} \end{aligned}$$

Properties about P:

(1). By construction, $P \neq \emptyset$.

(2). P is convex: because the sum of convex sets is convex.

(3). $w \notin P$: this follows from the Pareto optimality of $x^* = (x_1^*, ..., x_m^*)$.

As $\{w\}$ is convex and $\{w\} \cap P = \emptyset$, by the Separating Hyperplane Theorem 5.3, $\exists p \in \mathbb{R}^L, p \neq 0$ s.t. $p \cdot z \geq p \cdot w$ for all $z \in P$.

- *Fix j and suppose $u_j(x_j) > u_j(x_j^*)$ for some $x_j \in \mathbb{R}_+^L$. By continuity, $\exists \epsilon \in (0,1)$ sufficient small s.t. $u_j((1-\epsilon)x_j) > u_j(x_j^*)$. Let $y_j := (1-\epsilon)x_j$. For $i \neq j$: set $y := x_i^* + \frac{\epsilon}{m-1}x_j$. By* *strong monotonicity,* $u_i(y_i) > u_i(x_i^*), \forall i \neq j$*. So,* $\sum_{i=1}^m y_i \in P$ *by the definition of* P*.*

Then,

$$p \cdot \left(\sum_{i=1}^m y_i\right) \geq p \cdot w$$

$$\text{By } \sum_{i=1}^m x_i^* = w, \quad p \cdot \left(\sum_{i=1}^m y_i - \sum_{i=1}^m x_i^*\right) \geq 0$$

$$p \cdot (x_j - x_j^*) \geq 0$$

$$p \cdot x_j \geq p \cdot x_j^*$$

That is, with p, $u_j(x_j) > u_j(x_j^*) \Rightarrow p \cdot x_j \geq p \cdot x_j^*$*.*

By strong monotonicity, $u_j(x_j^* + (0,0,..,0,1,0,..,0)) > u_j(x_j^*)$*, hence,* $p \cdot (x_j^* + (0,0,..,0,1,0,..,0)) \geq p \cdot x_j^* \Rightarrow p \cdot (0,0,..,0,1,0,..,0) \geq 0$*. That is,* $p_i \geq 0, \forall i$*. By definition,* $p \neq 0$*,* $p > 0$*.*

By assumption $x_j^* >> 0$*,* $p \cdot x_j^* > 0$*. Now suppose* $\exists x_j \in \mathbb{R}_+^L$ *s.t.* $u_j(x_j) > u_j(x_j^*)$ *and* $p \cdot x_j = p \cdot x_j^*$*. By continuity,* $\exists \delta \in (0,1)$ *s.t.* $u_j(\delta x_j) > u_j(x_j^*)$*. By what we show above,* $u_j(x_j) > u_j(x_j^*) \Rightarrow p \cdot x_j \geq p \cdot x_j^*$*. We have* $p \cdot x_j > \delta p \cdot x_j = p \cdot (\delta x_j) \geq p \cdot x_j^* > 0$*. There is a contradiction. Hence, we prove that*

$$u_j(x_j) > u_j(x_j^*) \Rightarrow p \cdot x_j > p \cdot x_j^*$$

- *Let the transfers be* $T_i := p \cdot x_i^* - p \cdot w_i, \forall i$ *such that* $\sum_i T_i = p \cdot (\sum_i x_i^* - \sum_i w_i) = 0$*.*

All in all,

$$x_i^* \in \arg \max_{x \in \mathbb{R}_+^L \text{ s.t. } p^* \cdot x_i \leq p^* \cdot w_i + T_i} u_i(x_i)$$

x^* *is a price equilibrium with transfers* $\{T_i\}_i$ *and the price vector* p*.*

5.1.8 Second Welfare Theorem: P.O. with Endowments Used $\Rightarrow$ CE

Theorem 5.4 (Second Welfare Theorem (corollary))

Suppose that interior x^* *is Pareto efficient and consumers receive endowment worth* $p \cdot w^i = p \cdot x^{i^*}$ *for all* $i = 1, ..., m$*. Then, if a competitive equilibrium exists for such* w*, then* x^* *is a competitive equilibrium allocation.*

Proof 5.5

By the Second Welfare Theorem 5.2, interior P.O. allocation x^* *can be supported by transfers* $\{T_i\}_{i=1}^m$*. Then,* $p \cdot x_i \leq p \cdot w_i + T_i$*. Because* $p \cdot w^i = p \cdot x^{i^*}$*,* $T_i = 0, \forall i$*. So,* x^* *is exactly a competitive equilibrium allocation.*

5.1.9 Walras' Law in Competitive Equilibrium

Recall that "p is a competitive equilibrium price vector" ⇔ "$0 \in \sum_{i=1}^m x_i(p) - w$."

Note *Only relative prices matter, as the Marshallian demand has homogeneity of degree zero:* $x(\lambda p) = x(p)$. Hence, if p^* is a competitive equilibrium price vector, so is $\lambda p^*, \forall \lambda$, which correspond to the same competitive equilibrium.

Remark **Are markets independent?** No.

If $\succeq_i$ is locally non-satiated for all i, then $\forall i$: $p \cdot x_i(p) = p \cdot w_i, \forall p$. Adding over agents: $p \cdot \sum_{i=1}^m x_i(p) = p \cdot w, \forall p$. This is **Walras' Law** in aggregate level: $p \cdot [\sum_{i=1}^m x_i(p) - w] = 0, \forall p$.

Remark If Walras' Law holds and there exists $p^* >> 0$ such that all markets but one clear, then the p^* must clear the last market too.

Let $Z(p) = \sum_{i=1}^m x_i(p) - w$. By Walras' Law, $p \cdot Z(p) = 0, \forall p$. Suppose that exists $p^* >> 0$ such that $Z_l(p^*) = 0, l = 1, ..., L-1$. Then,

$$0 = p^* \cdot Z(p^*) = \sum_{l=1}^{L} p_l^* \cdot Z_l(p^*) = p_L^* Z_L(p^*)$$

$$p_L^* > 0 \Rightarrow Z_L(p^*) = 0$$

5.2 Private Ownership Production Economy

1. There are L perfectly divisible goods. The commodity space is $\mathbb{R}^L$.
2. There are m consumers. Each consumer $i = 1, ..., m$ has a preference relation $\succeq_i$ represented by a utility function $u_i : \mathbb{R}_+^L \to \mathbb{R}$, an initial endowment $w_i \in \mathbb{R}_+^L$, and owns shares $\{\theta_{ij} : j = 1, ..., J\}$ in the J firms, where $\theta_{ij} \geq 0, \forall i, j$ and $\sum_{i=}^m \theta_{ij} = 1, \forall j$.
3. There are J firms. Each firm $j = 1, ..., J$ has a production set $Y_j \subseteq \mathbb{R}^L$ that is nonempty, (representing the constraints of production).

   Note *Standard sign convention regarding net output vectors: y represents net output;*

   $y_k \leq 0 \Rightarrow$ *good k is a net input in y;*

   $y_k \geq 0 \Rightarrow$ *good k is a net output in y.*
4. The set of allocation is

$$\mathcal{A} := \left\{ (x, y) = (\underbrace{x_1, ..., x_m}_{\text{consumption}}, \underbrace{y_1, ..., y_J}_{\text{production}}) \in \mathbb{R}^{L \times m} \times \mathbb{R}^{L \times J} : \sum_{i=1}^m x_i = \sum_{j=1}^J y_j + \sum_{i=1}^m w_i, y_j \in Y_j, \forall j \right\} \tag{A}$$

Given $p \in \mathbb{R}^L$, firm j's problem is to choose production plan y_j^* s.t. $y_j^* \in y_j(p) = \arg\max p \cdot y_j$

$$y_j^* \in y_j(p) = \arg\max_{y_j \in Y_j} \quad p \cdot y_j \tag{ystar}$$

Given $p \in \mathbb{R}^L$ and production plans in $\{y_j(p), j = 1, ..., J\}$, consumer i's problem is to choose x_i^* s.t.

$$\begin{aligned} x_i^* \in x_i(p) = \arg\max_{x_i \in \mathbb{R}_+^L} \quad & u_i(x_i) \\ s.t. \quad & p \cdot x_i \leq p \cdot w_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j(p) \end{aligned} \tag{xstar}$$

5.2.1 Competitive Equilibrium

Definition 5.10 (Competitive Equilibrium)

An allocation (x^*, y^*) and a price vector $p^* \in \mathbb{R}^L$ are a *competitive equilibrium* in a private ownership production economy if

(i). $x_i^* \in x_i(p^*)$ (given by (xstar)) for all agent i. That is,

$$\begin{aligned} x_i^* \in x_i(p^*) = \arg\max_{x_i \in \mathbb{R}_+^L} \quad & u_i(x_i) \\ s.t. \quad & p^* \cdot x_i \leq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*(p^*) \end{aligned}$$

(ii). $y_j^* \in y_j(p^*)$ (given by (ystar)) for all firm j. That is,

$$y_j^* \in y_j(p^*) = \arg\max_{y_j \in Y_j} \quad p^* \cdot y_j$$

(iii). Market clearing: $(x^*, y^*) \in \mathcal{A}$ (given by (A)). That is

$$\sum_{i=1}^{m} x_i^* = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i$$

♣

Example 5.3 (Representative Agent Model) There is a single consumer ($m = 1$) and a single firm ($J = 1$).

For example, suppose there are $L = 2$ goods: time (label/leisure) x_l and consumption x_c.

Suppose the firm's production set is defined by a production function $f : \mathbb{R}_+ \to \mathbb{R}$, so

$$Y := \{(-y_l, y_c) \in \mathbb{R}^2 : y_l \geq 0, y_c \leq f(y_l)\}$$

The set of feasible consumption bundles is

$$\hat{Y} := (\underbrace{Y + \{\omega\}}_{\{y+w : y \in Y\}}) \cap \mathbb{R}_+^L$$

5.2.2 Pareto Optimal

Definition 5.11 (Pareto Optimal)

An allocation (x^*, y^*) in a private ownership production economy is **Pareto optimal** if there is no other allocation (x, y) s.t. $x_i \succeq_i x_i^*, \forall i$ and $x_h \succ_h x_h^*$ for some h.

♣

5.2.3 First-Welfare Theorem (production)

Theorem 5.5 (First-Welfare Theorem)

If each consumer's preference relation is locally non-satiated, then every competitive equilibrium allocation in a private ownership production economy is Pareto optimal.

♡

Proof 5.6

Let (x^*, y^*) be a competitive equilibrium allocation with corresponding equilibrium price vector p^*. Suppose by the way of contradiction that (x, y) is not Pareto optimal. That is, $\exists$ an allocation (x, y) s.t. $x_i \succeq_i x_i^*, \forall i$ and $x_h \succ_h x_h^*$ for some h. Then, by the (xstar),

$$p^* \cdot x_h > p^* \cdot w_h + \sum_{j=1}^{J} \theta_{hj} p^* \cdot y_j^*$$

and by local non-satiation,

$$p^* \cdot x_i \geq p^* \cdot w_i + \sum_{j=1}^{J} \theta_{ij} p^* \cdot y_j^*$$

Adding together

$$\sum_{i=1}^{m} p^* \cdot x_i > \sum_{i=1}^{m} p^* \cdot w_i + \sum_{j=1}^{J} p^* \cdot y_j^*$$

$$\Rightarrow \sum_{i=1}^{m} p^* \cdot x_i - \sum_{i=1}^{m} p^* \cdot w_i = p^* \cdot \left[\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} w_i \right] > \sum_{j=1}^{J} p^* \cdot y_j^*$$

As $\sum_{i=1}^{m} x_i = \sum_{j=1}^{J} y_j + \sum_{i=1}^{m} w_i$, we have $\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} w_i = \sum_{j=1}^{J} y_j$,

$$\sum_{j=1}^{J} p^* \cdot y_j = p^* \cdot \left[\sum_{i=1}^{m} x_i - \sum_{i=1}^{m} w_i \right] > \sum_{j=1}^{J} p^* \cdot y_j^*$$

There is a contradiction, since this implies there is a firm j and $y_j \in Y_j$ s.t. $p^* \cdot y_j > p^* \cdot y_j^*$, which contradicts to the assumption that y_j^* maximizes profits for form j at p^* ((ystar)).

5.2.4 Equilibrium with Transfers

Definition 5.12 ("Supportable" as a Price Equilibrium with Transfers)

An allocation (x^*, y^*) in a private ownership production economy is **supportable** as a **price equilibrium with transfers** if there exists a price vector $p^* \in \mathbb{R}^L$ and lump-sum budget-balancing transfers $\{T_i : i = 1, ..., m\}$ so that $\sum_{i=1}^m T_i = 0$, such that:

1. $\forall i$,
$$x_i^* \in \arg\max_{x \in \mathbb{R}_+^L} u_i(x_i)$$
$$\text{s.t. } p^* \cdot x_i \leq p^* \cdot w_i + \sum_{j=1}^J \theta_{ij} p^* \cdot y_j^* + T_i$$
2. $\forall j$,
$$y_j^* \in \arg\max_{y_j \in Y_j} p^* \cdot y_j$$
3. Feasibility:
$$\sum_{i=1}^m x_i^* = \sum_{j=1}^J y_j^* + \sum_{i=1}^m w_i$$

♣

5.2.5 Second Welfare Theorem (production)

Production economy is a more general form of exchange economy. It is the same that not every P.O. allocation can be supported as a price signal with transfers.

Theorem 5.6 (Second Welfare Theorem (production))

If each consumer's preference relation is continuous, strongly monotone, and convex, and each firm's production set is convex, then every interior P.O. allocation in a private ownership production economy can be supported as a price equilibrium with transfers.

♡

Proof 5.7

Let (x^*, y^*) be an interior P.O. allocation, so $x_i^* >> 0, \forall i$. The same as exchange economy, for each agent i, let $P_i := \{x_i \in \mathbb{R}_+^L : u_i(x_i) > u_i(x_i^*)\}$ and let

$$\begin{aligned} P :&= P_1 + \cdots + P_m \\ &= \left\{ z \in \mathbb{R}_L^+ : z = \sum_{i=1}^m x_i \text{ for some } (x_1, ..., x_m) \in \mathbb{R}_+^{L \times m} \text{ s.t. } u_i(x_i) > u_i(x_i^*), \forall i \right\} \end{aligned}$$

Then P is non-empty and convex.

In production side, let

$$Y := \sum_{j=1}^{J} Y_j = \left\{ y \in \mathbb{R}^L : y = \sum_{j=1}^{J} y_j \text{ for some } (y_1, ..., y_J) \text{ s.t. } y_j \in Y_j, \forall j \right\}$$

Then $Y + \{w\}$ is non-empty and convex.

Claim 5.2

$P \cap (Y + \{w\}) = \emptyset$. This follows from the assumption that (x^*, y^*) is P.O. (There is no allocation gives higher utilities while satisfies constraints).

By the Separating Hyperplane Theorem 5.3, $\exists p \in \mathbb{R}^L, p \neq 0$, s.t.

$$p \cdot z \geq p \cdot (y + w), \forall z \in P \text{ and } y \in Y \tag{p:SHT}$$

For each i, set

$$T_i := p \cdot x_i^* - p \cdot w_i - \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^*$$

Then,

$$p \cdot x_i^* = p \cdot w_i + \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^* + T_i$$

and

$$\begin{aligned} \sum_{1=1}^{m} T_i &= \sum_{i=1}^{m} \left(p \cdot x_i^* - p \cdot w_i - \sum_{j=1}^{J} \theta_{ij} p \cdot y_j^* \right) \\ &= p \cdot \left(\sum_{i=1}^{m} x_i^* - \sum_{i=1}^{m} w_i - \sum_{j=1}^{J} y_j^* \right) \\ &= p \cdot 0 = 0 \end{aligned}$$

So, $\{T_i : i = 1, ..., m\}$ are budget balancing.

Claim 5.3

$$p \cdot z \geq p \cdot (\sum_{i=1}^{m} x^*) = p \cdot (\sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i) \geq p \cdot (y + w), \forall z \in P \text{ and } y \in Y$$

To prove this, first note the feasibility, $\sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i \in Y + \{w\}$ and $\sum_{i=1}^{m} x_i^* = \sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i$. By the p:SHT,

$$p \cdot z \geq p \cdot \left(\sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i \right) = p \cdot \left(\sum_{i=1}^{m} x_i^* \right), \forall z \in P$$

Now using the strong monotonicity, for each i, we can choose a sequence $\{x_i^n\} \subseteq P_i$ s.t. $x_i^n \to x_i^*$ (e.g. $x_i^n = \left(1 + \frac{1}{n}\right) x_i^*$). Let $z^n = \sum_{i=1}^{m} x_i^n$ for all n. Then, $z^n \in P$ for all n and $z^n \to \sum_{i=1}^{m} x_i^*$.

Let $y \in Y$ be arbitrary. By the p:SHT,

$$\begin{aligned}
&p \cdot z^n \geq p \cdot (y + w), \forall n \\
&\Rightarrow \lim_{n \to \infty} p \cdot z^n = p \cdot \left(\sum_{i=1}^{m} x^* \right) \geq p \cdot (y + w) \\
&\Rightarrow p \cdot \left(\sum_{j=1}^{J} y_j^* + \sum_{i=1}^{m} w_i \right) = p \cdot \left(\sum_{i=1}^{m} x_i^* \right) \geq p \cdot (y + w)
\end{aligned}$$

That is, claim 5.3 is proved.

Claim 5.4

$\forall j$: $p \cdot y_j^* \geq p \cdot y_j, \forall y_j \in Y_j$

To show this, we fix k and $y_k \in Y_k$, such that $y_k + \sum_{j \neq k} y_j^* \in Y$. By claim 5.3,

$$\begin{aligned}
&p \cdot (\sum_{j=1}^{J} y_j^* + w) \geq p \cdot \left(y_k + \sum_{j \neq k} y_j^* + w \right) \\
&\Rightarrow p \cdot y_k^* \geq p \cdot y_k
\end{aligned}$$

Hence, claim 5.4 is proved.

Claim 5.5

$\forall i$: $u_i(x_i) > u_i(x_i^*) \Rightarrow p \cdot x_i > p \cdot x_i^*$.

Note that in the proof for the SWT in exchange economy, it is sufficient to show $\forall i$: $u_i(x_i) > u_i(x_i^*) \Rightarrow p \cdot x_i \geq p \cdot x_i^*$. Fix h and let $x_h \in P_h$. So, $u_h(x_h) > u_h(x_h^*)$. By the continuity and strong monotonicity of preference, we have $x_h + \sum_{i \neq h} x_i^* \in P$ (we can increase each x_i^* a little and reduce x_h). Hence, by 5.3,

$$\begin{aligned}
&p \cdot (x_h + \sum_{i \neq h} x_i^*) \geq p \cdot \left(\sum_{i=1}^{m} x_i^* \right) \\
&\Rightarrow p \cdot x_h \geq p \cdot x_h^*
\end{aligned}$$

Hence, claim 5.5 is proved.

All in all, SWT is proved.

5.3 Existence of Competitive Equilibrium

5.3.1 Excess Demand in Exchange Economies

Assumption *Suppose*

- *each consumer's preference relation is continuous, strongly monotone, and strictly convex, and*

- $\sum_i w_i >> 0$.

Based on this assumption 5.3.1, we have

- ∘ Each agent's demand function $x_i : \mathbb{R}^L_{++} \to \mathbb{R}^L_+$ is well-defined, continuous, homogeneous of degree 0, and satisfies Walras' Law (for individual).
- ∘ Excess demand function $Z : \mathbb{R}_{++} \to \mathbb{R}^L$ given by

$$Z(p) = \sum_{i=1}^{m} x_i(p) - \sum_{i=1}^{m} w_i$$

is

Definition 5.13 (Condition (1) to (4))

Given the Assumption 5.3.1, following conditions are satisfied:

(1). Continuous;

(2). Homogeneous of degree 0;

(3). Satisfies Walras' Law: $p \cdot Z(p) = 0, \forall p$;

(4). Bounded below: $\exists s > 0$ s.t. $Z_l(p) \geq -s, \forall p, \forall l = 1, ..., L$. ♣

5.3.2 Excess Demand in Production Economies

We use the same assumption 5.3.1 for consumers' preferences, and we add a assumption on the production side.

Assumption *Suppose each firm's production set Y_j is*

- *closed,*
- *strictly convex ($y, y' \in Y_j \Rightarrow \alpha y + (1-\alpha)y' \in \text{int}Y_j, \forall \alpha \in (0,1)$),*
- *bounded above ($\exists \bar{y}_j \in \mathbb{R}^L$ s.t. $y \leq \bar{y}_j, \forall y_j \in Y_j$), and*
- $0 \in Y_j$.

Based on this assumption 5.3.2, we have

- ∘ Each firm's supply function $y_j : \mathbb{R}^L_{++} \to \mathbb{R}^L$ is well-defined, continuous, and homogeneous of degree 0.

Based on assumption 5.3.1 and 5.3.2, we have

- ∘ Excess demand function $Z : \mathbb{R}_{++} \to \mathbb{R}^L$ given by

$$Z(p) = \sum_{i=1}^{m} x_i(p) - \sum_{j=1}^{J} y_j(p) - \sum_{i=1}^{m} w_i$$

is

Definition 5.14 (Condition (1) to (4))

Given the Assumption 5.3.2, following conditions are satisfied:

(1). Continuous;

(2). Homogeneous of degree 0;

(3). Satisfies Walras' Law: $p \cdot Z(p) = 0, \forall p$;

(4). Bounded below: $\exists s > 0$ s.t. $Z_l(p) \geq -s, \forall p, \forall l = 1, ..., L$.

- If $Z(p^*) = 0$, then p^* is a competitive equilibrium price vector, with corresponding equilibrium allocation $(x_1(p^*), ..., x_m(p^*), y_1(p^*), ..., y_J(p^*))$.

5.3.3 Boundary Condition

Since Z is homogeneous of degree 0, we can normalize prices, set

$$\Delta := \left\{ p \in \mathbb{R}_+^L : \sum_{l=1}^{L} p_l = 1 \right\}$$

We give other notations:

$$\partial\Delta : = \{p \in \Delta : p_l = 0 \text{ for some } l\}$$

$$\text{int}\Delta : = \{p \in \Delta : p_l > 0, \forall l\} = \Delta \cap \mathbb{R}_{++}^L$$

Consider an exchange economy. Let $p \in \partial\Delta$. Let $w_i >> 0$. If $\succeq_i$ is strongly monotone on $\mathbb{R}_{++}^L$, then demand of agent i is undefined at p_i (infinity for the zero price good).

So, we add a condition for excess demand Z:

Definition 5.15 (Condition (5))

(5). If $p^n \in \text{int}\Delta, \forall n$ and $p^n \to p$, where $p \in \partial\Delta$, then $\max_l\{Z_l(p^n)\} \to +\infty$.

Note

- *Condition (5) holds in an exchange economy with assumption 5.3.1;*
- *Condition (5) holds in a production economy with assumption 5.3.1 and 5.3.2;*
- *Condition (5) is not true in general, and the condition (5) does not imply $p_l^n \to 0 \Rightarrow Z_l(p^n) \to +\infty$ (relative prices matter!)*
- *By Walras' Law and lower bound on Z, then the converse holds:*

$$Z_l(p^n) \to +\infty \Rightarrow p_l^n \to 0$$

So, if condition (1) to (5) hold and $p^n \to p$ where $p^n >> 0$ and $p_l > 0$, then $\{Z_l(p^n)\}$ is bounded.

5.3.4 Existence of Competitive Equilibrium

Theorem 5.7 (Condition (1) to (5) $\Rightarrow \exists$ a competitive equilibrium)

Let $Z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ be a function s.t. condition (1) to (5) are satisfied, that is

(1). Continuous;

(2). Homogeneous of degree 0;

(3). Satisfies Walras' Law: $p \cdot Z(p) = 0, \forall p$;

(4). Bounded below: $\exists s > 0$ s.t. $Z_l(p) \geq -s, \forall p, \forall l = 1, ..., L$.

(5). If $p^n \in \text{int}\Delta, \forall n$ and $p^n \to p$, where $p \in \partial\Delta$, then $\max_l\{Z_l(p^n)\} \to +\infty$.

Note *(Fulfilled when Assumption 5.3.1 and 5.3.2 are satisfied).*

Then, $\exists \bar{p} \in \mathbb{R}^L_{++}$ s.t. $Z(\bar{p}) = 0$.

Proof 5.8

First, restrict attention to $p \in \Delta = \left\{p \in \mathbb{R}^L_+ : \sum_{l=1}^L p_l = 1\right\}$.

- *For $p \in \text{int}\Delta$, define a subset of good $\Lambda(p) \subseteq \{1, ..., L\}$ by*

$$\Lambda(p) := \left\{l \in \{1, ..., L\} : Z_l(p) = \max_k Z_k(p)\right\}$$

- *For $p \in \Delta \backslash \text{int}\Delta = \partial\Delta$, let*

$$\Lambda(p) := \{l \in \{1, ..., L\} : p_l = 0\}$$

Then, note $\Lambda(p) \neq \emptyset, \forall p \in \Delta$.

Define the correspondence $\varphi : \Delta \to 2^\Delta$ that maps a price vector p to a set of prices by

$$\varphi(p) := \{q \in \Delta : q_l = 0, \forall l \notin \Lambda(p)\}$$

As $\Lambda(p) \neq \emptyset, \forall p \in \Delta$, we have $\varphi(p) \neq \emptyset$ for all p and

$$\varphi(p) := \begin{cases} \{q \in \Delta : q \in \arg\max_{\tilde{q} \in \Delta} \tilde{q} \cdot Z(p)\} & \text{if } p \in \text{int}\Delta \\ \{q \in \Delta : q \cdot p = 0\} & \text{if } p \in \partial\Delta \end{cases}$$

Note

1. $\varphi(p) \subseteq \partial\Delta \Leftrightarrow \Lambda(p) \neq \{1, ..., L\}$
2. $p \in \partial\Delta \Rightarrow p$ *is not a fixed point of* φ.
3. p *is a fixed point of* $\varphi \Leftrightarrow p \in \text{int}\Delta$ *and* $\Lambda(p) = \{1, ..., L\} \Leftrightarrow p \in \text{int}\Delta$ *and* $\exists m \in \mathbb{R}$ *s.t.* $Z_l(p) = m, \forall l = \{1, ..., L\}$.
4. p *is a fixed point of* $\varphi \Leftrightarrow p \in \text{int}\Delta$ *and* $Z(p) = 0$. *(By Walras' Law:* $0 = p \cdot Z(p) = m \sum_l p_l = m$.*)*
5. $Z(p) \neq 0 \Rightarrow \Lambda(p) \neq \{1, ..., L\} \Rightarrow \varphi(p) \subseteq \partial\Delta$.

Now it suffices to show φ has a fixed point. Note that $\forall p \in \Delta$, $\varphi(p)$ is non-empty, convex, and compact, and Δ is non-empty, convex, and compact.

Claim 5.6

φ has closed graph.

Let $p^n \to p \in \Delta$ and $q^n \to q \in \Delta$ where $(p^n, q^n) \in$ graph φ $\forall n$. We want to show $(p, q) \in$ graph φ (i.e., $q \in \varphi(p)$):

- *Case 1: Suppose $p \in \text{int}\Delta$. Since $p >> 0$, assume without losing generality, $p^n >> 0$ $\forall n$. Suppose $l \notin \Lambda(p)$, we must show $q_l = 0$. Since $p >> 0$, $l \notin \Lambda(p) \Rightarrow Z_l(p) < \max_k Z_k(p)$. Since Z is continuous, $\exists N$, such that $\forall n \geq N$, $Z_l(p^n) < \max_k Z_k(p^n) \Rightarrow l \notin \Lambda(p^n), \forall n \geq N$ $\Rightarrow q_l^n = 0, \forall n \geq N \Rightarrow q_l = \lim_{n\to\infty} q_l^n = 0$. So, $q \in \varphi(p)$.*
- *Case 2: Suppose $p \in \partial\Delta$. Without losing generality, we write $p = (0, ..., 0, p_{r+1}, ..., p_L)$, where $p_l > 0$ for all $l = r+1, ..., L$. So, $\Lambda(p) = \{1, ..., r\}$ and $\varphi(p) = \{\tilde{q} \in \Delta : \tilde{q}_l = 0, l = r+1, ..., L\}$.*
  - *Case 2A: Suppose $\{p^n\}$ has a subsequence in $\text{int}\Delta$. Without losing generality, let $\{p^n\}$ denote this subsequence. Since $p^n \to p \in \partial\Delta$, $\max_k Z_k(p^n) \to +\infty$. Also, by Walras' Law and lower bound in Z, $\{Z_l(p^n)\}$ is bounded for $l = r+1, ..., L$.*

    Since $p^n \in \text{int}\Delta, \forall n$, $\exists N_2$ s.t. $\forall n \geq N_2$, $\Lambda(p^n) \subseteq \{1, ..., r\}$. Since $q^n \in \varphi(p^n), \forall n$ and $q_l^n = 0, \forall l = r+1, ..., L, \forall n \geq N_2$, we have $q_l = \lim_n q_l^n = 0$. Hence, $q \in \varphi(p)$.
  - *Case 2B: No subsequence of $\{p^n\}$ lies in $\text{int}\Delta$. Without losing generality, take $\{p^n\} \subseteq \partial\Delta$. Now, because $p_l > 0$ for $l = r+1, ..., L$, $\exists N_3$ s.t. $\forall n \geq N_3$, $p_l^n > 0$ for $l = r+1, ..., L$. Then, $\Lambda(p^n) \subseteq \{1, ..., r\}, \forall n \geq N_3$.*

    By the same argument above (in Case 2A), we have $q^n \in \varphi(p^n), \forall n \Rightarrow q_l^n = 0, \forall l = r+1, ..., L, \forall n \geq N_3 \Rightarrow q_l = \lim_{n\to\infty} q_l^n = 0, \forall l = r+1, ..., L$. Hence, $q \in \varphi(p)$.

All in all, φ has closed graph. By Kakutani's Fixed Point Theorem, φ has a fixed point. By above augment, $\bar{p} \in \text{int}\Delta$ and $Z(\bar{p}) = 0$.

Corollary 5.2

If an exchange economy satisfies assumption 5.3.1, then it has a competitive equilibrium. If a private ownership production economy satisfies assumption 5.3.1 and 5.3.2, then it also has a competitive equilibrium.

5.4 Uniqueness of Equilibrium

When is the equilibrium unique?

One condition:

Definition 5.16 (Strong Weak Axiom)

The function $Z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies the strong weak axiom if for any $\bar{p} \in \mathbb{R}^L_{++}$ s.t. $Z(\bar{p}) = 0$ and any $p \in \mathbb{R}^L_{++}$ s.t. $p \neq \alpha\bar{p}, \forall\alpha > 0$,

$$\bar{p} \cdot Z(p) > 0$$

♣

Theorem 5.8 (Uniqueness of Equilibrium)

If $Z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ satisfies condition (1)-(5) and strong weak axiom, then there is a unique $p^* \in \text{int}\Delta$ s.t. $Z(p^*) = 0$.

♡

Proof 5.9

Since Z satisfies condition (1)-(5), $\exists p^* \in \text{int}\Delta$ s.t. $Z(p^*) = 0$. By the strong weak axiom, if $p \in \text{int}\Delta$ and $p \neq p^*$, then $p^* \cdot Z(p) > 0 \Rightarrow Z(p) \neq 0$. So, there is a unique $p^* \in \text{int}\Delta$ s.t. $Z(p^*) = 0$.

Example 5.4

1. In an exchange economy with a representative consumer with strictly quasi-concave, strongly monotone, and C^1 (first-order continuously differentiable) utility function and $\omega >> 0$, the excess demand function satisfies the strong weak axiom.
2. If Z satisfies gross substitutes (for each l, Z_l is increasing in $p_k, \forall k \neq l$), then Z satisfies the strong weak axiom.

5.5 Market Demand and Observable Implications

Given an outcome, can we say it is obtained from an economy?

Restrict to exchange economies for simplicity.

Theorem 5.9 (Sonnenschein-Mantel-Debreu (SMD) Theorem)

Let $Z : \mathbb{R}^L_{++} \to \mathbb{R}^L$ be a function that is continuous and satisfies Walras' Law ($p \cdot Z(p) = 0, \forall p$). Then $\forall\epsilon > 0$ there is an exchange economy with L consumers having continuous, strictly convex, strongly monotone preferences, and endowments $\{w_i : i = 1, ..., L\} \subseteq \mathbb{R}^L_{+}$, s.t., the excess demand function for this economy is equivalent to Z on $\Delta^\epsilon = \{p \in \Delta : p_l \geq \epsilon, \forall l\}$.

♡

Theorem 5.10 (Mas-Colell Theorem)

Let $E \subseteq \mathrm{int}\Delta$ be compact. Then there exists an exchange economy with L consumers for which E is the set of competitive equilibrium prices. ♡

Example 5.5 Suppose $L = 2$, $Z : \mathbb{R}^2_{++} \to \mathbb{R}^2$ satisfies condition (1)-(5). Choose good 2 as numéraire, and set $p = (p_1, 1)$. Then,

1. If p_1 is close to 0, then $Z_1(p) > 0$ by the boundary condition (5) and lower bound condition (4).
2. If p_1 is large, then $Z_2(p) > 0$ by the boundary condition (5) and lower bound condition (4). So, $Z_1(p) = -\frac{1}{p_1} Z_2(p) < 0$ by Walras' Law (condition (3)).
3. $\exists p_1^*$ s.t. $Z_1(p^*) = 0$ by continuous Z (condition (1)), and p^* is an equilibrium price vector by Walras' Law (condition (3)).

5.6 Comparative Statics and Local Uniqueness

Restrict to exchange economies for simplicity.

Let $\vec{w} = (w_1, ..., w_m) \in \mathbb{R}^{L\times m}_{+}$ denote profile of initial endowments.

Let $\mathcal{E}(\vec{w})$ denote the economy with fixed preference relations $\{\succeq_i : i = 1, ..., m\}$ and endowment profile $\vec{w}$.

Let $Z : \mathbb{R}^L_{++} \times \mathbb{R}^{L\times m}_{++} \to \mathbb{R}^L$ denote excess demand as a function of $(p, \vec{w})$, so

$$Z(p, \vec{w}) := \sum_{i=1}^{m} x_i(p, p \cdot w_i) - \sum_{i=1}^{m} w_i$$

Let

$$Z_{-L}(p, \vec{w}) := (Z_1(p, \vec{w}), ..., Z_{L-1}(p, \vec{w}))$$

Normalize $p_L = 1$. Given $\vec{w}$, equilibrium in $\mathcal{E}(\vec{w})$ corresponds to p s.t. $Z_{-L}(p, \vec{w}) = 0$.

Assume $Z(\cdot, \vec{w})$ is C^1 and satisfies conditions (1)-(5), $\forall \vec{w}$.

Definition 5.17 (Regular Equilibrium)

Given $\vec{w}$, an equilibrium price vector p is a **regular equilibrium** if $D_p Z_{-L}(p, \vec{w})$ is non-singular (has full rank $L - 1$). ♣

Definition 5.18 (Regular/Critical Economy)

If every equilibrium in the economy $\mathcal{E}(\vec{w})$ is regular, then $\mathcal{E}(\vec{w})$ is a **regular economy**. An economy that is not regular is a **critical economy**. ♣

Proposition 5.3

1. *Regular equilibria are locally unique.*

   ($\exists$ open set V with $p^* \in V$ s.t. if $p \in V$, then $Z_{-L}(p, \vec{w}) = 0 \Leftrightarrow p = p^*$)

2. *A regular economy has finitely many equilibria.*

3. *In a regular economy, local equilibrium comparative statics are determinate.*

4. *$\mathcal{E}(\vec{w})$ is a regular economy if 0 is a regular value of $Z(\cdot, \vec{w})$. ($D_p Z(0, \vec{w})$ is non-singular).*

Theorem 5.11 (C^1 Demand Functions $\Rightarrow$ Regular Economy)

Suppose each agent $i = 1, ..., m$ has a C^1 demand function: $x_i : \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \to \mathbb{R}^L_+$. Then almost all economy are regular. That is,

$$\left\{\vec{w} \in \mathbb{R}^{L\times m}_{++} : \mathcal{E}(\vec{w}) \text{ is a critical economy}\right\}$$

has Lebesgue measure zero in $\mathbb{R}^{L\times m}$.

Proof 5.10

To prove DZ_{-L} has full rank, it is sufficient to show the sub-matrix has rank $L-1$.

Denote the derivative matrix of $DZ_{-L}(p, \vec{\omega})$ by A. The (j,k) item of the matrix is

$$\frac{\partial x_{jk}}{\partial p_k} + \frac{\partial x_{jk}}{\partial w}\omega_k$$

Compute the derivatives of Z_{-L} with respect to the initial endowment of consumer 1 (ω_1). Denote the derivative matrix by A. The (j,k) item of the matrix is

$$A_{jk} = \begin{cases} \frac{\partial x_{1j}}{\partial w}(p, p\cdot\omega_1)p_k - 1, & k = j \\ \frac{\partial x_{1j}}{\partial w}(p, p\cdot\omega_1)p_k, & k \neq j \end{cases}, j = 1, ..., L-1, k = 1, ..., L$$

Minus $k = 1, ..., L-1$ column by $\frac{p_k}{p_L}$ times L^{th} column. We can get

$$\begin{bmatrix} & A_{1L} \\ -I_{L-1\times L-1} & \vdots \\ & A_{L-1,L} \end{bmatrix}$$

which has rank $L-1$. Hence, A has rank $L-1$.

Hence, 0 is a regular value of Z_{-L}. So, the result follows from the Transversality Theorem.

5.7 General Equilibrium with Uncertainty

Set-up:

1. There are L (physical) goods.
2. There are two time periods, $t = 0, t = 1$.

3. At date $t = 0$, the state of nature is determinate.
4. At date $t = 1$, there are S ($s \in \{1, ..., S\}$) possible states of nature, and all uncertainty resolves at $t = 1$.
5. Hence, the commodity space is $\mathbb{R}^{L\times(S+1)}$.
6. Each consumer has preference relation $\succeq_i$ over $\mathbb{R}_+^{L\times(S+1)}$ represented by utility function u_i and initial endowment under different states $w_i = (w_{i0}, w_{i1}, ..., w_{iS}) \in \mathbb{R}_+^{L\times(S+1)}$, where w_{i0} is the endowment at $t = 0$ and w_{is} is the endowment at $t = 1$ with state $s \in \{1, ..., S\}$.

5.7.1 Basic Settings: (Complete) Contingent Commodities, Arrow-Debreu Equilibrium

Definition 5.19 (Contingent Commodity)

A unit of the **contingent commodity** (or **Arrow security**) l_s is a claim to receive a unit of good l if and only if state s occurs at date $t = 1$ ♣

Suppose at date $t = 0$, there are markets for "date $t = 0$ consumption" and "a complete set of Arrow securities". Given price vector $p \in \mathbb{R}^{L\times(S+1)}$, agent i's budget set is

$$B_i(p) = \left\{x \in \mathbb{R}_+^{L\times(S+1)} : p \cdot x \leq p \cdot w_i\right\}$$

Definition 5.20 (Arrow-Debreu Equilibrium)

A competitive equilibrium in this model is an **Arrow-Debreu equilibrium**. ♣

5.7.2 General: Asset Markets and Radner Equilibrium

There is a market of spots. Assets are using for the trading across different stages.

Note *Simplify: assume all assets payoff in units of good* 1.

An asset is defined by its return vector $r \in \mathbb{R}^S$.

Example 5.6

1. Arrow securities $l_{\bar{s}}$ (for good 1): $r_s = \begin{cases} 1, & s = \bar{s} \\ 0, & s \neq \bar{s} \end{cases}$
2. Riskless bond: $r_s = 1, \forall s$.
3. Another asset: $r_s = \begin{cases} 1, & s \text{ is even} \\ -1, & s \text{ is odd} \end{cases}$. Hence, $r = \left(-1, 1, ..., (-1)^{|S|}\right)$.

Suppose there are K assets traded at date $t = 0$, indexed by $k = 1, ...K$. Summarize their payoffs in an $S \times K$

matrix.

$$R := \begin{bmatrix} r_{11} & \cdots & r_{K1} \\ \vdots & \vdots & \vdots \\ r_{1S} & \cdots & r_{KS} \end{bmatrix}$$

where r_{ks} is the asset k's return in state s.

Assume assets are in zero total supply. Let $z_i \in \mathbb{R}^k$ denote agent i's portfolio. So,

$$z_{ik} = \text{\# units of asset } k \text{ bought/sold by agent } i$$

Let $q = (q_1, ..., q_K) \in \mathbb{R}^K$ denote the vector of asset prices.

Given an asset payoff structure R, the payoff from the portfolio z_i is $Rz_i \in \mathbb{R}^S$.

Let $p_s \in \mathbb{R}^L$ denote price vector expected at date 0 to hold in the spot market at date $t = 1$ if state s occurs.

Definition 5.21 (Radner Equilibrium)

A consumption allocation $(x_1^*, ..., x_m^*)$, portfolio profile $(z_1^*, ..., z_m^*)$, spot price vectors $(p_0^*, p_1^*, ..., p_S^*)$, and asset price vector q^* are a **Radner equilibrium** if

1. For every agent i: (x_i^*, z_i^*) solves
$$\max_{(x_i, z_i) \in \mathbb{R}_+^{L\times(S+1)} \times \mathbb{R}^K} u_i(x_i)$$
$$\text{s.t.} \left.\begin{aligned} p_0^* \cdot x_{i0} + q^* \cdot z_i \leq p_0^* \cdot w_{i0} \\ p_s^* \cdot x_{is} \leq p_s^* \cdot w_{is} + p_{1s}^*(Rz_i)_s, \forall s \end{aligned}\right\} \triangleq B_i(p^*, q^*, R)$$
(reminds that we assume all assets are payoff in good 1).
2. $\sum_{i=1}^m z_i^* = 0$, $\sum_{i=1}^m x_i^* = \sum_{i=1}^m w_i$.

♣

Note *Normalize:* $p_{1s} = 1, \forall s = 1, ..., S$.

Then consumer i's budget set is

$$B_i(p, q, R) := \left\{x_i \in \mathbb{R}_+^{L\times(S+1)} : \exists z \in \mathbb{R}^K \text{ s.t. } p_0 \cdot x_{i0} + q \cdot z \leq p_0 \cdot w_{i0} \text{ and } p_s \cdot (x_{is} - w_{is}) \leq (Rz_i)_s, \forall s\right\}$$

Definition 5.22 (Complete Asset Structure R)

The asset structure with return matrix R is **complete** if rank$R = S$. And the asset structure is **incomplete** if rank$R < S$.

♣

Theorem 5.12 (Arrow-Debreu Equilibrium ⇔ Radner Equilibrium)

Suppose preferences are strongly monotone and the asset structure is complete.

1. *If (x^*, p^*) is an Arrow-Debreu equilibrium, then there exists a portfolio profile z^* and asset price vector q^* such that (x^*, p^*, q^*, z^*) is a Radner equilibrium.*
2. *If (x^*, p^*, q^*, z^*) is a Radner equilibrium, then there exists a vector $\mu \in \mathbb{R}_{++}^S$ (i.e., common beliefs)*

such that $(x^*, (p_0^*, \mu_1 p_1^*, ..., \mu_S p_S^*))$ *is an Arrow-Debreu equilibrium.* ♡

Definition 5.23 (Arbitrage-free)

For asset structure with return matrix R, the asset price vector $q \in \mathbb{R}^K$ is **arbitrage-free** if $\nexists z \in \mathbb{R}^K$ s.t. $q \cdot z \leq 0$ and $Rz \geq 0$ with strict inequality for one. ♣

Proposition 5.4 (Strongly Monotone ⇒ Arbitrage-free)

If preferences are strongly monotone, then in any Radner equilibrium, asset prices must be arbitrage-free. ♠

We can also infer the components of assets from arbitrage-free prices.

Theorem 5.13 (Reconstruct Assets from Arbitrage-free Prices)

Consider an asset structure with return matrix R. If

- $q \in \mathbb{R}^K$ *is arbitrage-free, **or***
- $q \in \mathbb{R}^K$ *is a Radner equilibrium asset prices with* $r_k \geq 0, r_k \neq 0, \forall k$.

then there exists $\mu \in \mathbb{R}^S_{++}$ *such that* $q = \mu^T R$, *that is, s.t.*

$$q_k = \sum_s \mu_s r_{ks}, \forall k$$

♡

Chapter 6 Game Theory

Based on

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6.1 Basic Game Theory

6.1.1 Action and Domination Theorem

Let A be the finite set of possible actions and Ω be the finite set of possible states. A function can map the action and state to a value, $u(a, \omega)$. It can be represented by $\vec{u}(a) = \{u(a,\omega)\}_{\omega\in\Omega}$. It is common in game theory to assume the utility function is given or known.

A **mixed action** is a probability distribution over A, $\sigma \in \Delta(A)$.

A **belief** of the agent is a probability distribution over Ω, $\mu \in \Delta(\Omega)$.

Definition 6.1 (Optimal and Justifiable Mixed Action)

A mixed action $\sigma \in \Delta(A)$ is **optimal** given $\mu \in \Delta(\Omega)$ if

$$\mathbb{E}_\mu u(\sigma, \tilde{\omega}) \geq \mathbb{E}_\mu u(\sigma', \tilde{\omega}),\ \forall \sigma' \in \Delta(A)$$

A mixed action $\sigma \in \Delta(A)$ is **justifiable** if it is optimal for some belief $\mu \in \Delta(\Omega)$.

♣

Definition 6.2 (Dominant and Dominated Action)

A mixed action $\sigma \in \Delta(A)$ is **dominant** if

$$u(\sigma, \omega) > u(\sigma', \omega),\ \forall \omega \in \Omega, \sigma' \in \Delta(A), \sigma \neq \sigma'$$

A mixed action $\sigma \in \Delta(A)$ is **dominated** if

$$u(\sigma, \omega) < u(\sigma', \omega),\ \forall \omega \in \Omega,\ \text{and for some } \sigma' \in \Delta(A)$$

In this case we say σ' dominates σ.

♣

Theorem 6.1 (Domination Theorem: Justifiable = Not Dominated)

A mixed action is justifiable if and only if it is not dominated.

Proof 6.1

$\Rightarrow$ *is easily proved by the definition. We focus on proving* $\Leftarrow$*:*

Let $\mathcal{U} = \{\vec{u}(\sigma) : \sigma \in \Delta(A)\}$ *and* σ^* *be an undominated mixed action. Then, we have* $\mathcal{U} \cap (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++}) = \emptyset$*. Because* $\mathcal{U}$ *and* $\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++}$ *are disjoint, convex, and nonempty, we can use the Separating Hyperplane Theorem 5.3:* $\exists p \in \mathbb{R}^{\Omega}, p \neq 0$ *such that* $p \cdot a \leq p \cdot b, \forall a \in \mathcal{U}, b \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$*.*

Claim 6.1

$p \cdot \vec{u}(\sigma) \leq p \cdot \vec{u}(\sigma^*), \forall \sigma \in \Delta(A)$.

Proof 6.2

For any positive number m*,* $\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m}) \in \{\vec{u}(\sigma')\} + \mathbb{R}^{\Omega}_{++}$*. So, for any* $\sigma \in \Delta(A)$*,* $p \cdot \vec{u}(\sigma) \leq p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right)$*. By taking limit,* $p \cdot \vec{u}(\sigma^*) = \lim_{m \to \infty} p \cdot \left(\vec{u}(\sigma^*) + (\frac{1}{m}, ..., \frac{1}{m})\right) \geq p \cdot \vec{u}(\sigma)$*.*

Claim 6.2

$p > 0$.

Proof 6.3

Prove by the contradiction. Suppose $p_\omega < 0$ *for some* $\omega \in \Omega$*. Let* $z = (\epsilon, ..., \epsilon) + M\not{1}_\omega, M > 0, \epsilon > 0$*. So,* $\vec{u}(\sigma^*) + z \in (\{\vec{u}(\sigma^*)\} + \mathbb{R}^{\Omega}_{++})$*. We have* $p \cdot \vec{u}(\sigma^*) \leq p \cdot (\vec{u}(\sigma^*) + z)$ *by the result of SHT. There is a contradiction since* $p_\omega < 0$*. So, we have* $p \geq 0$*. Because* $p \neq 0$*,* $p > 0$ *is proved.*

Finally, we normalize p *to* $\mu = \frac{1}{\sum_\omega p_\omega} p$*. Then,* σ^* *is optimal for the belief* μ*, which means* σ^* *is justifiable.*

6.1.2 Extensive Game

Definition 6.3 (History)

The sequences of actions are called **histories**. $h' = (\underbrace{a_1, ..., a_n}_{h\text{:prefix of } h'}, a_{n+1}, ...) \in H$. We call h' is the **continuation** of h. h is a **terminal** of H if there is no continuation of h in H. ($\emptyset \in H$.)

Definition 6.4 (Extensive Perfect Information Game)

Am extensive game with prefect information is defined as $G = \{N, A, H, Z, P, O, o, \succ_{n \in N}\}$, where N is the set of players, A is the set of actions, H is the set of all histories, Z is the set of all histories that are terminals, $P : H/Z \to N$ is a mapping from a non-terminal histories to a player (who moves after a non-terminal history), O is the set of outcomes, and o is a function from Z to O.

A PIG is finite horizon if there is a bound on the length of its histories. ♣

We denote $A(h)$ as the actions available to player $P(h)$ after a history h.

Let $H_i = \{h \in H/Z : i = P(h)\}$ be the set of histories that player i moves after.

Definition 6.5 (Strategy)

A **strategy** is defined as a function $s_i : H_i \to A$ for which $s_i(h) \in A(h), \forall h \in H_i$. Let S_i be the set of all strategies available to the player i. A **strategy profile** is a collection of strategy $s = (s_i)_{i \in N}$. ♣

Definition 6.6 (Subgame)

A **subgame** of a PIG $G = \{N, A, H, Z, P, O, o, \succ_{n \in N}\}$ is a game (a PIG) that starts after a given finite history $h \in H$. Formally, the subgame $G(h)$ associated with $h = (h_1, ..., h_n) \in H$ is $G(h) = \{N, A, H_h, Z, P_h, O, o_h, \succ_{n \in N}\}$, where

$$H_h = \{(a_1, a_2, ...) : (h_1, ..., h_n, a_1, a_2, ...) \in H\}$$

$$o_h(h') = o(hh'), P_h(h') = P(hh')$$

A strategy s of G defines a strategy s_h of $G(h)$ by $s_h(h') = s(hh')$. ♣

Definition 6.7 (Subgame Perfect Equilibrium (SPNE))

A **subgame perfect equilibrium (SPNE)** of G is a strategy profile s^* such that for every subgame $G(h)$ it holds that $h' \mapsto s_i^*(hh')$ is an optimal strategy in $G(h)$, given beliefs that the rest of the players behave according to s_{-i}^* (or its restriction to $G(h)$). ♣

Definition 6.8 (Profitable Deviation)

Let s be a strategy profile. We say that s_i' is a **profitable deviation** from s for player i at history h if s_i' is a strategy for G such that

$$o_h\left(s_i', s_{-i}\right) \succ_i o_h(s)$$

♣

Note that a strategy profile has no profitable deviations iff it's a SPNE.

Theorem 6.2 (The one-deviation principle)

Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a finite horizon, extensive form game with perfect information. Let s be a strategy profile that is not a subgame perfect equilibrium. There exists some history h and a profitable deviation $\bar{s}_i$ for player $i = P(h)$ in $G(h)$ such that $\bar{s}_i(k) = s_i(k)$ for all $k \neq h$. ♡

- Let $G = \left(N, A, H, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a PIG.
- $A(\emptyset)$ is the set of allowed initial actions for player $i = P(\emptyset)$. For each $b \in A(\emptyset)$, let $s^{G(b)}$ be some

strategy profile for the subgame $G(b)$.

- Given some $a \in A(\emptyset)$, we denote by s^a the strategy profile for G in which player $i = P(\emptyset)$ chooses the initial action a, and for each action $b \in A(\emptyset)$ the subgame $G(b)$ is played according to $s^{G(b)}$.
- So $s_i^a(\emptyset) = a$ and for every player $j, b \in A(\emptyset)$ and $bh \in H \backslash Z$, $s_j^a(bh) = s_j^{G(b)}(h)$.

Lemma 6.1 (Backward Induction)

Let $G = \left(N, A, H, Z, O, o, P, \{\preceq_i\}_{i \in N}\right)$ be a finite PIG. Assume that for each $b \in A(\emptyset)$ the subgame $G(b)$ has a subgame perfect equilibrium $s^{G(b)}$. Let $i = P(\emptyset)$ and let a be the $\succ_i$-maximizer over $A(\emptyset)$ of $o_a\left(s^{G(a)}\right)$. Then s^a is a subgame perfect equilibrium of G. ♡

6.1.3 Strategic Form Game

Definition 6.9 (Normal Form Game)

A game in **normal form** is denoted by

$$G = \left(\underbrace{N}_{\text{players}}, \underbrace{\{S_i\}_{i \in N}}_{\text{Strategy Set}}, \underbrace{\{u_i(\cdot)\}_{i \in N}}_{\text{VNM utility}}\right)$$

$u_i : \prod_{i \in I} S_i \to \mathbb{R}$ is the utility function that maps all players' strategies to a player's utilities.

A <u>finite</u> game is a normal-form game in which the set of players N is a finite set, and the set of strategy profiles S is finite. ♣

Definition 6.10 (Mixed/Pure Strategy)

A mixed strategy for player i is a probability distribution $\sigma_i \in \Delta(S_i)$.

Elements of S_i are called pure strategies. ♣

Definition 6.11 (Dominant/Dominated Strategy)

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominant strategy** for i in G, if we have $u_i(\sigma_i, \sigma_{-i}) > u_i(\sigma_i', \sigma_{-i}), \forall \sigma_i' \neq \sigma_i, \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i, u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$.

A strategy $\sigma_i \in \Delta(S_i)$ is a **weakly dominated strategy** for i in G, if $\exists \sigma_i' \neq \sigma_i$, $u_i(\sigma_i, \sigma_{-i}) \leq u_i(\sigma_i', \sigma_{-i}), \forall \sigma_{-i} \in \times_{j \neq i} \Delta(S_j)$ and there is a $\sigma_{-i} \in \in \times_{j \neq i} \Delta(S_j)$, $u_i(\sigma_i, \sigma_{-i}) < u_i(\sigma_i', \sigma_{-i})$ ♣

Lemma 6.2

1. *A dominant strategy is always pure.*
2. *A strategy σ_i' dominates σ_i iff $u_i(\sigma_i', s_{-i}) > u_i(\sigma_i, s_{-i})$, for all pure strategy profiles $s_{-i} \in S_{-i}$.* ♡

Definition 6.12 (Belief, Best Response)

A **belief** for player i is a probability distribution $\mu \in \Delta(S_{-i})$.

A strategy $\sigma_i \in \Delta(S_i)$ is the **best response** to beliefs μ if it solves the problem of $\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, s_{-i})$.

Denote the set of all best responses to μ by $\beta_i(\mu)$. ♣

Lemma 6.3 (Mixed Strategy is BR iff its Pure Strategies are Indifferent)

A mixed strategy σ_i is in $\beta_i(\mu)$ iff every pure strategy in the support of σ_i is in $\beta_i(\mu)$. In particular, every strategy in the support of σ_i yields the same payoff to i. ♡

Theorem 6.3 (Domination Theorem rephrased)

In a finite game, a strategy is dominated iff there is no belief to which it is a best response. ♡

Definition 6.13 (Algorithm: Iterated Elimination of Dominated Strategies (IEDS))

Let $(N, (S_i), (u_i))$ be a finite game; $N = [n]$.

- We define (inductively) n sequences of sets of mixed strategies.
- Let $D_i^0 = \Delta(S_i)$.
- Given $D_1^{k-1}, \ldots, D_n^{k-1}$, let

$$D_i^k = \left\{\sigma_i : \nexists \bar{\sigma}_i : u_i(\sigma_i, \sigma_{-i}) < u_i(\bar{\sigma}_i, \sigma_{-i}) \, \forall \sigma_{-i} \in \times_{j \neq i} D_j^{k-1}\right\}.$$

- Note that $\left\{D_i^k\right\}$ is a decreasing sequence of sets.
- Let $D_i = \cap_{k=0}^{\infty} D_i^k$.
- The set $D = \times_{i=1}^{n} D_i$ be the set of strategies that survive the iterated elimination of dominated strategies.

A game is called **dominance-solvable** if D is a singleton. ♣

Definition 6.14 (Rationalizable Strategies)

- $R_i^0 = \Delta(S_i)$.
- Given $R_1^{k-1}, \ldots, R_n^{k-1}$, Let

$$Z_i^k = \left\{s_i \in S_i : \sigma_i(s_i) > 0 \text{ for some } \sigma_i \in R_i^{k-1}\right\}$$
$$R_i^k = \left\{\sigma_i \in \Delta(S_i) : \exists \mu \in \Delta\left(\times_{j \neq i} Z_j^k\right) \text{ s.t. } \sigma_i \in \beta_i(\mu)\right\}$$

Note: $\left\{R_i^k\right\}_{k=0}^{\infty}$ is a decreasing sequence of sets.

Let $R_i = \cap_{k=0}^{\infty} R_i^k$.

The **rationalizable strategies** are the elements of $R = \times_{i=1}^{n} R_i$. ♣

Lemma 6.4

In a finite game, R is always non-empty and contains a pure strategy profile.

Proposition 6.1

$\sigma_i \in \Delta(S_i)$ *is* ***rationalizable*** *iff there are sets* $Z_1, ..., Z_n, Z_j \subseteq S_j$ *such that*

1. $\sigma_i \in \beta_i(\mu_i)$ *for some* $\mu_i \in \Delta(\times_{h \neq i} Z_h)$.
2. *for every* $s_j \in Z_j$ *there is* $\mu_j \in \Delta(\times_{h \neq j} Z_h)$ *such that* $s_j \in \beta_j(\mu_j)$.

Corollary 6.1 (Rationalizable = IEDS)

Rationalizable strategies are exactly the strategies survive the iterated elimination of dominated strategies,

$$R = D$$

6.1.4 Nash Equilibrium and Existence

Definition 6.15 (Nash Equilibrium)

A strategy profile $\Sigma = (\sigma_1, ..., \sigma_I)$ is a **Nash** equilibrium of the game G if for every $i \in I$, we have: $u_i(\sigma_i^*, \sigma_{-i}^*) \geq u_i(\sigma_i', \sigma_{-i}^*), \forall \sigma_i' \in \Delta(S_i)$ (no profitable deviation). In other words,

1. σ_i is the best response to beliefs $\mu_i \in \Delta(S_{-i})$
2. $\mu_i = \sigma_{-i}$ (correct beliefs).

1. In rationalizable strategies, beliefs can be incorrect.
2. In a Nash equilibrium, beliefs are correct. Any strategy in a Nash equilibrium is rationalizable.

Definition 6.16 (Best Response Correspondence)

In a Nash equilibrium the player i's best response correspondence $\beta_i : \Delta(S_{-i}) \to 2^{\Delta(S_i)}$ is defined as $\beta_i(\sigma_{-i}) = \arg\max_{\sigma_i \in \Delta(S_i)} u_i(\sigma_i, \sigma_{-i})$. Let $\beta(\sigma) = \times_{i \in I} \beta_i(\sigma_{-i})$. Then σ is a Nash equilibrium iff $\beta(\sigma) = \sigma$. β is called the **best response correspondence** of the game.

Theorem 6.4 (Existence of Nash Equilibrium)

A Nash equilibrium exists in a finite game Γ, *if for all* $i \in I$,

(i). S_i *is non-empty, convex, compact, subset of* $\mathbb{R}^m$ *(i.e., for some finite dimensions of real numbers).*

(ii). $u_i(s_i, ..., s_I)$ *is continuous in* $(s_i, ..., s_I)$ *and quasi-concave in any* s_i.

Proof 6.4

We prove a lemma for the best response correspondence $\beta_i(s_{-i}) = \text{argmax}_{s_i \in S_i} u(s_i, s_{-i})$ *firstly.*

Lemma 6.5

Suppose $\{S_i\}_{i\in I}$ are non-empty. Suppose that S_i is compact and convex and u_i is continuous in $(s_i, ..., s_I)$ and quasi-concave in any s_i, then best response correspondence $\beta_i(s_{-i})$ is non-empty, convex-valued and uhc.

Proof 6.5

This lemma is proved by Berge's Maximum Theorem (Theroem 1.2).

Consider the best response correspondence of the game β with $\beta(s_i, ..., s_I) = \{\beta_1(s_{-1}), ..., \beta_I(s_{-I})\}$. As we proved β is non-empty, convex-valued and uhc from S to S where S is non-empty, compact, and convex. By the Kakutani's Fixed Point Theorem (Theorem 1.6), we have β has a fixed point $s \in S$, which should be the Nash equilibrium.

6.1.5 Bayesian Game

Definition 6.17 (Bayesian Game)

A **Bayesian game** is defined by

$$\Gamma = (I, \Omega, \{A_i\}_{i\in I}, \{u_i(\cdot)\}_{i\in I}, \{\Theta_i\}_{i\in I}, \{F_i\}_{i\in I})$$

where Ω is the state space, $u_i(a_i, a_{-i}, \theta_i)$ maps the actions of players and player i's type $\theta_i \in \Theta_i$ to player i's utilities, and $F_i \in \Delta(\Omega \times \Theta_i)$ is the (prior) distribution of the player i's type.

Definition 6.18 (Normal-form Bayesian game)

Assume a finite game. The **normal-form game** can be represented by

$$(I, (S_i, U_i)_{i\in I})$$

defined by letting S_i be the set of strategies based on types $s_i : \Theta_i \to A_i$ and

$$U_i(s) = \sum_{\omega\in\Omega} \sum_{(\theta_i)_{i\in I}\in\Theta} p(\omega, \theta_1, ..., \theta_I) u_i(s_1(\theta_1), ..., s_I(\theta_I), \omega)$$

for all $s \in S$.

A **Bayesian Nash equilibrium** of a Bayesian game is a strategy profile $(s_1, ..., s_n)$ that is a Nash equilibrium of the derived normal-form game.

Definition 6.19 (Best Response, Interim Payoff)

s_i is a BR to s_{-i} iff for all θ_i, $s_i(\theta_i)$ maximizes the **interim payoff** of player i. The interim payoff is defined by the expected payoff given the type θ_i of player i by playing action a_i.

$$\mathbb{E}_{\omega\in\Omega,\tilde{\theta}_{-i}\in\Theta_{-i}}[u_i(a_i, s_{-i}(\tilde{\theta}_{-i}), \omega)|\theta_i]$$

♣

6.1.6 Zero-sum Game

Let $G = (\{1,2\}, (S_1, S_2), (u_1, u_2))$. Suppose there is a constant c so that $u_1(s) + u_2(s) = c, \forall s \in S$. Then G is equivalent to a zero-sum game.

Definition 6.20 (Saddle Point)

Let X, Y be sets and $f : X \times Y \to \mathbb{R}$ a real function. $(x^*, y^*) \in X \times Y$ is a **saddle point** of f if $x^* \in \operatorname{argmax}_{x\in X} f(x, y^*)$ and $y^* \in \operatorname{argmin}_{y\in Y} f(x^*, y)$. That is,

$$f(x, y^*) \leq f(x^*, y^*) \leq f(x^*, y), \forall x \in X, y \in Y$$

♣

Consider a zero-sum game of two players. The strategy of the player 1 is max-min strategy, which is given by

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2)$$

and the strategy of the player 2 is min-max strategy, which is given by

$$\min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

Proposition 6.2 (min-max≥max-min)

Min-max strategy is always better than max-min strategy. That is,

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

♠

Proof 6.6

As $u(\sigma_1', \sigma_2) \leq \max_{\sigma_1} u(\sigma_1, \sigma_2)$ *for all* σ_1'*, we have*

$$\min_{\sigma_2} u(\sigma_1', \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2), \forall \sigma_1'$$

$$\Rightarrow \max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) \leq \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

We shall prove that these are in fact equal in the zero-sum game.

Definition 6.21 (Value of a zero-sum game)

A **value** for a zero-sum game G is a number $v \in \mathbb{R}$ for which there exists a strategy profile $(\bar{\sigma}_1, \bar{\sigma}_2)$ such that

$$u(\bar{\sigma}_1, \sigma_2) \geq v \quad \text{for all } \sigma_2$$
$$u(\sigma_1, \bar{\sigma}_2) \leq v \quad \text{for all } \sigma_1$$

Note: $v = u(\bar{\sigma}_1, \bar{\sigma}_2)$. The value is unique (if its exists), and represents a guaranteed payoff for the players. (Uniqueness: Suppose there are two values, v and $v' > v$, achieved by profiles σ and σ'. Then when 2 plays σ_2 we have that $u(\sigma_1', \sigma_2) \geq v'$ because σ_1' guarantees v'. And when 1 plays σ_1' we have $v \geq u(\sigma_1', \sigma_2)$ because σ_2 guarantees v. So $v' > v$ leads to a contradiction.)

Proposition 6.3

The following statements are equivalent

1. $(\bar{\sigma}_1, \bar{\sigma}_2)$ *is a Nash equilibrium.*
2. $v = u(\bar{\sigma}_1, \bar{\sigma}_2)$.

Corollary 6.2

If (σ_1^*, σ_2^*) *and* $(\bar{\sigma}_1, \bar{\sigma}_2)$ *are Nash equilibria of G, then so are the profiles* $(\bar{\sigma}_1, \sigma_2^*)$ *and* $(\sigma_1^*, \bar{\sigma}_2)$.

Theorem 6.5

Let G be a zero-sum game. There is a strategy profile (σ_1^*, σ_2^*) *s.t.*

$$\max_{\sigma_1} \min_{\sigma_2} u(\sigma_1, \sigma_2) = v = \min_{\sigma_2} \max_{\sigma_1} u(\sigma_1, \sigma_2)$$

where $v = u(\sigma_1^*, \sigma_2^*)$ *is the value of the game and* (σ_1^*, σ_2^*) *is a Nash equilibrium.*

Proof 6.7

Let $n_i = |S_i|$. *Let* $\vec{u}(\sigma_2) := \{u(s_1, \sigma_2) : s_1 \in S_1\}$. *Let* $\mathbb{C} = \{\vec{u}(\sigma_2) : \sigma_2 \in \Delta(S_2)\}$. *We can find* $\mathbb{C}$ *is convex and compact.*

Let $m(x) := \max\{x_i : i = 1, ..., n_1\}$. *Then, the player 2's min max payoff is given by* $v := \inf\{m(x) : x \in \mathbb{C}\}$. *By compactness, exists strategy* σ_2^* *such that* $\vec{u}(\sigma_2^*) = (v, v, ..., v)$. *That is,* $u(s_1, \sigma_2^*) = v, \forall s_1 \in S_1$.

Let $\mathbb{A} := \{z \in \mathbb{R}^{n_1} : z << (v, v, ..., v)\} = (v, v, ..., v) - \mathbb{R}^{n_1}_{++}$. *As* $\mathbb{C}$ *and* $\mathbb{A}$ *are disjoint. By SHT, we can find a* $p \neq 0$ *s.t.* $p \cdot \mathbb{A} \leq p \cdot \mathbb{C}$. $p > 0$ *since* $\mathbb{A}$ *can have arbitrary small elements in any dimension. Then, we normalize* p *to be in* $\Delta(S_1)$, *denote it by* σ_1^*.

By limitation, $v = \sigma_1^* \cdot (v, v, ..., v) = \lim_{\epsilon \to 0^+} \sigma_1^* \cdot (v - \epsilon, v - \epsilon, ..., v - \epsilon) \leq \sigma_1^* \cdot \vec{u}(\sigma_2), \forall \sigma_2 \in \Delta(S_2)$.

Hence, $u(s_1, \sigma_2^*) \leq m(\vec{u}(\sigma_2^*)) = v = u(\sigma_1^*, \sigma_2^*) \leq u(\sigma_1^*, \sigma_2)$

6.1.7 Correlated equilibrium

Suppose there is a mediator that give advices to each player based on a distribution $p \in \Delta(S)$. A player doesn't know other players' advices but knows the distribution p.

Definition 6.22 (Correlated Equilibrium)

A **correlated equilibrium** of G is any probability distribution $p \in \Delta(S)$ such that, for all i and $s_i, s_i' \in S_i$, the player i can't get a higher expected profit than by following the advice,

$$\sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) u_i(s_i, s_{-i}) \geq \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i}) u_i(s_i', s_{-i})$$

where LHS is the expected profit of player i when he receives an advice s_i from the mediator. ♣

Let G be a finite game,

Proposition 6.4

If we identify a Nash equilibrium σ of G with a probability distribution on $\Delta(S)$, then any Nash equilibrium of G is also a correlated equilibrium. ♠

Proposition 6.5

The set of correlated equilibria is a non-empty, convex, compact subset of $\Delta(S)$. ♠

For $p \in \Delta(S)$, the **marginal distribution** on S_i is given by

$$p_i(s_i) = \sum_{s_{-i}\in S_{-i}} p(s_i, s_{-i})$$

Proposition 6.6

A correlated equilibrium that is the independent mixture of its marginal distributions is a Nash equilibrium. ♠

6.1.8 Quantal Response Equilibrium

Imagine choosing $a \in A$.

Definition 6.23 (Quantal Response (softmax))

A **quantal response** is a function $\gamma : \mathbb{R}^A \to \Delta(A)$ mapping from a vector of utility values v to a probability distribution over actions, which satisfies that

- $\gamma(v) \gg 0$ for all v (interior);
- γ is continuous,
- γ is monotonic $(v_h < v_j \to \gamma_h(v) < \gamma_j(v))$

∘ γ is responsive $\left(v_j < v_j' \to \gamma_j(v) < \gamma_j\left(v_j', v_{-j}\right)\right)$

Interpret $\gamma(v)$ as the probability of choosing each of $a \in A$ alternatives when v is the vector of utility values of the alternatives in A.

Interpretation: mistakes or random utility.

One common quantal response function is the logistic function:

$$\gamma_j(v) = \frac{e^{\lambda v_j}}{\sum_{h \in I} e^{\lambda v_h}}$$

for $\lambda > 0$, where λ captures how close the quantal response is to choosing according to the largest values of v.

Fix a normal-form game $G = \{N, \{S_i, i \in N\}, \{u_i, i \in N\}\}$. Let $\gamma_i : \mathbb{R}^{S_i} \to \Delta(S_i)$ be a quantal response for player i.

Definition 6.24 (Quantal Response Equilibrium)

A **quantal response equilibrium** of G is a strategy profile σ^* such that

$$\sigma_i^* = \gamma_i\left(\vec{u}_i(\sigma_{-i}^*)\right)$$

where $\vec{u}_i(\sigma_{-i}^*) = (u_i(s_i, \sigma_{-i}^*))_{s_i \in S_i}$.

Proposition 6.7

Every finite normal-form game, with any profile of quantal responses, has a quantal response equilibrium.

Observation: λ measures the distance to Nash.

Proposition 6.8

Let $\{\lambda^k\}$ be a sequence such that $\lim_{k \to \infty} \lambda^k = \infty$ and $\sigma^*(\lambda^k)$ be a QRE when the logistic quantal responses take parameter value λ^k. If $\{\sigma^*(\lambda^k)\}$ is a convergent sequence, it converges to a Nash equilibrium.

6.2 Knowledge and Common Knowledge

6.2.1 Knowledge and Information

1. Let Ω be a (finite) set of possible states of the world.
2. Subsets $E \subseteq \Omega$ are **events**.
3. **Knowledge** is modeled through a function

$$K : 2^\Omega \to 2^\Omega$$

Interpretation: $K(E)$ (which we write as KE) is the event (the set of states) at which the agent knows that the event E has occurred.

4.

Definition 6.25 (Information Function given Knowledge)

Given knowledge function $K : 2^\Omega \to 2^\Omega$, we define the function $P : \Omega \to 2^\Omega$ by,

$$P(\omega) = \cap\{E \subseteq \Omega : KE \ni \omega\}$$

an **information function**. ♣

When the state is ω the decision-maker knows only that the state is in the set $P(\omega)$. Means that, if $\omega' \in P(\omega)$, then when the state is ω, information doesn't allow one to distinguish between ω and ω'. Given our interpretation of an information function, a decision-maker for whom $P(\omega) \subseteq E$ knows, in the state ω, that some state in the event E has occurred.

Definition 6.26 (Knowledge given Information Function)

Given an information function $P : \Omega \to 2^\Omega$, we define the **knowledge** $K : 2^\Omega \to 2^\Omega$ by,

$$KE = \{\omega \in \Omega : P(\omega) \subseteq E\}$$

♣

These two equations provide the back and fourth relationship between knowledge and information. However, we don't give any restrictions for the settings of the knowledge or the information function, so they can be any forms.

6.2.2 Partitional Information Function

Definition 6.27 (P1&P2 Conditions for Information Function)

Usually, we assume following two conditions of a information function:

P1. $\omega \in P(\omega)$ for every $\omega \in \Omega$.

P2. If $\omega' \in P(\omega)$ then $P(\omega') = P(\omega)$. ♣

Definition 6.28 (S5 Conditions for Knowledge)

There are 5 axioms of knowledge that can restrict the form of knowledge.

1. $K\Omega = \Omega$

   Alice knows that some state of the world has occurred.

2. $KA \cap KB = K(A \cap B)$

   Alice knows A and knows B iff she knows A and B.

3. $KA \subseteq A$ (Axiom of knowledge)

   If Alice knows A, then A has indeed occurred (some state in A is true).

4. $KKA = KA$ (Axiom of positive introspection)

   If Alice knows A then she knows that she knows A.

5. $(KA)^c = K((KA)^c)$ (Axiom of negative introspection)

   If Alice doesn't know A then she knows that she doesn't know A.

They are not independent. ♣

Definition 6.29 (Partitional)

Information function P (and associated knowledge operator K) is **partitional** if $\{P(\omega) : \omega \in \Omega\}$ constitute a partition of Ω. ♣

When P is partitional we abuse notation and denote the partition by P as well.

Theorem 6.6 (S5 ⇔ P1&P2 ⇔ Partitional)

For an information function P (and associated knowledge operator K), following conditions are equivalent:

- *P/K is partitional;*
- *P satisfies P1 and P2.*
- *K satisfies S5.*

K satisfies S5 iff it is partitional. ♡

Example 6.1 (Non-Partitional) $\Omega = \{\text{bark, don't bark}\}$. $P(\omega) = \begin{cases} \{\text{bark}\} & \text{if } \omega = \text{bark} \\ \{\text{bark, don't bark}\} & \text{if } \omega = \text{don't bark} \end{cases}$

Then, $K(\{\text{bark}\}) = \{\text{bark}\}$ and $K(\{\text{don't bark}\}) = \emptyset$. Axiom 5 is violated.

6.2.3 Self-evident and Algebra

Definition 6.30 (Self-evident)

An event $A \in 2^\Omega$ is **self-evident** if $KA = A$. ♣

Proposition 6.9 (Self-evident ⇔ Unions of Elements of Partition)

A is self-evident iff it is the union (1 or more) of elements of the partition P. ♠

Proof 6.8

A is self-evident iff $A = KA = \{\omega : P(\omega) \subseteq A\}$.

Definition 6.31 (Algebra)

A collection of events Σ is an **algebra** if it satisfies:

1. $\Omega \in \Sigma$;
2. If $A \in \Sigma$ then $A^c \in \Sigma$;
3. If $A, B \in \Sigma$ then $A \cup B \in \Sigma$.

♣

Corollary 6.3 (Set of Self-Evident Events is an Algebra)

Let Σ be the collection of self-evident events. Then Σ is an algebra (in fact, $\Sigma = \Sigma_P$, the algebra generated by the partition P).

♡

Corollary 6.4

For any A, $KA \in \Sigma$. In fact,

$$KA = \cup\{S \in \Sigma : S \subseteq A\}$$

♡

So KA is always self-evident. And, since any algebra is closed under unions, it follows that KA is the largest element of Σ that is contained in A.

Definition 6.32 (Refinement and Coarsening)

Let Σ, Π be two sub-algebras of some algebra.We say that Σ is a **refinement** of Σ and Π is a **coarsening** of Σ if $\Pi \subseteq \Sigma$.

♣

Definition 6.33

The **meet** of two algebras $\Sigma_1, \Sigma_2 \subseteq \Sigma$ is the finest sub-algebra of Σ that is a coarsening of each Σ_i.

♣

6.3 Mechanism Design

Design incentives for agents to reveal their types or achieve particular society outcomes.

Given the set of agents, alternatives (for the society), and types (of agents) are I, X, Θ and a social choice function $f : \Theta = (\Theta_1, ..., \Theta_I) \to X$.

Definition 6.34 (Mechanism)

A **mechanism** is represented as

$$\Gamma = \Big((S_1, ..., S_I), g : S \triangleq (S_1, ..., S_I) \to X\Big)$$

where S_i represents the strategy set of agent i.

♣

Definition 6.35 (Implement)

Γ (indirectly) **implements** a social choice function f if $\exists\,(s_1^*(\cdot), ..., s_I^*(\cdot))$ of a game induced by Γ such that $g(s_1^*(\theta_1), ..., s_I^*(\theta_I)) = f(\theta_1, ..., \theta_I)$ for all $(\theta_1, ..., \theta_I) \in \Theta$ ♣

Definition 6.36 (Direct Mechanism)

A mechanism is **direct mechanism** if $S_i = \Theta_i$ for all $i \in I$ and $g(\theta) = f(\theta)$ for all $\theta = (\theta_1, ..., \theta_I) \in \Theta$. So, a direct mechanism can be represented by $\Gamma = (\Theta, f(\cdot))$. ♣

Definition 6.37 (Weak Dominance)

A strategy is weakly dominant if for all $\theta_i \in \Theta_i$ and all $s_{-i}(\cdot) \in S_{-i}$, we have:

$$u_i(g(s_i(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

for all $s_i' \neq s_i$. ♣

Definition 6.38 (Dominant Strategy Equilibrium)

Strategy profile $s^* = (s_1^*(\cdot), ..., s_I^*(\cdot))$ is a **dominant strategy (D-S) equilibrium** of $\Gamma = (S, g(\cdot))$ if for all $i \in I$ and $\theta_i \in \Theta$, we have:

$$u_i(g(s_i^*(\theta_i), s_{-i}), \theta_i) \geq u_i(g(s_i', s_{-i}), \theta_i)$$

for all $s_i' \in S_i$ and $s_{-i} \in S_{-i}$. ♣

Definition 6.39 (Implement in dominant strategies)

Γ **implements** f in **dominant strategies** if $\exists$ a dominant strategy (D-S) equilibrium s^* of Γ such that $g(s^*(\theta)) = f(\theta)$. ♣

Definition 6.40 (Strategy-Proof, DSIC)

f is **strategy-proof** (also called dominant-strategy-incentive-compatible, DSIC) if "$s_i^*(\theta_i) = \theta_i$ for all $\theta_i \in \Theta_i$ and all $i \in I$" is a dominant strategy (D-S) equilibrium of the direct mechanism $\Gamma = (\Theta, f(\cdot))$. ♣

Theorem 6.7 (Revelation Principle)

Suppose that $\exists\Gamma = (S, g(\cdot))$ that (indirectly) implements f in dominant strategies. Then f is strategy-proof (DSIC). ♡

Theorem 6.8 (Gibbard-Satterthwaite theorem)

Suppose that $|X| \geq 3$ and a social choice function f is surjective. Then,

$$f \text{ is strategy-proof (DSIC)} \Leftrightarrow f \text{ is dictatorial (3.14)}$$

♡

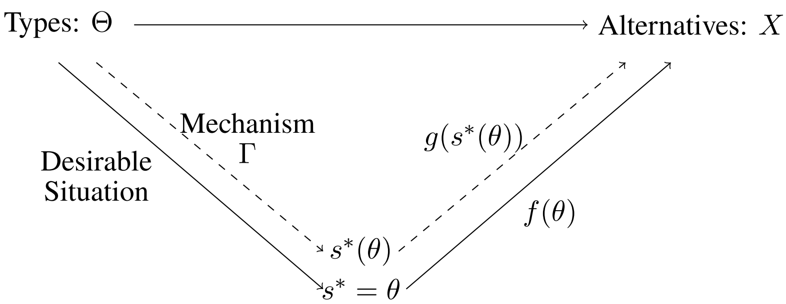

Figure 6.1: How Mechanism Design works

Some lemmas can help to prove the theorem.

Lemma 6.6

If f is strategy-proof (DSIC) and $f(\succeq) = x$ and $x \succeq_i y \Rightarrow x \succeq_i' y$ for all $i \in I$ and all $x \neq y \in X$, then $f(\succeq') = x$. ♡

Lemma 6.7 (Pareto Effeciency)

If f is strategy-proof (DSIC) and $x \succ_i y$ for all $i \in I$, then $f(\succeq') \neq y$. ♡

Example 6.2 Define $\succeq = \begin{pmatrix} x & y \\ y & x \\ z & z \end{pmatrix}$ and $\succeq' = \begin{pmatrix} x & y \\ y & z \\ z & x \end{pmatrix}$, each column 1/2 represents player 1/2's preferences.

6.4 Signaling Game

6.4.1 Canonical Game

Definition 6.41 (Canonical Game)

1. There are two players: **S** (sender) and **R** (receiver).
2. **S** holds more information than **R**: the value of some random variable t with support $\mathcal{T}$. (We say that t is the **type** of **S**)
3. Prior belief of **R** concerning t are given by a probability distribution ρ over $\mathcal{T}$ (common knowledge)
4. **S** sends a **signal** $s \in \mathcal{S}$ to **R** drawn from a signal set $\mathcal{S}$.
5. **R** receives this signal, and then takes an **action** $a \in \mathcal{A}$ drawn from a set $\mathcal{A}$ (which could depend on the signal s that is sent).
6. **S**'s payoff is given by a function $u : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$ and **R**'s payoff is given by a function $v : \mathcal{T} \times \mathcal{S} \times \mathcal{A} \to \mathbb{R}$.

♣

6.4.2 Nash Equilibrium

Definition 6.42 (Strategy)

A **behavior strategy** for **S** is given by a function $\sigma : \mathcal{T} \times \mathcal{S} \to [0,1]$ such that $\sum_s \sigma(t,s)$ for each t.

A **behavior strategy** for **R** is given by a function $\alpha : \mathcal{S} \times \mathcal{A} \to [0,1]$ such that $\sum_a \alpha(s,a)$ for each t. ♣

Definition 6.43 (Nash Equilibrium)

Behavior strategies α and σ form a **Nash equilibrium** if and only if

1. For all $t \in \mathcal{T}$,

$$\sigma(t,s) > 0 \text{ implies } \textstyle\sum_a \alpha(s,a)u(t,s,a) = \max_{s' \in \mathcal{S}} \left(\sum_a \alpha(s',a)u(t,s',a)\right)$$

2. For each $s \in \mathcal{S}$ such that $\sum_t \sigma(t,s)\rho(t) > 0$,

$$\alpha(s,a) > 0 \text{ implies } \textstyle\sum_t \mu(t;s)v(t,s,a) = \max_{a'} \sum_t \mu(t;s)v(t,s,a')$$

   where $\mu(t;s)$ is the $\mathbb{R}$'s posterior belief about t given s, $\mu(t;s) = \frac{\sigma(t,s)\rho(t)}{\sum_{t'} \sigma(t',s)\rho(t')}$ if $\sum_t \sigma(t,s)\rho(t) > 0$ and $\mu(t;s) = 0$ otherwise. ♣

Definition 6.44 (Separating & Pooling Equilibrium)

An equilibrium (σ, α) is called a **separating** equilibrium if each type t sends different signals; i.e., the set $\mathcal{S}$ can be partitioned into (disjoint) sets $\{\mathcal{S}_t; t \in \mathcal{S}\}$ such that $\sigma(t, \mathcal{S}_t) = 1$. An equilibrium (σ, α) is called a **pooling** equilibrium if there is a single signal s^* that is sent by all types; i.e., $\sigma(t, s^*) = 1$ for all $t \in \mathcal{T}$. ♣

6.4.3 Single-crossing

6.4.3.1 Situation over real line

Consider the situation that $\mathcal{T}, \mathcal{S}, \mathcal{A} \subseteq \mathbb{R}$ and $\geq$ is the usual "greater than or equal to" relationship.

1. We let $\Delta\mathcal{A}$ denote the set of probability distributions on $\mathcal{A}$.
2. For each $s \in \mathcal{S}$ and $\mathcal{T}' \subseteq \mathcal{T}$, we let $\Delta\mathcal{A}(s, \mathcal{T}')$ be the set of mixed strategies that are the best responses by **R** to $s \in \mathcal{S}$ for some probability distribution with support $\mathcal{T}'$.
3. For $\alpha \in \Delta\mathcal{A}$, we write $u(t,s,\alpha) \triangleq \sum_{a \in \mathcal{A}} u(t,s,a)\alpha(a)$.

Definition 6.45 (Single-crossing)

The data of the game are said to satisfy the **single-crossing property** if the following holds: If $t \in \mathcal{T}$, $(s,\alpha) \in \mathcal{S} \times \Delta\mathcal{A}$ and $(s',\alpha') \in \mathcal{S} \times \Delta\mathcal{A}$ are such that $\alpha \in \Delta\mathcal{A}(s,\mathcal{T})$, $\alpha' \in \Delta\mathcal{A}(s',\mathcal{T})$, $s > s'$ and $u(t,s,\alpha) \geq u(t,s',\alpha')$, then for all $t' \in T$ such that $t' > t$, $u(t',s,\alpha) \geq u(t',s',\alpha')$. ♣

6.5 Equilibrium Refinement

6.5.1 Cho-Kreps Intuitive Criterion

Definition 6.46 (Equilibrium Dominated Message)

A message is **equilibrium dominated** for a type if the type must do strictly worse by sending the message than it does in equilibrium (i.e., payoff in eq. is strictly better than maximum payoff from deviating). ♣

Definition 6.47 (Cho-Kreps Intuitive Criterion)

If an information set is off the eq. path and a message is eq. dominated for a type, then beliefs should assign zero probability to the message coming from that type (if possible). ♣

Chapter 7 Market Design

Based on

- Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis, Roth, Alvin E.& Sotomayor, Matilda, 1990.
- Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. *Mathematics of Operations research*, 28(1), 103-126.
- Hatfield, J. W., & Kominers, S. D. (2017). Contract design and stability in many-to-many matching. *Games and Economic Behavior*, 101, 78-97.

7.1 Matching One-to-One

Suppose there are doctors (D) and hospitals (H). For a doctor d, define a relation $\succeq_d$ over $H \cup \{d\}$; for a hospital h, define a relation $\succeq_h$ over $D \cup \{h\}$. A matching market is defined by

$$(D, H, \{\succeq_i\}_{i \in D \cup H})$$

Note *Given a matching $\mu : D \cup H \to D \cup H$, we would call $\mu(d)$ be "d's match".*

Definition 7.1 (Involution)

A matching $\mu : D \cup H \to D \cup H$ is **involution** such that

$$\mu(d) \neq d \Rightarrow \mu(d) \in H, \forall d \in D$$

and

$$\mu(h) \neq h \Rightarrow \mu(h) \in D, \forall h \in H$$

♣

Definition 7.2 (Stable)

A matching $\mu : D \cup H \to D \cup H$ is **stable** if it is

- Individually Rational: $\nexists\, i$ for whom $i > \mu(i)$.
- (Pairwise) Unblocked: $\nexists\, (d, h)$ such that $d \succ_h \mu(h)$ and $h \succ_d \mu(d)$.

♣

Theorem 7.1 (Gale-Shapley, 1962)

For any matching market, a stable matching μ exists.

♡

Proof 7.1

We prove it by an algorithm:

Definition 7.3 (Deferred Acceptance Algorithm (DA))

At each round, every doctor applies for his most preferred hospital among those haven't rejected him. Each hospital chooses its most preferred doctors among its applicants and the one on the previous waitlist, and then rejects others. ♣

Observation: DA terminates μ. We want to prove

1. *μ is IR (obviously);*
2. *μ is unblocked.*

   Suppose there is a block (d, h) such that $d \succ_h \mu(h)$ and $h \succ_d \mu(d)$. That is impossible, because the $d \neq \mu(h)$, the d must be rejected by h, which means $h \preceq_d \mu(d)$.

Note *We call "h is **achievable** for d" if $\mu(d) = h$ for some stable matching μ.*

7.1.1 Matching Markets: One-to-One

Definition 7.4 (D-Optimal Matching)

A matching $\mu : D \cup H \to D \cup H$ is **D-optimal**, denoted by μ^D, if for any stable μ' we have that $\mu^D \succeq_D \mu'$ (the best stable matching for all doctors). ♣

Theorem 7.2 (Deferred Acceptance Algorithm $\Rightarrow$ D-Optimal Matching)

Deferred Acceptance Algorithm (with D proposing) terminates in μ^D. ♡

Proof 7.2

...Theorem 2.12 (Gale and Shapley)

Theorem 7.3 (Lone-Wolf Theorem)

The set of matched agent is identical in every stable μ. ♡

Proof 7.3

$|\mu^D(H)| \geq |\mu(H)| \geq |\mu^H(H)|$*; by symmetry,* $|\mu^H(D)| \geq |\mu(D)| \geq |\mu^D(D)|$*. Because* $|\mu^D(H)| = |\mu^D(D)|$ *and* $|\mu^H(H)| = |\mu^H(D)|$ *by one-to-one, so everything is equal.*

7.1.2 Joint and Meet

Definition 7.5 (Joint and Meet)

1. **Join** $\mu \vee_D \mu'$ assign the more preferred match to every d and the less preferred match to every h, that is,

$$\mu \vee_D \mu'(d) = \begin{cases} \mu(d), & \text{if } \mu(d) >_d \mu'(d) \\ \mu'(d), & \text{otherwise} \end{cases}, \forall d \in D$$

$$\mu \vee_D \mu'(h) = \begin{cases} \mu(h), & \text{if } \mu(h) <_h \mu'(h) \\ \mu'(h), & \text{otherwise} \end{cases}, \forall h \in H$$

2. **Meet** $\mu \wedge_D \mu'$ assign the less preferred match to every d and the more preferred match to every h, that is,

$$\mu \wedge_D \mu'(d) = \begin{cases} \mu(d), & \text{if } \mu(d) <_d \mu'(d) \\ \mu'(d), & \text{otherwise} \end{cases}, \forall d \in D$$

$$\mu \wedge_D \mu'(h) = \begin{cases} \mu(h), & \text{if } \mu(h) >_h \mu'(h) \\ \mu'(h), & \text{otherwise} \end{cases}, \forall h \in H$$

♣

Theorem 7.4 (Join and Meet of Stable Matchings are Stable)

If μ and μ' are stable, then $\mu \vee_D \mu'$ and $\mu \wedge_D \mu'$ are stable.

♡

7.1.3 Strategic Incentives

- Type $=$ preference list.
- SCF: $f : \Theta \to \mathcal{M}$, where $\mathcal{M}$ is a set of stable matchings;
- Is f strategy-proof?
- Does there exist a stable and strategy-proof (direct) mechanism?

Definition 7.6

We say a mechanism φ is strategy-proof (SP) if $\varphi(\succ_i, \succ_{-i}) \geq \varphi(\succ_i', \succ_{-i})$ for all $i \in I$ and $\succ_i'$ and $\succ_{-i}$. ♣

Theorem 7.5 (Impossibility theorem (Roth))

There is no stable and strategy-proof (SP) mechanism.

♡

The mechanism that yields the D-optimal stable matching (in terms of the stated preferences) makes it a dominant strategy for each doctor to state his true preferences. (Similarly, the mechanism that yields the H-optimal stable matching makes it a dominant strategy for every hospital to state its true preferences.)

Theorem 7.6 (Dubins and Freedman; Roth)

The doctor(D)-optimal stable mechanism is strategy-proof for doctors.

Proof 7.4

7.2 Matching Many-to-Many

Contracts are denoted by $x \in X, x_D \in D, x_H \in H$. $F \triangleq D \cup H$.

Consider a set of contracts $Y \subseteq X$,

- Y_D = doctors listed in Y;
- Y_d = the contract in Y that list the doctor d;
- $\succ_d$ over set of contracts that name the doctor d;
- The set of contracts $f \in F$ chooses from Y: $C_f(Y) = \max_{\succ_f}\{Z \subseteq X : Z \subseteq Y_f\} \subseteq Y_f$;
- The set of contracts doctors choose from Y: $C_D(Y) = \cup_{d \in D} C_d(Y)$.
- The set of contracts doctors reject from Y: $R_D(Y) = Y \backslash C_D(Y)$.

The outcome of matching is $Y \subseteq X$.

Definition 7.7 (Stable Contracts)

$A \subseteq X$ is **stable** if

- Individually Rational (IR): for all $f \in F$: $C_f(A) = A_f$;
- Unblocked: $\nexists$ non-empty $Z \subseteq X$ such that $Z \cap A = \emptyset$ and for all $f \in F$, $Z_f \subseteq C_f(A \cup Z)$.

Example 7.1 Preferences over doctor d: $\{x, y\} > \{x\} > \emptyset > \{y\}$; Preferences over hospital h: $\{y\} > \{x, y\} > \{x\} > \emptyset$.

$\{x\} \Rightarrow \{x, y\} \Rightarrow \{y\} \Rightarrow \emptyset \Rightarrow \{x\}$.

Definition 7.8 (Substitutability Condition)

Preference of f satisfies the **substitutability condition** if for all $Y \subseteq X$ and $x, z \in X \backslash Y$:

$$z \notin C_f(Y \cup \{z\}) \Rightarrow z \notin C_f(Y \cup \{z\} \cup \{x\})$$

($Y' \subseteq Y \subseteq X \Rightarrow R_f(Y') \subseteq R_f(X)$, where R is the rejection choice.)

If z is rejected given a set, then it should also be rejected given a larger set.

Theorem 7.7

If contracts are substitutes, then $Y \subseteq X$ is stable <u>if and only if</u> pairwise stable.

Proof 7.5

Prove $\Leftarrow$: *(If not pairwise stable* $\Rightarrow$ *not stable)*

Suppose that Z *is a block. So,* $Z \subseteq C_f(A \cup Z)$ *for all* f *listed in* Z.

We can pick a $z \in Z$ *such that* $z \in C_f(A \cup Z)$. *By the substitutability condition,* $z \in C_f(A \cup \{z\})$. *So, it is stable.*

Theorem 7.8

If contracts are substitutes, then a stable outcome exists.

♡

Definition 7.9 (Lattice)

On a **lattice**, $L = (X, <, \wedge, \vee)$ (or we just use $L = (X, <)$), $<$ is a partial order on X in such a way that any two elements x and y of X have a unique greatest lower bound (glb) $x \wedge y$ (meet) and a unique lowest upper bound (lub) $x \vee y$ (join).

♣

Definition 7.10 (Complete Lattice)

A lattice $L = (X, <)$ is **complete** if there are both a meet (i.e. a greatest lower bound) and a join (i.e. a least upper bound) for any subset $Y \subseteq X$.

These generalized meet and join operations on Y are denoted by $\wedge Y$ and $\vee Y$.

♣

Definition 7.11 (Monotone Function over Lattice)

A function from one lattice to another lattice $f : (X, <) \to (X', <')$ is **monotone** if $x \leq y \Rightarrow f(x) \leq' f(y)$ for any $x, y \in X$.

♣

Theorem 7.9 (Tarski 1955)

Let $L = (X, <)$ *be a complete lattice and* $f : L \to L$ *be monotone* $(\leq)$ *function on* L. *Then, the set* $\{x \in L : f(x) = x\}$ *of fixed points is a non-empty, complete lattice with order* $\leq$.

♡

Proof 7.6

Fleiner, T. (2003). A fixed-point approach to stable matchings and some applications. Mathematics of Operations research, 28(1), 103-126.

If some contracts are not substitute, there are no stable outcomes exist.

7.3 Matching Many-to-One

<u>Settings</u>

- Doctors, D; Hospitals, H; Contracts $X = D \times H \times$ terms;
- Hospitals preference $\succ_h$ over 2^X;

- Doctors preference $\succ_d$ over X (compare one contract with another one contract, not compare over sets of contracts);
- Outcome is $Y \subseteq X$ s.t. $|Y_d| \leq 1$ for all $d \in D$ (a doctor signs at most one contract).

What restriction do we need to have a stable matching? Not as strong as substitute.

Corollary 7.1

Doctor-proposing DA algorithm produces a doctor-optimal stable matching. ♡

Example 7.2 The preferences of agents are

- $d_1 : h_1 \succ h_2$; $d_2 : h_1 \succ h_2$; $d_3 : h_2 \succ h_1$;
- $h_1 : d_3 \succ d_1, d_2 \succ d_1 \succ d_2$; $h_2 : d_1 \succ d_2 \succ d_3$.

There are two stable outcomes

1. (d_1, h_2), (d_3, h_1);
2. $(d_1, h_1), (d_2, h_1), (d_3, h_2)$.

Remark Lone-Wolf Theorem doesn't hold.

Assume the d_2's true preference is $h_2 \succ h_1$ and he reveals it, there is only one stable matching: (d_1, h_2), (d_3, h_1). So, the d_2 may benefit from lying.

Remark Strategy-proof doesn't hold.

Definition 7.12 (Law of Aggagate Demand/ Cardianlity Monotomicity (CM))

For h, $Y \subseteq Y' \subseteq X \Rightarrow |C_h(Y)| \leq |C_h(Y')|$ ♣

Theorem 7.10

Under substitutes and CM, doctor-proposing DA is strategy-proof and LWT holds. ♡

Theorem 7.11 (Rural Hosptial Theorem)

Under substitutes / CM, hospitals have same numbers of contracts in every stable outcome. ♡

Cadets-branch matching

Can be found in:

- Jagadeesan, R. (2019). Cadet-branch matching in a Kelso-Crawford economy. *American Economic Journal: Microeconomics*, 11(3), 191-224.

Remark Contracts are not substitutes.

Definition 7.13 (Unilateral Substitute)

Contracts are **unilateral substitutes** if for all $z, x \in X$ and $Y \subseteq X$ <u>such that $z_D \notin Y_D$</u> if $z \notin C_h(Y \cup \{z\}) \Rightarrow z \notin C_h(Y \cup \{z\} \cup \{x\})$ ♣

Remark Preferences of branches satisfying unilateral substitute.

Remark The outcome of doctor-proposing DA algorithm is doctor-optimal and stable.

7.4 Networks

Based on

- Fleiner, T., Jankó, Z., Tamura, A., & Teytelboym, A. (2015). Trading networks with bilateral contracts. arXiv preprint arXiv:1510.01210.
- Fleiner, T., Jankó, Z., Schlotter, I., & Teytelboym, A. (2023). Complexity of stability in trading networks. *International Journal of Game Theory*, 1-20.

Considering a trading network represented by a directed graph, where nodes are firms F and edges X are contracts (income arrow can be understood as buying products and outcome arrow can be understood as selling products).

The choice function of $f \in F$ is represented by C^f, the choice of f over $Y_f \subseteq X_f$ is $C^f(Y_f) \subseteq Y_f$, where X_f is the set of contracts involving f.

The choice sets of buyer side (B) and seller side (S) are defined as

$$C_B^f(Y|Z) \triangleq C^f(Y_f^B \cup Z_f^S) \cap X_f^B$$

$$C_S^f(Z|Y) \triangleq C^f(Z_f^S \cup Y_f^B) \cap X_f^S$$

where Y is the contracts from buyer side and Z is the contratcts from seller side.

Definition 7.14 (Irrelevance of Rejected Contracts)

Irrelevance of Rejected Contracts (IRC): $C(A) \subseteq B \subseteq A \Rightarrow C(A) = C(B)$ ♣

Definition 7.15 (Fully Substitute)

C^f is **fully substitute** if for $Y' \subseteq Y \subseteq X$ and $Z' \subseteq Z \subseteq X$,

$$R_B^f(Y'|Z) \subseteq R_B^f(Y|Z)$$

$$R_S^f(Z'|Y) \subseteq R_S^f(Z|Y)$$

and

$$R_B^f(Y|Z) \subseteq R_B^f(Y|Z')$$

$$R_S^f(Z|Y) \subseteq R_S^f(Z|Y')$$

♣

Define partial order, $(Y, Z) \geq (Y', Z')$ if $Y \subseteq Y'$ and $Z \supseteq Z'$.

Definition 7.16 (Stable Outcome, Hatfield and Kominers (2012))

An outcome $A \subseteq X$ is stable if it is

1. Individual Rational: $\forall f \in F, C^f(A_f) = A_f$;
2. Unblocked: there is no non-empty set $Z \subseteq X$ s.t. $Z \cap A = \emptyset$ and $\forall f \in F(Z), Z_f \subseteq C^f(A \cup Z)$, where $F(Z)$ is the set of the firms are lined to Z.

♣

Definition 7.17 (Trail)

Trail is the set of distinct edges $T = (X^1, X^2, ..., X^M)$ such that the buyer side (the firm who is the buyer in the edge) of X^i is exactly the seller side (the firm who is the seller in the edge) of X^{i+1}, which is denoted by $b(X^i) = s(X^{i+1}), i = 1, ..., M-1$.

♣

Definition 7.18 (Trail-stable Outcome)

An outcome $A \subseteq X$ is **trail-stable** if its is

1. Individual Rational;
2. There is no locally blocking trail $T = (X^1, X^2, ..., X^M)$ such that

   $X^1 \in C^{S(X^1)}(A \cup X^1)$;

   $\{X^i, X^{i+1}\} \in C^{b(X^i)}(A \cup X^i \cup X^{i+1})$;

   $X^M \in C^{b(X^M)}(A \cup X^M)$.

♣

Theorem 7.12 (Fleiner et al. 2016)

If C^f is fully substitute and IRC for all $f \in F$, then a trail-stable outcome exists.

♡

Proof 7.7

$Y \subseteq X$ and $Z \subseteq X$,

$$\Phi(Y, Z) = (X \backslash R_S(Z|Y), X \backslash R_B(Y|Z))$$

where $R_B(Y|Z) = \cup_{f \in F} R_B^f(Y|Z)$.

Claim 7.1

If (Y, Z) is a fixed point of Φ, then $A = Y \cap Z$ is trail-stable outcome.

♠

Lemma 7.1

C^f is fully substitute and IRC, and (Y, Z) such that $Y \cap Z = A$, $C_S(Z|Y) = A$, $C_B(Y|Z) = A$. Then, for a contract $x \in X \backslash A$ and $A \subseteq A' \subseteq X$ if $C_S^{S(x)}(A \cup x|A')$ then $x \in C_S^{S(x)}(Z \cup x|A')$.

♡

Φ will be monotone for the partial order $\geq$. As $(Y, Z) \geq (Y', Z')$, then $\Phi(Y, Z) \geq \Phi(Y', Z')$. Using Tarski fixed-point theorem, there is a (Y, Z) fixed point. .....

Read Fleiner, T., Jankó, Z., Tamura, A., & Teytelboym, A. (2015). Trading networks with bilateral contracts. arXiv preprint arXiv:1510.01210.

Proposition 7.1

A is trail-stable $\Rightarrow \exists\,(Y, Z)$ such that $Y \cap Z = A$ and (Y, Z) is a fixed point of Φ. ♠

7.5 Corporate Game Theory

There is a set of players $N = \{1, ..., n\}$. The subset of players $S \subseteq N$ is called coalition.

There is a value function about coalition $v : 2^N \to \mathbb{R}$, which assumes $v(N) \geq \max_{S\subseteq N} v(S)$.

Definition 7.19 (Cooperative Game)

A cooperative game is described by the pair $\langle N, v\rangle$. ♣

Definition 7.20 (Transferable Utility)

Utility is transferable if one player can losslessly transfer part of its utility to another player. ♣

Assume a TU (transferable utility) Economy. Consider a payoffs vector for all players, $x \in \mathbb{R}^n$. The efficiency requires $\sum_{i\in N} x_i = v(N)$. Individual Rational (IR) requires $x_i \geq v(\{i\})$.

7.5.1 Core of Corporate Game and Farkas' lemma

Definition 7.21 (Core)

The **core** is the set of feasible allocations where no coalition of agents can benefit by breaking away from the grand coalition.

$$C(v, N) = \left\{ x \in \mathbb{R}^n : \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S), \forall S \subseteq N \right\}$$

♣

Theorem 7.13 (Bondareva-Shapley Theorem)

?? *The core of $\langle N, v\rangle$ is non-empty ($C(v, N) \neq \emptyset$) <u>if and only if</u> for every function $\alpha : 2^N \backslash\{\emptyset\} \to [0, 1]$ where $\forall i \in N : \sum_{S:i\in S} \alpha(S) = 1$, the following condition holds:*

$$\sum_{S\in 2^N\backslash\{\emptyset\}} \alpha(S)v(S) \leq v(N)$$

♡

Consider $B(N)$ be the solutions to: $\begin{cases} \sum_{S:i\in S} y_S = 1, & \forall i \in N \\ y_S \geq 0, & \forall S \subseteq N \end{cases}$

Lemma 7.2 (Farkas' lemma)

*Let $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$. Then, **exactly one** of the following statement is true*

(1). There exists $x \in \mathbb{R}^n$ such that $Ax = b$ and $x \geq 0$

(2). There exists $y \in \mathbb{R}^n$ such that $A^T y \geq 0$ and $b^T y < 0$.

♡

Lemma 7.3

♡

Proof 7.8

Lemma 7.4 ((Alternative) Farkas' lemma)

Let A be $m \times n$ matrix, $b \in \mathbb{R}^m$ and $F = \{x \in \mathbb{R}^n : Ax \geq b, x \geq 0\}$. Then, either $Cx = d$ or $\exists z$ such that for $y_S \geq 0$, $C^T z - A^T y_S = 0$ and such that $d^T z - b^T y_S < 0$, but not both.

♡

By using this lemma, we can conclude $\begin{cases} v(N)z - \sum_S v(S)y_S < 0 \\ z - \sum y_S = 0 \\ y_S \geq 0 \end{cases}$ *must hold at the same time, (let $z = 1$, the last two lines are $B(N)$).*

Hence, $\forall y_S \in B(N)$, we have $v(N) \geq \sum_S v(S)y_S$.

7.5.2 Doubly stochastic matrix and Birkhoff-von Neumann Theorem

Consider a matching game between sellers and buyers: $v(\{i,j\}) = v_{ij}$, $v(\{i\}) = 0$ for buyer i and $v(\{j\}) \geq 0$ for seller j.

Core:

$$\begin{aligned} \max_{\alpha} \quad & \sum_i \sum_j v_{ij}\alpha_{ij} \\ \text{s.t.} \quad & \sum_i \alpha_{ij} = 1, \forall j \\ & \sum_j \alpha_{ij} = 1, \forall i \\ & \alpha_{ij} \geq 0 \end{aligned}$$

Definition 7.22 (Doubly Stochastic Matrix)

A **doubly stochastic matrix** is a square matrix $X = (x_{ij})$ of non-negative real numbers, each of whose rows and columns sums to 1.

The class of $n \times n$ doubly stochastic matrices is a convex polytope (convex set in euclidean space) known as the **Birkhoff polytope**.

♣

Theorem 7.14 (Birkhoff-von Neumann Theorem)

A matrix is doubly stochastic if and only if it is a convex combination of permutation matrices.

By this theorem, we can set efficient "integer" assignment.

Can the efficient allocation be competitive equilibrium (CE)?

Theorem 7.15

The core of assignment of game is non-empty.

Proof 7.9

The duality of core can be written as

$$\begin{aligned} \min \quad & \sum_j u_j^S + \sum_i u_i^B \\ \text{s.t.} \quad & u_j^S + u_i^B \geq v_{ij}, \forall i, j \end{aligned}$$

By strong duality, the minimum value should be equal to $V(N)$.

Hence, $\sum_{j \in T} u_j^S + \sum_{i \in T} u_i^B \geq V(T)$ for a subset $T \subseteq N$. That is, the core is non-empty.

Corollary 7.2

For an assignment game, outcome is in the core <u>if and only if</u> the outcome is CE outcome.

7.6 Constrained Demand Theory

7.6.1 Substitutes Valuation

There are buyers $i \in N$ and goods $j \in J$ with quantities $S \in \mathbb{Z}^J$ sold by a seller.

A buyer's utility is $v(x) - p \cdot x$, where $v(0) = 0$, $p \in \mathbb{R}^J$, and $x \in \{0,1\}^J$. The buyer's demand is represented by $D(p) \operatorname{argmax}_x \{v(x) - p \cdot x\}$.

The competitive equilibrium $\left(p^*, (x^{*i})_{i \in N}\right)$ here are

1. $x^{*i} \in D^i(p^*)$ for every $i \in N$ and
2. $\sum_i x^{*i} \leq S_i$, where the equality holds for $p_i > 0$.

Definition 7.23 (Substitutes Valuation)

A valuation v_i is a **substitutes valuation** if $\forall p : p' = p + \lambda e^j$ $(\lambda > 0)$, where $D^i(p) = \{x\}$ and $D^i(p') = \{x'\}$, we have that $x'_k \geq x_k$ for all $k \neq j$. (The increase of product j's price increases other product's demand).

Theorem 7.16 (Substitutes Valuation ⇒ Competitive Equilibrium Exists)

If agents have substitutes valuations, then a competitive equilibrium exists.

Theorem 7.17

If there exists an agent without substitutes valuation, then we can construct unit-demand preferences for other agents such that no competitive equilibrium exists.

7.6.2 Income Effect

There are buyers $i \in N$ and goods $j \in J$. The endowments (money and goods) of agents are denoted by $w = (w_0, w_I)$.

Outcome: The indivisible (bought) goods is represented by $x_I \in \{0,1\}^J$ and the (left) divisible money is represented by $x_0 \in (\underline{m}, \infty)$.

$$w_0 = x_0 + p_I \cdot x_I$$

must hold, where p_I is the vector of prices of goods.

Utility Function: An agent's utility function is defined by $u^i : (\underline{m}, \infty) \times \{0,1\}^J \to (-\infty, +\infty)$ with assumptions of strictly increasing in x_0, $\lim_{x_0 \to \underline{m}} u^i(x_0, x_I) = -\infty$, and $\lim_{x_0 \to \infty} u^i(x_0, x_I) = +\infty$.

Example 7.3 Examples of feasible utility functions:

1. $u^i(x) = v(x) - p \cdot x$ with $\underline{m} = -\infty$;
2. $u^i(x_0, x_I) = \log(x_0) - \log(-V_Q^i(x_I))$ with $V_Q^i : \{0,1\}^J \to (-\infty, 0)$.

Demand:

- $D^i_{\text{Marshallian}}(p, w) = \{x^* : x^* \in \arg\max_x u^i(x) \text{ s.t. } p \cdot x \le p \cdot w\}$
- $D^i_{\text{Hicksian}}(p, u) = \{x^* : x^* \in \arg\min_x p \cdot x \text{ s.t. } u^i(x) \ge u\}$ which is the dual of $D^i_{\text{Marshallian}}$.

Definition 7.24 (Competitive Equilibrium)

Given $(w^i)_{i \in I}$ s.t. $\sum_{i \in N} w_I^i = y_I$. A **competitive equilibrium** is a price vector $p_I^* \in \mathbb{R}^J$ and $x_I^{i\,*} \in D_{\text{Marshallian}}(p_I^*, w^i)$ for each $i \in N$ such that $\sum_{i \in N} x_I^{i\,*} = y_I$.

Based on the idea of duality, we can analyze problem based on the dual demand, Hicksian demand.

Definition 7.25 (Hicksian Valuation)

Hicksian valuation is defined by -1 times "the money that can lead to the utility u with goods x_I":

$$V^i_{\text{Hicksian}}(x_I, u) = -(u^i(\cdot, x_I))^{-1}(u)$$

Proposition 7.2 (Using Hicksian Valuation to Represent Hicksian Demand)

$D^i_{\text{Hicksian}}(p_I, u) = \arg\max_{x_I} \{v^i_{\text{Hicksian}}(x_I, u) - p_I \cdot x_I\}$

Proof 7.10

$D^i_{\text{Hicksian}}(p_I, u) = \arg\min_{x_I} \{(u^i(\cdot, x_I))^{-1}(u) + p_I \cdot x_I\} = \arg\max_{x_I} \{V^i_{\text{Hicksian}}(x_I, u) - p_I \cdot x_I\}$

Definition 7.26 (Hicksian Economy)

Hicksian economy: for a profile $(u^i)_{i\in N}$ is a transferable utility (TU) economy in which each agent's "valuation" is a Hicksian valuation V^i_{Hicksian}. ♣

Hicksian Economy works in finding Competitive Equilibrium

Theorem 7.18 (Equilibrium Existence Duality(EED))

Competitive Equilibrium exists for all feasible endowment profiles if and only if Competitive Equilibrium exists in the Hicksian economies for all profiles of utility levels. ♡

Marshallian	Hicksian
Housing Market	Assignment Game
Utility is not Quasi-linear	Utility is Quasi-linear
Unit Demand	Unit Demand
Existence in Housing Market	Existence in Assignment Game
$\times$	Lattice structure and Convexity of structure of CE prices
Net-substitutes	$\Rightarrow$ Substitutes

Like the Theorem 7.16, we want the Hicksian valuations be "substitutes".

Definition 7.27 (Net-Substitutes)

A agent's utility u^i is net-substitutes if $\forall u$, $D^i_H(p;u) = \{x\}$ and $D^i_H(p'_j, p_{-j}; u) = \{x'\}$, $p'_j > p_j \Rightarrow x'_k \geq x_k$ for all $k \neq j$. ♣

Theorem 7.19

Net-Substitutes Valuation $\Rightarrow$ competitive equilibrium exists. ♡

Proof 7.11

Net-substitutes $\Rightarrow$ substitutes holds in Hicksian economy. Hence, CE exists. By 7.18, CE exists in original economy.

Definition 7.28 (Gross-Substitutes)

A agent's utility u^i is gross-substitutes if $\forall w$, $D^i_M(p;w) = \{x\}$ and $D^i_M(p'_j, p_{-j}; w) = \{x'\}$, $p'_j > p_j \Rightarrow x'_k \geq x_k$ for all $k \neq j$. ♣

Example 7.4 In quasi housing market, we consider an example, of holding a house which price increases, the demand of another bad house doesn't change under Hicksian demand, which makes net-substitutes hold. But, the Marshallian demand decreases, which makes gross-substitutes don't hold.

Example 7.5 Net, but not gross:

Suppose there is a firm f thinking about workers s_1, s_2. f values worker at \$5 each, and the hiring budget is \$6;

- $p_1 = 2, p_2 = 4$;
- $p_1 = 3, p_2 = 4$

Obviously, the gross-substitutes (Marshallian Demand) leads to hiring both under $p_1 = 2, p_2 = 4$ and only hiring s_1 under $p_1 = 3, p_2 = 4$.

Let's consider the net-substitutes (Hicksian Demand): As the utility given under $p_1 = 2, p_2 = 4$ is \$10. We can find hiring two workers is still the optimal strategy.

Example 7.6 Net, but no auction:

Suppose there are two identical firms f_1, f_2 and workers s_1, s_1', s_2. The value of workers is \$5 each, but a firm want at most one of s_1, s_1' and has hiring budget \$6. A worker has reservation wage of \$1.

Equilibrium: \$1 for worker s_1, s_1' and \$5 for s_2; One firm hires one of s_1, s_1' and the other hires s_2.

7.7 Object Allocation

Exchange: $i \in N$ agent; Agents have strict preference $\succ_i$ over objects. (We use $\succ$ denote $\{\succ_i\}_{i\in N}$).

Two settings:

1. Exchange: an agent shows up with exactly one object.
2. Allocation: One planner owns N objects; agents have $\emptyset$.

A **mechanism** $\Phi(\succ)$ gives a outcome μ. We want the final outcome μ be

1. Individual Rationality (IR): for all $i \in N$, $\mu_i \succeq i$ (Exchange) and $\mu_i \succeq \emptyset$ (Allocation).
2. Pareto Efficient (PE): $\nexists \mu'$ such that $\mu_i' \succeq \mu_i$ for all $i \in N$, strict for at least one.
3. Strategy-Proof (SP): Φ induces a game. We want that, in this game, truth-telling is a weakly dominant strategy for all agent $i \in N$.

7.7.1 Allocation

(Random) Serial Dictatorship: Randomly order the agents, ask one by one, and allocate a remaining object. $\Rightarrow$ it satisfies IR, PE, SP, but unfair(?).

7.7.2 Exchange

Definition 7.29 (Core)

The **core** is the set of all allocations μ such that there is no $S \subseteq N$ and μ' for which:

- for $i \in S$, $\mu'_i = j$ for some $j \in S$
- $\mu'_i \succeq \mu_i$ for all $i \in S$, at least one strict.

Core: IR+PE. ♣

Theorem 7.20 (Core is a Singleton)

There is a unique element in the core. ♡

Proof 7.12

Run the algorithm: Top Trading Cycles (TTC).

Definition 7.30 (Top Trading Cycles (TTC))

Agent = node.

1. Step 1: every agent point at her favorite object/agent.

   (1A): Find cycles.

   (1B): Allocate object to agent who is pointing at it in cycle.

   (1C): Remove the cycle.

2. Step 2: every (remaining) agent point at her favorite object/agent.

   (2A): Find cycles.

   (2B): Allocate object to agent who is pointing at it in cycle.

   (2C): Remove the cycle.

3. Repeat $\cdots$

♣

Proposition 7.3

TTC produces an allocation that satisfies IR, PE, SP. ♠

Theorem 7.21 (TTC ⇔ IR, PE, SP (Ma, 1999))

There is at most 1 *IR, PE, SP mechanism (TTC).* ♡

Proof 7.13

Definition 7.31

The **size** of a preference profile $\succ$ is the total number of objects agents find acceptable in $\succ$:

$$S(\succ) = \sum_{i \in N} \#\text{acceptable objects in } \succ_i$$

♣

Consider two Φ and Ψ that disagree for some $\succ$, the $\succ$ is defined to be bad.

We define the minimal bad profile as a bad profile of minimal size. Consider the two outcomes given by these mechanisms:

$\Phi(\succ)$	same	$A(\Phi)$
$\Psi(\succ)$	same	$A(\Psi)$

the sum of different parts are $A \triangleq A(\Phi) + A(\Psi)$.

Lemma 7.5

If Φ and Ψ are SP, and $\succ$ is a minimal bad profile, then each agent in A has exactly two acceptable objects.

Proof 7.14

Suppose there exists $i \in A$ such that she has > 2 acceptable objects.

Without losing generality, we consider $\Phi_i(\succ) \succ_i \Psi_i(\succ)$.

Remove all objects from his preference list except $\Phi_i(\succ)$ and endowment of i (call it $\{i\}$). The new preference profile is denoted by $\succ_i'$.

Since Φ is SP, $\Phi_i(\succ') = \Phi_i(\succ)$; since Ψ is SP, $\Psi_i(\succ') \prec_i \Phi_i(\succ)$.

So, we have $\succ'$ is a bad profile and $S(\succ') < S(\succ)$, a contradiction.

7.8 School Choice

Model:

1. There is a set of school S; a school is denoted by $s \in S$; Quota for each s is q_s;
2. I is the set of all students; A student is denoted by $i \in I$; Student i has preference $\succ_i$.
3. School places = objects.
4. Each school has a priority order over students π_s.
5. Matching $\mu : I \to S$ such that $\forall s \in S : \#\mu^{-1}(s) \leq q_s$.
6. Matching violates priority if $\exists s \in S$ such that

   (i). $s \succ_i \mu(i)$ and

   (ii). either "Wastefulness: $\#\mu^{-1}(s) < q_s$" or "Justified Envy: $i\pi_s j$ for some $j \in \mu^{-1}(s)$"

   $\approx$ existence of a blocking pair.

   A matching is **stable** if there are no priority violates.

   (As we don't consider the preference of j in (ii), it is not true stable $\Rightarrow$ (Pareto) efficient.)

Example 7.7 Boston (Immediate Acceptance)

(1). Step 1: students apply for favorite schools; school accepts applicants up to capacity and reject rest permanently.

(2). Step k: students apply for favorite schools among those with capacity and hasn't already rejected them; schools accept applicants up to capacity q_s and reject rest permanently.

Proposition 7.4

DA gives a matching that satisfies stability and SP (not PE).

Run TTC:

Definition 7.32 (Top Trading Cycles (TTC))

Schools and Students (agents) = nodes.

1. Step 1: every agent point at her favorite object/agent.

   (1A): Find cycles.

   (1B): Allocate object (school) to agent (student) who is pointing at it in cycle. (Usually based on the students' preference.)

   (1C): Remove the cycle.

2. Step 2: every (remaining) agent point at her favorite object/agent.

   (2A): Find cycles.

   (2B): Allocate object to agent who is pointing at it in cycle.

   (2C): Remove the cycle.

3. Repeat $\cdots$

Proposition 7.5

TTC produces an allocation that satisfies PE and SP (not stable).

Hence, we need to make a trade-off between priority violation and efficiency.

Theorem 7.22 (Keslen)

For all $S, \{q_s\}_{s\in S}$, there exists $I, \succ_i, \{\pi_s\}_{s\in S}$ s.t. in the SOSM, every student gets either their last choice or second-last choice.

Theorem 7.23 (Abdulkadiroğlu, Pathak, Roth, AER)

There is no (PE+)SP mechanism that Pareto-dominates SOSM.

Theorem 7.24

There is no PE+SP mechanism that selects a PE+stable matching whenever it exists.

Definition 7.33 (Kesten/Tang+Yu Algorithm)

Suppose the number of student is not larger than the total capacity $\#I \leq \sum_s q_s$.

(i). Step 0: Run DA, set SOSM μ_0. Find under-demanded schools = a school that doesn't reject any students.

Assign $\mu^{-1}(s)$ permanently. Call these schools/students "settled". Remove all settled schools and students.

(ii). Step k: Rerun DA on everyone unsettled.

♣

Definition 7.34 (Priority-Neutral(PN), Reny 2022)

μ is **priority-neutral**(PN) iff $\exists$ no matching u that can make any student whose priority is violated at μ better off unless u violates the priority of some student and make them worse off.

We call μ is **priority-efficient** if it is PN and PE.

♣

Theorem 7.25 (Reny 2022)

1. *$\exists$ a unique Priority-efficient matching;*
2. *Priority efficient $\Leftrightarrow$ SO priority neutral matching;*
3. *It can be found by the CUTE Algorithm;*
4. *μ is priority efficient $\Leftrightarrow$ no matching u can make any student better off unless u unless u violates the priority of some student and make them worse off.*

♡

7.9 School Choice with Reserves

Consider a school choice model, students can be divided into majority (M) and minority (m), $I = I^M \cup I^m$. Quotas of schools are represented by $q_s = (q, q^M), s \in S$, where q^M is the quota for majority.

Definition 7.35 (Stability)

A matching is stable if, for all $s \in S$ such that $s \succ_i \mu(i)$,

1. Either: "No Wastefulness: $|\mu^{-1}(s)| = q_s$" and "No Justified Envy: $i' \pi_s i$ for all $i' \in \mu^{-1}(s)$"
2. Or: $i \in I^M$, "$|\mu^{-1}(s) \cap I^M| = q_s^M$" and "$i' \pi_s i$ for all $i' \in \mu^{-1}(s) \cap I^M$"

♣

Definition 7.36 (Stronger Quota)

A $\tilde{setting}$ (with $\tilde{q}_s$) has **stronger quota** than setting (with q_s) if $\tilde{q}_s = q_s$ but $q_s^M \geq \tilde{q}_s^M$.

♣

Definition 7.37 (Good Mechanism)

Mechanism Φ is **good**, if whenever a $\tilde{setting}$ has stronger quotas than its setting, it doesn't make all minority students worse off.

♣

Theorem 7.26 (Kojima 2012)
There is no stable good mechanism. ♡

7.9.1 Minority Reserves (slot-specific priority)

Suppose r_s^m is reserved for minority only. That is $q_s = q_s^M + r_s^m$.

Definition 7.38 (Minority Reserves)
School has minority reserve r_s^m whenever # of admitted minority students is less than r_s^m, then any minority students is admitted ahead of majority students. ♣

Definition 7.39 (No Blocking Pair)
No blocking pair if $s \succ_i \mu(i)$, then $|\mu(s)| = q_s$ and,

1. Either: $i \in I^m$ and "$i' \pi_s i$ for all $i' \in \mu^{-1}(s)$"
2. Or: $i \in I^M$, "$|\mu^{-1}(s) \cap I^m| > r_s^m$" and "$i' \pi_s i$ for all $i' \in \mu^{-1}(s)$"
3. Or: $i \in I^M$, "$|\mu^{-1}(s) \cap I^m| \leq r_s^m$" and "$i' \pi_s i$ for all $i' \in \mu^{-1}(s) \cap I^M$" ♣

Theorem 7.27 (Smart Reserves)
Suppose μ is a stable matching without affirmative action. Let r_s^m be such that

$$r_s^m \geq |\mu^{-1}(s) \cap I^m|, \forall s \in S$$

Then, either μ is stable w.r.t. r^m or $\exists$ stable matching under r^m that Pareto-dominates μ for I^m. ♡

7.10 Random Assignment

Suppose there are agents $i \in I$ and objects $j \in J$, where $|I| = |J|$. Agents have preferences $\succ_i$ over objects, and objects have priorities $\triangleright_j$ over agents.

An allocation is represented by a matrix that each row and each column has sum to 1 probability.

There are two mechanism can be used:

(i). RSD (Random: draw a priority order $\triangleright$ uniformly.)

(ii). TTC with uniform random endowment.

Theorem 7.28
These two mechanisms are equivalent (bijection). ♡

RSD is not Pareto-efficiently.

Proposition 7.6

For a row of an allocation matrix ($\tilde{\mu}$) for agent i, $\tilde{\mu}_i \succ_i \tilde{\mu}'_i$

- *if and only if $\tilde{\mu}_i \succ_{FOSD} \tilde{\mu}'_i$ (first-order stochastic dominance).*
- *if and only if $\mathbb{E}U(\tilde{\mu}_i) \geq \mathbb{E}U(\tilde{\mu}'_i)$ under expected utility.*

Definition 7.40

$\tilde{\mu}$ is **ordinally efficient (sd-efficient)** if there is no $\tilde{\mu}'$ which is $\succ_{FOSD}$ by all agents. (*ex-ante efficient* with respect to cardinal utility)

$\tilde{\mu}$ is **ex-post efficient** if those are only Pareto efficient outcome in the support.

Definition 7.41

$\tilde{\mu}$ is **ordinally envy-free** if $\tilde{\mu}_i \succ_{FOSD} \tilde{\mu}_j, \forall i, j$.

RSD is not envy-free.

There exists ordinally efficient and envy-free mechanism.

Definition 7.42 (Probabilitistic Serial Algorithm)

Based on the preference of agents:

1. Give each agent his most preferred object with the same proportion such that the sum of each object is at most 1.
2. Repeat by using remaining objects.

Example 7.8 Preference: A: $Obj1 \succ Obj3 \succ Obj2$; B: $Obj1 \succ Obj2 \succ Obj3$; C: $Obj2 \succ Obj3 \succ Obj1$

$t = \frac{1}{2}$ A: $\frac{1}{2}Obj1$; B: $\frac{1}{2}Obj1$; C: $\frac{1}{2}Obj2$.

$t = \frac{3}{4}$ A:$\frac{1}{2}Obj1 + \frac{1}{4}Obj3$; B: $\frac{1}{2}Obj1 + \frac{1}{4}Obj2$; C: $\frac{3}{4}Obj2$.

$t = 1$:$\frac{1}{2}Obj1 + \frac{1}{2}Obj3$; B: $\frac{1}{2}Obj1 + \frac{1}{4}Obj2 + \frac{1}{4}Obj3$; C: $\frac{3}{4}Obj2 + \frac{1}{4}Obj3$.

Theorem 7.29 (Welfare Theorem)

Probabilistic Serial Algorithm gives ordinally efficient and envy-free outcome.

Definition 7.43 (Equal Treatment of Equals (ETE))

Equal Treatment of Equals: if same preference $\succ_i \Rightarrow$ the same bundle $\tilde{\mu}_i$.

Proposition 7.7

For $n = 3$, RSD is ordinally efficient, ETE, Strategy-Proof. (These three properties are incompatible when $n > 3$).

7.11 Random Assignment in School Choice

Example 7.9

- Preference of Agents: $A : s_2 \succ s_3 \succ s_1$; $B : s_2 \succ s_3 \succ s_1$; $C : s_1 \succ s_2 \succ s_3$.
- Priority of Schools: $s_1 : A \succ B \succ C$, $s_2 : C \succ (A, B)$, $s_3 : C \succ B \succ A$.

There are two stable outcomes: $\mu : A - s_2, B - s_3, C - s_1$; $\mu' : A - s_3, B - s_2, C - s_1$.

It can't be strategy proof. In μ, B can lie: $s_2 \succ s_1 \succ s_3$, to make the outcome become μ'. In μ', A can lie: $s_2 \succ s_1 \succ s_3$, to make the outcome become μ.

Definition 7.44 (Stable Imporovement Cycle (S.I.C.))

Each student points at schools they prefer and where he doesn't have a lower priority among those students who prefer students to their assignment. ♣

Theorem 7.30

If a stable matching is not in the student-optimal stable set, then $\exists$ a S.I.C. ♡

Example 7.10

- Preference of Agents: $A : s_2 \succ s_1 \succ s_3$; $B : s_3 \succ s_2 \succ s_1$; $C : s_2 \succ s_3 \succ s_1$.
- Priority of Schools: $s_1 : A \succ (B, C)$, $s_2 : B \succ (A, C)$, $s_3 : C \succ (A, B)$.

DA: $A : s_1, B : s_2, C : s_3$. Another allocation: $A : s_1, B : s_3, C : s_2$.

Consider DA, A wants s_2: C also wants s_2, which has the same priority as A, so A can point at s_2. B points at s_3. C can also point at s_2. So, there is a S.I.C.

7.12 Pseduomarket (FF)

Consider an example that agent A_1 wants a, b for 0.9, A_2 wants a, c for 0.9, A_3 wants b, c for 2. Suppose the budget for each agent is 1.

Reminds that utility is only meaningful for the agent itself. Here, as the budget is the same, the demand of each agent is the same.

7.12.1 Problem of Implementability

An equilibrium (but can't be implemented): A_1 gets $\{\frac{1}{2} : \emptyset; \frac{1}{2} : a + b\}$; A_2 gets $\{\frac{1}{2} : \emptyset; \frac{1}{2} : a + c\}$; A_3 gets $\{\frac{1}{2} : \emptyset; \frac{1}{2} : b + c\}$.

Transfer Utility Economy	Pseduomarket
Allocation $x_j \in X_j, j = 1, ..., J$	Lottery $\tilde{x}_j \in \mathcal{L}(X_j)$
Price $p \in \mathbb{R}^I$	Budget b_j and Price $p \in \mathbb{R}^I$
$u_j(x) = v_j(x) - p \cdot x$	$V_j(\tilde{x}_j) = \sum_x v_j(x)\mathbb{P}(\tilde{x}_j = x)$
Demand $D_j(p) = \arg\max_x u_j(x)$	$\tilde{D}_j(p) = \arg\max_{\tilde{x}: p\cdot\tilde{x} \leq b_j} V_j(\tilde{x})$
CE: $(p^*, x^*) : x_j^* \in D_j(p^*), \sum_j x_j^* \leq S$	RE: $(p^*, \tilde{x}^*) : \tilde{x}_j^* \in \tilde{D}_j(p^*), \sum_j \tilde{x}_j^* \leq S$
(equality holds for no zero p^*)	

S is supply, which equals to $\sum_i \omega_i$ if the economy with endowments.

We want an allocation being implementable that an allocation (a set of lotteries over agents) $\{w_1, ..., w_J\} = \mathcal{W} \in \mathcal{L}(\prod_j X_j)$ (feasible bundles for each agent).

Define $\bar{w}_j = \mathbb{E}[w_j]$ and $\bar{\mathcal{W}} = \mathbb{E}[\mathcal{W}]$

Definition 7.45 (Implementable)

A random equilibrium $(p^*, \tilde{x}^*)$ is **implementable** if there exists $\mathcal{W}$ over feasible allocations such that $w_j \in D_j(p^*)$ and $\bar{x}_j^* = \bar{w}_j, \forall j = 1, ..., J$.

can be implemented by a distribution of allocations. (BvN)

Proposition 7.8

Random equilibrium always exists.

Definition 7.46 (Rich)

A set of valuations $\mathcal{V}^j = \{v_j(x) : x \in X_j\}$ is **rich** if whenever $v_j(x) \in \mathcal{V}^j$ then $v_j(x) + a \cdot x \in \mathcal{V}^j$ for all $a \in \mathbb{R}^I$. That is $\exists x'$ such that $v_j(x') = v_j(x) + a \cdot x$.

Complement may induce unimplementable problem.

Suppose value functions live in V and are rich.

Theorem 7.31

CE exists for all valuations in V $\Leftrightarrow$ RE is implementable for all budgets profiles and all valuations in V.

Chapter 8 Auction

Based on

- Klemperer, P. (1998). Auctions with almost common values: The Wallet Game'and its applications. *European Economic Review*, 42(3-5), 757-769.

8.1 Examples

8.1.1 Auctions with Common-value

(1). Financial assets;

(2). Oilfields;

(3). A takeover target has a common value if the bidders are financial acquirers (e.g. LBO firms) who will follow similar management strategies if successful;

(4). The Personal Communications Spectrum (PCS) licenses sold by the U.S. Government in the 1995 "Airwaves Auction".

8.2 Revenue Equivalence Theorem

Consider the Optimal Auctions in an Independent Private Values Setting. There is one object and N bidders.

1. Bidders are risk-neutral;
2. Bidders have private valuations;
3. each bidder i's valuation independently drawn from a strictly increasing c.d.f. $F_i(v)$ (with p.d.f. $f_i(v), v \in \mathcal{X}_i$) that is continuous and bounded below;
4. Seller knows each F_i (use F and f to represent all distributions) and have no value for the object.

Definition 8.1 (General Auction)

A general auction mechanism: bidders have values x and **strategies** $\beta : \mathcal{X} \triangleq \prod_i^N \mathcal{X}_i \to \mathcal{B}$ generate message (bids) based on their values, then there is an **allocation rule** based on bids $\pi : \mathcal{B} \to \Delta N$ generates a distribution over all bidders and a **payment rule** $\mu : \mathcal{B} \to \mathbb{R}^N$ generates payment for all bidders. ♣

Definition 8.2 (Direct Mechansim)

Consider a situation that bidders follow *revelation principle* that provide their true values. Then the outcome can directly base on the true values.

Then a **direct mechanism** can be represented as (Q, T), where $Q : \mathcal{X} \to \Delta N$ is the allocation rule and $T : \mathcal{X} \to \mathbb{R}^N$ is the payment rule, such that

$$Q(x) = \pi(\beta(x)), \quad T(x) = \mu(\beta(x))$$

♣

Proposition 8.1 (Revelation Principle)

Take any equilibrium of any auction mechanism $(\mathcal{B}, \pi, \mu)$. There is a distinct direct mechanism (Q, T) that produces the same outcome.

♠

Consider a direct mechanism (Q, T), an agent i reports v_i while others report their values.

$$\textbf{Expected allocation: } q_i(z_i) = \int_{\mathcal{X}_{-i}} Q_i(z_i, x_{-i}) dF_{-i}(x_{-i})$$

$$\textbf{Expected payment: } t_i(z_i) = \int_{\mathcal{X}_{-i}} T_i(z_i, x_{-i}) dF_{-i}(x_{-i})$$

where Q_i, T_i are i^{th} item of Q, T.

The bidder wants to maximize

$$q_i(z_i)x_i - t_i(z_i)$$

Define the maximum value is

$$u_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - t_i(z_i)\}$$

Assumption *The condition for direct mechanism being incentive competitive (IC) is:*

$$u_i(x_i) \equiv q_i(x_i)x_i - t_i(x_i) \geq q_i(z_i)x_i - t_i(z_i), \forall x_i, z_i \in \mathcal{X}_i \tag{Ass 1}$$

Firstly, we can compute, for any $z_i \in \mathcal{X}_i$

$$\begin{aligned} &q_i(x_i)z_i - t_i(x_i) \\ =&q_i(x_i)x_i - t_i(x_i) + q_i(x_i)(z_i - x_i) \\ =&u_i(x_i) + q_i(x_i)(z_i - x_i) \end{aligned}$$

Based on the assumption Ass 1, we have

$$u_i(z_i) \geq q_i(x_i)z_i - t_i(x_i) = u_i(x_i) + q_i(x_i)(z_i - x_i)$$

which shows that $u_i(\cdot)$ is convex.

If u_i is differentiable, $u_i'(x_i) = q_i(x_i)$. Then, we can write the **Envelope theorem/condition**:

$$u_i(x_i) = u_i(0) + \int_0^{x_i} q_i(y_i) dy_i$$

which only depends on the allocation rule.

Theorem 8.1 (Revenue Equivalence Theorem)

If the direct mechanism (Q, T) is incentive competitive (IC), then for all i, x, the ***expected payment*** *is*

$$t_i(x_i) = \underbrace{t_i(0)}_{e.g.=0} + q_i(x_i)x_i - \int_0^{x_i} q_i(y_i)dy_i$$

Proof 8.1

$$u_i(x_i) = q_i(x_i)x_i - t_i(x_i) = u_i(0) + \int_0^{x_i} q_i(y_i)dy_i$$
$$\Rightarrow t_i(x_i) = q_i(x_i)x_i - u_i(0) - \int_0^{x_i} q_i(y_i)dy_i$$

Set $u_i(0) = -t_i(0)$, that is, if i's value is zero, he pays zero.

Corollary 8.1 (Standard Revenue Equivalence Theorem)

Suppose that values are i.i.d. and bidders are risk-neutral.

Consider any auction and its symmetric and increasing equilibrium, in which the expected payment of bidders have 0 value is 0. Then the expected revenue to the seller is the same.

Proof 8.2

If equilibrium, is symmetric and increasing, then object is always allocated to the bidder with the highest value. Set $t_i(0) = 0$.

Standard Revenue Equivalence Theorem is based on symmetric, independent, and private (uncorrelated) values.

8.3 Optimal Auctions

Goal: Find the **optimal auction** that maximizes the seller's expected revenue subject to individual rationality (IR) and Bayesian incentive compatibility for the buyers.

Definition 8.3 (Virtual Valuation)

Bidder i's **virtual valuation** is $\phi_i(v_i) = v_i - \frac{1-F_i(v_i)}{f(v_i)}$

Definition 8.4 (Bidder-Specific Reserve Price)

Bidder i's bidder-specific reserve price r_i^* is the value for which $\phi_i(r_i^*) = 0$.

Theorem 8.2 (Myerson (1981))

The optimal (single-good) auction in terms of a direct mechanism: The good is sold to the agent $i = \arg\max_i \phi_i(\hat{v}_i)$, as long as $v_i \geq r_i^*$. If the good is sold, the winning agent i is charged the smallest *valuation that he could have declared while still remaining the winner:*

$$\inf\{v_i^* : \phi_i(v_i^*) \geq 0 \text{ and } \forall j \neq i, \phi_i(v_i^*) \geq \phi_j(\hat{v}_j)\}$$

♡