2021 Final Exam for Analysis

約束タワー

Question.1 Suppose μ, ν are σ-finite measures.

(1) State the definition of $\mu \ll \nu, \mu \perp \nu$ and Radon-Nikodym derivative $\dfrac{\mathrm{d}\mu}{\mathrm{d}\nu}$.

(2) Prove that $\mu \perp \nu$ if and only if $\dfrac{\mathrm{d}\nu}{\mathrm{d}(\nu+\mu)} \cdot \dfrac{\mathrm{d}\mu}{\mathrm{d}(\mu+\nu)} = 0$.

(3) Denote $\mu \simeq \nu$ if $\mu \ll \nu, \nu \ll \mu$. Prove that $\mu \simeq \nu$ is equivalent to $\dfrac{\mathrm{d}\nu}{\mathrm{d}(\nu+\mu)}, \dfrac{\mathrm{d}\mu}{\mathrm{d}(\mu+\nu)} > 0$.

Question.2 For measurable set E and $x_0 \in E$, define

$$\rho(x_0, E) = \lim_{r \searrow 0} \frac{\mathcal{L}(E \cap (x_0 - r, x_0 + r))}{2r}$$

if the limit exists, we say that E has **point density** at x_0 and call $\rho(x_0, E)$ the **point density** of x_0 in E.

(1) State the Riesz representation theorem for Radon measure.

(2) Does a has point density in $[a, b]$? If so, compute its value.

(3) For any $\lambda \in (0, 1)$, construct E such that $\rho(x_0, E) = \lambda$.

Question.3

(1) Show that $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\mathrm{d}x}{(1+t/n)^n\, t^{1/n}} = 1$ and $\displaystyle\lim_{n\to\infty} \int_0^\infty \frac{\log^p(x+n)}{n} \,\mathrm{e}^{-x} \cos x \,\mathrm{d}x = 0$.

(2) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a Lebesgue measurable function and is integrable on any open interval $(a, b) \subseteq \mathbb{R}$, i.e. $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. If $g \in C_c^n(\mathbb{R})$, prove that $h(y) := \displaystyle\int_{\mathbb{R}} f(x+y) g(x) \,\mathrm{d}x$ is well-defined in $C^n(\mathbb{R})$.

(3) Suppose f is integrable on $[a, b] \subset \bar{\mathbb{R}}$. Show that for any $\epsilon > 0$, there exists $\varphi \in C^0[a, b]$ such that $\displaystyle\int_a^b |f - \varphi| \,\mathrm{d}x < \epsilon$.

Question.4 $f : \mathbb{R} \to \mathbb{R}$ is Lebesgue measurable.

(1) Define the Lebesgue σ-algebra on $\mathbb{R}$ and state the definition of Lebesgue measurable function.

(2) If f satisfies $f(x+y) = f(x) + f(y), \forall x, y \in \mathbb{R}$, show that there exists a real number c such that $f(z) = cz, \forall z \in \mathbb{R}$.

Question.5 X is a Banach space.

(1) Z is a closed subspace of X, show that Z is Banach.

(2) We say Y is a **hyperplane** of X if $\operatorname{codim} Y = \dim X/Y = 1$. Prove Y is a closed hyperplane if and only if there exists a non-zero $f \in X^*$ such that $Y = f^{-1}(0)$.

Question.6

(1) State the definition of the spectrum of a linear operator.

(2) Define $S : \ell^1 \to \ell^1$ as $S(\alpha_1, \alpha_2, \cdots) = (\alpha_2, \alpha_3, \cdots)$, show that the spectrum of S is the closed unit ball $\bar{\mathbb{D}}$ on $\mathbb{C}$, i.e. $\operatorname{Spec}(S) = \bar{\mathbb{D}} := \{z : |z| \le 1\}$.

(3) Suppose $T((\alpha_j)) = (\alpha_j - 2\alpha_{j+1} + \alpha_{j+2}) : \ell^1 \to \ell^1$, show that the spectrum of T is the heart curve on $\mathbb{R}^2$: $\{(r, \theta) : r \le 2 + 2\cos\theta, 0 \le \theta < 2\pi\}$.

Question.7 $f \in L^\infty(\mathbb{R})$, define $T_n(f) = \dfrac{1}{2n}\displaystyle\int_{-n}^{n} f(x)\,\mathrm{d}x$.

(1) State the definition of operator norm and compute $||T_n||$.

(2) Show that any operator T with form

$$T(f) = \int_{\mathbb{R}} h(x) f(x)\,\mathrm{d}x$$

where $h(x) \in L^1(\mathbb{R})$ is not the weak*-limit of T_n. Specially, $[L^\infty(\mathbb{R})]^* \ncong L^1(\mathbb{R})$.

Question.8 Denote $X = \ell^\infty$, $Y = \{x = (x_1, x_2, \cdots) \in X : f = \lim\limits_{n\to\infty} \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} x_i \text{ exists.}\}$ is the subspace of sequences whose Cesáro sums convergent.

(1) State any version of Hahn-Banach theorem.

(2) Show that there exists a function $F \in X^*$ s.t. $||F|| = 1$ and $F|_Y = f$.

(3) Prove there's no $T \in \mathcal{B}(\ell^2, \ell^1)$ s.t. T is surjective.

Question.9 X is a Banach space.

(1) X is separable if and only if the unit sphere $\mathbb{S}_X = \{x \in X : ||x|| = 1\}$ of X is separable.

(2) Suppose $Y, X/Y$ are separable, prove X is separable.

(3) Y is a subspace of X, X is separable, prove that there exists a sequences of subspaces $Y = L_0 \le L_1 \le L_2 \le \cdots$ such that $L_\infty := \cup_{i=0}^{\infty} L_i$ is a dense subspace of X.

Question.10 $\mathcal{H}$ is a Hilbert space.

(1) State and prove Bessel inequality or Parseval indentity. State the Parallelogram law.
(2) Show that $\mathcal{H}$ is uniformly convex.
(3) Suppose $\mathcal{H}$ is a infinity dimensional separable space, show that $\mathcal{H}$ is isometric isomorphic to ℓ^2, i.e. $\mathcal{H} \underset{\text{isometr.}}{\xrightarrow{\sim}} \ell^2$.

Question.11
(1) State open mapping theorem and closed graph theorem.
(2) X, Y is Banach, $T \in \mathcal{B}(X, Y)$. Suppose $\operatorname{ran} T$ is a subspace of Y with infinity codimension. Prove $\operatorname{ran} T$ is closed.
(3) Conversely, show that every infinity dimensional Banach space has a finite codimensional subspace which is not closed.

Here's some questions from Final Exam-B.

Question.1
(1) State open mapping theorem and closed graph theorem.
(2) $T \in \mathcal{B}(X, Y)$. Prove following statements are equivalent.
 i) T is open;
 ii) T is open at 0;
 iii) Denote $y = Tx$, then there exists $M > 0$ s.t. $||x|| \leq M||y||$.
(3) If $B_Y \subseteq T(B_X) \subseteq \overline{B_Y}$, then T is surjective.

Question.2 $\mathcal{H}$ is a Hilbert space.
(1) Suppose Y is closed, show that $\mathcal{H} = Y \oplus Y^\perp$.
(2) Give an example to show that $\mathcal{H}$ can not be written as the direct sum of a subspace Y and its orthogonal complement.
(3) Suppose $|| \cdot ||$ is a norm which satisfies the parallelogram law, show that bilinear form $\langle \, , \, \rangle$ given by polarisation indentity

$$\langle x, y \rangle = \frac{1}{4} \left(||x + y||^2 - ||x - y||^2 + \mathrm{i} \, ||x + \mathrm{i}y||^2 - \mathrm{i} \, ||x - \mathrm{i}y||^2 \right)$$

is an inner product on $\mathcal{H}$.

Question.3

The same as **Question.1** in **A**.

Question.4

The same as **Question.2** in **A**.

Question.5

(1) State the definition of weak topology on X.

(2) If $F \subseteq X$ is a closed convex set (in norm topology), show that F is weak closed.

(3) Suppose x_n is weakly convergent, prove that there exists a convex combination y_n is strongly convergent.

(4) Prove X is reflexive if and only if X^* is reflexive.

Question.6

(1) and (3) are the same as **Question.3** in **A**.

(2) Prove $\displaystyle\int_\Omega \phi(f(x))\,\mu(\mathrm{d}x) = \int_0^\infty \mu(\{x \in \Omega : f(x) \geq t\})\,\nu(\mathrm{d}t)$ where $\phi(t) = \nu([0,t])$ μ, ν are measures.

Question.7

The same as **Question.6** in **A**.

Question.8

The same as **Question.7** in **A**.

Question.9 X is a Banach space.

(1) X is separable if and only if the unit sphere $\mathbb{S}_X = \{x \in X : ||x|| = 1\}$ of X is separable.

(2) Show that $C^0(0,1)$ under max-value norm is not separable.

(3) t.b.a.

Question.10

(1) Prove F.Riesz lemma: Let X be NVS. If Y is a proper closed subspace of X, then $\forall \epsilon > 0$, $\exists\, x \in \mathbb{S}_X$ s.t. $\mathrm{dist}(x, Y) \geq 1 - \epsilon$.

(2) Let X be infinity dimensional NVS, show that any open set of X contains uncountable disjoint balls with same radius.

(3) Use (2) to prove that there's no translation invariant measure on any infinity dimensional NVS.