Analysis of European Put Option as a Risk Measure

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1.Introduction

This report investigates the feasibility of European put option price as a risk measure and corresponding capital allocation based on Euler's principle. We first defined European put option as a risk measure and verified its qualification to be applied to do capital allocation, then we compared it with other typical risk measures (VaR and Expected Shortfall). Finally we analyzed capital allocation to different kinds of assets with correlations in a portfolio with RORAC as a performance indicator.

2.Model Assumptions

Basic assumptions for assets throughout this report:

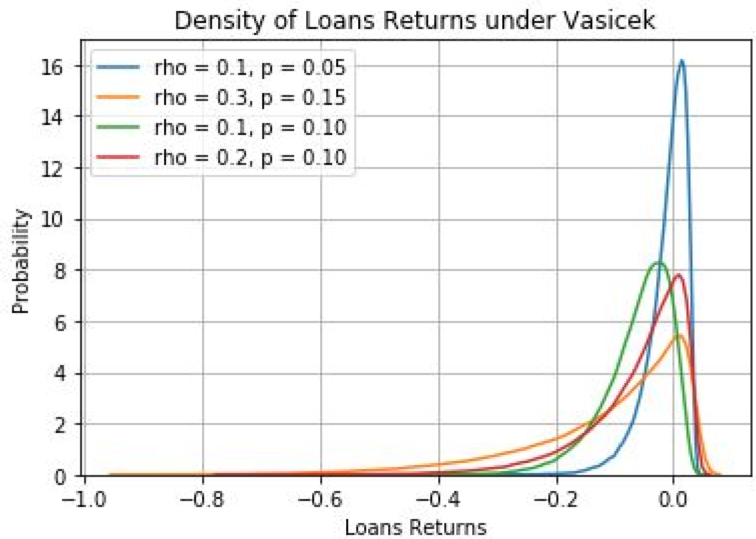
For simplicity, assume there are only two categories of investment instruments: loans and stocks.

The loss distribution of former follows Vasicek distribution while the price of latter follows geometric Brownian motion (GBM). Both models possess leptokurtic features and are extensively used in practice.

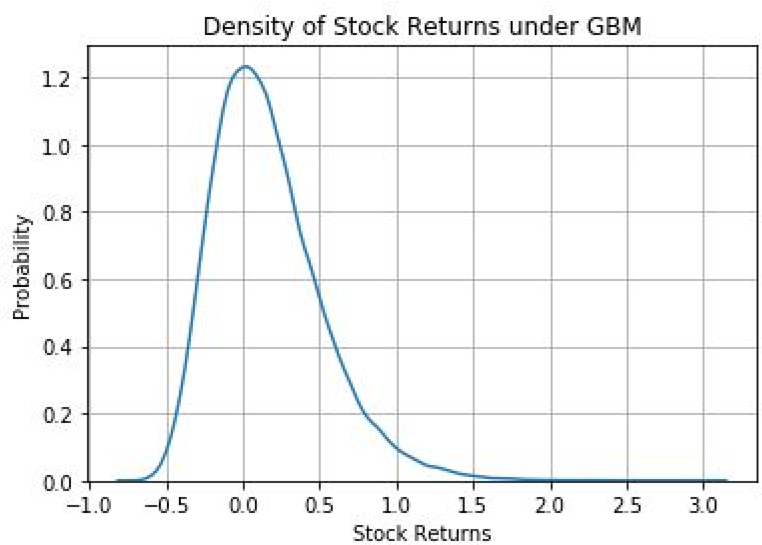
Over a time period of 1 year, we simulated the returns of both assets with the Monte Carlo method.

Vasicek model parameters, default threshold = $N^{-1}(p)$, where p is the unconditional default rate, and denote rho as the correlation coefficient to systematic factor. 0<rho<1.

For loans, suppose stipulated return is 4%(higher than risk free rate, because of risk premium), then the probability density plot for returns:



For stocks, supposed parameters are chosen as (sigma = 30%, r = 2%, mu = 15%), and simple return (S1-S0)/S0 is used here to compare with loans.



3.Concepts of Risk measures

According to definition in Merton Perold(1993): We define risk capital as the smallest amount that can be invested to insure the value of the firm's net assets against a loss in value relative to the risk-free investment of those net assets.

And risk capital is determined by a risk measure function ρ(X). Let X denote the portfolio-wide profit/loss.

$$X_i = g_i - L_i,$$

where g is the stipulated return on loans and L is the loss.

Then, a European put option with strike price equals the risk free future value of underlying assets can be considered as an insurance to secure riskless returns with the same payoff structure.

Thus, the price of such a European put option with the same maturity as the holding portfolio can be used as a risk measure. And we proved this risk measure satisfied some requirements for capital allocation later in this report.

Value-at-risk (VaR) is defined as the loss level that will not be exceeded with a certain confidence level during a certain period of time.

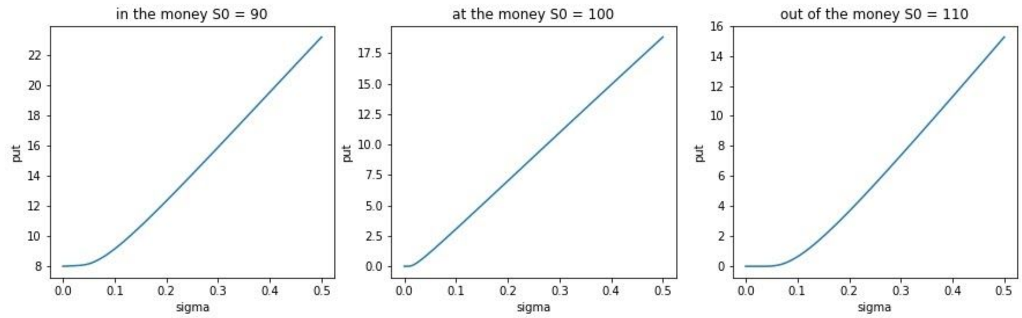
Expected shortfall (ES) is a function of two parameters: N (the time horizon in days) and X% (the confidence level). It is the expected loss during an N-day period, conditional that the loss is greater than the Xth percentile of the loss distribution.

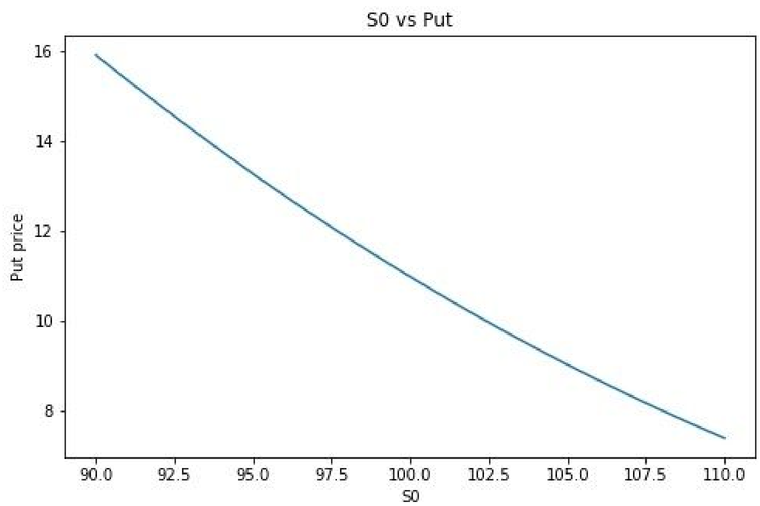
RORAC is the return on risk adjusted capital, defined below:

$$\mathrm{RORAC}(X_i \,|\, X) = \frac{\mathrm{E}[X_i]}{\rho(X_i \,|\, X)} = \frac{\mu_i}{\rho(X_i \,|\, X)}$$

4.Euler's principle: capital allocation

European put option (at the money) price can be considered as a qualified risk measure for Euler 's principle for capital allocation. As required by Euler's principle, the risk measure should be positive homogeneous of order 1. We apply the numerical method to prove that the put option price is positive homogeneous on implied volatility and initial underlying asset price. Following diagrams display the first degree homogeneous relationship.





5. Numerical Implementations

To implement these concepts numerically, we applied Monte Carlo simulation technique to obtain approximate values of each risk measure in both steps in Euler's capital allocation.

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;=\; -\mathrm{E}[X_i \,|\, X = -\mathrm{VaR}_\alpha(X)],$$

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;=\; \mathrm{E}[L_i \,|\, L = q_\alpha(L)] - g_i.$$

Often, it is not VaR itself that is of interest but rather *Unexpected Loss:*

$$\mathrm{UL}_{\mathrm{VaR},\alpha}(X) \;=\; \mathrm{VaR}_\alpha(X - \mathrm{E}[X]) \;=\; \mathrm{VaR}_\alpha(X) + \mathrm{E}[X]. \tag{3.8a}$$

In terms of $X_i = g_i - L_i$, Equation (3.8a) reads

$$\mathrm{UL}_{\mathrm{VaR},\alpha}(X) \;=\; \mathrm{VaR}_\alpha(\mathrm{E}[L] - L) \;=\; q_\alpha(L) - \mathrm{E}[L]. \tag{3.8b}$$

Having in mind that the Euler contribution of X_i to $\mathrm{E}[X]$ is obviously $\mathrm{E}[X_i]$, the formulae for the Euler contributions to $\mathrm{UL}_{\mathrm{VaR},\alpha}(X)$ are obvious from Equations (3.7a), (3.7b), (3.8a), and (3.8b).

In general, the conditional expectation of X_i given X cannot easily be calculated or estimated. For some exceptions from this observation see Tasche (2004b) or Tasche (2006). As the conditional expectation of X_i given X can be interpreted as the *best prediction* of X_i by X in a least squares context, approximation of $\mathrm{VaR}_\alpha(X_i \,|\, X)$ by *best linear predictions* of X_i by X has been proposed. Linear approximation of the right-hand side of (3.7a) by X and a constant yields

$$\mathrm{VaR}_\alpha(X_i \,|\, X) \;\approx\; \frac{\mathrm{cov}[X_i,\, X]}{\mathrm{var}[X]}\, \mathrm{UL}_{\mathrm{VaR},\alpha}(X) - \mathrm{E}[X_i]. \tag{3.9}$$

Similar to Expected shortfall .

The formula for the Euler ES-contributions reads

$$\begin{aligned} \mathrm{ES}_\alpha(X_i \,|\, X) \;&=\; -\mathrm{E}[X_i \,|\, X \le -\mathrm{VaR}_\alpha(X)] \\ &=\; -(1-\alpha)^{-1}\mathrm{E}[X_i\, \mathbf{1}_{\{X \le -\mathrm{VaR}_\alpha(X)\}}]. \end{aligned} \tag{3.11a}$$

Note that $\mathrm{E}[X_i \,|\, X \le -\mathrm{VaR}_\alpha(X)]$, in contrast to $\mathrm{E}[X_i \,|\, X]$ from (3.7a), is an *elementary conditional expectation* because the conditioning event has got a positive probability to occur. In case that X_i is given as $g_i - L_i$ (cf. (2.3)), we have $\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 q_u(L)\,du - \sum_{i=1}^n g_i$ and

$$\mathrm{ES}_\alpha(X_i \,|\, X) \;=\; \mathrm{E}[L_i \,|\, L \ge q_\alpha(L)] - g_i. \tag{3.11b}$$

Often, it is not ES itself that is of interest but rather *Unexpected Loss:*

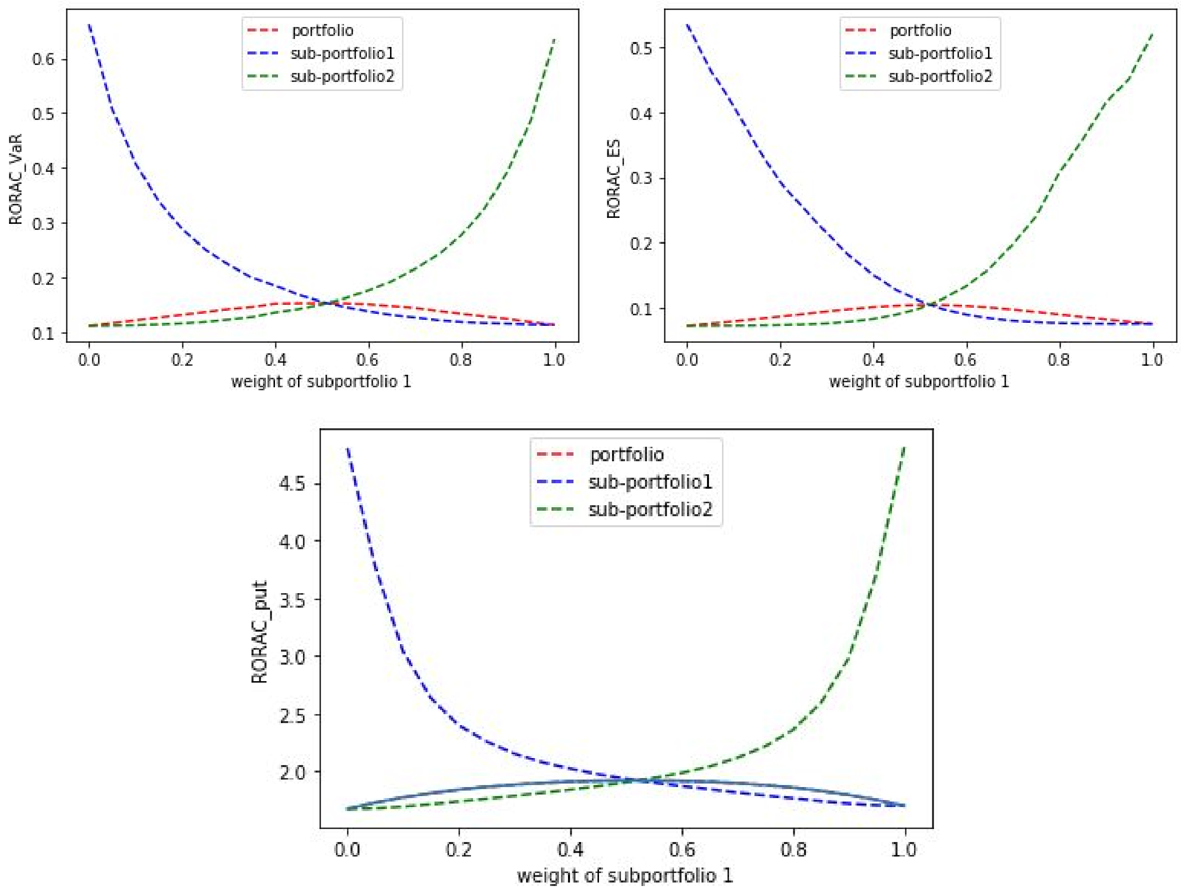
$$\mathrm{UL}_{\mathrm{ES},\alpha}(X) \;=\; \mathrm{ES}_\alpha(X - \mathrm{E}[X]) \;=\; \mathrm{ES}_\alpha(X) + \mathrm{E}[X]. \tag{3.12a}$$

One thing ES and put option price in common is that they are both calculated by integration within certain measures. Thus, their simulation methods are very similar. Here we assume risk neutral measure is the same as the physical measure for simplicity, however, the calculated values for put option price should be less than practical value because risk neutral measures overstates underlying risks. Please see specific technique details in the attached code snippets.

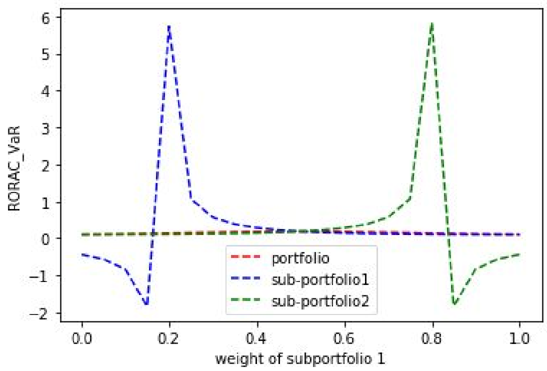
6.Comparison among different risk measures,

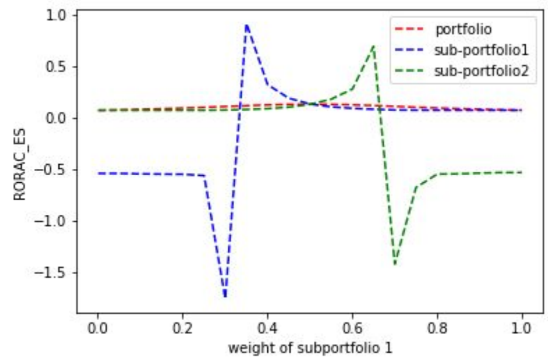
6.1 Loans & Loans

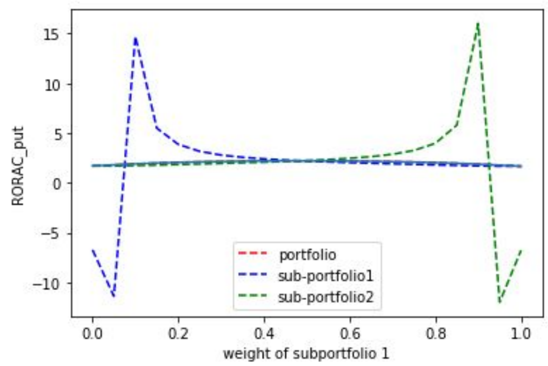
With rho = 0.4, p = 0.02, both under Vasicek model. Illustration with correlation = 0.4.
Plots of RORACs vs weights for different risk measures.



With rho = 0.4, p = 0.02, both under Vasicek model. Illustration with correlation = - 0.4.
Plots of RORACs vs weights for different risk measures.







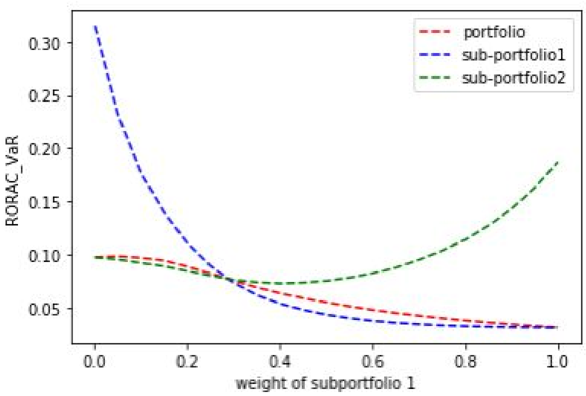
6.2 Loans & Stocks

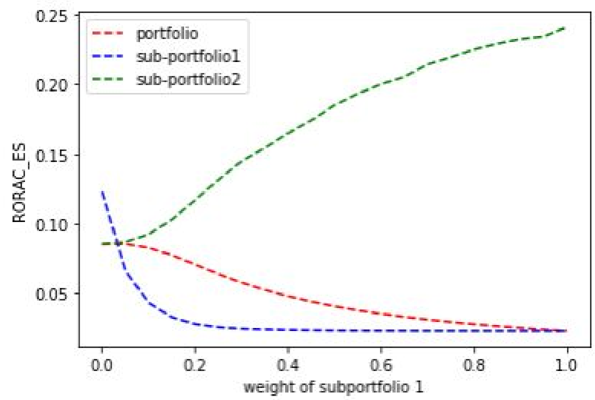
Assume a portfolio of two assets: loans and stocks, parameters are chosen to make both assets equally attractive for investors, otherwise the capital allocation will be extremely high in the more attractive one while almost zero capital allocated to the other. By ‘’attraction’’, it roughly means proportional return to risk.

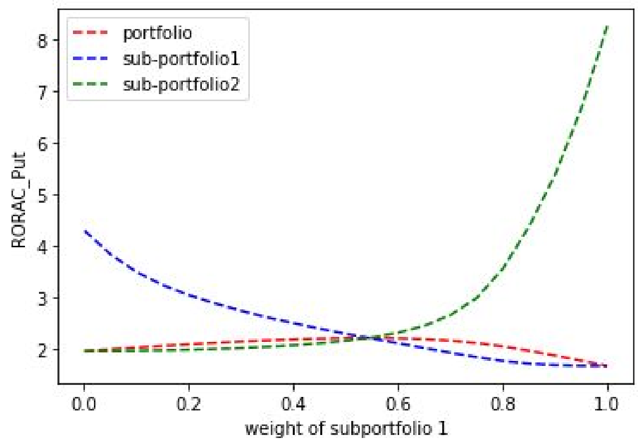
Set sub-portfolio 1 as loans while sub-portfolio 2 as stocks. Meanwhile, assume there is correlation between two assets, which can be either positive or negative. This assumption is for simulation of practical scenarios.

With Gaussian Coupla technique, we generated both positive and negative correlated returns between the two sub-portfolios to investigate the relationship between RORAC and correlation parameters for two risk measures VaR and ES respectively. Since the put option as a risk measure is special, we investigate it separately later.

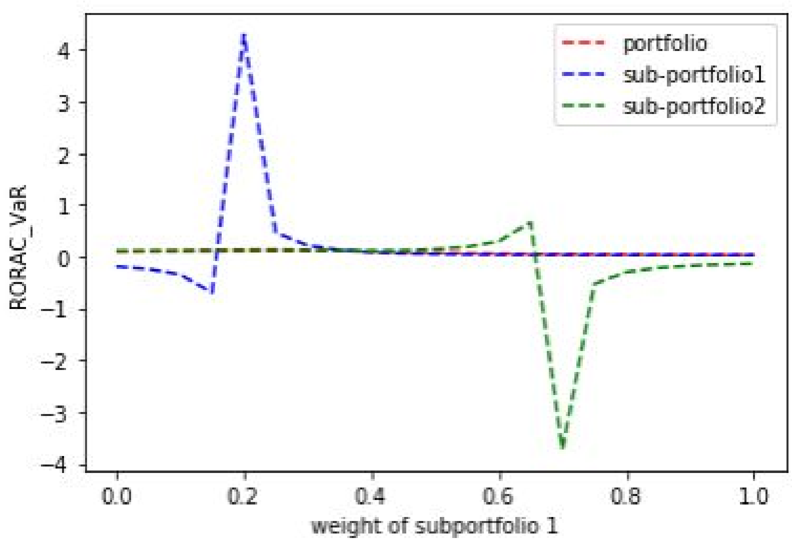
To illustrate different risk measures and corresponding RORACs with respected different weights, set positive correlation 0.4, with default threshold = $N^{-1}(0.03)$, rho = 0.5 in Vasicek model, GBM are unchanged.

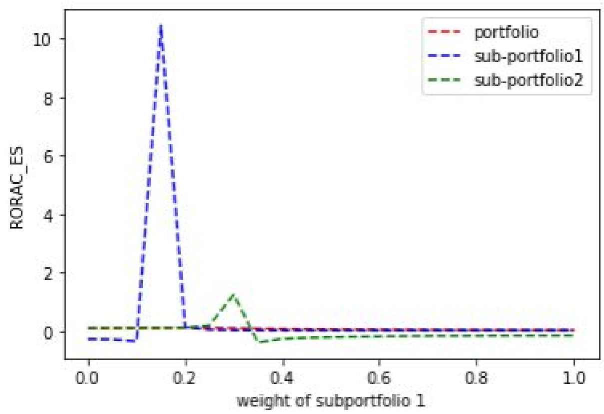


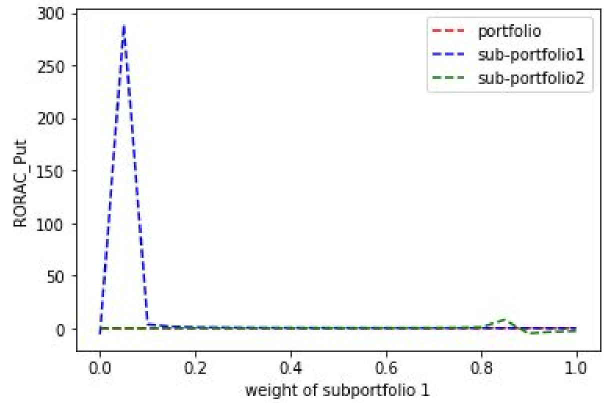




However, with negative correlations, in this case, correlation = -0.4

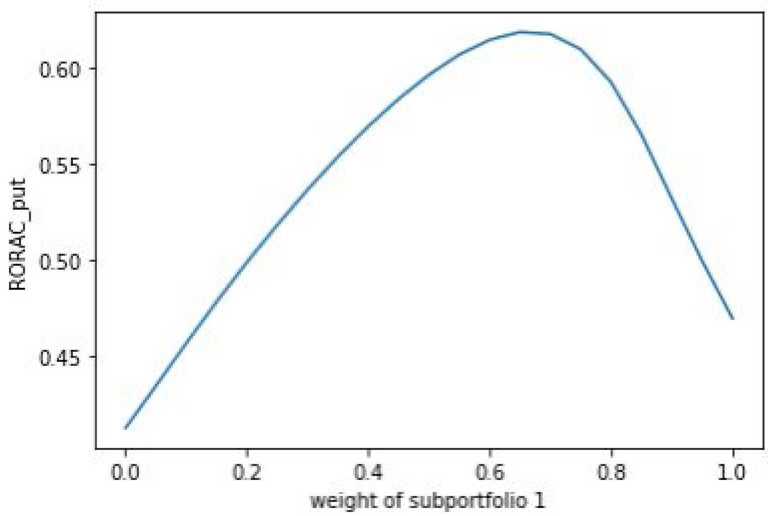




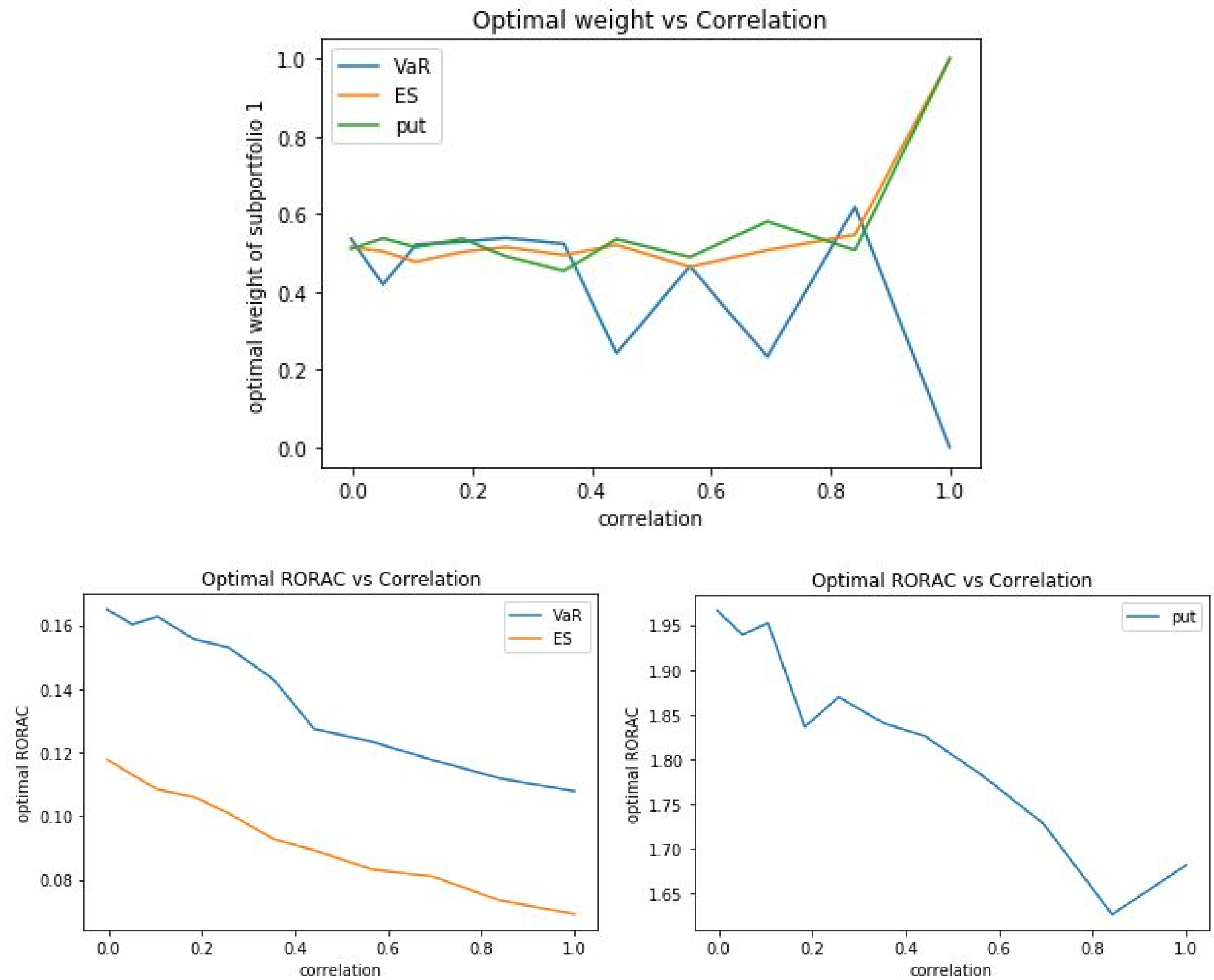


The spikes are caused because there exists some hedging effect with negative correlation, sometimes we did obtain negative or near zero risk contribution which is the denominator of RORAC, thus causing the value extremely high. And this is one of the drawbacks for Euler's capital allocation principle.

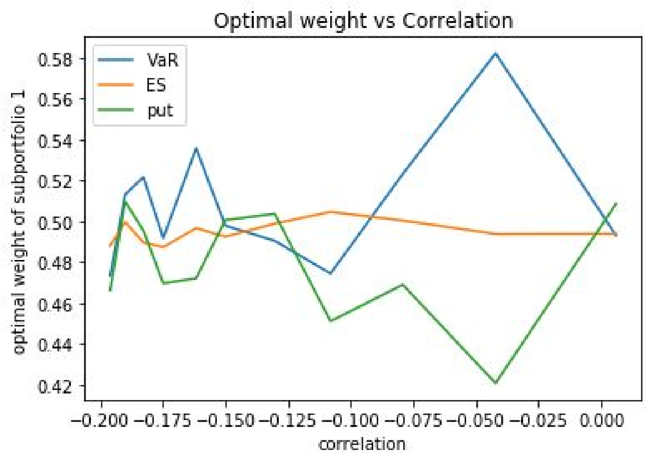
No matter how the returns are correlated, the trend of RORAC for the whole portfolio always has a parabolic curve shape: the illustration with correlation = -0.4, risk measure = put option price.

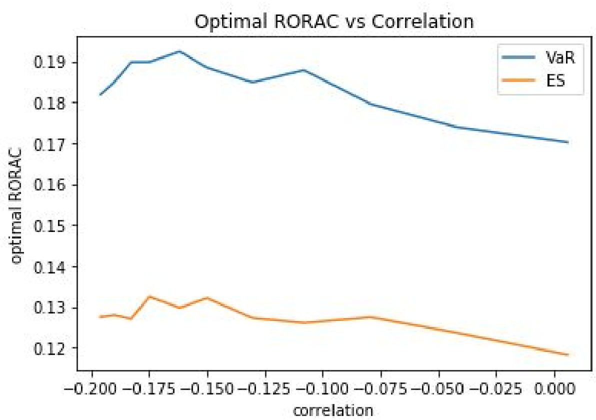


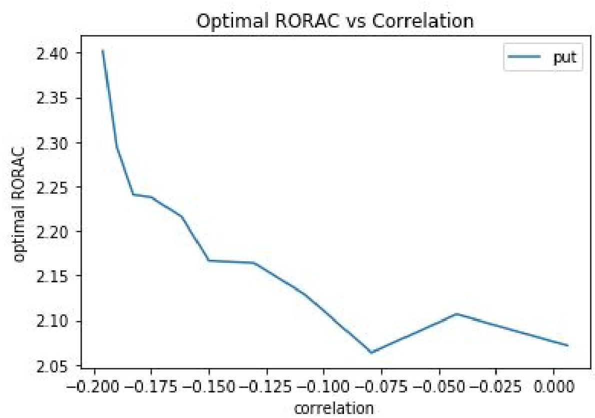
Plots are Optimal weights of sub-portfolio 1 as well as corresponding RORACs versus correlations with different risk measures:(case of positive correlations)



Plots are Optimal weights of sub-portfolio 1 as well as corresponding RORACs versus correlations with different risk measures:(case of negative correlations)

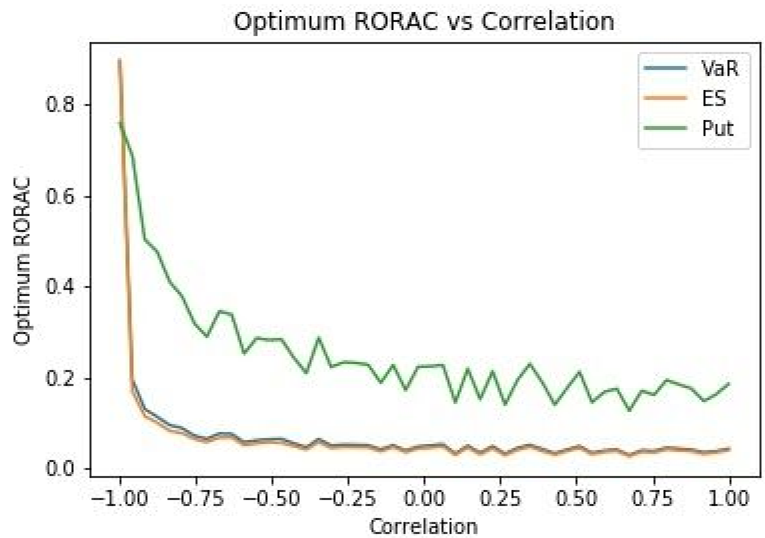




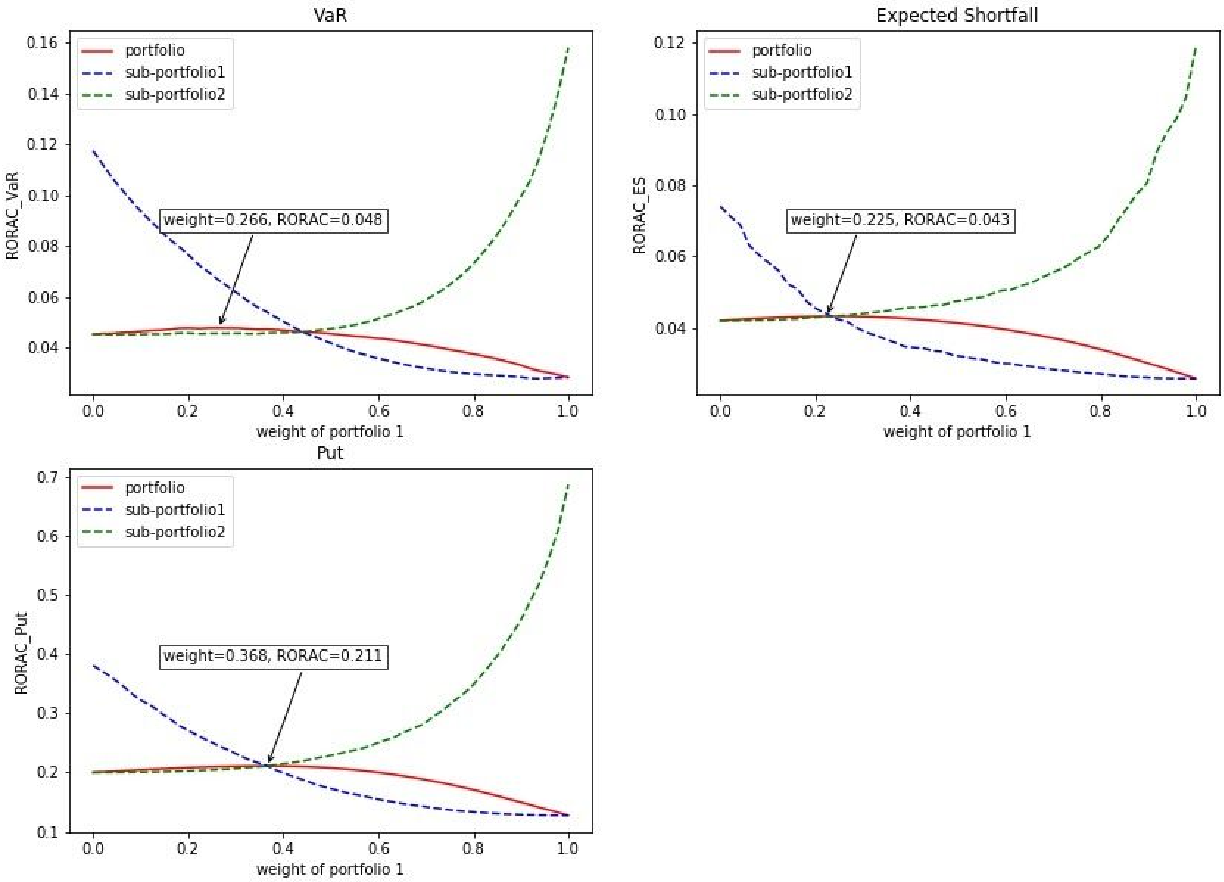


6.3 Stocks & Stocks

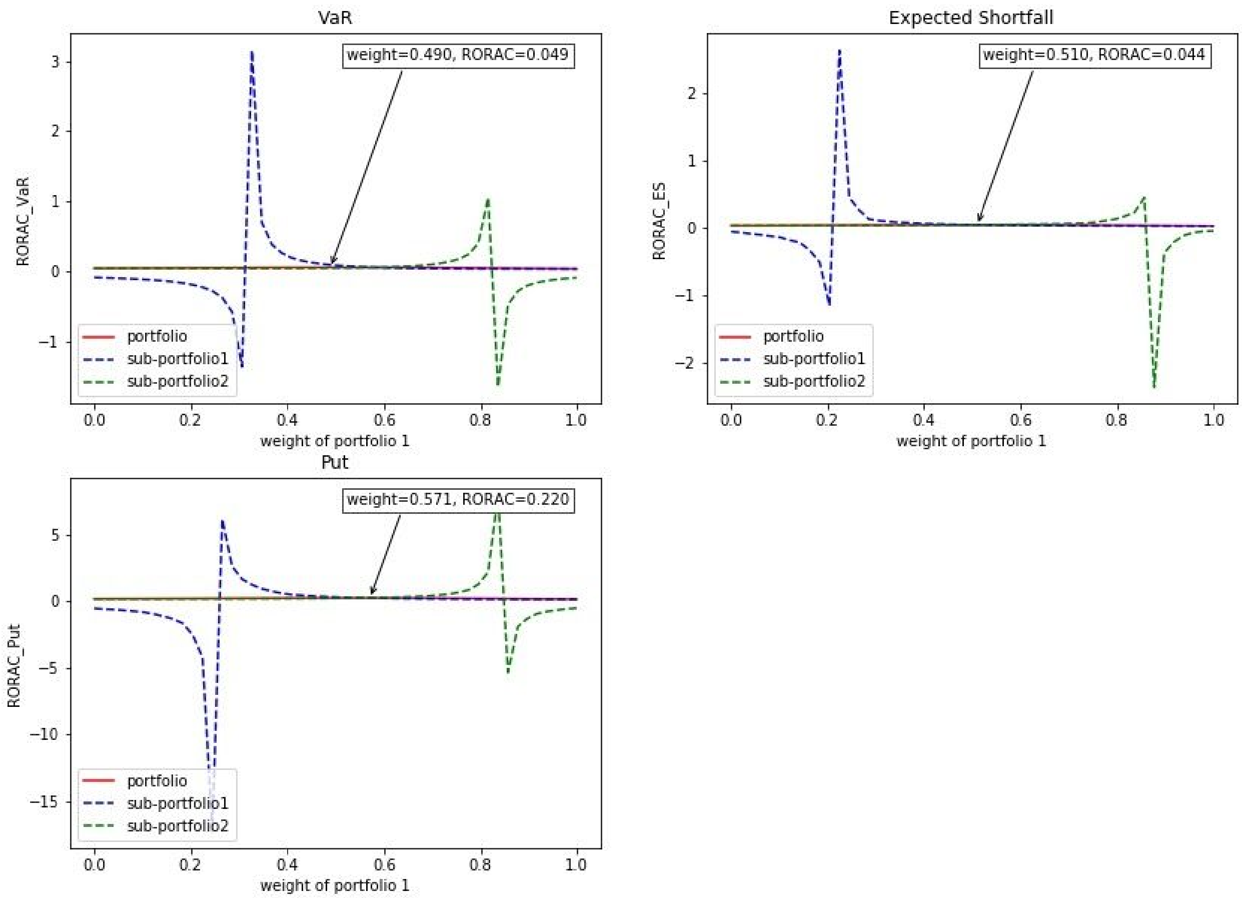
In the financial market, we use Black Scholes Model to simulate the price change of an underlying asset. The model assumes the price of the assets follows a Geometric Brownian motion with constant drift and volatility.



The graph above is the result under the Black Scholes assumptions. The graph shows the relationship between the correlation of the underlying assets and the portfolio optimum RORAC. During the simulation, we compared three risk measures: VaR, expected shortfall, put option price. As shown in the graph, when the two assets are perfectly negatively correlated, the optimum return for the portfolio is approaching 100% for VaR and Put price as the risk measure. As the correlation increases to the positive direction, the result of VaR and expected shortfall are synchronized and show no obvious difference. However, the result of the put price is higher than the other two, although the oscillating pattern is similar to the other two, the amplitude of the variation is way larger than the other two.



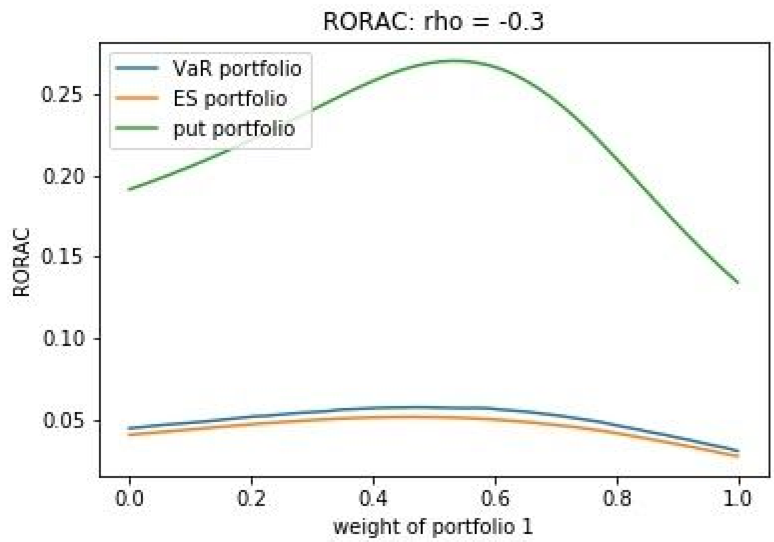
In our basic assumptions, there are two stocks in the portfolio with the exactly same initial value but different returns and violations. We set the correlation between the two stocks to be 0.3 and simulated the RORAC of each sub-portfolio with respect to different weights. The simulated curves are smooth for VaR and Put price but a slightly coarse for expected shortfall. The optimum RORAC is tagged on the graph and we observed similar value on VaR and expected shortfall. The optimum occurs at the similar level of weight of sub-portfolio 1 for VaR and expected shortfall at around 0.2 and optimum RORAC at around 0.04; the optimum value under put price measure is higher than the others at a weight level 0.37 and RORAC level 0.21.



As we adjusted the correlation between the two stocks to be -0.3, the result graphs became unstable. Although the optimum weight value/ RORAC value shown in the above graphs are not abnormal, there are huge jumps in both sub-portfolios' RORAC curves.

One thing noticeable is that RORAC for put is much larger than the other two risk measures. This is because the put option price as a risk measure is substantially smaller than the other two risk measures (VaR, ES), thus the denominator is small. We know that the VaR is definitely no larger than the Expected shortfall for a given confidence level, because ES is the conditional VaR within the area exceeding the benchmark. While for a European put option with strike the risk free return, the price is calculated under an assumed risk neutral measure with a given return distribution, which is a mathematical expectation without conditioning. It is straightforward that this expectation (put option price) is much higher than ES.

We observed strange spikes in the plots in the case of negative correlations, this phenomenon is caused by extreme values generated during the Monte Carlo simulation. (Since we do not have a closed form of risk measure function, we can only use numerical simulation.) With these several



extreme points removed, the plots become just as normal as they are in the cases of positive correlation. Thus, this reminds us to innovate our simulation methods to make the plot smooth. In the future research, a new implementable asymptotic expression for the risk measures and new sampling techniques will be desirable.

However, after we remove the sub-portfolio RORAC curves and focus on only the RORAC of the whole portfolio, we can get the reasonable result. The plot above is using the -0.3 as the correlation between the two stocks and the y-axis shows the value of the whole portfolio RORAC. We observed that there is only a tiny difference between the RORAC under the VaR measure and the expected shortfall measure, in this case, the VaR measure has a slightly higher portfolio return than the expected shortfall measure. The portfolio's return under put price measure is the highest among three but the peak RORAC value for the tree measures lies on the same vertical line. This graph is consistent with the first graph's result in this section.

7.Conclusion

We have verified that put option price as a risk measure can be used in capital allocation based on Euler's principle. However, one limit is that the put option price is not a coherent measure. And another issue involved in Euler's capital allocation is that sometimes negative risk capital values are generated, which does not coincide with practice. All above, there are further studies into new capital allocation rules as well as using American put option price as a new risk measure.

8.Acknowledgements

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9. References

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