L_0 Gradient Regularization and Scale Space Representation Model for Cartoon and Texture Decomposition

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1 Model and Algorithm

- The image decomposition model

$$\min_{\mathbf{u}} \mathcal{J}(\mathbf{u}) \equiv \frac{\lambda}{2} \|\boldsymbol{H}_\tau(\mathbf{u}-\mathbf{g})\|_2^2 + \|\nabla \mathbf{u}\|_0 . \tag{1}$$

$$\min_{\mathbf{z}=\nabla\mathbf{u}} \Psi(\mathbf{u},\mathbf{z}) \equiv \frac{\lambda}{2} \|\boldsymbol{H}(\mathbf{u}-\mathbf{g})\|_2^2 + \|\mathbf{z}\|_0 . \tag{2}$$

- Quadratic penalty method (QPM)

$$\mathcal{Q}(\mathbf{u},\mathbf{z};\beta) \equiv \frac{\lambda}{2} \|\boldsymbol{H}(\mathbf{g}-\mathbf{u})\|_2^2 + \frac{\beta}{2} \|\nabla\mathbf{u}-\mathbf{z}\|_2^2 + \|\mathbf{z}\|_0 . \tag{3}$$

1). Choose new penalty parameter $\beta_k > \beta_{k-1}$.
2). Choose new starting value $(\mathbf{u}_{k,0}, \mathbf{z}_{k,0})$.
3). Find an approximate minimizer $(\mathbf{u}_k, \mathbf{z}_k)$ of $\mathcal{Q}(\mathbf{u},\mathbf{z};\beta_k)$ starting at $(\mathbf{u}_{k,0}, \mathbf{z}_{k,0})$ and terminating when the stopping criterion is satisfied, i.e.,

$$(\mathbf{u}_k, \mathbf{z}_k) = \operatorname*{argmin}_{\mathbf{u},\mathbf{z}} \mathcal{Q}(\mathbf{u},\mathbf{z};\beta_k), \tag{4}$$

- The **algorithm AMS**(alternating minimization scheme)

$$\begin{aligned} \mathbf{u}_{k,i} &= \operatorname*{argmin}_{\mathbf{u}} \mathcal{Q}(\mathbf{u},\mathbf{z}_{k,i-1};\beta_k), & (5)\\ \mathbf{z}_{k,i} &= \operatorname*{argmin}_{\mathbf{z}} \mathcal{Q}(\mathbf{u}_{k,i},\mathbf{z};\beta_k). & (6) \end{aligned}$$

Theorem 1 *Let* $(\mathbf{u}_k, \mathbf{z}_k)$ *be the sequence generated by Algorithm QPM and assume that* $\lim_{k\to\infty} \epsilon_k \to 0$. *Then there exists an accumulation point* $\mathbf{u}^\dagger$ *of* $\{\mathbf{u}_k\}$ *such that* $\mathbf{u}^\dagger$ *is a local minimizer of* $\mathcal{J}(\mathbf{u})$.

Theorem 2 *Let* $(\mathbf{u}_{k,i}, \mathbf{z}_{k,i})$ *be the sequence generated by Algorithm AMS. Then, for any* ϵ_k, *there exists a finite integer* N *such that* $\operatorname{dist}(0, \partial\mathcal{Q}(\mathbf{u}_{k,N}, \mathbf{z}_{k,N};\beta_k)) < \epsilon_k$ *and* $\mathcal{Q}(\mathbf{u}_{k,N}, \mathbf{z}_{k,N};\beta_k) < C$.

2 Convergence Analysis

In this section, we show the convergence analysis of the proposed method. It includes an inner-outer iteration procedure, where inner iterations are handled with alternating minimization scheme, and the outer iterations update the penalty parameter. First, we give the subgradient of the function $\mathcal{Q}(\mathbf{u},\mathbf{z};\beta)$ as follows.

$$\partial\mathcal{Q}(\mathbf{u},\mathbf{z};\beta) = \begin{pmatrix} \lambda \boldsymbol{H}^T(\boldsymbol{H}\mathbf{u}-\mathbf{g}) \\ \partial_{\mathbf{z}} \|\mathbf{z}\|_0 \end{pmatrix} + \beta \begin{pmatrix} \nabla^T(\nabla\mathbf{u}-\mathbf{z}) \\ \mathbf{z}-\nabla\mathbf{u} \end{pmatrix} . \tag{7}$$

2.1 Convergence analysis of AMS

In this subsection, we will give the convergence of AMS. We first introduce that the Gaussian kernel is strictly positive definite. Bochner [2] proved that a function $\phi(x)$is positive definite in the sense employed in mathematics and RBF theory.

Lemma 3 *Let* $\boldsymbol{H}$ *be the matrix generalized by the Gaussian kernel, i.e.,* $\boldsymbol{H}_{ij} = \frac{1}{2\pi\sigma}\exp(-\frac{(i-j)^2}{2\sigma^2})$. *Then the matrix* $\boldsymbol{H}$ *is a symmetric positive definite matrix.*

Lemma 4 *Let* $(\mathbf{u}_{k,i}, \mathbf{z}_{k,i})$ *be the sequence generated by Algorithm AMS, then* $(\mathbf{u}_{k,i}, \mathbf{z}_{k,i})$ *is bounded.*

Proof: According to alternating minimization scheme, we have

$$\mathcal{Q}(\mathbf{u}_{k,i-1},\mathbf{z}_{k,i-1};\beta_k) \geq \mathcal{Q}(\mathbf{u}_{k,i},\mathbf{z}_{k,i-1},;\beta_k) \geq \mathcal{Q}(\mathbf{u}_{k,i},\mathbf{z}_{k,i};\beta_k)$$

for $i = 1, 2, \ldots$. Hence we obtain

$$\frac{1}{2} \|\boldsymbol{H}(\mathbf{u}_{k,i}-\mathbf{g})\|_2^2 \leq \mathcal{Q}(\mathbf{u}_{k,i},\mathbf{z}_{k,i};\beta_k) \leq \mathcal{Q}(\mathbf{u}_{k,0},\mathbf{z}_{k,0};\beta_k)$$

The last inequality is used that the last two terms in $\mathcal{Q}(\mathbf{u}_{k,i},\mathbf{z}_{k,i};\beta_k)$ is nonnegative. Notice that $\boldsymbol{H}$ is positive definite (see Lemma 3), hence $\mathbf{u}_{k,i}$ is bounded. According to $\mathbf{z}_{k,i} = \mathcal{T}(\nabla\mathbf{u}_{k,i}, \frac{2}{\beta_k})$, we know that $\mathbf{z}_{k,i}$ is also bounded. Therefore the lemma holds. $\square$

Lemma 5 *Let $(\mathbf{u}_{k,i}, \mathbf{z}_{k,i})$ be the sequence generated by Algorithm AMS and α be the smallest eigenvalue of the matrix $\lambda \boldsymbol{H}^T\boldsymbol{H} + \beta\nabla^T\nabla$. Then we have*

$$\mathcal{Q}(\mathbf{u}_{k,i-1}, \mathbf{z}_{k,i-1}; \beta_k) - \mathcal{Q}(\mathbf{z}_{k,i}, \mathbf{u}_{k,i}; \beta_k) \geq \frac{\alpha}{2} \|\mathbf{u}_{k,i} - \mathbf{u}_{k,i}\|_2^2. \tag{8}$$

Moreover, $\sum_{i=1}^{\infty} \|\mathbf{u}_{k,i} - \mathbf{u}_{k,i}\|_2^2$ is bounded and convergent, i.e.,

$$\lim_{i\to\infty} \|\mathbf{u}_{k,i} - \mathbf{u}_{k,i-1}\|_2^2 = 0.$$

Proof: Since $\mathbf{u}_{k,i} = \operatorname{argmin}_{\mathbf{u}} \mathcal{Q}(\mathbf{u}, \mathbf{z}_{k,i-1}; \beta_k)$ and $\mathcal{Q}(\mathbf{u}, \mathbf{z}_{k,i-1}; \beta_k)$ is a quadratic function with respect to $\mathbf{u}$, we have $\nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k) = 0$. By doing a Taylor series expansion of $\mathcal{Q}(\mathbf{u}, \mathbf{z}_{k,i-1}; \beta_k)$ around $\mathbf{u} = \mathbf{u}_{k,i}$, we have the following identify

$$\begin{aligned}\mathcal{Q}(\mathbf{u}, \mathbf{z}_{k,i-1}; \beta_k) &= \mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k) + \\ &\frac{1}{2}(\mathbf{u} - \mathbf{u}_{k,i})^T\nabla^2_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k)(\mathbf{u} - \mathbf{u}_{k,i}).\end{aligned} \tag{9}$$

Since $\nabla^2_{\mathbf{u}}\mathcal{Q}(\mathbf{u}, \mathbf{z}; \beta_k) = \lambda\boldsymbol{H}^T\boldsymbol{H} + \beta\nabla^T\nabla$ and α is the smallest eigenvalue of the matrix $\lambda\boldsymbol{H}^T\boldsymbol{H} + \beta\nabla^T\nabla$, we obtain

$$\begin{aligned}&(\mathbf{u}_{k,i} - \mathbf{u}_{k,i})^T\nabla^2_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k)(\mathbf{u}_{k,i} - \mathbf{u}_{k,i}) \\ \geq \;& \alpha \|\mathbf{u}_{k,i-1} - \mathbf{u}_{k,i}\|_2^2.\end{aligned}$$

Setting $\mathbf{u} = \mathbf{u}_{k,i-1}$ into (9) and using $\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k) \geq \mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k)$, we have

$$\mathcal{Q}(\mathbf{u}_{k,i-1}, \mathbf{z}_{k,i-1}; \beta_k) \geq \mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k) + \frac{\alpha}{2}\|\mathbf{u}_{k,i} - \mathbf{u}_{k,i-1}\|_2^2.$$

Summing the preceding inequality from $i = 1$ to ∞, we have

$$\mathcal{Q}(\mathbf{u}_{k,0}, \mathbf{z}_{k,0}; \beta_k) \geq \frac{\alpha}{2}\sum_{i=1}^{\infty}\|\mathbf{u}_{k,i} - \mathbf{u}_{k,i-1}\|_2^2.$$

Hence the lemma holds. □

Lemma 6 *Let $(\mathbf{u}_{k,i}, \mathbf{z}_{k,i})$ be the sequence generated by Algorithm AMS. Then*

$$\operatorname{dist}(0, \partial\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k)) = \beta_k \left\|\nabla^T(\mathbf{z}_{k,i} - \mathbf{z}_{k,i-1})\right\|_2.$$

Proof: According to the subgradient of $\mathcal{Q}(\mathbf{u}, \mathbf{z}; \beta_k)$ in (7) and the definition of the distance between a point and a set, we have

$$\begin{aligned}&\operatorname{dist}(0, \partial\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k))^2 \\ = \;& \|\nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k)\|_2^2 + \operatorname{dist}(0, \partial_{\mathbf{z}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k))^2.\end{aligned}$$

According to (6), we have $0 \in \partial_{\mathbf{z}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k)$. Then we obtain $\operatorname{dist}(0, \partial_{\mathbf{z}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}); \beta_k) = 0$. Now we compute the first term on the right-hand side of the above equation. Notice that

$$\nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k) - \nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k) = \beta_k\nabla^T(\mathbf{z}_{k,i} - \mathbf{z}_{k,i-1}).$$

Because $\mathbf{u}_{k,i} = \operatorname{argmin}_{\mathbf{u}} \mathcal{Q}(\mathbf{u}, \mathbf{z}_{k,i-1}; \beta_k)$, we have $\nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i-1}; \beta_k) = 0$. Hence we have $\nabla_{\mathbf{u}}\mathcal{Q}(\mathbf{u}_{k,i}, \mathbf{z}_{k,i}; \beta_k) = \beta_k\nabla^T(\mathbf{z}_{k,i} - \mathbf{z}_{k,i-1})$. Therefore, the lemma holds. □

2.1.1 Proof of Theorem 2.

Proof: According to (**??**), we have

$$\mathbf{u}_{k,i+1} - \mathbf{u}_{k,i} = \beta_k\left(\lambda\boldsymbol{H}^T\boldsymbol{H} + \beta_k\nabla^T\nabla\right)^{-1}\nabla^T(\mathbf{z}_{k,i} - \mathbf{z}_{k,i-1}).$$

Because $\boldsymbol{H}$ is a symmetric positive define matrix, $\lambda\boldsymbol{H}^T\boldsymbol{H} + \beta_k\nabla^T\nabla$ is also a symmetric positive define matrix. Assume that the minimum eigenvalue of the matrix $\left(\lambda\boldsymbol{H}^T\boldsymbol{H} + \beta_k\nabla^T\nabla\right)^{-1}$ is ρ, we obtain

$$\|\mathbf{u}_{k,i+1} - \mathbf{u}_{k,i}\|_2 \geq \rho\beta_k\left\|\nabla^T(\mathbf{z}_{k,i} - \mathbf{z}_{k,i-1})\right\|_2.$$

According to Lemma (5), for any ϵ_k, there exists a constant N such that

$$\|\mathbf{u}_{k,N+1} - \mathbf{u}_{k,N}\|_2 < \rho\epsilon_k.$$

Therefore, we have $\beta_k\left\|\nabla^T(\mathbf{z}_{k,N} - \mathbf{z}_{k,N-1})\right\|_2 < \epsilon_k$. Hence, the theorem holds. □

2.2 Convergence analysis of QPM

We consider the convergence of QPM.

Theorem 7 *Assume that* $\mathbf{u}$ *is not a constant value image. Let* $(\mathbf{u}_k, \mathbf{z}_k)$ *be the sequence generated by Algorithm QPM. Then, we have* $\lim_{k\to\infty}(\nabla \mathbf{u}_k - \mathbf{z}_k) = 0$. *There exists a subsequence* $\mathcal{K}$ *such that* $\{(\mathbf{u}_k, \mathbf{z}_k)\}_{k\in\mathcal{K}} \to (\mathbf{u}^\dagger, \mathbf{z}^\dagger)$, *here* $(\mathbf{u}^\dagger, \mathbf{z}^\dagger)$ *is a limit point of* $\{(\mathbf{u}_k, \mathbf{z}_k)\}$. *Moreover, we have* $\nabla \mathbf{u}^\dagger = \mathbf{z}^\dagger$.

Proof: By taking the limit as $k \to \infty$ on both side of (**??**) and using the fact $\beta_k \to \infty$, we obtain $\mathbf{z}_k - \nabla \mathbf{u}_k \to 0$.

By Lemma 4, we have that $\{\mathbf{z}_k\}$ and $\{\mathbf{u}_k\}$ are bounded, hence, there exists a subsequence $\mathcal{K}$ such that $\{(\mathbf{u}_k, \mathbf{z}_k)\}_{k\in\mathcal{K}} \to (\mathbf{u}^\dagger, \mathbf{z}^\dagger)$. Since $\lim_{k\to\infty}(\nabla \mathbf{u}_k - \mathbf{z}_k) = 0$, by taking limits as $k \to \infty$ for $k \in \mathcal{K}$, we obtain $\nabla \mathbf{u}^\dagger - \mathbf{z}^\dagger = 0$. Hence the lemma holds. □

The above lemma states that $\lim_{k\to\infty}(\nabla \mathbf{u}_k - \mathbf{z}_k) = 0$, however, is $\mathbf{r}_k = \beta_k(\nabla \mathbf{u}_k - \mathbf{z}_k)$ bounded? We have the following lemma.

Theorem 8 *Assume that* $\mathbf{u}$ *is not a constant value image. Let* $(\mathbf{u}_k, \mathbf{z}_k)$ *be the sequence generated by Algorithm QPM and assume that* $\lim_{k\to\infty} \epsilon_k \to 0$. *Let* $\mathbf{r}_k = \beta_k(\nabla \mathbf{u}_k - \mathbf{z}_k)$ *and suppose that* $\mathbf{z}^\dagger \neq 0$. *Then* $\{\mathbf{r}_k\}$ *is bounded and the limit point* $(\mathbf{u}^\dagger, \mathbf{z}^\dagger)$ *is a KKT point of* $\Psi(\mathbf{u}, \mathbf{z})$ *in* (2).

Proof: Because the approximate minimizer $(\mathbf{u}_k, \mathbf{z}_k)$ of $\mathcal{Q}(\mathbf{u}, \mathbf{z}; \beta_k)$ is obtained by the alternating minimization scheme, we have $\mathbf{r}_k \in \partial_{\mathbf{z}} \|\mathbf{z}_k\|_0$ and

$$\mathrm{dist}(0, \partial_{\mathbf{z}} \mathcal{Q}(\mathbf{u}_k, \mathbf{z}_k); \beta_k) = 0.$$

Hence we have

$$\mathrm{dist}(0, \partial \mathcal{Q}(\mathbf{u}_k, \mathbf{z}_k; \beta_k)) = \|\nabla_{\mathbf{u}} \mathcal{Q}(\mathbf{u}_k, \mathbf{z}_k; \beta_k)\|_2 .$$

Denote $\mathbf{y}_k = \nabla_{\mathbf{u}} \mathcal{Q}(\mathbf{u}_k, \mathbf{z}_k; \beta_k)$, we have $\mathbf{y}_k = \lambda \boldsymbol{H}^T(\boldsymbol{H}\mathbf{u}_k - \mathbf{g}) + \nabla^T \mathbf{r}_k$. Thus we have

$$\left\|\nabla^T \mathbf{r}_k\right\|_2 \leq \|\mathbf{y}_k\|_2 + \left\|\lambda \boldsymbol{H}^T(\boldsymbol{H}\mathbf{u}_k - \mathbf{g})\right\|_2 .$$

From the stopping criterion, we know $\|\mathbf{y}_k\|_2 < \epsilon_k$. Notice that the null space of ∇^T is the set of all constant vectors, $\mathbf{r}_k$ can be decomposed into the sum of a bounded vector $\widehat{\mathbf{r}}_k$ and a constant-valued vector $\tau_k \mathbf{1}$ with scalar parameter τ_k, i.e., $\mathbf{r}_k = \widehat{\mathbf{r}}_k + \tau_k \mathbf{1}$.

Because the sequence $\{(\mathbf{u}_k, \mathbf{z}_k)\}$ is bounded, it is easily to show that $\nabla^T \mathbf{r}_k$ is bounded and there exists a subsequence $\mathcal{K}$ such that $\{(\mathbf{u}_k, \mathbf{z}_k)\}_{k\in\mathcal{K}} \to (\mathbf{u}^\dagger, \mathbf{z}^\dagger)$. By contradiction, if $\{\mathbf{r}_k\}$ is unbounded, then we have $\tau_k \to \infty$ of $k \in \mathcal{K}$. Denote $\mathcal{I} = \{i : \mathbf{z}^\dagger[i] \neq 0\}$ and $\mathcal{I}^c = \{i : \mathbf{z}^\dagger[i] = 0\}$, and let $\mathbf{z}[\mathcal{I}] = \{\mathbf{z}[i] : i \in \mathcal{I}\}$ and $\mathbf{z}[\mathcal{I}^c] = \{\mathbf{z}[i] : i \in \mathcal{I}^c\}$. By $\mathbf{r}_k[\mathcal{I}] = \widehat{\mathbf{r}}_k[\mathcal{I}] + \tau_k \mathbf{1}$, we have

$$\lim_{k\in\mathcal{K}} \mathbf{r}_k[\mathcal{I}] = \infty. \tag{10}$$

Because $\lim_{k\in\mathcal{K}} \mathbf{z}_k = \mathbf{z}^\dagger$, there exists some k_1 such that $\mathbf{z}_k[i] \neq 0$ for all $k \geq k_1$ and $k \in \mathcal{K}$, and $i \in \mathcal{I}$. However, according to the definition of $\|\cdot\|_0$ and $\mathbf{r}_k \in \partial_{\mathbf{z}} \|\mathbf{z}_k\|_0$, we have $\mathbf{r}_k[\mathcal{I}] = 0$ for $k \in \mathcal{K}$. This contradicts with (10).

Now we show that the limit point $(\mathbf{u}^\dagger, \mathbf{z}^\dagger)$ is a KKT point of $\Psi(\mathbf{u}, \mathbf{z})$. Because the sequence $\{\mathbf{r}_k\}$ is bounded, there exists a subsequence $\mathcal{K}_1 \subseteq \mathcal{K}$ such that $\mathbf{r}_k$ converges to some $\mathbf{r}^\dagger$ for $k \in \mathcal{K}_1$. Hence we have $\mathbf{r}^\dagger \in \lambda \partial \left\|\mathbf{z}^\dagger\right\|_0$. By taking the limit as $k \in \mathcal{K}_1$ in the equations $\mathbf{y}_k = \lambda \boldsymbol{H}^T(\boldsymbol{H}\mathbf{u}_k - \mathbf{g}) + \nabla^T \mathbf{r}_k$, we have

$$\lambda \boldsymbol{H}^T(\boldsymbol{H}\mathbf{u}^\dagger - \mathbf{g}) + \nabla^T \mathbf{r}^\dagger = 0.$$

Together with $\nabla \mathbf{u}^\dagger = \mathbf{z}^\dagger$, the result holds. □

2.2.1 Proof of Theorem 1.

Proof: The Frechet sub-differential of Ψ is defined assume

$$\widehat{\partial} \Psi(\mathbf{x}) = \left\{ \boldsymbol{\xi} : \lim_{\mathbf{y}\to\infty} \inf_{\mathbf{y}\neq\mathbf{x}} \frac{\Psi(\mathbf{y}) - \Psi(\mathbf{x}) - \langle \boldsymbol{\xi}, \mathbf{y} - \mathbf{x}\rangle}{\|\mathbf{y} - \mathbf{x}\|_F} \right\} \geq 0.$$

Let $(\mathbf{u}^\dagger, \mathbf{z}^\dagger)$ be a limit point of $(\mathbf{u}_k, \mathbf{z}_k)$ and denote $\mathcal{I} = \{i : \mathbf{z}^\dagger[i] \neq 0\}$ and $\mathcal{I}^c = \{i : \mathbf{z}^\dagger[i] = 0\}$. The number of the set $\mathcal{I}^c$ is denoted as L_1 and the size of $\mathbf{z}$ is denoted as L. By the condition $\mathbf{z}^\dagger \neq 0$, we have $\mathcal{I} \neq \emptyset$, hence $L_1 < L$. Let $\mathbf{z}[\mathcal{I}] = \{\mathbf{z}[i] : i \in \mathcal{I}\}$ and $\mathbf{z}[\mathcal{I}^c] = \{\mathbf{z}[i] : i \in \mathcal{I}^c\}$, we have $\nabla^T \partial \left\|\mathbf{z}^\dagger\right\| = \{\nabla_{\mathcal{I}^c}^T \mathbf{x} : \mathbf{x} \in \mathbb{R}^{L_1}\}$ According to $\mathbf{z}^\dagger = \nabla \mathbf{u}^\dagger$, we have $\nabla \mathbf{u}^\dagger[\mathcal{I}] = \mathbf{z}^\dagger[\mathcal{I}] \neq 0$ and $\nabla \mathbf{u}^\dagger[\mathcal{I}^c] = \mathbf{z}^\dagger[\mathcal{I}^c] = 0$. Denote $\mathcal{S}_\mathcal{I} = \{\mathbf{u} : \nabla \mathbf{u}[\mathcal{I}] \neq 0\}$ and $\mathcal{S}_{\mathcal{I}^c} = \{\mathbf{u} : \nabla \mathbf{u}[\mathcal{I}^c] = 0\}$. According to $\mathbf{z}^\dagger = \nabla \mathbf{u}^\dagger \neq 0$, we have that the set $\mathcal{S}_\mathcal{I}$ is not empty. Hence if $\mathbf{u} \in \mathcal{S}_\mathcal{I} \bigcap \mathcal{S}_{\mathcal{I}^c}$, we have $\|\nabla \mathbf{u}\|_0 = \left\|\nabla \mathbf{u}^\dagger\right\|_0$, and if $\mathbf{u} \in \mathcal{S}_\mathcal{I} \backslash \mathcal{S}_{\mathcal{I}^c}$, we have $\|\nabla \mathbf{u}\|_0 > \left\|\nabla \mathbf{u}^\dagger\right\|_0$. Thus, for any $\boldsymbol{\xi} \in \nabla^T \partial \left\|\mathbf{z}^\dagger\right\|_0$, there exists a vector $\widehat{\boldsymbol{\xi}} \in \mathbb{R}^{L_1}$ such that $\boldsymbol{\xi} = \nabla_{\mathcal{I}^c} \widehat{\boldsymbol{\xi}}$. Hence, for all $\boldsymbol{\xi} \in \nabla^T \partial \left\|\mathbf{z}^\dagger\right\|_0$,we have

$$
\begin{aligned}
& \lim_{\mathbf{x} \to \mathbf{u}^\dagger} \inf_{\mathbf{x} \neq \mathbf{u}^\dagger} \frac{\|\nabla \mathbf{x}\|_0 - \left\|\nabla \mathbf{u}^\dagger\right\|_0 - \left\langle \boldsymbol{\xi}, \mathbf{x} - \mathbf{u}^\dagger \right\rangle}{\left\|\mathbf{x} - \mathbf{u}^\dagger\right\|_2} \\
= & \lim_{\mathbf{x} \to \mathbf{u}^\dagger} \inf_{\mathbf{x} \neq \mathbf{u}^\dagger, \mathbf{x} \in \mathcal{S}_{\mathcal{I}^c}} \frac{\|\nabla \mathbf{x}\|_0 - \left\|\nabla \mathbf{u}^\dagger\right\|_0 - \left\langle \nabla \widehat{\boldsymbol{\xi}}[\mathcal{I}^c], \mathbf{x} - \mathbf{u}^\dagger \right\rangle}{\left\|\mathbf{x} - \mathbf{u}^\dagger\right\|_2} = 0.
\end{aligned}
$$

The last equation is obtained by the facts $\left\langle \nabla_{\mathcal{I}^c} \widehat{\boldsymbol{\xi}}, \mathbf{x} - \mathbf{u}^\dagger \right\rangle = \left\langle \widehat{\boldsymbol{\xi}}, \nabla_{\mathcal{I}^c}^T (\mathbf{x} - \mathbf{u}^\dagger) \right\rangle$ and $\mathbf{u}^\dagger \in \mathcal{S}_{\mathcal{I}^c}$. Therefore, we have $\nabla^T \partial \left\|\mathbf{z}^\dagger\right\|_0 \subseteq \widehat{\partial} \left\|\nabla \mathbf{u}^\dagger\right\| \subseteq \partial \left\|\nabla \mathbf{u}^\dagger\right\|$. Notice that $\mathbf{r}^\dagger \in \partial \left\|\mathbf{z}^\dagger\right\| = \partial \left\|\nabla \mathbf{u}^\dagger\right\|$, we have

$$
0 \in \boldsymbol{H}(\boldsymbol{H}\mathbf{u}^\dagger - \mathbf{g}) + \lambda \nabla^T \partial \left\|\nabla \mathbf{u}^\dagger\right\|.
$$

Hence $\mathbf{u}^\dagger$ is a critical point of the function $\mathcal{J}(\mathbf{u})$ in (1). Applying the results in , we have $\mathbf{u}^\dagger$ is a local minimizer of $\mathcal{J}(\mathbf{u})$ in (1). □

3 Numerical experiments

3.1 Natural images

We show the decomposition results of the Barbara image obtained using different methods, and the decomposition results for the cartoon and texture components are shown in Figure 1 and 2, respectively. To investigate the influence of the parameters on decomposition results obtained by RTV and CIF, Figure 3 shows the decompositions of Barbara image with varying the parameters.

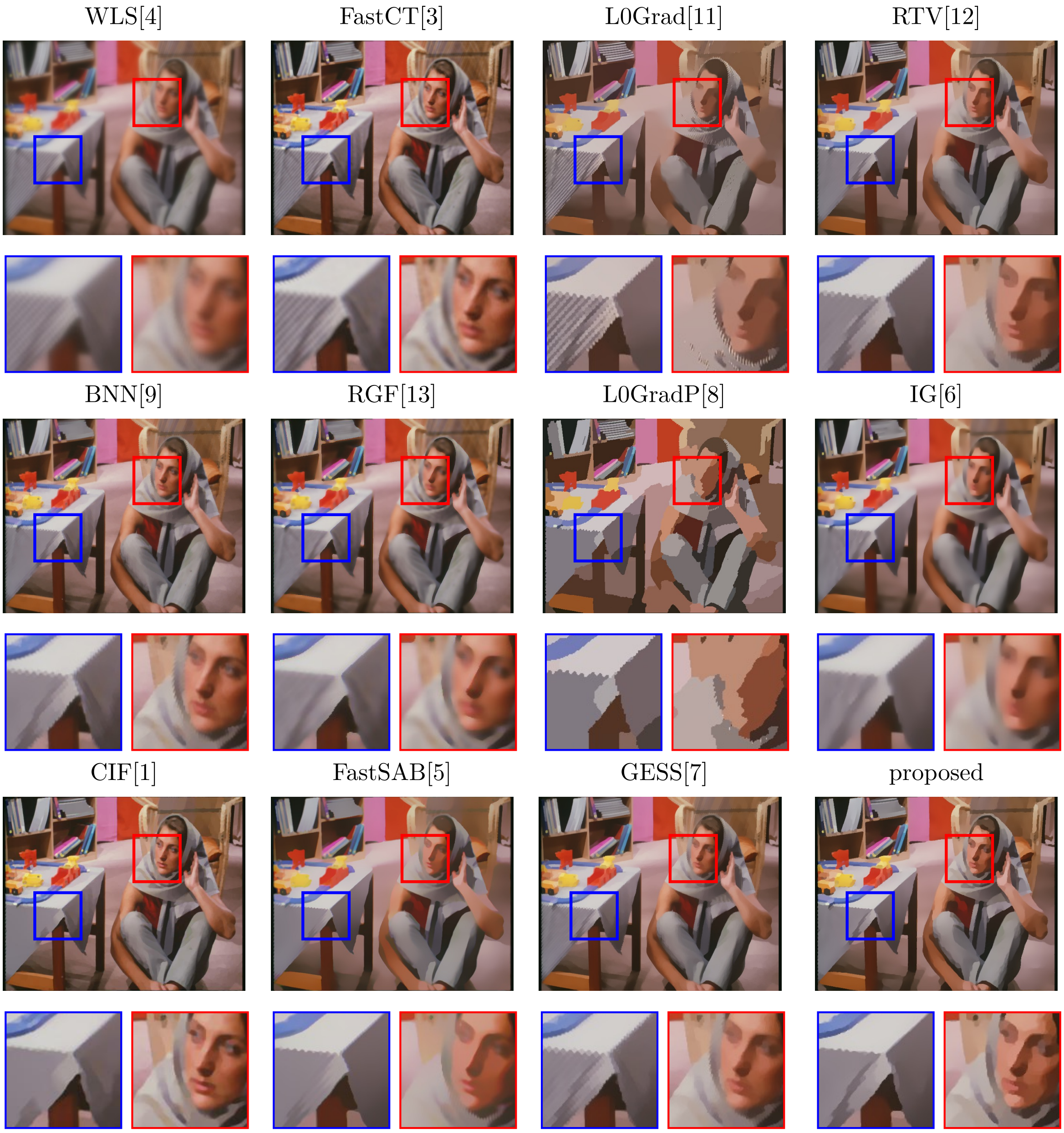


Figure 1: Comparison of cartoon components of Barbara image.

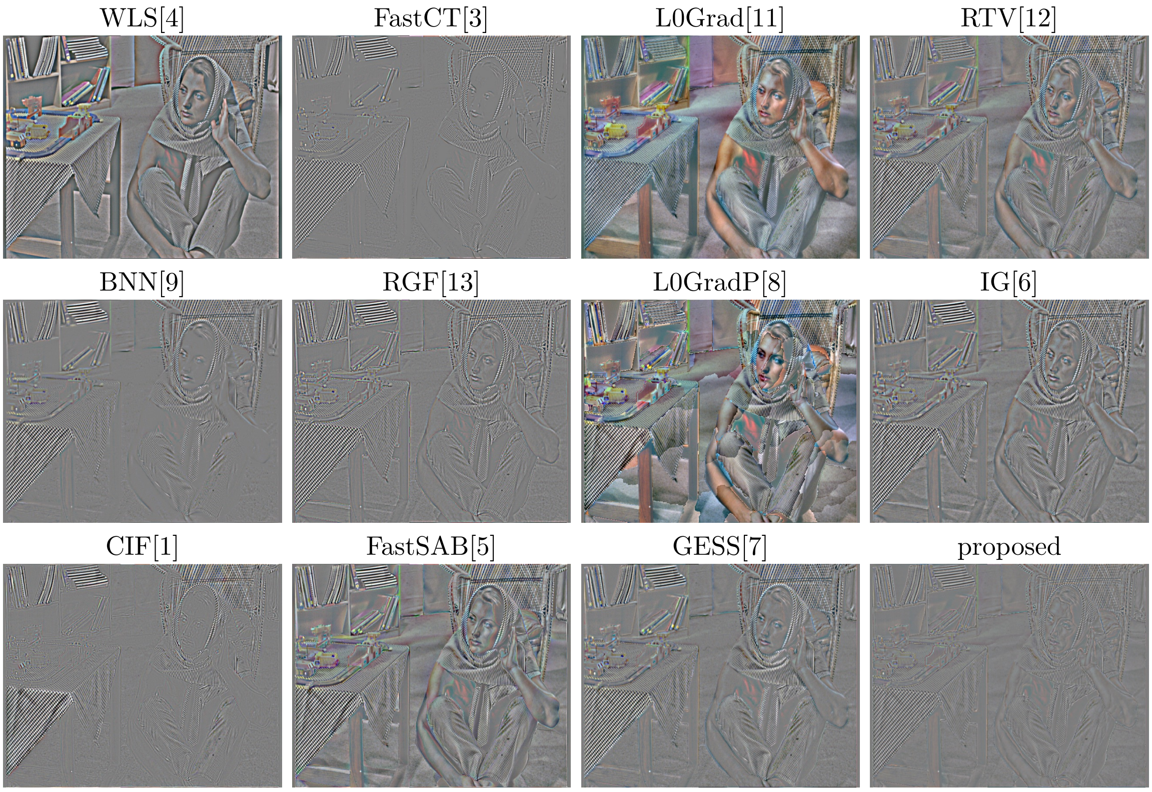
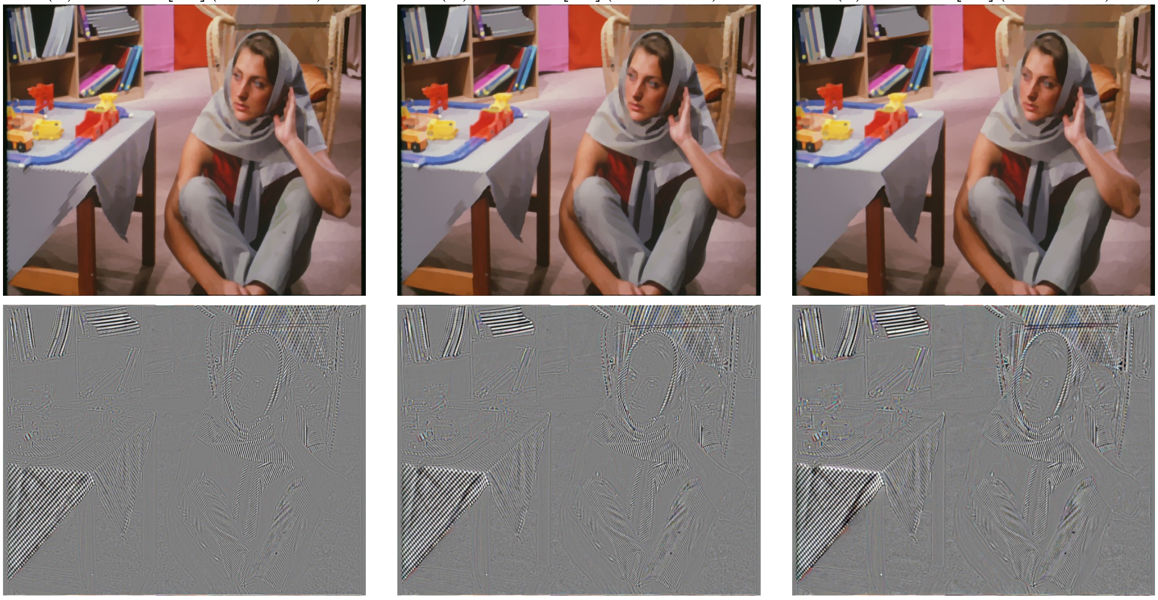


Figure 2: Comparison of texture components of Barbara image.

(a)L0Grad[11]($\lambda = 0.04$) (b)L0Grad[11]($\lambda = 0.08$) (c)L0Grad[11]($\lambda = 0.16$)



(d)CIF[1]($\lambda = 70, \sigma = 2$) (e)CIF[1]($\lambda = 70, \sigma = 4$) (f)CIF[1]($\lambda = 70, \sigma = 8$)

Figure 3: Cartoon images obtained by L0Grad and CIF with different parameters.

Figure 4: Test images: “Mountains” and “fish” .

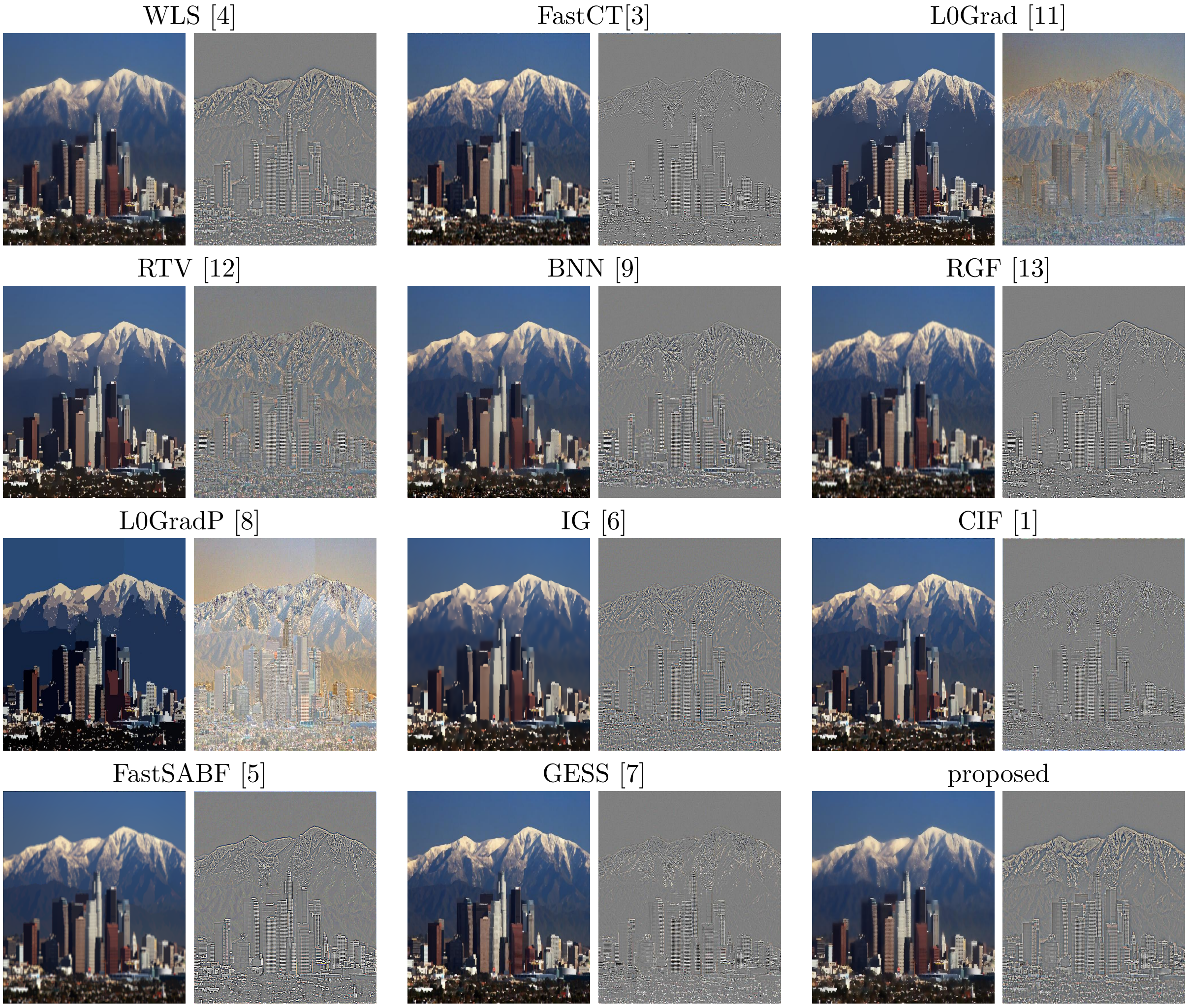


Figure 5: Comparison of the decomposition results using the proposed method and other methods.

3.2 Application: Edge detection

The original image and its edge map are shown in Figure 7. The cartoon images and the corresponding edge maps are shown in Figure 8.

Figure 7: The original “Love” image and its edge maps

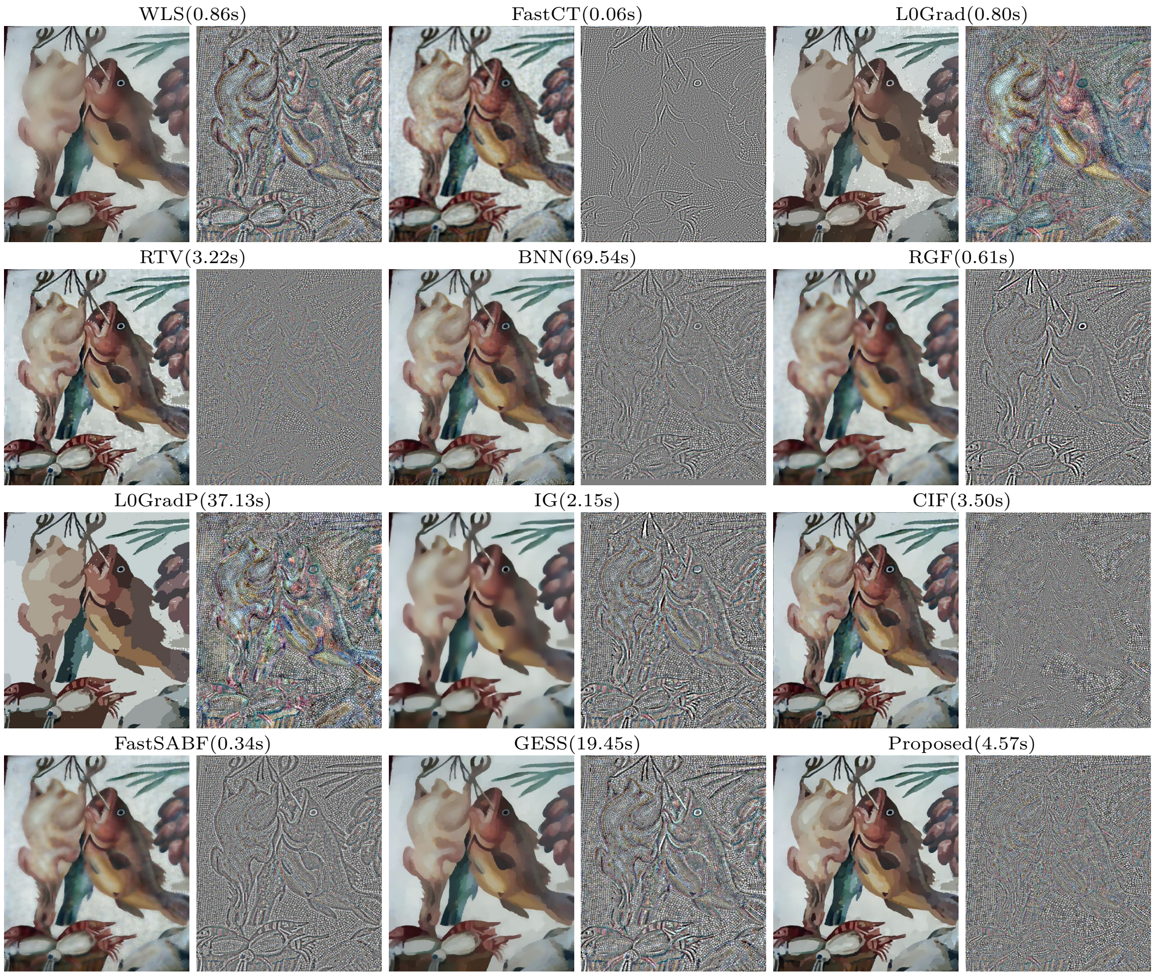


Figure 6: Comparison of the decomposition results using the proposed method and other methods on "Fish" image.

Figure 8: Comparison of the cartoon images obtained by different methods and the corresponding edge maps.

3.3 Application: Compression artifact removal

We evaluate the performance in removing compression artifacts from two images: "teddy" and "elephant". The uncompressed original images and their JPEG compressed image with a compression quality factor of 10% are shown in Figure 9. We show the visual comparison for different methods in Figure 10 and 11.

We also consider to remove compression artifacts from two compressed images: "watermelon" and "bus" shown in Figure 12. We don't have the background images. We show the visual comparison for different methods in Figure 13 and 14.

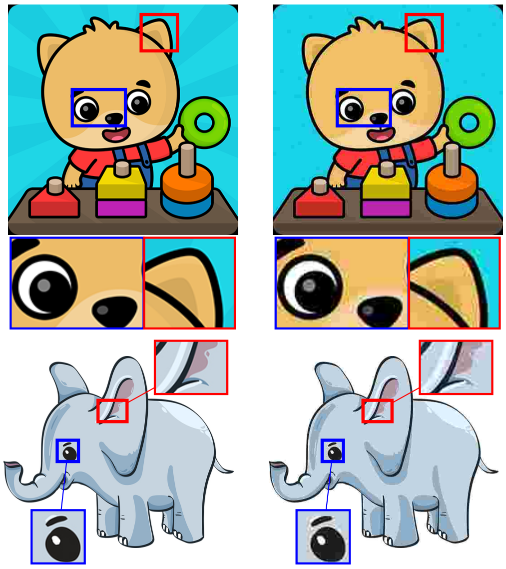


Figure 9: JPEG compression artifact removal. From left to right: the original high-quality image, compressed low-quality image with compression quality factor is 10%.

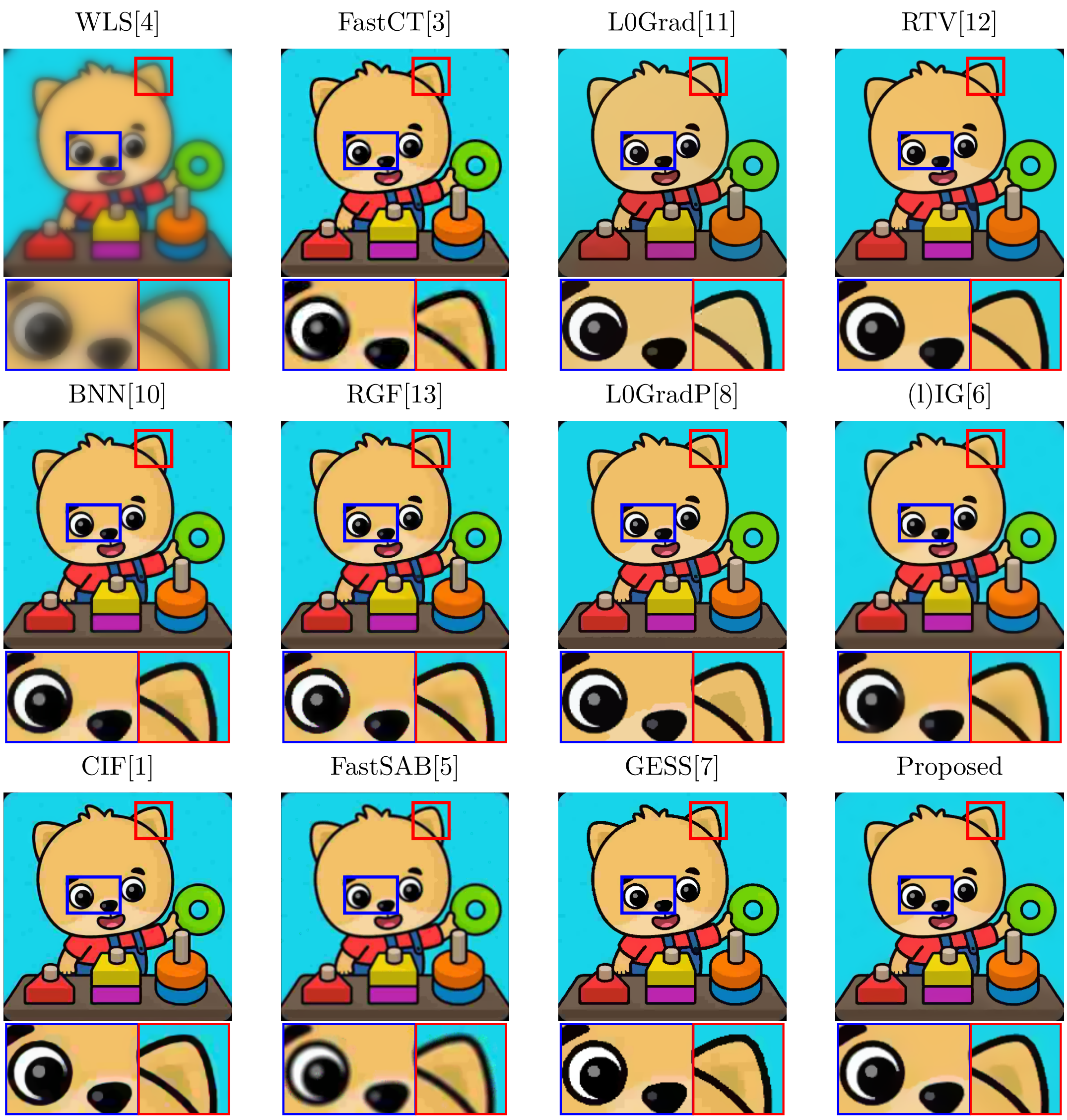


Figure 10: Comparison of JPEG compression artifact removal.

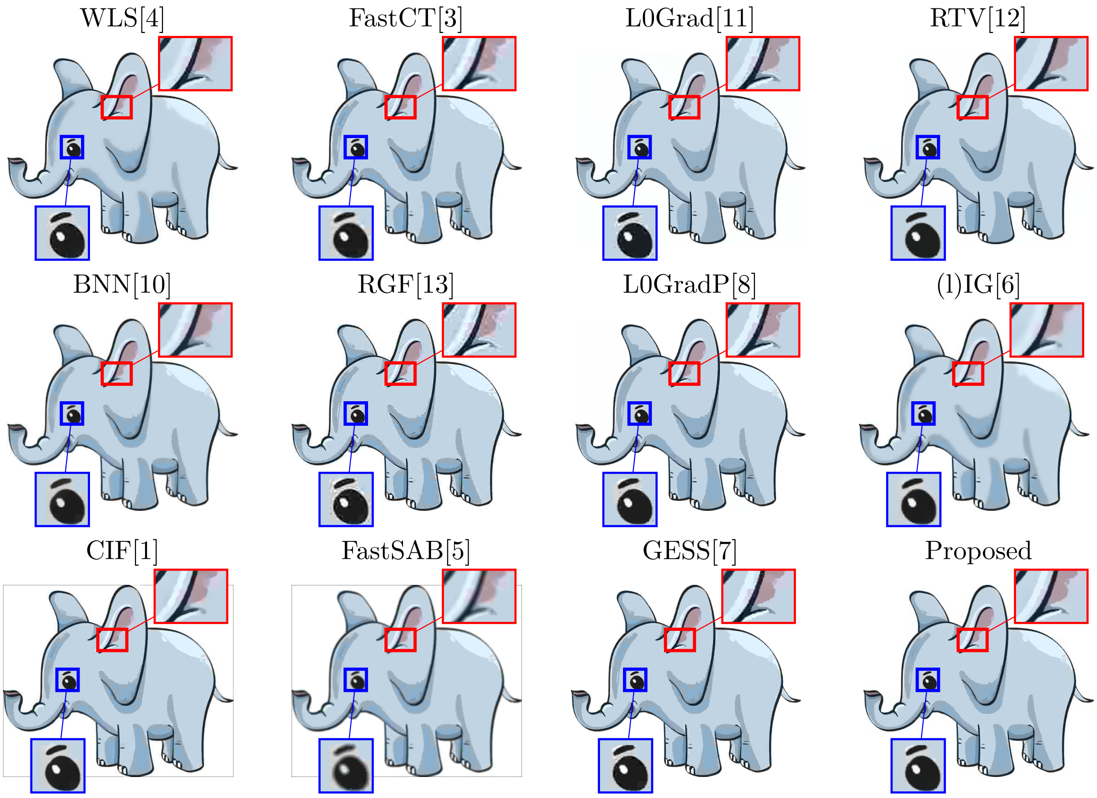


Figure 11: Comparison of JPEG compression artifact removal.

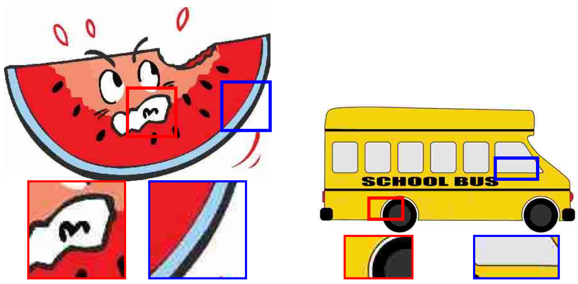


Figure 12: JPEG compression images.

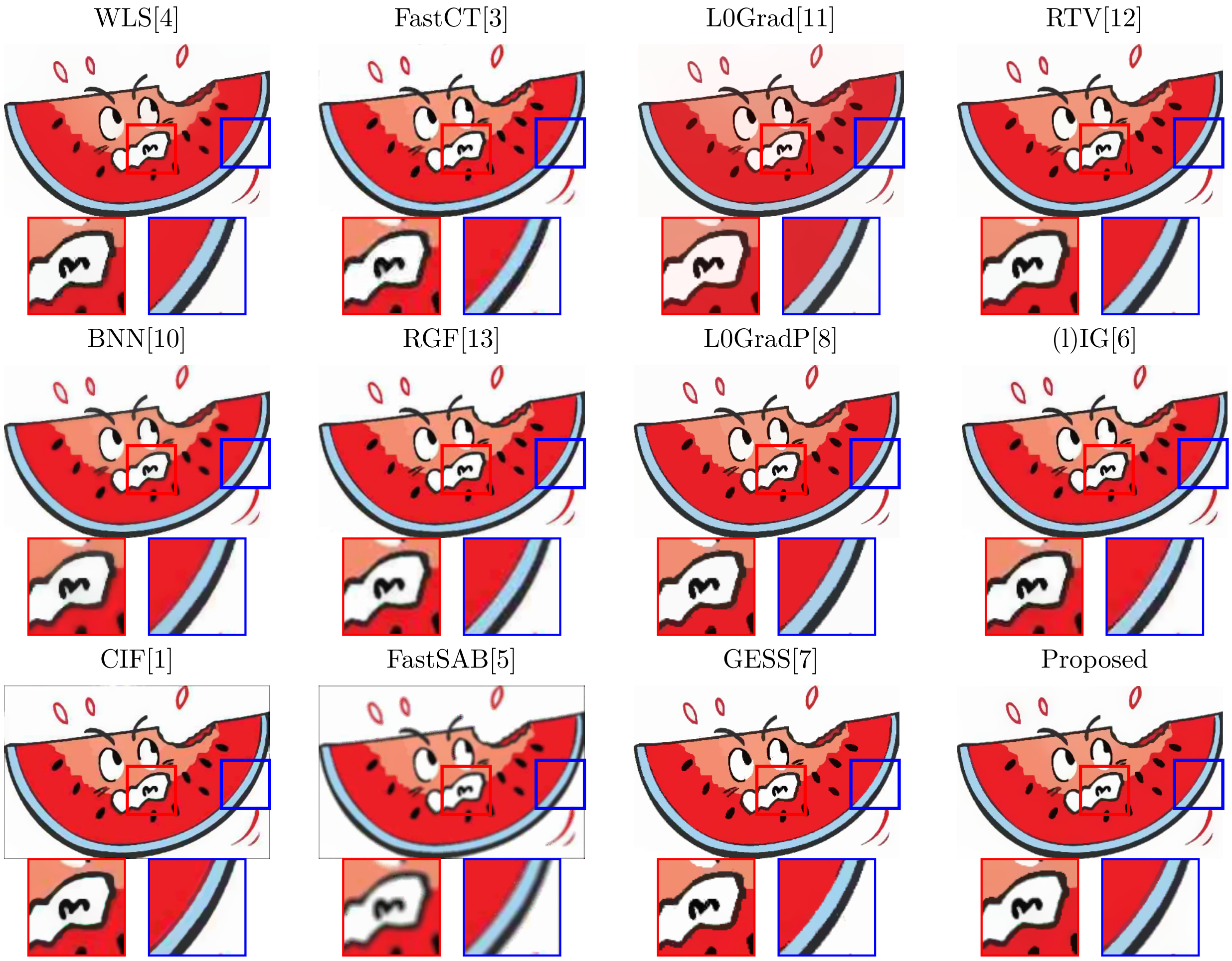


Figure 13: Comparison of JPEG compression artifact removal.

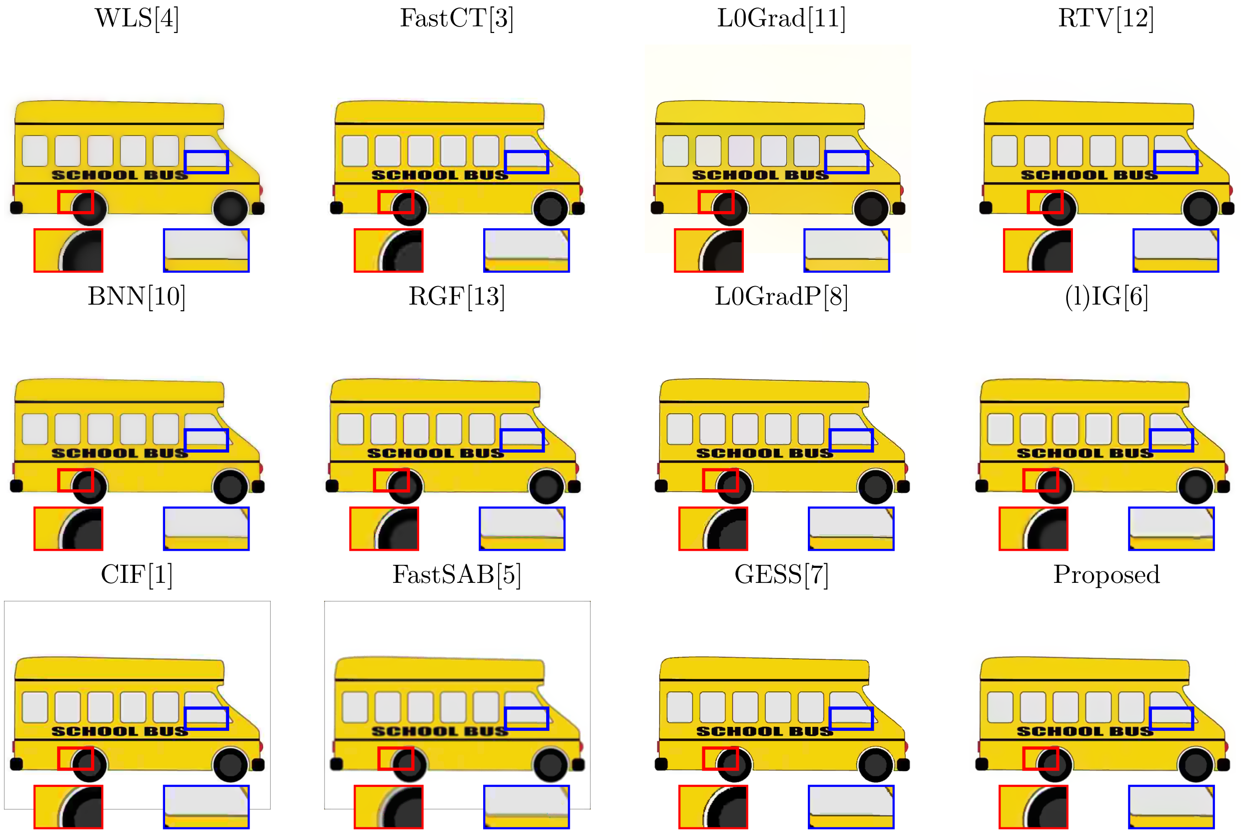


Figure 14: Comparison of JPEG compression artifact removal.

3.4 Application: Pencil sketching

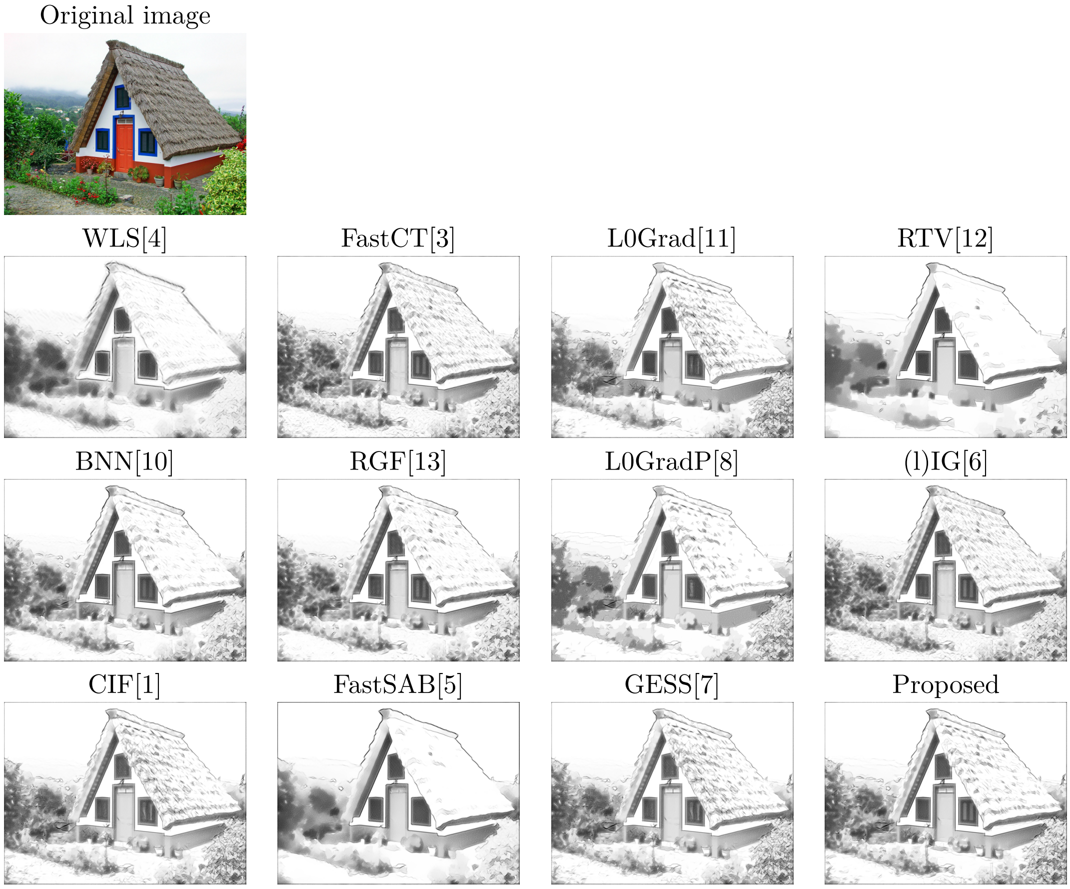


Figure 15: Pencil sketching.

3.5 Application: Image Sharpening

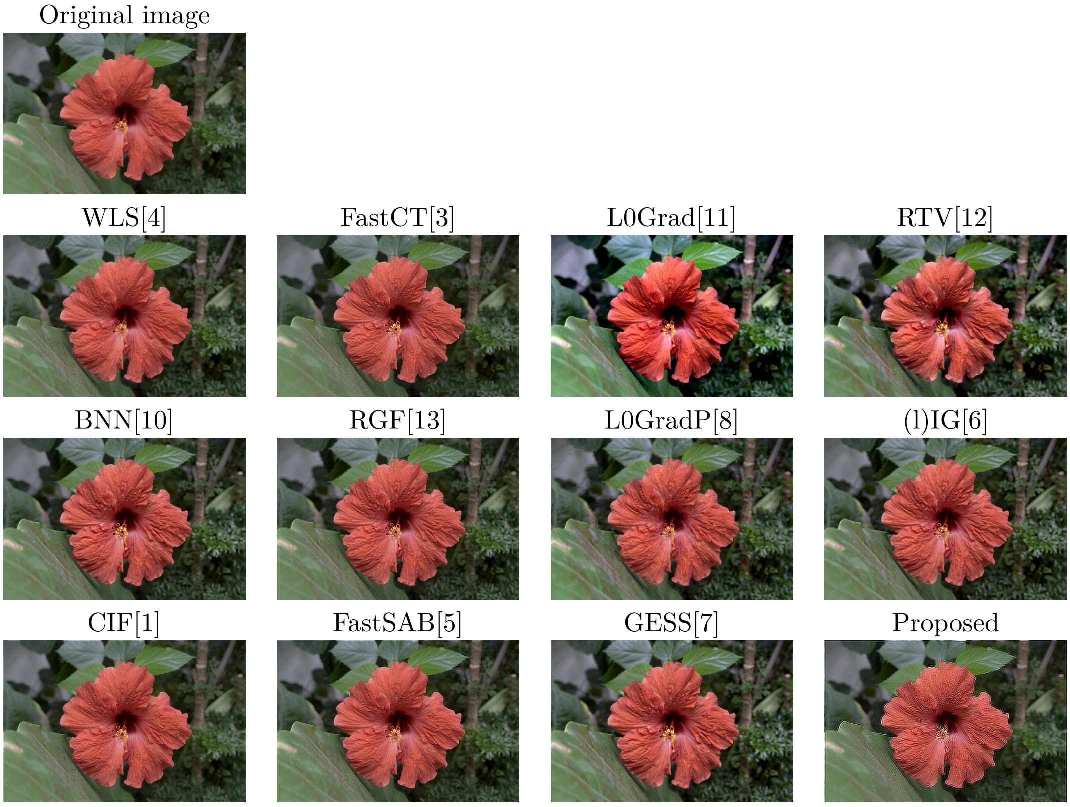


Figure 16: Image Sharpening.

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