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Lecture 2 - Calibration of interest rate models and optimization

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Outline

Introduction to financial instruments

Introduction to interest rate models

Nonlinear optimization

Introduction to financial instruments

The bank account:

The value at time t of the bank account B_t is defined by

$$dB_t = r_t B_t dt$$

$$\Rightarrow$$

$$B_t = B_0 e^{\int_0^t r_s ds},$$

where r_t is a short interest rate, governed by a short rate model.

Zero-coupon bond:

A zero-coupon bond $Z(t, T)$ with time of maturity T is a contract that guarantees the holder 1 unit of currency at time T.

The value of this contract at time $t \leq T$ is

$$Z(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left(e^{-\int_t^T r_s ds} \right). \tag{1}$$

Specifically, $Z(T, T) = 1$.

The value of the bank account $B_t = B_0 e^{\int_0^t r_s ds}$ at time t is computed from historical values of the interest rate r_s, $0 \leq s \leq t$.

The value of the zero-coupon bond $Z(t, T) = \mathbb{E}_t^{\mathbb{Q}} \left(e^{-\int_t^T r_s ds F} \right)$ at time t with time of maturity T is determined by the expectations on the interest rate r_s, $t \leq s \leq T$.

We will soon look at some interest rate models.

Day-count convention

There are different day-count conventions existing to account for different number of days of the months etc.

We will let $\tau(t_1, t_2)$, $t_1 < t_2$ represent the time-difference between t_1 and t_2 when the actual day-count convention is taken into account.

Example:

Actual/360 $\Rightarrow \tau(t, T) = \frac{T-t \text{ in days}}{360}$.

Simply-compunded spot interest rate

A simply-compounded spot interest rate $L(t, T)$ with time of maturity T at time t is defined by

$$L(t, T) = \frac{1 - Z(t, T)}{\tau(t, T)Z(t, T)}. \tag{2}$$

Rearranging of (2) gives

$$Z(t, T)(1 + L(t, T)\tau(t, T)) = 1,$$

i.e. the simply-compounded interest rate is the constant interest rate at time t, if the interest rate is paid out at time T.

LIBOR

LIBOR – The London Interbank Offered Rate – is a simply-compunded spot interest rate.

LIBOR is the average interest rate estimated by leading banks in London that the average leading bank would be charged if borrowing from other banks.

LIBOR rates are calculated for 5 currencies and 7 different maturities ranging from 1 day to 1 year.

For most currencies the day-count convention used is Actual/360.

STIBOR and more

STIBOR – The Stockholm Interbank Offered Rate is the average interest rate that SEB, Nordea, Svenska Handelsbanken, Swedbank och Danske Bank would charge if borrowing from each other.

STIBOR rates are calculated for 6 different maturities ranging from 1 day to 6 months and the day-count convention used is Actual/360.

Also other IBOR rates exist, e.g. CIBOR (Copenhagen Interbank Offered Rate).

LIBOR, STIBOR or other IBOR rates can be used to calibrate the interest models that we shall soon define.

Swap rates

Swap rates can also be used in the calibration of interest rate models. The interest rate swap prices are related to the zero-coupon prices as

$$s(t,T) = \frac{1 - Z(t,T)}{\sum_{j=1}^{N} \tau(T_{j-1},T_j) Z(t,T_j)} \tag{3}$$

where T_j, $j = 1, \ldots, N$ are reference dates with $T_N = T$ denoting the maturity of the swap.

Example: At $t = 0$, for a swap with maturity $T = 1$ year, and reference dates T_j that are 3 months apart, we have

$$\tau(T_{j-1},T_j) \equiv \Delta = 0.25,$$

$$T_0 = t = 0,\, T_1 = 0.25,\, T_2 = 0.5,\, T_3 = 0.75,\, T_4 = T = 1.$$

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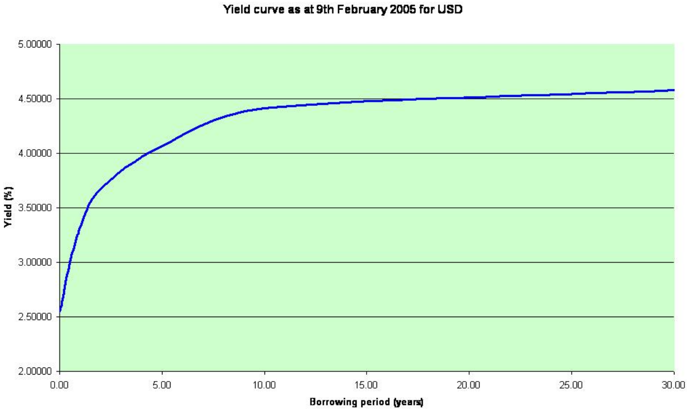
Yield curve

The yield curve is showing several yields or interest rates across the time to maturity (the "term").

This is also refered to as the *term structure of interest rates*.



Introduction to interest rate models

Vasicek model:

$$dr_t = \kappa(\phi - r_t)dt + \sigma dW_t \tag{4}$$

- ϕ is the long term mean of the short rate,
- κ is the mean reversion rate,
- σ is the volatility of the interest rate.

Note that the volatility is equally large independent of r_t.

This is a one factor short rate model.

For the Vasicek model it holds

$$
\begin{aligned}
Z(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\right) = e^{A(t,T)-B(t,T)r_t},\\
B(t,T) &= \frac{1-e^{-\kappa\tau(t,T)}}{\kappa},\\
A(t,T) &= (\phi - \tfrac{\sigma^2}{2\kappa^2})\left(B(t,T)-\tau(t,T)\right) - \tfrac{\sigma^2}{4\kappa}B^2(t,T).
\end{aligned}
$$

The parameters in this model are $\theta = \{\kappa, \phi, \sigma, r_t\}$.

From e.g. IBOR rates and swap rates, we can calibrate the model-parameters to the observed rates.

Cox – Ingersoll – Ross (CIR) model:

$$dr_t = \kappa(\phi - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{5}$$

- ϕ is the long term mean of the short rate,
- κ is the mean reversion rate,
- σ is the volatility of the interest rate.

If the Feller condition $2\kappa\phi > \sigma^2$ holds, then $r_t > 0$ for the CIR model.

In the CIR-model, the size of the volatility is scaled with $\sqrt{r_t}$, i.e., the smaller r_t is, the smaller is the actual volatility.

For the CIR model it holds

$$
\begin{aligned}
Z(t,T) &= \mathbb{E}_t^{\mathbb{Q}}\left(e^{-\int_t^T r_s ds}\right) = e^{A(t,T)-B(t,T)r_t},\\
B(t,T) &= \frac{2}{\kappa + \gamma \coth(\gamma\tau(t,T)/2)},\\
A(t,T) &= \frac{2\kappa\phi}{\sigma^2}\ln\left(\frac{e^{\kappa(T-t)/2}}{\cosh(\gamma\tau(t,T)/2)+\frac{\kappa}{\gamma}\sinh(\gamma\tau(t,T)/2)}\right),\\
\gamma &= \sqrt{\kappa^2+2\sigma^2}.
\end{aligned}
$$

Again, the parameters are $\theta = \{\kappa, \phi, \sigma, r_t\}$.

For general models, no closed-form solutions exist for $Z(t,T)$.

Other one-factor models:

Ho-Lee

$$dr_t = \phi(t)dt + \sigma dW_t \tag{6}$$

Hull-White

$$dr_t = \kappa(\phi(t) - r_t)dt + \sigma dW_t \tag{7}$$

Also multi-factor short rate models exist such as multi-factor Vasicek and multi-factor CIR.

Formulation of the calibration problem

When calibrating the short rate models to, e.g., IBOR rates and swap rates, we compute the parameters θ from

$$\inf_{\theta \in \Theta} \left(\sum_{i=1}^{N_1} H(L_i^{\theta} - L_i^{M}) + \sum_{i=1}^{N_2} H(s_i^{\theta} - s_i^{M}) \right), \qquad (8)$$

where N_1 is the number of market observations of IBOR rates, and N_2 is the number of market observations of swap rates and Θ is the space for all possible parameter sets.

For the Vasicek model we have

$$
\begin{aligned}
L_i^\theta &= \frac{1 - Z(0, T_i)}{\tau(0, T_i) Z(0, T_i)}, \\
s_i^\theta &= \frac{1 - Z(0, T_{i,N})}{\sum_{j=1}^{N} \tau(T_{i,j-1}, T_{i,j}) Z(t, T_{i,j})} \\
Z(0, T_i) &= e^{A(0,T_i) - B(0,T_i) r_0}, \\
B(0, T_i) &= \frac{1 - e^{-\kappa \tau(0, T_i)}}{\kappa}, \\
A(0, T_i) &= \left(\phi - \frac{\sigma^2}{2\kappa^2}\right)\left(B(0, T_i) - \tau(0, T_i)\right) - \\
&\quad \frac{\sigma^2}{4\kappa} B^2(0, T_i),
\end{aligned}
$$

and similarily for the CIR model.

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Nonlinear optimization

For simplicity of notation we rewrite Equation (8) as

$$\inf_{\bar{x}} f(\bar{x}) \quad , \quad \bar{x} = (x_1, \ldots, x_N). \tag{9}$$

We will here describe the

- Nelder-Mead simplex method
- Quasi-Newton method

to solve (9).

The Nelder-Mead simplex method

The algorithm uses a simplex of $n+1$ points for an n-dimensional vector $\bar{x}$.

Let x_i denote the list of points in the current simplex, $i = 1, \dots, n+1$.

1. Sort the vertex values $f(x_1) \leq f(x_2) \leq \dots \leq f(x_{n+1})$.
2. Calculate the centroid x_0 of the n best points $x_0 = (\sum_{i=1}^{n} x_i)/n$.
3. Generate the *reflected* point $x_r = x_0 + \alpha(x_0 - x_{n+1})$. *(In the direction away from the worst point.)*

4. If $f(x_1) \leq f(x_r) \leq f(x_n)$ then replace x_{n+1} with x_r and return to 1. *(New point is neither worst nor best.)*
5. If $f(x_r) < f(x_1)$ then compute the *expanded* point $x_e = x_0 + \gamma(x_0 - x_{n+1})$. *(New point best, go further.)* If $f(x_e) < f(x_r)$ then replace x_{n+1} with x_e, otherwise replace x_{n+1} with x_r and return to 1. *(Choose one.)*
6. If $f(x_r) \geq f(x_n)$ then compute the *contracted* point $x_c = x_0 + \rho(x_0 - x_{n+1})$. *(New point bad, go shorter.)* If $f(x_c) < f(x_{n+1})$ then replace x_{n+1} with x_c and return to 1.
7. Else shrink the simplex by $x_i = x_1 + \sigma(x_i - x_1)$, $i = 2, \ldots, n+1$ and return to 1. *(Shrink towards best corner.)*

A quasi-Newton method aims at finding the values $\bar{x}$ for which $\nabla f = 0$.

A Taylor-expansion of $f(\bar{x})$ around an iterate $\bar{x}_k$ gives

$$f(\bar{x}_k + \Delta x) \approx f(\bar{x}) + \Delta x^T \nabla f(\bar{x}_k) + \frac{1}{2}\Delta x^T B \Delta x \quad (10)$$

where B is an approximation to the Hessian matrix of f.

From (10) we get

$$\nabla f(\bar{x}_k + \Delta x) \approx \nabla f(\bar{x}_k) + B\Delta x$$

and setting this gradient to 0 we obtain

$$\Delta x = -B^{-1}\nabla f(x_k).$$

Nonlinear optimization in MATLAB

- A variant of the Nelder-Mead simplex method is implemented in `fminsearch`.
- `fminsearchcon` that allows for bounds on the parameters is available for download at `www.mathworks.com` .
- The Broyden–Fletcher–Goldfarb–Shanno (BFGS) algorithm – one variant of quasi-Newton – with cubic line search is implemented in `fminunc`.