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Abstract

This paper describes a maximum likelihood method for estimating the parameters of Heston's model of stochastic volatility using data on an underlying market index and the prices of options written on that index. Parameters of the physical measure (associated with the index) and the parameters of the risk-neutral measure (associated with the options) are identified including the equity and volatility risk premia. The estimation is implemented using a particle filter. The computational load of this estimation method, which previously has been prohibitive, is managed by the effective use of parallel computing using Graphical Processing Units. A byproduct of this focus on easing the computational burden is the development of a simplification of the closed-form approximation used to price European options in Heston's model. The efficacy of the filter is demonstrated under simulation and an empirical investigation of the fit of the model to the S&P 500 Index is undertaken. All the parameters of the model are reliably estimated and, in contrast to previous work, the volatility premium is well estimated and found to be significant.

1   Introduction

This paper describes and implements a maximum likelihood method for estimating the parameters of a stochastic volatility model by using time series data on index returns and cross section data on options written on the index. Specifically the data set comprises the S&P500 index from 2 January 1990 to 30 June 2012 (4,459,751 observations over 5672 trading days) and a total of 600,764 observations of traded European call and put options written on the index over this period. The fundamental advantage of this approach is that all the parameters of the model, including those characterising the equity and volatility premia, can be reliably identified in a way that maintains the internal consistency of the physical and risk-neutral measures.

The estimation procedure is described and applied in the context of Heston's model of stochastic volatility (Heston, 1993) The choice of Heston's model is motivated by the fact that it has a closed-form expression for the characteristic function of its transitional probability density function from which options can be efficiently priced, a feature of Heston's model that has received considerable attention in the literature (see, for example, Bakshi, Cao and Chen 1997). It should be stressed at the outset that the Heston model is used only as a specific example to allow the econometric methodology to be fully developed. The proposed method itself is, however, not limited to any particular model and the extension to models involving jumps is a matter of detail alone and requires no further significant conceptual development.

In a time-series context, fitting stochastic volatility models to index returns is a well developed field of research. For example, Aït-Sahalia and Kimmel (2007; 2010) develop closed-form approximations to the log-likelihood function; MCMC methods are used by Eraker (2001, 2004), Eraker, Johannes and Polson (2003) Jacquier, Polson and Rossi (2004); and filtering methods are used by Bates (1996), Johannes, Polson and Stroud (2009), Christoffersen, Jacobs and Mimouni (2010) and Hurn, Lindsay and McClelland (2012).

Similarly, the use of both time series data on the underlying's price as well as information on unobserved volatility encoded in options prices to identify the dynamics of stochastic volatility models is not new. In a non-parametric setting, Aït-Sahalia, Wang and Yared (2001) compare the risk-neutral densities estimated separately from spot prices and option prices. In a parametric environment there are two approaches. The first approach uses both

returns and option prices or information derived from option prices over time (Chernov and Ghysels, 2000; Pan, 2002; Jones, 2003; Eraker, 2004). The advantage of such an approach is that it appropriately weights each data source, simultaneously addressing a model's ability to fit the time series of returns and the cross section of options. However, there is a crucial drawback to this approach in the way in which it has been implemented to date, namely, that the computational burden severely constrains how much and what type of data can be used. The usual compromise it to limit either the time-span of the returns index or the cross section of the options or both. The second approach strikes a compromise between the competing constraints of computational feasibility and model consistency. The method is a two-stage method which uses estimates of the physical parameters of the model from time-series data and estimates the risk-neutral parameters from the cross section of options, where the restrictions imposed by moving between the two measures are sometimes imposed (Broadie, Chernov and Johannes, 2007; Christoffersen, Jacobs and Mimouni, 2010).

The ability to combine a long time series of returns data with all available options is, however, an area which is less well developed. Typically between one and three at-the-money options are used in studies in which the physical model and the risk-neutral model are estimated simultaneously. This is not surprising given that the computationally demanding nature of the task. The estimation procedure is based on a particle filter in which each particle is a single realisation of the unobserved state (volatility) and integration over the unobserved states is achieved by Monte Carlo integration. The filter described here is closest in spirit to that described by Johannes, Polson and Stroud (2009). The computational complexity of this approach is driven by the requirement that each particle in the filter must be used to price all the required options. Consequently each function evaluation of a search algorithm involves the calculation of around 1 billion model option prices when estimating Heston's model with 18,000 particles[1] and observations of 10 option prices on each day of the entire data set. On a desktop PC a realistic outcome is one function evaluation per day, which implies one full model estimation every 6 months thereby rendering any simulation exercise infeasible on a desktop machine.

The innovations that allow this kind of computation to be reduced to a matter of hours

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[1]The limitation of the algorithm is not on the number of particles but rather on the memory requirements of the host to hold the random numbers required by the estimation process. As many as 50,000 particles were used in experiments without any serious degradation in the speed of the algorithm other than an increase in computation time in rough proportion to the number of particles.

as opposed to months stems from the fact that the particles are independent draws from the distribution of the latent variable which means that a particle filtering algorithm lends itself to parallel computation. Indeed, in the parlance of computer theorists the problem is "embarrassingly parallel". In this work, the computational burden of each filtering step is dealt with by parallelisation on a graphical processing unit (GPU). Even so, to be a practical procedure an improvement in computational speed of several orders of magnitude is required. Accordingly, the calculations were performed on two supercomputers[2] both running running CUDA C version 4.1.

The fundamental contribution of this paper is to demonstrate that parallel computing using GPUs renders estimation of stochastic volatility models by means of a particle filter, which uses large numbers of observations on both the underlying asset and options written on the asset, a task that is now well within the scope of existing hardware. Not only can the parameters of both the physical model and the risk-neutral model be estimated but simulation studies designed to explore the efficacy of the particle filter are also possible within a reasonable time frame. The results presented here indicate that particle filtering is a powerful estimation method that offers substantial advances over existing methods, particularly when options prices are used.

The method is illustrated using the S&P 500 Index from 2 January 1990 to 30 June 2012 and all options written on the index over that period, subject to a set of internal consistency conditions being satisfied. All the parameters of the Heston's model of stochastic volatility are estimated with good precision. The most interesting result to emerge is that the volatility premium of the risk-neutral model is found to be statistically significant and that the risk-neutral dynamics are not explosive. This result contrasts sharply with existing evidence on the volatility premium (Pan, 2002; Jones, 2003; Eraker, 2004; Broadie, Chernov and Johannes, 2007).

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2 The Stochastic Volatility Model

Let $W_1(t)$ and $W_2(t)$ be independent Wiener processes with respective differentials dW_1 and dW_2. The prototypical stochastic volatility model proposed by Heston (1993), extended to include an equity premium, posits that the index level, S, and the volatility, V, evolve according to the stochastic differential equations

$$\begin{aligned} \frac{dS}{S} &= \left(r - q + \xi_S V\right) dt + \sqrt{V}\left(\sqrt{1 - \rho^2}\, dW_1 + \rho\, dW_2\right), \\ dV &= \kappa_{\mathrm{P}}(\gamma_{\mathrm{P}} - V)\, dt + \sigma\sqrt{V}\, dW_2, \end{aligned} \tag{1}$$

where r is the risk-free rate, q is the dividend-price ratio, ξ_S is a constant parameter governing the equity premium, σ governs the diffusion of volatility and ρ is the instantaneous correlation between innovations in returns and volatility. The quantities κ_{P} and γ_{P} denote respectively the rate of mean reversion of volatility and the long-run level of volatility, where the subscript is used to indicate that these parameters are associated with the physical measure, $\mathbb{P}$. Evidently, the expectation of returns under the physical measure satisfies

$$\mathbb{E}\left[\frac{dS}{S}\right] + (q - r)\, dt = \xi_S\, V\, dt$$

so that $\xi_S V$ is an additional premium earned by the asset in excess of the risk free rate of interest net of dividends. The equity premium is regarded as compensation for accepting the diffusive risk associated with the innovations dW_1 and dW_2. Economic reasoning indicates that these compensations take the respective forms $\lambda_1\sqrt{V}$ and $\lambda_2\sqrt{V}$ per unit of dW_1 and dW_2 risk, where the values of λ_1 and λ_2 are to be estimated from real data. The equity premium $\xi_S V$ is simply the sum of the compensations for dW_1 and dW_2 diffusive risks, and consequently

$$\xi_S V \equiv \lambda_1\sqrt{V}\left(\sqrt{1 - \rho^2}\sqrt{V}\right) + \lambda_2\sqrt{V}\left(\rho\sqrt{V}\right)$$

from which it follows immediately that

$$\xi_S = \lambda_1\sqrt{1 - \rho^2} + \lambda_2\,\rho\,. \tag{2}$$

While the value of ξ_S can be estimated from asset prices alone, clearly the partitioning of ξ_S between risk associated with dW_1 and dW_2 cannot be undertaken. What is needed is another class of observation, one which embodies another premium that is constructed from a different combination of λ_1 and λ_2. Observations of option prices meet this objective because they

are priced under the risk-neutral measure, $\mathbb{Q}$, and incorporate a volatility premium which depends of a different combination of the parameters λ_1 and λ_2. The risk-neutral model corresponding to equations (1) is given by

$$
\begin{aligned}
\frac{dS}{S} &= (r - q)\, dt + \sqrt{V}\big(\sqrt{1 - \rho^2}\, dW_1 + \rho\, dW_2\big), \\
dV &= \kappa_{\mathrm{P}}(\gamma_{\mathrm{P}} - V)\, dt - \xi_V V\, dt + \sigma\sqrt{V}\, dW_2\,,
\end{aligned}
\tag{3}
$$

where $\xi_V V = \lambda_2 \sigma V$ is known as the volatility premium. The volatility equation in (3) is usually written as

$$
dV = \kappa_{\mathrm{Q}}(\gamma_{\mathrm{Q}} - V)\, dt + \sigma\sqrt{V}\, dW_2\,,
\tag{4}
$$

where

$$
\kappa_{\mathrm{Q}} = \kappa_{\mathrm{P}} + \xi_V = \kappa_{\mathrm{P}} + \lambda_2 \sigma\,, \qquad \gamma_{\mathrm{Q}} = \frac{\kappa_{\mathrm{P}} \gamma_{\mathrm{P}}}{\kappa_{\mathrm{Q}}}\,.
$$

It is anticipated that $\xi_V < 0$ (and $\lambda_2 < 0$) so that $\gamma_{\mathrm{Q}} > \gamma_{\mathrm{P}}$, *i.e.* the risk-neutral measure captures the risk-averse nature of the physical world by increasing the mean value of risk-neutral volatility.

It is important to note that the absolute continuity requirement implies that certain model parameters, or combinations of parameters, are the same under both measures. This is a mild but important economic restriction on the parameters. A comparison of the equations governing the evolution of S and V under the measures $\mathbb{P}$ and $\mathbb{Q}$ indicates that σ and the product $\kappa\gamma$ take the same values under both measures. This implies that these parameters can be estimated using either equity index returns or option prices, but that the estimates should be the same from either data source. One way to impose this theoretical restriction is to constrain these parameters to be equal under both measures, as advocated by Bates (2000). We impose this constraint and refer to it as time-series consistency.

3   Parameter Estimation

Let $X_0, \cdots, X_T$ be observations of a system at discrete times $t_0, \cdots, t_T$ respectively, and let $Z_k = \{X_j\}_{j=0}^{j=k}$ denote the full history of observed data up to and including the time t_k. The general parameter estimation problem may be represented succinctly in the form

$$
\widehat{\theta} \equiv \arg\max_{\theta} \frac{1}{T}\Big(\log f(Z_0) + \sum_{k=1}^{T} \log f(X_k \mid Z_{k-1}) \Big).
\tag{5}
$$

Consider first the case in which $X_k = (S_k, V_k, H_k^{(1)} \cdots, H_k^{(M)})$, where S_k and V_k are respectively the index and volatility at time t_k and $H_k^{(1)}, \cdots, H_k^{(M)}$ are the prices of M options written on the index at that time. Because Heston's model is a Markovian system, then the history Z_{k-1} is captured by the state X_{k-1} and consequently $f(X_k \mid Z_{k-1})$ is just the value of the probability density function $f(X_k \mid X_{k-1})$. In particular, if $X_k = \{S_k, V_k\}$, then $f(X_k \mid X_{k-1})$ is the transitional probability density function $f_{\mathrm{P}}(X_k, \Delta_k \mid X_{k-1}; \theta)$ of the process $X = \{S, V\}$ under the physical measure for a transition of duration Δ_k starting at X_{k-1} and ending at X_k. The maximum likelihood estimate of θ is

$$\widehat{\theta} = \arg\max_{\theta} \frac{1}{T} \sum_{k=1}^{T} \log f_{\mathrm{P}}(X_k, \Delta_k \mid X_{k-1}; \theta) \tag{6}$$

in which no information has been associated with the initial state $X_0 = \{S_0, V_0\}$.

Although conceptually straightforward, the parameter estimation problem posed in (6) presents severe difficulties in practice. First and foremost among these is that f_{P} is not known in closed form even for simple multivariate affine problems, although the characteristic function of the density may be known in some cases. In all cases a full inversion of the characteristic function to obtain the complete density is a non-trivial numerical operation. Aït-Sahalia and Kimmel (2007) provide a close approximation for $\log f_{\mathrm{P}}$ which gives reasonable results.

3.1 Introducing Options Data

Suppose that X_k also includes observations of the prices $\{H_k^{(1)}, \cdots, H_k^{(M)})$ of M options and let $\{\widetilde{H}_k^{(1)}, \cdots, \widetilde{H}_k^{(M)}\}$ be the respective model prices of these options at time t_k. Because the system is fully observed each option price provides an independent piece of information which is incorporated within the maximum likelihood framework through the specification

$$f(X_k \mid X_{k-1}) = f_{\mathrm{P}}(X_k, \Delta_k \mid X_{k-1}; \theta) \prod_{j=1}^{M} g(H_k^{(j)} \mid \widetilde{H}_k^{(j)}; \theta),$$

where $f_{\mathrm{P}}(X_k, \Delta_k \mid X_{k-1}; \theta)$ is the usual transitional probability density function of the physical model associated with the transition of duration Δ_k from $\{S_{k-1}, V_{k-1}\}$ to $\{S_k, V_k\}$, and $g(H \mid \widetilde{H}; \theta)$ is the density of the observed option price H conditional on the model price $\widetilde{H}$. Whereas the contribution $\log f(Z_0)$ to the likelihood function was ignored in the absence of

observations of option prices, now this term contributes to the log-likelihood function via the information contained in the observed option prices at t_0. When index value, volatility and option prices are simultaneously observed, the maximum likelihood parameters estimates now satisfy

$$
\begin{aligned}
\widehat{\theta} \;=\; & \arg\max_{\theta} \frac{1}{T} \Big[\sum_{j=1}^{M} \log g(H_0^{(j)} \,|\, \widetilde{H}_0^{(j)}; \theta) \\
& + \sum_{k=1}^{T} \Big(\log f_{\mathrm{P}}(\mathbf{X}_k, \Delta_k \,|\, \mathbf{X}_{k-1}; \theta) + \sum_{j=1}^{M} \log g(H_k^{(j)} \,|\, \widetilde{H}_k^{(j)}; \theta) \Big) \Big],
\end{aligned}
\tag{7}
$$

in which all the parameters of the model $\theta = \{\rho, \kappa_{\mathrm{P}}, \gamma_{\mathrm{P}}, \sigma, \rho, \xi_S, \xi_V, \alpha\}$ are identifiable.

To complete the specification of the maximum likelihood estimation problem the distribution of pricing errors for options must be specified.[3] Technically there is an over-identification problem, since conditional on all state variables and parameters, option prices are known. To circumvent the possibility of singularity it is necessary to assume a pricing error which is either additive in the sense that the difference between the observed and model option price is a zero mean Gaussian deviate or multiplicative in the sense that it is the difference between the logarithms of the observed and model option prices that is a zero mean Gaussian deviate (Eraker, Johannes and Polson, 2003; Eraker, 2004; Forbes, Martin and Wright, 2007; Johannes and Polson, 2009).

In fact, there four at least possible different distributions of pricing errors that can be sensibly specified. As before, let H be the observed option price and $\widetilde{H}$ be the price of the option under the model. The four error specifications are as follows.

Error Specification 1 (ES1) This is an additive pricing error in which the observed price H is Gaussian distributed about the model prices $\widetilde{H}$ with a standard deviation proportional to the model price, that is

$$
H \sim \frac{1}{\sqrt{2\pi}\,\alpha\widetilde{H}} \exp\left[-\frac{\left(H - \widetilde{H}\right)^2}{2\alpha^2\widetilde{H}^2} \right].
\tag{8}
$$

In this specification α is a non-dimensional scaling factor.

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[3]This contrasts with approaches that minimize the squared deviations between model and market option prices. Christoffersen and Jacobs (2004) provide a detailed discussion of the objective function choice.

Error Specification 2 (ES2) The pricing error remains additive but now the observed price H is Gaussian distributed about the model prices $\widetilde{H}$ with constant standard deviation α, that is

$$H \sim \frac{1}{\sqrt{2\pi}\,\alpha} \exp\left[-\frac{\left(H - \widetilde{H}\right)^2}{2\alpha^2} \right].$$

(9)

This is the formulation favoured by Forbes, Martin and Wright (2007) in which α now represents a dollar value.

Error Specification 3 (ES3) This is a multiplicative pricing error specification in which $\log\left(H/\widetilde{H}\right)$ is Gaussian distributed with mean value zero and constant standard deviation α, that is

$$\log\frac{H}{\widetilde{H}} \sim \frac{1}{\sqrt{2\pi}\,\alpha} \exp\left[-\frac{\left(\log(H/\widetilde{H}\right)^2}{2\alpha^2} \right].$$

(10)

Error Specification 4 (ES4) This is again a multiplicative pricing error specification in which H is log-normally distributed with mean value $\widetilde{H}$ and parameter α, that is

$$H \sim \frac{1}{\sqrt{2\pi}\,\alpha H} \exp\left[-\frac{\left(\log(H/\widetilde{H}) + \alpha^2/2\right)^2}{2\alpha^2} \right].$$

(11)

The performance of the estimation method will be investigated for each of these pricing error distributions.

3.2 The Particle Filter

Hitherto it has been assumed that the system is fully observed, but that additional observations in the form of option prices may be available. In practice, however, the state $X = \{S, V\}$ is not fully observed because volatility V is a latent variable. Consider now the case in which the system retains its Markovian property but volatility V, a system variable, is not observed. Historical observations Z_k now provide information about both the historical behaviour of volatility, its current value V_k and the future evolution of index returns and option prices *i.e.* on the dependence of X_{k+1} upon Z_k. The procedure by which information in Z_{k-1} is "folded" recursively into the transitional probability density function $f(X_k \,|\, X_{k-1})$ in equation (5) is called a filtering rule. The complexity of the model in combination with the fact that observed option prices are strongly nonlinear functions of the underlying state variables means that the associated filtering procedure must be implemented numerically.

In overview the recursive filtering procedure advancing the calculation of the log-likelihood function from t_{k-1} to t_k proceeds in two steps taking as its starting point the filtered probability density function $f(V_{k-1} \mid Z_{k-1})$ of the unobserved volatility at time t_{k-1}. The steps are as follows.

1. Incorporate the observation X_k at time t_k into the filtered probability density function $f(V_{k-1} \mid Z_{k-1})$ of unobserved volatility at time t_{k-1} and compute the contribution $f(X_k \mid Z_{k-1})$ to the likelihood function from the observed data X_k at time t_k. This objective is achieved by first computing $f(X_k, V_k \mid Z_{k-1})$ as the value of the integral

$$f(X_k, V_k \mid Z_{k-1}) = \int_{\mathcal{V}} f(X_k \mid V_k, V_{k-1}) f(V_k \mid V_{k-1}) f(V_{k-1} \mid Z_{k-1}) \, dV_{k-1}, \qquad (12)$$

where $\mathcal{V}$ is the sample space of volatility. Thereafter, the contribution to the log-likelihood function at time t_k is

$$f(X_k \mid Z_{k-1}) = \int_{\mathcal{V}} f(X_k, V_k \mid Z_{k-1}) \, dV_k. \qquad (13)$$

2. Use Bayes Theorem to construct $f(V_k \mid Z_k)$, the updated filtered probability density function of the unobserved volatility at t_k. The filtered probability density function of unobserved volatility at t_k is given by

$$f(V_k \mid Z_k) = \frac{f(X_k, V_k \mid Z_{k-1})}{f(X_k \mid Z_{k-1})} \qquad (14)$$

thereby completing the second stage of the two-stage procedure.

Options data is introduced into the calculation through the recognition that the conditional probability density function $f(X_k \mid V_k, V_{k-1})$ appearing in integral (12) is the product of a contribution from the index value alone, namely $f(Y_k \mid V_k, V_{k-1})$ and option prices conditioned on the volatility V_k at time t_k. Specifically,

$$f(X_k \mid V_k, V_{k-1}) = f(Y_k \mid V_k, V_{k-1}) \prod_{j=1}^{M} f(H_j^{(k)} \mid V_k). \qquad (15)$$

The integrals (12 - 14) must be approximated numerically, a task which is achieved using the technique of particle filtering. Setting aside for the moment the question of initialisation, consider a cloud of M particles available at t_{k-1} in which each particle represents an

independent draw from the sample space of volatility (or in general from the space of latent variables). It is assumed that the particle cloud is an efficient summary of the distribution of the latent variables taking account of the information Z_{k-1}. In effect, the particle cloud is regarded as an efficient summary of the probability density function $f(X_k | Z_{k-1})$. The first step in the calculation of $f(X_k, V_k | Z_{k-1})$ is to use a numerical scheme to advance the solution of the physical model from the state $\{S_{k-1}, V_{k-1}\}$ at t_{k-1} in m small steps of size $\Delta t = (t_{k+1} - t_k)/m$. This procedure is repeated for each particle in the cloud, with each particle starting at the same index value S_{k-1} but a different value of V_{k-1}. In effect, this phase of the updating procedure constructs the transitional density $f(V_k | V_{k-1})$. The final stage in the computation of expression (12) calculates $f(X_k | V_k, V_{k-1})$ for each particle from the asset price S_k and option prices $H_k^{(1)}, \cdots, H_k^{(M)}$ using formula (15), $i.e.$

$$f(X_k | V_k, V_{k-1}) = f_{\mathrm{P}}(S_k | Z_{k-1}; \theta) \prod_{j=1}^{M} g(H_k^{(j)} | \widetilde{H}_k^{(j)}; \theta).$$

The density $f_{\mathrm{P}}(S_k | V_k, V_{k-1}; \theta)$ is estimated by the method suggested by Pedersen (1995) and Brandt and Santa-Clara (2002). In this approach the transitional probability density function $f_{\mathrm{P}}(S_k | V_k, V_{k-1}; \theta)$ is well approximated by a Gaussian probability density function with mean value and variance determined by the penultimate state of the integration procedure, namely a step of size Δt, (see, for example, Jensen and Poulsen, 2002). The contributions to the likelihood made by the options are calculated from expression (8 - 11) in which the model option price is determined from the index S_k and the volatility V_k, of course, a different value for each particle. Given θ, the value of $f(X_k, V_k | Z_{k-1})$ for the j-th particle is denoted by $\mathcal{L}_j(\theta)$.

The value of the integral (13) is then estimated from $\mathcal{L}_1(\theta), \cdots, \mathcal{L}_M(\theta)$ by Monte Carlo simulation taken over the cloud of particles at time t_{k-1}, $i.e.$ the contribution to the likelihood at t_k for the observations X_k is

$$f(X_k | Z_{k-1}) = \sum_{j=1}^{M} \mathcal{L}_j(\theta)$$

4 Computing Option Prices

It is well known that Heston's model of stochastic volatility admits close-form approximations for pricing European options. The primary computational effort when calculating a

model option price is largely expended in evaluating exponential, logarithmic and trigono-metric functions, *i.e.* transcendental functions. While the calculation of the exponential and logarithmic functions is unavoidable, the conventional calculation of the model price requires the evaluation of four trigonometric functions for each frequency. This section develops a re-arrangement of the pricing formula that requires the computation of only two trigonometric functions. While this may seem like a trivial theoretical development, its impact is more substantial in practice given the vast number of times that options must be priced in this estimation problem.

In order to price options, the marginal distribution of asset prices under the risk neutral measure is required, but a closed form expression for this marginal density is not available even for the simple affine Heston model. What is available, however, is a procedure for calculating the characteristic function of the transitional probability density function of the Heston model with respect to current values of the state variables. The procedure is based on the solution of the partial differential equation that is constructed by taking the Fourier transform of the backward Kolmogorov equation with respect to the forward state variables. In affine models such as Heston's model, the equation derived in this way from the backward Kolmogorov equation has a semi-closed form solution for the characteristic function (Fourier transform) of the transitional density in the sense that the generic form of the characteristic function can be written down in closed-form although certain features of the closed for expression must be determined by the numerical solution of differential equations. The beauty of this approach is that in general if the characteristic function of the transitional density is known then it can be used to price options without the need to compute the transitional density itself.

To appreciate how options may be priced from knowledge of the characteristic function of the transitional probability density finction, recall that the characteristic function of the risk-neutral Heston model is defined by the formula

$$F_{\mathbb{Q}}(Y, V, t, \omega_y, \omega_v; \theta) = \int_{\mathbb{R}^2} f_{\mathbb{Q}}(Y, V, t \,|\, y, v; \theta) e^{i\omega_y y + \omega_v v} \, dy \, dv \,. \tag{16}$$

In general $F_{\mathbb{Q}}(Y, V, t, \omega_y, \omega_v; \theta)$ satisfies a backward Kolmogorov equation. In the particular case of Heston's model it can be shown that the solution of the backward Kolmogorov

equation corresponding to the initial state $X = (Y, V)$ has the closed form expression

$$F_Q(Y, V, t, \omega_y, \omega_v; \theta) = \exp\left[i\omega_y\left(Y + (r - \xi)t - \frac{\rho\kappa\gamma t}{\sigma}\right) + \frac{\kappa^2\gamma t}{\sigma^2}\right.$$
$$+ V\left(\frac{i\omega_v\eta\sinh(\eta t/2) - (i\omega_v(K - i\rho\sigma\omega_y) + i\omega_y + \omega_y^2)\cosh(\eta t/2)}{(K - i\rho\sigma\omega_y - i\omega_v\sigma^2)\sinh(\eta t/2) + \eta\cosh(\eta t/2)}\right)\right] \qquad (17)$$
$$\times\left(\frac{\eta}{(K - i\rho\sigma\omega_y - i\omega_v\sigma^2)\sinh(\eta t/2) + \eta\cosh(\eta t/2)}\right)^{2\kappa\gamma/\sigma^2}$$

in which $\eta = \left[\kappa^2 + (1 - \rho^2)\sigma^2\omega^2 + i\sigma\omega_y(\sigma - 2\rho\kappa)\right]^{1/2}$. In principle the underlying transitional probability density function may be reconstructed from expression (17) using the inverse Fourier transform. Interestingly it is possible to analytically invert $F_Q(Y, V, t, \omega_y, \omega_v; \theta)$ in the frequency ω_v but not for the frequency ω_y. However with respect to the pricing of call option contracts, it is not strictly necessary to know the transitional density function itself, but rather an integral of the transitional probability density function. The expected payoff from a call option contract with maturity T and strike price K is

$$\int_K^\infty (S - K) \int_{-\infty}^\infty \widetilde{f}_Q(S_0, V, T \mid S, v; \theta)\, dS\, dv\,,$$

where $\widetilde{f}_Q(S_0, V, T \mid S, v; \theta)$ is the transitional density of asset price and volatility at maturity. When expressed in terms of the transitional probability of the logarithm of the ratio of asset price to spot price, the expected payoff from a call option contract with maturity T and strike price K is

$$S_0 \int_{\log\xi}^\infty (e^y - \xi) \int_{-\infty}^\infty f_Q(0, V, T \mid y, v; \theta)\, dy\, dv\,. \qquad (18)$$

where $\xi = K/S_0$ and it has been noted that the initial value of $Y = \log(S/S_0)$ is zero. The efficacy of the approach based on the characteristic function relies on the observation that $F(Y, V, T, \omega_y, 0; \theta)$ is the characteristic function of the marginal density of the logarithm of asset price, namely

$$F(Y, V, T, \omega_y, 0; \theta) = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(Y, V, T \mid y, v; \theta)\, dv\right) e^{i\omega_y y}\, dy\,. \qquad (19)$$

In order to manage data comprising observations of large numbers of daily option prices over many years we need the rapid evaluation of expression (18), ideally via a closed-form approximation for $f_Q(0, V, T \mid y, v; \theta)$. This approximation is based on the recognition that the transitional probability density function can be efficiently treated as a function of compact support. In the analysis to follow it will be assumed that this support is the interval

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$[S_0 e^{-\beta}, S_0 e^{\beta}]$, where S_0 is the spot price. Consequently the transitional probability density function of the logarithm of the ratio of asset price to spot price has support $[-\beta, \beta]$ over $\mathbb{R}$, and therefore the integrand of identity (19) is also efficiently approximated as a function of compact support over the interval $[-\beta, \beta]$. With this approximation in place,

$$\int_{-\infty}^{\infty} f(Y, V, T \,|\, y, v; \theta) \, dv = \sum_{k=-\infty}^{\infty} c_k \, e^{-k\pi i y/\beta} \,, \tag{20}$$

where the Fourier coefficients are given by the formula

$$c_k \approx \frac{1}{2\beta} \int_{-\beta}^{\beta} \Big(\int_{-\infty}^{\infty} f(Y, V, T \,|\, y, v; \theta) \, dv \Big) e^{k\pi i y/\beta} \, dy \,. \tag{21}$$

Taking account of the assumption that $f(Y, V, T \,|\, y, v; \theta)$ is well approximated as a function of compact support over $[-\beta, \beta]$, it follows that the contributions made to the characteristic function of $f(Y, V, T \,|\, y, v; \theta)$ from values of y satisfying $|y| > \beta$ are negligible so that the Fourier coefficients c_k are accurately approximated by the formula

$$c_k = \frac{1}{2\beta} \, F(Y, V, T, \omega_k, 0; \theta) \,, \qquad \omega_k = \frac{k\pi}{\beta} \,, \tag{22}$$

where it follows from expression (17) that

$$F_{\mathrm{Q}}(Y, V, t, \omega_y, 0; \theta) = \exp \Big[i\omega_y \Big(Y + (r - \xi)t - \frac{\rho \kappa \gamma t}{\sigma} \Big) + \frac{\kappa^2 \gamma t}{\sigma^2}$$
$$- \frac{\omega_y V(i + \omega_y)}{(K - i\rho\sigma\omega_y)\tanh(\eta t/2) + \eta} \Big] \times \Big(\frac{\eta}{(K - i\rho\sigma\omega_y)\sinh(\eta t/2) + \eta \cosh(\eta t/2)} \Big)^{2\kappa\gamma/\sigma^2} \,. \tag{23}$$

Furthermore, the fact that the left hand side of equation (20) is real-valued requires that $c_{-k} = \bar{c}_k$ which in turn simplifies equation (20) to give

$$\int_{-\infty}^{\infty} f(Y, V, T \,|\, y, v; \theta) \, dv = c_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k) \, \cos \omega_k y + 2 \sum_{k=1}^{\infty} \mathrm{Im}(c_k) \, \sin \omega_k y \,, \tag{24}$$

where $\mathrm{Re}(c_k)$ and $\mathrm{Im}(c_k)$ denote respectively the real part and imaginary part of c_k. Expression (18) for the expected payoff from a call option now becomes

$$S_0 \int_{\log \xi}^{\beta} (e^y - \xi) \Big[c_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k) \, \cos \omega_k y + 2 \sum_{k=1}^{\infty} \mathrm{Im}(c_k) \, \sin \omega_k y \Big] \, dy \,. \tag{25}$$

in which the upper limit of infinity has been replaced by $S_{\mathrm{max}} = S_0 e^{\beta}$, the value of the asset price beyond which the contributions to the value of the integral have been assumed to be

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zero. Each integral in expression (25) is now replaced by its value to get

$$c_0\Big(S_{\max} - K - K\log(S_{\max}/K)\Big) + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k) \left(\frac{K(\sin\beta_k - \omega_k\cos\beta_k) + S_{\max}\omega_k(-1)^k}{\omega_k(1+\omega_k^2)}\right)$$
$$+ 2\sum_{k=1}^{\infty} \mathrm{Im}(c_k) \left(\frac{((1+\omega_k^2)K - \omega_k^2 S_{\max})(-1)^k - K(\omega_k\sin\beta_k + \cos\beta_k)}{\omega_k(1+\omega_k^2)}\right),$$
$$\tag{26}$$

where $\beta_k = \omega_k \log\xi$. We emphasize again the importance of a concise representation of the expression (26) by noting that the calculation of a single likelihood function from market put and call prices over a period of 10 years will require in excess of one billion (*i.e.* 10^9) evaluations of expression (26). A direct calculation of each term of the summation in expression (26) requires the evaluation of one exponential function and four trigonometric functions, two of which are latent but arise in the computation of $\mathrm{Re}(c_k)$ and $\mathrm{Im}(c_k)$ from the characteristic function. The efficient computation of expression (26) is of paramount importance if parameter estimation based on real market put and call data is to be a feasible procedure. We now demonstrate how expression (26) may be restructured to avoid the calculation of two trigonometric functions.

The first stage of this calculation begins by noting that the assumption that transitional probability density in the region $|y| \geq \beta$ may be neglected manifests itself in the mathematically requirement that

$$c_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k) \cos\omega_k\beta + 2\sum_{k=1}^{\infty} \mathrm{Im}(c_k) \sin\omega_k\beta = c_0 + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k)(-1)^k = 0 \tag{27}$$

bearing in mind that $\omega_k\beta = k\pi$. The constant term of expression (26) is now divided into the components $c_0(S_{\max} - K)$ and $-c_0 K\log(S_{\max}/K)$. The occurrence of c_0 in the former expression is replaced using expression (27) to rewrite the original constant term in the form

$$-c_0 K\log(S_{\max}/K) - 2\big(S_{\max} - K\big)\sum_{k=1}^{\infty} \mathrm{Re}(c_k)(-1)^k. \tag{28}$$

With this representation of the constant term in place, expression (26) now becomes

$$-c_0 K\log(S_{\max}/K) + 2\sum_{k=1}^{\infty} \mathrm{Re}(c_k) \left(\frac{K(\sin\beta_k - \omega_k\cos\beta_k) + \omega_k\big((1+\omega_k^2)K - \omega_k^2 S_{\max}\big)(-1)^k}{\omega_k(1+\omega_k^2)}\right)$$
$$+ 2\sum_{k=1}^{\infty} \mathrm{Im}(c_k) \left(\frac{((1+\omega_k^2)K - \omega_k^2 S_{\max})(-1)^k - K(\omega_k\sin\beta_k + \cos\beta_k)}{\omega_k(1+\omega_k^2)}\right),$$

which in turn can be rearranged into the simplified form

$$-c_0 K \log(S_{\max}/K) + 2K \sum_{k=1}^{\infty} \frac{\mathrm{Re}(c_k)(\sin\beta_k - \omega_k \cos\beta_k) - \mathrm{Im}(c_k)(\omega_k \sin\beta_k + \cos\beta_k)}{\omega_k(1+\omega_k^2)}$$
$$+ 2\sum_{k=1}^{\infty} \frac{\left((1+\omega_k^2)K - \omega_k^2 S_{\max}\right)(-1)^k}{\omega_k(1+\omega_k^2)}\left(\omega_k \mathrm{Re}(c_k) + \mathrm{Im}(c_k)\right). \tag{29}$$

Let the phase angle ϕ_k be defined by the equations

$$\sin\phi_k = \frac{\omega_k}{(1+\omega_k^2)^{1/2}}, \qquad \cos\phi_k = \frac{1}{(1+\omega_k^2)^{1/2}},$$

then expression (29) further simplifies to get

$$-c_0 K \log(S_{\max}/K) - 2K \sum_{k=1}^{\infty} \frac{\mathrm{Re}(c_k)\sin(\phi_k - \beta_k) + \mathrm{Im}(c_k)\cos(\phi_k - \beta_k)}{\omega_k(1+\omega_k^2)^{1/2}}$$
$$+ 2\sum_{k=1}^{\infty} \frac{\left((1+\omega_k^2)K - \omega_k^2 S_{\max}\right)(-1)^k}{\omega_k(1+\omega_k^2)^{1/2}}\left(\sin\phi_k \mathrm{Re}(c_k) + \cos\phi_k \mathrm{Im}(c_k)\right). \tag{30}$$

The second stage of the simplification begins by noting that the generic form of the characteristic function is the exponential of a complex number. It follows immediately from equation (21) that $c_k = \mathrm{Re}(c_k) + i\mathrm{Im}(c_k) = \exp(\eta_k + i\psi_k)/2\beta$ in which η_k and ψ_k are real functions of frequency ω_k and the parameters θ. Consequently $\mathrm{Re}(c_k) = e^{\eta_k}(\cos\psi_k)/2\beta$ and $\mathrm{Im}(c_k) = e^{\eta_k}(\sin\psi_k)/2\beta$, together with the observation that $c_0 = 1/2\beta$ further simplifies expression (30) to give

$$-\frac{K}{2\beta}\log\left(\frac{S_{\max}}{K}\right) - \frac{K}{\beta}\sum_{k=1}^{\infty} e^{\eta_k} \frac{\cos\psi_k \sin(\phi_k - \beta_k) + \sin\psi_k \cos(\phi_k - \beta_k)}{\omega_k(1+\omega_k^2)^{1/2}}$$
$$+ \frac{1}{\beta}\sum_{k=1}^{\infty} \frac{\left((1+\omega_k^2)K - \omega_k^2 S_{\max}\right)(-1)^k}{\omega_k(1+\omega_k^2)^{1/2}}\left(\cos\psi_k \sin\phi_k + \sin\psi_k \cos\phi_k\right),$$

which in turn further simplifies to gives

$$C = -\frac{K}{2\beta}\log\left(\frac{S_{\max}}{K}\right) - \frac{K}{\beta}\sum_{k=1}^{\infty} e^{\eta_k} \frac{\sin(\phi_k + \psi_k - \beta_k)}{\omega_k(1+\omega_k^2)^{1/2}}$$
$$+ \frac{1}{\beta}\sum_{k=1}^{\infty} e^{\eta_k} \frac{\left((1+\omega_k^2)K - \omega_k^2 S_{\max}\right)\sin(\phi_k + \psi_k)(-1)^k}{\omega_k(1+\omega_k^2)^{1/2}} \tag{31}$$

for the price of a call option with maturity T and strike K. Each term in this expression involves the computation of one exponential function and two trigonometric function.

5 Simulation Experiments

The efficacy of the particle filtering algorithm for estimating the parameters of the Heston stochastic volatility model is now examined in terms of a series of simulation experiments. The primary objective of the experiments are to gauge the efficacy of the particle filter algorithm in recovering the parameters of the stochastic volatility model and to investigate the correct choice of the distribution of pricing errors in the option.

The first experiment compares the parameter estimates obtained by using the particle filter with no option prices with those obtained using quasi-maximum likelihood estimation (Hurn, Lindsay and McClelland, 2012) in which volatility is treated as observed. For the particle filter, ten years of daily data ($T = 2520$) are simulated with $\Delta t = 1/252$ so that the parameters may be interpreted as annualised values and the parameter estimation problem is repeated 500 times. The particle filter estimates are based on 17920 particles. This number of particles is chosen with a view to the available hardware. Some of the computations were done on a Tesla 2070 card (448 cores) and others on a Tesla 2090 card (512 cores). The number of particles chosen in the simulation experiment (17920) represents 40 threads on a 2070 card and 35 threads on a 2090 card. For quasi-maximum likelihood estimation, parameter estimates are generated for $T = 2520, 5040, 7560$ transitions with $\Delta t = 1/252$ in each case.

The results of the simulation are reported in Table 1. Recall that quasi-maximum likelihood estimation is treating volatility as observed whereas it is not treated as an observed variable by the particle filter. Despite this disparity the particle filter estimates of the drift parameters κ_v and the equity premium ξ_s for $T = 2520$ are significantly better than the quasi-maximum likelihood estimates. Indeed, it is only when $T = 7560$ that the quasi-maximum likelihood estimates match the quality of those of the particle filter. There is, however, no such difference in performance for the remaining parameters γ_v, σ_v and ρ and if anything quasi-maximum likelihood is slightly superior. This is, of course, to be expected given that these parameters are related to the volatility which is being treated as observed in generating the quasi-maximum likelihood estimates.

Parameter	True Value	PF (T=2520)		QML (T=2520)		QML (T=5040)		QML (T=7560)	
		Bias	RMSE	Bias	RMSE	Bias	RMSE	Bias	RMSE
κ_v	3.00	0.1000	0.2203	0.4076	0.9312	0.1917	0.5640	0.1292	0.4466
γ_v	0.03	−0.0003	0.0037	−0.0007	0.0058	−0.0004	0.0038	−0.0002	0.0031
σ_v	0.30	−0.0010	0.0243	0.0003	0.0040	0.0001	0.0027	0.0001	0.0023
ρ	−0.60	0.0058	0.0341	−0.0001	0.0114	0.0007	0.0083	0.0010	0.0069
ξ_{s}	5.00	0.1727	0.5541	0.4881	1.9948	0.2238	1.3478	0.1221	1.1266

Table 1: Parameter estimates for Heston's model without option price data for $T = 2520$ with time step $\Delta t = 1/252$. The estimates from the particle filter (PF) using 17920 particles for simulation experiments involving 500 repetitions are compared with quasi-maximum likelihood estimation (QML) in which volatility is treated as an observed variable.

The next four simulation experiments investigate the issue of the distribution of pricing errors in options. To facilitate comparison, all the experiments were conducted on a Tesla 2070 card with $T = 2520$, $\Delta t = 1/252$, 17920 particles and 6, 12 and 18 options per day. The path of the simulated index and the volatility are identical in all cases, but the option prices differ only because of the differing distributional properties of the pricing errors. Options are spread uniformly across the money when initiated. Three levels of maturity, one month, two months and three months, are used and each option is followed through its lifetime until five days prior to maturity at which point the option is dropped from the set and a new option is generated. This generation procedure mimics the method proposed by Eraker (2004) for choosing options to use in estimation. Tables 2 to 5 report the results of the simulations for pricing-error distributions ES1 to ES4 in equations (8 - 11). In each case the synthetic option prices are generated using the same distribution that is used in the estimation to construct the likelihood.

The most important result to emerge from scrutiny of these results is that the introduction of six option prices into the estimation substantially reduces the bias on the parameters reported in Table 1. In addition, the volatility premium, ξ_{v}, is identified and accurately estimated. Interestingly enough, the volatility premium is recovered with very good precision but the estimate of the equity premium, ξ_{s}, is still somewhat imprecise. The pricing error,

$\alpha = 5\%$, that is introduced into the simulated option prices is recovered with high accuracy. Increasing the number of options from 6 to 18, by contrast, makes little significant difference to the root mean square errors particularly when interpreted in the context of the large amount of extra data that is processed. This is perhaps to be expected because the options are all priced using the correct model so that the addition of extra options adds little in terms of new information but significantly increases the amount of arithmetic to be done in the estimation process.

Parameter	True Value	6 Options		12 Options		18 Options	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
κ_v	3.00	0.0164	0.0339	0.0186	0.0313	0.0175	0.0272
γ_v	0.03	−0.0002	0.0003	−0.0002	0.0002	−0.0002	0.0002
σ_v	0.30	0.0007	0.0014	0.0009	0.0014	0.0007	0.0011
ρ	−0.60	0.0016	0.0031	0.0016	0.0026	0.0014	0.0021
ξ_{s}	5.00	0.4285	0.4761	0.4452	0.4799	0.4490	0.4823
ξ_{v}	−3.00	−0.0096	0.0108	−0.0097	0.0106	−0.0094	0.0103
α	0.05	0.0000	0.0005	0.0000	0.0004	0.0001	0.0004
λ_1	6.25	0.5204	0.5857	0.5433	0.5901	0.5470	0.5928
λ_2	−4.00	−0.0234	0.0323	−0.0208	0.0274	−0.0217	0.0279

Table 2: Parameter estimates for Heston's model with option pricing error ES1. Estimation uses 17920 particles for simulation experiments involving 500 repetitions with a fixed set of option prices.

Parameter	True Value	6 Options Bias	6 Options RMSE	12 Options Bias	12 Options RMSE	18 Options Bias	18 Options RMSE
κ_v	3.00	0.0165	0.0298	0.0149	0.0295	0.0163	0.0299
γ_v	0.03	-0.0002	0.0003	-0.0001	0.0002	-0.0001	0.0002
σ_v	0.30	0.0007	0.0013	0.0006	0.0012	0.0008	0.0013
ρ	-0.60	0.0020	0.0031	0.0015	0.0029	0.0020	0.0030
ξ_s	5.00	0.4283	0.4405	0.4068	0.4234	0.3945	0.4073
ξ_v	-3.00	-0.0108	0.0111	-0.0100	0.0104	-0.0106	0.0109
α	0.05	0.0003	0.0008	0.0009	0.0012	0.0009	0.0011
λ_1	6.25	0.5183	0.5350	0.4918	0.5145	0.4773	0.4947
λ_2	-4.00	-0.0268	0.0308	-0.0253	0.0302	-0.0250	0.0305

Table 3: Parameter estimates for Heston's model with option pricing error ES2. Estimation uses 17920 particles for simulation experiments involving 500 repetitions with a fixed set of option prices.

Parameter	True Value	6 Options Bias	6 Options RMSE	12 Options Bias	12 Options RMSE	18 Options Bias	18 Options RMSE
κ_v	3.00	0.0302	0.0430	0.0225	0.0353	0.0200	0.0329
γ_v	0.03	-0.0003	0.0004	-0.0002	0.0003	-0.0002	0.0002
σ_v	0.30	0.0011	0.0016	0.0007	0.0014	0.0006	0.0012
ρ	-0.60	0.0022	0.0034	0.0011	0.0027	0.0011	0.0022
ξ_s	5.00	0.4108	0.4182	0.3974	0.4055	0.4027	0.4128
ξ_v	-3.00	-0.0104	0.0106	-0.0106	0.0108	-0.0106	0.0108
α	0.05	0.0000	0.0006	0.0000	0.0005	0.0001	0.0005
λ_1	6.25	0.5014	0.5109	0.4785	0.4900	0.4847	0.4988
λ_2	-4.00	-0.0205	0.0265	-0.0264	0.0310	-0.0271	0.0312

Table 4: Parameter estimates for Heston's model with option pricing error ES3. Estimation uses 17920 particles for simulation experiments involving 500 repetitions with a fixed set of option prices.

Parameter	True Value	6 Options		12 Options		18 Options	
		Bias	RMSE	Bias	RMSE	Bias	RMSE
κ_v	3.00	0.0234	0.0400	0.0152	0.0288	0.0144	0.0263
γ_v	0.03	-0.0002	0.0003	-0.0002	0.0002	-0.0001	0.0002
σ_v	0.30	0.0009	0.0014	0.0006	0.0011	0.0005	0.0010
ρ	-0.60	0.0018	0.0029	0.0010	0.0021	0.0010	0.0018
ξ_s	5.00	0.4310	0.4514	0.4490	0.4828	0.4744	0.5067
ξ_v	-3.00	-0.0102	0.0107	-0.0097	0.0105	-0.0092	0.0100
α	0.05	0.0000	0.0006	0.0000	0.0004	-0.0001	0.0004
λ_1	6.25	0.5248	0.5525	0.5441	0.5906	0.5764	0.6208
λ_2	-4.00	-0.0223	0.0290	-0.0248	0.0301	-0.0239	0.0287

Table 5: Parameter estimates for Heston's model with option pricing error ES4. Estimation uses 17920 particles for simulation experiments involving 500 repetitions with a fixed set of option prices.

There is little to choose between the different distributions of pricing errors in terms of the quality of the parameter estimates. There are, however, significant differences in terms of the computational effort required to obtain these parameter estimates. Table 6 gives the time required for a single estimation of the various models implemented on one Xeon core hosting a single Tesla 2070 card. Recall that ES2 is different from the others in the respect that the misspricing parameter α is independent of the model price and has the interpretation of a monetary value. This model, although relatively easy to estimate in terms of the results reported in Table 6, is perhaps the least realistic of the models when dealing with traded options. By contrast, ES1, ES3 and ES4 are concerned with the properties of the ratio of the observed to model price and within this class, ES3 appears superior in terms of computational effort.

Error Specification	0 Opts	6 Opts	12 Opts	18 Opts
ES1	1438 (252)	939 (201)	1640 (813)	2661 (1259)
ES2	1438 (252)	931 (210)	1132 (183)	1944 (428)
ES3	1438 (252)	939 (201)	1117 (145)	1945 (229)
ES4	1438 (252)	1004 (355)	1707 (847)	2829 (1315)

Table 6: The mean and standard deviations of the time taken (measured in seconds) for a single estimation of Heston's model of stochastic volatility using 17920 particles are given based on a sample of 500 trials of each combination of option count and distribution of pricing error. Each estimation is performed using a Tesla 2070 card to facilitate comparison.

Of course, the results of these experiments do not offer any real guidance as to the correct choice for the distribution of pricing error, but they do indicate that within a group of pricing error specifications that superficially appear to require comparable amounts of calculation and embody comparable accuracy as determined by the value of α (with the possible exception of ES2), the numerical effort in estimation is not necessarily similar. In the absence of any evidence to the contrary, however, it would appear reasonable to conclude the using ES3 is a sensible choice and will be used in estimation of the model parameters based on observed data in Section 7.

Finally, in a simulation environment, it is easily verified that the filtered path of unobservable volatility accurately tracks the actual path of the observed simulated volatility. Figure 4 plots the known volatility under simulation against the average particle volatility for a single repetition of a simulation experiment involving six options. Specifically, pairs of options were initiated every 22 days with maturities of 70 days and with one option 10% in the money and the other 10% out of the money when written. Options were discontinued once they reached a maturity of five days. It is readily apparent from Figure 4 that the estimated volatility obtained from the particle filter is indistinguishable from the volatility realised in the simulation.
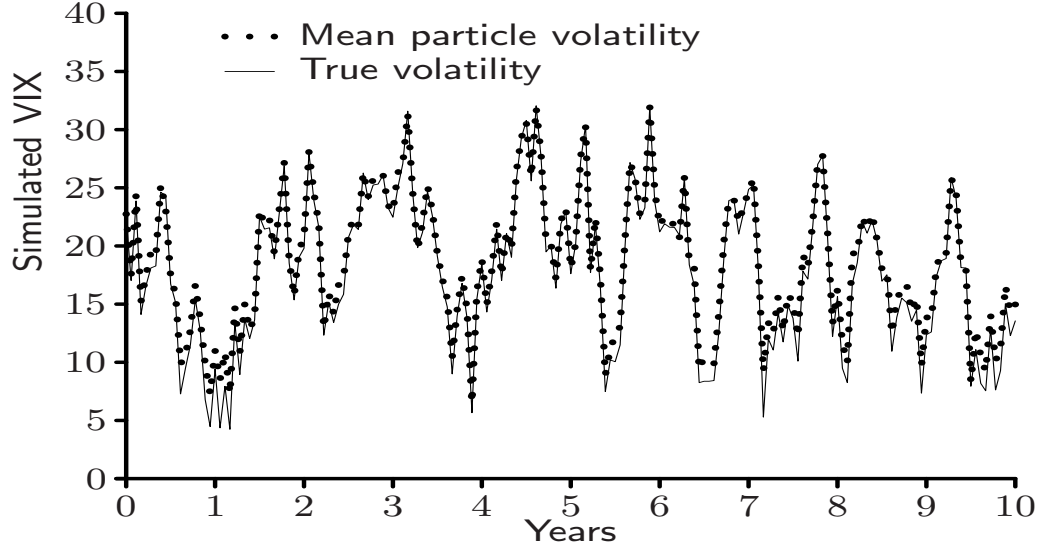
Figure 1: Comparison of the known volatility under simulation expressed as a percentage (solid line) and its estimated value obtained from the particle filter (dotted line) in a single repetition of a simulation experiment. The estimation uses 17920 particles and the simulation is for 10 years of daily data (2520 days).

6 Choice of Option Data

The parameters of Heston's model will be estimated from a data consisting of S&P500 index values and option prices. The market data consists of observations of bid and ask prices on all put and call options written on the S&P500 index from 2 January 1990 to 30 June 2012; a total of 4,459,751 observations over 5672 trading days. Prior to estimation, several criteria are used to filter the options data to ensure that only traded options are used and that the options are arbitrage free. A total of 3,165,523 observations were of untraded options, leaving a rump of 1,294,228 observations of traded options. To establish the arbitrage free condition, the options were subjected to a "bounds-test" and a "no-opportunity-for-arbitrage" test.

Suppose that C_{ask} and P_{ask} are respectively the ask prices for call and put options with strike price K and maturity T on an asset with spot price S. The *bounds-test* validates these prices against the theoretical bounds underlying the option, namely that

$$C_{\mathrm{ask}} \geq \max(0, S - Ke^{-rT}), \qquad\qquad P_{\mathrm{ask}} \geq \max(0, Ke^{-rT} - S). \qquad (32)$$

where r is the risk-free rate (assumed constant) during the period of the investment. Within the set of traded options, the total number of ask prices failing to satisfy the bounds-test was 25,228 (1.95%) with larger percentages in the years 1990 to 2008 and significantly smaller percentages from 2009 onwards. Curiously, 2003 had the lowest incidence of cases with only 285 (0.63%) observations of fundamental mis-pricing from a total of 45191 observations of traded option prices, whereas 2001 had the highest incidence of cases with 2020 (5.23%) observations of fundamental mis-pricing from a total of 38628 observations of traded option prices.

Suppose that call and put options with the same strike and maturity are traded on an asset with spot price S. Mis-pricing of these options can now introduce arbitrage opportunities when an investor takes opposite positions in call and put contracts with identical strikes and maturities. To establish the *no-opportunity-for-arbitrage* conditions let C_{bid} and P_{bid} be respectively the bid prices for call and put options with strike price K and maturity T on an asset with spot price S. First consider the trading strategy in which the investor simultaneously shorts the asset, buys a call option at price C_{ask} and shorts a put receiving premium P_{bid}. This trading strategy offers no opportunity for arbitrage provided $S + P_{\mathrm{bid}} - C_{\mathrm{ask}} \leq Ke^{-rT}$, that is, the cash flow resulting from the sale of the asset and the sale/purchase of options cannot exceed the discounted strike price. The alternative trading strategy of buying a put option at price P_{ask}, buying the asset at the spot price S and shorting a call option receiving premium C_{bid} will likewise provide no opportunity for arbitrage provided $S + P_{\mathrm{ask}} - C_{\mathrm{bid}} \geq Ke^{-rT}$, *i.e.* the present value of the portfolio exceeds its discounted value at the maturity of the options. Thus when call and put options with the same strike and maturity are traded on an asset, consistent bid and ask prices must satisfy the conditions

$$C_{\mathrm{ask}} + Ke^{-rT} - S - P_{\mathrm{bid}} \geq 0, \qquad S + P_{\mathrm{ask}} - C_{\mathrm{bid}} - Ke^{-rT} \geq 0. \qquad (33)$$

Within the set of traded options a total of 693,464 paired call and put options with identical strike prices and maturities were removed. Altogether 600,764 observations of traded options remained.

The final question to be resolved is which of the remaining options should be used in the estimation procedure. Of course, all the options can easily be incorporated in the estimation but it is likely that this will result in large misspricing errors. After all, there is no guarantee that the market options are priced using Heston's (or any other) model. Eraker (2004) uses a sampling strategy in which a contract that trades on the first day of the sample is chosen

at random. All observations of this contract are then included in the estimation. On the first trading day on which this option ceases to trade, another trading contract is drawn at random. The sample of observed option prices is then constructed in this way so that there is at least one traded option each day. By contrast, the method for choosing options use here follows Pan (2002) who concentrates on the most liquid instruments, the implicit assumption being that the prices of these contracts should convey the most precise information. Several estimation exercises are reported in Section 7 using one in-the-money and one out-of-the-money options (1-1), 2 out-of-the-money options (0-2), 4 out-of-the-money options (0-4), 6 out-of-the-money options (0-6) and 10 out-of-the-money options (0-10) from the most heavily traded option both in-the-money and out-of-the money. A strong preference for using out-of-the-money options is also motivated by the findings of Huang and Wu (2004) who suggest disregarding in-the-money options entirely.

7 Parameter Estimation Results

The parameters of the Heston model are now estimated using data consisting of daily data on the S&P 500 Index for the period 2 January 1990 to 30 December 2011. Daily cash dividends are ignored in the estimation. A plot of the annualised S&P 500 Index is given in Figure 4.

Figure 2: S&P 500 Index (annualised) from 2 January 1990 to 30 June 2012.

Observations of call and put option prices on S&P500 index with maturities between five and ninety days and chosen in the manner described in Section 6 are also used. Once again, due

to hardware considerations, each estimation is based on 17,920 particles. The construction of the likelihood function was based on the use of model ES3 in equation (10) for the pricing error between observed option prices and the model estimate of the option price.

Table 7 reports parameter estimates for various combinations of options, namely, one in-the-money and one out-of-the money option (1-1) and from two to ten out of the money options (0-2)-(0- 10). It is immediately apparent that all of the estimated parameters have the correct algebraic sign and all are statistically significant. The actual values of the estimated parameters are consistent with estimates previously reported in the literature. For example, estimates of the correlation parameter, ρ, are very similar to those previously reported. These include, for example, -0.48 (Eraker, Johannes, and Polson, 2003), -0.46 (Chernov, Ghysels, Gallant, and Tauchen, 2003) and -0.46 (Eraker, 2004). The one exception to this result is that the estimate of the correlation parameter for ten out-of-the-money options, $\rho = -0.2784$, looks to be a little on the low side. Not surprisingly, as more options are included in the estimation process, the standard deviation of the distribution of pricing errors, α, increases from 13% for the (1-1) case to 24% for the (0-10) case. Of course, in practice there is no knowing what model was actually used to price options and it is therefore inevitable that α increases as more options are included. Note however, that the increase is very slow beyond the inclusion of two out-of-the-money options.

The prices of Equity and Volatility risk, namely ξ_S and ξ_V are identified and statistically well determined. Existing evidence of the size and significance of the volatility premium in stochastic volatility models without jumps is mixed. Chernov and Ghysels (2000), Pan (2002) and Broadie, Chernov and Johannes (2007) find relatively small values for the volatility premium, approximately -0.25 to -1.25 respectively. The estimates reported here tend to support a value for of ξ_V of around -1.5. By contrast with the previous research, all the estimates of ξ_v are statistically significant. Furthermore, the relative sizes of κ_v and ξ_V is such that the dynamics of volatility under the risk-neutral measure, $\mathbb{Q}$, are stable and not explosive. This contradicts the findings of Pan (2002) and Jones (2003), when using post 1987 data.

Parameter	1-1	0-2	0-4	0-6	0-10
κ_v	3.2211	3.0221	4.1302	3.2194	3.4174
	(0.1473)	(0.0403)	(0.0259)	(0.0114)	(0.0278)
γ_v	0.0254	0.0356	0.0243	0.0325	0.0332
	(0.0009)	(0.0028)	(0.0002)	(0.0001)	(0.0001)
σ_v	0.2704	0.3980	0.3786	0.3300	0.4685
	(0.0101)	(0.0210)	(0.0038)	(0.0084)	(0.0026)
ρ	-0.4974	-0.4020	-0.4210	-0.4133	-0.2784
	(0.0195)	(0.0203)	(0.0079)	(0.0058)	(0.0010)
ξ_s	3.3665	4.6031	6.1632	6.5395	6.5984
	(0.0876)	(0.2047)	(0.0318)	(0.1795)	(0.0295)
ξ_v	-1.5549	-1.5964	-1.7112	-1.6181	-1.4435
	(0.0523)	(0.0099)	(0.0648)	(0.0429)	(0.0110)
α	0.1315	0.2001	0.2031	0.2319	0.2440
	(0.0012)	(0.0112)	(0.0020)	(0.0015)	(0.0003)

Table 7: Parameter estimates for Heston's model of stochastic volatility calculated using the S&P 500 index from 2 January 1990 to 30 December 2011 and various combinations of options written on the index over that period. Estimation uses 17920 particles and ES3.

To quantify the economic significance of the estimate of the risk premia, it is useful to consider the relative contribution of volatility risk to the equity risk premium. Broadie, Chernov and Johannes (2007) find that the equity premium is about 8% over the period of their sample and that the volatility risk premium contributes about 3% per year to this equity premium. Broadly speaking, the results reported here for the (0-4) option case suggest an equity of 15% per annum and the volatility contribution to that is approximately 5%. This effect is more marked when more out-of-the-money options are used.

A commonly used proxy for the volatility of the S&P 500 is the Implied Volatility Index (VIX) published by the Chicago Board of Exchange. The VIX is constructed from European put and call option prices such that at any given time it represents the risk-neutral expectation of integrated variance averaged over the next 30 calendar days (or 22 trading days). Additional evidence of the efficacy of the particle filter to track the path of volatility is given by Figure 3 which shows how well the filtered volatility from the estimated model tracks the VIX index.

The fact that the filtered volatility tracks the VIX so closely would indicate that the model is well estimated.
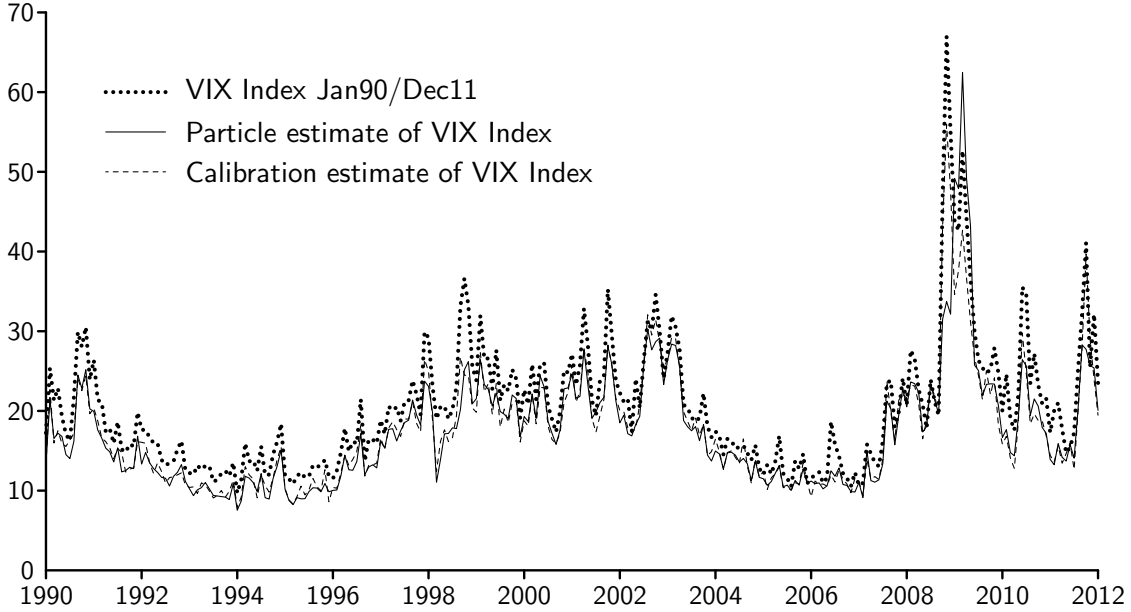

Figure 3: Plot of the VIX versus the filtered volatility obtained from the particle filter using 17920 options and 6 out-of-the-money options. Also shown is the estimate of the volatility obtained from least-squares calibration of the parameters of the risk-neutral model to a cross section of 10 option prices.

Also shown in Figure 3 is an estimate of the volatility obtained by calibration. Calibration involves estimating, for each day, the parameters of the risk-neutral model, κ_q, γ_q, σ_v and ρ, and the level of volatility using a different cross section of option prices on each day. Calibrated estimates are obtained by minimising a user-supplied positive definite function connecting the observed values of the chosen options with their corresponding values calculated from the stochastic volatility model. Since the objective here is to explore the nature of the parameter estimates generated by a calibration procedure rather than discuss how best to choose the function to be minimised, a naive least-squares fit is used. The sum of the squared discrepancies between observed and model option prices is applied each day to the ten out-of-the-money S&P 500 options closest to the money. The daily estimate of volatility obtained by calibration is almost identical to that calculated using the particle filtering algorithm and tracks the VIX very closely.

	κ_q	γ_q	σ_v	ρ	$\widehat{V}$	$\widehat{\mathrm{VIX}}$	VIX
First Quartile	0.5101	0.0281	0.0274	-0.9999	0.0075	12.4760	14.7100
Median	2.2775	0.0535	0.2878	-0.8475	0.0150	16.3739	19.0700
Third Quartile	11.0556	0.1125	0.6161	0.6356	0.0319	20.9021	24.1175

Table 8: The mean and standard deviations of the 95% value at risk level of returns for the first four months of 2012 using Heston's model of stochastic volatility with parameter estimates and risk premia calculated using the S&P 500 index from 2 January 1990 to 30 December 2011 and various combinations of options written on the index over that period.

The quartiles of the distribution of the calibrated parameters are presented in Table 8. The distribution of the estimated implied volatility index is similar to that of the actual VIX as demonstrated by the last two columns of Table 8, although a slight tendency for the estimated index to understate the true quartiles of the VIX is evident. It is also apparent from the first four columns of Table 8 that all the parameter estimates show strong variability from day-to-day. The most striking of these being the extreme variability exhibited by κ_q and the strong tendency for the correlation coefficient ρ to take the limiting values $\rho = -1$ and $\rho = 1$ as illustrated in Figure 4.
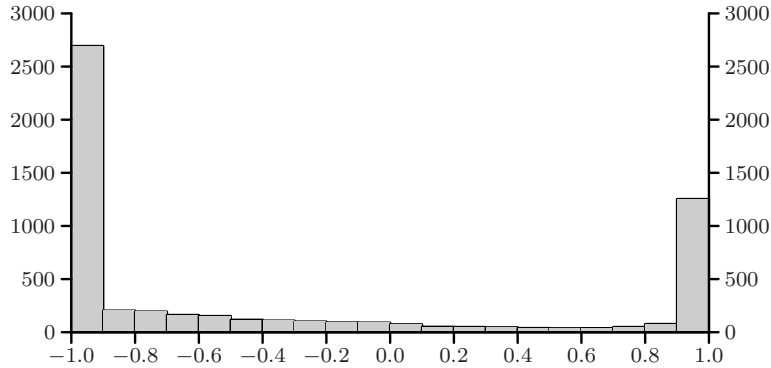

Figure 4: Histogram of calibrated values of the correlation parameter ρ from daily calibration of Heston's risk-neutral model using the 10 most heavily traded out-of-the-money options written on the S&P 500 Index from 2 January 1990 to 30 December 2011.

Although on the surface calibration appears to track volatility with good reliability, it is clear that the day-to-day variability inherent in the values of parameters constructed by calibration

severely limits the usefulness of the calibration procedure as a tool for the reliable pricing of exotic option contracts: the best that can be assumed is that the prices of the vanilla options used in the calibration procedure are accurately priced by the model parameters returned by the calibration exercise. Indeed, the excessive variability in the parameter estimates returned by a calibration procedure is nicely illustrated in Figure 4 which reveals a disturbing tendency for the calibrated value of ρ to flick between the extreme values of -1 and $+1$. The argument that the median parameters might be used for option pricing is also flawed. Note that the actual model estimates presented in Table 7 reveal that, for the (1-1) option case, the implied estimate of κ_q is 1.6662 which is substantially below the median value reported in Table 8. Similarly, the median value of ρ of -0.847 is much more negative than the estimated value of ρ in Table 7.

As a final check of the reliability of the estimates of the model parameters, the model is used to generate estimates of value-at-risk. Consider Table 9 which shows the 95% value-at-risk for returns at the end of January, February, March and April 2012. The calculation of these results begins by storing the optimal parameters and the final particle cloud that has been obtained on completion of the maximum likelihood estimation procedure. Heston's model with the optimal set of parameters is now used to compute the analytical distribution of returns for each particle of the final cloud. Of course, each particle provides a different distribution of returns due to the different level of final volatility captured by that particle, and consequently each particle identifies a different value of return for the 95% value-at-risk of returns. The values reported in Table **??** represent respectively the mean value and the standard deviation of the 95% value-at-risk of returns at the end of January 2012 (1 month), February 2012 (2 months), March 2012 (3 months) and April 2012 (4 months). This calculation was repeated for the various combinations of options described in Table 9. Interestingly each combination of S&P 500 options used in the construction of the 95% value-at-risk level of returns delivers estimates that are not statistically different at the end of January and February. Further out, it is clear that value-at-risk estimates based on two options per day seem to produce more aggressive value-at-risk levels that the comparable estimates based on four and six out-of-the-money options per day, but all paired estimates are within two standard deviations of each other and it may be argued, tentatively, that the differences are not statistically significant.

Combination of options	1 Month	2 Months	3 Months	4 Months
(1 - 1)	−18.57 (1.01)	−26.51 (1.16)	−32.19 (1.21)	−36.61 (1.21)
(0 - 2)	−19.31 (1.34)	−27.36 (1.48)	−32.82 (1.48)	−36.87 (1.41)
(0 - 4)	−18.54 (1.10)	−25.04 (1.15)	−28.99 (1.08)	−31.63 (1.00)
(0 - 6)	−17.32 (0.98)	−23.76 (1.03)	−27.84 (0.97)	−30.63 (0.89)

Table 9: The mean and standard deviations of the 95% value at risk level of returns for the first four months of 2012 using Heston's model of stochastic volatility with parameter estimates and risk premia calculated using the S&P 500 index from 2nd Jan 1990 to 30th Dec 2011 and various combinations of options written on the index over that period.

8   Conclusion

Observations on option prices represents a wealth of information pertinent to the parameter estimation problem and the associated filtering of the latent state variable(s) because of their dependence upon the parameters of the system under the risk-neutral measure. When used jointly with index returns data which depend on the model parameters under the physical measure, powerful advances can be made. Although considerable work has been undertaken to incorporate options data into estimation, to date, however, the computational burden imposed by the associated range of estimation procedures has significantly impeded progress. The main compromise appears to be to use a two-stage estimation procedure; the first stage fits the parameters of the physical model to the time series of index returns and at the second stage a least squares fitting procedure is used to fit the parameters of the risk-neutral model to the cross section of option prices. Although there are a few studies in which parameter estimates have been successfully provided by Markov Chain Monte Carlo methods the ability to provide maximum likelihood estimates using a particle filter have been thwarted by computational considerations. The main advance reported in this paper is the use of graphical processor technology and parallel computation to allow estimation of the parameters of the model under both risk-neutral and physical measures.

It is also shown that as many as one hundred thousand options prices can be handled with ease by the particle filter when run on appropriate graphical processing units. The Tesla 2070 graphical processing unit (GPU) used in the calculations of this paper has 448 cores each operating at 1.15GHz and has the approximate computational power of 12 Xeon processors. Setting aside the obvious physical differences in core structure and operating frequency between a Tesla card and a Xeon processor, another important practical distinction between computing with a GPU and computing with a CPU is that efficient GPU computing requires the user to manage the pipelining of data to the processor in order to unlock its power, whereas the CPU largely manages this process without user intervention thereby making the CPU significantly easier to program.

The method is illustrated using the Heston model of stochastic volatility. Notwithstanding its demonstrated shortcomings when modelling index returns, the Heston model may be regarded as the workhorse of option pricing and is still used as a benchmark in many practical instances. A simulation study reveals that the particle filter is equivalent to maximum likelihood based on a closed-form approximation to the log-likelihood function or quasi-maximum likelihood estimation. The parameters of the model under the physical measure returned by the particle filter algorithm are similar to many reported previously in the literature for the Heston model using S&P 500 data. Furthermore, the use of options data isolates statistically significant equity and volatility premia in the data. This is in sharp contrast to previous studies using two-stage procedures which yield statistically insignificant estimates of these risk premia. Specifically, the equity risk premium on average is shown to be of the order of 15% with the volatility premium amounting to about 5% of this figure. The internal consistency of the model and estimates is demonstrated by the fact that using the particle filter to extract an estimate of the unobserved state variable, volatility, yields an estimate that closely tracks the VIX index.

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