Differential Equations Review Sheet 1, Fall 2017

I. First Order DEs

(a) Separable

Form: $\frac{dy}{dx} = f(x)g(y)$

To solve: arrange it like so: $\frac{dy}{g(y)} = f(x)\,dx$, integrate both sides!

Don't forget the lost solutions $y = c$, where $g(c) = 0$.

(b) Linear

General Form: $a(x)y' + b(x)y = c(x)$

Standard form: $y' + p(x)y = f(x)$

To solve:

- Put in standard form (by dividing by $a(x)$ if necessary).
- Compute the homogeneous solution: $y_h(x) = e^{-\int p(x)dx}$.
- Use the variation of parameters formula: $y(x) = y_h(x)(\int \frac{q(x)}{y_h(x)}\,dx + C)$.
- There is also the definite integral solution: $y(x) = y_h(x)(\int_a^x \frac{q(u)}{y_h(u)}\,du + C)$, where $C = y(a)/y_h(a)$.

II. Linear Constant Coefficient Homogeneous DEs

(a) Linear, Constant Coefficient

1. What does it look like? Well, it's linear, and has constant coefficients. :)
   We write it as $P(D)x = 0$, where $P(r)$ is a polynomial.
2. How do I solve it?
   Get the characteristic polynomial: replace y by 1, y' by r, y'' by r^2 etc. (This comes from guessing $y = e^{rt}$ as a trial solution.)
3. Solve for the roots of the equation containing r (= characteristic equation).
4. Take roots, r_1, r_2 etc. and arrange as: $y = c_1e^{r_1t} + c_2e^{r_2t} + \ldots$
5. If roots are complex in the form of $a \pm bi$, and you want a real valued solution, then make them: $y = c_1e^{at}\cos(bt) + c_2e^{at}\sin(bt) + \ldots$
6. If r is a double root, then e^{rt} and te^{rt} are both homogeneous solutions.

(b) Damping (for $my'' + by' + ky = 0$)

1. Underdamping when $b^2 - 4mk < 0$, so roots are complex, solutions oscillate.
2. Overdamping when $b^2 - 4mk > 0$, so roots are real, solutions are exponentials.
3. Critical damping when $b^2 - 4mk = 0$, so roots are repeated,
   solution is $y = c_1\mathrm{e}^{-bt/2m} + c_2t\,\mathrm{e}^{-bt/2m}$.

(c) Stability

1. $y(t) = 0$ is the equilibrium solution.
2. Physics: the system is stable (really asymptotically stable) if the output to the unforced system always goes to the equilibrium as $t \to \infty$.
3. Math: the system is stable if all charactericstic roots have negative real part.
4. Equivalently: the system is stable if all homogeneous solutions to the DE go to 0 as $t \to \infty$.
5. For $my'' + by' + ky = 0$ the system is stable exactly when m, b and k all have the same sign (usually positive).

III. Complex Numbers

(a) Euler formula: $e^{i\theta} = \cos(\theta) + i\sin(\theta)$.

(b) Polar form: $a + ib = re^{i\theta}$, $r = \sqrt{a^2 + b^2}, \tan\theta = b/a$. (Remember how to draw the polar triangle!)

(c) n^{th} roots of $re^{i\theta}$: $z = r^{\frac{1}{n}} e^{\frac{i\theta}{n} + \frac{i2\pi k}{n}}$, $k = 0, 1, 2, \ldots, n-1$.

(d) Complexification: e.g. to solve $P(D)x = F_0 cos(\omega t)$ solve $P(D)\tilde{x} = F_0 e^{i\omega t}$ and then decomplexify: $x = \text{Re}(\tilde{x})$, or $\int e^{-x}\sin(\omega x)dx = \text{Im}(\int e^{(-1+i\omega)x}dx)$.

IV. Linear Constant Coefficient Inhomogenous DEs

(a) Preliminaries

1. I'm assuming you can solve the homogenous part, $P(D)y = 0$, already.
2. Inhomogenous CC linear DEs are of the form $P(D)y = f$, with a function $f(t) \neq 0$.
3. The general solution to $P(D)y = f$ is $y = y_p + y_h$.
   (y_p = particular solution, y_h = general homogeneous solution.)

(b) Exponential response formula (ERF), also called exponential input theorem

1. For solving $P(D)x = Be^{at}$.
2. Usual version: $x(t) = \dfrac{Be^{at}}{P(a)}$, if $P(a) \neq 0$.
   Note: a is allowed to be complex.
3. Extended version: if $P(a) = 0$ then the solution is $x(t) = \dfrac{Bte^{at}}{P'(a)}$, if $P'(a) \neq 0$.
4. How do you prove the ERF?
   • Try the solution ce^{at}. After substitution you find this works with $c = B/P(a)$.

(c) Sinusoidal response formula (SRF)

1. For solving $P(D)x = B\cos\omega t$.
2. Usual version: $x(t) = \dfrac{B\cos(\omega t - \phi)}{|p(i\omega)|}$, if $P(i\omega) \neq 0$.
   Here $\phi = \text{Arg}(P(i\omega)$. When writing ϕ using $\tan^{-1}$ don't forget to give the quadrants where $P(i\omega)$ might lie.
3. How do you prove the SRF?
   • Complexify $P(D)x = B\cos(\omega t)$ to $P(D)\tilde{x} = Be^{i\omega t}$. Then use the ERF.
4. Extended version: if $P(i\omega) = 0$ then you find the solution by complexifying $P(D)x = B\cos(\omega t)$ to $P(D)\tilde{x} = Be^{i\omega t}$. Then use the extended ERF.

(d) Undetermined coefficients

1. For solving $P(D)x =$ a polynomial.
2. Usual version: guess a solution $x(t) =$ a polynomial of the same degree. Then substitute and solve for the coefficients.
3. **Example.** Solve $x'' + 8x' + 7x = 2t$.
   answer: Try $x = At + B$.
   Substitution gives $7At + (8A + 7B) = 2t$
   Now equate coefficients: $7A = 2$, $8A + 7B = 0$. (So, $A = 2/7$, $B = -16/49$.)
4. Extended version: If the DE doesn't go all the way to x then multiply the guess by the right power of t
5. **Example.** Solve $x^{(4)} + 8x''' = 2t$.
   answer: This only goes to x''', so multiply the guess by t^3
   That is, guess $x = At^4 + Bt^3$.

V. Linear Operators in General

1. An operator T is linear if $T(c_1 f + c_2 g) = c_1 Tf + c_2 Tg$ for all functions f, g and constants c_1, c_2.

2. Our main examples of linear operators are D, $P(D)$.
3. Our main example of a non-linear operator is the squaring operator, $Tf = f^2$.
4. Linearity is almost always easy to check for.

VI. Physical Models

1. Exponential growth and decay: DE is $y' + ky = f(t)$.
2. Spring-mass-dashpot: DE is $my'' + by' + ky = F(t)$,
   where m = mass, b = damping, k = spring constant, F = external (driving) force.
3. LRC circuit: DE is $LI'' + RI' + \frac{1}{C}I = E'$,
   where L = inductance, R = resistance, C = capacitance, E = input voltage.
4. Mixing tanks
   Remember work with amounts not concentrations.
   Rate of change = rate in - rate out.

VII. Amplitude, Phase Lag, Resonance

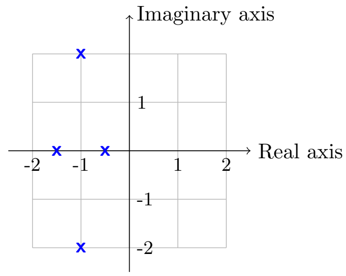
1. Consider the system
   $$my'' + by' + ky = kF_0 \cos(\omega t),$$
   where **we have declared** $F_0 \cos(\omega t)$ to be the input.
   - The characteristic polynomial is $P(r) = mr^2 + br + k$.
   - The input (angular) frequency is ω.
   - The periodic solution (response) is $y_p = g(\omega)F_0 \cos(\omega t - \phi)$.
   - The natural frequency of the system is $\omega_0 = \sqrt{k/m}$. This is the frequecy of oscillation of the the undamped unforced spring: $mx'' + kx = 0$.
   - $A = g(\omega)F_0$ is called the amplitude, where $g(\omega) = k/|P(i\omega)|$.
     The function $g(\omega)$ is called the gain or amplitude response of the system. It depends on ω (and m, b and k).
   - ϕ also depends on ω. The function $\phi(\omega)$ is called the phase lag or the phase response of the system.
   - Practical resonance occurs if $g(\omega)$ has a maximum value at ω_r (for $\omega_r > 0$).
     If there is no such maximum then the system does not have practical resonance.
   - Pure resonance can only happen if $b = 0$. In this case, at $\omega = \omega_0$ we say the gain $g(\omega_0)$ is infinite. Really, when $\omega = \omega_0$ the ERF gives $y_p = \dfrac{t \sin \omega_0 t}{2m\omega_0}$. This is not a sinusoid, rather it is a 'growing' oscillation.
2. Remember the gain depends on what we consider the input.
   For example, consider the DE: $my'' + by' + ky = bF_0 \cos(\omega t)'$,
   but still consider $F_0 \cos(\omega t)$ to be the input. Then the gain is $g(\omega) = \dfrac{b\omega}{|P(i\omega)|}$.
   There are many variations on this.

VIII. Pole diagrams

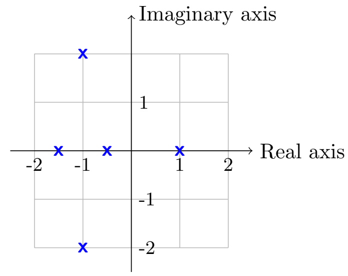
1. For our systems $P(D)x = f$, the pole diagram is drawn in the complex plane.
   The pole diagram tells us a lot about the homogeneous system.
   - We call the charactistic roots poles.
   - You put an $\times$ at each pole.
   - By counting the poles you can determine the order of the system.
   - If all the poles are in the left half-plane then the system is stable because all the exponents in the homogeneous solutions have negative real part.
   - If there are complex poles then the system is oscillatory.

- For a stable system the exponential rate that the unforced (homogeneous) system returns to equilibrium is determined by the real part of the right-most pole.

2. Examples



4 poles, stable, oscillatory



5 poles, unstable, oscillatory

End of review sheet 1