Probe Spectroscopy of Many-Body Quantum Chaos

Setup

Consider a few- or many-body matter system M , to which we **do not have direct or unrestricted access**. A realistic way to investigate such a system is via a **quantum probe** P. Fortunately, one can engineer or tune the interaction between the probe P and **any chosen subsystem of** M at a time. All information about M can only be **indirectly inferred** from measurements performed on P .

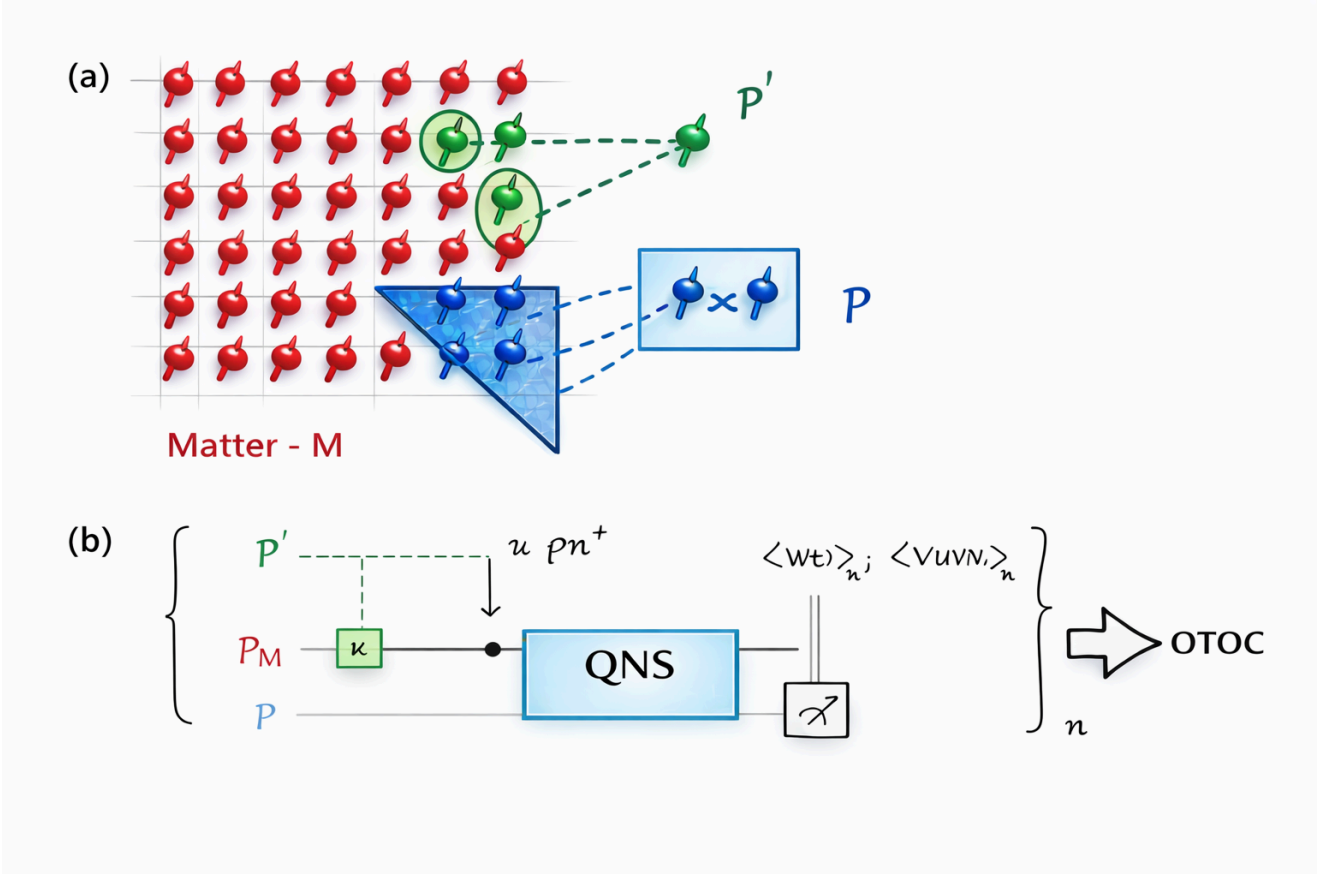
Question:
Can we use the probe P to measure the **OTOC** (out-of-time-ordered correlator), which is related to **quantum chaos**, of the system M ?

Strategy: Random Circuits $\times$ QNS (Quantum Noise Spectroscopy)

The strategy to probe M is through a "marriage" between

- the **random circuits method** (as in Vermersch *et al.*, PRX),
- and **Quantum Noise Spectroscopy (QNS)**.

The **full OTOC protocol** consists of a series of **mutually correlated random-circuit-based QNS experiments**. Each prescribed QNS experiment is **not** a regular unitary evolution on M alone (cf. PRX), but instead on the **joint probe–matter system** $P-M$. In the **QNS language**, the system M acts as a **"noise source"** (environment), since its details are unknown -or, in this case, hard to efficiently simulate- and need to be inferred. We adopt this perspective of an **open quantum system** to learn properties of M .



(a) Schematic of Probe Spectroscopy of Many-Body Quantum Chaos. The matter system M's information (OTOC) can be learnt through probes. The ancillary P' is used to apply random unitaries on M, and probe P couples to system of M to extract information using environmental QNS experiment by treating M as environment. (b) A Circuit model-based illustration of random-circuit QNS to learn OTOC from mutually correlated smaller-correlations reconstructed from single QNS.

Correlation Functions in QNS

In QNS, if we have a well-controlled **multi-qubit probe**, one can probe correlation functions of the environment M of the form

$$\langle B_{p_1}(t_1)\, B_{p_2}(t_2) \cdots B_{p_n}(t_n) \rangle,$$

where:

- p_i labels the **spatial degree of freedom** on the probe P ,
- B_i is the **operator of** M that couples to probe qubit p_i .

The causality limits the time ordering such that

$$t_1 \leq t_2 \cdots \leq t_j \geq \cdots \geq t_n, \quad \text{with } 1 \leq j \leq n.$$

We will check that OTOC are correlators that out of the league of standard QNS.

Time-Ordered Correlators

For a **trivial two-point time-ordered correlator**

$$\langle W(t)\, V \rangle,$$

this can be probed by **standard two-point Gaussian QNS**, if using two-qubit case probe ($q_1 = 1, q_2 = 2$)

$$B_1 = W, \quad B_2 = V,$$

with interaction times

$$t_1 = t, \quad t_2 = 0.$$

Out-of-Time-Ordered Correlators

For a **true OTOC**, e.g.

$$\langle W(t)\, V\, W(t)\, V \rangle,$$

the **violation of causality** prevents the use of standard QNS protocols directly.

Therefore, **randomization and correlations between multiple QNS experiments** are required.

OTOC Definition

The **standard infinite-temperature OTOC** (we focus lowest order OTOC, see Google's Quantum Echo) is defined as

$$C(t) \;=\; \mathrm{Tr}[\rho\, W(t)\, V\, W(t)\, V],$$

where

$$W(t) = U^{\dagger}(t)\, W\, U(t),$$

and $U(t)$ is the many-body time-evolution operator, which is **classically hard to simulate** for interacting systems.

Typically:

- W, V are **local operators**,
- for a many-qubit system, W, V can be **single-Pauli operators**.

The fundamental difficulty in computing $C(t)$ arises from the complexity of $U(t)$.
Random-circuit experiments allow extraction of $C(t)$ when the initial state to be computed is at infinite temperature ($\beta = 0$), or $\rho \propto \mathbb{I}$.

Finite-Temperature OTOC

We now focus on **finite temperature** ($\beta \neq 0$) thermal states.

Define

$$\rho = \frac{e^{-\beta H}}{\mathrm{Tr}(e^{-\beta H})}.$$

We use the **regularized finite-temperature OTOC**

$$F_\beta(t) = \mathrm{Tr}\Big[\rho^{1/4}\, W(t)\, \rho^{1/4}\, V\, \rho^{1/4}\, W(t)\, \rho^{1/4}\, V\Big]. \tag{1}$$

(Equivalent form written in the notes:)

$$F_\beta(t) = \mathrm{Tr}\Big[\rho^{1/2}\, W(t)\, \rho^{1/2}\, V\, W(t)\, V\Big]. \qquad (\text{------- omit it -------})$$

Let $N = \dim(\mathcal{H})$ represents the Hilbert space dimension of M, one can do the high-temperature expansion:

$$\rho^{1/4} = \frac{1}{N}\left[\mathbb{I} - \frac{\beta}{4}H + \frac{\beta^2}{32}H^2 - \frac{\beta^2}{8N}\mathrm{Tr}(H^2)\,\mathbb{I}\right] + O(\beta^3). \tag{2}$$

Hereafter, we will focus on the truncation at **quadratic** order, which is a bit trivial extension beyond linear-order truncation demonstrated in Vermersh PRX.

Plugging into $F_\beta(t)$, we obtain

$$\begin{aligned} F_\beta(t) = \frac{1}{N}\Bigg[& \mathrm{Tr}(W(t)\,V\,W(t)\,V) - \frac{\beta}{2}\,\mathrm{Tr}(H\,W(t)\,V\,W(t)\,V) \\ & + \frac{\beta^2}{8}\,\mathrm{Tr}(H\,W(t)\,H\,V\,W(t)\,V) + \frac{\beta^2}{8}\,\mathrm{Tr}\big(W(t)\,V\,W(t)\,H^2\,V\big) \\ & + \frac{\beta^2}{4}\,\mathrm{Tr}(W(t)\,H\,V\,W(t)\,H\,V) - \frac{\beta^2}{2N}\,\mathrm{Tr}(H^2)\;\mathrm{Tr}(W(t)\,V\,W(t)\,V)\Bigg] + O(\beta^3). \end{aligned}$$

Define

$$A = W(t), \quad B = V\,W(t)\,V, \quad C = H\,V\,W(t)\,H\,V.$$

Then

$$F_\beta(t) \approx \frac{1}{N}\left[\mathrm{Tr}(AB) - \frac{\beta}{2}\,\mathrm{Tr}([H,A]B) + \frac{\beta^2}{8}\,\mathrm{Tr}([H,[H,A]]B) + \frac{\beta^2}{4}\,\mathrm{Tr}(AC) - \frac{\beta^2}{2N}\,\mathrm{Tr}(H^2)\;\mathrm{Tr}(AB)\right]. \tag{3}$$

Equivalently,

$$F_\beta(t) \approx \frac{1}{N}\left\{\mathrm{Tr}[W(t)VW(t)V] - \frac{\beta}{2}\mathrm{Tr}(\widetilde{W}(t)VW(t)V) + \frac{\beta^2}{2}\mathrm{Tr}(\widetilde{\widetilde{W}}(t)VW(t)V) + \frac{\beta^2}{4}\mathrm{Tr}(W(t)\widetilde{V}W(t)\widetilde{V}) - \frac{\beta^2}{2N}\mathrm{Tr}(H^2)\,\mathrm{Tr}[W(t)VW(t)V]\right\},$$

with

$$\widetilde{W} = \{H, W\}, \quad \widetilde{\widetilde{W}} = \{H, \{H, W\}\}, \quad \widetilde{V} = HV.$$

Interpretation

Thus, the **finite-temperature OTOC** reduces to knowing a **complete set of infinite-temperature OTOCs** of the form

$$\langle K(t)\,V\,W(t)\,V\rangle, \quad K \in \{W,\ [H,W],\ [H,[H,W]]\}.$$

If $V \to HV$, then for **local** H, V, this still produces **local operators**, the above infinite-temperature OTOCs represents the algebraically complete form that our protocol needs to address.

Random Circuits Protocol

Random circuits can probe

$$\langle K(t)\,V\,W(t)\,V\rangle = \mathrm{Tr}[\rho\,K(t)\,V\,W(t)\,V]|_{\rho\propto\mathbb{I}}.$$

In the real experiment, M is prepared in the **thermal state** whose temperature (β) is known.

Define

$$X_u = \langle V\,K(t)\,V\rangle_{\rho_u} = \mathrm{Tr}\big[u\rho u^\dagger\,VK(t)V\big], \tag{4}$$

$$Y_u = \langle W(t)\rangle_{\rho_u} = \mathrm{Tr}\big[u\rho u^\dagger\,W(t)\big]. \tag{5}$$

where $\rho_u = u\rho u^\dagger$. (One can also let $X_u =< K(t) >$ and $Y_u =< VW(t)V >$ that is operational)

Then

$$X_u Y_u = \mathrm{Tr}\big[(\rho\otimes\rho)\,(u^\dagger\otimes u^\dagger)\,(VK(t)V\otimes W(t))\,(u\otimes u)\big]. \tag{6}$$

We assume the sampling of random unitary $\{u\}$ satisfies the **2-design**, so the above algebra after ensemble average can be represented by the **Haar measure** on the unitary group. By denoting **Haar twirling map** Φ (2nd order) as

$$\Phi^{(2)}(X) \equiv \int du\,(u^\dagger\otimes u^\dagger)\,X\,(u\otimes u).$$

Haar Twirling projects X onto the subspace of operators that commute with every $U\otimes U$. This subspace is exactly the **commutant** of the representation $U\otimes U$. The **Schur–Weyl duality** gives that the commutant of $U\otimes U : U\in U(N)\{U\otimes U : U\in U(N)\}$ is spanned by only two operators: $\mathbb{I}\otimes\mathbb{I}$ and SWAP.

After averaging,

$$\mathbb{E}_u(X_u Y_u) = \overline{X_u Y_u} = \mathrm{Tr}\Big[(\rho\otimes\rho)\,\Phi^{(2)}(VK(t)V\otimes W(t))\Big], \tag{7}$$

$$\Phi^{(2)}(\cdot) = \alpha(\cdot)\,\mathbb{I} + \beta(\cdot)\,\hat{S}, \tag{8}$$

with $\hat{S}$ the **swap operator**.

Let

$$\mathcal{M} = VK(t)V\otimes W(t).$$

Solving for $\alpha(\mathcal{M}), \beta(\mathcal{M})$:

$$\alpha(\mathcal{M}) = \frac{\mathrm{Tr}(\mathcal{M}\hat{S}) - N\,\mathrm{Tr}(\mathcal{M})}{N(1-N^2)}, \quad \beta(M) = \frac{\mathrm{Tr}(\mathcal{M}) - N\ \mathrm{Tr}(\mathcal{M}\hat{S})}{N(1-N^2)}.$$

Hence,

$$\mathbb{E}_u(X_u Y_u) = \frac{1}{N(1-N^2)}\left[1 - N\ \mathrm{Tr}(\rho^2)\right]\langle K(t)\,V\,W(t)\,V\rangle. \tag{9}$$

In the above, the quantity $X_u = \langle u^\dagger VK(t)Vu\rangle_\rho$, $Y_u =$. . is measured using **QNS**.

Uniform Clifford group

The random unitary $\{u\}$ ensemble must satisfy unitary 2-design to form the fully-symmetrized Haar measure on the whole unitary group. In our probe spectroscopy, application of $\{u\}$ on M is nontrivial and need elaboration here.

First of all, we notice that the limited access to M and finite locality of $P-M$ will rule out of straightforward application of any unitary in $SU(N)$ on M whenever N beyond a few. Instead, the $SU(N)$ will be implemented on M through a shallow circuit of local unitaries, each of which only involves smallest few subsystem of M and P as a coupler. The shallow circuit approximate arbitrary u and more importantly, we can simply let the unitaried on M to be sampled from the uniform distribution over the Clifford group, as it forms an **exact unitary 2-design** [PRA 80, 012304 2009].

All you need is a shallow Clifford circuit that is drawn randomly from the ensemble.

- Implementing “random u” via shallow Clifford circuits is realistic ~*$\mathrm{poly}(n)$
- There are rigorous bounds showing they converge to an approximate unitary t-design after polynomial depth[Comm. Math. Phys. 346, 397 (2016)]

Operationally, we can add another coupler P' to implement the Clifford group on M, and P' can be orthogonal degree of freedom to P. Once the random unitary is done, P' is left idle and totally trivial.

Hereafter, we focus on $P-M$ interaction that is at the heart of QNS.

QNS Measurement & Interaction Design

For each QNS experiment, one must estimate:

$$\langle u^\dagger VK(t)Vu\rangle_\rho = \langle VK(t)V\rangle_{\rho_u}, \quad \langle u^\dagger W(t)u\rangle = \langle W(t)\rangle_{\rho_u}.$$

We clarify the details by focusing on the former.

QNS is designed to estimate

$$\langle B_{p_1}(t_1)\,B_{p_2}(t_2)\,B_{p_3}(t_3)\rangle = \langle V\,K(t)\,V\rangle,$$

and

$$\langle B(t)\rangle = \langle W(t)\rangle$$

The single-point correlator is easily handle by linear QNS.

Probe–Matter Coupling

In the probe–matter interaction, one can engineer a **three-qubit probe** coupled to M **in the lab** as

$$H_{PM} = \sum_p \frac{1}{2}\sigma_p^z \otimes B_p = Z\otimes\mathbb{I}\otimes\mathbb{I}\otimes V_a + \mathbb{I}\otimes Z\otimes\mathbb{I}\otimes K + \mathbb{I}\otimes\mathbb{I}\otimes Z\otimes V_b.$$

With such coupling, together with unitary probe control, one can probe

$$\langle B_{p_1}(t_1)\,B_{p_2}(t_2)\,B_{p_3}(t_3)\rangle = \langle V_a\,K(t)\,V_b\rangle.$$

Let

$$B_{p_1} = V_a, \quad B_{p_2} = K, \quad B_{p_3} = V_b,$$

with

$$t_1 = t_3 = 0, \quad t_2 = t.$$

Remark:
Since the target at hand is $V_a = V_b = V$, so one can also use a two-qubit probe, where $p_1 = p_3 = 2$; but notice it is still a three-point correlator

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Complexity

Recall that the QNS layer measures quantities of the form

$$\langle U^\dagger V K(t) V U \rangle_{\rho_0}, \qquad \langle U^\dagger W(t) U \rangle_{\rho_0},$$

where ρ_0 is the (unknown) initial state of the matter system M .

Equivalently, in the random-circuit notation used earlier:

$$\langle V K(t) V \rangle_{\rho_u}, \qquad \langle W(t) \rangle_{\rho_u}.$$

As focus on

$$\langle \cdot \rangle_{\rho_u} \quad \text{instead of} \quad \langle U^\dagger(\cdot) U \rangle_{\rho_0}.$$

That is, we regard the randomization as absorbed into the effective state ρ_u .

From the finite-temperature expansion, the operators K that appear are:

$$K \in \{W,\ [H,W],\ [H,[H,W]]\}.$$

Define the **degree** (or distance) $d(\cdot)$ of an operator as its **Hamming distance to the identity**, i.e. the number of non-identity local factors in its operator support.

For local Pauli operators:

$$d(W) = d(V) = 1,$$

since W, V are single-Pauli operators.

Structure of the Hamiltonian

Assume the Hamiltonian decomposes as

$$H = \sum_{s=1}^{n_H} H_s,$$

where:

- s indexes local subsystems,
- each H_s is a local Hamiltonian term.

Example:
For an L -site nearest-neighbor Ising chain,

$$n_H = L - 1,$$

and each H_s is of ZZ -type.

Define

$$d_H := \max_{s \in H} d(H_s),$$

i.e. the **maximum degree** of any local Hamiltonian term.

Degree Growth Under Commutators

Then the degree of the operators K satisfies the bounds:

$$d(K) \ \le \ \begin{cases} 1, & K = W, \\ d_H + 1, & K = [H, W], \\ 2d_H + 1, & K = [H,[H,W]]. \end{cases}$$

Consequence for Probe–Matter Interaction Size

Thus, in the **probe–matter (P–M) interaction**, the **size of the matter subsystem** that must be accessed (i.e. coupled to the probe) can be **minimally upper-bounded** by

$$2d_H + 1.$$

> *Put in a theorem environment — make it a theorem.*

(Your note here is preserved verbatim in intent.)

Learning / Sample Complexity

We now focus on the **learning complexity** of estimating

$$\langle VK(t)V\rangle, \quad \text{in particular } \langle VW(t)V\rangle.$$

For the first commutator level:

$$\langle V[H, W(t)]V\rangle = \sum_{s=1}^{n_H}\langle V[H_s, W(t)]V\rangle = \sum_{s=1}^{n_H}\langle V\tilde{W}_s(t)V\rangle,$$

where

$$\tilde{W}_s = [H_s, W].$$

For the second commutator level:

$$\langle V[H,[H,W(t)]]V\rangle = \sum_{s_1,s_2}\langle V\tilde{W}_{s_1s_2}(t)V\rangle,$$

with

$$\tilde{W}_{s_1s_2} = [H_{s_1},[H_{s_2},W]].$$

Equivalently, one can write

$$\langle \tilde{V}\, W(t)\, \tilde{V}\rangle = \sum_{s_1,s_2}\langle V_{s_1}\, W(t)\, V_{s_2}\rangle, \qquad V_s := H_s V.$$

Similarly,

$$\langle H^2\rangle = \sum_{s_1,s_2}\langle H_{s_1}H_{s_2}\rangle,$$

which is **classically simulable**.

(Your marginal note “classically simulable” is preserved.)

For **each QNS-based correlator** (each RHS term above),
a number of shots

$$\tilde{N} \gg 1$$

is required to estimate it to fixed precision.

Collecting all terms, the **total QNS sampling cost** scales as

$$\tilde{N}\left(2n_H^2 + n_H + 2\right).$$

Therefore, the **full sample complexity** to learn an
n_H -size system OTOC is

$$\mathcal{O}\left(\tilde{N}\, n_H^2\right),$$

i.e. **only quadratic** in the number of local Hamiltonian terms.

> *Put in theorem environment.*

(This remark is also preserved.)