

Characterization of Arbitrary-Order Correlations in Quantum Baths by Weak Measurement

Ping Wang,¹ Chong Chen,¹ Xinhua Peng,^{2,3} Jörg Wrachtrup,^{4,5} and Ren-Bao Liu^{1,6,*}

¹*Department of Physics, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong, China*

²*Hefei National Laboratory for Physical Sciences at Microscale and Department of Modern Physics, University of Science and Technology of China, Hefei 230026, China*

³*CAS Key Laboratory of Microscale Magnetic Resonance and Synergetic Innovation Center of Quantum Information & Quantum Physics, University of Science and Technology of China, Hefei 230026, China*

⁴*3rd Institute of Physics, Research Center SCoPE and IQST, University of Stuttgart, 70569 Stuttgart, Germany*

⁵*Max Planck Institute for Solid State Research, 70569 Stuttgart, Germany*

⁶*The Hong Kong Institute of Quantum Information Science and Technology, The Chinese University of Hong Kong, Shatin, New Territories, Hong Kong, China*



(Received 16 February 2019; published 1 August 2019)

Correlations of fluctuations are the driving forces behind the dynamics and thermodynamics in quantum many-body systems. For qubits embedded in a quantum bath, the correlations in the bath are key to understanding and combating decoherence—a critical issue in quantum information technology. However, there is no systematic method for characterizing the many-body correlations in quantum baths beyond the second order or the Gaussian approximation. Here we present a scheme to characterize the correlations in a quantum bath to arbitrary order. The scheme employs a weak measurement of the bath via the projective measurement of a central system. The bath correlations, including both the “classical” and the “quantum” parts, can be reconstructed from the correlations of the measurement outputs. The possibility of full characterization of many-body correlations in a quantum bath forms the basis for optimizing quantum control against decoherence in realistic environments, for studying the quantum characteristics of baths, and for the quantum sensing of correlated clusters in quantum baths.

DOI: [10.1103/PhysRevLett.123.050603](https://doi.org/10.1103/PhysRevLett.123.050603)

Introduction.—The correlations of fluctuations are the driving forces underlying the quantum dynamics and thermodynamic processes (such as critical phenomena) of quantum many-body systems. Conventionally the correlations in many-body physics are considered at the second order or in the Gaussian approximation (which amounts to taking into consideration the quasiparticle excitations around the mean field), with the assumption that the higher order correlations are usually much smaller than the second order ones. Recent studies have revealed the importance of higher order correlations, especially in mesoscopic quantum systems [1,2]. The study of higher-order correlations, however, is challenging due to their many-body nature.

The quantum many-body correlations are particularly important to quantum information technology for their relevance in decoherence of central quantum systems coupled to quantum baths [3–7]. Here, and often in literature, the system is referred to as “central” (not in a geometrical sense) to be differentiated from the environment or bath. Recently, Gasbarri and Ferialdi [8] show that the dynamics of a central quantum system is determined by the correlations in the quantum bath and the effects of a quantum bath can be fully simulated by *complex* classical noises. This remarkable work paves the way of optimal quantum control for quantum gates and quantum memory in realistic environments. For classical noises, once the

noise spectra are known, the quantum control of the central quantum systems can be designed to combat the decoherence [9–11]. Outstanding examples are dynamical decoupling [12–16] and dynamically optimized quantum gates [17–20]. For quantum baths, the backaction of the central system means that the bath correlations may need to be characterized each time for each new quantum operation and yet the optimization would be extremely time consuming, if not impossible at all, due to the notoriously difficult quantum many-body problems. Now thanks to the progress in Ref. [8], the quantum control optimization can be applied to quantum baths as well, as long as the bath correlations can be characterized.

Thus, both for studying many-body physics and for applications in quantum information technology, what is highly desirable is a systematic method to measure the many-body correlations in a quantum bath. With the assumption of Gaussian noises, the noise spectroscopy (e.g., by dynamical decoupling or frequency combing) [21–26] can be employed to obtain the noise correlation spectra. The applications of the frequency comb approach, e.g., to higher-order correlations, however, are tricky due to the interference of nonlinear effects [21,23,24] and spurious signals [27] and yet are often limited to the case of pure dephasing [24].

In this Letter, we present a general scheme for completely characterizing the correlations in a quantum bath. The scheme is based on the weak measurement of the bath via projective measurement of the central system. By designing the measurement sequence, the bath correlations at arbitrary orders can be reconstructed from the correlations of the measurement outputs. Quantum weak measurement has been used to monitor quantum coherent oscillations [28,29], characterize spectral diffusion [30], and measure the nonsymmetric correlations [31,32]. Multitime correlations of continuous weak measurements have also been studied [33–35]. Recently, weak measurement was considered for improving spectral resolution in quantum sensing [36,37]. The application of weak measurement enabled the high-spectral-resolution magnetic resonance spectroscopy of single nuclear spins [37], which is possible due to the fact that the disturbance to the system caused by the weak measurement (i.e., measurement induced decoherence) is negligible. The weak disturbance feature of weak measurement is exploited in our scheme of characterizing bath correlations.

Before proceeding to present our scheme, here we first summarize the main results of Ref. [8]. A general Hamiltonian $H = H_0 + V$ is considered, where $H_0 = H_S(t) + H_B$ contains the system Hamiltonian $H_S(t)$ (which may be time dependent due to external control) and the bath Hamiltonian H_B , and $V = \sum_\alpha S_\alpha B_\alpha$ is the coupling between the bath operators B_α (the noise fields) and the system operators S_α . In the interaction picture,

$$\hat{V}(t) = \sum_\alpha \hat{S}_\alpha(t) \hat{B}_\alpha(t),$$

where the operator in the interaction picture is given by $\hat{A}(t) \equiv U_0^\dagger(t) A U_0(t)$ with $U_0(t) \equiv \mathcal{T} e^{-i \int_0^t H_0(\tau) d\tau}$ and \mathcal{T} denotes time ordering. The initial state of the system and the bath is assumed to be separable, described by the density operator $\rho(0) = \rho^S(0) \otimes \rho^B$. The density operator in the interaction picture, $\hat{\rho}(t) \equiv U_0^\dagger(t) \rho(t) U_0(t)$, evolves according to $\hat{\rho}(t) = \mathcal{T} e^{\int_0^t \mathcal{L}(\tau) d\tau} \hat{\rho}(t_0)$, with the Liouville superoperator \mathcal{L} defined by $\mathcal{L}(t)\hat{\rho} = -i[\hat{V}(t)\hat{\rho} - \hat{\rho}\hat{V}(t)]$. Defining the superoperators \mathcal{A}^\pm as $\mathcal{A}^-\hat{B} = -i(\hat{A}\hat{B} - \hat{B}\hat{A})/2$ (essentially a commutator) and $\mathcal{A}^+\hat{B} = (\hat{A}\hat{B} + \hat{B}\hat{A})/2$ (essentially an anticommutator) and using the identity $-i[\hat{A}\hat{B}, \hat{C}] = 2(\mathcal{A}^+\mathcal{B}^- + \mathcal{A}^-\mathcal{B}^+)\hat{C}$, one obtains the reduced density operator of the central system $\hat{\rho}^S(t) \equiv \text{Tr}_B \hat{\rho}(t)$ as

$$\begin{aligned} \hat{\rho}^S(t) = & \sum_{N=0}^{+\infty} \frac{2^N}{N!} \sum_{\{\alpha_i\}, \{\eta_i=\pm\}} \int_0^t dt_1 dt_2 \cdots dt_N C_{\alpha_N, \dots, \alpha_2, \alpha_1}^{\eta_N, \dots, \eta_2, \eta_1} \\ & \times [\mathcal{T} S_{\alpha_N}^{\eta_N}(t_N) \cdots S_{\alpha_2}^{\eta_2}(t_2) S_{\alpha_1}^{\eta_1}(t_1)] \hat{\rho}^S(0), \end{aligned} \quad (1)$$

determined by the bath field correlations

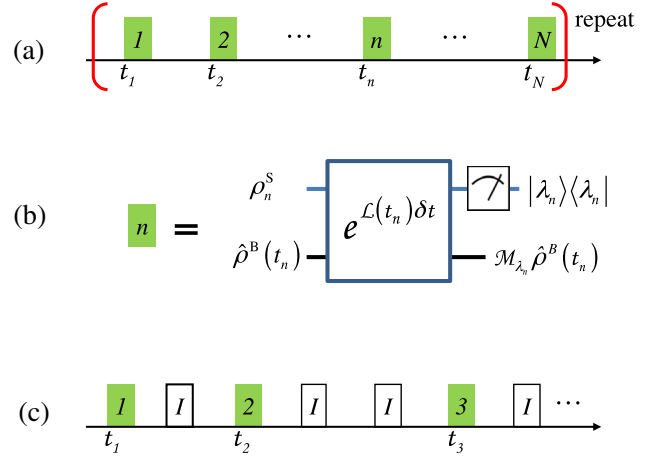


FIG. 1. Weak measurement for reconstruction of bath correlations. (a) A unit sequence (to be repeated many times) of N weak measurements at different times for reconstructing the bath correlations at the N th order. (b) Realization of a weak measurement on the bath via the projective measurement of the central system. (c) Reconstruction of the bath correlations by selecting a subset of outputs from a long measurement sequence, with the unused outputs in between taken as “idle” (I).

$$C_{\alpha_N, \dots, \alpha_2, \alpha_1}^{\eta_N, \dots, \eta_2, \eta_1} = \text{Tr}[\mathcal{T} \mathcal{B}_{\alpha_N}^{\eta_N}(t_N) \cdots \mathcal{B}_{\alpha_2}^{\eta_2}(t_2) \mathcal{B}_{\alpha_1}^{\eta_1}(t_1) \rho^B], \quad (2)$$

where $\bar{\eta}_n = -\eta_n$. For t_N being the latest time, the correlation $C_{\alpha_N, \dots, \alpha_1}^{\eta_N, \dots, \eta_1}$ ($\eta_N = -$) vanishes since that the last superoperator $\mathcal{B}_{\alpha_N}^-$ is a commutator and the trace of a commutator is always zero. In terms of the irreducible bath correlations (cumulants) $\tilde{C}_{\alpha_N, \dots, \alpha_1}^{\eta_N, \dots, \eta_1}$ [8], the central system dynamics can be written as

$$\hat{\rho}^S(t) = \mathcal{T} e^{\sum_{N=1}^{+\infty} (2^N/N!) \int_0^t dt_N \cdots dt_1 \tilde{C}_{\alpha_N, \dots, \alpha_1}^{\eta_N, \dots, \eta_1} S_{\alpha_N}^{\eta_N}(t_N) \cdots S_{\alpha_1}^{\eta_1}(t_1)} \hat{\rho}^S(0) \quad (3)$$

Hereafter summation over the repeated indices η_n and α_n is assumed. The effects of a quantum bath can be fully simulated by complex classical noises $b_\alpha(t) = b_\alpha^+(t) + i b_\alpha^-(t)$ that have the correlations $\langle b_{\alpha_N}^{\eta_N}(t_N) \cdots b_{\alpha_2}^{\eta_2}(t_2) b_{\alpha_1}^{\eta_1}(t_1) \rangle = C_{\alpha_N, \dots, \alpha_2, \alpha_1}^{\eta_N, \dots, \eta_2, \eta_1}$. The equivalence between a quantum bath and complex classical noises in their effects on central system dynamics offers an interesting venue for studying non-Hermitian quantum dynamics and thermodynamics in complex plane [38–40].

Measurement of bath correlations.—We present a protocol for measuring the bath correlations to an arbitrary order. The scheme is based on weak measurement of the bath via projective measurement of the central system (see Fig. 1). To measure an N th order correlation, a unit sequence of N weak measurements [Fig. 1(a)] is applied to the quantum bath. In each unit, the quantum bath is prepared in the initial state ρ^B at $t = 0$ and then evolves under the bath Hamiltonian H_B . At time t_n (for

$n = 1, 2, \dots, N$), the central system is prepared in the state ρ_n^S , and then is coupled to the bath through the interaction $V = \sum_\alpha S_\alpha B_\alpha$ for a small period δt of evolution. The state of the central system and the bath, in the interaction picture, becomes $\hat{\rho}(t_n + \delta t) \approx e^{\mathcal{L}(t_n)\delta t} \hat{\rho}^B(t_n) \otimes \rho_n^S$, where $\mathcal{L}(t) = 2 \sum_\alpha [\mathcal{S}_\alpha^+ \mathcal{B}_\alpha^-(t) + \mathcal{S}_\alpha^- \mathcal{B}_\alpha^+(t)]$. A quantity Λ_n of the central system is measured at $t_n + \delta t$. The output would be randomly an eigenvalue λ_n of Λ_n corresponding to the eigenstate $|\lambda_n\rangle$. Note that here the subscript n indicates that the eigenstate belongs to the observable Λ_n under the n th measurement and λ_n can take different eigenvalues of Λ_n . The unit sequence of N measurements is repeated many times. The outputs of the N measurements, averaged over the repeated units, yield the measurement correlation $G^{(N)} = \langle \lambda_N \cdots \lambda_2 \lambda_1 \rangle$.

The projective measurement of the system operator Λ_n constitutes a weak measurement of the bath (due to the weak entanglement during the interaction in the small period of time). See Fig. 1(b) for an illustration. The weak measurement is characterized by the Kraus superoperator $\mathcal{M}_{\lambda_n} = \text{Tr}_S[|\lambda_n\rangle\langle\lambda_n| e^{\mathcal{L}(t_n)\delta t} \rho_n^S]$ corresponding to the output λ_n . The probability of the output λ_n is $p(\lambda_n) = \text{Tr}_B[\mathcal{M}_{\lambda_n} \hat{\rho}^B(t_n)]$. The bath state after the measurement is $\mathcal{M}_{\lambda_n} \hat{\rho}^B(t_n) / p(\lambda_n)$. The joint probability of a sequence of N outputs is $p(\lambda_N, \dots, \lambda_1) = \text{Tr}_B[\mathcal{T} \mathcal{M}_{\lambda_N} \cdots \mathcal{M}_{\lambda_1} \rho^B]$. The measurement correlation is $G^{(N)} = \sum_{\{\lambda_n\}} p(\lambda_N, \dots, \lambda_1) \lambda_N \cdots \lambda_1$. For small δt , the evolution during the interaction $e^{\mathcal{L}(t_n)\delta t} \approx 1 + \mathcal{L}(t_n)\delta t$. To pick up the signal proportional to the noise fields B_α (hence proportional to the interaction \mathcal{L}), we choose the initial state ρ_n^S and the observable Λ_n such that the background term $\text{Tr}[\Lambda_n \rho_n^S] = 0$. Thus, the measurement correlation up to the leading order of δt is

$$G^{(N)} \approx \delta t^N \sum A_{\alpha_N}^{\eta_N}(t_N) \cdots A_{\alpha_2}^{\eta_2}(t_2) A_{\alpha_1}^{\eta_1}(t_1) C_{\alpha_N, \dots, \alpha_2, \alpha_1}^{\eta_N, \dots, \eta_2, \eta_1}, \quad (4)$$

where the coefficient

$$A_{\alpha}^{\eta}(t_n) = 2 \text{Tr}_S[\Lambda_n \mathcal{S}_{\alpha}^{\eta} \rho_n^S]. \quad (5)$$

Equation (4) defines a linear equation for the bath correlations of the N th order. Since the coefficients can be independently set by choosing the system state ρ_n^S and the observable Λ_n , a set of linearly independent equations can be established. By solving the set of linear equations, the bath correlations can be reconstructed.

With the cumulant expansion in Eq. (3), only the irreducible bath correlations are needed. Below we use the shorthand notation $C(N, \dots, 1) \equiv C_{\alpha_N, \dots, \alpha_1}^{\eta_N, \dots, \eta_1}$. The irreducible bath correlations (cumulants) $\tilde{C}(N, \dots, 2, 1)$ can be recursively obtained by $\tilde{C}(1) = C(1)$, $\tilde{C}(2, 1) = C(2, 1) - \tilde{C}(2)\tilde{C}(1)$, $\tilde{C}(3, 2, 1) = C(3, 2, 1) - \tilde{C}(3, 2)\tilde{C}(1) - \tilde{C}(3, 1)\tilde{C}(2) - \tilde{C}(2, 1)\tilde{C}(3) - \tilde{C}(3)\tilde{C}(2)\tilde{C}(1)$, and so on.

The cumulant expansion can often be truncated at a rather low order. In particular, in the case of Gaussian baths (such as a quadratic boson bath [8]), the truncation at the second order irreducible correlations is exact. The truncation approximation would greatly reduce the number of measurements required to reconstruct the bath correlations.

We remark that both the “classical” and the “quantum” parts of the bath correlations can be extracted from the weak measurements. The “quantum” correlations refer to the terms that contain at least one bath superoperator $\mathcal{B}_{\alpha_n}^{\eta_n}$ with $\eta_n = -$ (a commutator) and the “classical” correlations contain only bath superoperators with $\eta_n = +$ (anti-commutators). This classification of bath correlations is based on the observation that the commutator \mathcal{B}_{α}^- would vanish if B_α is a classical noise field. As shown in Eq. (5), to extract a quantum correlation, one just needs to choose the central system state ρ_n^S and observable Λ_n such that $A_{\alpha}^+(t_n) = 2 \text{Tr}_S[\Lambda_n \mathcal{S}_{\alpha}^+ \rho_n^S] \neq 0$. It should be noted that for a bath at infinitely high temperature (such as a nuclear spin bath at room temperature [37]), $\rho^B \propto 1$ and $\mathcal{B}_{\alpha}^- \rho^B = 0$, so the quantum correlations at the second order $C_{\alpha_2, \alpha_1}^{+, -} = 0$. In this case, one needs to examine at least the third order to extract the quantum correlations in a quantum bath. Such higher-order, “quantum” correlations are signatures of coherent clusters in baths [5]. For example, these signatures can be employed for quantum sensing of correlated nuclear spins in nuclear spin baths [5, 41]. The higher-order quantum correlations may also be used to study the quantum characteristics (such as the Leggett-Garg inequality [42]) of quantum baths.

In practice, the protocol for reconstructing the bath correlations can be simplified by exploiting the facts that the perturbation of the weak measurement to the bath is small and the bath is usually in a thermal equilibrium state. One can perform an indefinitely long sequence of weak measurements on the bath at t_k (with, e.g., $t_k = k\tau$ for $k = 1, 2, \dots$). In each shot of measurement, the central system is prepared in state ρ_k^S at t_k , coupled to the bath through V for time δt , and then measured on the observable Λ_k at $t_k + \delta t$. No preparation of the bath state is needed. See Fig. 1(c) for illustration. The measurement correlations at a given order N and for a given timing (t_1, t_2, \dots, t_N) are obtained by selecting a subset of the measurements. The data from the other measurements (taken as idle) are discarded (but would be used for constructing correlations at other orders and/or for other timings). For the measurement whose output λ_k is discarded, the evolution of the bath averaged over all possible outputs is $\sum_{\lambda_k} \mathcal{M}_{\lambda_k} \hat{\rho}^B(t_k) \equiv \mathcal{M}_k \hat{\rho}^B(t_k)$, which amounts to measurement-induced decoherence. If the measurement is weak ($|V\delta t| \ll 1$), the measurement-induced decoherence is negligible, i.e., $\mathcal{M}_k \approx 1$ and $\mathcal{M}_k \hat{\rho}^B(t_k) \approx \hat{\rho}^B(t_k)$. Furthermore, if the bath is in the thermal equilibrium state $\rho^B \propto e^{-H_B/(k_B T)}$ at temperature T , the bath Hamiltonian H_B induces no evolution on it. Therefore, under the conditions

that the bath is initially in a thermal equilibrium state and the measurement is sufficiently weak, the measurement correlations extracted from a subset of the measurements are the same, in the leading order of δt , as those obtained without the idle measurements [that is, the same as those in Eq. (4)]. In this simplified protocol, the sequential weak measurements can be carried out with a simple timing (e.g., equally spaced in time), there is no need to prepare the bath state in each unit sequence of measurement, and the output data can be reused for constructing correlations at different orders and for different timings [37].

Special case of central spin-1/2.—As an example, we present the explicit protocol for reconstructing the correlations in a quantum bath of a central spin-1/2 (qubit). The qubit-bath coupling can be written as $V = \frac{1}{2} \sum_{\alpha=x,y,z} \sigma_{\alpha} B_{\alpha}$, where σ_{α} is the Pauli matrix of the qubit along the α axis and B_{α} is the magnetic noise operator. Without loss of generality, we assume $t_N > t_{N-1} > \dots > t_1$ in the correlation functions.

Let us consider the weak measurement of the bath at t_1 first. The central spin is polarized to be along, e.g., the x axis, described by the density operator $\rho_1^S = (1 + \sigma_x)/2$ at $t = t_1$. After the interaction with the bath through V for time δt , a spin operator Λ_1 is measured. To make the background term $\text{Tr}[\Lambda_1 \rho_1^S]$ vanish, we choose the measurement axis to be along a direction perpendicular to the initial polarization, e.g., $\Lambda_1 = \sigma_y$. With the definition in Eq. (5), the coefficient in Eq. (4) becomes

$$A_y^+(t_1) = \text{Tr}[\sigma_y(\sigma_y \rho_1^S + \rho_1^S \sigma_y)]/2 = 1,$$

$$A_z^-(t_1) = -i \text{Tr}[\sigma_y(\sigma_z \rho_1^S - \rho_1^S \sigma_z)]/2 = 1,$$

and else = 0. Therefore the measurement correlation becomes

$$G^{(N)} = \delta t^N A_{\alpha_N}^{\eta_N}(t_N) \cdots A_{\alpha_2}^{\eta_2}(t_2) (C_{\alpha_N, \dots, \alpha_2, y}^{\eta_N, \dots, \eta_2, -} + C_{\alpha_N, \dots, \alpha_2, z}^{\eta_N, \dots, \eta_2, +}).$$

Or if we choose $\rho_1^S = (1 - \sigma_x)/2$ (central spin initially polarized along the $-x$ direction) and $\Lambda_1 = \sigma_y$, we have $A_y^+(t_1) = -A_z^-(t_1) = 1$ and else = 0. The measurement correlation would be

$$\bar{G}^{(N)} = \delta t^N A_{\alpha_N}^{\eta_N}(t_N) \cdots A_{\alpha_2}^{\eta_2}(t_2) (C_{\alpha_N, \dots, \alpha_2, y}^{\eta_N, \dots, \eta_2, -} - C_{\alpha_N, \dots, \alpha_2, z}^{\eta_N, \dots, \eta_2, +}).$$

The summation and difference of $G^{(N)}$ and $\bar{G}^{(N)}$ pick up the bath correlations $C_{\alpha_N, \dots, \alpha_2, y}^{\eta_N, \dots, \eta_2, -}$ and $C_{\alpha_N, \dots, \alpha_2, z}^{\eta_N, \dots, \eta_2, +}$, respectively. That is

$$G^{(N)} + \bar{G}^{(N)} = 2\delta t^N A_{\alpha_N}^{\eta_N}(t_N) \cdots A_{\alpha_2}^{\eta_2}(t_2) C_{\alpha_N, \dots, \alpha_2, y}^{\eta_N, \dots, \eta_2, -}, \quad (6a)$$

$$G^{(N)} - \bar{G}^{(N)} = 2\delta t^N A_{\alpha_N}^{\eta_N}(t_N) \cdots A_{\alpha_2}^{\eta_2}(t_2) C_{\alpha_N, \dots, \alpha_2, z}^{\eta_N, \dots, \eta_2, +}. \quad (6b)$$

The procedure can be similarly applied to the measurements at other times. For the latest time t_N , since the

correlation function vanishes for $\eta_N = -$, only one set of (ρ_N^S, Λ_N) is needed to pick up the correlation function $C_{\alpha_N, \alpha_{N-1}, \dots, \alpha_1}^{+\eta_{N-1}, \dots, \eta_1}$. Thus, using measurement correlation functions for 2^{N-1} configurations of central spin initialization and measurement directions $\{(\rho_N^S, \Lambda_N)\}$, one can determine 2^{N-1} bath correlation functions $C_{\alpha_N, \alpha_{N-1}, \dots, \alpha_1}^{+\eta_{N-1}, \dots, \eta_1}$ with $\alpha_n = y$ or z corresponding to $\eta_n = -$ or $+$ for each n . The correlations of noise fields along other directions can be similarly determined [e.g., correlations with $(\alpha_n, \eta_n) = (x/z, \pm)$ can be extracted from measurements with $\rho_n^S = (1 \pm \sigma_y)/2$ and $\Lambda_n = \sigma_z$]. For example, the third-order correlation (for $t_3 > t_2 > t_1$)

$$C_{x,y,z}^{+-+} = (G_{z,y,y}^{y,x,x} + G_{z,y,y}^{y,\bar{x},x} - G_{z,y,y}^{y,x,\bar{x}} - G_{z,y,y}^{y,\bar{x},\bar{x}})/(4\delta t^3),$$

where $G_{z,y,y}^{y,\bar{x},x}$ denotes the measurement correlations for the central spin initialized along x and measured along y at t_1 , initialized along $-x$ and measured along y at t_2 , and initialized along y and measured along z at t_3 (similarly for G with other indices).

Among different types of decoherence, pure dephasing is often the most relevant to quantum information technology since it does not involve the slow energy dissipation process. For pure dephasing, the qubit-bath coupling assumes the form $\hat{V}(t) = S_z \hat{B}_z(t)$. The qubit dynamics is determined by the bath correlations as

$$\hat{\rho}^S(t) = \sum_{N=0}^{+\infty} \frac{2^N}{N!} \int_0^t dt_N \cdots dt_1 C_{z, \dots, z}^{+, \dots, +} S_z^- \cdots S_z^- \hat{\rho}^S(0).$$

Here we have used the fact that $C_{z, z, \dots, z}^{-\eta_{N-1}, \dots, \eta_1} = 0$ and $S_z^+ S_z^- = 0$. In the case of pure dephasing, the effects of quantum bath is fully determined by the correlation $C_{z, \dots, z}^{+, \dots, +}$, which is directly related to the weak measurement correlations through, e.g.,

$$C_{z, z, z}^{+++} = G_{y, y, y}^{x, x, x}/\delta t^3.$$

Here we have used the fact that in the pure dephasing case, $C_{z, y, z}^{+-+} = C_{z, z, y}^{+-+} = C_{z, y, y}^{+-+} = 0$ (for $B_y = 0$) and therefore $C_{y, y, y}^{x, x, x} = -C_{y, y, y}^{x, \bar{x}, x} = -C_{y, y, y}^{x, x, \bar{x}} = G_{y, y, y}^{x, \bar{x}, \bar{x}}$ [according to Eq. (6)]. It should be noted that even though the pure dephasing is determined only by the “classical” bath correlations $C_{z, \dots, z}^{+, \dots, +}$, the “quantum” correlations (those that contain at least one commutator) can still be measured by weak measurements. For example,

$$C_{z, z, z}^{+--} = C_{y, z, y}^{x, x, x}/\delta t^3.$$

Conclusion.—We propose a general scheme for complete characterization of arbitrary order correlations in a quantum bath, based on weak measurement of the bath realized by projective measurement of a central system embedded in the bath. From the weak measurement

correlations at the N th order, one can reconstruct the N th order bath correlations. The weak measurement has the advantage of negligible disturbance (i.e., measurement-induced decoherence) to the bath—this advantage allows the measurement data be collected at a simple timing and the correlations be extracted by selecting certain subsets of the data, which greatly reduces the time consumption for reconstructing the correlation functions [37]. Once the bath correlations are characterized, they can be used for optimizing quantum controls under all circumstances [9–11,17–20]. Characterizing arbitrary-order correlations in quantum baths may provide an approach to studying the quantum characteristics (such as the Leggett-Garg inequality [42]) of many-body environments and enable quantum sensing of nuclear spin clusters of different types of correlations [5,41]. We expect that the experimental demonstration of the protocol is feasible in solid spin systems such as nitrogen-vacancy center spins [43], donor spins in silicon [44], and quantum dots [45].

This work was supported by Hong Kong RGC-NSFC Joint Scheme—Project N_CUHK403/16, National Key Research and Development Program of China (No. 2018YFA0306600), Anhui Initiative in Quantum Information Technologies (Grant No. AHY050000), National Natural Science Foundation of China (No. 11425523 and No. 11661161018), and the European Union funding via EU ERC grant SMeL and EU project ASTERIQS as well as the Volkswagen Foundation.

*Corresponding author.
rbliu@cuhk.edu.hk

- [1] G. A. Alvarez, D. Suter, and R. Kaiser, Localization-delocalization transition in the dynamics of dipolar-coupled nuclear spins, *Science* **349**, 846 (2015).
- [2] T. Schweigler, V. Kasper, S. Erne, I. Mazets, B. Rauer, F. Cataldini, T. Langen, T. Gasenzer, J. Berges, and J. Schmiedmayer, Experimental characterization of a quantum many-body system via higher-order correlations, *Nature (London)* **545**, 323 (2017).
- [3] W. M. Witzel and S. Das Sarma, Quantum theory for electron spin decoherence induced by nuclear spin dynamics in semiconductor quantum computer architectures: Spectral diffusion of localized electron spins in the nuclear solid-state environment, *Phys. Rev. B* **74**, 035322 (2006).
- [4] L. Cywiński, W. M. Witzel, and S. Das Sarma, Electron Spin Dephasing due to Hyperfine Interactions with a Nuclear Spin Bath, *Phys. Rev. Lett.* **102**, 057601 (2009).
- [5] W. Yang and R.-B. Liu, Quantum many-body theory of qubit decoherence in a finite-size spin bath, *Phys. Rev. B* **78**, 085315 (2008).
- [6] W. Yao, R.-B. Liu, and L. J. Sham, Theory of electron spin decoherence by interacting nuclear spins in a quantum dot, *Phys. Rev. B* **74**, 195301 (2006).
- [7] W. Yang, W.-L. Ma, and R.-B. Liu, Quantum many-body theory for electron spin decoherence in nanoscale nuclear spin baths, *Rep. Prog. Phys.* **80**, 016001 (2017).
- [8] G. Gasbarri and L. Ferialdi, Stochastic unravelings of non-Markovian completely positive and trace-preserving maps, *Phys. Rev. A* **98**, 042111 (2018).
- [9] A. G. Kofman and G. Kurizki, Universal Dynamical Control of Quantum Mechanical Decay: Modulation of the Coupling to the Continuum, *Phys. Rev. Lett.* **87**, 270405 (2001).
- [10] A. G. Kofman and G. Kurizki, Unified Theory of Dynamically Suppressed Qubit Decoherence in Thermal Baths, *Phys. Rev. Lett.* **93**, 130406 (2004).
- [11] L. Cywinski, R. M. Lutchyn, C. P. Nave, and S. Das Sarma, How to enhance dephasing time in superconducting qubits, *Phys. Rev. B* **77**, 174509 (2008).
- [12] L. Viola and S. Lloyd, Dynamical suppression of decoherence in two-state quantum systems, *Phys. Rev. A* **58**, 2733 (1998).
- [13] M. Ban, Photon-echo technique for reducing the decoherence of a quantum bit, *J. Mod. Opt.* **45**, 2315 (1998).
- [14] P. Zanardi, Symmetrizing evolutions, *Phys. Lett. A* **258**, 77 (1999).
- [15] L. Viola, E. Knill, and S. Lloyd, Dynamical Decoupling of Open Quantum Systems, *Phys. Rev. Lett.* **82**, 2417 (1999).
- [16] K. Khodjasteh and D. A. Lidar, Fault-Tolerant Quantum Dynamical Decoupling, *Phys. Rev. Lett.* **95**, 180501 (2005).
- [17] M. Grace, C. Brif, H. Rabitz, I. A. Walmsley, R. L. Kosut, and D. A. Lidar, Optimal control of quantum gates and suppression of decoherence in a system of interacting two-level particles, *J. Phys. B* **40**, S103 (2007).
- [18] G. Gordon, G. Kurizki, and D. A. Lidar, Optimal Dynamical Decoherence Control of a Qubit, *Phys. Rev. Lett.* **101**, 010403 (2008).
- [19] J. R. West, D. A. Lidar, B. H. Fong, and M. F. Gyure, High Fidelity Quantum Gates via Dynamical Decoupling, *Phys. Rev. Lett.* **105**, 230503 (2010).
- [20] G.-Q. Liu, H. C. Po, J. Du, R.-B. Liu, and X.-Y. Pan, Noise-resilient quantum evolution steered by dynamical decoupling, *Nat. Commun.* **4**, 2254 (2013).
- [21] G. A. Álvarez and D. Suter, Measuring the Spectrum of Colored Noise by Dynamical Decoupling, *Phys. Rev. Lett.* **107**, 230501 (2011).
- [22] K. C. Young and K. B. Whaley, Qubits as spectrometers of dephasing noise, *Phys. Rev. A* **86**, 012314 (2012).
- [23] G. A. Paz-Silva and L. Viola, General Transfer-Function Approach to Noise Filtering in Open-Loop Quantum Control, *Phys. Rev. Lett.* **113**, 250501 (2014).
- [24] L. M. Norris, G. A. Paz-Silva, and L. Viola, Qubit Noise Spectroscopy for Non-Gaussian Dephasing Environments, *Phys. Rev. Lett.* **116**, 150503 (2016).
- [25] G. A. Paz-Silva, L. M. Norris, and L. Viola, Multiqubit spectroscopy of Gaussian quantum noise, *Phys. Rev. A* **95**, 022121 (2017).
- [26] C. Ferrie, C. Granade, G. Paz-Silva, and H. M. Wiseman, Bayesian quantum noise spectroscopy, *New J. Phys.* **20**, 123005 (2018).
- [27] M. Loretz, J. M. Boss, T. Rosskopf, H. J. Mamin, D. Rugar, and C. L. Degen, Spurious Harmonic Response of Multipulse Quantum Sensing Sequences, *Phys. Rev. X* **5**, 021009 (2015).
- [28] A. N. Korotkov, Selective quantum evolution of a qubit state due to continuous measurement, *Phys. Rev. B* **63**, 115403 (2001).

- [29] A. N. Korotkov, Output spectrum of a detector measuring quantum oscillations, *Phys. Rev. B* **63**, 085312 (2001).
- [30] G. Sallen, A. Tribu, T. Aichele, R. Andre, L. Besombes, C. Bougerol, M. Richard, S. Tatarenko, K. Kheng, and J.-P. Poizat, Subnanosecond spectral diffusion measurement using photon correlation, *Nat. Photonics* **4**, 696 (2010).
- [31] A. Bednorz, C. Bruder, B. Reulet, and W. Belzig, Nonsymmetrized Correlations in Quantum Noninvasive Measurements, *Phys. Rev. Lett.* **110**, 250404 (2013).
- [32] J. Bülte, A. Bednorz, C. Bruder, and W. Belzig, Noninvasive Quantum Measurement of Arbitrary Operator Order by Engineered Non-Markovian Detectors, *Phys. Rev. Lett.* **120**, 140407 (2018).
- [33] D. Hägele and F. Schefczik, Higher-order moments, cumulants, and spectra of continuous quantum noise measurements, *Phys. Rev. B* **98**, 205143 (2018).
- [34] J. Atalaya, S. Hacoen-Gourgy, L. S. Martin, I. Siddiqi, and A. N. Korotkov, Multitime correlators in continuous measurement of qubit observables, *Phys. Rev. A* **97**, 020104(R) (2018).
- [35] A. Tilloy, Exact signal correlators in continuous quantum measurements, *Phys. Rev. A* **98**, 010104(R) (2018).
- [36] T. Gefen, M. Khodas, L. P. McGuinness, F. Jelezko, and A. Retzker, Quantum spectroscopy of single spins assisted by a classical clock, *Phys. Rev. A* **98**, 013844 (2018).
- [37] M. Pfender, P. Wang, H. Sumiya, S. Onoda, W. Yang, D. B. R. Dasari, P. Neumann, X.-Y. Pan, J. Isoya, R.-B. Liu, and J. Wrachtrup, High-resolution spectroscopy of single nuclear spins via sequential weak measurements, *Nat. Commun.* **10**, 594 (2019).
- [38] B.-B. Wei and R.-B. Liu, Lee-Yang Zeros and Critical Times in Decoherence of a Probe Spin Coupled to a Bath, *Phys. Rev. Lett.* **109**, 185701 (2012).
- [39] X. Peng, H. Zhou, B.-B. Wei, J. Cui, J. Du, and R.-B. Liu, Experimental Observation of Lee-Yang Zeros, *Phys. Rev. Lett.* **114**, 010601 (2015).
- [40] B.-B. Wei, S.-W. Chen, H.-C. Po, and R.-B. Liu, Phase transitions in the complex plane of physical parameters, *Sci. Rep.* **4**, 5202 (2014).
- [41] W.-L. Ma and R.-B. Liu, Proposal for Quantum Sensing Based on Two-Dimensional Dynamical Decoupling: NMR Correlation Spectroscopy of Single Molecules, *Phys. Rev. Applied* **6**, 054012 (2016).
- [42] A. J. Leggett and A. Garg, Quantum mechanics versus macroscopic realism: Is the flux there when nobody looks? *Phys. Rev. Lett.* **54**, 857 (1985).
- [43] M. W. Doherty, N. B. Manson, P. Delaney, F. Jelezko, J. Wrachtrup, and L. C. Hollenberg, The nitrogen-vacancy colour centre in diamond, *Phys. Rep.* **528**, 1 (2013).
- [44] R. E. George, W. Witzel, H. Riemann, N. V. Abrosimov, N. Nötzel, M. L. W. Thewalt, and J. J. L. Morton, Electron Spin Coherence and Electron Nuclear Double Resonance of Bi Donors in Natural Si, *Phys. Rev. Lett.* **105**, 067601 (2010).
- [45] R. Hanson, L. P. Kouwenhoven, J. R. Petta, S. Tarucha, and L. M. K. Vandersypen, Spins in few-electron quantum dots, *Rev. Mod. Phys.* **79**, 1217 (2007).