Probes Reveal Fluctuational Information Beyond Response

Wenzheng Dong[1] and Jinzhao Sun[1]
[1] *Queen Mary University of London*

A fundamental question in quantum physics is whether the physical properties of a quantum system can be fully characterized by perturbing and measuring the system alone, or whether introducing an auxiliary quantum degree of freedom—a probe—is essential to access certain classes of information. Traditional response theory, both linear and nonlinear, treats the probe as classical and characterizes the system through causal susceptibilities expressed as nested commutators. By contrast, modern probe-based approaches couple a controllable quantum system to the target and infer properties of the target from the probe's reduced dynamics.

In this work we provide a rigorous operator-level analysis of the distinction between these two paradigms. We show that system-only response theory is structurally restricted to commutator (retarded) correlators, encoding causal response, whereas probe-based measurements generically access anti-commutator and mixed operator orderings that encode fluctuations, noise, occupations, decoherence, and irreversibility. We present a complete second-order derivation for a standard dephasing spectroscopy protocol, demonstrating explicitly that probe coherence decay is governed by the symmetrized (anti-commutator) correlator, while the probe phase shift is governed by the commutator correlator with a causal kernel. We then generalize the analysis to arbitrary order using a contour-ordered formulation, introducing a basis of nested commutator/anti-commutator "braketor" correlators and showing which subsets are accessible to classical response versus quantum probes. Finally, we discuss energy-exchange probes that directly access unsymmetrized correlators and population information. Our results clarify, in a mathematically precise way, why auxiliary quantum degrees of freedom make fluctuation-related properties operationally measurable.

I. INTRODUCTION

Response theory has long provided a unifying framework for understanding how quantum systems react to external perturbations. In its standard formulation, a system is perturbed by a classical control field and observables of the same system are measured. Linear and nonlinear response functions obtained in this way encode transport coefficients, susceptibilities, and spectral properties.

At the same time, contemporary quantum platforms increasingly employ *quantum probes*: ancillary qubits, spins, cavities, or modes that couple to a target system and are measured directly. Such probe-based protocols underpin quantum noise spectroscopy, decoherence spectroscopy, thermometry, and many forms of open-system characterization. It is often stated that quantum probes "reveal more information" than classical probes, but this statement is rarely made precise at the operator level.

The purpose of this work is to give a *structural answer* to the question:

> *What physical information is inaccessible to system-only response theory but becomes operationally measurable when a quantum probe is introduced?*

The central result is that the distinction is not merely technological but algebraic. System-only response theory is confined to retarded, commutator-based correlators, whereas probe-based measurements naturally produce Keldysh contour objects whose real-time expansions contain anti-commutators and mixed operator orderings. These additional sectors encode fluctuations, noise, occupations, and irreversibility, which are independent of response functions out of equilibrium.

II. CORRELATORS AND OPERATOR ORDERINGS

Let M be a quantum system with density operator ρ_M, and let $B(t)$ be a Hermitian system operator in the Heisenberg picture with respect to the system Hamiltonian H_M.

A. Greater and lesser correlators

We define the two basic two-point correlators

$$C^>(t,t') := \langle B(t)B(t')\rangle\,, \tag{1}$$
$$C^<(t,t') := \langle B(t')B(t)\rangle\,, \tag{2}$$

where $\langle\cdot\rangle = \mathrm{Tr}(\rho_M\,\cdot)$.

B. Commutator and anti-commutator correlators

From these we define the commutator and anti-commutator correlators

$$C^-(t,t') := \langle[B(t),B(t')]\rangle = C^>(t,t') - C^<(t,t'), \tag{3}$$
$$C^+(t,t') := \langle\{B(t),B(t')\}\rangle = C^>(t,t') + C^<(t,t'). \tag{4}$$

These relations can be inverted as

$$C^> = \tfrac{1}{2}\big(C^+ + C^-\big)\,, \qquad C^< = \tfrac{1}{2}\big(C^+ - C^-\big)\,. \tag{5}$$

C. Time-ordered and anti-time-ordered correlators

The time-ordered and anti-time-ordered correlators are

$$\begin{aligned} C^T(t,t') &= \langle \mathcal{T}\, B(t)B(t')\rangle \\ &= \Theta(t-t')\, C^>(t,t') + \Theta(t'-t)\, C^<(t,t'), \\ C^{\tilde{T}}(t,t') &= \left\langle \tilde{\mathcal{T}}\, B(t)B(t')\right\rangle \\ &= \Theta(t-t')\, C^<(t,t') + \Theta(t'-t)\, C^>(t,t'). \end{aligned} \tag{6}$$

A key identity used repeatedly below is

$$C^T(t,t') + C^{\tilde{T}}(t,t') = C^+(t,t'). \tag{7}$$

III. SYSTEM-ONLY PROBING: RESPONSE THEORY AND ITS ALGEBRAIC RESTRICTION

We now recall why classical response theory accesses only commutator structures.

A. Linear response

Consider a classical perturbation $\lambda(t)$ coupled to B,

$$H(t) = H_M - \lambda(t)\, B. \tag{8}$$

In the interaction picture,

$$U_I(t) = \mathcal{T} \exp\left(\frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}s\, \lambda(s)\, B(s)\right). \tag{9}$$

The expectation value of an observable A is

$$\langle A(t)\rangle_\lambda = \mathrm{Tr}\Big[U_I^\dagger(t)\, A(t)\, U_I(t)\, \rho_M\Big]. \tag{10}$$

Expanding to first order gives the standard Kubo formula,

$$\delta\,\langle A(t)\rangle = \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}s\, \lambda(s)\, \langle [A(t), B(s)]\rangle. \tag{11}$$

The retarded susceptibility is therefore

$$\chi^R_{AB}(t,s) = \frac{\mathrm{i}}{\hbar}\Theta(t-s)\, \langle [A(t), B(s)]\rangle. \tag{12}$$

B. Nonlinear response

Higher-order response functions are obtained by expanding U_I to higher order. At second order one finds

$$\begin{aligned} \delta^{(2)}\,\langle A(t)\rangle \propto \int \mathrm{d}s_1 \mathrm{d}s_2 \Theta(t-s_1)\Theta(s_1-s_2) \\ \langle [[A(t), B(s_1)]\,, B(s_2)]\rangle. \end{aligned} \tag{13}$$

Thus classical response theory generates only nested commutators with causal ordering. Anti-commutators never appear unless additional assumptions (such as equilibrium relations) are imposed.

IV. QUANTUM PROBE: DEPHASING SPECTROSCOPY AND EXACT SECOND-ORDER ANALYSIS

We now introduce a quantum probe and show explicitly how anti-commutators enter measured quantities.

A. Model and interaction picture

Let P be a qubit probe coupled to the system via

$$H_I(t) = \frac{1}{2}\,\sigma_z \otimes y(t)\, B(t), \tag{14}$$

where $y(t)$ is a known modulation determined by probe control.

Assume an initial product state

$$\rho(0) = |+\rangle\langle +| \otimes \rho_M, \qquad |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}. \tag{15}$$

B. Conditional system evolutions

The interaction-picture propagator is

$$U_I(T) = \mathcal{T} \exp\left[-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\, \frac{1}{2}\,\sigma_z \otimes y(t)\, B(t)\right]. \tag{16}$$

Since $\sigma_z\,|0\rangle = +\,|0\rangle$ and $\sigma_z\,|1\rangle = -\,|1\rangle$,

$$U_I(T) = |0\rangle\langle 0| \otimes U_+(T) + |1\rangle\langle 1| \otimes U_-(T), \tag{17}$$

with

$$U_\pm(T) = \mathcal{T} \exp\left[\mp\frac{\mathrm{i}}{2\hbar}\int_0^T \mathrm{d}t\, y(t)\, B(t)\right]. \tag{18}$$

C. Probe coherence as a forward–backward object

The probe coherence evolves as

$$\langle \sigma_+(T)\rangle = \langle \sigma_+(0)\rangle\; \mathcal{L}(T), \tag{19}$$

where

$$\mathcal{L}(T) := \mathrm{Tr}_M\Big[U_+(T)\, \rho_M\, U_-^\dagger(T)\Big]. \tag{20}$$

This is a Keldysh contour object, involving forward and backward evolution of the system.

V. SECOND-ORDER EXPANSION: SEPARATION INTO FLUCTUATION AND RESPONSE SECTORS

Define

$$X := \frac{1}{2\hbar}\int_0^T \mathrm{d}t\, y(t)\, B(t). \tag{21}$$

Then

$$U_+(T) = \mathcal{T}\, \mathrm{e}^{-\mathrm{i}X}, \qquad U_-^\dagger(T) = \tilde{\mathcal{T}}\, \mathrm{e}^{-\mathrm{i}X}. \tag{22}$$

Expanding to second order and assuming $\langle B(t)\rangle = 0$,

$$\mathcal{L}(T) = 1 - \frac{1}{2}\left\langle \mathcal{T}X^2\right\rangle - \frac{1}{2}\left\langle \tilde{\mathcal{T}}X^2\right\rangle - \left\langle X^2\right\rangle + O(B^3). \tag{23}$$

Using the identities from Sec. **??**[1] one finds

$$\left\langle \mathcal{T}X^2\right\rangle + \left\langle \tilde{\mathcal{T}}X^2\right\rangle = \frac{1}{4\hbar^2}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t'\, y(t)y(t')\, C^+(t,t'), \tag{24}$$

$$\left\langle X^2\right\rangle = \frac{1}{4\hbar^2}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t'\, y(t)y(t')\, C^>(t,t'). \tag{25}$$

After reorganizing terms and performing a cumulant expansion, one may write

$$\ln \mathcal{L}(T) = -\chi(T) + \mathrm{i}\phi(T) + O(B^3), \tag{26}$$

with the key separation

$$\boxed{\chi(T) = \frac{1}{8\hbar^2}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t'\, y(t)y(t')\, C^+(t,t')} \tag{27}$$

and

$$\boxed{\phi(T) = \frac{1}{8\hbar^2}\int_0^T \mathrm{d}t \int_0^t \mathrm{d}t'\, y(t)y(t')\, \frac{1}{\mathrm{i}}\, C^-(t,t')} \tag{28}$$

Equivalently, the phase may be written symmetrically as

$$\phi(T) = \frac{1}{16\hbar^2}\int_0^T \mathrm{d}t \int_0^T \mathrm{d}t'\, y(t)y(t')\, \mathrm{sgn}(t-t')\, \frac{1}{\mathrm{i}}\, C^-(t,t'). \tag{29}$$

Thus:

- the decoherence functional $\chi(T)$ depends solely on the anti-commutator correlator C^+ (fluctuations/noise sector);
- the phase shift $\phi(T)$ depends on the commutator correlator C^- with a causal kernel (response sector). The i in ϕ will be cancelled in $\mathcal{L}$, so it is not clear ϕ is phase term.

[1] In the final draft, replace this placeholder by an explicit internal reference if you choose to add labels.

VI. FREQUENCY-DOMAIN FORMULATION

Assuming stationarity, define Fourier transforms

$$F(\omega) = \int_{-\infty}^{\infty} \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} f(\tau), \quad f(\tau) = \int \frac{\mathrm{d}\omega}{2\pi}\, \mathrm{e}^{-\mathrm{i}\omega\tau} F(\omega). \tag{30}$$

Define the symmetrized and antisymmetrized spectra

$$S^+(\omega) := \int \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau}\, \frac{1}{2}\, C^+(\tau), \quad A(\omega) := \int \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau}\, \frac{1}{2\mathrm{i}}\, C^-(\tau). \tag{31}$$

With the filter function

$$Y(\omega) := \int_0^T \mathrm{d}t\, y(t)\, \mathrm{e}^{\mathrm{i}\omega t}, \tag{32}$$

the decoherence functional becomes

$$\chi(T) = \frac{1}{4\hbar^2}\int \frac{\mathrm{d}\omega}{2\pi}\, \left|Y(\omega)\right|^2 S^+(\omega). \tag{33}$$

VII. ENERGY-EXCHANGE PROBES AND UNSYMMETRIZED CORRELATORS

For probes that exchange energy with the system, e.g.

$$H_I = (\sigma_+ + \sigma_-) \otimes B, \tag{34}$$

the upward and downward transition rates satisfy

$$\Gamma_\uparrow \propto C^<(\omega_0), \qquad \Gamma_\downarrow \propto C^>(\omega_0), \tag{35}$$

where

$$C^>(\omega) = \int \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} C^>(\tau), \qquad C^<(\omega) = \int \mathrm{d}\tau\, \mathrm{e}^{\mathrm{i}\omega\tau} C^<(\tau). \tag{36}$$

Thus energy-exchange probes directly access emission and absorption spectra, and hence population information.

VIII. EQUILIBRIUM AS A SPECIAL CASE: KMS AND FLUCTUATION–DISSIPATION

In thermal equilibrium,

$$C^>(\omega) = \mathrm{e}^{\beta\hbar\omega}\, C^<(\omega). \tag{37}$$

This implies

$$C^+(\omega) = \coth\left(\frac{\beta\hbar\omega}{2}\right) C^-(\omega), \tag{38}$$

so fluctuation and response sectors are no longer independent. Out of equilibrium this relation fails, and C^+ contains genuinely new information.

IX. HIGHER-ORDER STRUCTURE: CONTOUR EXPANSION AND BRAKETORS

The influence functional admits the exact contour expansion

$$\mathcal{L}(T) = \left\langle \mathcal{T}_c \exp\left[-\frac{\mathrm{i}}{2\hbar}\int_c \mathrm{d}t\, \eta(t)\, y(t)\, B(t)\right]\right\rangle, \qquad (39)$$

where $\eta = \pm 1$ labels the forward/backward branch.

Expanding to order k yields contour-ordered k-point correlators. When mapped to real time, these generate nested combinations of commutators and anti-commutators.

Define the nested bracket

$$[X, Y]_- := [X, Y]\,, \qquad [X, Y]_+ := \{X, Y\}\,. \qquad (40)$$

For $k \geq 2$, define the "braketor"

$$\mathcal{B}^{(k)}_{\vec{\mu}}(\vec{t}) = \left\langle [\cdots [[B(t_1), B(t_2)]_{\mu_1}, B(t_3)]_{\mu_2} \cdots, B(t_k)]_{\mu_{k-1}} \right\rangle, \qquad (41)$$

with $\vec{\mu} \in \{\pm\}^{k-1}$.

Classical response accesses only the fully retarded sector $\vec{\mu} = (-, -, \ldots, -)$. Probe-based measurements access linear combinations spanning the full braketor space.

X. CONCLUSION

We have shown, with explicit derivations, that the distinction between system-only response theory and probe-based measurement is algebraic and fundamental. Classical response theory accesses only commutator-based, causal correlators. Quantum probes, by introducing a second quantum branch, render anti-commutator and mixed-ordering correlators operationally measurable.

These additional sectors encode fluctuations, noise, occupations, decoherence, and irreversibility—quantities that are independent of response functions out of equilibrium. Auxiliary quantum degrees of freedom therefore do not arbitrarily add information; they enable access to relational and statistical properties that are otherwise hidden.