Letter

Building Krylov complexity from circuit complexity

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Krylov complexity has emerged as a probe of operator growth in a wide range of nonequilibrium quantum dynamics. However, a fundamental issue remains in such studies: the definition of the distance between basis states in Krylov space is ambiguous. Here we show that Krylov complexity can be rigorously established from circuit complexity when dynamical symmetries exist. Whereas circuit complexity characterizes the geodesic distance in a multidimensional operator space, Krylov complexity measures the height of the final operator in a particular direction. The geometric representation of circuit complexity thus unambiguously designates the distance between basis states in Krylov space. This geometric approach also applies to time-dependent Liouvillian superoperators, where a single Krylov complexity is no longer sufficient. Multiple Krylov complexity may be exploited jointly to fully describe operator dynamics.

Significant progress has recently been made in the study of nonequilibrium quantum phenomena, ranging from eigenstate thermalization to quantum information scrambling [1–8]. A central task is to explore how local quantum information spreads to the vastly large degree of freedom in a many-body system [7–11]. Commonly used theoretical tools include the out-of-time-order correlator and Loschmidt echoes [12–18]. Krylov (K) complexity has been introduced as a probe of quantum information spreading, where a local operator may evolve into a nonlocal one in quantum dynamics [19]. K complexity traces such operator size growth and thus provides physicists with deep insights to many fundamental questions in nonequilibrium dynamics [19–35].

The study of K complexity is based on the Lanczos algorithm [19,36]. The Lanczos theorem allows us to find an orthonormal set of operators, termed the Krylov basis, that an initial operator spreads into in the Heisenberg picture. K complexity measures the averaged displacement of the final operator in the Krylov basis. In the Heisenberg picture, an operator is denoted by $\hat{O}(t) = \sum_{m,n} O_{mn}|m\rangle\langle n|$, where t is the time, $|m\rangle$ is a set of orthogonal eigenstates, and O_{mn} are the matrix elements. $\hat{O}(t)$ is regarded as a state in the operator space and is denoted as $|O(t)) \equiv \sum_{m,n} O_{mn}|m\rangle \otimes |n\rangle$. Using the Krylov basis $|O_n)$, an ordered set of mutual orthogonal operators,

$$|O(t)) = \sum_n \phi_n(t)|O_n), \tag{1}$$

where $|O_0) = |O(t=0))$. $|O_{n\neq 0})$ is generated by commutators of $\hat{O}_0$ and $\hat{H}$ recursively using the Lanczos algorithm. Here, we note that $\hat{H}$ is a generic Hermitian Hamiltonian. Both the Krylov basis $|O_n)$ and K complexity depend on the initial operator $\hat{O}_0$ and the Hamiltonian $\hat{H}$. $|O_1) = b_0^{-1}|[H, O_0])$, where the normalization constant b_0 is introduced such that $(O_n|O_n) = (O_0|O_0)$. The inner product in operator space is defined as $(A|B) = \text{Tr}(A^\dagger B)$. The other Krylov basis states are defined recursively by $|O_n) = b_{n-1}^{-1}(|[H, O_{n-1}]) - b_{n-2}|O_{n-2}))$. K complexity is defined as

$$C_K(t) = \sum_n n|\phi_n(t)|^2, \tag{2}$$

which can be viewed as the expectation value of the Krylov operator:

$$\hat{K}_O = \sum_n n|O_n)(O_n|. \tag{3}$$

Intuitively, Eq. (2) describes the mean width of a wave packet in the Krylov space and thus quantitatively measures how the size of the operator increases as time goes by. Whereas it was hypothesized that C_K has the fastest growth in a chaotic system [19], recent studies have shown that similar behaviors of C_K may arise in nonchaotic systems [23,24].

Despite the exciting development in the study of K complexity, a fundamental question remains. The Lanczos algorithm only provides an ordered Krylov basis but does not supply a distance among them. To obtain the operator size described by Eq. (2), $|O_n)$ are assumed to be equally spaced. As such, any quantitative results of K complexity are built upon the choice that the distance between $|O_n)$ and $|O_0)$ has been chosen as n. In practice, substituting n by any other function $h(n)$ shall change all results of K complexity. At a more fundamental level, without prior knowledge of the geometry of the operator space, any choice of the distance between $|O_n)$

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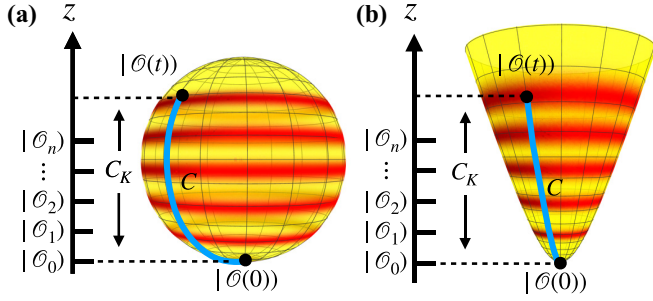†Contact author: zhou753@purdue.edu

FIG. 1. A schematic of the Krylov basis as strips on a two-sphere (a) and a hyperboloid (b) for SU(2) and SU(1,1), respectively. The curves on the surfaces denote the geodesics and the geodesic length is state C complexity C. The height of the final operator is K complexity C_K.

could be regarded as ambiguous. It is thus desirable to define K complexity rigorously by specifying the geometry of the Krylov space where $|\mathcal{O}_n)$ live.

Here, we show K complexity can be systematically established from circuit (C) complexity, a concept that originated from quantum computation and is now being widely used in many other areas including high energy physics and condensed matter physics [37–43]. In quantum computation, C complexity describes the smallest number of gates to reach a target state from a reference state. It can be visualized using Nielsen's geometric approach, which provides a metric for operators such that C complexity is given by the geodesic in the space of circuits [37]. Compared to other definitions of the metric in operator space, such as the trace distance [44], a unique advantage of circuit complexity is that the distance between operators has a clear physical interpretation. A longer (shorter) path corresponds to more (fewer) gates required to evolve from one operator to the other. Applying circuit complexity to nonequilibrium quantum dynamics, it can be understood as the shortest time to reach a desired state and thus allows experimentalists to optimize quantum controls [45]. In previous studies, C complexity was explored using coherent states in systems with dynamical symmetries [40,46]. Coherent states have also been implemented to study K complexity [24]. However, the relation between K complexity and C complexity had never been established before our work, to our knowledge.

We have found that K complexity measures the height of the time-dependent operator along a particular direction in the operator space. For instance, in systems with SU(2) symmetry, the metric of all operators involved in quantum dynamics forms a sphere. As shown in Fig. 1(a), the initial operator $|\mathcal{O}(0))$ is placed at the south pole. The length of a trajectory starting from the south pole along a big circle provides us with the circuit complexity. Projecting this trajectory to the z axis, the K complexity is the height of the final operator measured from the south pole. Our results also show that the distance between Krylov basis states is rigorously determined by the circuit complexity. Here, the Krylov basis states correspond to strips along latitudes on the sphere. Such a distance turns out to be variable once it becomes easier or more difficult to change the operators in some directions in the operator space.

We note that the geometric interpretation of K complexity has recently been studied in an elegant work by Caputa *et al.* [24]. It was argued that the K complexity corresponds to the area of a certain region, using the Fubini-Study metric to define the metric of the operator space [47–49]. Compared to such an inner product of operators, the physical meaning of C complexity used here is more clear in quantum dynamics. We find that it is more appropriate to interpret K complexity as a length rather than an area. Furthermore, our approach applies to time-dependent Liouvillian superoperators, a largely unexplored problem in the study of K complexity. As we will show, it requires multiple K complexity to fully describe the operator growth in the most generic case where Liouvillian superoperators are time-dependent.

We need to emphasize that we consider SU(2) and SU(1,1) in this paper for the purpose of demonstrating the methodology. Our scheme is directly applicable to any physical systems with arbitrary dynamical symmetries. The Lie algebra underlying the dynamical symmetry allows one to compute C complexity and the underlying geometry [37–42]. Furthermore, any point in the geometry corresponds to a coherent state [50], and coherent states form an overcomplete basis. The Krylov basis thus can be expressed in terms of the coherent states, providing a transparent visualization and a geometric interpretation of K complexity.

In the operator space, the Heisenberg equation of motion is recast into a similar form as the Schrödinger equation,

$$-i\partial_t |\mathcal{O}(t)) = \hat{\mathcal{L}} |\mathcal{O}(t)), \qquad (4)$$

where the Liouvillian superoperator becomes $\hat{\mathcal{L}} = \hat{H} \otimes \hat{\mathbb{I}} - \hat{\mathbb{I}} \otimes \hat{H}^T$. When a dynamical symmetry exists, $\hat{\mathcal{L}}$ can be written as a generator of symmetry groups, and this equation has simple analytical solutions. In the following, we will consider the SU(2) and SU(1,1) dynamical symmetry. While the former can be realized by an initial spherical tensor operator in a noninteracting spin system subjected to magnetic field [51], the latter appears when one considers the initial single Majorana operator in the large-q limit of the SYK model [19].

When $\hat{\mathcal{L}}$ is written as $\hat{\mathcal{L}} = B\hat{\mathcal{S}}_x$,

$$\hat{\mathcal{S}}_x = \sum_{n=0}^{2\ell-1} \frac{1}{2} b_n |\mathcal{O}_{n+1})(\mathcal{O}_n| + \text{H.c.}, \qquad (5)$$

where $b_n = \sqrt{(n+1)(2\ell-n)}$, 2ℓ is an integer denoting the total number of Krylov basis states, and B is a constant. One can define $\hat{\mathcal{S}}^+ = \sum_n b_n |\mathcal{O}_{n+1})(\mathcal{O}_n|$ and $\hat{\mathcal{S}}^- = \sum_n b_n |\mathcal{O}_n)(\mathcal{O}_{n+1}|$. $\hat{\mathcal{K}}_{\mathcal{O}}$ and $\hat{\mathcal{S}}^{\pm}$ provide three generators of SU(2), as recognized by Caputa *et al.* [24]:

$$[\hat{\mathcal{K}}_{\mathcal{O}}, \hat{\mathcal{S}}^{\pm}] = \pm\hat{\mathcal{S}}^{\pm}, \quad [\hat{\mathcal{S}}^+, \hat{\mathcal{S}}^-] = 2(\hat{\mathcal{K}}_{\mathcal{O}} - \ell). \qquad (6)$$

This dynamical symmetry is seen by noting that Eq. (5) corresponds to the spin operator $\hat{\mathcal{S}}_x$ of a total spin-ℓ, $\hat{\mathcal{S}}_y = (\hat{\mathcal{S}}^+ - \hat{\mathcal{S}}^-)/(2i)$ and $\hat{\mathcal{S}}_z = \hat{\mathcal{K}}_{\mathcal{O}} - \ell$, and the constant B corresponds to the strength of the magnetic field acting on the spin. Equation (4) thus can be viewed as the Schrödinger equation for a spin-ℓ subject to a constant magnetic field in the x direction. Alternatively, $\phi_n(t)$ in Eq. (1) can be regarded as the time-dependent wave function in a lattice model where the tunneling amplitude is denoted by b_n.

For Liouvillians with dynamical symmetry, the propagator $\hat{\mathcal{U}} = \mathcal{T} e^{i \int_0^t \hat{\mathcal{L}}(t')dt'}$, where $\mathcal{T}$ is the time-ordering operator, can be parameterized using the generators of the symmetry group in the same manner as the evolution of a quantum state. For SU(2):

$$\hat{\mathcal{U}}(t) = e^{i\psi(t)\hat{\mathcal{S}}_z} e^{i\theta(t)\hat{\mathcal{S}}_x} e^{i\varphi(t)\hat{\mathcal{S}}_z}. \qquad (7)$$

It evolves the initial operator $|\mathcal{O}(0))$ to a generalized coherent state $|\mathcal{O}(t)) = \hat{\mathcal{U}}(t)|\mathcal{O}(0)) = |\mathcal{C}_{\theta(t),\varphi(t),\psi(t)})$ [50].

In the most generic case, the Liouvillian superoperator in Eq. (4) is given by $\hat{\mathcal{L}} = B_x\hat{\mathcal{S}}_x + B_y\hat{\mathcal{S}}_y + B_z\hat{\mathcal{S}}_z$. The operator dynamics is viewed as the evolution of a spin ℓ in a magnetic field $\vec{B}(t) = (B_x, B_y, B_z)$. The circuit complexity denotes the shortest time to reach a target state when the strength of the magnetic field is fixed, i.e., $|\vec{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2} \equiv B$ [45]. This amounts to fixing the metric of the operator space [42,52]:

$$ds^2 = \ell^2 dt^2 (B_x^2 + B_y^2 + B_z^2). \qquad (8)$$

Noting that $d\hat{\mathcal{U}}\hat{\mathcal{U}}^{-1} = i\hat{\mathcal{L}}dt$, putting Eq. (7) to the left-hand side shows that $d\theta$, $d\psi$ and $d\varphi$ are determined by $\vec{B}$. Using $\mathrm{Tr}(\hat{\mathcal{S}}_i\hat{\mathcal{S}}_j^\dagger) = \delta_{i,j}$, $B_{i=x,y,z}$ can be expressed in terms of ψ, θ, φ, $d\psi$, $d\theta$, and $d\varphi$. In the content of circuit complexity, it amounts to defining a cost function $F^2 = \ell^2\mathrm{Tr}(d\hat{\mathcal{U}}\hat{\mathcal{U}}^{-1}(d\hat{\mathcal{U}}\hat{\mathcal{U}}^{-1})^\dagger)$ [42]. Substituting such expressions to Eq. (8), we obtain the following metric:

$$ds^2 = \ell^2(d\theta^2 + d\psi^2 + d\varphi^2 + 2\cos(\theta)d\varphi d\psi). \qquad (9)$$

This is the metric of a three-sphere.

Equation (1) can be understood as an expansion of the coherent state using $|\mathcal{O}_n)$, which plays the roles of eigenstates of $\hat{\mathcal{S}}_z$ in the usual spin problem. To be explicit,

$$\phi_n(t) = e^{-i\ell(\varphi+\psi)}\cos^{2\ell}(\theta/2)\sqrt{\frac{(2\ell)!}{n!(2\ell-n)!}}\mu^n, \qquad (10)$$

where $\mu = i\tan(\theta/2)e^{i\psi}$. ψ, θ, φ as functions of t are defined in Eq. (7) from expanding the propagator, and $\phi_n(t)$ is obtained by directly applying Eq. (7) to $|\mathcal{O}(0))$. As such, φ controls the global phase, ψ denotes the relative phase between ϕ_n, and θ determines the amplitude distribution of $|\phi_n|^2$. Very often, two operators differ by a global phase can be identified as the same one. As such, the state C complexity is defined as the minimum of C complexity to the final state with different φ but the same ψ and θ. Since Eq. (9) is a quadratic form of $d\varphi$, extremizing Eq. (9) by setting $d\varphi = -\cos(\theta)d\psi$ reduces a three-sphere to a two-sphere:

$$ds^2 = \ell^2(d\theta^2 + \sin^2(\theta)d\psi^2). \qquad (11)$$

Each point on the two-sphere is equivalent to a coherent state $|\mathcal{C}'_{\theta,\psi}) \equiv |\mathcal{C}_{\theta,\psi,0})$. Since the geodesics of a two-sphere are big circles, the state C complexity could be visualized as the length of the arc along a big circle connecting the initial and the final operators or, equivalently, the polar angle θ, as shown in Fig. 1.

We expand $|\mathcal{O}_n)$ using the overcomplete basis $|\mathcal{C}'_{\theta,\psi})$,

$$|\mathcal{O}_n) = \frac{2\ell+1}{4\pi}\int \sin(\theta)d\theta d\psi f_n(\theta,\psi)|\mathcal{C}'_{\theta,\psi}), \qquad (12)$$

where $|f_n(\theta,\psi)|^2 = |\phi_n^*|^2_{\varphi=0}$ is centered around $\theta_n = \arccos(1-n/\ell)$. This is precisely the Husimi Q representation of $|\mathcal{O}_n)(\mathcal{O}_n|$ using the coherent state of operators [50]. On the two-sphere, $|\mathcal{O}_0)$ and $|\mathcal{O}_{2\ell})$ are placed at the south and the north poles, respectively. Each other $|\mathcal{O}_{n\neq0})$ corresponds to a strip along a latitude with a finite width. The projection of the strip to the z axis is equally spaced and the height of the strip measured from the south pole linearly increases with n. In other words, each $|\mathcal{O}_n)$ can be assigned a unique coordinate on the z axis, n. This is directly a consequence of the metric tensor in Eq. (11). We conclude that once the metric of the operator space is fixed in the study of C complexity, the distance between basis states in the Krylov space is uniquely determined.

Since the weight of $|\mathcal{O}(t))$ in $|\mathcal{O}_n)$ is given by $|\phi_n(t)|^2$, K complexity defined in Eq. (2) could be viewed as the height of $|\mathcal{O}(t))$ measured from the south pole, similar to $\langle\hat{\mathcal{S}}_z\rangle$ of a spin. This is the physical meaning of K complexity in the geometric representation of C complexity. For the dynamics generated by $\hat{\mathcal{L}} = B\hat{\mathcal{S}}_x$ in Eq. (5), the evolution of the operator is equivalent to a spin processing about the x axis. The state C complexity, which equals the arc length at polar angle θ, grows linearly as a function of time, $C = \ell\theta(t) = \ell Bt$. K complexity, the height of the final operator is equivalent to the average value of $\hat{\mathcal{S}}_z$ of a spin, $C_K = \ell - \ell\cos(Bt)$.

Similar discussions apply to SU(1,1), where b_n in Eq. (5) is replaced by $\sqrt{(n+1)(2k+n)}$, and k is the Bargmann index, a counterpart of ℓ in SU(2). The state C complexity could be visualized using a hyperbolic surface embedded in Minkowski space. As shown in Fig. 1(b), each point on the hyperbolic surface is an SU(1,1) coherent state in the operator space,

$$|\mathcal{C}'_{\theta,\psi}) = \frac{e^{ik\psi}}{\cosh^{2k}(\theta/2)}\sum_n \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}}\tilde{\mu}^n|\mathcal{O}_n), \qquad (13)$$

where $\tilde{\mu} = i\tanh(\theta/2)e^{i\psi}$. C is the length of the arc connecting the initial and final operators on the hyperbolic surface and grows linearly as a function of time, $C = kBt$. The Krylov basis states again correspond to strips on this surface. On the z axis, $|\mathcal{O}_n)$ are equally spaced, and C_K is the height of the final operator measured from the initial operator placed at the bottom of the hyperbolic surface. A characteristic feature of a hyperbolic surface is that the height (projection of the geodesic on the z axis) increases exponentially while the proper distance (the length of the geodesic) on the surface has a linear growth. As such, C complexity increases linearly and K complexity grows exponentially. This provides a geometric interpretation of the exponentially growing K complexity:

$$C_K(t) = k\cosh(Bt) - k. \qquad (14)$$

In a finite system, it was found that a linear growth of C is followed by a saturation [53]. Whereas our paper mainly considers an infinite system, a finite system will impose a cutoff in the sum of n in Eqs. (2) and (3) and C will also saturate eventually.

The metric of operator space, for instance, the two-sphere in Eq. (11), depends on ds defined in Eq. (8). In the study of C complexity, it is known that the metric of the operator space could change if a different ds is used. Physically, this originates from that it may be easier (or more difficult) to
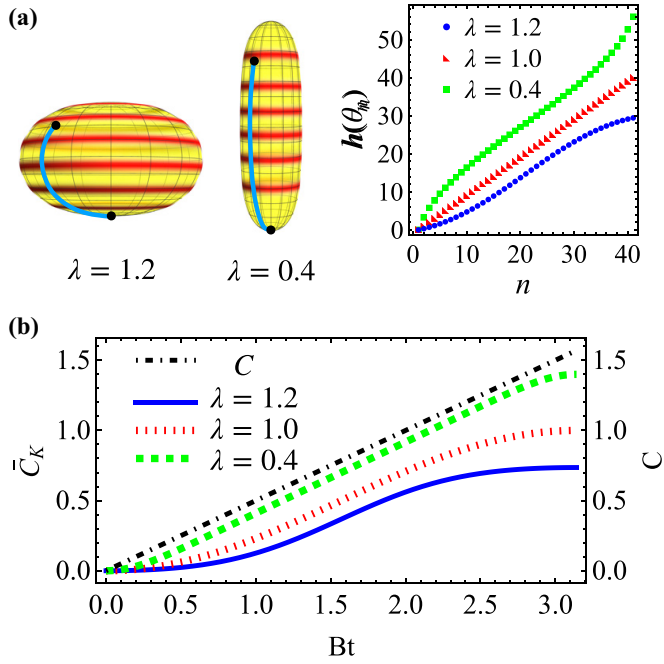
FIG. 2. (a) Squashed ($\lambda > 1$) or stretched ($\lambda < 1$) spheres for SU(2). The Krylov basis has unequally spaced heights $h(\theta_n)$ when $\lambda \neq 1$. (b) Generalized K-complexity (in unit of 2ℓ) as a function of time t depends on how the sphere is deformed. C-complexity always grows linearly.

implement certain gates than others [42]. For instance, in SU(2), one may choose

$$ds^2 = \ell^2 dt^2 (B_x^2 + B_y^2 + \lambda^2 B_z^2). \quad (15)$$

$\lambda > 1$ ($\lambda < 1$) means that it is more difficult (easier) to implement the gate in the z direction $\hat{\mathcal{L}}_z$. An alternative understanding is that one seeks the least time to access the target state under a different constraint. The new choice means that $\sqrt{B_x^2 + B_y^2 + \lambda^2 B_z^2}$ other than $|B|$ is fixed. The metric of the operator space becomes

$$ds^2 = \ell^2 (d\theta^2 + d\psi^2 + d\varphi^2 + 2\cos(\theta)d\varphi d\psi)$$
$$+ \ell^2 (\lambda^2 - 1)(\cos(\theta)d\varphi + d\psi)^2. \quad (16)$$

The state C complexity is obtained by minimizing the above metric, $d\varphi = -\frac{\lambda^2 \cos(\theta)}{1+(\lambda^2-1)\cos^2(\theta)}d\psi$. We obtain

$$ds^2 = \ell^2 \left(d\theta^2 + \frac{\lambda^2 \sin^2(\theta)}{1+(\lambda^2-1)\cos^2(\theta)} d\psi^2 \right). \quad (17)$$

When $\lambda = 1$, Eq. (17) reduces to the previous result of a two-sphere in Eq. (11). When $\lambda \neq 1$, Eq. (17) describes a deformed sphere, as shown in Fig. 2(a). The height of a point on the sphere with a polar angle θ measured from the south pole now becomes

$$h(\theta) = \ell \int_0^\theta \sqrt{1 - \frac{\cos^2(\theta')}{(\lambda^{-2}\sin^2(\theta') + \cos^2(\theta'))^3}} d\theta'. \quad (18)$$

When $\lambda \neq 1$, the Krylov basis states $|\mathcal{O}_n\rangle$ are no longer equally spaced. Under this situation, we can define a generalized K complexity $\bar{C}_K(t)$ such that it gives rise to the height
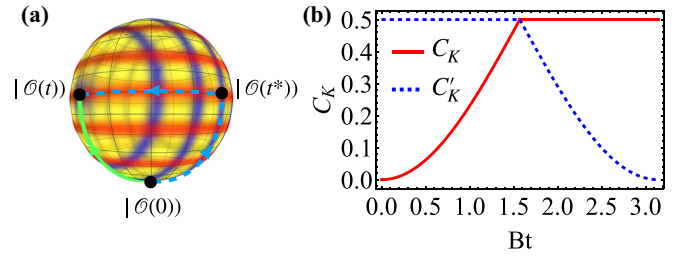

FIG. 3. (a) The operator dynamics (blue dashed arrows) when the Liouvillian superoperator is time dependent, changing from $\hat{\mathcal{L}} = B\hat{S}_x$ to $\hat{\mathcal{L}}' = B\hat{S}_z$ at $Bt^* = \pi/2$. Red and blue strips denote $|\mathcal{O}_n\rangle$ and $|\mathcal{O}_n'\rangle$, respectively. The green curve with an arrow represents the shortest path connecting the initial and final operators. (b) The K complexity (in the unit of 2ℓ) as functions of time of the quench dynamics.

of the final operator $|\mathcal{O}(t)\rangle$:

$$\bar{C}_K(t) = \sum_n h(\theta_n) |\phi_n(t)|^2. \quad (19)$$

As a result, the time dependence of $\bar{C}_K$ changes, as shown in Fig. 2(b). The originally defined K complexity as shown in Eq. (2) corresponds to the situation where it is of the same difficulty to apply all three gates, and C complexity is controlled by the metric tensor in Eq. (11). The generalized K complexity takes into account more generic scenarios where it is easier (or more difficult) to apply certain gates than others. Both the K complexity and the generalized K complexity have transparent geometric interpretations as the height of the final operator. We emphasize that the quantitative mismatch between C complexity and K complexity reflects the properties of the underlying geometry in C complexity. For instance, in the left panel (right panel) of Fig. 2(a), such a quantitative difference increases (decreases). This reflects that certain gates are more difficult to implement than others.

Our geometric approach also applies to time-dependent Liouvillian superoperators. The simplest case is a quantum quench, where the dynamics is determined by $\hat{\mathcal{L}}$ and $\hat{\mathcal{L}}'$ before and after t^*, respectively. For instance, in the case of SU(2), $\hat{\mathcal{L}} = B\hat{S}_x$ ($\hat{\mathcal{L}}' = B\hat{S}_z$) when $t < t^*$ ($t > t^*$). The trajectory on the two-sphere is no longer along the longitude but the latitude when $t > t^*$. For simplicity, we consider only undeformed spheres. The generalization to $\lambda \neq 1$ is straightforward. Though the operator $\mathcal{O}(t)$ is still evolving, C_K remains a constant and thus cannot capture the quantum dynamics when $t > t^*$. This can be simply seen from the definition of C_K in Eq. (2), in which only the amplitude of ϕ_n is included. Nevertheless, the relative phase between ϕ_n must be included, since the full dynamics happens on a two-sphere and the polar angle ψ of this two-sphere is precisely the relative phase between ϕ_n. State C complexity C that fully exploits the metric of a two-sphere can naturally capture the dynamics under an arbitrary time-dependent Liouvillian superoperator. For instance, the shortest path connecting the initial and final operators in this quench dynamics is along another longitude, as shown in Fig. 3(a). Apparently, this path can be accessed using another time-independent Liouvillian superoperator $\hat{\mathcal{L}}'' = B\hat{S}_y$.

Since a single K complexity is no longer sufficient when the Liouvillian superoperator is time-dependent, multiple K complexities may be used. In SU(2), we can define another set of Krylov basis states,

$$|\mathcal{O}'_n\rangle = e^{-i\pi\hat{\mathcal{S}}_y/2}|\mathcal{O}_n\rangle, \qquad (20)$$

which play the same role as eigenstates of $\hat{\mathcal{S}}_x$ in a spin problem. On the two-sphere, $|\mathcal{O}'_n\rangle$ corresponds to another set of strips, as shown in Fig. 3(a). Correspondingly, $|\mathcal{O}(t)\rangle$ may be expanded using $|\mathcal{O}'_n\rangle$ as $|\mathcal{O}(t)\rangle = \sum_n \phi'_n(t)|\mathcal{O}'_n\rangle$ and another K complexity is defined, $C'_K = \sum_n n|\phi'_n|^2$. As shown in Fig. 3(b), when $t < t^*$, C_K increases with increasing t while C'_K remains unchanged. After $t > t^*$, whereas C_K stops growing, C'_K begins to change. As such, at any time t, C_K, and C'_K could jointly determine $|\mathcal{O}(t)\rangle$ on the two-sphere, in the same manner as uniquely determining a spin on the Bloch sphere using the expectation values of both $\hat{S}_z$ and $\hat{S}_x$ in

spin tomography [54]. Similar results can be straightforwardly obtained for SU(1, 1).

We have rigorously established K complexity from C complexity in the presence of dynamical symmetry. Our method provides a clear geometrical picture and physical meaning of K complexity. Our results can be generalized to other symmetry groups, which are expected to bring even richer physics bridging geometry and quantum information spreading in nonequilibrium dynamics.

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