Cake Eating Problem

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1. Model Setting

Suppose that you are presented with a cake of size W_0 at time 0. At each period of time $t = 0, 1, \ldots,$ you can eat some of the cake but must save the rest. Let c_t be your consumption in period t, and let $u(c_t)$ represent the flow of utility from this consumption. Here, we use $u(c) = \log(c)$, which is real valued, differentiable, strictly increasing, and strictly concave, and $\lim_{c\to 0^+} u'(c) = +\infty$. Therefore, the problem is

$$\begin{aligned}
&\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t u(c_t) \\
&\text{s.t. } W_{t+1} = W_t - c_t, \\
&\qquad c_t \geq 0,\ W_{t+1} \geq 0,\ t = 0, 1, \ldots.
\end{aligned}$$

The Bellman equation associated with this problem is

$$V(W_t) = \max_{0 \leq c_t \leq W_t} \left\{ u(c_t) + \beta V(\underbrace{W_{t+1}}_{=W_t - c_t}) \right\}$$

1.1. Analytical Solution

We start with a guess that

$$V(W) = A + B\log(W),$$

where A and B are coefficients to be determined. Given this conjecture, we can write the Bellman equation as

$$A + B\log(W) = \max_c \left\{ \log c + \beta \left(A + B\log(W - c) \right) \right\}.$$

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The FOC is

$$\frac{1}{c} - \frac{\beta B}{W-c} = 0 \quad \Rightarrow \quad c = \frac{W}{1+\beta B},\ W - c = \frac{\beta B W}{1+\beta B}.$$

Then we have

$$\begin{aligned} A + B\log(W) &= \log(W) + \log\frac{1}{1+\beta B} + \beta A + \beta B \log(W) + \beta B \log \frac{\beta B}{1+\beta B} \\ &\Rightarrow \begin{cases} A = \beta A + \log\frac{1}{1+\beta B} + \beta B \log\frac{\beta B}{1+\beta B} \\ B = 1 + \beta B \end{cases} \end{aligned}$$

After some algebraic calculation, we can obtain the analytic solution to this simple cake-eating problem:

$$\begin{aligned} c^\star(W) &= (1-\beta)W \\ V(W) &= \frac{\log(W)}{1-\beta} + \frac{\log(1-\beta)}{1-\beta} + \frac{\beta \log(\beta)}{(1-\beta)^2} \end{aligned}$$

2. Implementation 1

2.1. VFI

Value function iteration means that we start with an arbitrary guess $V_0(W)$. At iteration $i = 1, 2, \ldots$, we compute

$$\begin{aligned} V_i(W) = T(V_{i-1})(W) &= \max_{0 \le c \le W} \left\{ u(c) + \beta V_{i-1}(W-c) \right\} \\ c_{i-1}(W) &= \operatorname*{arg\,max}_{0 \le c \le W} \left\{ u(c) + \beta V_{i-1}(W-c) \right\}. \end{aligned}$$

To put this idea into practice, we first need to **discretize** W, that is, to construct a grid of cake-sizes $\overrightarrow{W} \in \mathscr{W} := \{0, \ldots, \overline{W}\}$, and then calculate the following maximization problem in iteration i:

$$V_i(\overrightarrow{W}) = \max_{0 \le c \le \overrightarrow{W}} \left\{ u(c) + \beta V_{i-1}(\overrightarrow{W} - c) \right\}.$$

However, one important complication is that we need to evaluate V_{i-1} ***off-the-grids***, that is, there is no guarantee that $\overrightarrow{W} - c \in \mathscr{W}$. Therefore, we need to exploit some interpolation or function approximation techniques.

2.2. Avoiding interpolation: Solving the maximization problem "on-the-grid"

We can re-organize the Bellman equation as

$$V_i(\overrightarrow{W}) = \max_{0 \le \overrightarrow{W}' \le \overrightarrow{W}} \left\{ u\left(\overrightarrow{W} - \overrightarrow{W}'\right) + \beta V_{i-1}\left(\overrightarrow{W}'\right) \right\}.$$

By choosing $\overrightarrow{W}, \overrightarrow{W}' \in \mathscr{W}$, the last-iteration value function V_{i-1} is always evaluated on the grid, thus avoiding interpolation.