Wavelet Image Denoising with Dimension Reduction

Yunrong Wan, Yuxuan Li
Department of AMS
Johns Hopkins University
Baltimore, MD 21218
ywan13@jh.edu,yli441@jh.edu

July 1, 2023

1 Introduction

Digital images are widely used in various fields such as medical imaging, satellite imaging, and computer vision. However, images obtained from real-world scenarios are often corrupted with noise, which can significantly degrade their quality and affect their usefulness in applications. Wavelet image denoising methods have been shown to be effective in removing noise from images while preserving their structural information. However, these methods often require a large number of coefficients, which can lead to high computational complexity and memory requirements. To address these issues, dimension reduction techniques have been proposed to reduce the number of coefficients required for wavelet-based denoising.

In this paper, we propose a novel Wavelet image denoising method that incorporates dimension reduction using principal component analysis (PCA) and Singular Value Decomposition (SVD). Our method aims to achieve high de-noise performance with reduced computational complexity and memory requirements.

We establish two sets of picture experiments, cats and handwriting image. In each experiment, we add Gaussian noise to original images to obtain noisy images. Then we apply Wavelet Transform, including the BayesShrink and the VisuShrink with different thresholds, on noisy images and that after dimension reduction. By comparing efficiency of denoising based on the peak signal to the noise ratio (PSNR) and the structural similarity index (SSIM), our results demonstrate that in the denoising process, to preserve the important information of the subject, direct BayesShrink is more suitable; to reduce the interference from noise of useless information on the whole, the VisuShrink after PCA is a more suitable approach. We also conclude dimension reduction's effects on wavelet denoising and denoising effects on image information. The experiment results can be a valuable reference to image denoising, signal processing and other image-based applications.

2 Method

2.1 Dimension Reduction

2.1.1 PCA

Principal Component Analysis (PCA) works to transform high-dimensional data into a low-dimensional representation while preserving the most important information by transforming a given dataset into a new coordinate system such that the variance of the data in each axis is maximized. The new coordinate system is defined by the eigenvectors of the covariance matrix of the dataset. The eigenvectors with the highest eigenvalues correspond to the principal components, which represent the directions of maximum variation in the data.

PCA has numerous applications in image processing, including image compression, image denoising, and image recognition. In image compression, PCA is used to reduce the size of an image by retaining only the most important eigenvectors. In image denoising, PCA can be used to remove noise from an image by projecting the noisy image onto the eigenvectors corresponding to the largest eigenvalues. In image recognition, PCA can be used to extract the most relevant features from an image, which can then be used to classify the image.

2.1.2 SVD

Singular Value Decomposition (SVD) is a mathematical technique used to decompose a matrix into its constituent parts. In SVD, a given matrix is decomposed into three matrices, the left singular matrix, the singular value matrix, and the right singular matrix. The left and right singular matrices contain the eigenvectors of the matrix, while the singular value matrix contains the singular values, which represent the relative importance of each eigenvector.

Similar with PCA, SVD has applications in image denoising. In image compression, SVD is used to reduce the size of an image by retaining only the most important eigenvectors.In image denoising, SVD can be used to remove noise from an image by decomposing the noisy image into its eigenvectors and removing the noise from the singular values. In image recognition, SVD can be used to extract the most relevant features from an image, which can then be used to classify the image.

PCA and SVD are closely related because the eigenvectors and eigenvalues of the covariance matrix used in PCA can be computed using the SVD of the data matrix. Specifically, the singular values of the data matrix correspond to the square roots of the eigenvalues of the covariance matrix, and the right singular vectors of the data matrix correspond to the eigenvectors of the covariance matrix. Therefore, SVD can be used as an alternative method for computing the principal components of a dataset, and the projection matrix can be constructed using the right singular vectors of the data matrix.

2.2 Wavelet Transform

A function $\psi \in L^2(R)$ is called an orthonormal wavelet if it can be used to define a Hilbert basis, that is a complete orthonormal system, for the Hilbert space $L^2(R)$ of square integrable functions.The Hilbert basis is constructed as the family of functions $\{\psi_{jk} : j, k \in Z\}$ by means of dyadic translations and dilations of ψ, $\psi_{jk}(x) = 2^{\frac{j}{2}}\psi(2^j x - k)$ for integers $j, k \in Z$. Completeness is satisfied if every function $f(x) \in L^2(R)$ may be expanded in the basis as

$$f(x) = \sum_{j,k=-\infty}^{+\infty} c_{jk}\psi_{jk}(x)$$

with convergence of the series understood to be convergence in norm. Such a representation of f is known as a wavelet series. The integral wavelet transform is the integral transform defined as

$$[W_\psi f](a, b) = \frac{1}{\sqrt{|a|}} \int_{-\infty}^{+\infty} \overline{\psi(\frac{x-b}{a})} f(x)dx$$

The wavelet coefficients c_{jk} are then given by

$$c_{jk} = [W_\psi f]\left(2^{-j}, k2^{-j}\right)$$

Table 1: Comparison with Fourier Transform and Time-Frequency Analysis.

Transform	Representation	Input
Fourier Transform	$X(f) = \int_{-\infty}^{+\infty} x(t)e^{-i2\pi ft}dt$	f:frequency
Time-Frequency Analysis	$X(t, f)$	t:time; f:frequency
Wavelet Transform	$X(a, b) = \frac{1}{\sqrt{a}} \int_{-\infty}^{+\infty} \overline{\psi(\frac{t-b}{a})} x(t)d(t)$	a:scaling; b:time shift factor

Wavelet transform is a mathematical tool used for analyzing signals or images in the time-frequency domain. It allows for a multi-resolution analysis of the signal by decomposing it into a set of wavelet functions at different scales.

Different from Fourier Transform, Wavelet Transform has the following obvious advantages. Firstly, it has two variables scaling and time shift factor. Scaling controls the expansion of the wavelet function, and shifting controls the location. The scaling corresponds to frequency, and the shifting corresponds to time. Secondly, there are a wide variety of wavelets to choose from to best match the function shape.

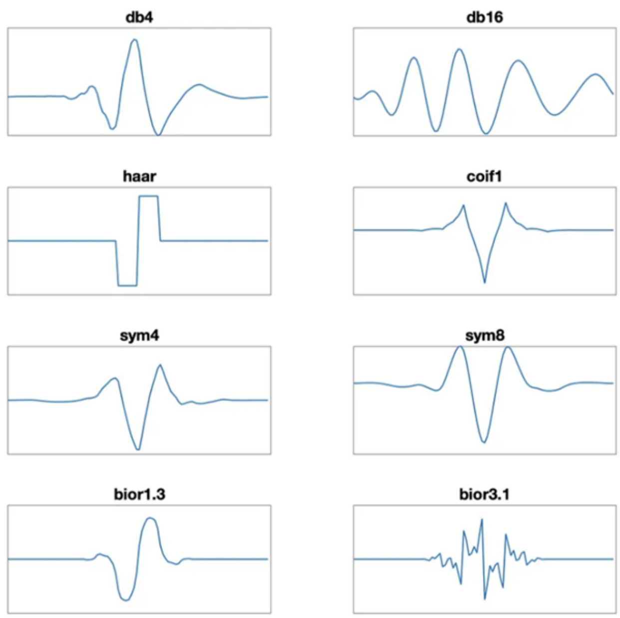

Figure 1: Wavelet

2.3 Wavelet Denoising

Wavelet denoising depends on the wavelet representation of the image. Gaussian noise tends to be represented by small values in the wavelet domain and can be eliminated by setting coefficients below a given threshold to zero (hard threshold) or shrinking gradually all coefficients towards zero by a given amount (soft threshold). The main idea of wavelet threshold denoising is that after the image is transformed by wavelet transform, the wavelet coefficients generated by this step contain both the important and unimportant information of the image. By selecting an appropriate threshold, the wavelet coefficients larger than the threshold are considered to be generated by this image and should be retained, while the wavelet coefficients smaller than the threshold are considered to be generated by noise and set to zero so as to achieve the purpose of denoising. In this paper, we apply two different methods for wavelet coefficient threshold selection: BayesShrink and VisuShrink.

2.3.1 BayesShrink

The BayesShrink is algorithm is an adaptive approach to wavelet soft thresholding where a unique threshold is estimated for each wavelet subband. This generally results in an improvement over what can be obtained with a single threshold. It is a soft threshold processing method. It can compress signals that are smaller than a certain threshold and retain signals that are larger than a threshold. Unlike the hard threshold, the soft threshold does not compress the signal below the threshold directly to zero, but shrinks it to a manageable degree. The processing result is smoother, which can effectively retain the details of the image and avoid the distortion of the image.

2.3.2 VisuShrink

The VisuShrink approach employs a single, universal threshold to all wavelet detail coefficients. This threshold is designed to remove additive Gaussian noise with high probability, which tends to result in overly smooth picture appearance. By specifying a sigma that is smaller than the true noise standard deviation, a more visually agreeable result can be obtained. It is a hard threshold means that the image signal less than a certain threshold is forcibly compressed to zero, while the signal larger than the threshold remains unchanged. This method is simple and practical, but it is easy to cause image distortion and detail loss.

3 Image process

After preliminary attempts, we found that dimension reduction after wavelet denoising has little effect on image denoising, so we mainly focus on the effect of image dimension reduction first on wavelet denoising. We added Gaussian noise, which is a common distribution of wavelet noise, with $\sigma = 0.15$ on the original noise-free image, and processed the noisy image to explore the impact of dimension reduction on the wavelet denoising effect. Three different image processing methods are applied on the noisy image. The first one is to perform wavelet denoising processing on the noisy image without any dimension reduction; the second one is to firstly perform PCA dimension reduction on the noisy image, and then perform wavelet denoise processing; the last one is to perform SVD dimension reduction processing on noisy images first, and then perform wavelet denoise processing. Finally, PSNR and SSIM value of each denoising image will be used to show the efficiency. A specific cat picture, as fig2 will be used as an example to explain the steps and effects in detail.

Figure 2: Cat

3.1 Direct Wavelet Denoising

In the common selection of the coefficient threshold for Wavelet denoising, BayesShrink and VisuShrink are methods which have better effects on Gaussian noise. Therefore, we apply these methods as threshold algorithm to denoise the noisy image. For the selection of noise variance of VisuShrink, we firstly estimate the variance of the noisy image to get σ_{est}, and then take σ_{est}, $\frac{1}{2}\sigma_{est}$ and $\frac{1}{4}\sigma_{est}$ as the variance of VisuShrink to try. For our example, σ_{est}=0.1445. The function `denoise_wavelet, estimate_sigma, peak_signal_noise_ratio`, as well as `structural_similarity` from package `skimage` are used to implement the process. The results are as 3:

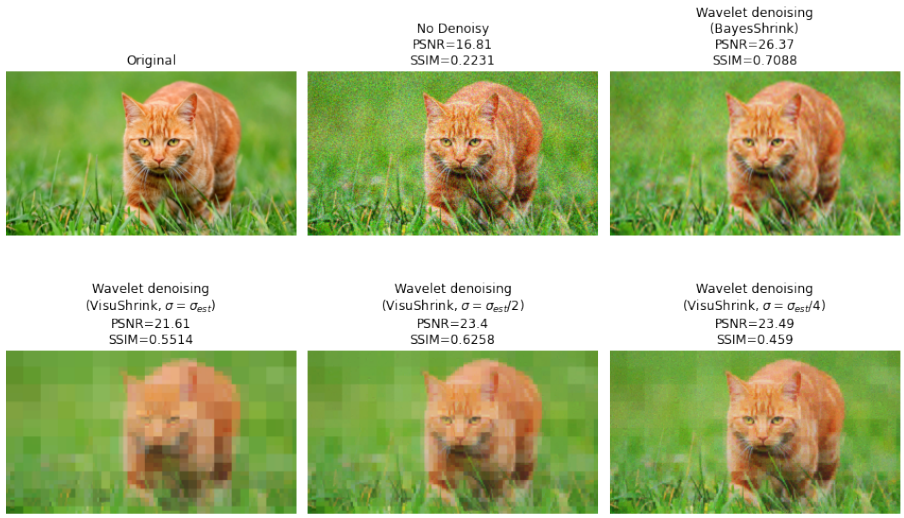

Figure 3: Direct Wavelet Denoising

3.2 Wavelet Denoising with Dimension Reduction

A colorful image is combined with three channels of RGB, so the dataset of it is a three-dimensional array, as fig4. In order to obtain a dimension reduction process that is more suitable for each channel, we perform dimension reduction on the matrix data corresponding to each of RGB channels, and then combine them as a three-dimensional data matrix as dimensionality reduction for the entire image.

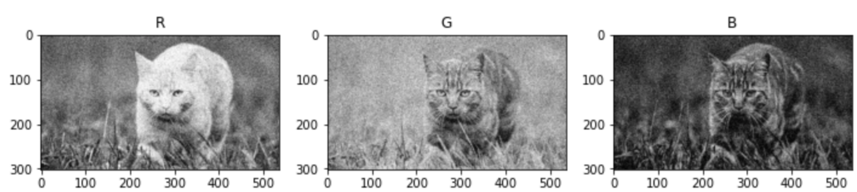

Figure 4: RGB of cat

3.2.1 PCA

The principle of PCA denoising is to judge the amount of information based on the size of the eigenvalues, and treat those small eigenvalues as noise interference and remove them.

For the number of principal components to be retained in PCA, we preliminary try it from 20 to 200. The results shows that, with the increasing of number, the image becomes clearer, while PSNR and SSIM become worse, which shows bad denoising effort, as fig5 and 6.So we select 50 principal components for trade off between denoising effort and deblurring. Then we apply the wavelet denoising after dimension reduction. The function `PCA` from package `sklearn.decomposition` is used, and the results are as 7:

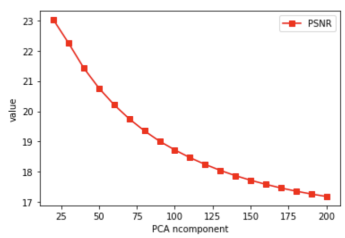

Figure 5: PSNR on various components number

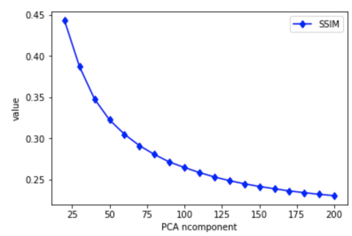

Figure 6: SSIM on various components number

Figure 7: Wavelet Denoising after PCA

3.2.2 SVD

Generally speaking, the important information of a matrix can be expressed by fewer singular values. So discarding some of the singular values to a common way to achieve compression. In image processing, the part with a small singular value in image matrix often represents noise, then the SVD algorithm is a good way to reduce dimension and achieve denoising.

Similar with the PCA logic, the number of retained singular values of SVD is smaller, the redundant information in the picture is less, but, corresponsively, main information will also be missing more, which makes the image blurred. Therefore, in the selection of the retained number, after preliminary attempts, 10% of the singular values are kept to achieve a tradeoff between denoising and deblurring. Then we apply the wavelet denoising after dimension reduction. The results are as 8:

Figure 8: Wavelet Denoising after SVD

3.3 Other examples

We do the same processing on another handwriting image fig9, the results are as fig10,11 and 12

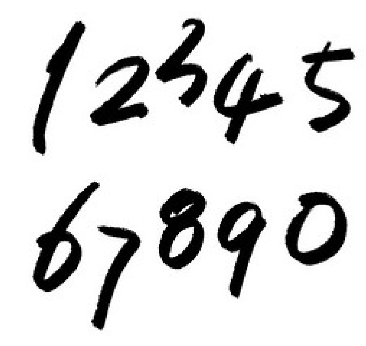

Figure 9: Original image

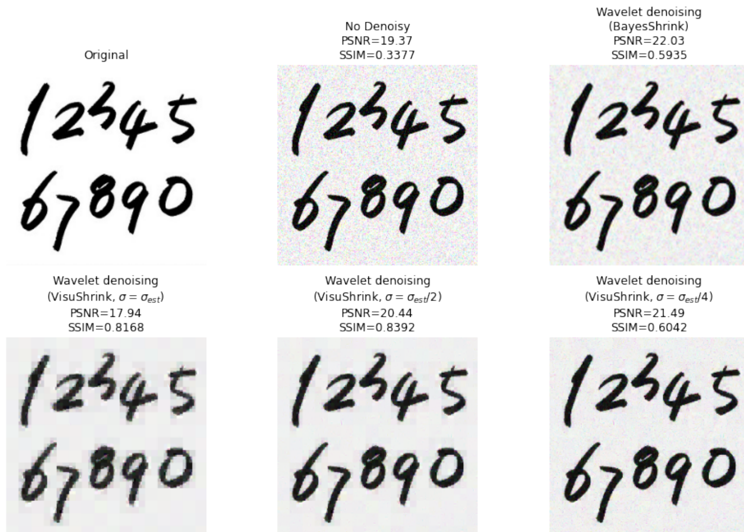

Figure 10: Direct Wavelet Denoising

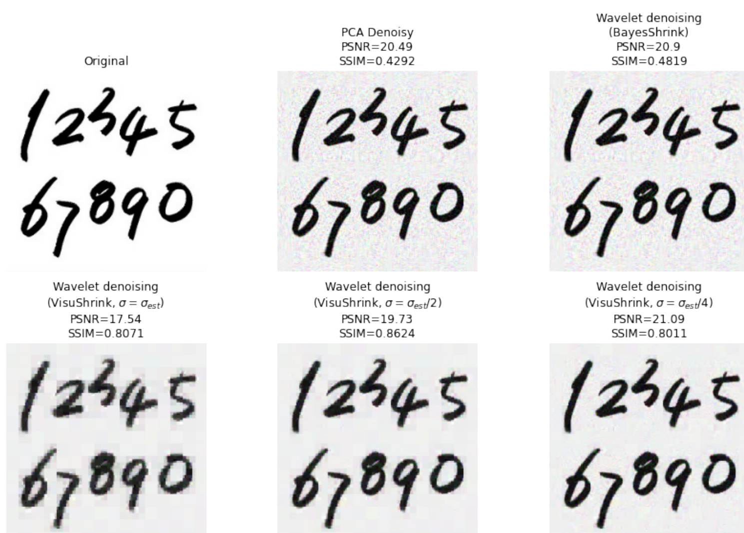

Figure 11: Wavelet Denoising after PCA

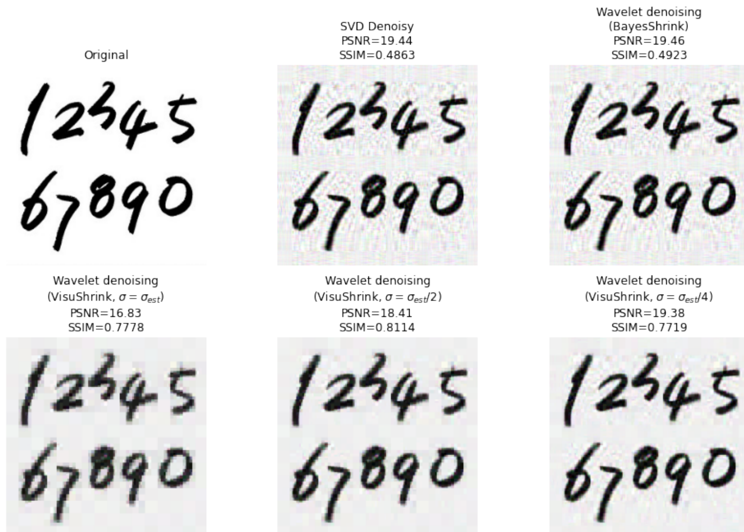

Figure 12: Wavelet Denoising after SVD

4 Conclusion

4.1 Dimension reduction effect on wavelet denoising

Compared with the PSNR and SSIM reults of three methods, see Table2,we find that:
(1) For BayesShrink algorithm, directly applying it on noisy images can remove noise points very well and get the best denoising efficiency, while reducing dimension first and then denoising will reduce its effect.

(2) For VisuShrink threshold, setting variance as $\frac{1}{4}\sigma_{est}$ will perform better than the others, and reducing dimension first will improve its denoising effect. Also the PCA has better improvement for VisuShrink denoising than SVD

Table 2: Summary of denoising effect for Cat (PSNR/SSIM)

Methods	Dimension Reduction	BayesShrink	VisuShrink σ_{est}	VisuShrink $\frac{1}{2}\sigma_{est}$	VisuShrink $\frac{1}{4}\sigma_{est}$
None	16.81 0.2231	26.37** 0.7088	21.61 0.5514	23.40 0.6258	23.49 0.4590
PCA	20.76 0.3237	24.20 0.5134	21.47 0.5442	22.97 0.6161	24.52** 0.6737
SVD	22.28 0.3895	23.50 0.4720	21.42 0.5424	22.81 0.6078	24.23** 0.6756

The top three performers are marked by double asterisks(**).

Table 3: Summary of denoising effect for Handwriting (PSNR/SSIM)

Methods	Dimension Reduction	BayesShrink	VisuShrink σ_{est}	VisuShrink $\frac{1}{2}\sigma_{est}$	VisuShrink $\frac{1}{4}\sigma_{est}$
None	19.37 0.3377	22.03** 0.5935	17.94 0.8168	20.44 0.8392	21.49** 0.6042
PCA	20.49 0.4292	20.90 0.4819	17.54 0.8071	19.73 0.8624	21.09** 0.8011
SVD	19.44 0.4863	19.46 0.4923	16.83 0.7778	18.41 0.8114	19.38 0.7719

The top three performers are marked by double asterisks(**).

4.2 Denoise effects on specific information

Compared the direct BayesShrink wavelet denoising and VisuShrink wavelet denoising after PCA, which are the top 2 denoising preformance, on processing specific information, the two denoising processing shows different effects.

When denoising the main information with a large amount of information, by zooming in picture and comparing the denoising effect of the main part, we found that direct BayesShrink has a better denoising performance, which means that it keeps more accurate important information about the main content, see fig13 and 14.

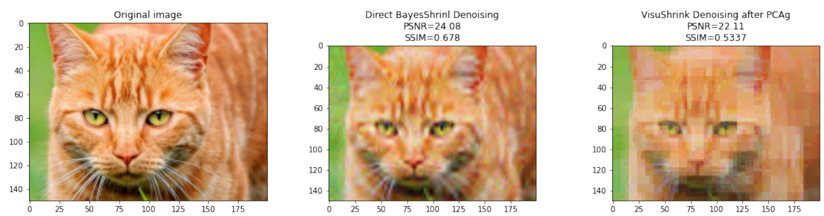

Figure 13: Main information of Cat

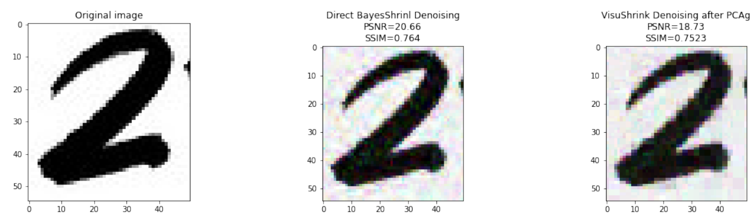

Figure 14: Main information of Handwriting

In terms of comparing the denoising effects on unimportant information, we enlarge the background of the image without main information and compare the two performances with the original image. We found that VisuShrink after PCA performs better in reducing the agitation effect of unimportant information, which means that it can reduce the the interference of useless information to the whole, see fig15 and 16.

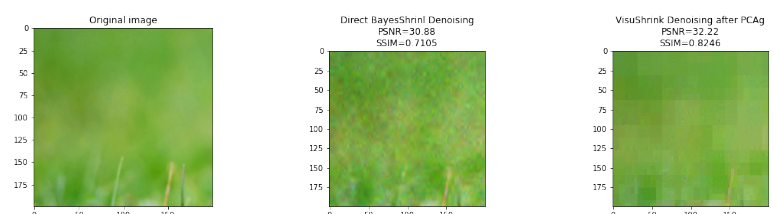

Figure 15: Background of Cat

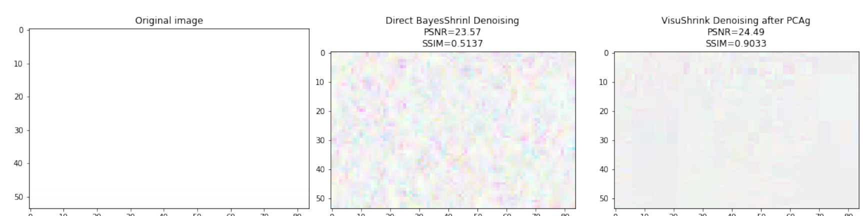

Figure 16: Background of Handwriting

Based on the above analysis, we can draw the following conclusions:

(1) In the denoising process whose purpose is preserving the important information of the subject as much as possible, Direct BayesShrink is a more suitable method of eliminating noise.
(2) In the denoising process whose purpose is reducing the interference from noise of useless information on the whole, VisuShrink after PCA is a more suitable method of removing noise.

This also applies to data processing, signal processing and other researches involving noise interference.

References

[1] K. Dabov, A. Foi, V. Katkovnik and K. Egiazarian, "Image Denoising by Sparse 3-D Transform-Domain Collaborative Filtering," in IEEE Transactions on Image Processing, vol. 16, no. 8, pp. 2080-2095, Aug. 2007, doi: 10.1109/TIP.2007.901238.

[2] Koushik Chandrasekaran, "2D-Discrete Wavelet Transformation and its applications in Digital Image Processing using MATLAB", https://medium.com/@koushikc2000/2d-discrete-wavelet-transformation-and-its-applications-in-digital-image-processing-using-matlab-1f5c68672de3

[3] Bnou, K., Raghay, S. & Hakim, A. A wavelet denoising approach based on unsupervised learning model. EURASIP J. Adv. Signal Process. 2020, 36 (2020). https://doi.org/10.1186/s13634-020-00693-4

[4] Collince Odhiambo, "Wavelet-based Denoising of the 1-D signal using Python", https://www.section.io/engineering-education/wavelet-transform-analysis-of-1d-signals-using-python/

[5] Christopher Schölzel, learn-wavelets, 2016, https://github.com/CSchoel/learn-wavelets#readme