VE401 Assignment 3

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Exercise 10. Bivariate Normal Distribution as a Mixture of Independent Normal Distributions

i) **Proof.** First we want to say that, since $\phi(X) = AX$ is linear, we can directly see that $E(AX) = AE(X)$.

Of course, we can still multiply things together and still prove this formula. Assume that $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ and $X = \binom{x_1}{x_2}$. Then

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix}$$

And then

$$E\begin{pmatrix} A_{11}x_1 + A_{12}x_2 \\ A_{21}x_1 + A_{22}x_2 \end{pmatrix} = \begin{pmatrix} E(A_{11}x_1 + A_{12}x_2) \\ E(A_{21}x_1 + A_{22}x_2) \end{pmatrix} = \begin{pmatrix} A_{11}E(x_1) + A_{12}E(x_2) \\ A_{21}E(x_1) + A_{22}E(x_2) \end{pmatrix} = AE(X)$$

□

ii) For our convenience, we denote $VarX_1$ as σ_1^2, $VarX_2$ as σ_2^2, $Cov(X_1, X_2)$ as †. Now we are going to take a deep breadth and multiply these matrix together.

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} \sigma_1^2 & \dagger \\ \dagger & \sigma_2^2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} = \begin{pmatrix} A_{11}\sigma_1^2 + A_{12}\dagger & A_{11}\dagger + A_{12}\sigma_2^2 \\ A_{21}\sigma_1^2 + A_{22}\dagger & A_{21}\dagger + A_{22}\sigma_2^2 \end{pmatrix} \begin{pmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{pmatrix} =$$
$$\begin{pmatrix} A_{11}^2\sigma_1^2 + 2A_{11}A_{12}\dagger + A_{12}^2\sigma_2^2 & A_{11}A_{21}\sigma_1^2 + A_{21}A12\dagger + A_{12}A_{22}\sigma_2^2 + A_{22}A_{11}\dagger \\ A_{11}A_{21}\sigma_1^2 + A_{21}A_{12}\dagger + A_{12}A_{22}\sigma_2^2 + A_{22}A_{11}\dagger & A_{21}^2\sigma_1^2 + 2A_{21}A_{22}\dagger + A_{22}^2\sigma_2^2 \end{pmatrix}$$

We can see exactly that main diagonal is just $Var(A_{11}x_1 + A_{12}x_2)$ and $Var(A_{21}x_1 + A_{22}x_2)$. For the other diagonal, we can see that $\sigma_1^2 = Cov(x_1, x_1)^2$, hence they are exactly $Cov(A_{11}x_1 + A_{12}x_2, A_{21}x_1 + A_{22}x_2)$ and $Cov(A_{21}x_1 + A_{22}x_2, A_{11}x_1 + A_{12}x_2)$.

Therefore, $Var(AX) = A(VarX)A^T$.

iii) X_1 and X_2 follow independent normal distributions, thus

$$\begin{aligned} f_X(x_1, x_2) &= f_{X_1}(x_1) \cdot f_{X_2}(x_2) \\ &= \frac{1}{2\pi} \cdot \frac{1}{\sigma_1} e^{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2} \cdot \frac{1}{\sigma_2} e^{-\frac{1}{2}(\frac{x_2-\mu_2}{\sigma_2})^2} \\ &= \frac{1}{2\pi\sqrt{det \sum_x}} \cdot e^{-\frac{1}{2}(\frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2})} \end{aligned}$$

For our convenience, we now focus on the exponential. Note that

$$\begin{aligned} \frac{(x_1-\mu_1)^2}{\sigma_1^2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2} &= \left\langle \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sigma_1^2} \cdot (x_1 - \mu_1) \\ \frac{1}{\sigma_2^2} \cdot (x_2 - \mu_2) \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}, \begin{pmatrix} \frac{1}{\sigma_1^2} & 0 \\ 0 & \frac{1}{\sigma_2^2} \end{pmatrix} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} \right\rangle \\ &= \langle x - \mu_x, \Sigma_x^{-1}(x - \mu_x) \rangle \end{aligned}$$

iv) 5

v) **Solution.** We know the m.g.f. of normal distribution is

$$m_x(t) = exp\left(u \times t - 0.5t^2 \times D\right)$$

where D is the variance. We send it in matrix terms.

Let $X = (x1, x2)$. $M_x(t) = exp(u^T t - 0.5t^T \cdot D \cdot t)$, D is

$$\begin{pmatrix} \sigma_1^2 & ,0 \\ 0 & ,\sigma_2^2 \end{pmatrix}$$

, A is the transform matrix for Y=AX, u is matrix of expression.

We already know X follows bi-normal distribution for

$$\begin{aligned} f_{x_1x_2}(X_1, X_2) &= P(X_1 = x1, X_2 = x_2) \\ &= P(X_1 = x_1) \cdot P(X_2 = x_2) \\ &= f_{x_1}(x) \cdot f_{x_2}(x) \\ &= \sqrt{A} \cdot \int e^{(n^2)^B} dx_1 \cdot \sqrt{A} \cdot \int e^{(m^2)^B} dx_2 \\ &= A \cdot \int e^{(m^2+n^2)^B} dx_1 dx_2 \end{aligned}$$

It's Bi-normal distribution's pdf when $\rho = 0$.Since X_1,X_2 are independent, $\rho = 0$. So X follows Bi-normal distribution. Y=AX

In matrix term,

$$\begin{aligned} m_Y(t) &= E(e^Y t^T) = E(e^A X t^T) = E(e^{(A^T t)^T} X) \\ &= exp(u^T A^T t - 0.5(A^T t)^T \cdot D \cdot A^T t) \\ &= exp((Au)^T t - 0.5(A^T t)^T \cdot D \cdot A^T t) \\ &= exp((Au)^T t - 0.5(t)^T \cdot (A \cdot D \cdot A^T) \cdot t) \\ &= exp((u')^T t - 0.5(t)^T \cdot (D') \cdot t) \end{aligned}$$

Obviously Y follows the same kind of distribution with only different parameters. Y follows bi-normal distribution.

According to 3.10.3,

$$f_{x_1x_2}(x_1, x_2) = A \cdot exp(-0.5 < x - ux, \sum{}^{-1} x(x - ux) >)$$

Hence

$$f_{y_1y_2}(y_1, y_2) = A \cdot exp(-0.5 < y - uy, \sum{}^{-1} y(y - uy) >)$$

And $fy(y) = Ae^(m^2 - 2\rho mn + n^2)^B$

with:

$A = 1/2p\sigma_1\sigma_2\sqrt{1 - \rho_2}$

$B = -0.5/(1 - \rho_2)$

$m = (y_2 - u_2)/\sigma_2$

$n = (y_1 - \mu_1)/\sigma_1$ □