

# VE401 Assignment

Yang, Tiancheng 517370910259

Qiu, Yuqing 518370910026

Chang, Siyao 517370910024

March 8, 2020

**Exercise 1.** Elementary Probability

**Solution.** We use Cardano's principle to get the probability. The number of ways to pick 120 people from 2000 individuals is

$$n_1 = \frac{2000!}{120! \times (2000 - 120)!}$$

The number of ways that me and my friend are both chosen is equal to the number of ways to choose 118 people from 1998 individuals, which is

$$n_2 = \frac{1998!}{118! \times (1998 - 118)!}$$

Therefore the probability that me and my friend will both be chosen is

$$\frac{n_2}{n_1} = \frac{\frac{1998!}{118! \times (1998 - 118)!}}{\frac{2000!}{120! \times (2000 - 120)!}} = 0.357\%$$

□

**Exercise 2.** Some Routine Calculations

- i) **Proof.** Since  $A \subset B$ ,  $B = A + B \setminus A$ . Note that  $A \cap B \setminus A = \emptyset$ . Thus  $P[B] = P[A] + P[B \setminus A] \geq P[A]$ . Therefore  $P[A] \leq P[B]$ . □
- ii) **Proof.** Since A and B are independent, we have  $P[A \cap B] = P[A]P[B]$ . We know that  $P[A]P[B] > 0$  so  $P[A \cap B] > 0$ . Thus  $P[A \cap B] = \frac{|A \cap B|}{|S|} > 0$ , which means that  $|A \cap B| > 0$ . Therefore,  $A \cap B \neq \emptyset$  and hence they are not mutually exclusive. □
- iii) **Proof.** First we have two trivial equations:

$$P[A] = P[A \setminus (A \cap B)] + P[A \cap B]$$

$$P[B] = P[B \setminus (A \cap B)] + P[A \cap B]$$

We also have

$$P[A \cup B] = P[A \setminus (A \cap B)] + P[B \setminus (A \cap B)] + P[A \cap B]$$

. Therefore,

$$P[A \cup B] = P[A] - P[A \cap B] + P[B] - P[A \cap B] + P[A \cap B] = P[A] + P[B] - P[A \cap B]$$

□

**Exercise 3.** D'Alembert's Coins

- i) **Solution.** No, it is not possible. If the coin is fair, we know that

$$P[\text{two heads}] = P[\text{no heads}] < P[\text{onehead}]$$

. Now if the coin is biased, for that coin  $P[\text{head}] \neq P[\text{nohead}]$ . And hence if it is tossed twice,

$$P[\text{two heads}] \neq P[\text{no heads}]$$

. Therefore, even though the coin can be biased, the three outcomes cannot have the same probability. □

ii) **Solution.** No, it is not possible as well. Denote one coin as  $A$ , with  $P[A, head] = a$ . We know that  $P[A, head] + P[A, tail] = 1$  so  $P[A, tail] = 1 - a$ . Similarly, we also have coin  $B$  with  $b$  and  $1 - b$ . Now, if the three outcomes have same probability, then

$$a \times b = (1 - a) \times (1 - b) = 1/3$$

. From here we get that  $a + b = 1$ . Now we calculate

$$a \times b = a \times (1 - a) = \frac{1}{3}$$

which has no real solution. Therefore, it is impossible to make the coins so that D'Alembert's claim is true.  $\square$

#### Exercise 4. Independence

i) **Solution.** Denotation of events:

P1: a participant from the first group is chosen;

P2: a participant from the second group is chosen;

A: the participant replies "yes" to the second question;

Now we list the known probabilities:  $P[P1] = 50\%$ ,  $P[P2] = 50\%$ ,  $P[A|P1] = 17\%$ ,  $P[A|P2] = 3\%$ . We want to know the total probability of  $P[A]$ .

$$P[A] = P[A|P1] \times P[P1] + P[A|P2] \times P[P2] = 17\% \times 50\% + 3\% \times 50\% = 10\%$$

Therefore, this probability is 10%.  $\square$

ii) **Solution.** No. We have  $P[A|P1] > P[A]$ . Therefore it is not independent.  $\square$

#### Exercise 5. This one may need a little thinking about... Though it doesn't.

**Solution.** We denote the event that a chip is defective by  $D$ . The event that a chip is stolen is  $S$ .  $\neg S$  means that a chip is not stolen. Thieves stole the chips before inspection, so  $P[D|S] = 50\%$ . We know that  $P[S] = 1\%$  of the chips is illegal, hence  $P[\neg S] = 99\%$  of the chips is legally marketed. For those chips that are legally marketed, their probability of being defective is 5%, because of the inspection. This means that if the chips is not stolen, the probability of being defective is  $P[D|\neg S] = 5\%$ . Now we apply the Bayes' theorem to get the probability we want to calculate, i.e.,  $P[S|D]$ .

$$P[S|D] = \frac{P[D|S]P[S]}{P[D|S]P[S] + P[D|\neg S]P[\neg S]} = \frac{0.5 \times 0.01}{0.5 \times 0.01 + 0.05 \times 0.99} = 9.17\%$$

$\square$

#### Exercise 6. Mounty Hall in Prison?

**Solution.** They are both wrong. Conclusions first: the prisoner is wrong because the warden would have given him some information; the warden is wrong because the one who have a bigger chance of dying would not have been prisoner  $A$ .

Suppose that the warden tells  $A$  that prisoner  $B$  is not to be executed. We denote this event by  $B^*$ . The event that prisoner  $X$  is to be executed is denoted by  $DX$ . By Bayes' Theorem, we have

$$P[DA|B^*] = \frac{P[B^*|DA]P[DA]}{P[B^*|DA]P[DA] + P[B^*|DB]P[DB] + P[B^*|DC]P[DC]} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 + 1 \times \frac{1}{3}} = \frac{1}{3}$$

However, for prisoner  $C$ , the probability that he is to be executed is then

$$P[DC|B^*] = \frac{P[B^*|DC]P[DC]}{P[B^*|DC]P[DC] + P[B^*|DB]P[DB] + P[B^*|DA]P[DA]} = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 + \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3}$$

Therefore, neither of them is right.  $\square$

### Exercise 7. Two Children Paradox - Birthday Party

**Solution.** Denote the events as:

A: The lady's other child is a girl;

B: The lady's boy is born in July;

Then, according to the problem, what we need to calculate is  $P[A|B]$ .

We define the sample space of the gender as  $\{(b, b), (b, g), (g, b)\}$ , where 'b' means the child is a boy and 'g' means the child is a girl and the former child is older than the latter. Each sample point has the same probability, i.e.

$$P[(b, b)] = P[(b, g)] = P[(g, b)] = \frac{1}{3}$$

For  $(b, g)$  and  $(g, b)$ , the probability of the boy born in July should be  $\frac{1}{12}$ .

For  $(b, b)$ , the probability of at least one boy born in July can be calculated as follows:

i) Only one boy is born in July:  $\frac{1}{12} \times \frac{11}{12} \times 2 = \frac{22}{144}$

ii) Two boys are both born in July:  $\frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$

Then, the probability of at least one boy born in July is  $\frac{22}{144} + \frac{1}{144} = \frac{23}{144}$

Here we calculate  $P[A \cap B]$ . We use the total probability,

$$\begin{aligned} P[A \cap B] &= P[A \cap B|(b, b)]P[(b, b)] + P[A \cap B|(b, g)]P[(b, g)] + P[A \cap B|(g, b)]P[(g, b)] \\ &= 0 \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} \\ &= \frac{1}{18} \end{aligned}$$

Similarly, we can calculate  $P[B]$ .

$$\begin{aligned} P[B] &= P[B|(b, b)]P[(b, b)] + P[B|(b, g)]P[(b, g)] + P[B|(g, b)]P[(g, b)] \\ &= \frac{23}{144} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} \\ &= \frac{47}{432} \end{aligned}$$

Then, using the formula of conditional formula,

$$\begin{aligned} P[A|B] &= \frac{P[A \cap B]}{P[B]} \\ &= \frac{\frac{1}{18}}{\frac{47}{432}} \\ &= \frac{24}{47} \end{aligned}$$

□