VE 401 Assignment 3 Yuqing Qiu

Problem 1.

i) $E[V] = \int_{-\infty}^{\infty} v \cdot f_V(v)\, dv = \int_0^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{KT}\right)^{3/2} v^3 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}}\, dv + \int_{-\infty}^{0} v \cdot 0\, dv$.

$= \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{KT}\right)^{3/2} \int_0^{\infty} v^3 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv$

For $\int_0^{\infty} v^3 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv$, we denote $t = \frac{m}{KT}\cdot\frac{v^2}{2}$ then $dt = \frac{mv}{KT}\cdot dv$, $v^2 = t \cdot \frac{2KT}{m}$

then $\int_0^{\infty} v^3 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv = \int_0^{\infty} \frac{2K^2T^2}{m^2} t\, e^{-t} dt = \frac{2K^2T^2}{m^2} \int_0^{\infty} t^{2-1} e^{-t} dt = \frac{2K^2T^2}{m^2} \Gamma(2) = \frac{2K^2T^2}{m^2}$

then $E[V] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} \cdot \frac{2K^2T^2}{m^2} = \boxed{2\sqrt{\frac{2KT}{\pi m}}}$

$E[V^2] = \int_{-\infty}^{\infty} v^2 \cdot f_V(v)\, dv = \int_0^{\infty} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} v^4 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv + \int_{-\infty}^{0} v^2 \cdot 0\, dv$

$= \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} \int_0^{\infty} v^4 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv$

For $\int_0^{\infty} v^4 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv$, similary, we denote $t = \frac{m}{KT}\cdot\frac{v^2}{2}$, then $dv = \frac{KT}{mv}\cdot dt$, $v^3 = t^{\frac{3}{2}} \cdot \left(\frac{2KT}{m}\right)^{\frac{3}{2}}$

then $\int_0^{\infty} v^4 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv = 2^{\frac{3}{2}} \cdot \left(\frac{KT}{m}\right)^{\frac{5}{2}} \int_0^{\infty} t^{\frac{5}{2}-1} e^{-t} dt = 2^{\frac{3}{2}} \left(\frac{KT}{m}\right)^{\frac{5}{2}} \Gamma\left(\frac{5}{2}\right) = 2^{\frac{3}{2}} \left(\frac{KT}{m}\right)^{\frac{5}{2}} \times \frac{3}{2} \times \frac{1}{2} \times \sqrt{\pi} = \frac{3\sqrt{\pi}}{\sqrt{2}} \left(\frac{KT}{m}\right)^{\frac{5}{2}}$

then $E[V^2] = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} \cdot \frac{3\sqrt{\pi}}{\sqrt{2}} \cdot \left(\frac{KT}{m}\right)^{\frac{5}{2}} = \frac{3KT}{m}$

Then, $V[V] = E[V^2] - E[V]^2$

$= \frac{3KT}{m} - 4 \times \frac{2KT}{\pi m}$

$= \boxed{\frac{(3\pi - 8)KT}{\pi m}}$

ii) $E\left[\frac{mV^2}{2}\right] = \int_{-\infty}^{\infty} \frac{mv^2}{2} f_V(v)\, dv = \frac{m}{2} \int_{-\infty}^{\infty} v^2 \cdot f_V(v)\, dv = \frac{m}{2} E[V^2] = \frac{\cancel{m}}{2} \cdot \frac{3KT}{\cancel{m}} = \boxed{\frac{3KT}{2}}$

iii) $F_E(\varepsilon) = P[E \le \varepsilon] = P\left[\frac{mV^2}{2} \le \varepsilon\right] = P\left[V^2 \le \frac{2\varepsilon}{m}\right]$

① if $\varepsilon < 0$, then $P\left[V^2 \le \frac{2\varepsilon}{m}\right] = 0$, i.e. $F_E(\varepsilon) = 0$, therefore, $f_E(\varepsilon) = F_E(\varepsilon)' = 0$.

② if $\varepsilon \geq 0$, then $F_E(\varepsilon) = P\left[V^2 \le \frac{2\varepsilon}{m}\right] = P\left[-\sqrt{\frac{2\varepsilon}{m}} \le V \le \sqrt{\frac{2\varepsilon}{m}}\right] = \int_0^{\sqrt{\frac{2\varepsilon}{m}}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} v^2 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv + \int_{-\sqrt{\frac{2\varepsilon}{m}}}^{0} 0\, dv$.

$= \int_0^{\sqrt{\frac{2\varepsilon}{m}}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} v^2 e^{-\frac{m}{KT}\cdot\frac{v^2}{2}} dv$.

then $f_E(\varepsilon) = F_E(\varepsilon)' = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \left(\frac{m}{KT}\right)^{\frac{3}{2}} \frac{2\varepsilon}{m} e^{-\frac{m}{KT}\cdot\frac{\varepsilon}{m}} = \left(\frac{2m}{\pi}\right)^{\frac{1}{2}} \cdot \left(\frac{1}{KT}\right)^{\frac{3}{2}} \cdot 2\varepsilon \cdot e^{-\frac{\varepsilon}{KT}}$

In conclusion, $f_E(\varepsilon) = \begin{cases} 2\varepsilon e^{-\frac{\varepsilon}{KT}} \left(\frac{2m}{\pi}\right)^{\frac{1}{2}} \left(\frac{1}{KT}\right)^{\frac{3}{2}}, & \varepsilon \geq 0 \\ 0, & \varepsilon < 0 \end{cases}$

Problem 2.

$\Gamma(\frac{2n+1}{2}) = \Gamma(n+\frac{1}{2})$

we now prove that $\Gamma(x+1) = x\Gamma(x) \quad (x>0)$.

$\Gamma(x+1) = \int_0^\infty t^x e^{-t} dt = [-t^x e^{-t}]_0^\infty + x\int_0^\infty t^{x-1} e^{-t} dt = [-t^x e^{-t}]_0^\infty + x\Gamma(x)$

$[-t^x e^{-t}]_0^\infty = -\lim_{t\to\infty} \frac{t^x}{e^t} = -\lim_{t\to\infty} \frac{C\cdot t^{x-[x]-1}}{e^t}$, where C is a constant and $[x]$ is the maximum integer which is not larger than x

then $[-t^x e^{-t}]_0^\infty = 0$

then $\Gamma(x+1) = x\Gamma(x)$

Using this property, we can get that $\Gamma(n+\frac{1}{2}) = \frac{2n-1}{2}\Gamma(n-1+\frac{1}{2}) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\Gamma(n-2+\frac{1}{2})$

$= \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$

$\Gamma(\frac{1}{2}) = \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt$, now we change t as a^2, then $dt = 2a\,da$

$\Rightarrow \Gamma(\frac{1}{2}) = \int_0^\infty (a^2)^{-\frac{1}{2}} e^{-a^2}\cdot(2a)\,da = 2\int_0^\infty e^{-a^2} da$

since $(\int_0^\infty e^{-a^2} da)^2 = \int_0^\infty e^{-a^2} da \int_0^\infty e^{-b^2} db = \int_0^\infty\int_0^\infty e^{-(a^2+b^2)} da\,db$, if we change a as $r\cos\theta$, b as $r\sin\theta$

then $(\int_0^\infty e^{-a^2} da)^2 = \int_0^{\frac{\pi}{2}}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{2}} d\theta\cdot[-\frac{1}{2}e^{-r^2}]_0^\infty = \frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi}{4}$

then $2\int_0^\infty e^{-a^2} da = 2\times\sqrt{\frac{\pi}{4}} = \sqrt{\pi}$, i.e. $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

Finally, $\Gamma(n+\frac{1}{2}) = \frac{(2n-1)(2n-3)\cdots 3\cdot 1}{2^n}\sqrt{\pi}$

$= \frac{(2n)!}{(2n\cdot(2n-2)\cdots 2)\cdot 2^n}\sqrt{\pi}$

$= \boxed{\frac{(2n)!}{2^{2n}\cdot n!}\sqrt{\pi}}$

Problem 3.

i) $P[X < 6250] = P[\frac{X-\mu}{\sigma} < \frac{6250-6000}{100}] = P[Z < 2.5] = \Phi(2.5) = \boxed{0.9938}$

ii) $P[5800 \le X \le 5900] = P[\frac{5800-6000}{100} \le \frac{X-\mu}{\sigma} \le \frac{5900-6000}{100}] = P[-2 \le Z \le -1] = \Phi(-1) - \Phi(-2) = \boxed{0.1359}$

iii) $P[X \le x] = P[\frac{X-\mu}{\sigma} \le \frac{x-6000}{100}] = P[Z \le \frac{x-6000}{100}] = 1-0.95 = 0.05$

$\Rightarrow \frac{x-6000}{100} = -1.6449 \Rightarrow x = \boxed{5835.5\ kg/cm^2}$

Problem 4.

i) Denote X as a discrete random variable representing as the number of errors in these random 50 pages, $X \in N$

then $P[X \ge 1] = 1 - P[X < 1]$ since when $X < 0$, $P[X] = 0$, then $P[X \ge 1] = 1 - P[X = 0]$

$P[X=0]$ means there is no error in these 50 pages, i.e. all of the five errors are in the rest 150 pages.

For each error, it has the probability of $\frac{50}{200}$ appearing in the chosen 50 pages and has the probability of $\frac{150}{200}$ appearing in the rest 150 pages.

Then we can consider it as a binomial distribution

Then $P[X=0] = \binom{5}{5}\left(\frac{50}{200}\right)^0 \cdot \left(\frac{150}{200}\right)^5 = 0.2373$

then $P[X \geq 1] = 1 - P[X=0] = \boxed{0.7627}$

ii)

$P[X \geq 3] = 1 - P[X \leq 2] = 0.9 \Rightarrow P[X \leq 2] = 0.1$.

Then we use the normal approximation, $P[X \leq y] = \sum_{x=0}^{y} \binom{n}{x} p^x (1-p)^{n-x} \approx \phi\left(\frac{y+\frac{1}{2}-np}{\sqrt{np(1-p)}}\right)$

where n should be 5, p should be $\frac{m}{200}$ (m is the number of pages we choose) and $1-p$ should be $\frac{200-m}{200}$

$\Rightarrow P[X \leq 2] = \phi\left(\frac{2.5 - 5 \times \frac{m}{200}}{\sqrt{5 \times \frac{m(200-m)}{200^2}}}\right) = 0.1 \Rightarrow \frac{2.5 - 5 \times \frac{m}{200}}{\sqrt{5 \times \frac{m(200-m)}{200^2}}} = -1.28$

Then, we get that $m = \boxed{150}$

Problem 5.

Since X, Y are independent standard normal distribution,

then $f_{XY}(x,y) = f_X(x) \cdot f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$

since $U = \frac{X}{Y}$, then using the lemma $f_U(u) = \int_{-\infty}^{\infty} f_{XY}(uv, v) \cdot |v| \, dv$

we get that $f_U(u) = \int_{-\infty}^{\infty} \frac{1}{2\pi} \cdot e^{-\frac{u^2v^2+v^2}{2}} \cdot |v| \, dv = \frac{1}{2\pi} \int_{-\infty}^{\infty} |v| e^{-\frac{u^2+1}{2} \cdot v^2} dv$

$$= \frac{1}{2\pi}\left(\int_0^{\infty} v e^{-\frac{u^2+1}{2} \cdot v^2} dv - \int_{-\infty}^{0} v e^{-\frac{u^2+1}{2} \cdot v^2} dv\right)$$

$$= \frac{1}{2\pi}\left(\left[-\frac{1}{u^2+1} e^{-\frac{u^2+1}{2} v^2}\right]_0^{\infty} - \left[-\frac{1}{u^2+1} e^{-\frac{u^2+1}{2} v^2}\right]_{-\infty}^{0}\right)$$

$$= \frac{1}{2\pi}\left(\frac{1}{u^2+1} + \frac{1}{u^2+1}\right)$$

$$= \boxed{\frac{1}{\pi(u^2+1)}}$$

$\cancel{E[U] = \int_{-\infty}^{\infty} u \cdot f_U(u) du = \int_{-\infty}^{\infty} u \cdot \frac{1}{\pi(u^2+1)} du = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{u}{u^2+1} du = \frac{1}{\pi}\left[\frac{1}{2} \ln(u^2+1)\right]_{-\infty}^{\infty} = \frac{1}{2\pi}\left[\ln(u^2+1)\right]_{-\infty}^{\infty}}$

The expectation of U does not exist because it's not absolutely convergent.

Proof. $\int_{-\infty}^{\infty} |u \cdot f_U(u)| du = \int_{-\infty}^{\infty} |u| \cdot \frac{1}{2(u^2+1)} du = \int_0^{\infty} u \cdot \frac{1}{\pi} \cdot \frac{1}{u^2+1} du + \int_{-\infty}^{0} -u \cdot \frac{1}{\pi} \cdot \frac{1}{u^2+1} du$

$$= \frac{1}{2\pi}\left(\left[\ln|u^2+1|\right]_0^{\infty} - \left[\ln|u^2+1|\right]_{-\infty}^{0}\right)$$

$$= \frac{1}{2\pi}\left(\lim_{u \to \infty} \ln(u^2+1) + \lim_{u \to -\infty} \ln(u^2+1)\right)$$

$$= \infty$$

Therefore, it's not absolutely convergent. Thus, $E[U]$ does not exist.

Problem 6.

Consider the transformation $\varphi: (X,Y) \to (U,V)$ where $\varphi(x,y) = (x+y, y)$.

then, $\varphi^{-1}(u,v) = (u-v, v)$

we calculate $D_{\varphi^{-1}}(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

so $|\det D_{\varphi^{-1}}(u,v)| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1$

Then $f_{UV}(u,v) = f_{XY}(u-v, v)$.

According to marginal density, f_U is given by

$$f_U(u) = \int_{-\infty}^{\infty} f_{UV}(u,v)\,dv = \int_{-\infty}^{\infty} f_{XY}(u-v,v)\,dv.$$

Problem 7.

Since X and Y are exponentially distributed random variable with $\beta_1 = \frac{1}{3}$, $\beta_2 = 1$

Then $f_X(x) = \begin{cases} \frac{1}{3}e^{-\frac{1}{3}x}, & x>0 \\ 0, & x \le 0 \end{cases}$, $f_Y(y) = \begin{cases} e^{-y}, & y>0 \\ 0, & y\le 0 \end{cases}$

Since they are independent, then $f_{XY}(x,y) = f_X(x)\cdot f_Y(y) = \begin{cases} \frac{1}{3}e^{-\frac{1}{3}x-y}, & x>0 \text{ and } y>0 \\ 0, & \text{otherwise.} \end{cases}$

Using the conclusion proved in 3.6, since $U = X+Y$, then $f_U(u) = \int_{-\infty}^{\infty} f_{XY}(u-v,v)\,dv$.

then for $u>0$, since $\begin{cases} u-v>0 \\ v>0 \end{cases} \Rightarrow 0<v<u$, then $f_U(u) = \int_0^u \frac{1}{3}e^{-\frac{1}{3}(u-v)-v}\,dv = e^{-\frac{1}{3}u}\int_0^u \frac{1}{3}e^{-\frac{2}{3}v}\,dv$

$$= e^{-\frac{1}{3}u}\cdot\left(-\frac{1}{2}\right)\cdot\left[e^{-\frac{2}{3}v}\right]_0^u = -\frac{1}{2}e^{-\frac{1}{3}u}\left(e^{-\frac{2}{3}u}-1\right)$$

$$= \frac{1}{2}\left(e^{-\frac{1}{3}u} - e^{-u}\right)$$

for $u \le 0$, then $f_U(u) = \int_{-\infty}^{\infty} 0\cdot dv = 0$.

In conclusion, $f_U(u) = \begin{cases} (e^{-\frac{u}{3}} - e^{-u})/2, & u>0 \\ 0, & u\le 0. \end{cases}$

~~Problem 8.~~

~~It's given that, $f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi\sigma_1^2}}e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$, $f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi\sigma_2^2}}e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$~~

~~Since the cumulative distribution function of Y is given by~~

~~$F_Y(y) = P[Y\le y] = P[\lambda_1 X_1 + \lambda_2 X_2 \le y]$~~

Problem 8.

since X_1, X_2 are both normal distributions with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2

then $f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2}$ $\quad f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}(\frac{x_2-\mu_2}{\sigma_2})^2}$

For random variable, $Y = \lambda_1 X_1 + \lambda_2 X_2$, we can calculate its MGF as

$$M_Y(t) = E[e^{tY}] = E[e^{t(\lambda_1 X_1 + \lambda_2 X_2)}] = E[e^{t\lambda_1 X_1} \cdot e^{t\lambda_2 X_2}]$$

Now we prove that random variables $V_1 = e^{\lambda_1 X_1}$, $V_2 = e^{\lambda_2 X_2}$ are independent

$$P[V_1 = v_1 \text{ and } V_2 = v_2] = P[e^{\lambda_1 X_1} = v_1 \text{ and } e^{\lambda_2 X_2} = v_2]$$

$$= P[X_1 = \frac{\ln|v_1|}{\lambda_1} \text{ and } X_2 = \frac{\ln|v_2|}{\lambda_2}]$$

since X_1 and X_2 are independent, i.e. $P[X_1 = x_1 \text{ and } X_2 = x_2] = P[X_1 = x_1] \cdot P[X_2 = x_2]$

then $P[V_1 = v_1 \text{ and } V_2 = v_2] = P[X_1 = \frac{\ln|v_1|}{\lambda_1}] \cdot P[X_2 = \frac{\ln|v_2|}{\lambda_2}]$

$$= P[e^{\lambda_1 X_1} = v_1] \cdot P[e^{\lambda_2 X_2} = v_2]$$

$$= P[V_1 = v_1] \cdot P[V_2 = v_2]$$

then, V_1 and V_2 are independent

then $M_Y(t) = E[e^{t\lambda_1 X_1} \cdot e^{t\lambda_2 X_2}] = E[e^{t\lambda_1 X_1}] \cdot E[e^{t\lambda_2 X_2}]$

$$= \int_{-\infty}^{\infty} e^{t\lambda_1 x_1} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}(\frac{x_1-\mu_1}{\sigma_1})^2} dx_1 \cdot \int_{-\infty}^{\infty} e^{t\lambda_2 x_2} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}(\frac{x_2-\mu_2}{\sigma_2})^2} dx_2$$

$$= e^{t^2\lambda_1^2\sigma_1^2/2 + \mu_1 t\lambda_1} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(x_1-(\mu_1+t\lambda_1\sigma_1^2))^2}{2\sigma_1^2}} dx_1 \cdot e^{t^2\lambda_2^2\sigma_2^2/2 + \mu_2 t\lambda_2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{(x_2-(\mu_2+t\lambda_2\sigma_2^2))^2}{2\sigma_2^2}} dx_2$$

$$= e^{t^2\lambda_1^2\sigma_1^2/2 + \mu_1 t\lambda_1} \times 1 \times e^{t^2\lambda_2^2\sigma_2^2/2 + \mu_2 t\lambda_2} \times 1$$

$$= e^{(\mu_1\lambda_1 + \mu_2\lambda_2)t + (\sigma_1^2\lambda_1^2 + \sigma_2^2\lambda_2^2)\cdot\frac{t^2}{2}}$$

we can find that $M_Y(t)$ is in the form of MGF of normal distribution: $m_X(t) = e^{\mu t + \sigma^2 t^2/2}$

Then, according to the uniqueness of moment generating function, we can conclude that $Y = \lambda_1 X_1 + \lambda_2 X_2$ follows a normal distribution with $E[Y] = \lambda_1\mu_1 + \lambda_2\mu_2$ and $V[Y] = \lambda_1^2\sigma_1^2 + \lambda_2^2\sigma_2^2$.

Problem 9.

i) The magrinal density of X is given by

$$\int_{-\infty}^{\infty} f_{X_1X_2}(x_1,x_2)\,dx_2 = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]} dx_2$$

To simplify calculation, here we denote $A = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} > 0$, $B = \frac{1}{2(1-\rho^2)} > 0$, $y_1 = \frac{x_1-\mu_1}{\sigma_1}$, $y_2 = \frac{x_2-\mu_2}{\sigma_2}$

we can derive that $dy_2 = \frac{1}{\sigma_2} dx_2 \Rightarrow dx_2 = \sigma_2\, dy_2$.

Then, we can change the formula to $\int_{-\infty}^{\infty} A e^{-B[y_1^2 - 2\rho y_1 y_2 + y_2^2]}\ \sigma_2\, dy_2$.

$$= A\sigma_2 \int_{-\infty}^{\infty} e^{-B(y_2-\rho y_1)^2} \cdot e^{-B(y_1^2-\rho^2 y_1^2)}\, dy_2.$$

$$= A\sigma_2 e^{-B(y_1^2-\rho^2y_1^2)} \int_{-\infty}^{\infty} e^{-[\sqrt{B}(y_2-\rho y_1)]^2} dy_2$$

$$= A\sigma_2 e^{-B(y_1^2-\rho^2y_1^2)} \frac{1}{\sqrt{B}} \int_{-\infty}^{\infty} e^{-s^2} ds \quad (s = \sqrt{B}(y_2-\rho y_1))$$

$$= A\sigma_2 e^{-B(y_1^2-\rho^2y_1^2)} \cdot \frac{1}{\sqrt{B}} \cdot \sqrt{\pi}$$

$$= \frac{1}{2\pi\sigma_1\cancel{\sigma_2}\sqrt{1-\rho^2}} \cdot \cancel{\sigma_2} \cdot e^{-\frac{1}{2(1-\rho^2)} \cdot y_1^2(1-\rho^2)} \cdot \sqrt{2(1-\rho^2)} \cdot \sqrt{\pi}.$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}.$$

Therefore, $f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_1} \cdot e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$, which is of a normal distribution with mean μ_1 and variance σ_1^2.

ii).

$$E[X_1X_2] = \iint_{\mathbb{R}^2} x_1x_2 \cdot f_{X_1X_2}(x_1,x_2)\,dx_1dx_2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{x_1x_2}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]} dx_1dx_2.$$

Similarly, to simplify calculation, we denote $A = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} > 0$, $B = \frac{1}{2(1-\rho^2)} > 0$, $y_1 = \frac{x_1-\mu_1}{\sigma_1}$, $y_2 = \frac{x_2-\mu_2}{\sigma_2}$

then $dy_1 = \frac{1}{\sigma_1}dx_1$, $dy_2 = \frac{1}{\sigma_2}dx_2 \Rightarrow dx_2 = \sigma_2 dy_2$, $dx_1 = \sigma_1 dy_1$ and $x_1 = \sigma_1 y_1 + \mu_1$, $x_2 = \sigma_2 y_2 + \mu_2$.

then
$$E[X_1X_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} A\sigma_1\sigma_2\, e^{-B(y_1^2-\rho^2y_1^2)} \Big[(\sigma_1y_1+\mu_1)\,\sigma_2(y_2-\rho y_1)\, e^{-B(y_2-\rho y_1)^2} + (\rho\sigma_2y_1+\mu_2)(\sigma_1y_1+\mu_1)\, e^{-B(y_2-\rho y_1)^2}\Big] dy_2\, dy_1$$

For $\int_{-\infty}^{\infty} \sigma_2(y_2-\rho y_1) e^{-B(y_1-\rho y_1)^2} dy_2$, we can finally transform it into $c\left[e^{-B(y_2-\rho y_1)^2}\right]_{-\infty}^{\infty} = 0$.
(c is a constant).

then $E[X_1X_2] = \int_{-\infty}^{\infty} A\sigma_1\sigma_2\, e^{-B(y_1^2-\rho^2y_1^2)} \int_{-\infty}^{\infty} (\rho\sigma_2y_1+\mu_2)(\sigma_1y_1+\mu_1)\, e^{-B(y_2-\rho y_1)^2} dy_2\, dy_1.$

Since $\int_{-\infty}^{\infty} (\rho\sigma_2y_1+\mu_2)(\sigma_1y_1+\mu_1) e^{-B(y_2-\rho y_1)^2} dy_2$

$$= (\rho\sigma_2y_1+\mu_2)(\sigma y_1+\mu_2) \cdot \frac{1}{\sqrt{B}} \int_{-\infty}^{\infty} e^{-s^2} ds \quad (s := \sqrt{B}(y_2-\rho y_1))$$

$$= (\rho\sigma_2y_1+\mu_2)(\sigma y_1+\mu_2) \cdot \frac{1}{\sqrt{B}} \cdot \sqrt{\pi}.$$

then
$$E[X_1X_2] = \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \sigma_1\sigma_2 \cdot e^{-\frac{1}{2(1-\rho^2)} y_1^2(1-\rho^2)} \cdot (\rho\sigma_2y_1+\mu_2)(\sigma_1y_1+\mu_1) \cdot \sqrt{\pi} \cdot \sqrt{2(1-\rho^2)}\, dy_1$$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y_1^2}{2}} (\rho\sigma_2y_1+\mu_2)(\sigma_1y_1+\mu_1)\, dy_1$$

$$= \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty} \rho\sigma_1\sigma_2 y_1^2 e^{-\frac{y_1^2}{2}} dy_1 + \int_{-\infty}^{\infty} (\mu_2\sigma_1 + \rho\sigma_2\mu_1) y_1 e^{-\frac{y_1^2}{2}} dy_1 + \int_{-\infty}^{\infty} \mu_1\mu_2 e^{-\frac{y_1^2}{2}} dy_1\right].$$

$$= \frac{1}{\sqrt{2\pi}}\left[\rho\sigma_1\sigma_2 \cdot \sqrt{2} \cdot \left(\int_0^{\infty} u^{\frac{1}{2}} e^{-u} du + \int_{-\infty}^{0} -u^{\frac{1}{2}} e^{-u} du\right) + 0 + \mu_1\mu_2 \cdot \sqrt{2} \int_{-\infty}^{\infty} e^{-s^2} ds\right].$$

$(u = \frac{y_1^2}{2},\ du = y_1 dy_1)$ $(s = \frac{y_1}{\sqrt{2}},\ dy_1 = \sqrt{2} ds_1)$.

$$= \frac{1}{\sqrt{2\pi}}\left[\rho\sigma_1\sigma_2 \cdot \sqrt{2} \cdot \left(2\int_0^{\infty} u^{\frac{1}{2}} e^{-u} du\right) + 0 + \mu_1\mu_2 \cdot \sqrt{2} \cdot \sqrt{\pi}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left(2\sqrt{2}\rho\sigma_1\sigma_2 \Gamma(\tfrac{3}{2}) + \sqrt{2\pi}\mu_1\mu_2\right).$$

$$= \frac{1}{\sqrt{2\pi}}\left(2\sqrt{2}\rho\sigma_1\sigma_2 \cdot \tfrac{1}{2} \cdot \Gamma(\tfrac{1}{2}) + \sqrt{2\pi}\mu_1\mu_2\right).$$

$$= \frac{1}{\sqrt{2\pi}}\left(\cancel{2}\sqrt{2}\rho\sigma_1\sigma_2 \cdot \tfrac{1}{\cancel{2}} \cdot \sqrt{\pi} \cdot \sqrt{2\pi}\mu_1\mu_2\right).$$

$$= \rho\sigma_1\sigma_2 + \mu_1\mu_2.$$

In the first question, we have proved that X_1 follows a normal distribution with mean μ_1 and variance σ_1^2

Similarly, we can prove that X_2 follows a normal distribution with mean μ_2 and variance σ_2^2

Then we can get that $E[X_1] = \mu_1$ and $E[X_2] = \mu_2$, $Var[X_1] = \sigma_1^2$ and $Var[X_2] = \sigma_2^2$

then $Cov[X_1, X_2] = E[X_1 X_2] - E[X_1] \cdot E[X_2]$

$$= \rho\sigma_1\sigma_2 + \mu_1\mu_2 - \mu_1\mu_2$$

$$= \rho\sigma_1\sigma_2.$$

then $\rho = \frac{Cov[X_1, X_2]}{\sqrt{Var[X_1]\, Var[X_2]}}$, which is the coefficient of correlation between X_1 and X_2.

(4)

Sufficient condition:

If $\rho = 0$, then $f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - \frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$

$$= \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-\frac{1}{2}\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2}$$

$$= f_{X_1}(x_1) \cdot f_{X_2}(x_2).$$

$\Rightarrow$ X_1, X_2 are independent

Necessary condition:

If X_1, X_2 are independent, then $E[X_1X_2] = E[X_1]E[X_2]$

then $\rho\sigma_1\sigma_2 + \mu_1\mu_2 = \mu_1\mu_2$

$\rho\sigma_1\sigma_2 = 0$ since $\sigma_1, \sigma_2 > 0$

then $\rho = 0$.

$\boxed{\text{No.}}$ It's true that if X_1 and X_2 are independent, then $\rho = 0$ because $Cov[X_1, X_2] = E[X_1X_2] - E[X_1]E[X_2] = 0$

then $\rho = \frac{Cov[X_1, X_2]}{\sigma_1\sigma_2} = 0$.

But it's not true that if $\rho = 0$, then X_1 and X_2 are independent

If $\rho = 0$, then $Cov[X_1, X_2] = \rho\sigma_1\sigma_2 = 0$, but X_1 and X_2 are not necessarily independent.

iv).

$$\mu_{X_2|x_1} = E[X_2|x_1] = \int_{-\infty}^{\infty} x_2 \cdot f_{X_2|x_1}(x_2)\, dx_2 = \int_{-\infty}^{\infty} x_2 \cdot \frac{f_{X_1X_2}(x_1,x_2)}{f_{X_1}(x_1)}\, dx_2 .$$

since $f_{X_1}(x_1) = \frac{1}{\sqrt{2\pi}\,\sigma_1} e^{-\frac{1}{2}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$

$$f_{X_1X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right]}$$

then $\frac{f_{X_1X_2}(x_1,x_2)}{f_{X_1}(x_1)} = \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_2-\mu_2}{\sigma_2}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right] - \frac{\rho^2}{2(1-\rho^2)}\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2}$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x_2-\mu_2}{\sigma_2} - \rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right]^2}$$

then $\mu_{X_2|x_1} = \int_{-\infty}^{\infty} \frac{x_2}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x_2-\mu_2}{\sigma_2} - \rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\right]^2} dx_2$

Similarly, to simplify calculation, we denote $A = \frac{1}{\sqrt{2\pi}\,\sigma_2\sqrt{1-\rho^2}}$, $B = \frac{1}{2(1-\rho^2)}$, $Y_2 = \frac{x_2-\mu_2}{\sigma_2}$, $Y_1 = \frac{x_1-\mu_1}{\sigma_1}$

then $x_2 = \sigma_2 Y_2 + \mu_2$, $dx_2 = \sigma_2\, dY_2$

then $\mu_{X_2|x_1} = \int_{-\infty}^{\infty} A(\sigma_2 Y_2 + \mu_2) e^{-B[Y_2 - \rho Y_1]^2} \sigma_2\, dY_2$.

$$= A\sigma_2\left[\int_{-\infty}^{\infty} \sigma_2(Y_2 - \rho Y_1) e^{-B[Y_2-\rho Y_1]^2} dY_2 + \int_{-\infty}^{\infty} (\rho\sigma_2 Y_1 + \mu_2) e^{-B[Y_2-\rho Y_1]^2} dY_2\right]$$

$$= A\sigma_2\left(0 + \frac{\rho\sigma_2 Y_1 + \mu_2}{\sqrt{B}} \int_{-\infty}^{\infty} e^{-s^2} ds\right) \quad (s := \sqrt{B}(Y_2 - \rho Y_1))$$

$$= A\sigma_2\left(\frac{\rho\sigma_2 Y_1 + \mu_2}{\sqrt{B}} \cdot \sqrt{\pi}\right)$$

$$= \frac{1}{\cancel{\sqrt{2\pi}}\,\cancel{\sigma_2}\,\cancel{\sqrt{1-\rho^2}}} \cdot \cancel{\sigma_2} \cdot \left(\rho\sigma_2 \cdot \frac{x_1-\mu_1}{\sigma_1} + \mu_2\right) \cdot \cancel{\sqrt{\pi}} \cdot \cancel{\sqrt{2(1-\rho^2)}}$$

$$= \boxed{\mu_2 + \rho\frac{\sigma_2}{\sigma_1}(x_1 - \mu_1).}$$

v)

$x_2 = 0.098$ $\mu_2 = 0.1$ $\sigma_2^2 = 0.01$. $\Rightarrow \frac{x_2-\mu_2}{\sigma_2} = \frac{0.098-0.1}{0.1} = -0.02$

then $f_{X_1X_2}(x_1, 0.098) = \frac{1}{10\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\cdot(-0.02) + (-0.02)^2\right]}$

the $P[x_1 \geq 1950 |_{x_2=0.098}] = \int_{1950}^{\infty} f_{X_1X_2}(x_1, 0.098)\, dx_1$

Similarly, we denote $A = \frac{1}{10\pi\sqrt{1-\rho^2}}$, $B = \frac{1}{2(1-\rho^2)}$, $Y_1 = \frac{x_1-\mu_1}{\sigma_1}$, $dx_1 = \sigma_1 dY_1$, $Y_1 \geq -1$

$\Rightarrow P[x_1 \geq 1950 |_{x_2=0.098}] = \int_{-1}^{\infty} A e^{-B[Y_1^2 + 0.04\rho Y_1 + 0.02^2]} \sigma_1\, dY_1$

$$= A\sigma_1\left[\int_{-1}^{\infty} e^{-(\sqrt{B}(Y_1+0.02\rho))^2} \times e^{0.02^2 B(\rho^2-1)} dY_1\right].$$

$$= A\sigma_1 e^{0.02^2 B(\rho^2-1)} \frac{1}{\sqrt{2B}}\left[\int_{1.991}^{\infty} e^{-\frac{s^2}{2}} ds\right] \quad (s := \sqrt{2B}(Y_1 + 0.02\rho))$$

$$= \frac{1}{10\pi\cancel{\sqrt{1-\rho^2}}} \times \sigma_1 \times e^{0.02^2\frac{\rho^2-1}{2(1-\rho^2)}} \times \cancel{\sqrt{1-\rho^2}} \times 0.9767 \times \sqrt{2\pi}$$

=

$\cancel{P[x_1 \geq 1950 \mid x_2 = 0.098] = \int_{1950}^{\infty} \frac{f_{X_1X_2}(x_1, 0.098)}{f_{X_2}(0.098)} dx_1}$

(v). $$\frac{f_{X_1X_2}(x_1,x_2)}{f_{X_2}(x_2)} = \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x_1-\mu_1}{\sigma_1} - \rho\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right]^2}$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_1\sqrt{1-\rho^2}} e^{-\frac{1}{2}\cdot\left[\frac{x_1-\mu_1-\frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)}{\sigma_1\sqrt{1-\rho^2}}\right]^2}$$

we can see that it follows a normal distribution with $\mu = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)$ and $\sigma = \sigma_1\sqrt{1-\rho^2}$

therefore, $\mu_{X_1|x_2} = \mu_1 + \frac{\rho\sigma_1}{\sigma_2}(x_2-\mu_2)$, $\sigma_{x_1|x_2} = \sigma_1\sqrt{1-\rho^2}$

then $\mu_{X_1|0.098} = 2000 + \frac{0.87\times 50}{0.1}(0.098-0.1) = 1999.13$

$\sigma_{x_1|0.098} = 50\times\sqrt{1-0.87^2} = 24.6526$.

then $P[x_1|_{0.098} \geq 1950] = P[z \geq \frac{1950-1999.13}{24.6526}] = P[z \geq -1.993] = \boxed{0.9767}$

扫描全能王 创建

Problem 11.

i) for the first component, $R_1(t) = e^{-\alpha t^{\beta}} = e^{-0.006 t^{0.5}}$ $(t>0)$

for the second component, $f_2(t) = \beta' e^{-\beta' t}$

$R_2(t) = 1 - F_2(t) = 1 - \int_0^t \beta' e^{-\beta' t} dt = 1 - [-e^{-\beta' t}]_0^t = e^{-\beta' t}$

then, since they are connected in series, then $R_s(t) = R_1(t) R_2(t) = e^{-0.006 t^{0.5}} \times e^{-\frac{1}{25000} t}$

$R_s(2500) = \boxed{0.67}$

ii) $R_s(2000) = 0.7059$.

then $F_s(2000) = 1 - R_s(2000) = \boxed{0.2941}$.

iii). If they are connected in parallel, then $R_s(t) = 1 - (1 - R_1(t))(1 - R_2(t))$.

$R_s(2500) = \boxed{0.975}$