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Problem 1.

i) $m_X(t) = E[e^{tX}] = \sum_{k=1}^{n} e^{tx_k} \cdot \frac{1}{n} = \frac{1}{n}\sum_{k=1}^{n} e^{tx_k}$

ii) $E[X] = \frac{dm_X(t)}{dt}\Big|_{t=0} = \frac{1}{n}\sum_{k=1}^{n} x_k e^{tx_k}\Big|_{t=0} = \frac{1}{n}\sum_{k=1}^{n} x_k$

$Var[X] = E[X^2] - E[X]^2$

$E[X^2] = \frac{d^2 m_X(t)}{dt^2}\Big|_{t=0} = \frac{1}{n}\sum_{k=1}^{n} x_k^2 e^{tx_k}\Big|_{t=0} = \frac{1}{n}\sum_{k=1}^{n} x_k^2$

$\Rightarrow Var[X] = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \left(\frac{1}{n}\sum_{k=1}^{n} x_k\right)^2$

$= \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \frac{1}{n^2}\left(\sum_{k=1}^{n} x_k\right)^2$

Problem 2.

$m_X(t) = m_Y(t) \Rightarrow \sum_{k=0}^{n} e^{kt} f_X(k) = \sum_{k=0}^{n} e^{kt} f_Y(k) \Rightarrow \sum_{k=0}^{n} e^{kt}[f_X(k) - f_Y(k)] = 0$

since for $\forall t \in (-\varepsilon, \varepsilon)$, $m_X(t) = m_Y(t)$

$\Rightarrow$ for $\forall e^t \in (e^{-\varepsilon}, e^{\varepsilon})$, $\sum_{k=0}^{n} (e^t)^k [f_X(k) - f_Y(k)] = 0$

$\Rightarrow$ the only solution should be all the coefficients $= 0$ $\Rightarrow f_X(k) = f_Y(k)$ for $\forall\ 0 \le k \le n$

$\Rightarrow f_X(x) = f_Y(x)$ for $x = 0, \dots, n$.

Problem 3.

i) $P[Z=z] = \underbrace{\sum_{x+y=z} P[X=x \text{ and } Y=y]}_{P[X+Y=z]}$ since $Z = X+Y$

$= \sum_{x+y=z} P[X=x] \cdot P[Y=y]$ since $X$ and $Y$ are independent.

ii) suppose $X$ and $Y$ are two independent and identical geometric random variables

s.t. $X: S \to \Omega = \mathbb{N}\setminus\{0\}$, $f_X(x) = (1-p)^{x-1} p$

$Y: S \to \Omega = \mathbb{N}\setminus\{0\}$, $f_Y(y) = (1-p)^{y-1} p$.

denote the sum of them as $Z = X+Y$ where $Z: S \to \Omega = \mathbb{N}\setminus\{0\}$.

then $\underbrace{P[Z=z]}_{f_Z(z)=} = \sum_{x+y=z} P[X=x] \cdot P[Y=y] = \sum_{x+y=z} f_X(x) \cdot f_Y(y)$ using the conclusion above.

$= \sum_{x+y=z} (1-p)^{x+y-2} \cdot p^2 = \sum_{x+y=z} (1-p)^{z-2} \cdot p^2$.

since $y \ge 1$, then $x \le z-1$ since $x \ge 1 \Rightarrow 1 \le x \le z-1 \in \mathbb{N}\setminus\{0\}$.

$\Rightarrow$ $x$ can have $z-1$ possible values ranging from $1$ to $z-1$ and we can get the corresponding

$y = z-x$

therefore, the ways of getting $x+y=z$ are in total $\binom{z-1}{1}$

$\Rightarrow f_Z(z) = \binom{z-1}{1}(1-p)^{z-2}p^2$, which is a Pascal distribution with $r=2$.

Problem 4.

the initial value should be $P_0(0)=1$

$P_0(0) \overset{=1}{}$ means in the time period $[0,0]$, the probability of no arrival should be 1, which means there must be no arrival.

then with conditions: $\underset{P_0'=-\lambda P_0}{P_0(0)=1}$, $P_x'+\lambda P_x = \lambda P_{x-1}$, we use induction to prove that $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$

① when $x=0$, since $\frac{dP_0}{dt} = -\lambda P_0 \Rightarrow \frac{dP_0}{P_0} = -\lambda dt \Rightarrow \ln|P_0| = -\lambda t + C \Rightarrow P_0 = Ce^{-\lambda t}$

since $P_0(0)=1$, then $P_0(0) = Ce^0 = C = 1 \Rightarrow P_0(0) = e^{-\lambda t}$

since $P_0(t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$, then the conclusion is right.

② for $x=k, k\geq 0$, suppose $P_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

then since $P_{k+1}' + \lambda P_{k+1} = \lambda P_k = \frac{\lambda^{k+1}t^k e^{-\lambda t}}{k!}$

$$\mu = e^{\int \lambda dt} = e^{\lambda t}, \quad \int \frac{\lambda^{k+1}t^k e^{-\lambda t}}{k!} \times e^{\lambda t} dt = \int \frac{\lambda^{k+1}t^k}{k!} dt = \frac{\lambda^{k+1}}{(k+1)!} t^{k+1} = \frac{(\lambda t)^{k+1}}{(k+1)!}$$

$$\Rightarrow P_{k+1} = e^{-\lambda t}\left(\frac{(\lambda t)^{k+1}}{(k+1)!} + C\right) = \frac{(\lambda t)^{k+1}e^{-\lambda t}}{(k+1)!} + Ce^{-\lambda t}$$

since $P_{k+1}(0) = 0$ because in $[0,0]$, the probability of $k+1$ arrivals should be 0.

then $P_{k+1}(0) = C = 0$

$\Rightarrow P_{k+1} = \frac{(\lambda t)^{k+1}e^{-\lambda t}}{(k+1)!}$, which accords with $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$

Based on ① and ②, we can conclude that $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$ holds true for $\forall x \in N$, then it gets proved.

Problem 5.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!}\left(\frac{k}{n}\right)^x \left(1-\frac{k}{n}\right)^{n-x}$$

when $n\to\infty$, using stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n\to\infty$.

then $f(x) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{x!(n-x)!}\left(\frac{k}{n}\right)^x\left(1-\frac{k}{n}\right)^{n-x}$ as $n\to\infty$.

$$= \frac{k^x}{x!} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-x)}\left(\frac{n-x}{e}\right)^{n-x}} \cdot n^{-x} \cdot \left(1-\frac{k}{n}\right)^n \cdot \left(1-\frac{k}{n}\right)^{-x}$$

since $\lim_{n\to\infty}\left(1-\frac{k}{n}\right)^n = e^{-k}$, then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!}\lim_{n\to\infty}\sqrt{\frac{n}{n-x}} \cdot \left(\frac{n}{n-x}\right)^{n-x} \cdot e^{-x} \cdot \left(1-\frac{k}{n}\right)^{-x}$.

when $n \to \infty$, it's obvious that $\sqrt{\frac{n}{n-x}} = \sqrt{\frac{1}{1-\frac{x}{n}}} \to 1$ and $(1-\frac{k}{n})^{-x} \to 1$.

then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!} \left( \lim_{n\to\infty} \left(\frac{n}{n-x}\right)^{n-x} \right) \cdot e^{-x}$.

since $\lim_{n\to\infty} \left(\frac{n}{n-x}\right)^{n-x} = \lim_{n\to\infty} \left(1+\frac{x}{n-x}\right)^{n-x} = \lim_{n-x\to\infty} \left(1+\frac{x}{n-x}\right)^{n-x} = e^x$.

then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!} e^x \cdot e^{-x}$

$= \frac{e^{-k}k^x}{x!}$

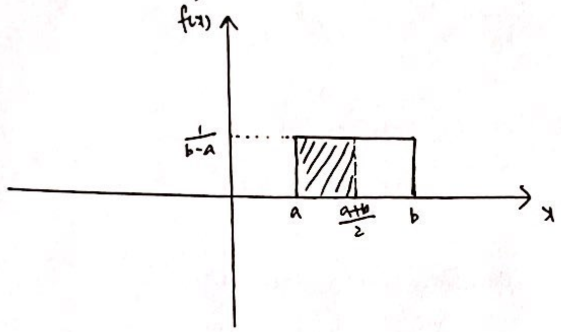
Problem 6.

i) since $\frac{1}{b-a} > 0$, then $f(x) \geq 0$ for all $x \in R$.

and $\int_{-\infty}^{\infty} f(x)dx = 0 + \int_a^b \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_a^b = \frac{b-a}{b-a} = 1$

therefore, it's a density for a continuous random variable.

ii)




iii) $P[X \leq \frac{a+b}{2}] = \int_{-\infty}^{\frac{a+b}{2}} f(x)dx = 0 + \int_a^{\frac{a+b}{2}} \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_a^{\frac{a+b}{2}} = \frac{\frac{a+b}{2}-a}{b-a} = \frac{\frac{b-a}{2}}{b-a} = \frac{1}{2}$.

iv)

$P[c \leq X \leq d] = \int_c^d f(x)dx = \int_c^d \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_c^d = \frac{d-c}{b-a}$, since $(c,d)$ is subinterval of $(a,b)$.

$P[e \leq X \leq f] = \int_e^f f(x)dx = \int_e^f \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_e^f = \frac{f-e}{b-a}$, since $(e,f)$ is subinterval of $(a,b)$.

since $(c,d)$ and $(e,f)$ are of equal length, then $d-c = f-e$. $\Rightarrow \frac{d-c}{b-a} = \frac{f-e}{b-a}$

then $P[c \leq X \leq d] = P[e \leq X \leq f]$

v). $F(x) = P[X \leq x] = \int_{-\infty}^{x} f(y)dy$

if $x < a$, then $f(y) = 0 \Rightarrow F(x) = 0$.

if $a < x < b$, then $F(x) = \int_{-\infty}^{a} 0\,dy + \int_a^x \frac{1}{b-a}dy = 0 + \left[\frac{y}{b-a}\right]_a^x = \frac{x-a}{b-a}$

if $x \geq b$, then $F(x) = \int_{-\infty}^{a} 0\,dy + \int_a^b \frac{1}{b-a}dy + \int_b^x 0\,dy = 0 + \frac{b-a}{b-a} + 0 = 1$.

$$\Rightarrow F(x) = \begin{cases} 0 & , x \leq a \\ \frac{x-a}{b-a} & , a < x < b \\ 1 & , x \geq b. \end{cases}$$

vi). $\cancel{m_x = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx = \int_{-\infty}^{a} e^{tx}\cdot 0\,dx + \int_a^b e^{tx}\cdot\frac{1}{b-a}\,dx + \int_b^{\infty} e^{tx}\cdot 0\cdot dx}$

$\cancel{= 0 + \left[\frac{te^{tx}}{b-a}\right]_a^b + 0}$

$\cancel{= \frac{e^{tb}-e^{ta}}{t(b-a)}}$

$\cancel{\Rightarrow E[X] = \frac{dm_x}{dt}\Big|_{t=0} = \frac{e^t\cdot t - e^t}{t^2}\cdot\frac{e^b-e^a}{b-a}}$

vi). $E[X] = \int_{-\infty}^{\infty} x\cdot f(x)\,dx = \int_a^b x\cdot\frac{1}{b-a}\,dx = \frac{1}{b-a}\cdot\left[\frac{1}{2}x^2\right]_a^b = \frac{1}{b-a}\cdot\frac{1}{2}\cdot(b^2-a^2) = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$

$E[X^2] = \int_{-\infty}^{\infty} x^2\cdot f(x)\,dx = \int_a^b x^2\cdot\frac{1}{b-a}\,dx = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$

$Var[X] = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \frac{a^2+b^2+2ab}{4} = \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$