VE401 Assignment 2

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Exercise 1. Discrete Uniform Distribution

i) **Solution.** With the parameter n, we have

$$\begin{aligned} m_x(t) = E(e^{xt}) &= \sum_{k=1}^{n} e^{x_k t}\frac{1}{n} \\ &= \frac{1}{n}\sum_{k=1}^{n} e^{x_k t} \end{aligned}$$

□

ii) **Solution.** From the moment generating function we get that

$$\begin{aligned} E[X] &= \frac{d}{dt}m_x(t)|_{t=0} \\ &= \frac{1}{n}\sum_{k=1}^{n}[\frac{d}{dt}e^{x_k t}|_{t=0}] \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k \end{aligned}$$

and

$$\begin{aligned} E[X^2] &= \frac{d^2}{dt^2}m_x(t)|_{t=0} \\ &= \frac{1}{n}\sum_{k=1}^{n}[\frac{d^2}{dt^2}e^{x_k t}|_{t=0}] \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 \end{aligned}$$

And hence the variance is given by

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \frac{1}{n^2}(\sum_{k=1}^{n} x_k)^2$$

□

Exercise 2. Uniqueness of Moment generating functions - Simple Case

Proof. With $m_X(t) = m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$ we can see that

$$\frac{d}{dt}m_X(t) = \frac{d}{dt}m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$$

This gives us

$$\frac{d}{dt}m_X(t)|_{t=0} = \frac{d}{dt}m_Y(t)|_{t=0}$$

which is

$$E[X] = E[Y]$$

By definition of the expectation,

$$\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$$

Now we prove by induction that $\forall n \in \mathbb{N}$, $f_X(x) = f_Y(x)$.

First when $n = 0$ we directly have $f_X(x) = f_Y(x) = 1$.Now we want to prove that $f_X(n+1) = f_Y(n+1)$ given that $f_X(n) = f_Y(n)$. This is simple. We first write

$$\sum_{x=0}^{n+1} x \cdot f_X(x) = \sum_{x=0}^{n+1} x \cdot f_Y(x)$$
$$\sum_{x=0}^{n} x \cdot f_X(x) + (n+1)f_X(n+1) = \sum_{x=0}^{n} x \cdot f_Y(x) + (n+1)f_Y(n+1)$$

Note that $\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$ given that $\forall N \leq n$, $f_X(N) = f_Y(N)$.Thus by cancelling the sums, we have our desired result

$$(n+1)f_X(n+1) = (n+1)f_Y(n+1)$$
$$f_X(n+1) = f_Y(n+1)$$

Therefore, by induction we have proved that $f_X(x) = f_Y(x)$ for $x = 0, ..., n$. □

Exercise 3. Sums of Independent Discrete Random Variables

i) **Proof.** We first divide $x + y \in ranZ$ into two parts: $x + y = z$ and $x + y \neq z$.

$$\begin{aligned} P[Z = z] &= P[X + Y = z] \\ &= \sum_{x+y \in ranZ} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &= \sum_{x+y \neq z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &+ \sum_{x+y=z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \end{aligned}$$

We note that if $x + y \neq z$, then $P[x + y = z] = 0$. Hence

$$\sum_{x+y \neq z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] = 0$$

Note that if $x + y = z$, then $P[x + y = z] = 1$. Thus

$$\begin{aligned} &\sum_{x+y=z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &= P[X = x \wedge Y = y] \\ &= P[X = x] \cdot P[Y = y] \end{aligned}$$

Therefore, $P[Z = z] = 0 + P[X = x] \cdot P[Y = y] = P[X = x] \cdot P[Y = y]$. □

ii) **Proof.** We denote the parameter of the geometric distribution as p, as usual. Now we have $X, Y \sim Geom(p)$. Applying the density function we write $P[X = x] = (1-p)^{x-1}p$ and $P[Y = x] = (1-p)^{x-1}p$. Now the sum of X and Y is $Z = X + Y$. The probability for Z is therefore

$$\begin{aligned} P[Z = z] &= \sum_{x+y=z} (1-p)^{x-1} \cdot (1-p)^{y-1}p \\ &= p^2 \sum_{x+y=z} (1-p)^{x+y-2} \end{aligned}$$

We know that $x, y \in \mathbb{N} \backslash 0$. Thus there are $z-1$ terms in the above sum, resulting in

$$\begin{aligned} P[Z = z] &= p^2(z-1)(1-p)^{z-2} \\ &= \binom{z-1}{1} p^2 (1-p)^{z-2} \end{aligned}$$

which is exactly a Pascal distribution with $r = 2$. $\square$

$\Rightarrow$ x can have $z-1$ possible values ranging from 1 to $z-1$ and we can get the corresponding $y = z-x$

therefore, the ways of getting $x+y=z$ are in total $\binom{z-1}{1}$

$\Rightarrow$ $f_Z(z) = \binom{z-1}{1}(1-p)^{z-2}p^2$, which is a Pascal distribution with $r=2$.

Problem 4.

the initial value should be $P_0(0)=1$

$P_0(0) \overset{=1}{}$ means in the time period $[0,0]$, the probability of no arrival should be 1, which means there must be no arrival.

then with conditions: $\overset{P_0(0)=1}{P_0' = -\lambda P_0}$, $P_x' + \lambda P_x = \lambda P_{x-1}$, we use induction to prove that $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$

① when $x=0$, since $\frac{dP_0}{dt} = -\lambda P_0 \Rightarrow \frac{dP_0}{P_0} = -\lambda dt \Rightarrow \ln|P_0| = -\lambda t + C \Rightarrow P_0 = Ce^{-\lambda t}$

since $P_0(0) = 1$, then $P_0(0) = Ce^0 = C = 1 \Rightarrow P_0(0) = e^{-\lambda t}$

since $P_0(t) = \frac{(\lambda t)^0 e^{-\lambda t}}{0!} = e^{-\lambda t}$, then the conclusion is right.

② for $x=k, k\geq 0$, suppose $P_k(t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$

then since $P_{k+1}' + \lambda P_{k+1} = \lambda P_k = \frac{\lambda^{k+1} t^k e^{-\lambda t}}{k!}$

$$\mu = e^{\int \lambda dt} = e^{\lambda t}, \quad \int \frac{\lambda^{k+1} t^k e^{-\lambda t}}{k!} \times e^{\lambda t} dt = \int \frac{\lambda^{k+1} t^k}{k!} dt = \frac{\lambda^{k+1}}{(k+1)!} t^{k+1} = \frac{(\lambda t)^{k+1}}{(k+1)!}$$

$$\Rightarrow P_{k+1} = e^{-\lambda t}\left(\frac{(\lambda t)^{k+1}}{(k+1)!} + C\right) = \frac{(\lambda t)^{k+1} e^{-\lambda t}}{(k+1)!} + Ce^{-\lambda t}$$

since $P_{k+1}(0) = 0$ because in $[0,0]$, the probability of $k+1$ arrivals should be 0.

then. $P_{k+1}(0) = C = 0$

$\Rightarrow P_{k+1} = \frac{(\lambda t)^{k+1} e^{-\lambda t}}{(k+1)!}$, which accords with $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$

Based on ① and ②, we can conclude that $P_x(t) = \frac{(\lambda t)^x e^{-\lambda t}}{x!}$ holds true for $\forall x \in N$, then it gets proved.

Problem 5.

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{n!}{x!(n-x)!}\left(\frac{k}{n}\right)^x \left(1-\frac{k}{n}\right)^{n-x}$$

when $n\to\infty$, using stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n\to\infty$.

then $f(x) = \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{x!(n-x)!}\left(\frac{k}{n}\right)^x\left(1-\frac{k}{n}\right)^{n-x}$ as $n\to\infty$.

$$= \frac{k^x}{x!} \frac{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}{\sqrt{2\pi(n-x)}\left(\frac{n-x}{e}\right)^{n-x}} \cdot n^{-x} \cdot \left(1-\frac{k}{n}\right)^n \cdot \left(1-\frac{k}{n}\right)^{-x}.$$

since $\lim_{n\to\infty}\left(1-\frac{k}{n}\right)^n = e^{-k}$, then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!} \lim_{n\to\infty} \sqrt{\frac{n}{n-x}} \cdot \left(\frac{n}{n-x}\right)^{n-x} \cdot e^{-x} \cdot \left(1-\frac{k}{n}\right)^{-x}$.

when $n \to \infty$, it's obvious that $\sqrt{\frac{n}{n-x}} = \sqrt{\frac{1}{1-\frac{x}{n}}} \to 1$ and $(1-\frac{k}{n})^{-x} \to 1$.

then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!}\left(\lim_{n\to\infty}\left(\frac{n}{n-x}\right)^{n-x}\right) \cdot e^{-x}$.

since $\lim_{n\to\infty}\left(\frac{n}{n-x}\right)^{n-x} = \lim_{n\to\infty}\left(1+\frac{x}{n-x}\right)^{n-x} = \lim_{n-x\to\infty}\left(1+\frac{x}{n-x}\right)^{n-x} = e^x$.

then $\lim_{n\to\infty} f(x) = \frac{e^{-k}k^x}{x!} e^x \cdot e^{-x}$

$$= \frac{e^{-k}k^x}{x!}$$

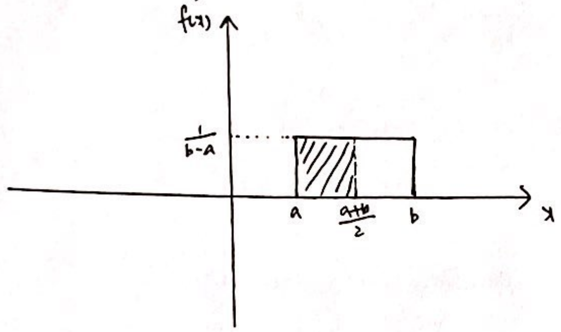
Problem 6.

i) since $\frac{1}{b-a} > 0$, then $f(x) \geq 0$ for all $x \in \mathbb{R}$.

and $\int_{-\infty}^{\infty} f(x)dx = 0 + \int_a^b \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_a^b = \frac{b-a}{b-a} = 1$

therefore, it's a density for a continuous random variable.

ii)



iii) $P[X \leq \frac{a+b}{2}] = \int_{-\infty}^{\frac{a+b}{2}} f(x)dx = 0 + \int_a^{\frac{a+b}{2}} \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_a^{\frac{a+b}{2}} = \frac{\frac{a+b}{2}-a}{b-a} = \frac{\frac{b-a}{2}}{b-a} = \frac{1}{2}$.

iv)

$P[c \leq X \leq d] = \int_c^d f(x)dx = \int_c^d \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_c^d = \frac{d-c}{b-a}$, since (c,d) is subinterval of (a,b).

$P[e \leq X \leq f] = \int_e^f f(x)dx = \int_e^f \frac{1}{b-a}dx = \left[\frac{x}{b-a}\right]_e^f = \frac{f-e}{b-a}$, since (e,f) is subinterval of (a,b).

since (c,d) and (e,f) are of equal length, then $d-c = f-e$. $\Rightarrow \frac{d-c}{b-a} = \frac{f-e}{b-a}$

then $P[c \leq X \leq d] = P[e \leq X \leq f]$

v). $F(x) = P[X \leq x] = \int_{-\infty}^{x} f(y)dy$

if $x < a$, then $f(y) = 0 \Rightarrow F(x) = 0$.

if $a < x < b$, then $F(x) = \int_{-\infty}^a 0\,dy + \int_a^x \frac{1}{b-a}dy = 0 + \left[\frac{y}{b-a}\right]_a^x = \frac{x-a}{b-a}$

if $x \geq b$, then $F(x) = \int_{-\infty}^a 0\,dy + \int_a^b \frac{1}{b-a}dy + \int_b^x 0\,dy = 0 + \frac{b-a}{b-a} + 0 = 1$.

$$\Rightarrow F(x) = \begin{cases} 0, & x \leq a \\ \frac{x-a}{b-a}, & a < x < b \\ 1, & x \geq b. \end{cases}$$

~~vi).~~ $\cancel{M_x = E[e^{tX}]} = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx = \int_{-\infty}^{a} e^{tx}\cdot 0\, dx + \int_a^b \cancel{e^{tx}\cdot \frac{1}{b-a}\, dx + \int_b^{\infty} e^{tx}\cdot 0\cdot dx}$

$$= 0 + \cancel{\left[\frac{te^{tx}}{b-a}\right]_a^b} + 0$$

$$= \cancel{\frac{e^{tb}-e^{ta}}{t(b-a)}}$$

$$\Rightarrow \cancel{E[X] = \frac{dM_x}{dt}\Big|_{t=0} = \frac{e^t\cdot t - e^t}{t^2}\cdot\frac{e^b-e^a}{b-a}}$$

vi). $E[X] = \int_{-\infty}^{\infty} x\cdot f(x)\, dx = \int_a^b x\cdot \frac{1}{b-a}\, dx = \frac{1}{b-a}\cdot\left[\frac{1}{2}x^2\right]_a^b = \frac{1}{b-a}\cdot\frac{1}{2}\cdot(b^2-a^2) = \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \frac{a+b}{2}$

$E[X^2] = \int_{-\infty}^{\infty} x^2\cdot f(x)\, dx = \int_a^b x^2\cdot\frac{1}{b-a}\, dx = \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$

$Var[X] = E[X^2] - E[X]^2 = \frac{b^2+ab+a^2}{3} - \frac{a^2+b^2+2ab}{4} = \frac{a^2+b^2-2ab}{12} = \frac{(b-a)^2}{12}$