VE401 Assignment 3

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Exercise 11. Bivariate Normal Distribution as a Mixture of Independent Normal Distributions

i) 1

ii) 2

iii) 3

iv) 5

v) **Solution.** We know the m.g.f. of normal distribution is

$$m_x(t) = exp\left(u \times t - 0.5t^2 \times D\right)$$

where D is the variance. We send it in matrix terms.
Let $X = (x1, x2)$. $M_x(t) = exp(u^T t - 0.5t^T \cdot D \cdot t)$, D is

$$\begin{pmatrix} \sigma_1^2 & ,0 \\ 0 & ,\sigma_2^2 \end{pmatrix}$$

, A is the transform matrix for Y=AX, u is matrix of expression.

We already know X follows bi-normal distribution for

$$\begin{aligned} f_{x_1x_2}(X_1, X_2) &= P(X_1 = x1, X_2 = x_2) \\ &= P(X_1 = x_1) \cdot P(X_2 = x_2) \\ &= f_{x_1}(x) \cdot f_{x_2}(x) \\ &= \sqrt{A} \cdot \int e^{(n^2)^B} dx_1 \cdot \sqrt{A} \cdot \int e^{(m^2)^B} dx_2 \\ &= A \cdot \int e^{(m^2+n^2)^B} dx_1 dx_2 \end{aligned}$$

It's Bi-normal distribution's pdf when $\rho = 0$.Since X_1,X_2 are independent, $\rho = 0$. So X follows Bi-normal distribution. Y=AX

In matrix term,

$$\begin{aligned} m_Y(t) &= E(e^Y t^T) = E(e^A X t^T) = E(e^{(A^T t)^T} X) \\ &= exp(u^T A^T t - 0.5(A^T t)^T \cdot D \cdot A^T t) \\ &= exp((Au)^T t - 0.5(A^T t)^T \cdot D \cdot A^T t) \\ &= exp((Au)^T t - 0.5(t)^T \cdot (A \cdot D \cdot A^T) \cdot t) \\ &= exp((u')^T t - 0.5(t)^T \cdot (D') \cdot t) \end{aligned}$$

Obviously Y follows the same kind of distribution with only different parameters. Y follows bi-normal distribution.

According to 3.10.3,

$$f_{x_1x_2}(x_1, x_2) = A \cdot exp(-0.5 < x - ux, \sum{}^{-1} x(x - ux) >)$$

Hence

$$f_{y_1y_2}(y_1, y_2) = A \cdot exp(-0.5 < y - uy, \sum{}^{-1} y(y - uy) >)$$

And $fy(y) = Ae^(m^2 - 2\rho mn + n^2)^B$

with:

$A = 1/2p\sigma_1\sigma_2\sqrt{1-\rho_2}$

$B = -0.5/(1-\rho_2)$

$m = (y_2 - u_2)/\sigma_2$

$n = (y_1 - \mu_1)/\sigma_1$ $\square$