Ve401 Probabilistic Methods in Engineering

Spring 2020 — Assigment 4

Date Due: 11:00 PM, Friday, the 3rd of April 2020

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This assignment has a total of **(27 Marks)**.

Exercise 4.1 Data Visualization
The shear strengths of 100 spot welds in a titanium alloy follow:

5408 5431 5475 5442 5376 5388 5459 5422 5416 5435 5420 5429 5401 5446 5487 5416 5382 5357 5388
5457 5407 5469 5416 5377 5454 5375 5409 5459 5445 5429 5463 5408 5481 5453 5422 5354 5421 5406
5444 5466 5399 5391 5477 5447 5329 5473 5423 5441 5412 5384 5445 5436 5454 5453 5428 5418 5465
5427 5421 5396 5381 5425 5388 5388 5378 5481 5387 5440 5482 5406 5401 5411 5399 5431 5440 5413
5406 5342 5452 5420 5458 5485 5431 5416 5431 5390 5399 5435 5387 5462 5383 5401 5407 5385 5440
5422 5448 5366 5430 5418

i) Construct a stem-and-leaf diagram for the weld strength data and comment on any important features that you notice.
(1 Mark)

ii) Construct a histogram. Comment on the shape of the histogram. Does it convey the same information as the stem-and-leaf display?
(2 Marks)

iii) Construct a box plot of the data and write an interpretation of the plot. How does the box plot compare in interpretive value to the original stem-and-leaf diagram?
(2 Marks)

Exercise 4.2 Unbiased Is Not Always Best
Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from a random variable with variance σ^2. We have seen that the sample variance

$$S_{n-1}^2 := \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \overline{X})^2$$

is an unbiased estimator for σ^2. It can be shown that

$$\operatorname{Var}(S_{n-1}^2) = \operatorname{MSE}(S_{n-1}^2) = \frac{1}{n}\left(\operatorname{E}[(X-\overline{X})^4] - \frac{n-3}{n-1}\sigma^4\right) = \frac{1}{n}\left(\gamma_2 + \frac{2n}{n-1}\right)\sigma^4$$

where $\gamma_2 := \operatorname{E}[(X-\mu)^4]/\sigma^4 - 3$ is called the *excess kurtosis* of a distribution. (You do not have to perform this tedious calculation!)

i) Show that if X follows a normal distribution with mean μ and variance σ^2,

$$\operatorname{MSE}(S_{n-1}^2) = \frac{2}{n-1}\sigma^4$$

(2 Marks)

ii) For $a > 0$ set

$$S_a^2 := \frac{n-1}{a}S_{n-1}^2.$$

Find $\operatorname{MSE}(S_a^2)$ and show that the mean square error is minimized for

$$a = n + 1 + \frac{n-1}{n}\gamma_2$$

In the case of a normal distribution with mean μ and variance σ^2, show that this reduces to $a = n+1$. Conclude that a biased estimator may be "better" overall than an unbiased estimator.
(3 Marks)

Exercise 4.3 Maximum-Likelihood Estimator
Suppose[1] that the random variable X has the probability density

$$f(x) = \begin{cases} (\gamma+1)x^{\gamma}, & 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Let $X_1, X_2, \ldots, X_n$ be a random sample of size n. Find the maximum likelihood estimator for γ.
(2 Marks)

Exercise 4.4 Maximum-Likelihood Estimate
A new material is being tested for possible use in the brake shoes of automobiles. These shoes are expected to last for at least 75,000 miles. Fifteen sets of four of these experimental shoes are subjected to accelerated life testing. The random variable X, the number of shoes in each group of 4 that fail early, is assumed to be binomially distributed with $n = 4$ and p unknown.

i) Find the maximum-likelihood estimate for p based on these data:

1	0	1	0	2	1	0	1
0	1	0	0	0	1	0	

(2 Marks)

ii) If an early failure rate in excess of 10% is unacceptable from a business point of view, would you have some doubts concerning the use of this new material? Explain.
(1 Mark)

Exercise 4.5 Transformed Normal Distributions
Let $X = (X_1, X_2)$ where X_1 and X_2 follow i.i.d. normal distributions with variance σ^2 and mean μ. Let $Y = (Y_1, Y_2) = AX$ with $A^T = A^{-1}$. Show that Y_1 and Y_2 are independent with mean $\mathrm{E}[Y] = A\,\mathrm{E}[X]$ and variance $\mathrm{Var}\,Y = \sigma^2 \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$.
(3 Marks)

Exercise 4.6 Symmetric Confidence Intervals Are Optimal
Consider a general two-sided $100(1-\alpha)\%$ confidence interval for the mean μ when σ is known:

$$\overline{x} - z_{\alpha_1}\sigma/\sqrt{n} \le \mu \le \overline{x} - z_{\alpha_2}\sigma/\sqrt{n}$$

where $\alpha_1 + \alpha_2 = \alpha$. Show that the length of the interval, $\sigma(z_{\alpha_1} + z_{\alpha_2})/\sqrt{n}$ is minimized when $\alpha_1 = \alpha_2 = \alpha/2$.
(2 Marks)

Exercise 4.7 Confidence Interval for the Standard Deviation
The percentage of titanium in an alloy used in aerospace castings is measured in 51 randomly selected parts. The sample standard deviation is $s = 0.37$. Construct a 95% two-sided confidence interval for σ.
(2 Marks)

Exercise 4.8 Non-Parametric Confidence Intervals
Let $X_1, \ldots, X_n$ be a random sample of size n from a continuous distribution with median M. Let

$$X_{\min} = \min_{1\le i\le n} X_i, \qquad X_{\max} = \max_{1\le i\le n} X_i.$$

i) Show that

$$P[X_{\min} \le M \le X_{\max}] = 1 - \left(\frac{1}{2}\right)^{n-1}.$$

(2 Marks)

ii) Suppose the sample data are arranged from smallest to largest, so that $X_1 \le X_2 \le \cdots \le X_n$. Calculate

$$P[X_{k+1} \le M \le X_{n-k}]$$

for $k = 0, \ldots, \lfloor n/2 \rfloor$.
(3 Marks)

[1] This exercise is taken from the second midterm exam of Fall 2008.