VE401 Assignment 2

Yang, Tiancheng 517370910259
Qiu, Yuqing 518370910026
Chang, Siyao 517370910024

March 14, 2020

Exercise 1. Discrete Uniform Distribution

i) **Solution.** With the parameter n, we have

$$m_x(t) = E(e^{xt}) = \sum_{k=1}^{n} e^{x_k t}\frac{1}{n}$$
$$= \frac{1}{n}\sum_{k=1}^{n} e^{x_k t}$$

□

ii) **Solution.** From the moment generating function we get that

$$E[X] = \frac{d}{dt}m_x(t)|_{t=0}$$
$$= \frac{1}{n}\sum_{k=1}^{n}[\frac{d}{dt}e^{x_k t}|_{t=0}]$$
$$= \frac{1}{n}\sum_{k=1}^{n} x_k$$

and

$$E[X^2] = \frac{d^2}{dt^2}m_x(t)|_{t=0}$$
$$= \frac{1}{n}\sum_{k=1}^{n}[\frac{d^2}{dt^2}e^{x_k t}|_{t=0}]$$
$$= \frac{1}{n}\sum_{k=1}^{n} x_k^2$$

And hence the variance is given by

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \frac{1}{n^2}(\sum_{k=1}^{n} x_k)^2$$

□

Exercise 2. Uniqueness of Moment generating functions - Simple Case

Proof. With $m_X(t) = m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$ we can see that

$$\frac{d}{dt}m_X(t) = \frac{d}{dt}m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$$

This gives us

$$\frac{d}{dt}m_X(t)|_{t=0} = \frac{d}{dt}m_Y(t)|_{t=0}$$

which is

$$E[X] = E[Y]$$

By definition of the expectation,

$$\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$$

Now we prove by induction that $\forall n \in \mathbb{N}$, $f_X(x) = f_Y(x)$.

First when $n = 0$ we directly have $f_X(x) = f_Y(x) = 1$.Now we want to prove that $f_X(n+1) = f_Y(n+1)$ given that $f_X(n) = f_Y(n)$. This is simple. We first write

$$\sum_{x=0}^{n+1} x \cdot f_X(x) = \sum_{x=0}^{n+1} x \cdot f_Y(x)$$
$$\sum_{x=0}^{n} x \cdot f_X(x) + (n+1)f_X(n+1) = \sum_{x=0}^{n} x \cdot f_Y(x) + (n+1)f_Y(n+1)$$

Note that $\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$ given that $\forall N \leq n$, $f_X(N) = f_Y(N)$.Thus by cancelling the sums, we have our desired result

$$(n+1)f_X(n+1) = (n+1)f_Y(n+1)$$
$$f_X(n+1) = f_Y(n+1)$$

Therefore, by induction we have proved that $f_X(x) = f_Y(x)$ for $x = 0, ..., n$. □

Exercise 3. Sums of Independent Discrete Random Variables

i) **Proof.** We first divide $x + y \in ranZ$ into two parts: $x + y = z$ and $x + y \neq z$.

$$\begin{aligned} P[Z = z] &= P[X + Y = z] \\ &= \sum_{x+y \in ranZ} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &= \sum_{x+y \neq z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &+ \sum_{x+y = z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \end{aligned}$$

We note that if $x + y \neq z$, then $P[x + y = z] = 0$. Hence

$$\sum_{x+y \neq z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] = 0$$

Note that if $x + y = z$, then $P[x + y = z] = 1$. Thus

$$\begin{aligned} &\sum_{x+y = z} P[X + Y = z | X = x \wedge Y = y] \cdot P[X = x \wedge Y = y] \\ &= P[X = x \wedge Y = y] \\ &= P[X = x] \cdot P[Y = y] \end{aligned}$$

Therefore, $P[Z = z] = 0 + P[X = x] \cdot P[Y = y] = P[X = x] \cdot P[Y = y]$. □

ii) **Proof.** We denote the parameter of the geometric distribution as p, as usual. Now we have $X, Y \sim Geom(p)$. Applying the density function we write $P[X = x] = (1-p)^{x-1}p$ and $P[Y = x] = (1-p)^{x-1}p$. Now the sum of X and Y is $Z = X + Y$. The probability for Z is therefore

$$\begin{aligned} P[Z = z] &= \sum_{x+y=z} (1-p)^{x-1} \cdot (1-p)^{y-1}p \\ &= p^2 \sum_{x+y=z} (1-p)^{x+y-2} \end{aligned}$$

We know that $x, y \in \mathbb{N} \backslash 0$. Thus there are $z-1$ terms in the above sum, resulting in

$$
\begin{aligned}
P[Z=z] &= p^2(z-1)(1-p)^{z-2} \\
&= \binom{z-1}{1} p^2 (1-p)^{z-2}
\end{aligned}
$$

which is exactly a Pascal distribution with $r = 2$. $\square$