VE401 Assignment

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Exercise 1. Elementary Probability

Solution. We use Cardano's principle to get the probability. The number of ways to pick 120 people from 2000 individuals is

$$n_1 = \frac{2000!}{120! \times (2000-120)!}$$

The number of ways that me and my friend are both chosen is equal to the number of ways to choose 118 people from 1998 individuals, which is

$$n_2 = \frac{1998!}{118! \times (1998-118)!}$$

Therefore the probability that me and my friend will both be chosen is

$$\frac{n_2}{n_1} = \frac{\frac{1998!}{118! \times (1998-118)!}}{\frac{2000!}{120! \times (2000-120)!}} = 0.357\%$$

□

Exercise 2. Some Routine Calculations

i) **Proof.** Since $A \subset B$, $B = A + B \backslash A$. Note that $A \cap B \backslash A = \emptyset$. Thus $P[B] = P[A] + P[B \backslash A] \geq P[A]$. Therefore $P[A] \leq P[B]$. □

ii) **Proof.** Since A and B are independent, we have $P[A \cap B] = P[A]P[B]$. We know that $P[A]P[B] > 0$ so $P[A \cap B] > 0$. Thus $P[A \cap B] = \frac{|A \cap B|}{|S|} > 0$, which means that $|A \cap B| > 0$. Therefore, $A \cap B \neq \emptyset$ and hence they are not mutually exclusive. □

iii) **Proof.** First we have two trivial equations:

$$P[A] = P[A \backslash (A \cap B)] + P[A \cap B]$$

$$P[B] = P[B \backslash (A \cap B)] + P[A \cap B]$$

We also have

$$P[A \cup B] = P[A \backslash (A \cap B)] + P[B \backslash (A \cap B)] + P[A \cap B]$$

. Therefore,

$$P[A \cup B] = P[A] - P[A \cap B] + P[B] - P[A \cap B] + P[A \cap B] = P[A] + P[B] - P[A \cap B]$$

□

Exercise 3. D'Alembert's Coins

i) **Solution.** No, it is not possible. If the coin is fair, we know that

$$P[two\ heads] = P[no\ heads] < P[onehead]$$

. Now if the coin is biased, for that coin $P[head] \neq P[nohead]$. And hence if it is tossed twice,

$$P[two\ heads] \neq P[no\ heads]$$

. Therefore, even though the coin can be biased, the three outcomes cannot have the same probability. □

ii) **Solution.** No, it is not possible as well. Denote one coin as A, with $P[A, head] = a$. We know that $P[A, head] + P[A, tail] = 1$ so $P[A, tail] = 1 - a$. Similarly, we also have coin B with b and $1 - b$. Now, if the three outcomes have same probability, then

$$a \times b = (1 - a) \times (1 - b) = 1/3$$

. From here we get that $a + b = 1$. Now we calculate

$$a \times b = a \times (1 - a) = \frac{1}{3}$$

which has no real solution. Therefore, it is impossible to make the coins so that D'Alembert's claim is true. □

Exercise 4. Independence

i) **Solution.** Denotion of events:
P1: a participant from the first group is chosen;
P2: a participant from the second group is chosen;
A: the participant replies "yes" to the second question;
Now we list the known probabilities: $P[P1] = 50\%$, $P[P2] = 50\%$, $P[A|P1] = 17\%$, $P[A|P2] = 3\%$. We want to know the total probability of $P[A]$.

$$P[A] = P[A|P1] \times P[P1] + P[A|P2] \times P[P2] = 17\% \times 50\% + 3\% \times 50\% = 10\%$$

Therefore, this probability is 10%. □

ii) **Solution.** No. We have $P[A|P1] > P[A]$. Therefore it is not independent. □

Exercise 5. This one may need a little thinking about... Though it doesn't.

Solution. We denote the event that a chip is defective by D. The event that a chip is stolen is S. $\neg S$ means that a chip is not stolen. Thieves stole the chips before inspection, so $P[D|S] = 50\%$. We know that $P[S] = 1\%$ of the chips is illegal, hence $P[\neg S] = 99\%$ of the chips is legally marketed. For those chips that are legally marketed, their probability of being defective is 5%, because of the inspection. This means that if the chips is not stolen, the probability of being defective is $P[D|\neg S] = 5\%$. Now we apply the Bayes' theorem to get the probability we want to calculate, i.e., $P[S|D]$.

$$P[S|D] = \frac{P[D|S]P[S]}{P[D|S]P[S] + P[D|\neg S]P[\neg S]} = \frac{0.5 \times 0.01}{0.5 \times 0.01 + 0.05 \times 0.99} = 9.17\%$$

□

Exercise 6. Mounty Hall in Prison?

Solution. They are both wrong. Conclusions first: the prisoner is wrong because the warden would have given him some information; the warden is wrong because the one who have a bigger chance of dying would not have been prisoner A.
Suppose that the warden tells A that prisoner B is not to be executed. We denote this event by B^*. The event that prisoner X is to be executed is denoted by DX. By Bayes' Theorem, we have

$$P[DA|B^*] = \frac{P[B^*|DA]P[DA]}{P[B^*|DA]P[DA] + P[B^*|DB]P[DB] + P[B^*|DC]P[DC]} = \frac{\frac{1}{2} \times \frac{1}{3}}{\frac{1}{2} \times \frac{1}{3} + 0 + 1 \times \frac{1}{3}} = \frac{1}{3}$$

However, for prisoner C, the probability that he is to be executed is then

$$P[DC|B^*] = \frac{P[B^*|DC]P[DC]}{P[B^*|DC]P[DC] + P[B^*|DB]P[DB] + P[B^*|DA]P[DA]} = \frac{1 \times \frac{1}{3}}{1 \times \frac{1}{3} + 0 + \frac{1}{2} \times \frac{1}{3}} = \frac{2}{3}$$

Therefore, neither of them is right. □

Exercise 7. Two Children Paradox - Birthday Party

Solution. Denote the events as:
A: The lady's other child is a girl;
B: The lady's boy is born in July;
Then, according to the problem, what we need to calculate is $P[A|B]$.
We define the sample space of the gender as $\{(b,b),(b,g),(g,b)\}$, where 'b' means the child is a boy and 'g' means the child is a girl and the former child is older than the latter. Each sample point has the same probability, i.e.

$$P[(b,b)] = P[(b,g)] = P[(g,b)] = \frac{1}{3}$$

For (b,g) and (g,b), the probability of the boy born in July should be $\frac{1}{12}$.
For (b,b), the probability of at least one boy born in July can be calculated as follows:

i) Only one boy is born in July: $\frac{1}{12} \times \frac{11}{12} \times 2 = \frac{22}{144}$

ii) Two boys are both born in July: $\frac{1}{12} \times \frac{1}{12} = \frac{1}{144}$

Then, the probability of at least one boy born in July is $\frac{22}{144} + \frac{1}{144} = \frac{23}{144}$ □

Here we calculate $P[A \cap B]$. We use the total probability,

$$\begin{aligned} P[A \cap B] &= P[A \cap B|(b,b)]P[(b,b)] + P[A \cap B|(b,g)]P[(b,g)] + P[A \cap B|(g,b)]P[(g,b)] \\ &= 0 \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} \\ &= \frac{1}{18} \end{aligned}$$

Similarly, we can calculate $P[B]$.

$$\begin{aligned} P[B] &= P[B|(b,b)]P[(b,b)] + P[B|(b,g)]P[(b,g)] + P[B|(g,b)]P[(g,b)] \\ &= \frac{23}{144} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} + \frac{1}{12} \times \frac{1}{3} \\ &= \frac{47}{432} \end{aligned}$$

Then, using the formula of conditional formula,

$$\begin{aligned} P[A|B] &= \frac{P[A \cap B]}{P[B]} \\ &= \frac{\frac{1}{18}}{\frac{47}{432}} \\ &= \frac{24}{47} \end{aligned}$$