VE401 Assignment 2

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Exercise 1. Discrete Uniform Distribution

1. **Solution.** With the parameter n, we have

$$\begin{aligned} m_x(t) = E(e^{xt}) &= \sum_{k=1}^{n} e^{x_k t}\frac{1}{n} \\ &= \frac{1}{n}\sum_{k=1}^{n} e^{x_k t} \end{aligned}$$

□

2. **Solution.** From the moment generating function we get that

$$\begin{aligned} E[X] &= \frac{d}{dt}m_x(t)|_{t=0} \\ &= \frac{1}{n}\sum_{k=1}^{n}[\frac{d}{dt}e^{x_k t}|_{t=0}] \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k \end{aligned}$$

and

$$\begin{aligned} E[X^2] &= \frac{d^2}{dt^2}m_x(t)|_{t=0} \\ &= \frac{1}{n}\sum_{k=1}^{n}[\frac{d^2}{dt^2}e^{x_k t}|_{t=0}] \\ &= \frac{1}{n}\sum_{k=1}^{n} x_k^2 \end{aligned}$$

And hence the variance is given by

$$Var[X] = E[X^2] - E[X]^2 = \frac{1}{n}\sum_{k=1}^{n} x_k^2 - \frac{1}{n^2}(\sum_{k=1}^{n} x_k)^2$$

□

Exercise 2. Uniqueness of Moment generating functions - Simple Case

Proof. With $m_X(t) = m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$ we can see that

$$\frac{d}{dt}m_X(t) = \frac{d}{dt}m_Y(t)\ \forall t \in (-\varepsilon, \varepsilon)$$

This gives us

$$\frac{d}{dt}m_X(t)|_{t=0} = \frac{d}{dt}m_Y(t)|_{t=0}$$

which is

$$E[X] = E[Y]$$

By definition of the expectation,

$$\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$$

Now we prove by induction that $\forall n \in \mathbb{N}$, $f_X(x) = f_Y(x)$.

First when $n = 0$ we directly have $f_X(x) = f_Y(x) = 1$.Now we want to prove that $f_X(n+1) = f_Y(n+1)$ given that $f_X(n) = f_Y(n)$. This is simple. We first write

$$\sum_{x=0}^{n+1} x \cdot f_X(x) = \sum_{x=0}^{n+1} x \cdot f_Y(x)$$
$$\sum_{x=0}^{n} x \cdot f_X(x) + (n+1)f_X(n+1) = \sum_{x=0}^{n} x \cdot f_Y(x) + (n+1)f_Y(n+1)$$

Note that $\sum_{x=0}^{n} x \cdot f_X(x) = \sum_{x=0}^{n} x \cdot f_Y(x)$ given that $\forall N \leq n$, $f_X(N) = f_Y(N)$.Thus by cancelling the sums, we have our desired result

$$(n+1)f_X(n+1) = (n+1)f_Y(n+1)$$
$$f_X(n+1) = f_Y(n+1)$$

Therefore, by induction we have proved that $f_X(x) = f_Y(x)$ for $x = 0, ..., n$. $\square$