1

Let the covolution mask be given by $w = [w(0); w(1); w(2); w(3); w(4); w(5); w(6)]$ and the image vector be given by $f = [f(0); f(1); \dots; f(n-1)]$.
Now, we will pad $\text{len}(w) - 1$ $(= 6)$ zeroes on both side of the image to get the same answer as convolving w with an f of infinite size. Let this new vector be

$$f_p = [\underbrace{f(-6); f(-5); \dots; f(-1)}_{0\text{'s}}; \underbrace{f(0); \dots; f(n-1)}_{f}; \underbrace{f(n); \dots; f(n+4); f(n+5)}_{0\text{'s}}]$$

Now, let $r = f * w$, then the equivalent truncated convolution $r_s = f_p * w$ where r_s contains $n+6$ elements $[r_s(0); r_s(1); \dots, r_s(n+5)]$ such that $r_s(i) = \sum_{j=0}^{6} f(i-j) \cdot w_j$.

This can be written in a matrix form as shown below

$$\underbrace{\begin{bmatrix} r_s(0) \\ r_s(1) \\ r_s(2) \\ \vdots \\ r_s(n+5) \end{bmatrix}}_{r_s} = \underbrace{\left[\overbrace{\begin{matrix} w(6) & w(5) & w(4) & w(3) & w(2) & w(1) \\ 0 & w(6) & w(5) & w(4) & w(3) & w(2) \\ 0 & 0 & w(6) & w(5) & w(4) & w(3) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}}^{f(-6); f(-5); \dots; f(-1)} \; \overbrace{\begin{matrix} w(0) & 0 \cdots\cdots\cdots\cdots\cdots 0 & & & & \\ w(1) & w(0) & 0 \cdots\cdots\cdots 0 & & & \\ w(2) & w(1) & w(0) & 0 \cdots\cdots 0 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 \cdots\cdots 0 & & & w(6) & w(5) & w(4) \\ 0 \cdots\cdots\cdots 0 & & & & w(6) & w(5) \\ 0 \cdots\cdots\cdots\cdots 0 & & & & & w(6) \end{matrix}}^{f(0); \dots; f(n-1)} \; \overbrace{\begin{matrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ w(3) & w(2) & w(1) & w(0) & 0 & 0 \\ w(4) & w(3) & w(2) & w(1) & w(0) & 0 \\ w(5) & w(4) & w(3) & w(2) & w(1) & w(0) \end{matrix}}^{f(n); \dots; f(n+5)} \right]}_{M} \underbrace{\begin{bmatrix} f(-6) \\ f(-5) \\ f(-4) \\ f(-3) \\ f(-2) \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \\ f(n-1) \\ f(n) \\ f(n+1) \\ f(n+2) \\ f(n+3) \\ f(n+4) \\ f(n+5) \end{bmatrix}}_{f_p} \quad (1)$$

This M is a $(n+6) \times (n+12)$ matrix. Moreover, we can remove first and last six columns of M corresponding to multiplication with zero-padded elements of f_p.

This gives us a $(n+6) \times n$ matrix M_t with same behaviour as M

$$
\underbrace{\begin{bmatrix} r_s(0) \\ r_s(1) \\ r_s(2) \\ \vdots \\ r_s(n+5) \end{bmatrix}}_{r_s}
=
\underbrace{\begin{bmatrix}
w(0) & 0 & \cdots & & & & & & & & 0 \\
w(1) & w(0) & 0 & \cdots & & & & & & & 0 \\
w(2) & w(1) & w(0) & 0 & \cdots & & & & & & 0 \\
w(3) & w(2) & w(1) & w(0) & 0 & \cdots & & & & & 0 \\
w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & & & & 0 \\
w(5) & w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & & & 0 \\
w(6) & w(5) & w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & & 0 \\
0 & w(6) & w(5) & w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & 0 & w(6) & w(5) & w(4) & w(3) & w(2) & w(1) & w(0) & 0 \\
0 & \cdots & & 0 & w(6) & w(5) & w(4) & w(3) & w(2) & w(1) & w(0) \\
0 & \cdots & & & 0 & w(6) & w(5) & w(4) & w(3) & w(2) & w(1) \\
0 & \cdots & & & & 0 & w(6) & w(5) & w(4) & w(3) & w(2) \\
0 & \cdots & & & & & 0 & w(6) & w(5) & w(4) & w(3) \\
0 & \cdots & & & & & & 0 & w(6) & w(5) & w(4) \\
0 & \cdots & & & & & & & 0 & w(6) & w(5) \\
0 & \cdots & & & & & & & & 0 & w(6)
\end{bmatrix}}_{M_t}
\underbrace{\begin{bmatrix} f(0) \\ \vdots \\ f(n-1) \end{bmatrix}}_{f}
\tag{2}
$$

But, this pattern requires an additional condition that $n \geq 6$, and the matrix for $n < 6$ case will be slightly different. Instead of making seperate cases, we can make size of f greater than six by padding it with three zeroes in both directions. Let this new vector be

$$
f_s = [\underbrace{f(-3); f(-2); f(-1)}_{0\text{'s}}; \underbrace{f(0); \ldots; f(n-1)}_{f}; \underbrace{f(n); f(n+1); f(n+2)}_{0\text{'s}}]
$$

Also, we might truncate the resultant convolution vector to the size of n by removing the first and last three entries and get r_t. So, r_s can also be indexed using elements of this truncated r_t as

$$
r_s = r_t(-3); r_t(-2); r_t(-1); r_t(0); \ldots; r_t(n-1); r_t(n); r_t(n+1); r_t(n+2)
$$

This gives us a square matrix M_s as follows given below

1 Answer

$$\underbrace{\begin{bmatrix} r_t(-3) \\ r_t(-2) \\ r_t(-1) \\ r_t(0) \\ r_t(1) \\ \vdots \\ r_t(n-1) \\ r_t(n) \\ r_t(n+1) \\ r_t(n+2) \end{bmatrix}}_{r_s} = \underbrace{\begin{array}{c} \overbrace{\qquad\qquad\qquad\qquad}^{f(-3);\, f(-2);\, f(-1)} \quad \overbrace{\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad}^{f(0);\, \ldots;\, f(n-1)} \quad \overbrace{\qquad\qquad\qquad\qquad}^{f(n);\, f(n+1);\, f(n+2)} \\ \begin{bmatrix} w(3) & w(2) & w(1) & w(0) & 0 & \cdots & \cdots & \cdots & 0 & 0 & 0 & 0 \\ w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ w(5) & w(4) & w(3) & w(2) & w(1) & w(0) & 0 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & w(6) & w(5) & w(4) & w(3) & w(2) & w(1) \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & w(6) & w(5) & w(4) & w(3) & w(2) \\ 0 & 0 & 0 & 0 & \cdots & \cdots & \cdots & 0 & w(6) & w(5) & w(4) & w(3) \end{bmatrix} \end{array}}_{M_s} \underbrace{\begin{bmatrix} f(-3) \\ f(-2) \\ f(-1) \\ f(0) \\ f(1) \\ \vdots \\ f(n-1) \\ f(n) \\ f(n+1) \\ f(n+2) \end{bmatrix}}_{f_s} \tag{3}$$

2 Properties

- M_s is a sparse matrix as each row can potentially contain atmost six non-zero entries among its $(n+6)$ elements.
- All diagonal entries are $w(3)$.
- From the fourth row till the last fourth row, each next row is a right rotation of the previous row.

3 Application

Usually by doing convolution we lose some infomation of our input. But with this matrix representation we can calculate when do we lose the information and when the information is recoverable using the invertability of M_s. If M_s is invertible, we can recover back f_s from r_s as follows else we can't recover f_s

$$f_s = M_s^{-1} r_s$$