4

Given a 2D operator defined by a matrix $G \in R^{n \times n}$, using SVD decomposition we can write it as:

$$G = \sum_{i=1}^{n} \sigma_i u_i v_i^T \tag{1}$$

Clearly, G is separable iff $\forall i > 1, \sigma_i = 0$. Hence the number of non-zero singular values should be equal to 1. Since that is equal to the rank of the matrix, the rank of the matrix should be 1.

(a) Given Matrix, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, clearly has rank 2. Hence it is not separable.

(b) Since the Matrix is not separable, it cannot be written as an outer product of 2,1-D vectors. Hence you cannot separate this kernel and 2 consecutive 1-D convolutions to get the same result. However, the given Laplacian operator is written as,

$$\begin{aligned} \nabla^2 f &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \\ &= f(x+1, y) + f(x-1, y) \\ &\quad + f(x, y+1) + f(x, y-1) - 4f(x, y) \end{aligned}$$

Hence, we can make two 2^{nd} order derivatives using 1D convolution with kernels $\begin{pmatrix} 1 & -2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ and sum their results to apply the Laplacian masks. Thus, the Laplacian mask can be implemented entirely using 1D convolutions.