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(a) Given a 2D operator defined by a matrix $G \in R^{n \times n}$, using SVD decomposition we can write it as:

$$G = \sum_{i=1}^{n} \sigma_i u_i v_i^T \tag{1}$$

Clearly, G is separable iff $\forall i > 1, \sigma_i = 0$. Hence the number of non-zero singular values should be equal to 1. Since that is equal to the rank of the matrix, the rank of the matrix should be 1.

The given matrix $\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, clearly has rank 2. Hence, it can not be realised as a separable filter.

(b) Since the Matrix is not separable, it cannot be written as an outer product of 2,1-D vectors. Hence you cannot separate this kernel and apply 1-D convolutions consecutively to get the same result. However, the given Laplacian operator is written as,

$$\begin{aligned} & f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y) \\ &= \underbrace{(f(x+1,y) + f(x-1,y) - 2f(x,y))}_{\frac{\partial^2 f}{\partial x^2}} + \underbrace{(f(x,y+1) + f(x,y-1) - 2f(x,y))}_{\frac{\partial^2 f}{\partial y^2}} \end{aligned}$$

We can make 2^{nd} order derivatives w.r.t x and y using 1D convolution with the kernels $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ & $\begin{bmatrix} 1 & -2 & 1 \end{bmatrix}^T$ and sum their results to get the same result as applying the Laplacian mask. Thus, the Laplacian mask can be implemented entirely using 1D convolutions.