6

Recall that, given a continuous function $f\colon \mathbb{R} \to \mathbb{R}$, the Fourier Transform of f is given by

$$\hat{f}(\xi) \coloneqq \mathcal{F}\{f(t)\} = \int\limits_{-\infty}^{\infty} f(t)e^{-i2\pi\xi t}\,\mathrm{d}t,$$

and the inverse Fourier Transform is given by

$$f(x) = \int\limits_{-\infty}^{\infty} \hat{f}(\xi)e^{i2\pi\xi t}\,\mathrm{d}\xi,$$

Throughout this answer, we will denote the Fourier Transform operator by $\mathcal{F}$.

As the hint suggests, we will first investigate what we get when we simplify $\mathcal{F}\{\mathcal{F}\{f(t)\}\}$. Since $\hat{f}(\xi) = \mathcal{F}\{f(t)\}$, we know that $\mathcal{F}\{\mathcal{F}\{f(t)\}\} = \mathcal{F}\{\hat{f}(\xi)\}$. Expanding the expression, we get

$$\mathcal{F}\{\hat{f}(\xi)\} = \int\limits_{-\infty}^{\infty} \hat{f}(\xi)e^{-i2\pi\xi t}\mathrm{d}\xi$$

By performing a change of variable with $u = -\xi$ (so that $\mathrm{d}u = -\mathrm{d}\xi$), we get

$$\begin{aligned}\int\limits_{-\infty}^{\infty} \hat{f}(\xi)e^{-i2\pi\xi t}\mathrm{d}\xi &= -\int\limits_{\infty}^{-\infty} \hat{f}(-u)e^{i2\pi ut}\mathrm{d}u \\ &= \int\limits_{-\infty}^{\infty} \hat{f}(-u)e^{i2\pi ut}\mathrm{d}u\end{aligned}$$

The last expression is, by definition, the inverse Fourier Transform of $f(-u)$. Hence, we have shown (sans some variable renaming) that

$$\mathcal{F}\{\mathcal{F}\{f(t)\}\} = f(-t)$$

With this lemma in mind, we can easily prove the given proposition. Given $\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{\mathcal{F}\{f(t)\}\}\}\}$, we can simplify it to $\mathcal{F}\{\mathcal{F}\{f(-t)\}\}$ by using the previous lemma. By applying the same lemma again, we get$\mathcal{F}\{\mathcal{F}\{f(-t)\}\} = f\big(-(-t)\big) = f(t)$, which proves the proposition. $\square$