5

(a) Covariance matrix in PCA is given as follows

$$\boldsymbol{C} = \frac{1}{N-1}\boldsymbol{X}\boldsymbol{X}^T \text{ where } \boldsymbol{X} = \begin{bmatrix}\overline{\boldsymbol{x}}_1 & \cdots & \overline{\boldsymbol{x}}_N\end{bmatrix} \text{ and } \overline{\boldsymbol{x}} \in \mathbb{R}^d \tag{1}$$

Now, this covariance matrix is symmetric as

$$\begin{aligned}\boldsymbol{C}^T &= \left(\frac{1}{N-1}\boldsymbol{X}\boldsymbol{X}^T\right)^T \\ &= \frac{1}{N-1}(\boldsymbol{X}^T)^T\boldsymbol{X}^T \\ &= \frac{1}{N-1}\boldsymbol{X}\boldsymbol{X}^T \\ &= \boldsymbol{C}\end{aligned} \tag{2}$$

Now, to prove that $\boldsymbol{C}$ is positive semi-definite we assume the opposite and show its infeasibility. That is, assume it is not positive semi-definite then there exists a vector $\boldsymbol{y} \in \mathbb{R}^d$ such that $\boldsymbol{y}^T\boldsymbol{C}\boldsymbol{y} < 0$.

$$\begin{aligned}\text{Now, } \exists \boldsymbol{y} \in \mathbb{R}^d | \boldsymbol{y}^T\boldsymbol{C}\boldsymbol{y} < 0 &\Leftrightarrow \boldsymbol{y}^T\frac{1}{N-1}\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{y} < 0 \\ &\Leftrightarrow \frac{1}{N-1}\boldsymbol{y}^T\boldsymbol{X}\boldsymbol{X}^T\boldsymbol{y} < 0 \\ &\Leftrightarrow \frac{1}{N-1}(\boldsymbol{X}^T\boldsymbol{y})^T\boldsymbol{X}^T\boldsymbol{y} < 0 \\ &\Leftrightarrow \frac{1}{N-1}\|\boldsymbol{X}^T\boldsymbol{y}\| < 0 \\ &\Leftrightarrow \|\boldsymbol{X}^T\boldsymbol{y}\| < 0 \\ &\Leftrightarrow \perp\end{aligned} \tag{3}$$

a contradiction.

(b) Let $\boldsymbol{v}_1, \boldsymbol{v}_2$ be two eigenvectors of $\boldsymbol{A}$ with distinct eigenvalues λ_1, λ_2 respectively. Now, consider

$$\begin{aligned}\lambda_1\boldsymbol{v}_1^T\boldsymbol{v}_2 &= \boldsymbol{A}\boldsymbol{v}_1^T\boldsymbol{v}_2 \\ &= \boldsymbol{A}\boldsymbol{v}_2^T\boldsymbol{v}_1 \quad (\boldsymbol{v}_1^T\boldsymbol{v}_2 \text{ is a scalar, so it is equal to its transpose}) \\ &= \lambda_2\boldsymbol{v}_2^T\boldsymbol{v}_1 \\ &= \lambda_2\boldsymbol{v}_1^T\boldsymbol{v}_2 \qquad\qquad\qquad\qquad \text{(taking transpose again)}\end{aligned} \tag{4}$$

This implies

$$\begin{aligned}&\lambda_1\boldsymbol{v}_1^T\boldsymbol{v}_2 - \lambda_2\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0 \\ &\Leftrightarrow (\lambda_1 - \lambda_2)\boldsymbol{v}_1^T\boldsymbol{v}_2 = 0 \\ &\Leftrightarrow \boldsymbol{v}_1^T\boldsymbol{v}_2 = 0 \qquad \text{(since eigenvalues are distinct)}\end{aligned} \tag{5}$$

Hence, the eigenvectors of a symmetric matrix are orthonormal.