3

$\boldsymbol{A} \in \mathbb{R}^{m\times n}$ where $m \leq n$.
Now, $\boldsymbol{P} = \boldsymbol{A}^T\boldsymbol{A} \in \mathbb{R}^{n\times n}$ and $\boldsymbol{Q} = \boldsymbol{A}\boldsymbol{A}^T \in \mathbb{R}^{m\times m}$

(a) Now for $\boldsymbol{y} \in \mathbb{R}^n$, we have

$$\begin{aligned} \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y} &= \boldsymbol{y}^T\boldsymbol{A}^T\boldsymbol{A}\boldsymbol{y} \\ &= (\boldsymbol{A}\boldsymbol{y})^T\boldsymbol{A}\boldsymbol{y} \\ &= \|\boldsymbol{A}\boldsymbol{y}\|_2^2 \\ \text{Hence } \boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y} &\geq 0 \qquad\qquad (\|\cdot\| \geq 0) \end{aligned} \tag{1}$$

Similarly, for $\boldsymbol{z} \in \mathbb{R}^m$, we have

$$\begin{aligned} \boldsymbol{z}^T\boldsymbol{Q}\boldsymbol{z} &= \boldsymbol{z}^T\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{z} \\ &= (\boldsymbol{A}^T\boldsymbol{z})^T\boldsymbol{A}\boldsymbol{z} \\ &= \|\boldsymbol{A}^T\boldsymbol{z}\|_2^2 \\ \text{Hence } \boldsymbol{z}^T\boldsymbol{Q}\boldsymbol{z} &\geq 0 \end{aligned} \tag{2}$$

Now, let $\boldsymbol{u} \in \mathbb{R}^n$ be an eigenvector of $\boldsymbol{P}$ with eigenvalue λ.

$$\begin{aligned} \lambda < 0 \Rightarrow \boldsymbol{u}^T\boldsymbol{P}\boldsymbol{u} &= \boldsymbol{u}^T(\boldsymbol{P}\boldsymbol{u}) \\ &= \boldsymbol{u}^T(\lambda\boldsymbol{u}) \\ &= \lambda\boldsymbol{u}^T\boldsymbol{u} \\ &= \lambda\|\boldsymbol{u}\|_2^2 \\ \text{Hence } \lambda < 0 \Rightarrow \boldsymbol{u}^T\boldsymbol{P}\boldsymbol{u} &< 0 \end{aligned} \tag{3}$$

But $\boldsymbol{y}^T\boldsymbol{P}\boldsymbol{y} \geq 0$ for any $\boldsymbol{y} \in \mathbb{R}^n$, hence the eigenvalues of $\boldsymbol{P}$ are non-negative.

Similarly, let $\boldsymbol{v} \in \mathbb{R}^m$ be an eigenvector of $\boldsymbol{Q}$ with eigenvalue μ.

$$\begin{aligned} \mu < 0 \Rightarrow \boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v} &= \boldsymbol{v}^T(\boldsymbol{Q}\boldsymbol{v}) \\ &= \boldsymbol{v}^T(\mu\boldsymbol{v}) \\ &= \mu\boldsymbol{v}^T\boldsymbol{v} \\ &= \mu\|\boldsymbol{v}\|_2^2 \\ \text{Hence } \mu < 0 \Rightarrow \boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v} &< 0 \end{aligned} \tag{4}$$

But $\boldsymbol{v}^T\boldsymbol{Q}\boldsymbol{v} \geq 0$ for any $\boldsymbol{y} \in \mathbb{R}^m$, hence the eigenvalues of $\boldsymbol{Q}$ are also non-negative.

(b) Here, $\boldsymbol{u}$ eigenvector of $\boldsymbol{P}$ with eigenvalue λ and $\boldsymbol{v}$ eigenvector of $\boldsymbol{Q}$ with eigenvalue μ. They have n and m elements respectively.

Now,

$$
\begin{aligned}
\boldsymbol{Q}(\boldsymbol{A}\boldsymbol{u}) &= \boldsymbol{A}\boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{u}) \\
&= \boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{A})\boldsymbol{u} \\
&= \boldsymbol{A}(\boldsymbol{P}\boldsymbol{u}) \\
&= \boldsymbol{A}\lambda\boldsymbol{u} \qquad\qquad (\boldsymbol{P}\boldsymbol{u} = \lambda\boldsymbol{u}) \\
\text{Hence } \boldsymbol{Q}(\boldsymbol{A}\boldsymbol{u}) &= \lambda(\boldsymbol{A}\boldsymbol{u})
\end{aligned}
\tag{5}
$$

Similarly,

$$
\begin{aligned}
\boldsymbol{P}(\boldsymbol{A}^T\boldsymbol{v}) &= \boldsymbol{A}^T\boldsymbol{A}(\boldsymbol{A}^T\boldsymbol{v}) \\
&= \boldsymbol{A}^T(\boldsymbol{A}\boldsymbol{A}^T)\boldsymbol{v} \\
&= \boldsymbol{A}^T(\boldsymbol{Q}\boldsymbol{v}) \\
&= \boldsymbol{A}^T\mu\boldsymbol{v} \qquad\qquad (\boldsymbol{Q}\boldsymbol{v} = \mu\boldsymbol{v}) \\
\text{Hence } \boldsymbol{P}(\boldsymbol{A}^T\boldsymbol{v}) &= \mu(\boldsymbol{A}^T\boldsymbol{v})
\end{aligned}
\tag{6}
$$

Hence, $\boldsymbol{A}\boldsymbol{u}$ is an eigenvector of $\boldsymbol{Q}$ with eigenvalue λ and $\boldsymbol{A}^T\boldsymbol{v}$ is an eigenvector of $\boldsymbol{P}$ with eigenvalue μ.

(c) Here, $\boldsymbol{v}_i$ eigenvector of $\boldsymbol{Q}$ with eigenvalue μ and $\boldsymbol{u}_i \triangleq \dfrac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$

Consider,

$$
\begin{aligned}
\boldsymbol{A}\boldsymbol{u}_i &= \boldsymbol{A}\frac{\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} \\
&= \frac{\boldsymbol{A}\boldsymbol{A}^T\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} \\
&= \frac{\boldsymbol{Q}\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2} \\
&= \frac{\mu\boldsymbol{v}_i}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}
\end{aligned}
\tag{7}
$$

Hence, there exists $\gamma_i = \dfrac{\mu}{\|\boldsymbol{A}^T\boldsymbol{v}_i\|_2}$ such that $\boldsymbol{A}\boldsymbol{u}_i = \gamma_i\boldsymbol{v}_i$. Also γ_i is non-negative as the eigenvalue of $\boldsymbol{Q}$ (μ) is non-negative as shown in (a).

(d) We know that $\boldsymbol{P}$ will have n eigenvectors and $\boldsymbol{Q}$ will have m eigenvectors. Since $m \leq n$, the following holds

$$
\boldsymbol{A}\boldsymbol{u}_i = \begin{cases} \gamma_i\boldsymbol{v}_i & \text{if } i \in \{1, 2, \ldots, m\} \text{ (from (c))} \\ 0_m & \text{if } i \in \{m+1, m+2, \ldots, n\} \end{cases}
\tag{8}
$$

Construct $\boldsymbol{U} = \begin{bmatrix}\boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_m\end{bmatrix}$, $\boldsymbol{V} = \begin{bmatrix}\boldsymbol{u}_1 & \boldsymbol{u}_2 & \cdots & \boldsymbol{u}_n\end{bmatrix}$ and $\boldsymbol{\Gamma}$ as a $m \times n$ rectangular diagonal matrix with diagonal entries γ_i as shown below

$$
\boldsymbol{\Gamma} = \begin{bmatrix}
\gamma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & \gamma_2 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & \gamma_m & 0 & \cdots & 0
\end{bmatrix}
\tag{9}
$$

Note that $\boldsymbol{U}$, $\boldsymbol{V}$ are orthogonal matrices as $\boldsymbol{u}_i^T \boldsymbol{u}_j = \delta_{ij}$, $\boldsymbol{v}_i^T \boldsymbol{v}_j = \delta_{ij}$ (proved in the footnote 1 of the homework sheet). Now, 8 can also be written as

$$\begin{aligned} \boldsymbol{A}\boldsymbol{V} &= \boldsymbol{U}\boldsymbol{\Gamma} \\ \boldsymbol{A} &= \boldsymbol{U}\boldsymbol{\Gamma}\boldsymbol{V}^T \quad (\boldsymbol{V}^{-1} = \boldsymbol{V}^T) \end{aligned} \tag{10}$$

Equation 10 gives us the singular value decomposition of any matrix $\boldsymbol{A}$, where $\boldsymbol{U}, \boldsymbol{V}$ are the corresponding orthogonal matrices and $\boldsymbol{\Gamma}$ corresponds to $\boldsymbol{\Sigma}$ matrix and γ_i indeed corresponds to singular values as they are also non-negative as proved in (c).