3

1 3a

The goal is to maximize f^TCf constrained to the condition that $f \perp e$ and $|f| = 1$
The Lagrangian for this optimization problem can be written as:

$$\mathscr{L} = f^TCf + \alpha_2(1 - |f|) + \beta(0 - f^Te)$$
$$= f^TCf + \alpha_2(1 - f^Tf) + \beta(0 - f^Te)$$

Differentiating the above equation w.r.t f, we get:

$$2Cf - 2\alpha_2(f) - \beta(e) = 0 \tag{1}$$

Left multiplying the equation by e^T:

$$2e^TCf - 2\alpha_2 e^Tf - \beta e^Te = 0$$

Since e is an eigenvector of C (say, with eigenvalue α_1), we have:

$$Ce = \alpha_1 e$$

or, transposing both sides:

$$e^TC = \alpha_1 e^T$$

Hence we obtain:

$$2\alpha_1 e^Tf - 2\alpha_2 e^Tf - \beta e^Te = 0$$

Clearly,$f \perp e \implies e^Tf = 0$ and $e^Te = 1$, hence:

$$\beta = 0$$

Equation (1) reduces to:

$$Cf = \alpha_2 f$$

Hence, we conclude that f is an eigenvector of C. In order to maximize f^TCf we need to maximize α_2. Because f must be orthogonal to e, it cannot be the eigenvector corresponding to the largest eigenvalue($\alpha_2 \neq \alpha_1$). As such, f must be an eigenvector corresponding to the second-largest eigenvalue.

2 3b

Let us now consider the expression $g^T C g$ where g is a vector perpendicular to both f and e. The Lagrangian for this optimization problem can be written as:

$$\begin{aligned}\mathscr{L} &= g^T C g + \alpha_3(1 - |g|) + \beta(0 - g^T e) + \gamma(0 - g^T f) \\ &= g^T C g + \alpha_3(1 - g^T g) + \beta(0 - g^T e) + \gamma(0 - g^T f))\end{aligned}$$

Differentiating the above equation w.r.t g, we get:

$$2Cg - 2\alpha_3(g) - \beta(e) - \gamma(f) = 0 \tag{2}$$

Left multiplying the equation by e^T:

$$2e^T C g - 2\alpha_3 e^T g - \beta e^T e - \gamma e^T f = 0$$

As seen before, $e^T C g = \alpha_1 e^T g = 0$. Also, $e^T g = 0$ and $e^T f = 0$. Hence:

$$\beta = 0$$

Similarly, we can left multiply (2) by f^T to obtain:

$$2f^T C g - 2\alpha_3 f^T g - \beta f^T e - \gamma f^T f = 0$$

Again, $f^T C g = \alpha_2 f^T g = 0$. Also, $f^T g = 0$ and $f^T e = 0$. Hence:

$$\gamma = 0$$

Thus, (2) reduces to:

$$Cg = \alpha_3 g$$

Hence, we conclude that g is an eigenvector of C. In order to maximize $g^T C g$ we need to maximize α_3. Because g must be orthogonal to e and f, it cannot be the eigenvector corresponding to the largest eigenvalue($\alpha_3 \neq \alpha_1$) or the second-largest eigenvalue($\alpha_3 \neq \alpha_2$). As such, f must be an eigenvector corresponding to the third-largest eigenvalue.
He can proceed inductively to prove the result for the j^{th} orthogonal vector.