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Let Ω be an open set in $\mathbb{R}^n$ for some n, and let $f\colon \Omega \to \mathbb{R}$ be a function such that $f \in C^2(\Omega)$ (i.e., f is twice-continuously differentiable).

Let

$$\begin{aligned} u &= x\cos\theta - y\sin\theta \\ v &= x\sin\theta + y\cos\theta \end{aligned}$$

be a rotation of the coordinate system by θ.

To show that the Laplacian is rotationally invariant, it suffices to show that $f_{xx} + f_{yy}$ is equal to $f_{uu} + f_{vv}$ for any arbitrary f.
We know that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$$

Substituting the expressions for $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial v}{\partial x}$, we get

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta$$

We need to differentiate the above expression w.r.t. x once more to get $\dfrac{\partial^2 f}{\partial x^2}$:

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\cos\theta + \frac{\partial f}{\partial v}\sin\theta\right) \\ &= (\cos\theta)\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right)}_{T_1} + (\sin\theta)\underbrace{\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right)}_{T_2} \end{aligned}$$

Now, we will calculate T_1 and T_2 separately:

$$\begin{aligned} T_1 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial u}\right) &= \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial x} \\ &= \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial x} \\ &= \frac{\partial^2 f}{\partial u^2}\cos\theta + \frac{\partial^2 f}{\partial v \partial u}\sin\theta \end{aligned}$$

$$T_2 = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial x}$$
$$= \frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial x}$$
$$= \frac{\partial^2 f}{\partial u \partial v}\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin\theta$$

Putting it all together, we get

$$\frac{\partial^2 f}{\partial x^2} = (\cos\theta)\left(\frac{\partial^2 f}{\partial u^2}\cos\theta + \frac{\partial^2 f}{\partial v \partial u}\sin\theta\right) + (\sin\theta)\left(\frac{\partial^2 f}{\partial u \partial v}\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin\theta\right)$$

Since our hypothesis is that f is a C^2 function, we can use Schwarz's theorem to simplify the above expression further:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial u^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\theta \tag{1}$$

Now, we will perform the same steps to compute $\frac{\partial^2 f}{\partial y^2}$, starting with the expression for $\frac{\partial f}{\partial y}$:

$$\frac{\partial f}{\partial y} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta$$

From this, we get

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$
$$= \frac{\partial}{\partial y}\left(-\frac{\partial f}{\partial u}\sin\theta + \frac{\partial f}{\partial v}\cos\theta\right)$$
$$= (-\sin\theta)\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right)}_{T_3} + (\cos\theta)\underbrace{\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right)}_{T_4}$$

Now, we will calculate T_3 and T_4 separately:

$$T_3 = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial u}\right)\frac{\partial v}{\partial y}$$
$$= \frac{\partial^2 f}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v \partial u}\frac{\partial v}{\partial y}$$
$$= -\frac{\partial^2 f}{\partial u^2}\sin\theta + \frac{\partial^2 f}{\partial v \partial u}\cos\theta$$

$$T_4 = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial f}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial f}{\partial v}\right)\frac{\partial v}{\partial y}$$
$$= \frac{\partial^2 f}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 f}{\partial v^2}\frac{\partial v}{\partial y}$$
$$= -\frac{\partial^2 f}{\partial u \partial v}\sin\theta + \frac{\partial^2 f}{\partial v^2}\cos\theta$$

Hence,

$$\frac{\partial^2 f}{\partial y^2} = -\sin\theta\left(-\frac{\partial^2 f}{\partial u^2}\sin\theta + \frac{\partial^2 f}{\partial v \partial u}\cos\theta\right) + \cos\theta\left(-\frac{\partial^2 f}{\partial u \partial v}\sin\theta + \frac{\partial^2 f}{\partial v^2}\cos\theta\right)$$

By using Schwarz's Theorem again, we can simplify the above expression to

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial u^2}\sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial v^2}\cos^2\theta \tag{2}$$

By adding (1) and (2), we get

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \left(\frac{\partial^2 f}{\partial u^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial v^2}\sin^2\theta\right) \\
&\quad + \left(\frac{\partial^2 f}{\partial u^2}\sin^2\theta - 2\frac{\partial^2 f}{\partial u \partial v}\sin\theta\cos\theta + \frac{\partial^2 f}{\partial v^2}\cos^2\theta\right) \\
&= \frac{\partial^2 f}{\partial u^2}\left(\cos^2\theta + \sin^2\theta\right) + \frac{\partial^2 f}{\partial v^2}\left(\sin^2\theta + \cos^2\theta\right) \\
&= \frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2} \quad \square
\end{aligned}$$