Sphere and Cylinder simulations

(Dated: November 23, 2021)

Abstract: We explore the hydrodynamic interactions of motors in curved fluidic membranes....

I. EXAMPLE ON THE FLAT MEMBRANE

Notations All boldface letters will denote vectors. In the following expressions, we will set the constants as

$$\eta_{2D} = 1, \;\; \kappa = 1 \tag{1}$$

and the orientation of dipoles, denoted by α, is always measured wrt $\hat{x}$ in the anticlockwise sense. The dynamical equations for motion of force dipoles on the xy plane (at $z = 0$) is given by

Dynamical Equations :

$$\dot{x}_i = \sum_{j\neq i}^{N} v_x^{dipole}[x_i, y_i, \alpha_i, x_j, y_j, \alpha_j], \qquad \dot{y}_i = \sum_{j\neq i}^{N} v_y^{dipole}[x_i, y_i, \alpha_i, x_j, y_j, \alpha_j]$$
$$\dot{\alpha}_i = \sum_{j\neq i}^{N} \underbrace{\frac{1}{2}\left(\nabla_i^{flat} \times \boldsymbol{v}^{dipole}[x_i, y_i, \alpha_i, x_j, y_j, \alpha_j]\right).\hat{z}_i}_{\text{Vorticity induced rotation}} \tag{2}$$

where $\nabla_i^{flat} \times \boldsymbol{v}$ is the usual curl of $\boldsymbol{v}$ in cartesian (x,y,z) coordinates and

$$\boldsymbol{v}^{dipole} \equiv v_x^{dipole} \quad \hat{x} \quad + \quad v_y^{dipole} \quad \hat{y} \tag{3}$$

$$v_x^{dipole}[x, y, \alpha, x_0, y_0, \alpha_0] = -\frac{\kappa}{4\pi\eta_{2D} r}\left(1 - 2\left(\frac{x - x_0}{r}\cos\alpha_0 + \frac{y - y_0}{r}\sin\alpha_0\right)^2\right)\frac{x - x_0}{r}$$
$$v_y^{dipole}[x, y, \alpha, x_0, y_0, \alpha_0] = -\frac{\kappa}{4\pi\eta_{2D} r}\left(1 - 2\left(\frac{x - x_0}{r}\cos\alpha_0 + \frac{y - y_0}{r}\sin\alpha_0\right)^2\right)\frac{y - y_0}{r} \tag{4}$$

where $r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$.
For two dipoles we will get nonlinear oscillatory dynamics like this

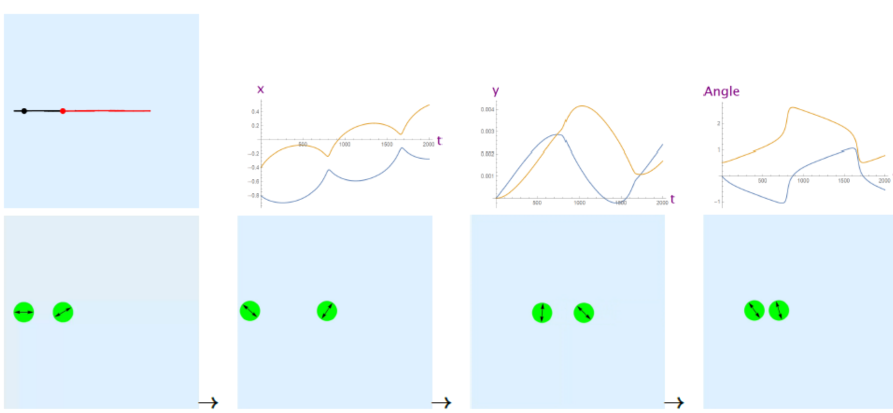

FIG. 1. Dynamics of 2 dipoles in a flat membrane. Initially we start with both dipoles along x axis, with $(x_1, y_1, \alpha_1) = (0, 0, 0)$ and $(x_2, y_2, \alpha_2) = (0.4, 0, \pi/6)$. In the top row starting from left, we show the entire trajectory of the two dipoles wrt time, followed by the evolution of (x_1, x_2) , (y_1, y_2) and (α_1, α_2) wrt time. In the second row, we show four snapshots of the simulation. The size of the dipoles for now is arbitrary (we choose it appropriately so that there is overlap of the dipoles while viewing the simulation. In the actual case, the size will be fixed by a separate physics which we will include later.

II. FORCE DIPOLES ON SPHERE

Notations, constants and conventions:
Let us consider a sphere of radius R. The location of the dipoles on the sphere surface will be given by the usual latitude and longitude ie. (θ, ϕ) and the orientation of the dipole will be denoted by α. Note that we will measure α wrt $\hat{\phi}$ in the local tangent plane where the dipole is located.

In the expressions below, several constants will appear. We will set those constants as

$$R = 1,\ \eta_{2D} = 0.1,\ \eta_+ = \eta_- = 1, \kappa = 1 \tag{5}$$

Also some derived constants are $\lambda_+ = \frac{\eta_{2D}}{\eta_+}$ and $\lambda_- = \frac{\eta_{2D}}{\eta_-}$.
We will also adopt the convention where gradient of a function is treated as a column vector and the vectors like $\hat{\boldsymbol{T}}$ is treated as a row vector.

Now we write the dynamical equations for motion of force dipoles on the sphere.

Dynamical Equations :

$$R\ \dot{\theta}_i = \textstyle\sum_{j\neq i}^{N} v_\theta^{dipole}[\theta_i, \phi_i, \alpha_i, \theta_j, \phi_j, \alpha_j], \qquad R\sin\theta_i\ \dot{\phi}_i = \textstyle\sum_{j\neq i}^{N} v_\phi^{dipole}[\theta_i, \phi_i, \alpha_i, \theta_j, \phi_j, \alpha_j]$$
$$\dot{\alpha}_i = \textstyle\sum_{j\neq i}^{N} \underbrace{\frac{1}{2}\left(\nabla_i^{sp} \times \boldsymbol{v}^{dipole}[\theta_i, \phi_i, \alpha_i, \theta_j, \phi_j, \alpha_j]\right).\hat{r}_i}_{\text{Vorticity induced rotation}} + \underbrace{\frac{1}{R}\cot\theta_i\ v_\phi^{dipole}[\theta_i, \phi_i, \alpha_i, \theta_j, \phi_j, \alpha_j]}_{\text{Curvature induced rotation}} \tag{6}$$

where $\nabla^{sp} \times \boldsymbol{v}$ is the curl in spherical co-ordinates and

$$\boldsymbol{v}^{dipole} \equiv v_\theta^{dipole}\quad \hat{\theta}\quad +\quad v_\phi^{dipole}\quad \hat{\phi} \tag{7}$$

$$\boldsymbol{v}^{dipole}[\theta, \phi, \alpha, \theta_0, \phi_0, \alpha_0] = \frac{\kappa}{4\pi\eta_{2D}R}\tilde{\boldsymbol{\nabla}}_{\theta,\phi}\left(\left[\hat{\boldsymbol{T}}\cdot\boldsymbol{\nabla}_{\theta_0,\phi_0}\right]\left(\hat{\boldsymbol{T}}\cdot\tilde{\boldsymbol{\nabla}}_{\theta_0,\phi_0}S[\gamma]\right) + (\hat{\boldsymbol{T}}\cdot\boldsymbol{M})\cdot\tilde{\boldsymbol{\nabla}}_{\theta_0,\phi_0}S[\gamma]\right) \tag{8}$$

where $\tilde{\boldsymbol{\nabla}}_{\theta,\phi}$ is the twisted gradient operator

$$\tilde{\boldsymbol{\nabla}}_{\theta,\phi} = \begin{bmatrix}\csc\theta\ \partial_\phi \\ -\partial_\theta\end{bmatrix} \tag{9}$$

and similar definition for $\tilde{\boldsymbol{\nabla}}_{\theta_0,\phi_0}$

$$\tilde{\boldsymbol{\nabla}}_{\theta_0,\phi_0} = \begin{bmatrix}\csc\theta_0\ \partial_{\phi_0} \\ -\partial_{\theta_0}\end{bmatrix} \tag{10}$$

Meanwhile gradients without the tildes $\boldsymbol{\nabla}_{\theta_0,\phi_0}$ are the usual ones

$$\boldsymbol{\nabla}_{\theta_0,\phi_0} = \begin{bmatrix}\partial_{\theta_0} \\ \csc\theta_0\ \partial_{\phi_0}\end{bmatrix} \tag{11}$$

The row vector $\hat{\boldsymbol{T}}$ dictates the orientation of the dipole situated at (θ_0, ϕ_0) denoted by α_0

$$\hat{\boldsymbol{T}} := (\sin\alpha_0, \cos\alpha_0) \tag{12}$$

The matrix $\boldsymbol{M}$ is given by

$$\boldsymbol{M} = \frac{1}{R}\begin{pmatrix} 0 & -\cot\theta_0\cos\alpha_0 \\ \cot\theta_0\cos\alpha_0 & 0\end{pmatrix} \tag{13}$$

and finally

$$S[\gamma] := \sum_{l=1}^{\infty}\frac{2l+1}{s_l l(l+1)}P_l[\cos\gamma] \tag{14}$$

where $\cos\gamma = \sin\theta\sin\theta_0\cos(\phi-\phi_0)+\cos\theta\cos\theta_0$ and

$$s_l = l(l+1) - 2 + \frac{R}{\lambda_-}(l-1) + \frac{R}{\lambda_+}(l+2) \tag{15}$$

The plots of the vector field defined in Eq.(8) in (θ,ϕ) plane is expected to be similar to the following figure, which can be used as a test while numerically storing the Legendre Polynomials

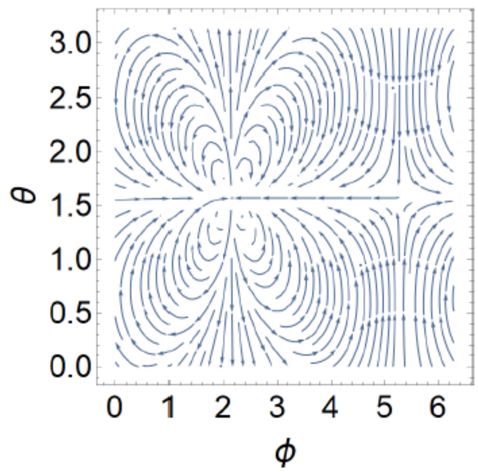

FIG. 2. Plot of the flow created by a force dipole situated at $\theta_0 = \pi/2, \phi_0 = 2.2$.

III. FORCE DIPOLES ON CYLINDER

Notations, constants and conventions:
Let us consider a cylinder of radius R, with its axis along $\hat{z}$ and infinitely extended along the z-axis. The location of the dipoles on the cylinder surface is given by the usual (θ, z) and the orientations of the dipole will be denoted by α, measured wrt $\hat{\theta}$ in the local tangent plane where the dipole is located.

In the expressions below, several constants will appear. We will set those constants as

$$R = 1,\ \eta_{2D} = 0.1,\ \eta_+ = \eta_- = 1, \kappa = 1 \tag{16}$$

Also some derived constants are $\lambda_+ = \frac{\eta_{2D}}{\eta_+}$ and $\lambda_- = \frac{\eta_{2D}}{\eta_-}$.
We will also adopt the shorthand notation Λ for the Fourier modes on the cylinder $(n, q = k/R)$ and also define

$$\sum_{n=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dk}{R} \equiv \int d\Lambda.$$

Now we write the dynamical equations for motion of force dipoles on the cylinder:

Dynamical Equations :

$$R\ \dot{\theta}_i = \textstyle\sum_{j\neq i}^{N} \boldsymbol{v}_\theta^{dipole}[\theta_i, z_i, \alpha_i, \theta_j, z_j, \alpha_j], \qquad \dot{z}_i = \textstyle\sum_{j\neq i}^{N} \boldsymbol{v}_z^{dipole}[\theta_i, z_i, \alpha_i, \theta_j, z_j, \alpha_j]$$

$$\dot{\alpha}_i = \textstyle\sum_{j\neq i}^{N} \underbrace{\frac{1}{2}\left(\nabla_i^{cyl} \times \boldsymbol{v}^{dipole}[\theta_i, z_i, \alpha_i, \theta_j, z_j, \alpha_j]\right).\hat{r}_i}_{\text{Vorticity induced rotation}} \tag{17}$$

where $\nabla^{cyl} \times \boldsymbol{v}$ denotes the curl of $\boldsymbol{v}$ in cylindrical co-ordinates, where

$$\boldsymbol{v}^{dipole} \equiv v_\theta^{dipole} \quad \hat{\theta} \quad + \quad v_z^{dipole} \quad \hat{z} \tag{18}$$

The force dipole flow is given by

$$\begin{aligned} v_\theta^{dipole}[\theta, z, \alpha, \theta_0, z_0, \alpha_0] &= \kappa \left(\frac{\cos\alpha_0}{R} \ \partial_{\theta_0} + \sin\alpha_0 \ \partial_{z_0} \right) (G_{\theta\theta} \cos\alpha_0 + G_{\theta z} \sin\alpha_0) \\ v_z^{dipole}[\theta, z, \alpha, \theta_0, z_0, \alpha_0] &= \kappa \left(\frac{\cos\alpha_0}{R} \ \partial_{\theta_0} + \sin\alpha_0 \ \partial_{z_0} \right) (G_{z\theta} \cos\alpha_0 + G_{zz} \sin\alpha_0) \end{aligned} \tag{19}$$

where κ is the dipole strength and

$$\begin{aligned} G_{\theta\theta} &= \tfrac{1}{4\pi^2 \eta_{2D}} \int R \ d\Lambda \ \tfrac{k^2}{c_n(k)} e^{i\left(n(\theta-\theta_0)+\frac{k}{R}(z-z_0)\right)}, \quad G_{\theta z} = -\tfrac{1}{4\pi^2 \eta_{2D}} \int R \ d\Lambda \ \tfrac{k \ n}{c_n(k)} e^{i\left(n(\theta-\theta_0)+\frac{k}{R}(z-z_0)\right)} \\ G_{z\theta} &= -\tfrac{1}{4\pi^2 \eta_{2D}} \int R \ d\Lambda \ \tfrac{k \ n}{c_n(k)} e^{i\left(n(\theta-\theta_0)+\frac{k}{R}(z-z_0)\right)}, \quad G_{zz} = \tfrac{1}{4\pi^2 \eta_{2D}} \int R \ d\Lambda \ \tfrac{n^2}{c_n(k)} e^{i\left(n(\theta-\theta_0)+\frac{k}{R}(z-z_0)\right)} \end{aligned} \tag{20}$$

where the function $c_n(k)$ appearing in the denominator is given by

$$c_n(k) = (n^2 + k^2)^2 - \tfrac{R}{\lambda_-} C^-(\Lambda) + \tfrac{R}{\lambda_+} C^+(\Lambda) \tag{21}$$

The functions $C^\pm$ are given as follows :

$$C^\pm[n, k] = \frac{2n^2 \rho_\pm^3 + (n^2+k^2)^2 \rho_\pm^2 + 2\rho_\pm (k^4 - n^4) - (k^2+n^2)^3}{\rho_\pm k^2 - (\rho_\pm - n)(\rho_\pm + n)(\rho_\pm + 2)} \tag{22}$$

with

$$\rho_+[n, k] = \frac{|k| \left. \frac{dK_n[u]}{du} \right|_{u=|k|}}{K_n[|k|]}, \qquad \rho_-[n, k] = \frac{|k| \left. \frac{dI_n[u]}{du} \right|_{u=|k|}}{I_n[|k|]} \tag{23}$$

where K_n and I_n are modified Bessel functions of order n of first and second kind respectively. Note the absolute values in the argument.
The plots of the vector field defined in Eq.(19) in (θ, z) plane is expected to be similar to the following figures, which can be used as a test while numerically storing the Bessel functions:

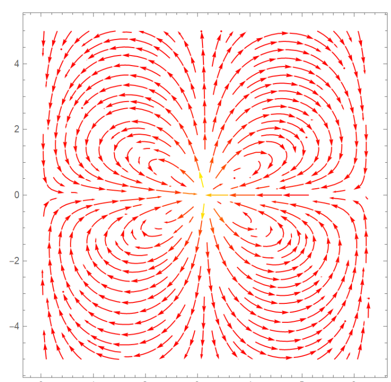

FIG. 3. Plot of the flow created by a force dipole situated at $\theta_0 = \pi, z_0 = 0$.

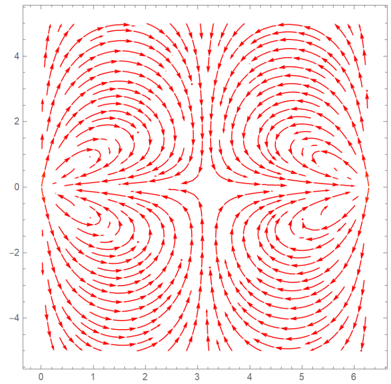

FIG. 4. Plot of the flow created by a force dipole situated at $\theta_0 = 0, z_0 = 0$.