Linear Reference Tracking MPC

Timothy Werner

***Abstract*— This paper briefly introduces the implemented Linear Reference Tracking MPC scheme.**

I. The Tracking Problem

We consider linear discrete time systems, sampled at rate δ of the form

$$\mathbf{x}_{k+1} = A\mathbf{x}_k + B\mathbf{u}_k \tag{1a}$$
$$\mathbf{y}_k = C\mathbf{x}_k, \tag{1b}$$

where $\mathbf{x} \in \mathcal{X} \subseteq \mathbb{R}^n$ are the states, $\mathbf{u} \in \mathcal{U} \subseteq \mathbb{R}^p$ are the inputs and $\mathbf{y} \in \mathcal{Y} \subseteq \mathbb{R}^q$ are the outputs of the system. The goal is to steer the output along a known reference trajectory $\mathbf{r} \in \mathcal{Y}$, i.e. to render the reference error

$$\mathbf{e}_k := \mathbf{y}_k - \mathbf{r}_k \triangleq C\mathbf{x}_k - \mathbf{r}_k \to 0,$$

as $k \to \infty$. Note that we consider the state-space, the input-space and the output-space to be constrained. We will also consider rate constraints because all physical systems are rate-constrained. This extension also appears natural for two more reasons which will be discussed briefly in the following. To this end, we define the rate as

$$\Delta\mathbf{u}_k := \mathbf{u}_k - \mathbf{u}_{k-1}. \tag{2}$$

II. The Linear Tracking MPC

It appears natural to consider the tracking error as defined above as the objective to be minimized. We can thus formulate the cost

$$\mathcal{J}_j := \sum_{k=0}^{N-1} \|\mathbf{e}_{j+k}\|_Q^2 + \|\Delta\mathbf{u}_{j+k}\|_{R_\Delta}^2 \tag{3}$$

where N is the discrete MPC horizon over which the optimal input sequence is to be found and we denoted $\|\mathbf{e}\|_Q^2 := \mathbf{e}^\top Q\mathbf{e}$. Note that a rate-regularization was introduced to the cost, which ensures a smooth tracking, which is one of the two main reasons to consider input rates.

III. Implementation

A. The Cost Function

To enforce stability on the controller, we introduce a terminal constraint that drives the predicted trajectory onto the desired trajectory $\mathbf{r}$ near the end of the prediction horizon. We further introduce a direct penalty on the inputs $\mathbf{u}$ due to a state augmentation to be introduced later. We have thus

$$\begin{aligned}\mathcal{J}_j := \sum_{k=0}^{N-1} \|\mathbf{e}_{j+k}\|_Q^2 + \|\Delta\mathbf{u}_{j+k}\|_{R_\Delta}^2 + \\ \|\mathbf{u}_{j+k}\|_R^2 + \|\mathbf{e}_{j+N}\|_P^2\end{aligned} \tag{4}$$

If we further investigate the terms, omitting the constant terms w.r.t to $\mathbf{x}$ and $\mathbf{u}$, we find

$$\begin{aligned}\mathcal{J}_j := \sum_{k=0}^{N-1} & \mathbf{x}_{j+k}^\top C^\top QC\mathbf{x}_{j+k} \\ & + \mathbf{u}_{j+k}^\top R\mathbf{u}_{j+k} \\ & + \Delta\mathbf{u}_{j+k}^\top R_\Delta \Delta\mathbf{u}_{j+k} \\ & - 2\mathbf{r}_{j+k}^\top QC\mathbf{x}_{j+k} \\ & + \mathbf{x}_{j+N}^\top C^\top PC\mathbf{x}_{j+N} \\ & - 2\mathbf{r}_{j+N}^\top PC\mathbf{x}_{j+N}\end{aligned} \tag{5}$$

B. State Augmentation

We define the new state $\tilde{\mathbf{x}}_k := \begin{bmatrix}\mathbf{x}_k^\top & \mathbf{u}_{k-1}^\top\end{bmatrix}^\top$ and combining (1a) and (2), we find the new augmented system

$$\tilde{\mathbf{x}}_{k+1} = \begin{bmatrix}A & B \\ & I\end{bmatrix}\tilde{\mathbf{x}}_k + \begin{bmatrix}B \\ I\end{bmatrix}\Delta\mathbf{u}_k, \tag{6}$$

which we will refer to as

$$\tilde{\mathbf{x}}_{k+1} = \tilde{A}\tilde{\mathbf{x}}_k + \tilde{B}\Delta\mathbf{u}_k.$$

This state augmentation was mainly performed to ensure zero steady state error independent of the weights chosen for error penalty and input penalty. With a mere error and input penalty, the steady state error would be directly dependent on the weights chosen for either term. Imagine to this end a high weight on the inputs and a low weight on the reference error. The controller would not put a lot of effort in tracking the reference, since in would cost a lot of input. A high weight on tracking error and a low weight on the inputs on the other hand would lead to a bang-bang behavior which is in general not a desired control strategy. Choosing the reference error and the rates to be minimized on the other hand resolves in a well posed problem, where the weights on error and input rates shift the pareto-solution between aggressive and smooth tracking. The drawback of this method is that the dimension of the problem is raised from $(N+1)n + Np$ to $(N+1)(n+p) + Np$. Rewriting the cost function in terms of (6), we find

$$\begin{aligned}\mathcal{J}_j := \sum_{k=0}^{N-1} & \tilde{\mathbf{x}}_{j+k}^\top \tilde{Q}\tilde{\mathbf{x}}_{j+k} \\ & + \Delta\mathbf{u}_{j+k}^\top R_\Delta \Delta\mathbf{u}_{j+k} \\ & - C_Q^\top \tilde{\mathbf{x}}_{j+k} \\ & + \tilde{\mathbf{x}}_{j+N}^\top \tilde{P}\tilde{\mathbf{x}}_{j+N} \\ & - C_P^\top \tilde{\mathbf{x}}_{j+N},\end{aligned} \tag{7}$$

with

$$\tilde{Q} := \begin{bmatrix} C^\top Q C & \\ & R \end{bmatrix},$$

$$\tilde{P} := \begin{bmatrix} C^\top P C & \\ & R \end{bmatrix},$$

$$C_Q^\top := \begin{bmatrix} 2\mathbf{r}_k^\top Q C & 0 \end{bmatrix},$$

and

$$C_P^\top := \begin{bmatrix} 2\mathbf{r}_k^\top P C & 0 \end{bmatrix}$$

C. The Optimization Problem

With this defined, we can formulate the QP to be solved at time instance $t = j$.

$$\begin{aligned}
\text{minimize} \quad & \mathcal{J}_j && \text{(8a)}\\
\text{subject to} \quad & \tilde{\mathbf{x}}_0^\top = \begin{bmatrix} \mathbf{x}_j & \left(\mathbf{u}_{0|j-1}^\top\right)^* \end{bmatrix}^\top && \text{(8b)}\\
& \tilde{\mathbf{x}}_{j+k+1} = \tilde{A}\tilde{\mathbf{x}}_{j+k} + \tilde{B}\Delta\mathbf{u}_{j+k} && \text{(8c)}\\
& \underline{\mathbf{x}} \leq \mathbf{x}_{j+k} \leq \overline{\mathbf{x}} && \text{(8d)}\\
& \underline{\mathbf{u}} \leq \mathbf{u}_{j+k} \leq \overline{\mathbf{u}} && \text{(8e)}\\
& \underline{\Delta\mathbf{u}} \leq \Delta\mathbf{u}_{j+k} \leq \overline{\Delta\mathbf{u}} && \text{(8f)}
\end{aligned}$$

The solution of this yields the optimal input sequence $\mathbf{\Omega}_j^*$ and the corresponding predicted state trajectory $\mathbf{\Xi}_j^*$ at time instance j

$$\mathbf{\Omega}_j^* := \left\{ \mathbf{u}_{0|j}^* \quad \mathbf{u}_{1|j}^* \quad \cdots \quad \mathbf{u}_{N-1|j}^* \right\} \tag{9}$$

$$\mathbf{\Xi}_j^* := \left\{ \mathbf{x}_{0|j}^* \quad \mathbf{x}_{1|j}^* \quad \cdots \quad \mathbf{x}_{N|j}^* \right\} \tag{10}$$

$$\tag{11}$$

where $\{k|j\}$ indicates the value at MPC stage $k \in [0, N]$ at time instance $t = j$. The controller will then output $\mathbf{u}(\mathbf{x}_j, j) = \mathbf{u}_{0|j}^*$ and apply it to (1a). Note that (8b) depends on the first stage of the optimal input sequence of the previous time step $\left(\mathbf{u}_{0|j-1}^\top\right)^*$. This ensures satisfaction of the rate constraints of the resulting closed-loop input sequence

$$\mathbf{\Omega}_j := \left\{ \mathbf{u}_{0|0}^* \quad \mathbf{u}_{1|\delta}^* \quad \cdots \right\}. \tag{12}$$