Kap 5. Pfadintegrale

Alternative Form der QM parallel zu

- Matrizenmechanik
- Wellenmechanik

Beispiel: 1 dim System
Impulsoperator

$$[\hat{q}, \hat{p}] = i\hbar$$

Ortsbasis: $\hat{|}q\rangle = q|q\rangle$ Schrödingerbild: $|\alpha t\rangle_S = \int dq|q\rangle \underbrace{\langle q|e^{-i\hat{H}t/\hbar}}_{\langle qt|} \underbrace{|\alpha, 0\rangle}_{|\alpha\rangle_H}$

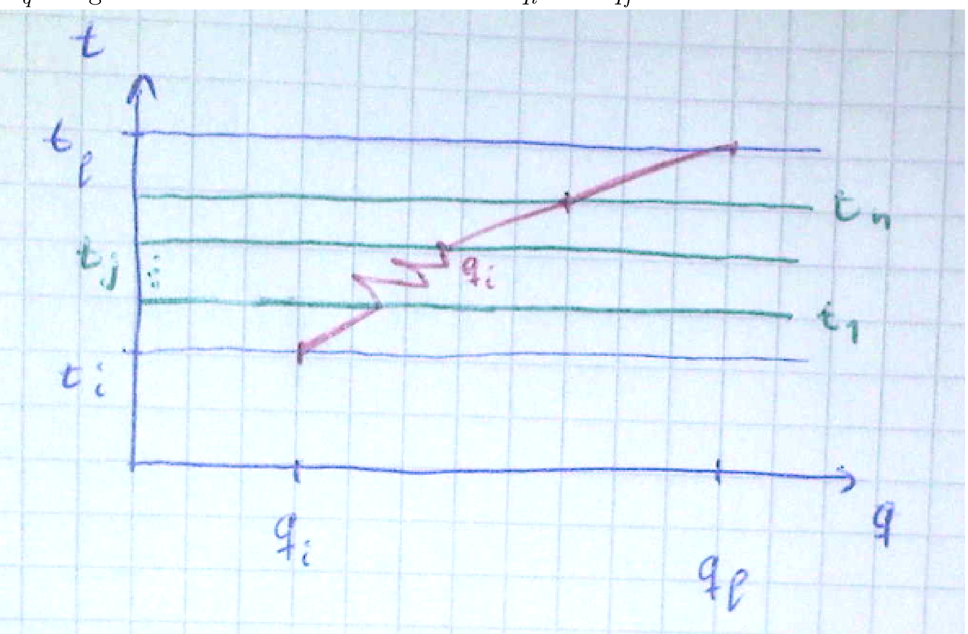
Def: $|qt\rangle = e^{+i\hat{H}t/\hbar}|q\rangle$
Wellenfkt: $\psi(q,t) = \langle q|\alpha t\rangle_S$

$$\psi(q_f, t_f) = \langle q_f t_f|\alpha\rangle_H = \int dq_i \underbrace{\langle q_f t_f|q_i t_i\rangle}_{K(q_f,t_f;q_i t_i} \underbrace{\langle q_i t_i|\alpha\rangle_H}_{\psi(q_i,t_i)}$$

K = 'Propagator' = Zeitenintegral operator in Ortsbasis
Feynman:

$$K(q_f t_f; q_i t_i) = \int \mathcal{D}_q exp(\frac{i}{\hbar}\int_{t_i}^{t_f} L(q,\dot{q})dt)\Big|_{q(t_i)=q_i;q(t_f)=q_f}$$

D_q Integral über alle Klassischen Pfade von q_i nach q_f



$t_f - t_i = \tau(n+1)$

$$\langle q_f t_f|q_i t_i\rangle = \int_{-\infty}^{\infty} dq_i dq_2 ... dq_n \langle \overbrace{q_f}^{q_{n+1}} \overbrace{t_f}^{t_{n+1}} |q_n t_n\rangle\langle q_n t_n|q_{n-1}t_{n-1}\rangle ... \langle q_{j+1}t_{j+1}|q_j t_j\rangle ... \langle q_1 t_1| \underbrace{q_i}_{q_o} \underbrace{t_i}_{t_o}\rangle$$

$$\langle q_{j+1} t_{j+1} \rangle = \langle q_{j+1}| \underbrace{e^{-i\hbar t_{j+1}/\hbar} e^{i\hbar t_{j+1}/\hbar}}_{e^{-i\hbar \hat{H}(t_{j+1}-t_j)/\hbar} = e^{-i\hat{H}\tau/\hbar}} |q_j\rangle \approx 1 - i\tau/\hbar \hat{H} + ...$$

$$= \underbrace{\delta(q_{j+1} - q_j)}_{\int_{-\infty}^{\infty} \frac{dp}{2\pi\hbar} e^{ip(q_{j+1}-q_j)/\hbar}} -$$

Annahme $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$

$\langle q_{j+1}|\frac{\hat{p}^2}{2m}|q_j\rangle = \int dp' dp \quad \underbrace{\langle q_{j+1}|p'\rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipq_{j+1}/\hbar}} \quad \langle p'|\frac{\hat{p}^2}{2m}|p\rangle p \quad \underbrace{\langle p|q_j\rangle}_{\frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}}$

$$= \int \frac{dp}{2\pi\hbar} e^{i\frac{p}{\hbar}(q_{j+1}-q_j} \frac{p^2}{2m}$$

Normierung der $|p\rangle$: $\langle p|p'\rangle \delta(p-p')$; $\langle p|q\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-ipq/\hbar}$

$$\langle q_{j+1}|V(\hat{q})|q_j\rangle = V(q_j)\delta(q_{j+1}-q_j) = \int \frac{dp}{2\pi\hbar} e^{ip/\hbar(q_{j+1}-q_j)} V(q_j)$$

$$\langle q_{j+1}|e^{-i\hat{H}\tau/\hbar}|q_j\rangle = \int \frac{dp}{2\pi\hbar} e^{ip(q_{j+1}-q_j)/\hbar} \overbrace{[1 - i\frac{\tau}{\hbar}(\frac{p^2}{2m} + V(q_i)) + ...]}^{e^{-iH(p,q_i)\tau/\hbar}}$$

Für Propagator:

$$\langle q_f t_f | q_i t_i \rangle = \lim_{n\to\infty} \int ... \int \prod_{j=1}^{n} (dq_j \frac{dp_j}{2\pi\hbar}) \frac{dp_o}{2\pi\hbar} exp(\frac{i}{\hbar} \sum_{j=0}^{n} [p_j(q_{j+1}-q_j) - \tau H(p_j, q_j)])$$

$$= \int D_p D_q exp(\frac{i}{\hbar} \int_{t_i}^{t_f} dt(p(t)\dot{q}(t) - H(p(t), q(t)))$$

Sei $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q})$

$$\langle q_f t_f | q_i t_i \rangle = \lim_{nto\infty} \int ... \int \prod_j dq_j \frac{dp_j}{2\pi\hbar} exp(i \sum_{j=0}^{\infty} (\underbrace{-\tau \frac{p_j^2}{2m} + \overbrace{p_j(q_{j+1}-q_j)/\hbar}^{2\sqrt{\frac{\tau}{2m\hbar}} p_j(q_{j+1}-q_j)/2\sqrt{\frac{\tau}{2m\hbar}} \frac{1}{\hbar}}}_{-(p_j\sqrt{\frac{\tau}{2m\hbar}} - (q_{j+1}-q_j)\sqrt{\frac{m}{2\tau\hbar}})^2 + \frac{m}{2\tau\hbar}(q_{j+1}-q_j)^2} - V(q_j)\frac{\tau}{\hbar})$$

$$= \lim_{nto\infty} \int \underbrace{\prod_{k=0}^{n} \frac{dp_k}{2\pi\hbar} e^{-\frac{i}{\hbar}\frac{\tau\hbar}{2m} p_k^2/\hbar^2}}_{\sqrt{\frac{2in\pi}{i\tau\hbar(2\pi)^2}}^{n+1} = \sqrt{\frac{m}{i2\pi\hbar\tau}}^{n+1}} \prod_{j=1}^{n} dq_j exp(\underbrace{\frac{i}{\hbar} \sum_{j=0}^{n} \tau(\frac{m}{2}(\frac{q_{j+1}-q_j}{\tau})^2 - V(q_j)))}_{\to \frac{i}{\hbar}\int_{t_i}^{t_f} dt(\frac{m}{2}(\dot{q}(t)^2 - V(q))}$$

mit $\int_{-\infty}^{\infty} dx e^{-\alpha x^2} = \sqrt{\frac{\pi}{\alpha}}$

$$\langle q_f t_f | q_i t_i \rangle = \lim_{nto\infty} \int \underbrace{(\frac{m}{2\pi i\tau\hbar})^{n+1} \prod_{j=1}^{n} dq_j}_{\mathcal{D}_q} e^{\frac{i}{\hbar}\int_{t_i}^{t_f} (\underbrace{\frac{m}{2}\dot{q}^2 - V(q)}_{L(q,\dot{q})})dt}$$

$$= \int \mathcal{D}_q e^{\frac{i}{\hbar}\int_{t_i}^{t_f} dt L(q,\dot{q})} \Big|_{q(t_i)=q_i; q(t_f)=q_f}$$

Klassisches freies Teilchen

$$L = \frac{1}{2}mv^2$$

2 Pfade sind z.B.

1. $x_i(t) = vt \; v = 1\frac{cm}{sec}$
2. $x_2(t) = gt^2 \; g = \frac{cm}{sec}$

Randbed. $t_i = 0, x_i = 0, t_f = 1sec, x_f = 1cm$

	$m = 1g$	$m = m_e$
$x_1 : S_1 = \frac{m}{2}v^2t_f$	$4.7 * 10^26\hbar$	$0.43\hbar$
$x_2 : S_2 = \frac{2}{3}mg^2t_f^3 = \frac{4}{3}S_1$		
$S_2 - S_1$	$1.6 * 10^26\hbar$	$0.14\hbar$

In $\int \mathcal{D}_q e^{\frac{i}{\hbar}S}$ extra $\frac{1.6\cdot 10^{26}}{2\pi} \approx 10^{25}$ Oszillationen gegen S_1
Oszillation dämpfen Pfade mit $S(q) >> S_{min}$ Wichtig sind extremale Pfade mit $\delta S = 0 \Leftrightarrow$ euler-Lagrange