

Local variable selection and coefficient estimation for spatially-varying-coefficients models in the context of geographically-weighted regression using regularization

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1. Introduction

Varying-coefficients regression (VCR) (Hastie and Tibshirani, 1993) is a technique used to model non-stationary regression processes. Whereas the coefficients in ordinary least squares (OLS) regression are scalar constants, the coefficients in a VCR equation are functions - often *smooth* functions - of some effect-modifying parameter. Where the effect-modifying parameter s represents spatial location, a VCR model implies that there is a local regression equation (e.g. $y_i = \mathbf{x}'_i \boldsymbol{\beta}(s_i) + \varepsilon_i$) at each location s_i . In this case, VCR's local models are an alternative to a global model where the coefficients are constant across the domain. Estimating the coefficient functions of a VCR model is therefore more complicated than estimating the coefficients in a global regression model. Geographically-weighted regression (GWR) (Brundson et al., 1998; Fotheringham et al., 2002) is a kernel-based technique for estimating VCR coefficient functions in the context of spatial data.

Spatial association - meaning that nearby locations are more alike than distant locations - is a key concept in spatial statistics. Common practice in the analysis of geostatistical data (Cressie, 1993) is to write a spatial model as the sum of systematic and random components, as in: $Y(s) = \mathbf{x}'(s)\boldsymbol{\beta} + W(s) + \varepsilon(s)$ where $\mathbf{x}'(s)$ is the spatial process that generates (possibly multivariate) covariates, $\boldsymbol{\beta}$ is a vector of regression coefficients, $\varepsilon(s)$ is a white noise process, and $W(s)$ is a second-order stationary mean-zero process that is independent of the white noise process, called the spatial random effect process. Fitting such a model then proceeds by specifying a covariance function for the spatial random effect (Diggle and Ribeiro, 2007). For example, $\text{Cov}(W(s), W(s')) = \exp(-\phi^{-1}\|s - s'\|_2)$ is the exponential covariance function.

The spatial random effect describes how deviations from the systematic part of the model (i.e. $\mathbf{x}'(s)\boldsymbol{\beta}$) are spatially clustered. Though the model may be unbiased globally, clustered deviations may indicate that the global model achieves this global unbiasedness by allowing local biases to offset each other. In that case, a VCR model might eliminate the need for a spatial random effect by being unbiased both globally and locally.

There are reasons other than bias to make spatial models local. Analysts may be interested, e.g., in local coefficient estimates, local variable selection, local mean-squared error (MSE), and local variance, to name a few.

Both spline-based (Wood, 2006) and kernel-based (Fan and Zhang, 1999; Loader, 1999) methods are available for estimating varying coefficient functions. Wood (2006) demonstrates that it is straightforward to modify a thin-plate regression spline model into a VCR model; Wang et al. (2008) introduced the SCAD penalty of Fan and Li (2001) for global variable selection in spline-based VCR models with a univariate effect-modifying parameter, and Antoniadis et al. (2012) use the nonnegative garrote of Breiman (1995) for global variable selection in P-spline-based VCR models having a univariate effect-modifying parameter.

This paper focuses on GWR, which is a kernel-based method of estimating the coefficients of a VCR model. GWR uses kernel-weighted regression with weights based on the distance between observation locations. The presentation of GWR in Fotheringham et al. (2002) follows the development of local likelihood in Loader (1999). GWR can be thought of as a kernel smoother for regression coefficients, and hence GWR coefficient estimates are likely to exhibit bias near the boundary of the region being modeled (Hastie and Loader, 1993). Modeling the coefficient surface as locally linear rather than locally constant (by including coefficient-by-location interactions) can reduce this boundary-effect bias (Hastie and Loader, 1993). Adding these interactions to the GWR model is analogous to a transition from kernel smoothing to local regression, and was introduced in Wang et al. (2008).

One reason to prefer GWR to spline-based VCR models for spatial data is the ability to do local variable selection. This paper describes local variable selection in GWR models using the adaptive lasso of Zou (2006). The idea first appears in the literature as the geographically-weighted lasso (GWL) of Wheeler (2009), which uses a jackknife criterion for selection of the lasso tuning parameters. Because the jackknife criterion can only be computed at locations where the response variable is observed, the GWL cannot be used for imputation of missing data nor for interpolation between observation locations. We avoid this limitation of the GWL by using a penalized-likelihood criterion to select the lasso tuning parameters (specifically the AIC, but in principle one could use the BIC, *et cetera*). The AIC allows us to easily adapt our method to the setting of a generalized linear model. The local AIC presented here is based on the local likelihood (Loader, 1999) and the total AIC is based on an *ad hoc* calculation of the sample size and degrees of freedom for estimating the spatially-varying coefficient surfaces.

2. Geographically-weighted regression

2.1. Model

Consider n data observations, made at sampling locations $\mathbf{s}_1, \dots, \mathbf{s}_n$ in a spatial domain $D \subset \mathbb{R}^2$. For $i = 1, \dots, n$, let $y(\mathbf{s}_i)$ and $\mathbf{x}(\mathbf{s}_i)$ denote the univariate response variable, and a $(p + 1)$ -variate vector of covariates measured at location \mathbf{s}_i , respectively. At each location \mathbf{s}_i , assume that the outcome is related to the covariates by a linear model where the coefficients $\boldsymbol{\beta}_i(\mathbf{s}_i)$ may be spatially-varying and $\varepsilon(\mathbf{s}_i)$ is random noise at location \mathbf{s}_i .

$$y(\mathbf{s}_i) = \mathbf{x}'(\mathbf{s}_i)\boldsymbol{\beta}(\mathbf{s}_i) + \varepsilon(\mathbf{s}_i) \tag{1}$$

Further assume that the error term $\varepsilon(\mathbf{s}_i)$ is normally distributed with zero mean and a possibly spatially-varying variance $\sigma^2(\mathbf{s}_i)$

$$\varepsilon(\mathbf{s}_i) \sim \mathcal{N}(0, \sigma^2(\mathbf{s}_i)) \tag{2}$$

In order to simplify the notation, let subscripts denote the values of data or parameters at the locations where data is observed. Thus, $\mathbf{x}(\mathbf{s}_i) \equiv \mathbf{x}_i \equiv (1, x_{i1}, \dots, x_{ip})'$, $\boldsymbol{\beta}(\mathbf{s}_i) \equiv \boldsymbol{\beta}_i \equiv (\beta_{i0}, \beta_{i1}, \dots, \beta_{ip})'$, $y(\mathbf{s}_i) \equiv y_i$, and $\sigma^2(\mathbf{s}_i) \equiv \sigma_i^2$. Let $\mathbf{X} = (\mathbf{x}_1, \dots, \mathbf{x}_n)'$ and $\mathbf{Y} = (y_1, \dots, y_n)'$. Equations (1) and (2) can now be rewritten as

$$y_i = \mathbf{x}_i' \boldsymbol{\beta}_i + \varepsilon_i \text{ and } \varepsilon_i \sim \mathcal{N}(0, \sigma_i^2) \quad (3)$$

Assume that, given the design matrix \mathbf{X} , observations of the response variable at different locations are statistically independent of each other. Then the total log-likelihood of the observed data is the sum of the log-likelihood of each individual observation.

$$\ell(\boldsymbol{\beta}) = -1/2 \sum_{i=1}^n \left\{ \log(2\pi\sigma_i^2) + \sigma_i^{-2} (y_i - \mathbf{x}_i' \boldsymbol{\beta}_i)^2 \right\} \quad (4)$$

With n observations and $n \times (p+1)$ free parameters, the model is not identifiable so it is not possible to directly maximize the total likelihood. One way to effectively reduce the number of parameters is to assume that the spatially-varying coefficients $\boldsymbol{\beta}(\mathbf{s})$ are smoothly varying, and use a kernel smoother to make pointwise estimates of the coefficients by maximizing the local likelihood. In the setting of spatial data and with the kernel smoother based on the physical distance between observation locations, this is ordinary GWR.

2.2. Estimation

Geographically-weighted regression estimates the value of the coefficient surface $\boldsymbol{\beta}(\mathbf{s})$ at each location \mathbf{s}_i . First calculate the euclidean distance $\delta_{ii'} \equiv \delta(\mathbf{s}_i, \mathbf{s}_{i'}) \equiv \|\mathbf{s}_i - \mathbf{s}_{i'}\|_2$ between locations \mathbf{s}_i and $\mathbf{s}_{i'}$ for all i, i' . A bisquare kernel is used to generate spatial weights based on the euclidean distances and a bandwidth ϕ :

$$w_{ii'} = \begin{cases} \left[1 - (\phi^{-1} \delta_{ii'})^2 \right]^2 & \text{if } \delta_{ii'} < \phi \\ 0 & \text{if } \delta_{ii'} \geq \phi \end{cases} \quad (5)$$

For the purpose of estimation, define the local likelihood at each location (Fotheringham et al., 2002):

$$\mathcal{L}_i(\boldsymbol{\beta}_i) = \prod_{i'=1}^n \left\{ (2\pi\sigma_i^2)^{-1/2} \exp \left[-\frac{1}{2}\sigma_i^{-2} (y_{i'} - \mathbf{x}_{i'}'\boldsymbol{\beta}_i)^2 \right] \right\}^{w_{ii'}} \quad (6)$$

Thus, the local log-likelihood function is:

$$\ell_i(\boldsymbol{\beta}_i) \propto -1/2 \sum_{i'=1}^n w_{ii'} \left\{ \log \sigma_i^2 + \sigma_i^{-2} (y_{i'} - \mathbf{x}_{i'}'\boldsymbol{\beta}_i)^2 \right\} \quad (7)$$

From which it is apparent that the GWR coefficient estimates $\hat{\boldsymbol{\beta}}_{i,\text{GWR}}$, which maximize the local likelihood at location \mathbf{s}_i , can be calculated using weighted least squares. Letting the diagonal weight matrix \mathbf{W}_i be:

$$\mathbf{W}_i = \text{diag} \{w_{ii'}\}_{i'=1}^n \quad (8)$$

We have:

$$\hat{\boldsymbol{\beta}}_{i,\text{GWR}} = (\mathbf{X}'\mathbf{W}_i\mathbf{X})^{-1} \mathbf{X}'\mathbf{W}_i\mathbf{Y} \quad (9)$$

And $\hat{\sigma}_i$, which maximizes (7), is:

$$\hat{\sigma}_i = (\mathbf{1}_n'\mathbf{w}_i)^{-1} \mathbf{w}_i' \left(\mathbf{Y} - \mathbf{X} (\mathbf{X}'\mathbf{W}_i\mathbf{X})^{-1} \mathbf{X}'\mathbf{W}_i\mathbf{Y} \right) \quad (10)$$

3. Model selection

3.1. Variable selection

Traditional GWR relies on *a priori* model selection to decide which variables should be included in the model. In the context of ordinary least squares regression, regularization methods such as the adaptive lasso (Zou, 2006) have been shown to have appealing properties for automating variable selection, sometimes including the “oracle” property of asymptotically selecting exactly the correct variables for inclusion in a regression model.

Three regularization methods were used in this work. The adaptive lasso was implemented in two ways - once via the lars algorithm (Efron et al., 2004) which uses least squares, and once via

coordinate descent using the R package `glmnet` (Friedman et al., 2010). The third regularization method implemented here uses the adaptive elastic net penalty (Zou and Zhang, 2009), also via coordinate descent using the `glmnet` package.

3.1.1. Adaptive lasso

The adaptive lasso is applied to GWR by first multiplying the design matrix \mathbf{X} by $\mathbf{W}_i^{1/2}$, the diagonal matrix of geographic weights centered at s_i . Since some of the weights $w_{ii'}$ may be zero, the matrix $\mathbf{W}_i^{1/2}\mathbf{X}$ is not of full rank. The matrices \mathbf{Y}_i^* , \mathbf{X}_i^* , and \mathbf{W}_i^* are formed by dropping the rows of \mathbf{X} and \mathbf{W}_i that correspond to observations with zero weight in the regression model at location s_i . Now, letting $\mathbf{U}_i^* = \mathbf{W}_i^{*1/2}\mathbf{X}_i^*$ and $\mathbf{V}_i^* = \mathbf{W}_i^{*1/2}\mathbf{Y}_i^*$, we seek the coefficients β_i of the regression model:

$$\mathbf{V}_i^* = \mathbf{U}_i^* \beta_i + \varepsilon \quad (11)$$

To apply the adaptive lasso for estimating these regression coefficients, each column of \mathbf{U}_i^* is centered around zero and rescaled to have an L_2 -norm of one. Let $\tilde{\mathbf{U}}_i^*$ be the centered-and-scaled version of \mathbf{U}_i^* . Adaptive weights are calculated using the OLS regression coefficients γ_i^* via ordinary least squares (OLS):

$$\gamma_i^* = \left(\tilde{\mathbf{U}}_i^{*'} \tilde{\mathbf{U}}_i^* \right)^{-1} \tilde{\mathbf{U}}_i^{*'} \mathbf{V}_i^* \quad (12)$$

Now a final scaling step is done: for $j = 1, \dots, p$, the j th column of $\tilde{\mathbf{U}}_i^*$ is multiplied by $(\gamma_i^*)_j$, the corresponding coefficient from (12). Call this rescaled matrix $\check{\mathbf{U}}_i^*$.

Finally, the adaptive lasso coefficient estimates at location s_i are found, either by using the `lars` algorithm (Efron et al., 2004) to model \mathbf{V}_i^* as a function of $\check{\mathbf{U}}_i^*$ or by using the `glmnet` package to implement coordinate descent. Either way, the objective being minimized is the same:

$$\sum_{i'=1}^n w_{ii'} (y_{i'} - \mathbf{x}_{i'}' \boldsymbol{\beta}_i)^2 + \lambda_i \sum_{j=1}^p |\beta_{ij} / \gamma_{ij}^*| \quad (13)$$

$$(14)$$

3.1.2. Adaptive elastic net

To implement the adaptive elastic net (Zou and Zhang, 2009), the adaptive weights γ_i^* are calculated as for the adaptive lasso, but there is an additional elastic net parameter α that controls the balance between the ℓ_1 and ℓ_2 penalties, so that the objective to be minimized is:

$$\sum_{i'=1}^n w_{ii'} (y_{i'} - \mathbf{x}_{i'}' \boldsymbol{\beta}_i)^2 + \alpha \lambda_i \sum_{j=1}^p |\beta_{ij} / \gamma_{ij}^*| + (1 - \alpha) \lambda_i \sum_{j=1}^p (\beta_{ij} / \gamma_{ij}^*)^2 \quad (15)$$

$$= \sum_{i'=1}^n w_{ii'} (y_{i'} - \mathbf{x}_{i'}' \boldsymbol{\beta}_i)^2 + \lambda_i \left(\alpha \sum_{j=1}^p |\beta_{ij} / \gamma_{ij}^*| + (1 - \alpha) \sum_{j=1}^p [\beta_{ij} / \gamma_{ij}^*]^2 \right) \quad (16)$$

In the simulation study (Section 4), α is calculated from the maximum global (i.e. for all data without weighting) Pearson correlation between any two covariates, ρ_{\max} : $\alpha = 1 - \rho_{\max}$.

3.2. Tuning parameter selection

At each location \mathbf{s}_i , it is necessary to select the lasso tuning parameter λ_i . To compare different values of λ_i , we propose a locally-weighted version of the Akaike information criterion (AIC) (Akaike, 1974) which we call the local AIC, or AIC_{loc} . The local AIC is calculated by adding a penalty to the local likelihood, with the sum of the weights around \mathbf{s}_i , $\sum_{i'=1}^n w_{ii'}$, playing the role of the sample size and the “degrees of freedom” (df_i) at \mathbf{s}_i given by the number of nonzero coefficients in $\boldsymbol{\beta}_i$ (Zou et al., 2007).

$$\text{AIC}_{\text{loc},i} = -2 \sum_{i'=1}^n \ell_{ii'} + 2\text{df}_i \quad (17)$$

$$= -2 \times \sum_{i'=1}^n \log \left\{ (2\pi \hat{\sigma}_i^2)^{-1/2} \exp \left[-\frac{1}{2} \hat{\sigma}_i^{-2} (y_{i'} - \mathbf{x}_{i'}' \hat{\beta}_i)^2 \right] \right\}^{w_{ii'}} + 2\text{df}_i \quad (18)$$

$$= \sum_{i'=1}^n w_{ii'} \left\{ \log(2\pi) + \log \hat{\sigma}_i^2 + \hat{\sigma}_i^{-2} (y_{i'} - \mathbf{x}_{i'}' \hat{\beta}_i)^2 \right\} + 2\text{df}_i \quad (19)$$

$$= \hat{\sigma}_i^{-2} \sum_{i'=1}^n w_{ii'} (y_{i'} - \mathbf{x}_{i'}' \hat{\beta}_i)^2 + 2\text{df}_i + C_i \quad (20)$$

Where the estimated local variance $\hat{\sigma}_i^2$ is the variance estimate from the unpenalized local model (Zou et al., 2007), so C_i does not depend on the choice of tuning parameter and can be ignored.

Wheeler (2009) proposed selecting the tuning parameter for the lasso at location \mathbf{s}_i to minimize the jackknife prediction error $|y_i - \hat{y}_i^{(i)}|$. Because the jackknife prediction error is undefined everywhere except for at observation locations, this choice restricts coefficient estimation to occur at the locations where data has been observed. By contrast, the local AIC can be calculated at any location where we can calculate the local likelihood. As a practical matter this allows for variable selection and coefficient surface estimation to be done at locations where no data was observed (interpolation) and for imputation of missing values of the response variable.

3.3. Bandwidth selection

The bandwidth parameter is global and so we need a global statistic for comparing prospective bandwidths. The objective minimized by GWL is:

$$\sum_{i'=1}^n w_{ii'} (y_{i'} - \mathbf{x}_{i'}' \beta_i)^2 + \sum_{j=1}^p \lambda_{ij} |\beta_{ij}| \quad (21)$$

Where $\lambda_{ij}, j = 1, \dots, p$ are penalties from the adaptive lasso (Zou, 2006). Taking the derivatives with respect to β and setting to zero, we see that

$$\hat{\beta}_{i,\text{GWL}} = (\mathbf{X}' \mathbf{W}_i \mathbf{X})^{-1} \mathbf{X}' \mathbf{W}_i \mathbf{Y} - \frac{1}{2} (\mathbf{X}' \mathbf{W}_i \mathbf{X})^{-1} \boldsymbol{\lambda}_i \quad (22)$$

$$\hat{y}_i = \mathbf{x}_i' \hat{\beta}_{i,\text{GWL}} = \mathbf{x}_i' (\mathbf{X}' \mathbf{W}_i \mathbf{X})^{-1} \mathbf{X}' \mathbf{W}_i \mathbf{Y} - \frac{1}{2} \mathbf{x}_i' (\mathbf{X}' \mathbf{W}_i \mathbf{X})^{-1} \boldsymbol{\lambda}_i \quad (23)$$

Unlike in the case of ordinary geographically-weighted regression, the fitted values $\hat{\mathbf{Y}}$ are not a linear combination of the observations \mathbf{Y} . Because GWL is not a linear smoother it is not possible to calculate the AIC as in Fotheringham et al. (2002) (Zou, 2006). We propose a statistic called the total AIC (AIC_{tot}) for the purpose of selecting the bandwidth parameter. Because of the kernel weights and the application of the adaptive lasso, the sample size and the degrees of freedom are different at each location. The total AIC is found by taking the sum over all of the observed data:

$$\text{AIC}_{\text{tot}} = -2 \times \sum_{i=1}^n \ell_i + 2\text{df} \quad (24)$$

$$= \sum_{i=1}^n \left\{ \log \hat{\sigma}_i^2 + \hat{\sigma}_i^{-2} \left(y_i - \mathbf{x}_i' \hat{\boldsymbol{\beta}}_i \right)^2 \right\} + 2\text{df} \quad (25)$$

What remains is to calculate df, the number of degrees of freedom used by the model. Ordinary GWR, as developed in Loader (1999) and Fotheringham et al. (2002) calculates df using the trace of the “hat” matrix, but because the GWL is not a linear smoother, there is no “hat” matrix associated with GWL. Instead, notice that df can be pulled into the summation in (25):

$$\text{df} = \sum_{i=1}^n (n^{-1} \text{df}) \quad (26)$$

Now, because we are considering the sum of local weights to be the sample size for the local models, we estimate df by $\sum_{i=1}^n \left\{ \left(\sum_{i'=1}^n w_{ii'} \right)^{-1} \text{df}_i \right\}$, and the total AIC is then:

$$\text{AIC}_{\text{tot}} = \sum_{i=1}^n \left\{ \log \hat{\sigma}_i^2 + \hat{\sigma}_i^{-2} \left(y_i - \mathbf{x}_i' \hat{\boldsymbol{\beta}}_i \right)^2 + 2 \left(\sum_{i'=1}^n w_{ii'} \right)^{-1} \text{df}_i \right\} \quad (27)$$

The bandwidth that minimizes (27) is found by a line search.

3.4. Confidence interval construction

Confidence intervals for the GWL’s coefficient estimates can be calculated either by the bootstrap (Efron and Tibshirani, 1986) or by exploiting an assumption of normally-distributed residuals.

Then the, e.g., 95% confidence interval for each regression coefficient is defined by the (2.5, 97.5) percentiles of the coefficient estimates from the bootstrap replicates.

To compute coefficient confidence intervals via the bootstrap, the observations with non-zero geographic weights are resampled uniformly with replacement for each of n_B bootstrap replicates. For each bootstrap replicate, the GWL is used to estimate regression coefficients. The local likelihood of the bootstrap replicates may be different from that of the original sample, so the adaptive lasso tuning parameter may differ for each bootstrap replicate. Since the GWL is applied independently to each bootstrap replicate, the variables selected by GWL may be different for each replicate.

Unshrunk coefficient estimates are found by using the GWL at each location for variable selection only and then estimating the coefficients for the selected variables by weighted least squares. An unshrunk bootstrap confidence interval is found by estimating the unshrunk coefficients for each of the n_B bootstrap replicates and then calculating the percentiles as above.

A third way to estimate the coefficient confidence intervals is to use the GWL for variable selection only and then to use weighted least squares for both coefficient estimation and confidence interval construction:

$$\hat{\text{se}}_{\beta_i} = \left(\tilde{\mathbf{X}}_i' \mathbf{W}_i \tilde{\mathbf{X}}_i \right)^{-1} \tilde{\mathbf{X}}_i' \mathbf{W}_i \mathbf{Y} \quad (28)$$

where $\tilde{\mathbf{X}}_i$ is the model matrix including only those variables that are selected by GWL at location i .

4. Simulation

4.1. Simulation setup

A simulation study was conducted to assess the performance of the method described in Sections 2-3. There were twelve simulation settings, each of which was simulated 100 times. For each of the twelve settings, β_1 , the true coefficient surface for \mathbf{Z}_1 , was nonzero in at least part of the simulation

domain. There were four other simulated covariates, but their true coefficient surfaces were zero across the area under simulation.

Data was simulated on $[0, 1] \times [0, 1]$, which was divided into a 30×30 grid. Each of $p = 5$ covariates Z_1, \dots, Z_p was simulated by a Gaussian random field (GRF) with mean zero and exponential spatial covariance $Cov(Z_{ji}, Z_{ji'}) = \sigma_z^2 \exp(-\tau_z^{-1} \delta_{ii'})$ where $\sigma_z^2 = 1$ is the variance, τ_z is the range parameter, and $\delta_{ii'}$ is the Euclidean distance $\|\mathbf{s}_i - \mathbf{s}_{i'}\|_2$. Correlation was induced between the covariates by multiplying the \mathbf{Z} matrix by \mathbf{R} , where \mathbf{R} is the Cholesky decomposition of the covariance matrix $\Sigma = \mathbf{R}'\mathbf{R}$. The covariance matrix Σ is a 5×5 matrix that has ones on the diagonal and ρ for all off-diagonal entries, where ρ is the between-covariate correlation.

The simulated response is $y_i = \mathbf{z}_i' \boldsymbol{\beta}_i + \varepsilon_i$ for $i = 1, \dots, 900$ where for simplicity the vector of additive errors $\boldsymbol{\varepsilon}$ were iid Gaussian.

The simulated data include the output y and five covariates Z_1, \dots, Z_5 . The true data-generating model uses only Z_1 , so Z_2, \dots, Z_5 are included to test the variable-selection properties of GWL.

The twelve simulation settings are described in Table 1. Three parameters were varied to produce the twelve settings: there were three functional forms for the coefficient surface β_1 (step, gradient, and parabola - see Figure 1); data was simulated both with ($\rho = 0.5$) and without ($\rho = 0$) correlation between the covariates; and simulations were made with low ($\sigma^2 = 0.25$) and high ($\sigma^2 = 1$) variance for the random noise term.

The performance of the penalized GWR methods (adaptive lasso via `lars` and via `glmnet`, and the adaptive elastic net (`enet`) was compared to that of oracular GWR (O-GWR), which is ordinary GWR with “oracular” variable selection, meaning that exactly the correct set of predictors was used to fit the GWR model at each location in the simulation. Also included in the comparison was the GWR algorithm of Fotheringham et al. (2002) without variable selection (`gwr`). Finally, there is a category of simulation results using the three penalized GWR methods for local variable selection and then ordinary GWR for coefficient estimation.

[Table 1 about here.]

[Figure 1 about here.]

[Figure 2 about here.]

4.2. Results

Results from the simulation were summarized at five locations on the simulated grid (see Figure 2). The five key locations were chosen because they represent interesting regions of the β_1 coefficient surfaces. The results of variable selection and coefficient estimation are presented in the tables below.

Selection: Tables 2 - 6

MSE of $\hat{Y}(s_i)$ ($i = 1, \dots, 5$): Tables 22 - 26

MSE of $\hat{\beta}_1(s_i)$ ($i = 1, \dots, 5$): Tables 7 - 11

Bias of $\hat{\beta}_1(s_i)$ ($i = 1, \dots, 5$): Tables 12 - 16

Variance of $\hat{\beta}_1(s_i)$ ($i = 1, \dots, 5$): Tables 17 - 21

4.3. Discussion

There is no clear winner among the methods tested. In most cases, the mean squared error of the penalized GWR coefficient estimates is quite similar to that of the ordinary GWR coefficient estimates for $\beta_1(\mathbf{s})$.

4.4. Tables

4.4.1. Selection

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

[Table 5 about here.]

[Table 6 about here.]

4.4.2. Estimation

[Table 7 about here.]

[Table 8 about here.]

[Table 9 about here.]

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5. Data analysis

5.1. Census poverty data

An example data analysis is presented to demonstrate application of penalized GWR. In this example we use penalized GWR to do local variable selection and coefficient estimation for a varying-coefficients model of how poverty is related to a list of demographic and social variables. The data is from the U.S. Census Bureau's decennial census from 1970. This analysis looks specifically at the upper midwest states of Minnesota, Iowa, Wisconsin, Illinois, Indiana, and Michigan. This is areal data, aggregated at the county level.

Table 27 lists the variables that were considered as potential predictors of county-level poverty rate. The outcome of interest (poverty rate) is a proportion and so takes values on $[0, 1]$, but to demonstrate the geographically-weighted lasso in a linear regression context, we model the logit-transformed poverty rate. The predictor variables were not transformed - raw proportions were used.

[Table 27 about here.]

5.2. Modeling

The adaptive elastic net was used for variable selection and coefficient estimation.

5.3. Figures

The coefficient estimates are plotted on maps of the upper midwest in Figures ?? - ?. It is immediately apparent that the estimated coefficient surfaces are non-constant for most variables.

[Figure 3 about here.]

5.4. Discussion

If the model is to be believed, then it is not uncommon for the same variable to have both positive and negative effects on poverty in different geographical areas - see, for instance, the coefficient surface for `pex` (mining employment) in the 1970 census. That surface indicates an interaction

whereby the proportion of people working in mining in southern parts of the studied area is associated with an increase in the poverty rate, while in northern parts of the studied area it is associated with a decrease in the poverty rate. Often, a variable is found to be associated with an effect on poverty in some counties but not in others (see, for instance, the coefficient surface for `pserve` (services employment) in 1980).

6. References

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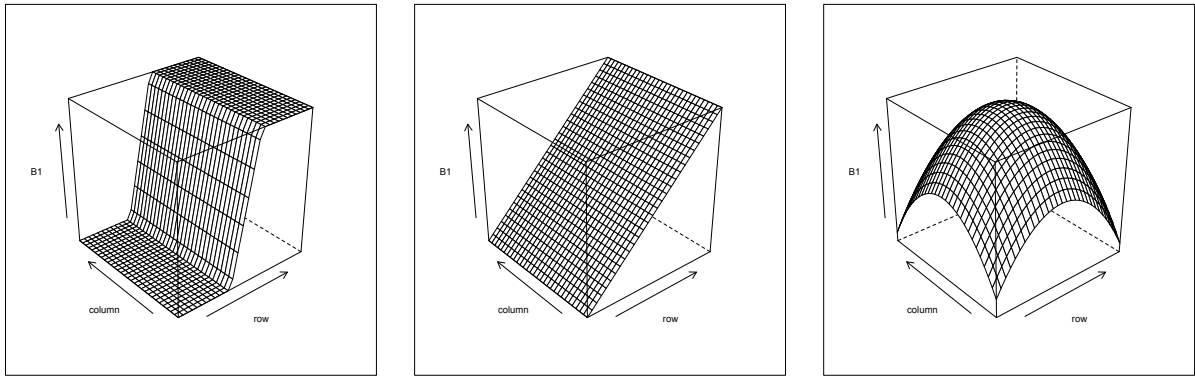


Figure 1: The actual β_1 coefficient surface used in the simulation.

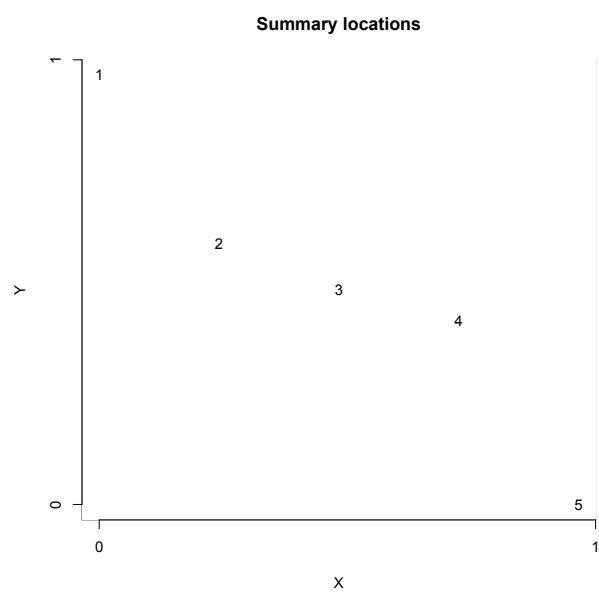


Figure 2: Locations where the variable selection and coefficient estimation of GWL were summarized.

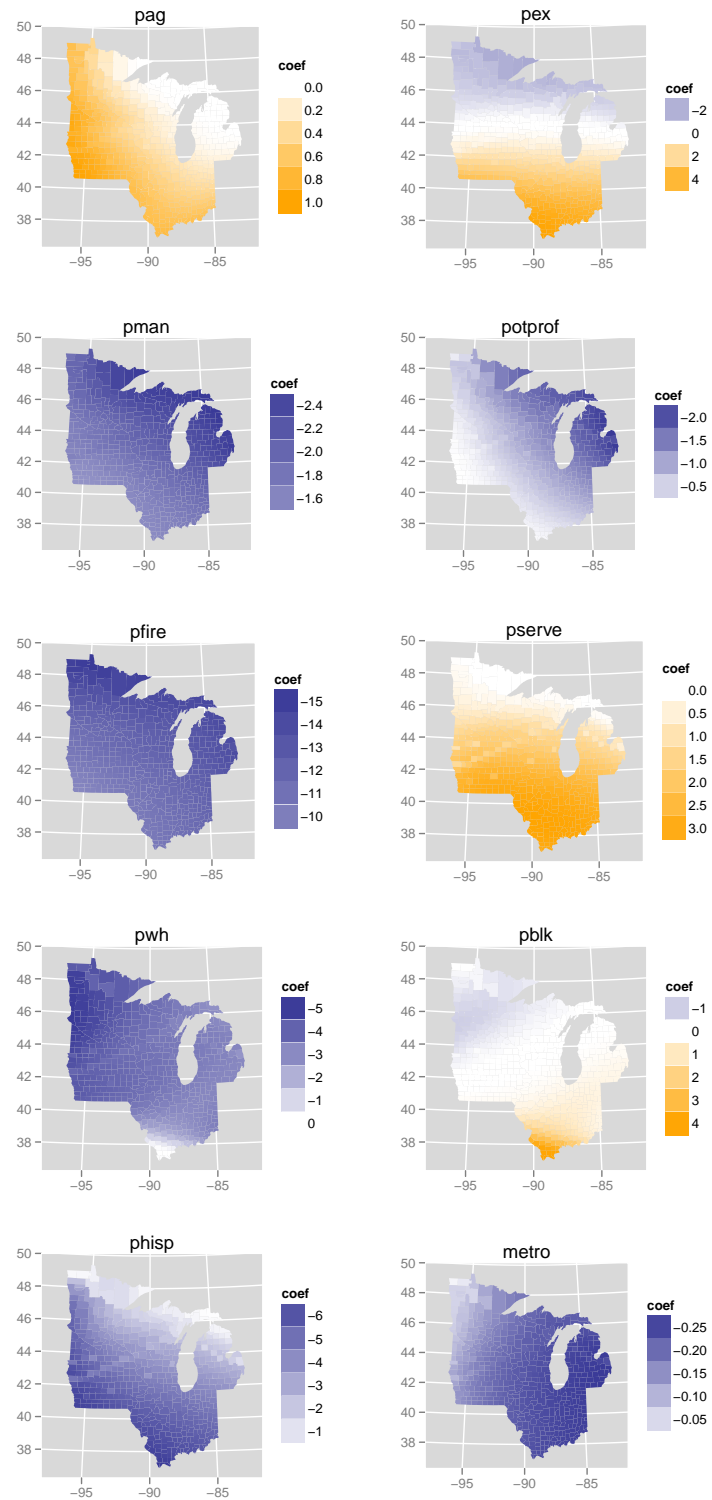


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Setting	function	ρ	σ^2
1	step	0	0.25
2	step	0	1
3	step	0.5	0.25
4	step	0.5	1
5	gradient	0	0.25
6	gradient	0	1
7	gradient	0.5	0.25
8	gradient	0.5	1
9	parabola	0	0.25
10	parabola	0	1
11	parabola	0.5	0.25
12	parabola	0.5	1

Table 1: Simulation parameters for each setting.

		lars		enet		glmnet	
		β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$
step		0.98	0.04	1.00	0.04	1.00	0.05
		0.89	0.09	0.86	0.09	0.82	0.07
		0.96	0.07	0.99	0.10	0.96	0.09
		0.84	0.04	0.84	0.07	0.88	0.05
gradient		1.00	0.04	1.00	0.03	1.00	0.03
		0.99	0.08	0.97	0.07	0.97	0.07
		1.00	0.07	1.00	0.06	1.00	0.04
		0.90	0.08	0.92	0.08	0.92	0.08
parabola		0.94	0.06	0.95	0.06	0.94	0.06
		0.80	0.06	0.81	0.07	0.80	0.06
		0.95	0.06	0.94	0.09	0.95	0.04
		0.78	0.12	0.79	0.12	0.80	0.12

Table 2: Selection frequency at location 1

	lars		enet		glmnet	
	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$
step	1.00	0.07	1.00	0.07	1.00	0.07
	1.00	0.06	1.00	0.06	1.00	0.07
	1.00	0.05	1.00	0.06	1.00	0.05
	0.99	0.03	1.00	0.07	0.99	0.04
gradient	1.00	0.10	1.00	0.08	1.00	0.07
	0.98	0.07	0.98	0.08	0.99	0.07
	1.00	0.07	1.00	0.06	1.00	0.05
	0.98	0.06	0.99	0.08	0.99	0.05
parabola	1.00	0.09	1.00	0.08	1.00	0.08
	0.97	0.12	0.98	0.11	0.98	0.10
	1.00	0.06	1.00	0.05	1.00	0.05
	0.94	0.08	0.94	0.10	0.94	0.08

Table 3: Selection frequency at location 2

		lars		enet		glmnet	
		β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$
step		0.99	0.05	0.99	0.06	0.99	0.06
		0.84	0.08	0.84	0.08	0.82	0.07
		0.96	0.05	0.97	0.08	0.92	0.04
		0.78	0.08	0.81	0.11	0.80	0.08
gradient		1.00	0.09	1.00	0.08	1.00	0.07
		0.98	0.08	0.95	0.08	0.96	0.07
		1.00	0.07	1.00	0.06	1.00	0.04
		0.93	0.09	0.95	0.09	0.94	0.09
parabola		1.00	0.09	1.00	0.09	1.00	0.09
		0.96	0.10	0.97	0.09	0.97	0.10
		1.00	0.08	1.00	0.07	1.00	0.07
		0.93	0.10	0.94	0.10	0.96	0.10

Table 4: Selection frequency at location 3

	lars		enet		glmnet	
	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$
step	0.57	0.08	0.64	0.06	0.59	0.06
	0.48	0.07	0.48	0.07	0.49	0.07
	0.45	0.08	0.51	0.12	0.40	0.07
	0.53	0.08	0.52	0.07	0.51	0.07
gradient	1.00	0.06	1.00	0.06	1.00	0.06
	0.98	0.07	0.95	0.07	0.93	0.06
	1.00	0.09	1.00	0.08	1.00	0.10
	0.96	0.07	0.95	0.11	0.95	0.08
parabola	1.00	0.09	1.00	0.08	1.00	0.08
	0.93	0.07	0.92	0.08	0.94	0.08
	1.00	0.08	1.00	0.08	1.00	0.08
	0.96	0.08	0.96	0.09	0.96	0.09

Table 5: Selection frequency at location 4

	lars		enet		glmnet	
	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$	β_1	$\beta_4 - \beta_5$
step	0.04	0.03	0.03	0.03	0.03	0.03
	0.07	0.05	0.06	0.04	0.04	0.05
	0.02	0.04	0.02	0.03	0.03	0.05
	0.05	0.04	0.04	0.03	0.06	0.06
gradient	0.92	0.05	0.93	0.05	0.94	0.04
	0.71	0.08	0.70	0.07	0.70	0.07
	0.93	0.10	0.95	0.14	0.95	0.10
	0.60	0.07	0.63	0.13	0.64	0.06
parabola	0.93	0.10	0.93	0.09	0.92	0.10
	0.80	0.05	0.81	0.05	0.79	0.05
	0.93	0.07	0.94	0.12	0.94	0.07
	0.81	0.09	0.81	0.11	0.83	0.08

Table 6: Selection frequency at location 5

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.046	0.025	<i>0.023</i>	0.151	0.127	0.124	0.082	0.005
	0.146	0.186	0.216	0.290	0.376	0.375	<i>0.134</i>	0.009
	0.072	<i>0.045</i>	0.073	0.172	0.134	0.205	0.101	0.011
	0.214	0.218	0.179	0.441	0.425	0.369	<i>0.154</i>	0.022
gradient	0.066	0.069	0.070	0.007	0.007	<i>0.007</i>	0.010	0.016
	0.084	0.094	0.096	0.161	0.078	0.085	<i>0.045</i>	0.042
	0.065	0.070	0.069	0.009	0.007	<i>0.008</i>	0.009	0.019
	0.161	0.149	0.144	0.149	0.123	0.121	0.040	<i>0.050</i>
parabola	0.074	0.075	0.074	0.020	<i>0.020</i>	0.020	0.022	0.105
	0.079	0.078	0.077	0.041	<i>0.040</i>	0.039	0.063	0.106
	0.077	0.069	0.076	0.024	0.018	0.023	<i>0.023</i>	0.099
	0.083	0.072	0.083	<i>0.048</i>	0.044	0.050	0.067	0.110

Table 7: Mean squared error of estimates for β_1 at location 1 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.024	0.024	0.024	0.020	0.021	<i>0.021</i>	0.021	0.042
	0.061	0.063	0.068	<i>0.050</i>	0.054	0.056	0.042	0.070
	0.022	0.027	0.021	0.017	0.021	<i>0.017</i>	0.018	0.044
	0.069	0.071	0.071	<i>0.057</i>	<i>0.056</i>	0.061	0.043	0.075
gradient	0.003	0.003	0.003	<i>0.001</i>	0.001	0.001	0.001	0.001
	0.014	0.013	0.008	0.015	0.013	0.009	<i>0.002</i>	0.002
	0.003	0.003	0.003	0.001	<i>0.001</i>	0.001	0.001	0.002
	0.015	0.012	0.012	0.014	0.011	0.011	0.003	<i>0.004</i>
parabola	<i>0.005</i>	0.005	0.005	0.007	0.007	0.007	0.004	0.007
	0.018	0.016	0.016	0.026	0.021	0.022	<i>0.008</i>	0.007
	0.007	0.007	<i>0.007</i>	0.011	0.010	0.009	0.004	0.008
	0.020	0.022	0.020	0.022	0.022	0.023	0.007	<i>0.009</i>

Table 8: Mean squared error of estimates for β_1 at location 2 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.011	0.011	0.010	0.007	0.007	0.007	0.004	<i>0.005</i>
	0.043	0.043	0.047	0.049	0.049	0.054	<i>0.009</i>	0.008
	0.016	0.014	0.022	0.013	0.011	0.021	0.005	<i>0.005</i>
	0.048	0.047	0.045	0.049	0.045	0.044	<i>0.008</i>	0.008
gradient	0.001	0.001	0.001	0.001	0.001	0.001	<i>0.001</i>	0.001
	0.007	0.017	0.015	0.007	0.019	0.017	<i>0.002</i>	0.002
	0.001	0.001	0.000	0.000	<i>0.000</i>	0.000	0.001	0.002
	0.022	0.017	0.019	0.023	0.017	0.021	0.002	<i>0.003</i>
parabola	0.015	0.015	0.015	0.015	0.015	<i>0.015</i>	0.005	0.022
	0.032	0.029	0.029	0.030	0.027	0.027	0.012	<i>0.023</i>
	0.019	0.018	0.019	0.018	<i>0.017</i>	0.018	0.005	0.024
	0.037	0.037	0.030	0.037	0.034	0.029	0.012	<i>0.024</i>

Table 9: Mean squared error of estimates for β_1 at location 3 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	<i>0.014</i>	0.014	0.014	0.017	0.019	0.018	0.021	0.042
	<i>0.037</i>	0.036	0.039	0.039	0.042	0.046	0.047	0.074
	0.010	0.012	<i>0.011</i>	0.013	0.016	0.014	0.020	0.044
	<i>0.038</i>	0.028	0.038	0.048	0.047	0.048	0.043	0.082
gradient	0.003	0.003	0.003	0.002	0.001	<i>0.001</i>	0.001	0.001
	0.009	0.014	0.016	0.007	0.012	0.015	0.002	<i>0.003</i>
	0.003	0.002	0.003	0.002	0.001	<i>0.001</i>	0.001	0.002
	0.013	0.015	0.014	0.013	0.014	0.014	0.003	<i>0.004</i>
parabola	0.006	0.006	<i>0.006</i>	0.009	0.009	0.009	0.004	0.008
	0.025	0.027	0.023	0.027	0.029	0.024	<i>0.010</i>	0.009
	<i>0.008</i>	0.008	0.008	0.011	0.011	0.011	0.004	0.010
	0.018	0.020	0.017	0.022	0.022	0.022	0.009	<i>0.010</i>

Table 10: Mean squared error of estimates for β_1 at location 4 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.002	<i>0.001</i>	0.002	0.006	0.004	0.004	0.000	0.007
	0.003	0.006	<i>0.002</i>	0.016	0.024	0.009	0.000	0.011
	<i>0.002</i>	0.002	0.003	0.009	0.009	0.009	0.000	0.010
	0.017	<i>0.004</i>	0.022	0.046	0.038	0.043	0.000	0.015
gradient	0.067	0.068	0.069	0.004	0.004	<i>0.004</i>	0.000	0.016
	0.054	0.051	0.052	<i>0.019</i>	0.019	0.019	0.000	0.044
	0.062	0.060	0.064	0.009	0.010	<i>0.007</i>	0.000	0.021
	0.050	0.047	0.053	<i>0.017</i>	0.020	0.017	0.000	0.051
parabola	0.074	0.075	0.075	0.018	0.020	0.021	<i>0.020</i>	0.104
	0.075	0.074	0.073	0.024	0.022	<i>0.023</i>	0.055	0.104
	0.077	0.069	0.076	<i>0.021</i>	0.023	0.020	0.025	0.099
	0.081	0.075	0.081	0.037	<i>0.036</i>	0.035	0.042	0.113

Table 11: Mean squared error of estimates for β_1 at location 5 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	-0.057	-0.029	-0.020	-0.003	0.038	0.033	0.034	<i>-0.004</i>
	-0.150	-0.195	-0.211	-0.004	-0.082	-0.075	0.053	<i>-0.017</i>
	-0.090	-0.077	-0.073	0.002	0.028	<i>-0.014</i>	0.050	-0.017
	-0.208	-0.214	-0.167	-0.053	-0.067	-0.068	<i>0.035</i>	0.006
gradient	-0.246	-0.248	-0.253	0.008	<i>0.010</i>	0.011	0.011	-0.111
	-0.233	-0.242	-0.248	0.026	<i>-0.014</i>	-0.022	-0.007	-0.182
	-0.245	-0.259	-0.255	-0.004	<i>-0.003</i>	-0.000	0.003	-0.112
	-0.306	-0.290	-0.287	-0.083	<i>-0.055</i>	-0.080	0.017	-0.197
parabola	0.252	0.254	0.252	0.091	0.090	<i>0.090</i>	0.029	0.323
	0.240	0.241	0.236	0.130	0.129	<i>0.126</i>	0.070	0.322
	0.262	0.245	0.259	0.091	<i>0.083</i>	0.101	0.041	0.313
	0.239	0.222	0.242	0.121	<i>0.095</i>	0.115	0.068	0.323

Table 12: Bias of estimates for β_1 at location 1 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	-0.117	-0.119	-0.119	-0.106	-0.110	<i>-0.110</i>	-0.124	-0.196
	-0.178	-0.178	-0.186	<i>-0.145</i>	-0.145	-0.150	-0.175	-0.253
	-0.103	-0.130	-0.099	-0.083	-0.103	<i>-0.083</i>	-0.110	-0.199
	-0.212	-0.221	-0.216	<i>-0.175</i>	-0.167	-0.184	-0.182	-0.263
gradient	-0.046	-0.047	-0.048	0.000	0.002	<i>0.001</i>	0.004	0.002
	-0.034	-0.044	-0.035	0.009	0.003	0.011	<i>0.008</i>	-0.011
	-0.039	-0.046	-0.043	0.003	0.003	0.003	0.002	<i>0.002</i>
	-0.058	-0.062	-0.056	-0.016	-0.011	<i>-0.009</i>	-0.002	-0.020
parabola	-0.062	-0.063	<i>-0.062</i>	-0.073	-0.074	-0.072	-0.048	-0.079
	-0.067	-0.066	-0.063	-0.069	-0.068	<i>-0.065</i>	-0.072	-0.078
	<i>-0.073</i>	-0.079	-0.076	-0.087	-0.091	-0.087	-0.052	-0.085
	-0.093	-0.104	-0.097	-0.101	-0.102	-0.101	-0.065	<i>-0.078</i>

Table 13: Bias of estimates for β_1 at location 2 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	-0.014	-0.014	-0.010	0.018	0.017	0.015	0.021	0.040
	-0.026	-0.027	-0.031	<i>0.006</i>	0.009	0.004	0.050	0.059
	-0.044	-0.056	-0.056	<i>-0.013</i>	-0.009	-0.030	0.017	0.034
	-0.083	-0.094	-0.077	-0.059	-0.056	-0.056	0.017	<i>0.055</i>
gradient	0.007	0.005	0.005	0.003	<i>0.002</i>	0.001	0.003	0.002
	-0.004	-0.017	-0.012	-0.003	-0.013	-0.007	<i>0.003</i>	0.006
	0.006	0.005	0.006	0.001	<i>0.000</i>	0.000	0.003	0.000
	-0.019	-0.029	-0.018	-0.012	-0.022	-0.013	-0.002	<i>0.003</i>
parabola	-0.107	-0.107	-0.106	-0.103	-0.103	<i>-0.102</i>	-0.057	-0.148
	-0.141	-0.136	-0.132	-0.129	-0.122	<i>-0.121</i>	-0.090	-0.147
	-0.125	-0.127	-0.125	-0.123	<i>-0.121</i>	-0.121	-0.060	-0.154
	-0.147	-0.156	-0.131	-0.137	-0.136	<i>-0.121</i>	-0.092	-0.147

Table 14: Bias of estimates for β_1 at location 3 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.047	0.059	<i>0.049</i>	0.058	0.074	0.065	0.129	0.196
	0.072	<i>0.075</i>	0.076	0.080	0.088	0.090	0.193	0.263
	<i>0.014</i>	0.027	0.010	0.027	0.043	0.020	0.129	0.199
	0.091	0.073	<i>0.089</i>	0.108	0.105	0.105	0.189	0.275
gradient	0.047	0.043	0.045	0.009	<i>0.006</i>	0.007	0.004	0.008
	0.021	0.006	0.002	<i>-0.003</i>	-0.014	-0.023	0.004	0.020
	0.039	0.038	0.039	<i>0.001</i>	-0.001	-0.001	0.000	-0.001
	0.000	-0.009	0.003	-0.009	-0.021	-0.013	<i>-0.003</i>	0.014
parabola	<i>-0.066</i>	-0.070	-0.069	-0.078	-0.083	-0.081	-0.051	-0.088
	-0.113	-0.119	-0.110	-0.119	-0.126	-0.115	-0.081	<i>-0.088</i>
	-0.080	-0.085	<i>-0.078</i>	-0.092	-0.095	-0.090	-0.055	-0.095
	-0.088	-0.099	-0.088	-0.094	-0.099	-0.094	-0.079	<i>-0.086</i>

Table 15: Bias of estimates for β_1 at location 4 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	-0.009	<i>-0.006</i>	-0.006	-0.015	-0.009	-0.010	0.000	-0.006
	0.001	-0.009	<i>-0.000</i>	-0.018	-0.025	-0.008	0.000	-0.011
	-0.006	<i>-0.005</i>	-0.009	-0.010	-0.009	-0.012	0.000	-0.009
	-0.012	-0.011	-0.011	-0.026	-0.036	-0.021	0.000	<i>-0.007</i>
gradient	0.246	0.249	0.253	0.002	<i>0.002</i>	0.003	0.000	0.113
	0.191	0.182	0.186	0.007	<i>-0.001</i>	0.004	0.000	0.187
	0.234	0.234	0.243	0.007	<i>-0.001</i>	0.008	0.000	0.115
	0.168	0.165	0.179	0.030	<i>0.024</i>	0.029	0.000	0.190
parabola	0.253	0.256	0.253	<i>0.085</i>	0.091	0.089	0.022	0.321
	0.234	0.233	0.228	0.095	<i>0.088</i>	0.089	0.004	0.319
	0.257	0.243	0.257	0.093	<i>0.079</i>	0.088	0.052	0.313
	0.243	0.232	0.246	0.120	<i>0.096</i>	0.110	0.069	0.328

Table 16: Bias of estimates for β_1 at location 5 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.043	0.024	<i>0.023</i>	0.152	0.127	0.124	0.081	0.005
	<i>0.125</i>	0.149	0.173	0.293	0.373	0.373	0.133	0.009
	0.064	<i>0.040</i>	0.068	0.173	0.134	0.207	0.099	0.011
	0.173	0.174	<i>0.153</i>	0.443	0.424	0.368	0.154	0.022
gradient	<i>0.006</i>	0.007	0.006	0.008	0.007	0.007	0.010	0.004
	<i>0.030</i>	0.035	0.035	0.162	0.079	0.085	0.046	0.009
	0.005	0.003	<i>0.004</i>	0.009	0.007	0.008	0.009	0.007
	0.068	0.066	0.062	0.143	0.121	0.116	<i>0.040</i>	0.011
parabola	0.010	<i>0.010</i>	0.010	0.012	0.012	0.012	0.022	0.001
	0.021	<i>0.020</i>	0.021	0.024	0.024	0.024	0.058	0.002
	<i>0.009</i>	0.009	0.009	0.015	0.011	0.013	0.021	0.001
	0.026	<i>0.023</i>	0.025	0.034	0.035	0.037	0.064	0.006

Table 17: Variance of estimates for β_1 at location 1 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.010	0.010	0.010	0.009	0.009	0.009	<i>0.006</i>	0.003
	0.029	0.032	0.034	0.029	0.033	0.034	<i>0.012</i>	0.006
	0.011	0.011	0.012	0.010	0.010	0.010	<i>0.006</i>	0.005
	0.024	0.022	0.025	0.026	0.028	0.028	<i>0.010</i>	0.006
gradient	<i>0.001</i>	0.001	0.001	0.001	0.001	0.001	0.001	0.001
	0.013	0.011	0.007	0.016	0.013	0.009	<i>0.002</i>	0.002
	0.001	<i>0.001</i>	0.001	0.001	0.001	0.001	0.001	0.002
	0.011	0.008	0.009	0.014	0.011	0.011	0.003	<i>0.003</i>
parabola	<i>0.001</i>	0.001	0.001	0.002	0.002	0.002	0.001	0.000
	0.014	0.012	0.012	0.021	0.017	0.018	<i>0.003</i>	0.001
	0.001	0.001	<i>0.001</i>	0.003	0.002	0.002	0.001	0.000
	0.012	0.011	0.011	0.012	0.012	0.013	<i>0.003</i>	0.003

Table 18: Variance of estimates for β_1 at location 2 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.011	0.011	0.010	0.007	0.007	0.007	<i>0.004</i>	0.003
	0.043	0.043	0.047	0.050	0.050	0.055	<i>0.007</i>	0.004
	0.014	0.011	0.019	0.013	0.011	0.020	<i>0.004</i>	0.004
	0.041	0.039	0.039	0.046	0.043	0.042	<i>0.008</i>	0.005
gradient	0.001	0.001	0.001	0.001	0.001	0.001	<i>0.001</i>	0.001
	0.007	0.017	0.015	0.007	0.019	0.017	<i>0.002</i>	0.002
	0.001	0.001	0.000	0.000	<i>0.000</i>	0.000	0.001	0.002
	0.022	0.016	0.019	0.023	0.016	0.021	0.002	<i>0.003</i>
parabola	0.004	0.004	0.004	0.005	0.005	0.005	<i>0.002</i>	0.000
	0.012	0.010	0.012	0.014	0.013	0.013	<i>0.004</i>	0.001
	0.003	0.002	0.003	0.003	0.003	0.003	<i>0.002</i>	0.000
	0.016	0.013	0.013	0.018	0.016	0.015	<i>0.004</i>	0.002

Table 19: Variance of estimates for β_1 at location 3 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.012	0.011	0.012	0.014	0.014	0.014	<i>0.004</i>	0.003
	0.032	0.030	0.033	0.033	0.035	0.038	<i>0.009</i>	0.005
	0.010	0.011	0.011	0.013	0.014	0.013	0.003	<i>0.004</i>
	0.029	0.023	0.031	0.037	0.037	0.037	<i>0.007</i>	0.006
gradient	0.001	0.001	0.001	0.002	0.001	<i>0.001</i>	0.001	0.001
	0.008	0.014	0.017	0.007	0.012	0.014	0.002	<i>0.002</i>
	0.001	0.001	0.001	0.002	0.001	<i>0.001</i>	0.001	0.002
	0.013	0.015	0.014	0.013	0.014	0.014	0.003	<i>0.004</i>
parabola	0.002	<i>0.001</i>	0.001	0.003	0.002	0.002	0.001	0.000
	0.012	0.013	0.011	0.013	0.013	0.011	<i>0.003</i>	0.001
	0.001	<i>0.001</i>	0.002	0.003	0.001	0.003	0.001	0.001
	0.011	0.010	0.010	0.014	0.012	0.013	<i>0.003</i>	0.003

Table 20: Variance of estimates for β_1 at location 4 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.002	<i>0.001</i>	0.002	0.006	0.004	0.004	0.000	0.007
	0.003	0.006	<i>0.002</i>	0.016	0.024	0.009	0.000	0.011
	<i>0.002</i>	0.002	0.003	0.009	0.009	0.009	0.000	0.010
	0.017	<i>0.004</i>	0.022	0.045	0.037	0.043	0.000	0.015
gradient	0.007	0.006	0.005	0.004	0.004	0.004	0.000	<i>0.003</i>
	0.018	0.018	0.018	0.019	0.020	0.020	0.000	<i>0.009</i>
	0.007	0.005	<i>0.005</i>	0.009	0.010	0.007	0.000	0.008
	0.022	0.020	0.021	0.016	0.020	0.016	0.000	<i>0.015</i>
parabola	0.010	<i>0.010</i>	0.011	0.011	0.012	0.013	0.019	0.001
	0.021	0.019	0.021	0.016	<i>0.014</i>	0.015	0.056	0.003
	0.011	0.010	<i>0.010</i>	0.013	0.017	0.012	0.023	0.001
	0.022	0.021	<i>0.021</i>	0.023	0.027	0.023	0.038	0.005

Table 21: Variance of estimates for β_1 at location 5 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.130	0.100	0.101	0.130	<i>0.100</i>	0.101	0.111	0.118
	0.483	0.594	0.564	<i>0.483</i>	0.594	0.564	0.694	0.850
	0.196	<i>0.151</i>	0.169	0.196	0.151	0.169	0.213	0.247
	0.563	0.559	0.552	0.563	0.559	<i>0.552</i>	0.757	0.895
gradient	0.235	0.224	0.232	0.235	0.224	0.232	<i>0.223</i>	0.222
	0.693	0.669	0.671	0.693	<i>0.669</i>	0.671	0.723	0.757
	0.257	0.258	0.260	0.257	0.258	0.260	<i>0.237</i>	0.210
	0.724	0.733	0.731	<i>0.724</i>	0.733	0.731	0.815	0.784
parabola	0.145	0.142	0.140	0.145	0.142	<i>0.140</i>	0.157	0.248
	1.275	<i>1.257</i>	1.266	1.275	1.257	1.266	1.153	1.466
	0.299	<i>0.285</i>	0.295	0.299	0.285	0.295	0.270	0.434
	0.835	0.801	0.806	0.835	<i>0.801</i>	0.806	0.862	0.986

Table 22: Mean squared error of estimates for Y at location 1 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.193	0.196	0.194	<i>0.193</i>	0.196	0.194	0.225	0.244
	1.023	1.019	1.001	1.023	1.019	<i>1.001</i>	1.171	1.123
	0.270	0.275	0.273	<i>0.270</i>	0.275	0.273	0.311	0.332
	0.973	0.897	0.953	0.973	<i>0.897</i>	0.953	1.000	1.048
gradient	0.218	0.216	0.218	0.218	<i>0.216</i>	0.218	0.221	0.210
	0.828	0.814	0.836	0.828	<i>0.814</i>	0.836	0.863	0.832
	0.257	0.257	0.257	0.257	0.257	0.257	<i>0.257</i>	0.247
	0.795	0.819	0.803	<i>0.795</i>	0.819	0.803	0.822	0.799
parabola	<i>0.192</i>	0.195	0.193	0.192	0.195	0.193	0.204	0.199
	1.139	1.193	1.152	<i>1.139</i>	1.193	1.152	1.204	1.214
	<i>0.248</i>	0.254	0.250	0.248	0.254	0.250	0.246	0.257
	1.165	1.150	1.181	1.165	<i>1.150</i>	1.181	1.180	1.199

Table 23: Mean squared error of estimates for Y at location 2 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.238	<i>0.232</i>	0.233	0.238	0.232	0.233	0.255	0.262
	0.852	0.850	0.833	0.852	0.850	<i>0.833</i>	1.025	1.020
	0.246	0.257	0.246	<i>0.246</i>	0.257	0.246	0.275	0.265
	0.622	0.620	0.652	0.622	<i>0.620</i>	0.652	0.673	0.664
gradient	0.241	0.241	<i>0.241</i>	0.241	0.241	0.241	0.249	0.229
	1.113	1.094	1.096	1.113	<i>1.094</i>	1.096	1.135	1.117
	0.311	0.311	0.313	0.311	<i>0.311</i>	0.313	0.314	0.305
	1.256	1.244	1.252	1.256	<i>1.244</i>	1.252	1.289	1.259
parabola	0.214	0.214	<i>0.213</i>	0.214	0.214	0.213	0.221	0.233
	1.022	1.024	1.029	<i>1.022</i>	1.024	1.029	1.075	1.081
	0.241	0.241	0.243	<i>0.241</i>	0.241	0.243	0.238	0.252
	0.982	0.977	<i>0.975</i>	0.982	0.977	0.975	0.990	1.006

Table 24: Mean squared error of estimates for Y at location 3 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.234	0.241	0.250	<i>0.234</i>	0.241	0.250	0.269	0.288
	0.984	0.950	0.950	0.984	0.950	<i>0.950</i>	1.045	1.053
	0.260	0.293	0.259	0.260	0.293	<i>0.259</i>	0.304	0.333
	0.715	0.748	0.743	<i>0.715</i>	0.748	0.743	0.815	0.802
gradient	0.277	0.276	0.277	0.277	<i>0.276</i>	0.277	0.281	0.262
	<i>0.874</i>	0.882	0.875	0.874	0.882	0.875	0.885	0.870
	0.204	0.204	0.202	0.204	0.204	<i>0.202</i>	0.206	0.201
	0.776	0.785	0.776	0.776	0.785	<i>0.776</i>	0.807	0.810
parabola	0.249	0.246	0.247	0.249	<i>0.246</i>	0.247	0.247	0.245
	1.417	1.405	1.378	1.417	1.405	<i>1.378</i>	1.387	1.383
	0.306	0.306	0.304	0.306	0.306	0.304	0.297	<i>0.303</i>
	1.031	0.999	1.022	1.031	<i>0.999</i>	1.022	1.072	1.058

Table 25: Mean squared error of estimates for Y at location 4 (**minimum**, *next best*).

	lars	enet	glmnet	unshrunk.lars	unshrunk.enet	unshrunk.glmnet	oracular	gwr
step	0.219	0.231	0.224	<i>0.219</i>	0.231	0.224	0.293	0.234
	0.701	0.675	0.697	0.701	<i>0.675</i>	0.697	0.782	0.716
	0.206	0.259	0.203	0.206	0.259	<i>0.203</i>	0.278	0.238
	<i>0.889</i>	0.961	0.915	0.889	0.961	0.915	1.127	0.972
gradient	0.198	0.197	0.202	0.198	<i>0.197</i>	0.202	0.222	0.202
	1.245	1.257	1.256	<i>1.245</i>	1.257	1.256	1.289	1.275
	<i>0.216</i>	0.219	0.220	0.216	0.219	0.220	0.231	0.204
	0.877	0.919	0.884	<i>0.877</i>	0.919	0.884	1.068	0.996
parabola	0.223	0.230	0.225	<i>0.223</i>	0.230	0.225	0.223	0.328
	0.950	0.952	<i>0.948</i>	0.950	0.952	0.948	0.963	1.037
	0.199	<i>0.192</i>	0.201	0.199	0.192	0.201	0.190	0.282
	0.842	0.861	0.848	<i>0.842</i>	0.861	0.848	0.870	1.016

Table 26: Mean squared error of estimates for Y at location 5 (**minimum**, *next best*).

Variable name	Description
<code>pag</code>	Proportion working in agriculture
<code>pex</code>	Proportion working in extraction (mining)
<code>pman</code>	Proportion working in manufacturing
<code>pserve</code>	Proportion working in services
<code>pfire</code>	Proportion working in finance, insurance, and real estate
<code>potprof</code>	Proportion working in other professions
<code>pwh</code>	Proportion who are white
<code>pblk</code>	Proportion who are black
<code>phisp</code>	Proportion who are hispanic
<code>metro</code>	Is the county in a metropolitan area?

Table 27: Description of the variables used in the census-data example