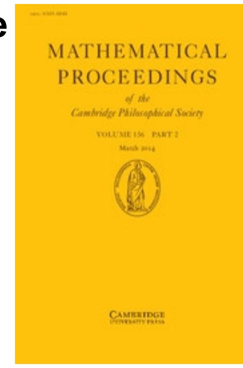

On a Gaussian process related to multivariate probability density estimation

B. W. Silverman

Mathematical Proceedings of the Cambridge Philosophical Society / Volume 80 / Issue 01 / July 1976, pp 135 - 144
DOI: 10.1017/S0305004100052762, Published online: 24 October 2008

Link to this article: http://journals.cambridge.org/abstract_S0305004100052762

How to cite this article:
B. W. Silverman (1976). On a Gaussian process related to multivariate probability density estimation. Mathematical Proceedings of the Cambridge Philosophical Society, 80, pp 135-144 doi:10.1017/S0305004100052762

Math. Proc. Camb. Phil. Soc. (1976), **80**, 135
MPCPS 80–16
Printed in Great Britain

On a Gaussian process related to multivariate probability density estimation

BY B. W. SILVERMAN
Statistical Laboratory, University of Cambridge

(*Received* 27 *May* 1975)

The multivariate Gaussian process with the same variance/covariance structure as the multivariate kernel density estimator in Euclidean space of dimension d is considered. An exact result is obtained for the limit in probability of the maximum of the normalized process. In addition weak and strong bounds are placed on the asymptotic behaviour of the maximum of the process over a multidimensional interval which is allowed to increase as the sample size increases. All the bounds obtained on the process are

$$O\left\{\left(\frac{1}{nh^d}\log\frac{1}{h}\right)^{\frac{1}{2}}\right\}.$$

Only the uniform continuity of the underlying density is assumed; the conditions on the kernel are also mild.

1. *Introduction and summary of results.* Suppose $f(\mathbf{t})$ is a d-dimensional multivariate probability density function, uniformly continuous on $\mathbb{R}^d$.

Suppose $\delta(\mathbf{t})$ is a kernel function on $\mathbb{R}^d$ with the following properties: (Unspecified integrals will be taken to be over $\mathbb{R}^d$ throughout.)

(i) δ is uniformly continuous with modulus of continuity w_δ.

(ii) $\int|\delta(\boldsymbol{\xi})|\,d\boldsymbol{\xi} < \infty$.

(iii) δ is bounded (and hence $\int\{\delta(\boldsymbol{\xi})\}^2\,d\boldsymbol{\xi} < \infty$).

(iv) $\delta(\mathbf{x})\to 0$ as $\|\mathbf{x}\|\to\infty$.

(v) Setting $\gamma(u) = \{w_\delta(u)\}^{\frac{1}{2}}$,

$$\int_0^1\left(\log\frac{1}{u}\right)^{\frac{1}{2}} d\gamma(u) < \infty.$$

These conditions on δ and f will be referred to as conditions (1).

Suppose $h(n)$ is a sequence of 'window widths', such that $h(n)\to 0$ as $n\to\infty$. The explicit dependence of $h(n)$ on n will generally be suppressed.

Consider a sequence of sample function continuous Gaussian processes $\rho_n(\mathbf{x})$ (with d-dimensional parameter $\mathbf{x}$) with $\mathbb{E}\rho_n(\mathbf{x}) = 0$ and

$$\begin{aligned}\operatorname{cov}(\rho_n(\mathbf{x}),\rho_n(\mathbf{y})) &= r_n(\mathbf{x},\mathbf{y})\\ &= h^{-2d}\int\delta\{h^{-1}(\mathbf{x}-\mathbf{t})\}\,\delta\{h^{-1}(\mathbf{y}-\mathbf{t})\}f(\mathbf{t})\,d\mathbf{t}\\ &\quad -h^{-2d}\int\delta\{h^{-1}(\mathbf{x}-\mathbf{t})\}f(\mathbf{t})\,d\mathbf{t}\int\delta\{h^{-1}(\mathbf{y}-\mathbf{t})\}f(\mathbf{t})\,d\mathbf{t}.\end{aligned} \qquad (2)$$

(The explicit dependence of ρ and r on n will generally be suppressed.)

The process ρ arises in the consideration of the kernel estimate f_n of the multivariate density f with 'window width' h defined by

$$f_n(\mathbf{x}) = \sum_{i=1}^{n} n^{-1} h^{-d} \delta\{h^{-1}(\mathbf{x}-\mathbf{X}_i)\},$$

where $\mathbf{X}_1, \ldots, \mathbf{X}_n$ are independent observations from the density f. This estimate was first considered by Rosenblatt (4) for the case $d = 1$. In recent work {cf. Rosenblatt (5) and Revesz (3)} the process $n^{\frac{1}{2}}(f_n - \mathbb{E}f_n)$ has been approximated by ρ. In this note the closeness of this approximation is not considered but limit theorems for the Gaussian process are proved which may be combined with results on the closeness of the approximation to yield properties of $f_n - \mathbb{E}f_n$.

The results of this note are as follows:

THEOREM A. *Suppose I is a bounded d-dimensional interval. Suppose conditions* (1) *hold on δ and f.*

Then, as $n \to \infty$,

$$\sup_{\mathbf{x}\in I} \left(\frac{1}{h^d}\log\frac{1}{h}\right)^{-\frac{1}{2}} \rho(\mathbf{x}) \xrightarrow{p} \{2d \textstyle\int \delta^2 \sup_I f(\mathbf{x})\}^{\frac{1}{2}}$$

and

$$\inf_{\mathbf{x}\in I} \left(\frac{1}{h^d}\log\frac{1}{h}\right)^{-\frac{1}{2}} \rho(\mathbf{x}) \xrightarrow{p} -\{2d \textstyle\int \delta^2 \sup_I f(\mathbf{x})\}^{\frac{1}{2}}.$$

THEOREM B. *Suppose conditions* (1) *hold. Suppose I_n is a sequence of d-dimensional intervals with $l(n) = \mathrm{Leb}\,(I_n)$. Suppose*

$$\{h(n)\}^{\epsilon}\, l(n) \to 0 \quad \textit{for some } \epsilon > 0; \tag{3}$$

let

$$\epsilon_0 = \inf\{\epsilon : h^{\epsilon} l \to 0 \textit{ as } n \to \infty\}.$$

Then

$$p\limsup_{n\to\infty} \sup_{I_n} \left(\frac{1}{h^d}\log\frac{1}{h}\right)^{-\frac{1}{2}} |\rho(\mathbf{x})| \leqslant \{2(d+\epsilon_0)\, M_1 \textstyle\int \delta^2\}^{\frac{1}{2}}$$

where $M_1 = \sup_{\cup I_n} f(\mathbf{x})$. (The notations $p\limsup$ and $p\liminf$ will be used throughout to denote the lim sup and lim inf in probability.)

THEOREM C. *Suppose conditions* (1) *hold. Defining I_n, $l(n)$ and M_1 as in theorem B, suppose*

$$\sum_n \{h(n)\}^{\epsilon}\, l(n) < \infty \quad \textit{for some } \epsilon. \tag{4}$$

Let

$$\epsilon_1 = \inf\{\epsilon : \Sigma h^{\epsilon} l < \infty\}.$$

Then

$$\limsup_{n\to\infty} \sup_{I_n} \left(\frac{1}{h^d}\log\frac{1}{h}\right)^{-\frac{1}{2}} |\rho(\mathbf{x})| \leqslant \{2(d+\epsilon_1)\, M_1 \textstyle\int \delta^2\}^{\frac{1}{2}} \quad \text{a.s.}$$

Note. Allowing the interval I to vary in Theorems B and C is of use when considering $\sup_{\mathbb{R}^d} |f_n - f|$. Given any reasonable dependence of h on n, and very mild conditions on the tail of f, it is possible to define a sequence of intervals I_n such that conditions (3) and (4) are satisfied and

$$\text{pr}\,(\text{all of } \mathbf{X}_1, \ldots, \mathbf{X}_n \text{ lie within } I_n) \to 1 \quad \text{as } n \to \infty$$

or indeed

$$\sum_{n=1}^{\infty} \text{pr}\,(\text{any of } \mathbf{X}_1, \ldots, \mathbf{X}_n \text{ lie outside } I_n) < \infty$$

so that weak and strong results for

$$\sup_{\mathbb{R}^d} |f_n - f|$$

may be deduced from those for

$$\sup_{I_n} |f_n - f|.$$

In any case, setting $I_n = I$ for all n in Theorem C yields a strong result for $\sup_I |\rho|$.

2. *Proofs of Theorems.* First of all we make some additional definitions. Let

$$\begin{aligned}
k_0(\boldsymbol{\xi}) &= \int \delta(\mathbf{t})\,\delta(\mathbf{t}+\boldsymbol{\xi})\,d\mathbf{t} \quad \text{for} \quad \boldsymbol{\xi} \in \mathbb{R}^d;\\
k_1(\boldsymbol{\xi}) &= \int |\delta(\mathbf{t})\,\delta(\mathbf{t}+\boldsymbol{\xi})|\,d\mathbf{t} \quad \text{for} \quad \boldsymbol{\xi} \in \mathbb{R}^d;\\
\kappa(a) &= \sup_{\|\mathbf{a}\| \leqslant a} \{k_0(\mathbf{0}) - k_0(\mathbf{a})\}^{\frac{1}{2}} \quad \text{for} \quad a \geqslant 0.\\
M_0 &= \sup_{\mathbb{R}^d} f(\mathbf{x}); \quad M_0 < \infty \text{ since } f \text{ is uniformly continuous};\\
\sigma^2(\mathbf{x}) &= r(\mathbf{x}, \mathbf{x}) = \text{var}\,\rho(\mathbf{x}).
\end{aligned}$$

Various properties of r, σ and δ will also be needed; these are collected in the following lemma. (Unless otherwise stated, the notations o and O apply uniformly over $\mathbb{R}^d$ throughout.)

LEMMA 1. *With σ, r, k_1, and κ defined as above,*

$$\sigma^2(\mathbf{x}) = h^{-d} f(\mathbf{x}) \int \delta^2 + o(h^{-d}) \quad \textit{as } n \to \infty, \tag{5}$$

$$|r(\mathbf{x}, \mathbf{x} + h\boldsymbol{\xi})| = k_1(\boldsymbol{\xi})\, O(h^{-d}) + O(1) \tag{6}$$

uniformly over $\boldsymbol{\xi}$ and $\mathbf{x}$ as $n \to \infty$, and

$$\int_0^1 \{\log(1/u)\}^{\frac{1}{2}}\, d\kappa(u) < \infty. \tag{7}$$

Proofs. Changing the variable in the integral and using the bound M_0 for f,

$$\left|h^{-d} \int \delta\{h^{-1}(\mathbf{x} - \mathbf{t})\} f(\mathbf{t})\, d\mathbf{t}\right| \leqslant M_0 \int |\delta|. \tag{8}$$

Substituting (8) into (2) and setting $\mathbf{x} = \mathbf{y}$,

$$\sigma^2(\mathbf{x}) = h^{-2d} \int [\delta\{h^{-1}(\mathbf{x} - \mathbf{t})\}]^2 f(\mathbf{t})\, d\mathbf{t} + O(1).$$

By changing the variable in the integral,

$$\begin{aligned}
\sigma^2(\mathbf{x}) &= h^{-d} \int \{\delta(\mathbf{s})\}^2 f(\mathbf{x} - h\mathbf{s})\, d\mathbf{s} + O(1)\\
&= h^{-d} f(\mathbf{x}) \int \delta^2 + h^{-d} \int \{\delta(\mathbf{s})\}^2 \{f(\mathbf{x} - h\mathbf{s}) - f(\mathbf{x})\}\, d\mathbf{s} + O(1).
\end{aligned} \tag{9}$$

Letting w_f be the modulus of continuity of f and then applying the dominated convergence theorem (using the fact that $\int \delta^2 < \infty$)

$$\begin{aligned}|\text{second term of (9)}| &\leqslant h^{-d}\int \{\delta(\mathbf{s})\}^2 w_f(h\|\mathbf{s}\|)\,d\mathbf{s}\\ &= o(h^{-d}).\end{aligned}$$

Thus it follows that $\sigma^2(\mathbf{x}) = h^{-d}f(\mathbf{x})\int \delta^2 + o(h^{-d})$ as required.

To prove (6), set $\mathbf{y} = \mathbf{x} + h\boldsymbol{\xi}$ in (2), use the result of (8) and then change the variable in the integral to get

$$\begin{aligned}|r(\mathbf{x}, \mathbf{x}+h\boldsymbol{\xi})| &= |h^{-2d}\int \delta\{h^{-1}(\mathbf{x}-\mathbf{t})\}\,\delta\{h^{-1}(\mathbf{x}-\mathbf{t})+\boldsymbol{\xi}\}f(\mathbf{t})\,d\mathbf{t}| + O(1)\\ &\leqslant M_0 h^{-d}\int |\delta(\mathbf{y})\,\delta(\mathbf{y}+\boldsymbol{\xi})|\,d\mathbf{y} + O(1)\\ &= k_1(\boldsymbol{\xi})\,O(h^{-d}) + O(1)\end{aligned}$$

as required. To prove (7), we first demonstrate a connexion between κ and γ.

For $\|\mathbf{a}\| \leqslant a$,

$$\begin{aligned}|k_0(\mathbf{0}) - k_0(\mathbf{a})| &= |\int \delta(\mathbf{t})\,\{\delta(\mathbf{t}) - \delta(\mathbf{t}+\mathbf{a})\}\,d\mathbf{t}|\\ &\leqslant w_\delta(a)\int |\delta|\\ &= \gamma(a)^2 \times \text{constant}.\end{aligned}$$

So $\kappa(a) \leqslant \text{constant} \times \gamma(a)$. So

$$\int_0^1 \{\log(1/u)\}^{\frac{1}{2}}\,d\kappa(u) < \infty \quad \text{as required.}$$

It is now possible to prove a proposition which places an asymptotic lower bound on the maximum of the Gaussian process.

PROPOSITION 1. *Suppose I_0 is an open set in $\mathbb{R}^d$ such that*

$$\sup_{\mathbf{x}\in I_0} f(\mathbf{x}) = M > 0.$$

Then

$$p\liminf_{n\to\infty} \{h^{-d}\log(1/h)\}^{-\frac{1}{2}} \sup_{\mathbf{x}\in I_0} \rho(\mathbf{x}) \geqslant (2dM\int \delta^2)^{\frac{1}{2}}$$

and

$$p\limsup_{n\to\infty} \{h^{-d}\log(1/h)\}^{-\frac{1}{2}} \inf_{\mathbf{x}\in I_0} \rho(\mathbf{x}) \leqslant -(2dM\int \delta^2)^{\frac{1}{2}}.$$

Proof. The processes ρ and $-\rho$ have the same probability structure, so the second result follows at once from the first.

Given any $c^2 < 2dM\int \delta^2$, with $c > 0$, let

$$1 - \lambda = c(2dM\int \delta^2)^{-\frac{1}{2}}.$$

Choose a d-dimensional interval I contained in I_0 such that $f(\mathbf{x}) > (1 - \frac{1}{8}\lambda)\,M$ for all $\mathbf{x}$ in I. This is possible because f is uniformly continuous. Let l be the Lebesgue measure of I. Because f is bounded away from zero on I, it follows from (5) that, uniformly over I as $n \to \infty$,

$$\sigma^2(\mathbf{x}) = h^{-d}f(\mathbf{x})\int \delta^2\{1 + o(1)\}$$

and therefore that, for sufficiently small h,

$$\inf_I \sigma(\mathbf{x}) \geqslant \{h^{-d}M\int \delta^2\}^{\frac{1}{2}}\,(1 - \tfrac{1}{2}\lambda). \tag{10}$$

Suppose it were true that

$$\sup_I \{\rho(\mathbf{x})/\sigma(\mathbf{x})\} \geqslant \{2d \log(1/h)\}^{\frac{1}{2}}(1-\tfrac{1}{2}\lambda).$$

It would follow from (10) and the definition of c that, for sufficiently small h,

$$\begin{aligned}\sup_{I_0} \rho(\mathbf{x}) &\geqslant \sup_I \rho(\mathbf{x}) \\ &\geqslant \inf_I \sigma(\mathbf{x}) \sup_I \{\rho(\mathbf{x})/\sigma(\mathbf{x})\} \\ &\geqslant (1-\tfrac{1}{2}\lambda)^2 \{2dM \textstyle\int \delta^2 h^{-d} \log(1/h)\}^{\frac{1}{2}} \\ &\geqslant c\{h^{-d}\log(1/h)\}^{\frac{1}{2}}.\end{aligned}$$

Therefore it suffices to prove that

$$\operatorname{pr}[\sup_I \{\rho(\mathbf{x})/\sigma(\mathbf{x})\} \geqslant (1-\tfrac{1}{2}\lambda)\{2d\log(1/h)\}^{\frac{1}{2}}] \to 1 \quad \text{as } n \to \infty. \tag{11}$$

It will then follow that

$$\operatorname{pr}[\sup_{I_0} \rho(\mathbf{x}) \geqslant c\{h^{-d}\log(1/h)\}^{\frac{1}{2}}] \to 1 \quad \text{as } n \to \infty \text{ as required.}$$

Proof of (11). Following Cramer and Leadbetter (1), section (13·5), define

$$\eta(\mathbf{x}) = \begin{cases} 1 & \text{if } \rho(\mathbf{x})/\sigma(\mathbf{x}) > (1-\tfrac{1}{2}\lambda)\{2d\log(1/h)\}^{\frac{1}{2}}, \\ 0 & \text{otherwise.}\end{cases}$$

Let
$$Z_0 = l^{-1}\int_I \eta(\mathbf{x})\,d\mathbf{x}.$$

Now use the same argument as (1) p. 296 (an application of Chebyshev's inequality followed by manipulation of the bivariate normal density function) to get

$$\begin{aligned}\operatorname{pr}[\sup_I &\{\rho(\mathbf{x})/\sigma(\mathbf{x})\} \leqslant (1-\tfrac{1}{2}\lambda)\{2d\log(1/h)\}^{\frac{1}{2}}] \\ &= \operatorname{pr}(Z_0 = 0) \\ &\leqslant \frac{\operatorname{var} Z_0}{\mathbb{E}Z_0^2} \\ &\leqslant Kl^{-2}(1-\tfrac{1}{2}\lambda)^2 \log(1/h) \\ &\qquad \times \iint_I |\chi| \exp\{2d\log(1/h)(1-\tfrac{1}{2}\lambda)^2 |\chi|/(1+|\chi|)\},\end{aligned} \tag{12}$$

where K is a constant and $\chi(\mathbf{x},\mathbf{y}) = r(\mathbf{x},\mathbf{y})/\sigma(\mathbf{x})\,\sigma(\mathbf{y})$. [Cf. expression (13·5·5) of (1).]

Because $|\chi|/(1+|\chi|) \leqslant \frac{1}{2}$, the integrand in (12) is bounded by

$$(\inf_I \sigma)^{-2} |r(\mathbf{x},\mathbf{y})|\, h^{-(1-\frac{1}{2}\lambda)^2 d}.$$

By substituting for $\mathbf{y}$ in the integral and then using (6), we have, putting

$$J(\mathbf{x}) = h^{-1}(I-\mathbf{x}),$$

a d-dimensional interval of Lebesgue measure $h^{-d}l$,

$$\iint_I |r(\mathbf{x},\mathbf{y})|\,d\mathbf{x}\,d\mathbf{y} = h^d\int_I d\mathbf{x}\int_J |r(\mathbf{x},\mathbf{x}+h\boldsymbol{\xi})|\,d\boldsymbol{\xi}$$
$$\leqslant h^d\int_I d\mathbf{x}\int_J \{O(1)+k_1(\boldsymbol{\xi})\,O(h^{-d})\}\,d\boldsymbol{\xi}$$
$$= h^d l^2 h^{-d}O(1) + h^d l O(h^{-d})\int_J k_1(\boldsymbol{\xi})\,d\boldsymbol{\xi}$$
$$= O(1) \quad \text{since} \quad \int k_1 \leqslant \left(\int|\delta|\right)^2 < \infty.$$

Now combine this result with the result of (10) to show that the double integral in (12) is

$$O(h^d)\,.\,O(1)\,.\,h^{-(1-\frac{1}{2}\lambda)^2 d}$$

and hence that

$$\operatorname{pr}[\sup_I\{\rho(\mathbf{x})/\sigma(\mathbf{x})\} < (1-\tfrac{1}{2}\lambda)\{2d\log(1/h)\}^{\frac{1}{2}}] = O\{h^{d(\lambda-\frac{1}{4}\lambda^2)}\log(1/h)\},$$

which tends to zero as h tends to zero and hence as n tends to infinity.

This completes the proof of (11) and hence of Proposition 1.

In order to be able to complete the proof of Theorem A, and to prove Theorems B and C, three lemmas will be required. The first of these is a trivial generalization of a result of Garsia (2) on the modulus of continuity of Gaussian processes with multi-dimensional 'time' parameter.

LEMMA 2. {Theorems 1 and 2 of (2).} *Suppose X is a continuous Gaussian process on a d-dimensional interval I of Lebesgue measure l. Suppose*

$$p(u) \geqslant \sup_{\|\mathbf{s}-\mathbf{t}\|\leqslant ud^{\frac{1}{2}}} [\mathbb{E}\{X(\mathbf{s})-X(\mathbf{t})\}^2]^{\frac{1}{2}}$$

and $p(0) = 0$. Then there exists a random variable B such that, for $\epsilon \leqslant 1$, the modulus of continuity of X on I is almost surely dominated by

$$16(\log B)^{\frac{1}{2}}p(\epsilon) + 16(2d)^{\frac{1}{2}}\int_0^\epsilon \{\log(1/u)\}^{\frac{1}{2}}\,dp(u) \tag{13}$$

and $\mathbb{E}B \leqslant 4l^2\sqrt{2}$.

Lemma 3 is obtained by applying Lemma 2 to the process ρ.

LEMMA 3. *Let w_ρ^I be the modulus of continuity of ρ on I, a d-dimensional interval of Lebesgue measure l. Then $w_\rho^I(\epsilon)$ is almost surely dominated by*

$$h^{-\frac{1}{2}d}\left[16(\log B_n)^{\frac{1}{2}}(2M_0)^{\frac{1}{2}}\kappa(\epsilon d^{\frac{1}{2}}/h) + 32(M_0 d)^{\frac{1}{2}}\int_0^{\epsilon d^{\frac{1}{2}}/h}\{\log(d^{\frac{1}{2}}/hu)\}^{\frac{1}{2}}\,d\kappa(u)\right],$$

where B_n is a random variable with $\mathbb{E}B_n \leqslant 4l^2\sqrt{2}$.

Proof. The proof depends on finding a suitable function p which is then substituted into (13) to obtain the result.

Suppose $\mathbf{x}$ and $\mathbf{y}$ are in I. By (2),

$$\begin{aligned}\mathbb{E}\{\rho(\mathbf{x})-\rho(\mathbf{y})\}^2 &= h^{-2d}\int[\delta\{h^{-1}(\mathbf{x}-\mathbf{s})\}-\delta\{h^{-1}(\mathbf{y}-\mathbf{s})\}]^2 f(\mathbf{s})\,d\mathbf{s}\\ &\quad -h^{-2d}[\int\delta\{h^{-1}(\mathbf{x}-\mathbf{s})\}f(\mathbf{s})\,d\mathbf{s}-\int\delta\{h^{-1}(\mathbf{y}-\mathbf{s})\}f(\mathbf{s})\,d\mathbf{s}]^2\\ &\leqslant M_0 h^{-2d}\int[\delta\{h^{-1}(\mathbf{x}-\mathbf{s})\}-\delta\{h^{-1}(\mathbf{y}-\mathbf{s})\}]^2 d\mathbf{s}\\ &= 2M_0 h^{-d}[k_0(\mathbf{0})-k_0\{h^{-1}(\mathbf{x}-\mathbf{y})\}].\end{aligned}$$

Therefore, by definition of κ,

$$\sup_{\|\mathbf{x}-\mathbf{y}\|\leqslant ud^{\frac12}}[\mathbb{E}\{\rho(\mathbf{x})-\rho(\mathbf{y})\}^2]^{\frac12}\leqslant(2M_0h^{-d})^{\frac12}\kappa(ud^{\frac12}/h).$$

Substituting $p(u)=(2M_0h^{-d})^{\frac12}\kappa(ud^{\frac12}/h)$ in Lemma 2 completes the proof.

From now on, allow I to depend on n as in (3) and (4) above. Let w_n be the modulus of continuity of the process $\{h^{-d}\log(1/h)\}^{-\frac12}\rho$ on the d-dimensional interval I_n. Lemma 3 is now used to prove a result about w_n.

Lemma 4. *Suppose I and h depend on n as in the statement of Theorem B. Define ϵ_0 as in* (3). *With the above definition of w_n*

$$p\limsup_{n\to\infty} w_n(h\epsilon)\leqslant 32(\epsilon_0^{\frac12}+d^{\frac12})\,M_0^{\frac12}\kappa(\epsilon d^{\frac12}).$$

If ϵ_1 exists as in (4),

$$\limsup_{n\to\infty} w_n(h\epsilon)\leqslant 32(\epsilon_1^{\frac12}+d^{\frac12})\,M_0^{\frac12}\kappa(\epsilon d^{\frac12})\quad \text{a.s.}$$

Proof. By Lemma 3, $w_n(h\epsilon)$ is a.s. dominated by

$$16\{\log(1/h)\}^{-\frac12}(\log B_n)^{\frac12}(2M_0)^{\frac12}\kappa(\epsilon d^{\frac12})+32(M_0d)^{\frac12}\int_0^{\epsilon d^{\frac12}}\{\log(d^{\frac12}/hu)/\log(1/h)\}^{\frac12}d\kappa(u). \tag{14}$$

To deal with the first term of (14), use standard theory to get, for any $\eta>0$,

$$\begin{aligned}\mathrm{pr}[\{\log(1/h)\}^{-\frac12}(\log B_n)^{\frac12}>\eta] &= \mathrm{pr}[\log B_n>\eta^2\log(1/h)]\\ &= \mathrm{pr}[B_n>h^{-\eta^2}]\\ &< h^{\eta^2}\mathbb{E}B_n.\end{aligned}\tag{15}$$

Provided $\eta>(2\epsilon_0)^{\frac12}$, we have, by definition of ϵ_0,

$$h^{\frac12\eta^2}l\to 0\quad \text{as } n\to\infty.$$

The reasoning of (15) will then imply that

$$\mathrm{pr}[\{\log(1/h)\}^{-\frac12}(\log B_n)^{\frac12}>\eta]\to 0\quad \text{as } n\to\infty$$

so that

$$p\limsup_{n\to\infty}[\text{first term of (14)}]\leqslant 32(\epsilon_0M_0)^{\frac12}\kappa(\epsilon d^{\frac12}). \tag{16}$$

To prove the strong analogue of this result suppose that ϵ_1 exists and choose any $\eta>(2\epsilon_1)^{\frac12}$. The reasoning of (15) will imply that

$$\sum_n \mathrm{pr}[\{\log(1/h)\}^{-\frac12}(\log B_n)^{\frac12}>\eta]<\infty.$$

An application of the first Borel Cantelli Lemma then gives

$$\limsup_{n\to\infty} [\text{first term of } (14)] \leqslant 32(\epsilon_1 M_1)^{\frac{1}{2}} \kappa(\epsilon d^{\frac{1}{2}}) \quad \text{a.s.} \tag{17}$$

To deal with the second term of (14), notice that, for $u \leqslant d^{\frac{1}{2}}$ and sufficiently small h,

$$\{\log (d^{\frac{1}{2}}/hu)/\log (1/h)\}^{\frac{1}{2}} \leqslant 1 + (\log d^{\frac{1}{2}})^{\frac{1}{2}} + \{\log (1/u)\}^{\frac{1}{2}}.$$

By (7), the dominated convergence theorem can now be applied to obtain

$$\begin{aligned}\lim_{n\to\infty} \{\text{second term of } (14)\} &= 32(M_0 d)^{\frac{1}{2}} \int_0^{\epsilon d^{\frac{1}{2}}} d\kappa(u) \\ &= 32(M_0 d)^{\frac{1}{2}} \kappa(\epsilon d^{\frac{1}{2}}).\end{aligned} \tag{18}$$

To complete the proof of Lemma 4, put together (16), (17) and (18).

Next we prove a proposition which provides an asymptotic upper bound on the maximum of the Gaussian process. The two parts of the proposition are Theorems B and C respectively. Theorem A will follow as an easy corollary.

PROPOSITION 2. *With I and h depending on n as in* (3), *and* $M_1 = \sup_{\cup I_n} f$,

$$p \limsup_{n\to\infty} \sup_{I_n} \{h^{-d} \log (1/h)\}^{-\frac{1}{2}} |\rho(\mathbf{x})| \leqslant \{2(d+\epsilon_0) M_1 \textstyle\int \delta^2\}^{\frac{1}{2}}$$

and if ϵ_1 *exists as in* (4)

$$\limsup_{n\to\infty} \sup_{I_n} \{h^{-d} \log (1/h)\}^{-\frac{1}{2}} |\rho(\mathbf{x})| \leqslant \{2(d+\epsilon_1) M_1 \textstyle\int \delta^2\}^{\frac{1}{2}} \quad \text{a.s.}$$

Proof. The proof makes use of the bounds of Lemma 4 and the device described next. This approach brings together Garsia's result and the 'exceedance measures' of Cramer and Leadbetter.

For given positive u and v, suppose we had (with w_n defined as in Lemma 4)

$$w_n(h\epsilon) < \tfrac{1}{2}v \tag{19}$$

and also

$$\sup_I \{h^{-d} \log (1/h)\}^{-\frac{1}{2}} |\rho(\mathbf{x})| > u+v. \tag{20}$$

Then there would be some $\mathbf{x}_0$ in I such that

$$\{h^{-d} \log (1/h)\}^{-\frac{1}{2}} |\rho(\mathbf{x}_0)| > u+v.$$

The bound (19) on the modulus of continuity of the process $\{h^{-d} \log (1/h)\}^{-\frac{1}{2}} \rho$ would then imply that

$$\{h^{-d} \log (1/h)\}^{-\frac{1}{2}} |\rho(\mathbf{x})| > u + \tfrac{1}{2}v$$

for all $\mathbf{x}$ in a neighbourhood (in I) of $\mathbf{x}_0$ of radius at least $h\epsilon$, so that

$$\text{Leb}\,\{\mathbf{x} \in I : [h^{-d} \log (1/h)]^{-\frac{1}{2}} |\rho(\mathbf{x})| > u + \tfrac{1}{2}v\} > ch^d \epsilon^d, \tag{21}$$

where c is a constant which depends on the dimension d.

The proof of the first part of the proposition now proceeds as follows. Choose any

$$u > \{2(d+\epsilon_0)\, M_1 \textstyle\int \delta^2\}^{\frac{1}{2}} \quad \text{and any } v > 0.$$

The truth of (7) implies that $\kappa(u) \to 0$ as $u \to 0$. Choose $\epsilon > 0$ such that

$$32 M_0^{\frac{1}{2}}(d^{\frac{1}{2}} + \epsilon_0^{\frac{1}{2}})\, \kappa(\epsilon d^{\frac{1}{2}}) < \tfrac{1}{2}v.$$

Then, by Lemma 4, the probability that (19) holds tends to one as n tends to infinity. We prove below that the probability that (21) holds tends to zero. Since (19) and (20) together imply (21), it will follow that the probability that (20) holds tends to zero.

The first part of Proposition 2 will then follow from the way that u and v were chosen.

To prove the second part of Proposition 2 a similar argument is used. Choose any

$$u' > \{2(d+\epsilon_1)\, M_1 \textstyle\int \delta^2\}^{\frac{1}{2}} \quad \text{and any } v > 0.$$

Choose $\epsilon' > 0$ such that

$$32 M_0^{\frac{1}{2}}(d^{\frac{1}{2}} + \epsilon_1^{\frac{1}{2}})\, \kappa(\epsilon' d^{\frac{1}{2}}) < \tfrac{1}{2}v.$$

Substitute u' and ϵ' for u and ϵ in (19), (20) and (21). Then, by Lemma 4, the probability that (19) holds eventually is one. It is proved below that the probability that (21) holds for infinitely many n is zero. It follows that, with probability one, (20) holds for only finitely many n, completing the proof of the second part of Proposition 2.

We prove that

$$\mathrm{pr}\,[\mathrm{Leb}\,\{\mathbf{x} \in I : [h^{-d} \log(1/h)]^{-\frac{1}{2}}\, |\rho(\mathbf{x})| > u + \tfrac{1}{2}v\} > c h^d \epsilon^d] \to 0 \tag{22}$$

(i.e. $\mathrm{pr}\,\{(21) \text{ holds}\} \to 0$).

Set

$$\eta(\mathbf{x}) = \begin{cases} 1 & \text{if } \quad |\rho(\mathbf{x})|/\sigma(\mathbf{x}) > u\,\{\log(1/h)/M_1 \textstyle\int \delta^2\}^{\frac{1}{2}}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$Z_0 = l^{-1} \int_I \eta(\mathbf{x})\, d\mathbf{x}.$$

We have, by (5),

$$\begin{aligned} \sigma^2(\mathbf{x}) &= h^{-d} f(\mathbf{x}) \textstyle\int \delta^2 + o(h^{-d}) \quad \text{uniformly on } \mathbb{R}^d, \\ &< M_1 h^{-d} \textstyle\int \delta^2 \{1 + o(1)\} \quad \text{uniformly on } \cup I_n, \end{aligned}$$

so

$$\{\sigma(\mathbf{x})\}^{-1} \geqslant (h^d / M_1 \textstyle\int \delta^2)^{\frac{1}{2}} \{1 + o(1)\} \quad \text{uniformly on } \cup I_n.$$

Therefore, for all sufficiently small h,

$$\begin{aligned} l Z_0 &= \mathrm{Leb}\,\{\mathbf{x} \in I : |\rho(\mathbf{x})|/\sigma(\mathbf{x}) > u[\log(1/h)/M_1 \textstyle\int \delta^2]^{\frac{1}{2}}\} \\ &\geqslant \mathrm{Leb}\,\{\mathbf{x} \in I : [h^{-d} \log(1/h)]^{-\frac{1}{2}}\, |\rho(\mathbf{x})| > u + \tfrac{1}{2}v\}. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{pr}\,\{(21) \text{ holds}\} &\leqslant \mathrm{pr}\,(l Z_0 > c h^d \epsilon^d) \\ &\leqslant l c^{-1} h^{-d} \epsilon^{-d}\, \mathbb{E} Z_0. \end{aligned} \tag{23}$$

By definition of Z_0 and standard approximations to the Normal distribution and density functions, {cf. (2·10·2) of (1)}

$$\begin{aligned}\mathbb{E}Z_0 &= l^{-1}\int_I \operatorname{pr}\{\eta(\mathbf{x}) = 1\} \\ &= 2(1-\Phi[u\{\log(1/h)/M_1\textstyle\int\delta^2\}^{\frac{1}{2}}]) \\ &\leqslant 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}u^{-1}\{\log(1/h)/M_1\textstyle\int\delta^2\}^{-\frac{1}{2}}\exp\{-\frac{1}{2}u^2\log(1/h)/M_1\textstyle\int\delta^2\}.\end{aligned} \tag{24}$$

Substituting (24) in (23), the probability that (21) holds is bounded by a constant multiple of

$$\{\log(1/h)\}^{-\frac{1}{2}}\,lh^{-d+\frac{1}{2}u^2/M_1\int\delta^2}. \tag{25}$$

The expression (25) tends to zero as n tends to infinity, since $\{\log(1/h)\}^{-\frac{1}{2}}\to 0$ and $-d+\frac{1}{2}u^2/M_1\int\delta^2 > \epsilon_0$ by definition of u. The proof of the first part of Proposition 2 is now completed.

To prove the strong analogue of (22), substitute u' and ϵ' for u and ϵ throughout, and use the fact that $\{\log(1/h)\}^{-\frac{1}{2}}\to 0$ to get

$$\sum_n \operatorname{pr}\{(21)\text{ holds}\} \leqslant \text{constant}\times\sum_n lh^{-d+\frac{1}{2}u'^2/M_1\int\delta^2}. \tag{26}$$

Now use Borel Cantelli, noticing that

$$-d+\tfrac{1}{2}u'^2/M_1\textstyle\int\delta^2 > \epsilon_1$$

so that, by definition of ϵ_1, the series in (26) converges. This implies that, almost surely,

$$\operatorname{Leb}\{\mathbf{x}\in I:[h^{-d}\log(1/h)]^{-\frac{1}{2}}\,|\rho(\mathbf{x})| > u'+\tfrac{1}{2}v\} > ch^d\epsilon'^d$$

for only finitely many n.

This completes the proof of the second part of Proposition 2.

To prove Theorem A, set all the I_n equal to I. Then $M_1 = M$, and, since $h\to 0$ and l is constant, $\epsilon_0 = 0$. Proposition 2 thus gives

$$p\limsup_{n\to\infty}\sup_I\{h^{-d}\log(1/h)\}^{-\frac{1}{2}}\,|\rho(\mathbf{x})| \leqslant (2dM\textstyle\int\delta^2)^{\frac{1}{2}}. \tag{27}$$

Combining (27) with Proposition 1 (applied to the interior of I) completes the proof of Theorem A.

REFERENCES

(1) CRAMER, H. and LEADBETTER, M. R. *Stationary and related stochastic processes.* (New York: J. Wiley & Sons, Inc., 1967.)

(2) GARSIA, A. M. Continuity properties of Gaussian processes with multidimensional time parameter. *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability, volume II.* (Berkeley: University of California Press, 1972), 369–374.

(3) REVESZ, P. On multivariate empirical density functions. (To appear.)

(4) ROSENBLATT, M. Remarks on some non-parametric estimates of a density function. *Ann. Math. Statist.* **27** (1956), 832–837.

(5) ROSENBLATT, M. A quadratic measure of deviation of two-dimensional density estimates and a test of independence. *Ann. Statist.* **3** (1975), 1–14.