Local variable selection and parameter estimation for spatially varying coefficient models

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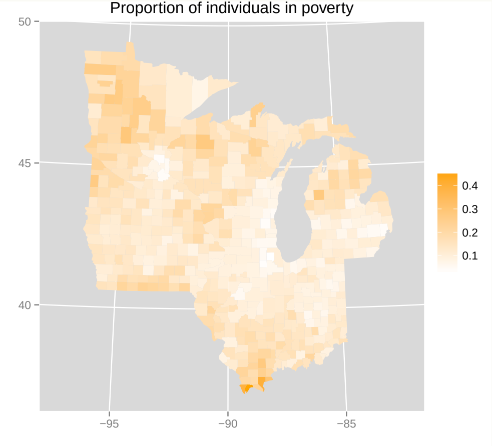
Motivation

Take a look at some data



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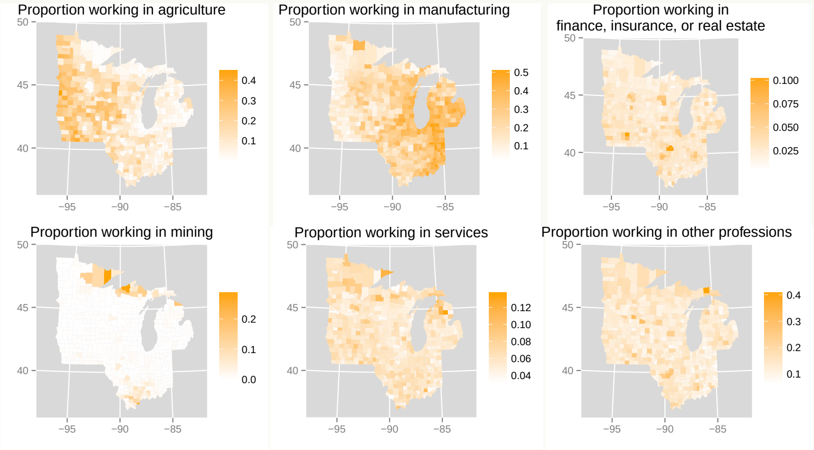

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Sensible questions about the data

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- What are the sign and magnitude of that association?
- Is poverty rate associated with the same economic-structure variables across the entire region?
- Are the sign and magnitude of the associations constant across the region?

Introduction

A review of existing methods

- Spatial regression

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- Varying coefficient regression

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- Spatial regression
- Varying coefficient regression
  - Splines
  - Kernels
  - Wavelets
- Model selection via regularization

Introduction

Some definitions

- Univariate spatial response process $\{Y(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{D}\}$
- Multivariate spatial covariate process $\{\boldsymbol{X}(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{D}\}$
- $n =$ number of observations
- $p =$ number of covariates
- Location (2-dimensional) $\boldsymbol{s}$
- Spatial domain $\mathcal{D}$

Introduction

Further definitions

- Geostatistical data:
  - Observations are made at sampling locations $\boldsymbol{s}_i$ for $i = 1, \ldots, n$
  - E.g. elevation, temperature
- Areal data:
  - Domain is partitioned into n regions $\{D_1, \ldots, D_n\}$
  - The regions do not overlap, and they divide the domain completely: $\mathcal{D} = \bigcup_{i=1}^{n} D_i$
  - Sampling locations $\boldsymbol{s}_i$ for $i = 1, \ldots, n$ are the centroids of the regions
  - E.g. poverty rate, population, spatial mean temperature

Introduction

Existing approaches: spatial regression

- The typical spatial regression (Cressie, 1993)

$$\begin{aligned} Y(\boldsymbol{s}) &= \boldsymbol{X}(\boldsymbol{s})'\boldsymbol{\beta} + W(\boldsymbol{s}) + \varepsilon(\boldsymbol{s}) \\ \text{cov}(W(\boldsymbol{s}), W(\boldsymbol{t})) &= \Gamma\left(\delta(\boldsymbol{s}, \boldsymbol{t})\right) \\ \delta(\boldsymbol{s}, \boldsymbol{t}) &= \sqrt{\|\boldsymbol{t} - \boldsymbol{s}\|_2} \\ \text{E.g. } \Gamma(\delta(\boldsymbol{s}, \boldsymbol{t})) &= \text{exp}\{-\phi^{-1}\delta(\boldsymbol{s}, \boldsymbol{t})\} \qquad (1) \end{aligned}$$

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- The coefficients $\boldsymbol{\beta}$ are constant
- Relies on *a priori* global variable selection

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- Coefficients are functions of S
- Generally assume that the coefficient functions are smooth
- This is a varying coefficient regression (VCR) model (Hastie and Tibshirani, 1993)
- If $\boldsymbol{s}$ is a spatial location then we have a spatially varying coefficient regression (SVCR) model

Introduction

Existing approaches: spatially varying coefficient process

- Making model more flexible: coefficients in a spatial regression model can be allowed to vary (Gelfand et al., 2003)

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- $\{\beta_1(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{D}\}, \ldots, \{\beta_p(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{D}\}$ are stationary spatial processes with Matèrn covariance functions
- Still relies on *a priori* global variable selection

Introduction

Existing approaches: spline-based VCR and SVCR models

- Splines are a way to parameterize smooth functions
- Splines can be incorporated into a generalized additive model (GAM):
  - $E\{Y(t)\} = f\{X_1(t)\} + \cdots + f\{X_p(t)\}$
- It is possible to parameterize a VCR model with splines for the coefficient functions:
  - $E\{Y(t)\} = \beta_1(t)X_1(t) + \cdots + \beta_p(t)X_p(t)$

Introduction

Existing approaches: Global selection in spline-based VCR models

Regularization methods for global variable selection in VCR models:

- The integral of a function squared (e.g. $\int \{f(t)\}^2 dt$) is zero if and only if the function is zero everywhere.
- Use regularization (maximize the likelihood plus a penalty) to encourage coefficient functions to be zero
- SCAD penalty (Fan and Li, 2001) on the integral of the square of the coefficient function (Wang et al., 2008)
- Non-negative garrote penalty (Breiman, 1995) on the integral of the square of the coefficient function (Antoniadas et al., 2012)

Introduction

Existing approaches: wavelet methods for VCR models

Wavelet methods involve decomposing a function into local frequency components. Wavelet methods for VCR models include using Bayesian variable selection or the Lasso to estimate which local frequency components have nonzero coefficients (Shang, 2011; Zhang and Clayton, 2011).

These methods achieve sparsity in the local frequency components but not in the local covariates, and so are not suitable for local variable selection.

Introduction

Existing approaches: Local regression

Local regression uses a kernel function at each sampling location to weight observations based on their distance from the sampling location. An example is the bisquare kernel:

$$w_{ii'} = \begin{cases} \left[1 - \left(\phi^{-1}\delta_{ii'}\right)^2\right]^2 & \text{if } \delta_{ii'} < \phi, \\ 0 & \text{if } \delta_{ii'} \geq \phi. \end{cases} \quad (2)$$

Where ϕ is a bandwidth parameter.
Given the weights, a local model is fit at each sampling location using the local likelihood (Loader, 1999)

Introduction

Existing approaches: Local likelihood

Calibrate the model by doing the following at each sampling location:

- Weight each observation's likelihood
- Weights are given by the kernel

$$L = \prod_{i'=1}^{n} \left(L_{i'}\right)^{w_{ii'}}$$

$$\ell = \sum_{i'=1}^{n} w_{ii'} \left\{ \log \sigma_i^2 + \sigma_i^{-2} \left(y_{i'} - \boldsymbol{x}_{i'}' \boldsymbol{\beta}_i\right)^2 \right\}$$

Where $\boldsymbol{\beta}_i = \boldsymbol{\beta}(\boldsymbol{s}_i)$.

Introduction

Existing approaches: geographically weighted regression

When the effect modifying variable s refers to spatial location, the method of local regression is called geographically weighted regression (GWR) (Brundson et al., 1998; Fotheringham et al., 2002)

Introduction

Existing approaches: bandwidth estimation for GWR

- Smaller bandwidth: less bias, more flexible coefficient surface
- Large bandwidth: less variance, less flexible coefficient surface
- Estimate the degrees of freedom used in estmating the coefficient surface (Hurvich et al., 1998):
  - $\hat{\boldsymbol{y}} = H\boldsymbol{y}$
  - $\nu = \mathrm{tr}(H)$
- Then the corrected AIC for bandwidth selection is:
- $\mathrm{AIC_c} = 2n \log \sigma + n \left\{ \frac{n+\nu}{n-2-\nu} \right\}$

Introduction

Existing approaches: geographically weighted Lasso

Within a GWR model, using the Lasso (Tibshirani, 1996) for local variable selection is called the geographically weighted Lasso (GWL) (Wheeler, 2009).

- The GWL requires estimating a Lasso tuning parameter for each local model
- Wheeler (2009) estimates the local Lasso tuning parameter at location s_i by minimizing a jacknife criterion: $|y_i - \hat{y}_i|$
- The jacknife criterion can only be calculated where data are observed, making it impossible to use the GWL to impute missing data or to estimate the value of the coefficient surface at new locations
- Also, the Lasso is known to be biased in variable selection and suboptimal for coefficient estimation

Methodology

Geographically weighted elastic net (GWEN)

- Local variable selection in a GWR model using the adaptive elastic net (AEN) (Zou and Zhang, 2009)
- Under suitable conditions, the AEN has an oracle property for selection

$$
\begin{aligned}
\mathcal{S}(\boldsymbol{\beta}_i) &= -2\ell_i(\boldsymbol{\beta}_i) + \mathcal{J}_2(\boldsymbol{\beta}_i) \\
&= \sum_{i'=1}^{n} w_{ii'} \left\{ \log \sigma_i^2 + \left(\sigma_i^2\right)^{-1} \left(y_{i'} - \boldsymbol{x}_{i'}' \boldsymbol{\beta}_i\right)^2 \right\} \\
&+ \alpha_i \lambda_i \sum_{j=1}^{p} |\beta_{ij}| / \gamma_{ij} \\
&+ (1 - \alpha_i) \lambda_i^* \sum_{j=1}^{p} \left(\beta_{ij} / \gamma_{ij}\right)^2
\end{aligned}
$$

Methodology

Geographically weighted elastic net (GWEN)

where $\sum_{i'=1}^{n} w_{ii'} \left(y_{i'} - \boldsymbol{x}'_{i'}\boldsymbol{\beta}_i\right)^2$ is the weighted sum of squares minimized by traditional GWR, and $\mathcal{J}_2(\boldsymbol{\beta}_i) = \alpha_i \lambda_i^* \sum_{j=1}^{p} |\beta_{ij}|/\gamma_{ij} + (1-\alpha_i)\lambda_i^* \sum_{j=1}^{p} \left(\beta_{ij}/\gamma_{ij}\right)^2$ is the AEN penalty.

Methodology

Geographically weighted elastic net (GWEN)

It is necessary to estimate an AEN tuning parameter for each local model. Using the local BIC allows fitting a local model at any location within the domain

$$
\begin{aligned}
\mathrm{BIC}_{\mathrm{loc},i} &= -2\sum_{i'=1}^{n} \ell_{ii'} + \log\left(\sum_{i'=1}^{n} w_{ii'}\right) \mathrm{df}_i \\
&= -2\sum_{i'=1}^{n} \log\left[\left(2\pi\hat{\sigma}_i^2\right)^{-1/2} \exp\left\{-\frac{1}{2}\hat{\sigma}_i^{-2}\left(y_{i'} - \boldsymbol{x}_{i'}'\hat{\boldsymbol{\beta}}_{i'}\right)^2\right\}\right]^{w} \\
&+ \log\left(\sum_{i'=1}^{n} w_{ii'}\right) \mathrm{df}_i
\end{aligned} \qquad (3)
$$

Methodology

Geographically weighted elastic net (GWEN)

$$= \sum_{i'=1}^{n} w_{ii'} \left\{ \mathsf{log}\,(2\pi) + \mathsf{log}\,\hat{\sigma}_i^2 + \hat{\sigma}_i^{-2} \left(y_{i'} - \boldsymbol{x}'_{i'} \hat{\boldsymbol{\beta}}_{i'} \right)^2 \right\}$$

$$+ \mathsf{log} \left(\sum_{i'=1}^{n} w_{ii'} \right) \mathsf{df}_i$$

Simulation study

Simulating covariates

Five covariates $\tilde{X}_1, \ldots, \tilde{X}_5$ were simulated by Gaussian random fields on the domain $[0,1] \times [0,1]$ on a 30×30 grid of sampling locations:

$$\tilde{X}_j \sim N(0, \Sigma) \text{ for } j = 1, \ldots, 5$$
$$\{\Sigma\}_{i,i'} = \exp\{-\tau^{-1}\delta_{ii'}\} \text{ for } i, i' = 1, \ldots, n$$

Where the covariates were simulated with colinearity, the colinearity was induced by multiplying the design matrix by the square root of the colinearity matrix:

$$\begin{aligned} \text{diag}(\Omega_{5\times5}) &= 1 \\ \text{off-diag}(\Omega_{5\times5}) &= \rho \\ X &= \tilde{X}R \end{aligned} \tag{4}$$

Where $\Omega_{5\times5} = R'R$ is the Cholesky decomposition.

Simulation study

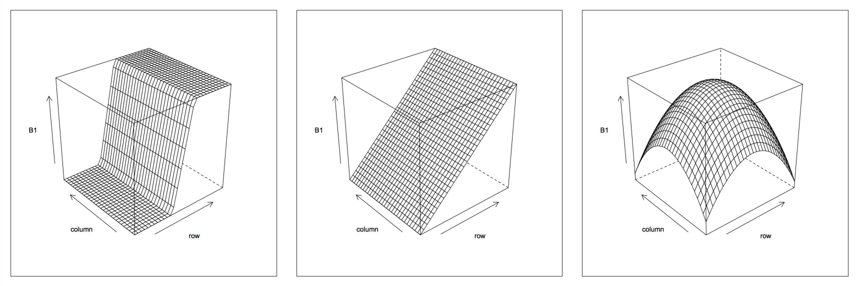
Simulating the response

- $Y(\boldsymbol{s}) = X(\boldsymbol{s})'\boldsymbol{\beta}(\boldsymbol{s}) = \sum_{j=1}^{5} \beta_j(\boldsymbol{s}) X_j(\boldsymbol{s}) + \varepsilon(\boldsymbol{s})$
- $\boldsymbol{\varepsilon} \sim \ iid \ \ N(0, \sigma^2)$
- $\beta_1(\boldsymbol{s})$, the coefficient function for X_1, is nonzero in part of the domain.
- Coefficients for $X_2, \ldots, X_5$ are zero everywhere

Simulation study

Coefficient functions

Call these functions step, gradient, and parabola:



Simulation study

Simulation settings

Setting	function	ρ	σ^2
1	step	0	0.25
2	step	0	1
3	step	0.5	0.25
4	step	0.5	1
5	gradient	0	0.25
6	gradient	0	1
7	gradient	0.5	0.25
8	gradient	0.5	1
9	parabola	0	0.25
10	parabola	0	1
11	parabola	0.5	0.25
12	parabola	0.5	1

Table : Simulation parameters for each setting.

Simulation results

Selection

	step				gradient				
	`enet`		`lasso`		`enet`		`lasso`		
location	β_1	β_2 - β_5	β_1	β_2 - β_5	β_1	β_2 - β_5	β_1	β_2 - β_5	β
1	0.99	0.00	0.99	0.00	1.00	0.00	1.00	0.00	0.3
	0.99	0.02	0.99	0.02	1.00	0.01	1.00	0.01	0.7
	0.99	0.00	1.00	0.00	1.00	0.00	1.00	0.00	0.2
	0.96	0.05	0.91	0.04	0.99	0.03	0.99	0.01	0.5
2	1.00	0.00	1.00	0.00	1.00	0.01	1.00	0.00	1.0
	1.00	0.03	1.00	0.03	1.00	0.02	1.00	0.02	1.0
	1.00	0.01	1.00	0.00	1.00	0.01	1.00	0.01	1.0
	0.99	0.05	0.97	0.04	1.00	0.02	0.99	0.01	0.9
3	0.91	0.01	0.91	0.00	1.00	0.01	1.00	0.01	1.0
	0.96	0.05	0.96	0.05	1.00	0.01	1.00	0.01	1.0
	0.92	0.05	0.95	0.02	1.00	0.02	1.00	0.01	1.0
	0.92	0.08	0.87	0.05	1.00	0.02	0.98	0.02	0.9
	0.48	0.01	0.43	0.01	1.00	0.01	1.00	0.01	1.0

Simulation results

MSE of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
step	1	0.026	*0.025*	0.057	0.057	0.062	**0.008**
		0.042	*0.040*	0.193	0.180	0.102	**0.016**
		0.036	**0.014**	0.055	0.067	0.080	*0.016*
		0.093	0.130	0.230	0.285	0.144	**0.030**
	2	0.063	0.058	0.043	*0.043*	**0.038**	0.055
		0.087	0.084	**0.064**	*0.064*	0.073	0.084
		0.068	0.049	0.045	*0.040*	**0.036**	0.052
		0.140	0.128	*0.082*	0.093	**0.074**	0.096
	3	0.025	0.025	0.019	0.019	**0.004**	*0.010*
		0.021	0.021	0.015	0.015	**0.007**	*0.011*
		0.027	0.021	0.018	*0.014*	**0.006**	0.019
		0.027	0.038	0.020	0.031	**0.007**	*0.016*
	4	*0.026*	0.026	0.028	**0.025**	0.034	0.054
		0.046	*0.050*	0.054	0.057	0.073	0.081
		0.025	0.030	0.030	*0.027*	0.036	0.063
		0.035	*0.036*	0.043	0.046	0.072	0.083
	5	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	0.008
		0.002	0.002	0.001	*0.000*	**0.000**	0.014
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.021

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Simulation results

Variance of $\beta_1(\boldsymbol{s})$

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	1	*0.036*	0.036	0.195	0.180	0.098	**0.016**
		0.027	**0.013**	0.056	0.068	0.080	*0.016*
		0.059	0.100	0.226	0.276	0.145	**0.030**
		0.014	0.013	**0.006**	*0.006*	0.006	0.008
	2	0.017	0.017	0.011	*0.011*	**0.008**	0.013
		0.012	0.010	*0.005*	**0.004**	0.006	0.010
		0.021	0.033	0.021	0.037	**0.008**	*0.014*
		0.022	0.023	0.019	0.019	**0.004**	*0.009*
step	3	0.021	0.021	0.014	0.014	**0.005**	*0.008*
		0.024	0.021	0.018	*0.014*	**0.005**	0.016
		0.024	0.036	0.020	0.032	**0.005**	*0.014*
		0.022	0.023	0.023	0.022	**0.006**	*0.007*
	4	0.025	0.024	0.025	0.022	**0.006**	*0.008*
		0.021	0.025	0.024	0.023	**0.005**	*0.013*
		0.026	0.027	0.029	0.032	**0.009**	*0.015*
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.007
	5	0.002	0.002	0.001	*0.000*	**0.000**	0.014
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.021

Simulation results

Variance of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
		0.023	*0.023*	0.057	0.058	0.063	**0.009**
	1	*0.036*	0.036	0.195	0.180	0.098	**0.016**
		0.027	**0.013**	0.056	0.068	0.080	*0.016*
		0.059	0.100	0.226	0.276	0.145	**0.030**
		0.014	0.013	**0.006**	*0.006*	0.006	0.008
	2	0.017	0.017	0.011	*0.011*	**0.008**	0.013
		0.012	0.010	*0.005*	**0.004**	0.006	0.010
		0.021	0.033	0.021	0.037	**0.008**	*0.014*
		0.022	0.023	0.019	0.019	**0.004**	*0.009*
step	3	0.021	0.021	0.014	0.014	**0.005**	*0.008*
		0.024	0.021	0.018	*0.014*	**0.005**	0.016
		0.024	0.036	0.020	0.032	**0.005**	*0.014*
		0.022	0.023	0.023	0.022	**0.006**	*0.007*
	4	0.025	0.024	0.025	0.022	**0.006**	*0.008*
		0.021	0.025	0.024	0.023	**0.005**	*0.013*
		0.026	0.027	0.029	0.032	**0.009**	*0.015*
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.007
	5	0.002	0.002	0.001	*0.000*	**0.000**	0.014
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.021

Simulation results

Variance of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
step	1	0.023	*0.023*	0.057	0.058	0.063	**0.009**
		0.036	0.036	0.195	0.180	0.098	**0.016**
		0.027	**0.013**	0.056	0.068	0.080	*0.016*
		0.059	0.100	0.226	0.276	0.145	**0.030**
	2	0.014	0.013	**0.006**	*0.006*	0.006	0.008
		0.017	0.017	0.011	*0.011*	**0.008**	0.013
		0.012	0.010	*0.005*	**0.004**	0.006	0.010
		0.021	0.033	0.021	0.037	**0.008**	*0.014*
	3	0.022	0.023	0.019	0.019	**0.004**	*0.009*
		0.021	0.021	0.014	0.014	**0.005**	*0.008*
		0.024	0.021	0.018	*0.014*	**0.005**	0.016
		0.024	0.036	0.020	0.032	**0.005**	*0.014*
	4	0.022	0.023	0.023	0.022	**0.006**	*0.007*
		0.025	0.024	0.025	0.022	**0.006**	*0.008*
		0.021	0.025	0.024	0.023	**0.005**	*0.013*
		0.026	0.027	0.029	0.032	**0.009**	*0.015*
	5	**0.000**	**0.000**	**0.000**	**0.000**	**0.000**	0.007
		0.002	0.002	0.001	*0.000*	**0.000**	0.014
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	0.021

Simulation results

Bias of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
step	1	-0.056	-0.049	**0.001**	*0.005*	0.015	-0.007
		-0.080	-0.069	*0.020*	0.040	0.072	**0.002**
		-0.093	-0.037	-0.010	-0.009	*-0.005*	**0.003**
		-0.185	-0.177	-0.075	-0.110	*0.032*	**-0.009**
	2	-0.222	-0.213	-0.193	*-0.191*	**-0.178**	-0.217
		-0.264	-0.259	**-0.231**	*-0.232*	-0.256	-0.268
		-0.236	-0.197	-0.199	*-0.188*	**-0.176**	-0.204
		-0.345	-0.309	*-0.248*	**-0.236**	-0.257	-0.286
	3	-0.057	-0.047	*-0.006*	**-0.006**	0.025	0.024
		-0.009	**0.004**	0.022	0.024	0.047	0.051
		-0.052	*-0.007*	**0.003**	0.020	0.039	0.055
		-0.062	-0.046	**-0.011**	*-0.014*	0.046	0.046
	4	0.066	*0.058*	0.071	**0.057**	0.168	0.218
		0.147	*0.165*	0.170	0.188	0.260	0.272
		0.062	0.071	0.077	*0.067*	0.174	0.223
		0.098	**0.098**	0.121	0.119	0.250	0.262
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	-0.022
		0.003	*0.001*	-0.005	-0.003	**0.000**	-0.018

Simulation results

Bias of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
		-0.056	-0.049	**0.001**	*0.005*	0.015	-0.007
	1	-0.080	-0.069	*0.020*	0.040	0.072	**0.002**
		-0.093	-0.037	-0.010	-0.009	*-0.005*	**0.003**
		-0.185	-0.177	-0.075	-0.110	*0.032*	**-0.009**
		-0.222	-0.213	-0.193	*-0.191*	**-0.178**	-0.217
	2	-0.264	-0.259	**-0.231**	*-0.232*	-0.256	-0.268
		-0.236	-0.197	-0.199	*-0.188*	**-0.176**	-0.204
		-0.345	-0.309	*-0.248*	**-0.236**	-0.257	-0.286
		-0.057	-0.047	*-0.006*	**-0.006**	0.025	0.024
step	3	*-0.009*	**0.004**	0.022	0.024	0.047	0.051
		-0.052	*-0.007*	**0.003**	0.020	0.039	0.055
		-0.062	-0.046	**-0.011**	*-0.014*	0.046	0.046
		0.066	*0.058*	0.071	**0.057**	0.168	0.218
	4	**0.147**	*0.165*	0.170	0.188	0.260	0.272
		0.062	0.071	0.077	*0.067*	0.174	0.223
		0.098	**0.098**	0.121	0.119	0.250	0.262
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	-0.022
		0.003	*0.001*	-0.005	-0.003	**0.000**	-0.018

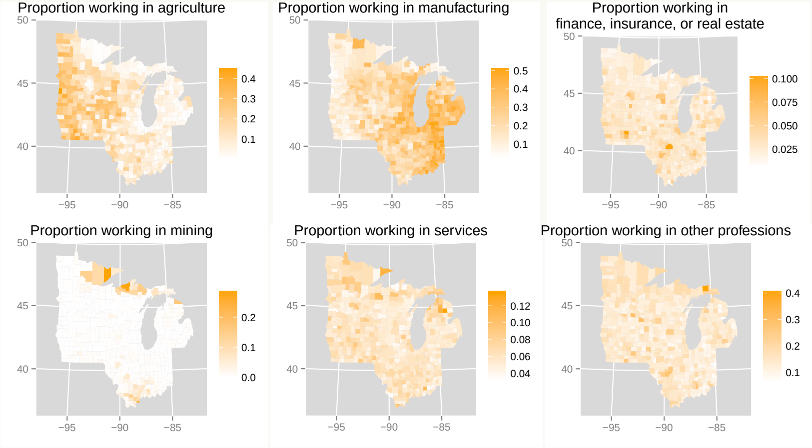
Simulation results

Bias of $\beta_1(\boldsymbol{s})$

function	location	GWEN	GWAL	u.enet	u.lasso	oracle	GWR
		-0.056	-0.049	**0.001**	*0.005*	0.015	-0.007
	1	-0.080	-0.069	*0.020*	0.040	0.072	**0.002**
		-0.093	-0.037	-0.010	-0.009	*-0.005*	**0.003**
		-0.185	-0.177	-0.075	-0.110	*0.032*	**-0.009**
		-0.222	-0.213	-0.193	*-0.191*	**-0.178**	-0.217
	2	-0.264	-0.259	**-0.231**	*-0.232*	-0.256	-0.268
		-0.236	-0.197	-0.199	*-0.188*	**-0.176**	-0.204
		-0.345	-0.309	*-0.248*	**-0.236**	-0.257	-0.286
		-0.057	-0.047	*-0.006*	**-0.006**	0.025	0.024
		-0.009	**0.004**	0.022	0.024	0.047	0.051
step	3	-0.052	*-0.007*	**0.003**	0.020	0.039	0.055
		-0.062	-0.046	**-0.011**	*-0.014*	0.046	0.046
		0.066	*0.058*	0.071	**0.057**	0.168	0.218
	4	**0.147**	*0.165*	0.170	0.188	0.260	0.272
		0.062	0.071	0.077	*0.067*	0.174	0.223
		0.098	**0.098**	0.121	0.119	0.250	0.262
		0.000	**0.000**	**0.000**	**0.000**	**0.000**	-0.022
		0.003	*0.001*	-0.005	-0.003	**0.000**	-0.018

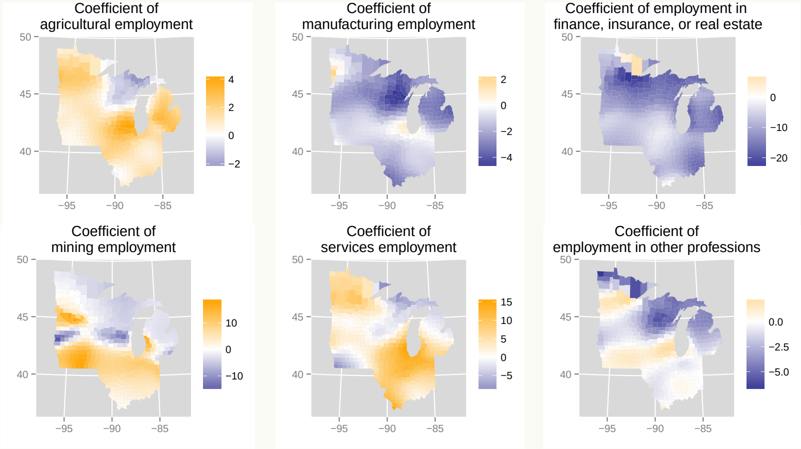
Data analysis

Revisiting the introductory example



Data analysis

Results from traditional GWR



Data analysis

Results from GWEN

Future work

- Apply the GWEN to data with non-Gaussian response variable
- Incorporate spatial autocorrelation in the model (simulated errors were iid)

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