If $\lambda^k = \lambda^{k+1} = \ldots = \lambda^{k+m-1}$ is an m-fold multiple eigenvalue of the problem (1) *then there exist positive real coefficients $\beta_h^k, \beta_h^{k+1}, \ldots, \beta_h^{k+m-1}$ such that for $n = k, k+1, \ldots, k+m-1$*

$$\left\| u^n - \sum_{i=0}^{m-1} \beta_h^{k+i} v_h^{k+i} \right\|_{1,R} < c_{15} h^{\gamma} \tag{32}$$

and

$$\max_{(x,y)\in R} \left| u^n(x,y) - \sum_{i=0}^{m-1} \beta_h^{k+i} v_h^{k+i}(x,y) \right| < c_{16} h^{\gamma^*} |\log h|^{1/2}. \tag{33}$$

Here $c_{12}, \ldots, c_{16}$ are positive constants, which depend only on k, m, the region R, and the coefficients of the operator L.

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(Received September 25, 1975)

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Periodica Mathematica Hungarica Vol. 8 (1), (1977), pp. 53—55

A REMARK ON THE APPROXIMATION OF THE SAMPLE DF IN THE MULTIDIMENSIONAL CASE

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Let $X_1, X_2, \ldots, X_n$ be i.i.d.r.v-s on the k-dimensional unit cube with

$$\mathsf{P}(X_1 < \mathbf{t}) = \lambda(\mathbf{t}) \quad \text{if} \quad \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}, \tag{1}$$

where $\mathbf{t} = (t_1, t_2, \ldots, t_k)$, and $\mathbf{1} = (1, 1, \ldots, 1)$ are k-dimensional vectors, $\lambda(\mathbf{t}) = \prod_{i=1}^{k} t_i$ is the k-dimensional volume of the rectangle determined by the origin and the point $\mathbf{t}$, and for two vectors $\mathbf{a}$ and $\mathbf{b}$ the inequality $\mathbf{a} < \mathbf{b}$ means that each coordinate of $\mathbf{a}$ is less than the corresponding coordinate of $\mathbf{b}$. The empirical distribution function $F_n(\mathbf{t})$ based on the sample $X_1, X_2, \ldots, X_n$ is the function

$$F_n(\mathbf{t}) = \frac{1}{n} \sum_{i : X_i < \mathbf{t}} 1 \qquad \text{if} \qquad \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}, \tag{2}$$

and the k-dimensional Brownian bridge $B(\mathbf{t})$ is defined by

$$B(\mathbf{t}) = W(\mathbf{t}) - \lambda(\mathbf{t}) W(\mathbf{1}) \quad \text{if} \quad \mathbf{0} \leq \mathbf{t} \leq \mathbf{1}, \tag{3}$$

where $W(\mathbf{t})$ is a k-dimensional Wiener process, i.e., $W(\mathbf{t})$ is a Gaussian process with independent increments, variance equal to the k-dimensional volume. In this remark we investigate the approximation of $F_n(\mathbf{t})$ by $B(\mathbf{t})$ in the case $k = 2$.

The one-dimensional case was investigated in [2], where we proved that there is a version of F_n and B such that

$$\mathsf{P}\left(\sup_{\mathbf{0} \leq \mathbf{t} \leq \mathbf{1}} | n(F_n(\mathbf{t}) - \lambda(\mathbf{t})) - n^{\frac{1}{2}} B(\mathbf{t}) | > C \log n + x \right) < K e^{-\lambda x} \tag{4}$$

olds for all x, where C, K, λ are positive absolute constants (Theorem 3). nvestigating the approximation of the whole sequence $\{F_n, n = 1, 2, \ldots\}$ we

AMS (MOS) subject classifications (1970). Primary 60B10; Secondary 62G10.
Key words and phrases. Invariance, embedding, testing multidimensional distributions.

got a theorem of two-dimensional character (Theorem 4). As we shall see, the result presented here is strongly connected with this latter theorem.

THEOREM. *In the case $k=2$ for any n there is a version of F_n and B such that*

$$(5)\qquad \mathsf{P}\left(\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}} |\, n(F_n(\mathbf{t})-\lambda(\mathbf{t})) - n^{\frac{1}{2}}B(\mathbf{t})\,| > (C\log n + x)\log n\right) < Ke^{-\lambda x}$$

holds for all x, where C, K, λ are positive absolute constants.

PROOF. The proof is based on the following version of Theorem 5 of [2]:

LEMMA. *In the case $k=1$ for any n there are measurable functions*

$$(6)\qquad e_i(w,f) \qquad i=1,2,\ldots,n$$

defined on the product of the spaces $C(0, 1)$ and $D(0, 1)$ such that for any independent pair of the one-dimensional Wiener process W and empirical DF F_n the random variables

$$(7)\qquad \varepsilon_i = e_i(W, F_n) \qquad\qquad i=1,2,\ldots,n$$

are i.i.d.r.v-s with distribution

$$(8)\qquad \mathsf{P}(\varepsilon_i=0)=\mathsf{P}(\varepsilon_i=1)=\frac{1}{2},$$

the set of random variables $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ is independent of F_n, and

$$(9)\qquad \mathsf{P}\left(\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}}\left|\frac{n}{2}(\widetilde{F}(\mathbf{t})-\lambda(\mathbf{t}))-\left(\frac{n}{2}\right)^{\frac{1}{2}}W(\mathbf{t})\right| > (C\log n + x)\right) < Ke^{-\lambda x}$$

holds for all x, where C, K, λ are positive absolute constants, and $\widetilde{F}_n$ is defined by

$$(10)\qquad \widetilde{F}_n(\mathbf{t}) = \frac{2}{n}\sum_{i=1}^{nF_n(\mathbf{t})}\varepsilon_i\,.$$

The proof of this lemma is similar to the proof of Theorem 5 of [2]. We shall use the conditional quantile transformation, and the dyadic scheme in the following way. Suppose $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ are arbitrary i.i.d.r.v-s of distribution (8), and the set $\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\}$ is independent of F_n. Let m and

$$\mathbf{0}=\mathbf{t}_0<\mathbf{t}_1<\ldots<\mathbf{t}_m=\mathbf{1}$$

be arbitrary and let $\widetilde{F}_n$ defined by (10). The conditional distribution of $\frac{n}{2}\widetilde{F}_n(\mathbf{t}_i)$ under the condition that $\{F_n(\mathbf{t}), \mathbf{0}\le\mathbf{t}\le\mathbf{1}\}$ and

$$\{\widetilde{F}_n(\mathbf{t}_0), \widetilde{F}_n(\mathbf{t}_1), \ldots, \widetilde{F}_n(\mathbf{t}_{i-1}), \widetilde{F}_n(\mathbf{t}_{i+1}), \ldots, \widetilde{F}_n(\mathbf{t}_m)\}$$

are given is a hypergeometric distribution with parameters depending on the condition, and the conditional distribution of $\widetilde{F}_n(\mathbf{1})$ given $\{F_n(\mathbf{t}), \mathbf{0}\le\mathbf{t}\le\mathbf{1}\}$ is a binomial distribution with parameters $\left(n, \frac{1}{2}\right)$. Hence we can transform the appropriate parts of $W(\mathbf{t})$ step by step to $\widetilde{F}_n(\mathbf{1})$, $\widetilde{F}_n\left(\frac{\mathbf{1}}{\mathbf{2}}\right)$, $\widetilde{F}_n\left(\frac{\mathbf{1}}{\mathbf{4}}\right)$, $\widetilde{F}_n\left(\frac{\mathbf{3}}{\mathbf{4}}\right)$ using in each step the conditional distribution of the new variable on the condition that $\{F_n(\mathbf{t}), \mathbf{0}\le\mathbf{t}\le\mathbf{1}\}$ and the just defined $\widetilde{F}_n(\mathbf{t}_i)$-s are given. This is the same construction as the construction of the proof of Theorem 5, the only difference is that here $\widetilde{F}_n(\mathbf{1})$ is also random variable. Hence the further details of the proof are omitted.

The theorem follows from the lemma in the same way as Theorem 4 follows from Theorem 5 in [2]. Hence its proof is also omitted.

REMARK 1. In the case $k=1$ we proved in [2] that our result is the best possible in the sense that there are positive absolute constants A, B such that for any n and any version of F_n, B

$$\mathsf{P}\left(\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}}|n(F_n(\mathbf{t})-\lambda(\mathbf{t}))-n^{\frac{1}{2}}B(\mathbf{t})|>A\log n\right)\ge B\,.$$

In the case $k=2$ the situation is different: we do not know whether for a given $\varepsilon>0$ are there positive constants A, B such that

$$(11)\qquad \mathsf{P}\left(\sup_{\mathbf{0}\le\mathbf{t}\le\mathbf{1}}|n(F_n(\mathbf{t})-\lambda(\mathbf{t}))-n^{\frac{1}{2}}B(\mathbf{t})|>A(\log n)^{1+\varepsilon}\right)\ge B\,.$$

REMARK 2. In case $k>2$ the best known available results are given by CSÖRGŐ and RÉVÉSZ [1]. They give an approximation of order $n^{\frac{k-1}{2k}}$. We do not know whether (11) is true for any $k>1$ or not.

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(Received October 15, 1975)

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