1. Asymptotic normality

Theorem 1.1. *If* $h^{-1}n^{-1/2}a_n \xrightarrow{p} 0$ *and* $hn^{-1/2}b_n \xrightarrow{p} \infty$ *then* $\hat{\boldsymbol{\beta}}(\boldsymbol{s}) - \boldsymbol{\beta}(\boldsymbol{s}) - \frac{\kappa_2 h^2}{2\kappa_0}\{\boldsymbol{\beta}_{uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{vv}(\boldsymbol{s})\} = O_p(n^{-1/2}h^{-1})$

Proof. The idea of the proof is to show that the objective being minimized achieves a unique minimum, which must be $\hat{\boldsymbol{\beta}}(\boldsymbol{s})$.

The order of convergence is $hn^{1/2}$ where $h = O(n^{-1/6})$.

We find the limiting distribution of the estimator:

$$
\begin{aligned}
V_4^{(n)}(\boldsymbol{u}) &= Q\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\} - Q\left\{\boldsymbol{\beta}(\boldsymbol{s})\right\} \\
&= (1/2)\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\}\right]^T \boldsymbol{W}(\boldsymbol{s})\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\}\right] \\
&\quad + \sum_{j=1}^{p} \lambda_j \|\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| \\
&\quad - (1/2)\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}^T \boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\} - \sum_{j=1}^{p} \lambda_j \|\boldsymbol{\beta}(\boldsymbol{s})\| \\
&= (1/2)\boldsymbol{u}^T\left\{h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s})\right\}\boldsymbol{u} - \boldsymbol{u}^T\left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}\right] \\
&\quad + \sum_{j=1}^{p} n^{-1/2}\lambda_j n^{1/2}\left\{\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right\} \qquad (1)
\end{aligned}
$$

Note the different limiting behavior of the third term between the cases $j \leqslant p_0$ and $j > p_0$:

Case $j \leqslant p_0$. If $j \leqslant p_0$ then $n^{-1/2}\lambda_j \to n^{-1/2}\lambda\|\boldsymbol{\beta}_j(\boldsymbol{s})\|^{-\gamma}$ and $|\sqrt{n}\left\{\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right\}| \leqslant h^{-1}\|\boldsymbol{u}_j\|$ so $\lim_{n\to\infty} \lambda_j\left(\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right) \leqslant h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| \leqslant h^{-1}n^{-1/2}a_n\|\boldsymbol{u}_j\| \to 0$

Case $j > p_0$. If $j > p_0$ then $\lambda_j\left(\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right) = \lambda_j h^{-1}n^{-1/2}\|\boldsymbol{u}_j\|$.

And note that $h = O(n^{-1/6})$ so that if $hn^{-1/2}b_n \xrightarrow{p} \infty$ then $h^{-1}n^{-1/2}b_n \xrightarrow{p} \infty$.

Now, if $\|\boldsymbol{u}_j\| \neq 0$ then $h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| \geqslant h^{-1}n^{-1/2}b_n\|\boldsymbol{u}_j\| \to \infty$. On the other hand, if $\|\boldsymbol{u}_j\| = 0$ then $h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| = 0$.

Thus, the limit of $V_4^{(n)}(\boldsymbol{u})$ is the same as the limit of $V_4^{*(n)}(\boldsymbol{u})$ where

$$V_4^{*(n)}(\boldsymbol{u}) = \begin{cases} (1/2)\boldsymbol{u}^T\left\{h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s})\right\}\boldsymbol{u} - \boldsymbol{u}^T\left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}\right] & \text{if } \|\boldsymbol{u}_j\| = 0 \; \forall j > p_0 \\ \infty & \text{otherwise} \end{cases}$$

And $V_4^{*(n)}(\boldsymbol{u})$ is convex and is minimized at $\hat{\boldsymbol{u}}^{(n)}$:

$$\begin{aligned} 0 &= \left\{h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s})\right\}\hat{\boldsymbol{u}}^{(n)} - \left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}\right] \\ \therefore \hat{\boldsymbol{u}}^{(n)} &= \left\{n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s})\right\}^{-1}\left[hn^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}\right] \end{aligned} \tag{2}$$

By the epiconvergence results of **?** and **?**, the minimizer of the limiting function is the limit of the minimizers $\hat{\boldsymbol{u}}^{(n)}$. And since, by Lemma 2 of **?**,

$$\hat{\boldsymbol{u}}^{(n)} \xrightarrow{d} N\left(0, f(\boldsymbol{s})\kappa_0^{-2}\nu_0\sigma^2\Psi^{-1}\right) \tag{3}$$

the result is proven. □

2. Selection

Theorem 2.1. *If* $hn^{-1/2}b_n \xrightarrow{p} \infty$ *then* $P\left\{\hat{\boldsymbol{\beta}}_{(b)}(\boldsymbol{s}) = 0\right\} \to 1.$

Proof. We showed in Theorem 1.1 that the

The proof is by contradiction.

Recall that the objective to be minimized by $\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})$ is

$$Q\left\{\hat{\boldsymbol{\beta}}(\boldsymbol{s})\right\} = (1/2)\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\hat{\boldsymbol{\beta}}(\boldsymbol{s})\right\}^T \boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\hat{\boldsymbol{\beta}}(\boldsymbol{s})\right\} + \sum_{j=1}^{p} \lambda_j \|\hat{\boldsymbol{\beta}}_p(\boldsymbol{s})\| \tag{4}$$

Let $\hat{\boldsymbol{\beta}}_p(\boldsymbol{s}) \neq 0$. Then (4) is differentiable w.r.t. $\boldsymbol{\beta}_p(\boldsymbol{s})$ and Q is maximized at

$$\begin{aligned}
0 &= \boldsymbol{Z}_{(p)}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}_{(-p)}(\boldsymbol{s})\hat{\boldsymbol{\beta}}_{(-p)}(\boldsymbol{s}) - \boldsymbol{Z}_{(p)}(\boldsymbol{s})\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\right\} - \lambda_p \frac{\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})}{\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\|} \\
&= \boldsymbol{Z}_{(p)}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) - \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{vv}(\boldsymbol{s})\right\}\right] \\
&\qquad + \boldsymbol{Z}_{(p)}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(-p)}(\boldsymbol{s})\left[\boldsymbol{\beta}_{(-p)}(\boldsymbol{s}) + \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{(-p),uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{(-p),vv}(\boldsymbol{s})\right\} - \hat{\boldsymbol{\beta}}_{(-p)}(\boldsymbol{s})\right] \\
&\qquad + \boldsymbol{Z}_{(p)}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(p)}(\boldsymbol{s})\left[\boldsymbol{\beta}_{(-p)}(\boldsymbol{s}) + \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{(p),uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{(p),vv}(\boldsymbol{s})\right\} - \hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\right] \\
&\qquad - \lambda_p \frac{\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})}{\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\|}
\end{aligned} \tag{5}$$

So

$$
\begin{aligned}
\frac{h}{\sqrt{n}}\lambda_p \frac{\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})}{\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\|} &= \boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\frac{h}{\sqrt{n}}\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) - \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{vv}(\boldsymbol{s})\right\}\right] \\
&+ \left\{n^{-1}\boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(-p)}(\boldsymbol{s})\right\} h\sqrt{n}\left[\boldsymbol{\beta}_{(-p)}(\boldsymbol{s}) + \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{(-p),uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{(-p),vv}(\boldsymbol{s})\right\} - \hat{\boldsymbol{\beta}}_{(-p)}(\boldsymbol{s})\right] \\
&+ \left\{n^{-1}\boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(-p)}(\boldsymbol{s})\right\} h\sqrt{n}\left[\boldsymbol{\beta}_{(-p)}(\boldsymbol{s}) + \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{(p),uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{(p),vv}(\boldsymbol{s})\right\} - \hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\right]
\end{aligned}
\tag{6}
$$

From Lemma 2 of **?**, $\left\{n^{-1}\boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(-p)}(\boldsymbol{s})\right\} = O_p(1)$ and $\left\{n^{-1}\boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}_{(p)}(\boldsymbol{s})\right\} = O_p(1)$.

From Theorem 3 of **?**, we have that $h\sqrt{n}\left[\hat{\boldsymbol{\beta}}_{(-p)}(\boldsymbol{s}) - \boldsymbol{\beta}_{(-p)}(\boldsymbol{s}) - \frac{h^2\kappa_2}{2\kappa_0}\left\{\beta_{(p),uu}(\boldsymbol{s}) + \beta_{(p),vv}(\boldsymbol{s})\right\}\right] = O_p(1)$ and $h\sqrt{n}\left[\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s}) - \boldsymbol{\beta}_{(p)}(\boldsymbol{s}) - \frac{h^2\kappa_2}{2\kappa_0}\left\{\beta_{(p),uu}(\boldsymbol{s}) + \beta_{(p),vv}(\boldsymbol{s})\right\}\right] = O_p(1)$.

So the second and third terms of the sum in (6) are $O_p(1)$.

We showed in the proof of 1.1 that $h\sqrt{n}\boldsymbol{Z}^T_{(p)}(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) - \frac{h^2\kappa_2}{2\kappa_0}\left\{\boldsymbol{\beta}_{uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{vv}(\boldsymbol{s})\right\}\right] = O_p(1)$.

The three terms of the sum to the right of the equals sign in (6) are $O_p(1)$, so for $\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})$ to be a solution, we must have that $hn^{-1/2}\lambda_p \frac{\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})}{\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\|} = O_p(1)$.

But since by assumption $\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s}_i) \neq 0$, there must be some $k \in \{1, \dots, d_p\}$ such that $|\hat{\beta}_{(p),k}(\boldsymbol{s})| = \max\{|\hat{\beta}_{(p),k'}(\boldsymbol{s})| : 1 \leqslant k' \leqslant d_p\}$. And for this k, we have that $|\hat{\beta}_{(p),k}(\boldsymbol{s})|/\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\| \geqslant 1/\sqrt{d_p} > 0$.

Now since $hn^{-1/2}b_n \to \infty$, we have that $hn^{-1/2}\lambda_p \frac{\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})}{\|\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s})\|} \geqslant hn^{-1/2}b_n d_p^{-1/2} \to \infty$ and therefore the term to the left of the equals sign dominates the sum to the right of the equals sign in (6). So for large enough n, $\hat{\boldsymbol{\beta}}_{(p)}(\boldsymbol{s}) \neq 0$ cannot maximize Q.

So $P\left\{\hat{\boldsymbol{\beta}}_{(b)}(\boldsymbol{s}) = 0\right\} \to 1$. □

3. A note on rates

To prove the oracle properties of LAGR, we assumed that $h^{-1}n^{-1/2}a_n \xrightarrow{p} 0$ and $hn^{-1/2}b_n \xrightarrow{p} \infty$. Therefore, $h^{-1}n^{-1/2}\lambda_n \to 0$ for $j \leqslant p_0$ and $hn^{-1/2}\lambda_n \|\boldsymbol{\beta}_j(\boldsymbol{s})\|^{-\gamma} \to \infty$ for $j > p_0$.

We require a λ_n that can satisfy both assumptions. Suppose $\lambda_n = n^{\alpha}$, and recall that $h = O(n^{-1/6})$. Then $h^{-1}n^{-1/2}\lambda_n = O(n^{-1/3+\alpha})$ and $hn^{-1/2}\lambda_n(h\sqrt{n})^{\gamma} = O(n^{-2/3+\alpha+\gamma/3})$.

So $(2-\gamma)/3 < \alpha < 1/3$, which can only be satisfied for $\gamma > 1$.