Local Variable Selection and Parameter Estimation of Spatially Varying Coefficient Regression Models

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1. Asymptotics

1.1. Asymptotic normality

Theorem 1.1. *If* $h^{-1}n^{-1/2}a_n \xrightarrow{p} 0$ *and* $hn^{-1/2}b_n \xrightarrow{p} \infty$ *then* $\hat{\boldsymbol{\beta}}(\boldsymbol{s}) - \boldsymbol{\beta}(\boldsymbol{s}) - \frac{\kappa_2 h^2}{2\kappa_0}\{\boldsymbol{\beta}_{uu}(\boldsymbol{s}) + \boldsymbol{\beta}_{vv}(\boldsymbol{s})\} = O_p(n^{-1/2}h^{-1})$

Proof. The idea of the proof is to show that the objective being minimized achieves a unique minimum, which must be $\hat{\boldsymbol{\beta}}(\boldsymbol{s})$.

The order of convergence is $hn^{1/2}$ where $h = O(n^{-1/6})$.

We find the limiting distribution of the estimator:

$$
\begin{aligned}
V_4^{(n)}(\boldsymbol{u}) &= Q\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\} - Q\left\{\boldsymbol{\beta}(\boldsymbol{s})\right\} \\
&= (1/2)\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\}\right]^T \boldsymbol{W}(\boldsymbol{s})\left[\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\left\{\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}\right\}\right] \\
&\quad + \sum_{j=1}^{p} \lambda_j \|\boldsymbol{\beta}(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| \\
&\quad - (1/2)\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}^T \boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\} - \sum_{j=1}^{p} \lambda_j \|\boldsymbol{\beta}(\boldsymbol{s})\| \\
&= (1/2)\boldsymbol{u}^T\left\{h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s})\right\}\boldsymbol{u} - \boldsymbol{u}^T\left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\left\{\boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s})\right\}\right] \\
&\quad + \sum_{j=1}^{p} n^{-1/2}\lambda_j n^{1/2}\left\{\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right\} \qquad (1)
\end{aligned}
$$

Note the different limiting behavior of the third term between the cases $j \leqslant p_0$ and $j > p_0$:

Case $j \leqslant p_0$. If $j \leqslant p_0$ then $n^{-1/2}\lambda_j \to n^{-1/2}\lambda\|\boldsymbol{\beta}_j(\boldsymbol{s})\|^{-\gamma}$ and $|\sqrt{n}\left\{\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right\}| \leqslant h^{-1}\|\boldsymbol{u}_j\|$ so $\lim\limits_{n\to\infty} \lambda_j\left(\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\|\right) \leqslant h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| \leqslant h^{-1}n^{-1/2}a_n\|\boldsymbol{u}_j\| \to 0$

Case $j > p_0$. If $j > p_0$ then $\lambda_j \left(\|\boldsymbol{\beta}_j(\boldsymbol{s}) + h^{-1}n^{-1/2}\boldsymbol{u}_j\| - \|\boldsymbol{\beta}_j(\boldsymbol{s})\| \right) = \lambda_j h^{-1} n^{-1/2} \|\boldsymbol{u}_j\|$.

And note that $h = O(n^{-1/6})$ so that if $hn^{-1/2}b_n \xrightarrow{p} \infty$ then $h^{-1}n^{-1/2}b_n \xrightarrow{p} \infty$.

Now, if $\|\boldsymbol{u}_j\| \neq 0$ then $h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| \geqslant h^{-1}n^{-1/2}b_n\|\boldsymbol{u}_j\| \to \infty$. On the other hand, if $\|\boldsymbol{u}_j\| = 0$ then $h^{-1}n^{-1/2}\lambda_j\|\boldsymbol{u}_j\| = 0$.

Thus, the limit of $V_4^{(n)}(\boldsymbol{u})$ is the same as the limit of $V_4^{*(n)}(\boldsymbol{u})$ where

$$V_4^{*(n)}(\boldsymbol{u}) = \begin{cases} (1/2)\boldsymbol{u}^T \left\{ h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s}) \right\} \boldsymbol{u} - \boldsymbol{u}^T \left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s}) \left\{ \boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) \right\} \right] & \text{if } \|\boldsymbol{u}_j\| = 0 \ \forall j > p_0 \\ \infty & \text{otherwise} \end{cases}$$

Now, $V_4^{*(n)}(\boldsymbol{u})$ is convex and is minimized at $\hat{\boldsymbol{u}}^{(n)}$:

$$\begin{aligned} 0 &= \left\{ h^{-2}n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s}) \right\} \hat{\boldsymbol{u}}^{(n)} - \left[h^{-1}n^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s}) \left\{ \boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) \right\} \right] \\ \therefore \hat{\boldsymbol{u}}^{(n)} &= \left\{ n^{-1}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s})\boldsymbol{Z}(\boldsymbol{s}) \right\}^{-1} \left[hn^{-1/2}\boldsymbol{Z}^T(\boldsymbol{s})\boldsymbol{W}(\boldsymbol{s}) \left\{ \boldsymbol{Y} - \boldsymbol{Z}(\boldsymbol{s})\boldsymbol{\beta}(\boldsymbol{s}) \right\} \right] \end{aligned} \tag{2}$$

By the epiconvergence results of **?** and **?**, the minimizer of the limiting function is the limit of the minimizers $\hat{\boldsymbol{u}}^{(n)}$. And since, by Lemma 2 of **?**,

$$\hat{\boldsymbol{u}}^{(n)} \xrightarrow{d} N\left(0, f(\boldsymbol{s})\kappa_0^{-2}\nu_0\sigma^2\Psi^{-1}\right) \tag{3}$$

the result is proven.

□