Deriving the eigenvalue sensitivity formula

Let v, w be col. vectors ...
w is right-eigenvector
v is left-eigenvector ... w/ eval λ.

By def: (1) $Aw = \lambda w$
(2) LEFT EVR: $A^T v = \lambda v$ (from class)

- Take $\frac{\partial}{\partial a_{ij}}$ of (1):

$$\underbrace{\left(\frac{\partial}{\partial a_{ij}} A\right)}_{\text{MATRIX}} \underbrace{w}_{\text{VECTOR}} + A \frac{\partial}{\partial a_{ij}} w = \frac{\partial \lambda}{\partial a_{ij}} w + \lambda \frac{\partial w}{\partial a_{ij}}$$

- Dot both sides with v: (Def: $x \cdot y = \sum_k x_k y_k$)

$$v \cdot \left(\frac{\partial}{\partial a_{ij}} A\right) w + v \cdot A \frac{\partial}{\partial a_{ij}} w = v \cdot \frac{\partial \lambda}{\partial a_{ij}} w + \lambda v \cdot \frac{\partial w}{\partial a_{ij}}$$

- NOTE:

USE FACT

$$v \cdot A \frac{\partial}{\partial a_{ij}} w = A^T v \cdot \frac{\partial}{\partial a_{ij}} w = \lambda v \cdot \frac{\partial}{\partial a_{ij}} w \qquad (*)$$

FACT: $v \cdot Aw = A^T v \cdot w$; pf. overleaf.

Therefore ~~~ terms cancel above, giving:

$$v \cdot \left(\frac{\partial}{\partial a_{ij}} A\right) w = \frac{\partial \lambda}{\partial a_{ij}} v \cdot w$$

Matrix of zeros ...
except for i,jth entry, which is 1

→

VSF FACT 2

$$v_i w_j = \frac{\partial}{\partial a_{ij}} \lambda \; v \cdot w$$

$$\boxed{\frac{\partial}{\partial a_{ij}} \lambda = \frac{v_i w_j}{v \cdot w}}$$

The Eigenvalue Sensitivity Formula.

FACT 2:

$$v \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} | \\ w \\ | \end{pmatrix} = v \cdot \begin{pmatrix} 0 \\ 0 \\ w_j \\ 0 \end{pmatrix} \leftarrow i^{th} \text{ row}$$

i, j

$3, 4$

$$= v_i w_j$$

(*) This is fact:

$v \cdot Aw = A^T v \cdot w$, which can be checked directly...

Pf: $v \cdot Aw = \sum_i v_i \sum_j A_{ij} w_j = \sum_j w_j \sum_i A_{ij} v_i$

$$= \sum_j w_j \sum_i A^T_{ji} v_i$$

$$= w \cdot A^T v$$