Numerical Approximation of Shock Formation

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1 Introduction

A shock wave occurs regularly in nature in traffic flow, glacier waves and supersonic aircraft. These shocks are modeled by nonlinear hyperbolic partial differential equations (PDE). In these nonlinear equations, the superposition principle used to construct solutions ceases to hold and new solution methods must be used. A shock wave arises by extending possible solutions for a PDE to include discontinuities.

The inviscid Burgers' equation,

$$u_t + uu_x = 0 \tag{1}$$

Is an example of a simple first-order nonlinear PDE. Shock wave solutions can be found for (1) once the initial conditions (IC) at time $t = 0$, have been specified. Using some unique properties of the characteristic curves of (1),

1. *Each characteristic curve is a straight line.*
2. *The solution is a constant on each such line.*
3. *The slope of each such line is equal to the value of $u(x,t)$ on it.*

and the IC's if they are sufficiently nice. An example of a IC leading to well defined shock wave can be given as,

$$\phi(x) = \begin{cases} 1, & x < 0 \\ 0, & x > 0 \end{cases} \tag{2}$$

From property 1, the characteristic curves for this PDE are straight, and from property 3, the slopes of the curves in the (x,t) plane are of slope 1 when $x < 0$ and slope 0 when $x > 0$. This is shown in Figure 1.

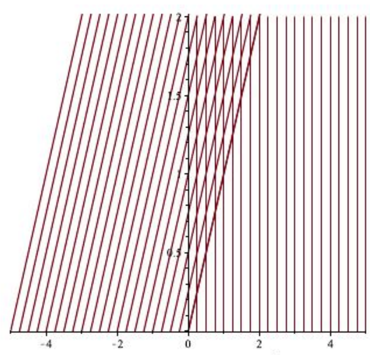


Figure 1: Characteristic curves from the initial conditions given in (2).

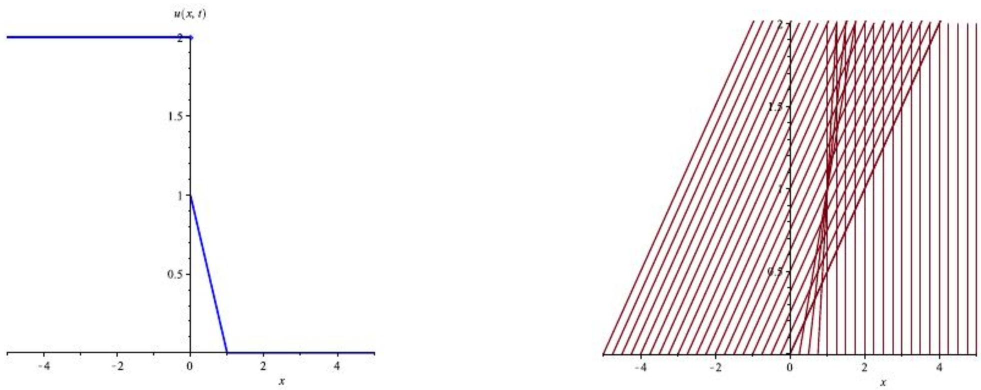
It is clear when analyzing Figure 1, there is nonlinearity in this PDE. Introducing the possibilities for a jump discontinuity in the solution to (1) with ICs (2), we can establish a boundary between both solutions and using the *Rankine-Hugoniot Formula* (HR), we can determine how this boundary moves over time. The HR formula is given by,

$$\frac{A(u^+) - A(u^-)}{u^+ - u^-} = s'(t). \tag{3}$$

where u^+ and u^- represent the solutions approaching the shock from the left and right respectively. The values of u^+ and u^-, since they are constant on either side of the shock wave, allows for a solution which is valid for all time $t > 0$. This is not always the case. Consider for (1) the IC,

$$\phi(x) = \begin{cases} 2\,, & x \;<\; 0 \\ 1-x, & 0 \;<\; x \;<\; 1 \\ 0, & x \;>\; 1 \end{cases} \tag{4}$$

the characteristics for which are depicted in Figure 2.



(a) (b)

Figure 2: The piecewise initial condition and the characteristic curves associated with the initial conditions from (4).

For the ICs (4), it is clear that there is a shock formed immediately at $x = 0$ and subsequently at some later time another shock emerges separating the 2 and 0 solutions. Once the second shock occurs we enter a region of the solution which can be determined for every subsequent time $t < t_*$ where t_* is the time where the second shock forms. Prior to this shock forming, there is another shock separating the solutions 2 and $\frac{1-x}{1-t}$. For this time interval of the solution $u(x,t)$ it is not clear what u^- to use in (3). In fact, u^- changes continuously until the time t_* when the solution $\frac{1-x}{1-t}$ is no longer included in $u(x,t)$. The following develops a numerical method to determine u^+ and u^- in situations such as this.

2 Determining the Properties of a Shock Wave

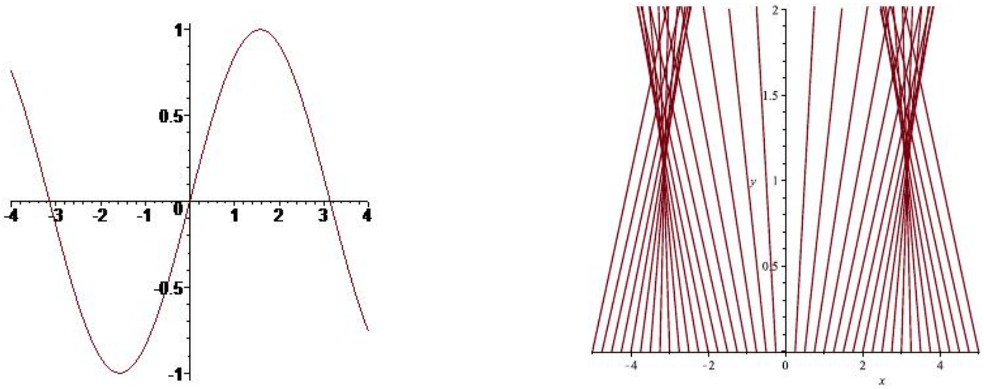
When determining a shock wave solution to a PDE, we are interested in two quantities, when the shock initiates and how quickly it moves through the solution as time progresses. The following discusses how these two properties are determined.

2.1 The Occurrence of Shock Waves

The time t_* of the occurrence of a shock wave in the cases where the IC for Burgers' equation are not simple, straight curves cannot easily be determined graphically. A method for calculating the t_* at which a shock occurs relies on the following property,

$$\frac{\delta u}{\delta x}(x_*, t) \to \infty \quad , \quad \text{as}\, t \to t_* \tag{5}$$

This can be interpreted as the slope at x_* on the IC curve becomes infinitely steep as time approaches t_*. Figure 3 shows the characteristic curves that arise from a sine wave initial condition. From 3b in Figure 3 we can see that the first intersection point of the characteristics curves cannot be found geometrically and we have to look to (4) to determine when the occur.



(a) $u(x, 0) \ = \ \sin(x)$ (b) Characteristic Curves

Figure 3: Burgers' equation with a sine wave initial condition.

Following the method of [2], noting that solution to (1) is $u(x, t) \ = \ f(x - ut)$ by the Method of Characteristics,

$$\frac{\delta u}{\delta x} \ = \ \frac{\delta}{\delta x} f(x - ut) \ = \ f'(\sigma)(1 - \frac{\delta u}{\delta x} t) \ = \ \frac{f'(\sigma)}{1 + t f'(\sigma)} \tag{6}$$

with $\sigma \ = \ x - ut$. From (5) is is clear to see that $\frac{\delta u}{\delta x} \ \rightarrow \ \infty$ when we have,

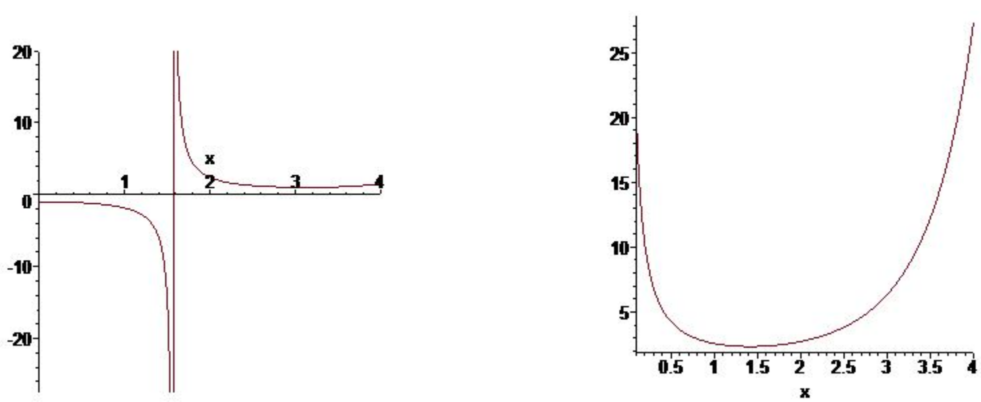
$$t \ = \ -\frac{1}{f'(\sigma)} \tag{7}$$

A shock will only occur if, from the IC $u(x, 0) \ = \ f(x)$ a point x_1 at larger initial speed u starts behind a point x_2 relative to the direction of motion. That is to say that,

$$\frac{\delta u}{\delta x} \ = \ \frac{u(x_1, 0) - u(x_2, 0)}{x_1 - x_2} \ < \ 0 \tag{8}$$

The critical time t_* for the first shock to occur is then given by,

$$t_* \ = \ min\{-\frac{1}{f'(x)} \mid f'(x) \ < \ 0\} \tag{9}$$



(a) t_* for a Sine IC (b) t_* for a Gaussian IC

Figure 4: Plots depicting the minimum occurrences following (8).

With (8) we can calculate the minimum values with several IC functions and therefore determine the time of each initial shock. Plots for a Sine and Gaussian IC are shown in Figure 4. For the Sine wave IC Figure 4a, we note that the minimum occurs as $x \to \pi$ and so the shock occurs at $t_* = 1$. For the Gaussian IC in Figure 4b, the point at which there is a minimum was calculated to be $t_* = 1.165821991$ as $x \to 1.414213562$. From this information we can infer initial condition when determining the form of the shock wave. The following section tackles a method for determining the speed of the wave.

2.2 Conservation Laws and the Shock Wave Speed

Once the point where the shock initiates has been determined, since at that point the solution to the PDE becomes multivalued, the problem turns to choosing which value is physically valid to include in the solution. To determine this, we note that (1) can be written in the form of a conservation law as,

$$\frac{\delta T}{\delta t} + \frac{\delta X}{\delta x} = 0 \Rightarrow \frac{\delta u}{\delta t} + \frac{\delta(\frac{1}{2}u^2)}{\delta x} = 0 \tag{10}$$

Following [2], T can be thought of as the *conserved* density while X is the flux. For the transport equation (1), we then have the following relationship,

$$\frac{d}{dt}\int_a^b u(x,t)dx \;=\; \int_a^b \frac{\delta u(x,t)}{\delta t}dx \;=\; -\int_a^b \frac{\delta(\frac{1}{2}[u(x,t)]^2)}{\delta x}dx \;=\; \frac{1}{2}[u(a,t)^2 - u(b,t)^2] \tag{11}$$

The left had side of the equation tells us the total density or mass contained within a region $[a, b]$, and is then in turn equal to the flux passing through the boundaries $[a, b]$ at any time. This gives a basic conservation law and holds under the assumption that mass is neither created nor destroyed within $[a, b]$. Introducing the shock discontinuity, we want this interval to simple be the shock, and so we look at the limiting case where a and b approach the shock,

$$u_-(t) = u(t, s^-(t)) = \lim_{a \to s^-(t)} u(a,t), \quad u_+(t) = u(t, s^+(t)) = \lim_{b \to s^+(t)} u(b,t) \tag{12}$$

Here now we require an expression for the mass or density passing through the shock at any time t. We can think of this as being the amount of mass entering the shock region times the speed of the shock over a certain time interval Δt. This is expressed from [2] as,

$$\frac{\delta m}{\delta t} = \lim_{\Delta t \to 0} \frac{m(t+\Delta t) - m(t)}{\Delta t} = \lim_{\Delta t \to 0}[\bar{u}_-(t) - \bar{u}_+(t)]\frac{s(t+\Delta t) - s(t)}{\Delta t} = [u_-(t) - u_+(t)]\frac{\delta s}{\delta t} \tag{13}$$

This expression gives us the mass passing though the shock at a time t. From the conservation law (9), the mass passing through the shock region at anytime must be equal to the flux passing though the boundaries of the shock region. Equating this leads to the familiar *Rankine-Hugoniot Formula*[1].

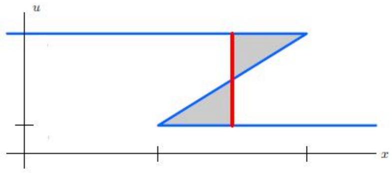
$$\frac{\delta s}{\delta t} = \frac{1}{2}\frac{[u_-(t)^2 - u_+(t)^2]}{[u_-(t) - u_+(t)]} \tag{14}$$

By allowing characteristics to only ever enter a shock region not leave, the causality condition is defined and further determines the proper physical solution to the PDE. The causality condition is,

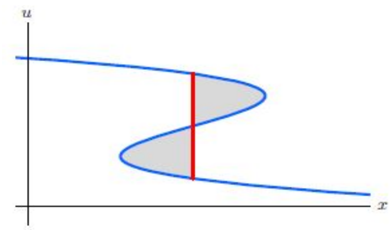
$$u_+(t) \;<\; \frac{\delta s}{\delta t} \;<\; u_-(t) \tag{15}$$

To interpret this graphically, once a shock has occurred in a solution $u(x,t)$, the x position of the shock wave at any time t is determined by the amount of flux, $u_-(t)$ and $u_+(t)$ which passes through the shock region. If the conservation law is to hold across the shock wave, that is $\frac{\delta m}{\delta t}$ is constant, then the total flux must equal zero. This may be represented as a vertical line in the (x, u) plane, representing the x position of this balance at a certain time t. This is known as the Equal Area rule (EAR) and with this, the conservation laws hold.

[1]Which is Equivalent to (3).



(a) The Equal Area rule applied to a simple constant IC.



(b) The Equal Area rule applied to an $arctan(x)$ IC.

Figure 5: Shock waves found by the Equal Area rule, as given in [2]

Figure 5 highlights the difficulty of specifying a shock wave solution to a PDE with non-constant initial conditions. In Figure 5a, the position of the shock wave is given by the EAR while u_- and u_+ define above as the limit of u approaching $s(t)$ from either side of the shock, remain constant for all time. In Figure 5b, for an instant of time t given the position of the shock wave by the EAR, u_- and u_+ may be specified but given a certain time interval Δt, the total mass moving through the shock will change which in turn changes u_- and u_+ at the new time $t + \Delta t$. It is clear then for an IC with constantly varying u, the shock solutions u_- and u_+ hold only for one instant in time in general and so the solution must be specified at even point in time.

3 Numerical Approximations to the Shock Wave Solutions at Individual Time Steps

As highlighted in previous sections, general shock solutions must include not only the speed of the shock wave through time, but also the values of u_- and u_+ at each individual time step. To calculate this by hand is unrealistic, as observed in Figure 5b the values of u_- and u_+ change for each infinitesimal time step dt. Numerically, u_- and u_+ may be determined to a close approximation with use of the EAR. The following demonstrates the application of the EAR to determine u_- and u_+ and results for various ICs.

3.1 Numerical Method

The numerical calculations of the shocks appearing from continuous initial conditions for Burgers' equation were calculated using the Python programming language. The following description attempts to follow in order the steps taken by the program to preform the calculations.

Firstly, the user is allowed to choose which function to analyze. Functions are predetermined in this program to allow for efficiency and speed but can be extended with minor changes to analyze a wide range of functions. Once a function has been selected, it is sectioned off into an array of x and u values such that x_1 is the leftmost x value in the specified domain D. For the predefined time step interval the new x values at each time t for every indexed point on the curve are shifted based on the associated u with,

$$x_{n+1} = ut + x_n \tag{16}$$

Applying (16) to each point on the function, the motion of the initial condition can be calculated for any time $t > 0$. The program runs through each individual time step continually checking the condition, $\mathrm{x}_n < \mathrm{x}_{n+1}$ $\forall\, \mathrm{x} \in D$. As long as this condition remains true, the program will continue to advance through time. Once this condition is found to be false for any indexed pair of x values in accordance with (6), $\frac{\delta u}{\delta x} = \infty$ and a shock has occurred. Once this has been found the position and times of the shocks are recorded. The shape of the function at time when the first shock occurs is displayed to give a glimpse into the distortion of the initial condition. An example of this is displayed in Figure 6,

Once a shock has occurred, the task is now turns to find where the shock occurs. The method for finding the shock position comes from the equal area rule described above. To do this the function is sectioned off

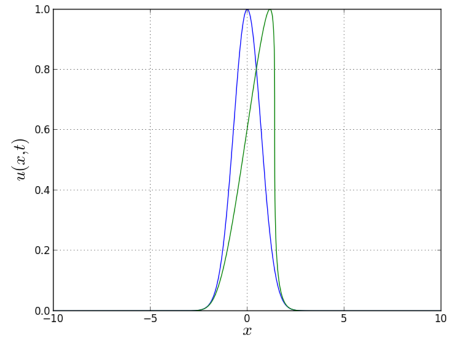


Figure 6: The Gaussian initial condition compared to its position at time $t > 0$ when the first shock has occurred.

into three regions. The first region contains all x values that are less then the lowest indexed value of x=x^* that fails the no shock condition[2]. The second region extends from x^* to the next highest indexed x^{**} that satisfies the no shock condition. The third region contains all x higher than x^{**}. Once the two x^* and x^{**} have been determined, their numerical average is found and this x value is chosen as the initial guess for the position of the shock. A depiction of these three regions along with the shock wave position is shown in Figure 7,

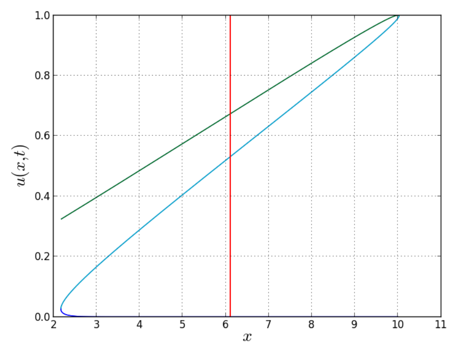


Figure 7: A Gaussian wave with the 3 regions used for calculations highlighted in separated colors. The position of the shock is also highlighted by the red vertical line.

Once the guess for the position of the shock is in place, the true position for a given time step is found by shifting the shock so that the areas in the left and right lobes are equal. Since the original function was initially divided into a list of points the integrating to find the area of each lobe must be done discretely. The widths of the Riemann sum rectangles are determine by averaging the distance between consecutive indexed pairs and then four integrations are preformed over the lobes. Using Figure 7 as an example, the first integration is done from the shock to the point x^* along the topmost line and then from x^* to the shock again along the line directly below. These two area values are then subtracted from each other giving an approximate value to the area of the right lobe. A similar process happens between the shock and x^{**} to obtain the area of the left lobe. We then aim to compare these two areas. If the area of the right lobe is greater than that of the left, then the position of the shock is moved right and the process is repeated. In the other case the shock is shifted left. This is repeated until the areas of both lobes are equal within a certain predetermined tolerance.

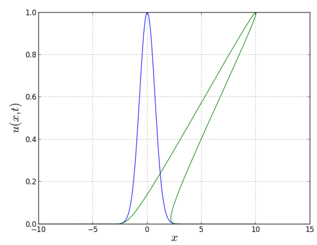
Once the position of the shock is determined for a given time step, the program will then move to the

[2]Recall indexes are assigned from left to right.

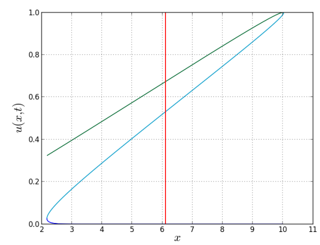
next time and repeat the process until it has evaluated to shock at every time left in the predetermined interval. To end the program, the final shape of the function is displayed along with the shock wave plotted over top of the characteristic curves of the original initial condition. As a final video is presented which displays a time lapse image of the motion of the initial condition. The program then closes and is prepared to run again.

3.2 Results and Discussion

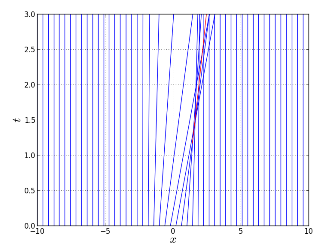
Once the program was complete, three initial conditions were analyzed to show that it was capable of accurately describing the initial time and position of shock creation as well as its movement though time. The three examples chosen where a Gaussian, a sine wave and a hyperbolic tangent curve. The final position of the Gaussian, the shock dividing line and shock curve over the characteristics are presented in Figure 8,



(a) Gaussian final position



(b) Divided Gaussian sections



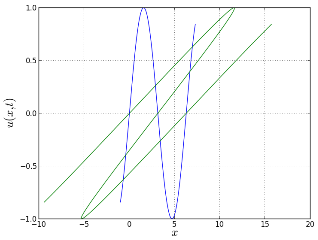
(c) Gaussian shock Curve

Figure 8: Three output plots depicting a Gaussian curve initial condition and the creation of a shock wave.

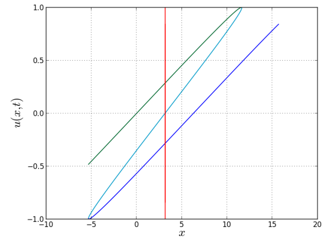
We can see from Figure 8c that the position and time of occurrence of the shock from the Gaussian initial condition appears where you would expect it based on the characteristic lines. Further examples follow in Figures 9 and 10. From these examples we find that the shocks for these functions occur close the positions and times predicted using (6). The values for each are given in Table 1,

IC	Position x	Time t
Gaussian	1.4173	1.1712
Sine Wave	3.1416	1.010
Hyperbolic Tangent	0.00	1.001

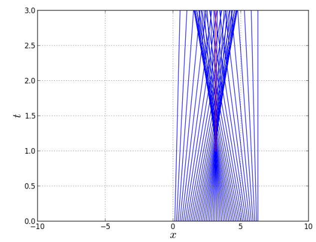
Table 1: The positions and times where each shock occurred for the ICs analyzed here.



(a) Sine wave final position

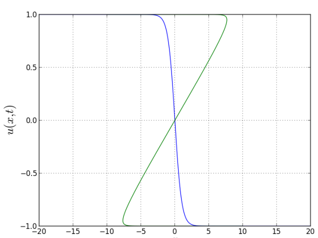


(b) Divided sine wave sections

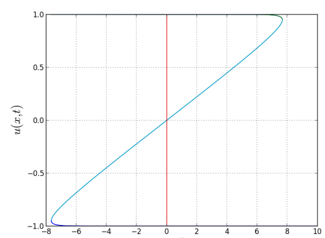


(c) Sine wave shock curve

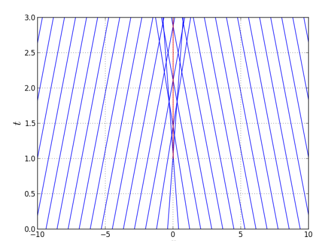
Figure 9: Three output plots depicting a Sine wave curve initial condition and the creation of a shock wave.



(a) Hyperbolic tangent final position



(b) Divided hyperbolic tangent sections



(c) Hyperbolic tangent shock curve

Figure 10: Three output plots depicting a Gaussian curve initial condition and the creation of a shock wave.

This program for determining a shock position does have limitation. The coding is only capable of dealing with one shock wave occurring, this leads to limiting the domain of certain functions that are analyzed. In the case of the sine wave, the domain of x values was restricted to 2π or one full period. Without issue the domain could be extended to $5\pi/2$ but beyond that we would introduce another section in the domain where the derivative is less than zero and hence a second shock occurs. Thus the code developed here can successfully determine the time and location of a shock from a continuous initial condition if there is only one continuous region in the domain with a negative derivative. With two separated regions[3] with negative

[3]The derivatives go from negative to positive and back to negative.

derivatives, two separate shocks will occur and the position of either given by this program will have little or no accuracy. It is then essential to determine beforehand the signs of the derivatives of a function in the domain for a shock analysis to be successful.

4 Conclusion

This project was meant to look into the problem of continuous initial condition to the inviscid Burgers' equation. The general theory for shock creation was discussed, highlighting the method to calculating the time and position of a shock occurrence. The equal area rule based on conservation laws from physics was employed to determine the position of a shock for a Gaussian, sine wave and hyperbolic tangent initial condition. The program method for calculating the shock was described in detail and the application of theory was highlighted in the process.

Results for the positions of the shocks from each initial condition were displayed and it was found that the numerical results for the time and position of shock creation match closely with the theoretical values. We can then conclude that the implementation of the equal area rule done here is a very accurate method to determine shock positions. Although successful in determining the time and location of one shock, the issue of two shock created separately was discussed. To begin to analyze multiple shock situations more would be necessary in the program to allow for separate regions of the domain of an initial condition to have negative derivatives. In a restricted manner, the work done here can be used to show a wide variety of shocks produced by continuous initial conditions.

References

[1] Strauss, Walter A., *Partial Differential Equation, an Introduction*, Second Ed., John Wiley and Sons, Inc. United States, 2008.

[2] Olver, Peter J., *Chapter 22, Nonlinear Partial Differential Equation*, University of Minnesota, http://www.math.umn.edu/ olver/am_/npd.pdf, Viewed March 9, 2013.