Lecture 12 (and 14 and 15)

Some Notes

Lectures

These slides are **long** - we definitely won't finish today. These slides will be used for Lecture 14 (Wednesday after reading week) and 15 (Friday after reading week) as well.

Midterm

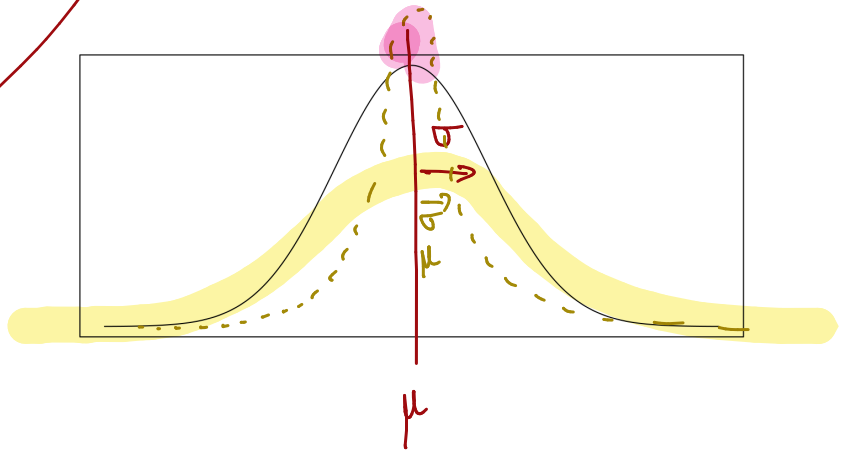
Reminder that our midterm is on Friday! Please show up on time, and bring at least two writing implements (pen or pencil is fine) and a hand calculator. It will make things easier if you don't bring a backpack - space will be an issue.

Further reminder: as you enter the room for the midterm, come to the front, and fill in the middle first. Do **not** sit on the ends of rows, or someone will snap at you for being in the way (that someone might be me …).

Normal Distribution

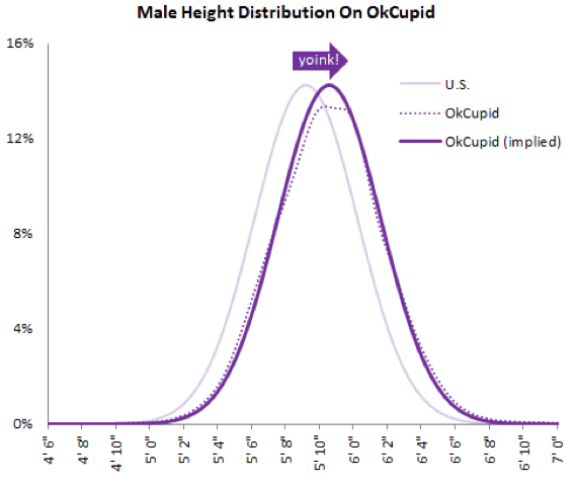
The Normal Distribution

- Unimodal and symmetric, bell shaped curve
- Many variables are nearly normal, but none are exactly normal
- Denoted as $N(\mu, \sigma)$ → Normal with mean μ and standard deviation σ



Binomial: n, p

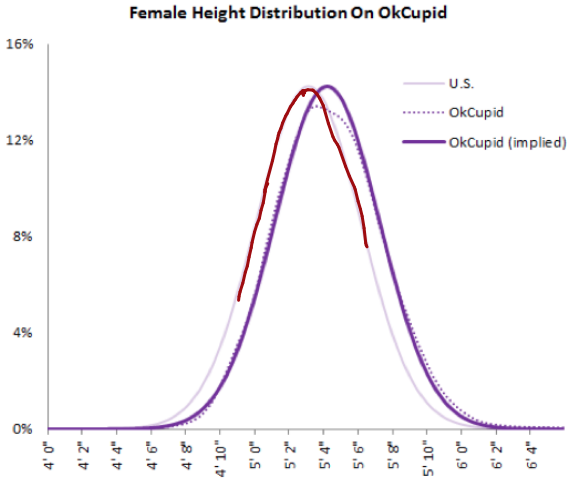
Heights of Males



"The male heights on OkCupid very nearly follow the expected normal distribution – except the whole thing is shifted to the right of where it should be. Almost universally guys like to add a couple inches."

"You can also see a more subtle vanity at work: starting at roughly 5'8", the top of the dotted curve tilts even further rightward. This means that guys as they get closer to six feet round up a bit more than usual, stretching for that coveted psychological benchmark."

Heights of Females

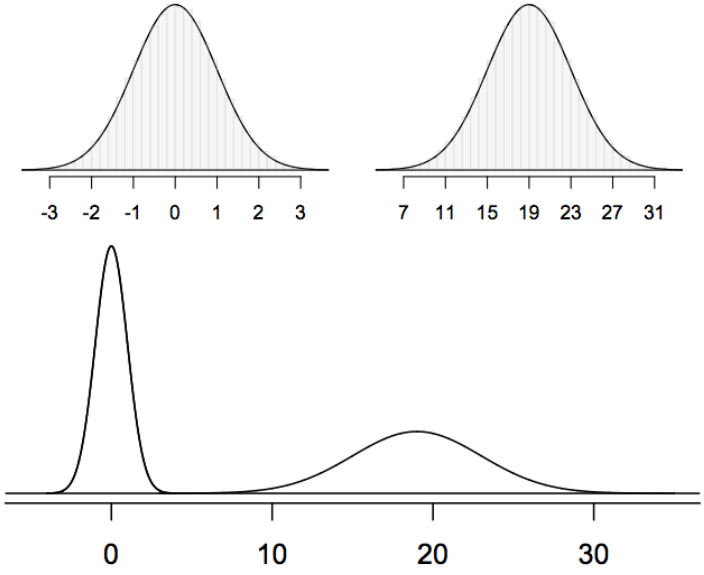


"When we looked into the data for women, we were surprised to see height exaggeration was just as widespread, though without the lurch towards a benchmark height."

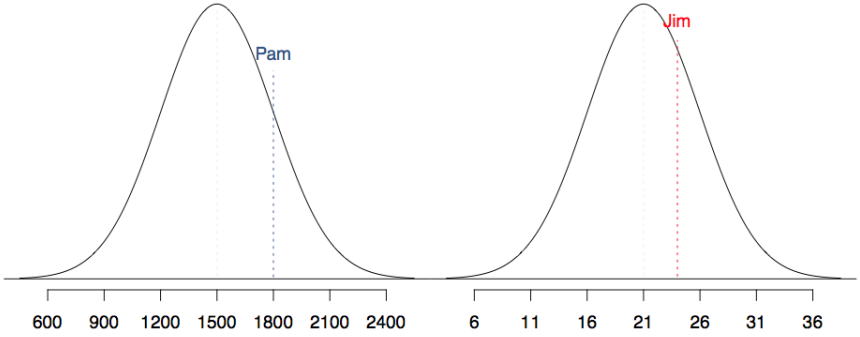
Normal distributions with different parameters

μ: mean, σ: standard deviation

$N(\mu = 0, \sigma = 1)$ $N(\mu = 19, \sigma = 4)$



SAT scores are distributed nearly normally with mean 1500 and standard deviation 300. ACT scores are distributed nearly normally with mean 21 and standard deviation 5. A college admissions officer wants to determine which of the two applicants scored better on their standardized test with respect to the other test takers: Pam, who earned an 1800 on her SAT, or Jim, who scored a 24 on his ACT?



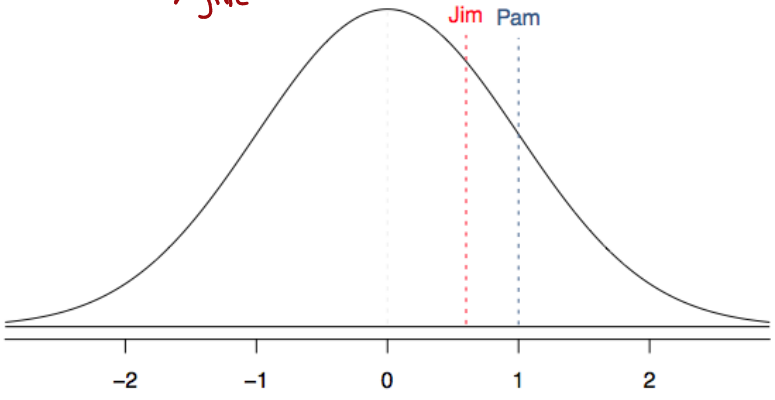
Standardizing with Z-scores

Since we cannot just compare these two raw scores, we instead compare how many standard deviations beyond the mean each observation is.

- Pam's score is $(1800 - 1500)/300 = 1$ standard deviation above the mean.
- Jim's score is $(24 - 21)/5 = 0.6$ standard deviations above the mean.

$z_{pam} = 1.00$

$z_{jim} = 0.60$



$N(0,1)$ — "standard normal"

Standardizing with Z-scores (continued)

These are called **standardized scores**, or **Z-scores** (or **Z scores**).

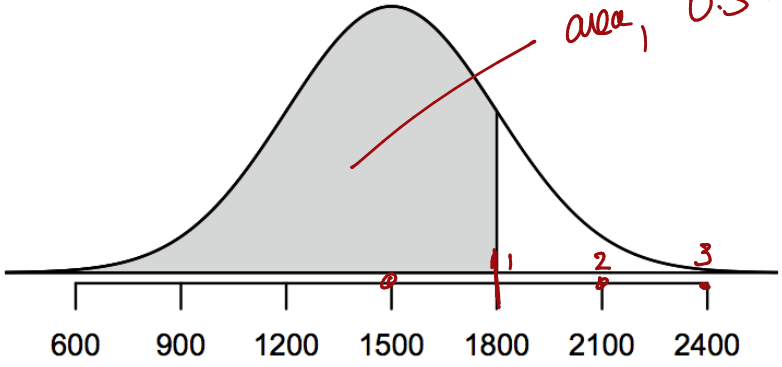
- Z score of an observation is the number of standard deviations it falls above or below the mean.

$$Z = (\text{observation} - \text{mean})/\text{SD}$$

- Z scores are defined for distributions of any shape, but only when the distribution is normal can we use Z scores to calculate percentiles.
- Observations that are more than 2 SD away from the mean ($|Z| > 2$) are usually considered unusual.

Percentiles

- **Percentile** is the percentage of observations that fall below a given data point
- Graphically, percentile is the area below the probability distribution curve to the left of that observation



eg 0.6 ⇒ 60th percentile is 1800

Calculating Percentiles using Computation

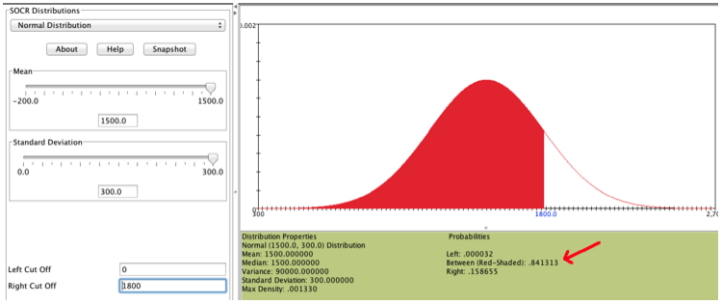
There are many ways to compute percentiles/areas under the curve.

R:

```
pnorm(1800, mean = 1500, sd = 300)
```

```
## [1] 0.8413447
```

Applets:



Calculating Percentiles - Look them Up!

	Second decimal place of Z									
Z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	*0.8413*	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015

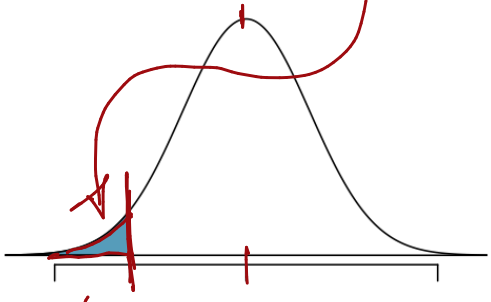
Six Sigma

The term six sigma process comes from the notion that if one has six standard deviations between the process mean and the nearest specification limit, as shown in the graph, practically no items will fail to meet specifications.

Example: Quality Control

At the Heinz ketchup factory, the amounts which go into bottles of ketchup are supposed to be normally distributed with mean 36 oz. and standard deviation 0.11 oz. Once every 30 minutes a bottle is selected from the production line, and its contents are noted precisely. If the amount of ketchup in the bottle is below 35.8 oz. or above 36.2 oz., then the bottle fails the quality control inspection. What percent of bottles have less than 35.8 ounces of ketchup?

- Let X = amount of ketchup in a bottle: $X \sim \mathcal{N}(\mu = 36, \sigma = 0.11)$



(1) $$Z = \frac{35.8 - 36}{0.11} = -1.82$$

what is this area??

Finding the exact probability - using the Z table

Second decimal place of Z										
0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	Z
0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019	−2.9
0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026	−2.8
0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035	−2.7
0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047	−2.6
0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062	−2.5
0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082	−2.4
0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107	−2.3
0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139	−2.2
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	−2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	−2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	−1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	−1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	−1.5

Finding the exact probability - using the Z table

Second decimal place of Z										
0.09	0.08	0.07	0.06	0.05	0.04	0.03	0.02	0.01	0.00	Z
0.0014	0.0014	0.0015	0.0015	0.0016	0.0016	0.0017	0.0018	0.0018	0.0019	−2.9
0.0019	0.0020	0.0021	0.0021	0.0022	0.0023	0.0023	0.0024	0.0025	0.0026	−2.8
0.0026	0.0027	0.0028	0.0029	0.0030	0.0031	0.0032	0.0033	0.0034	0.0035	−2.7
0.0036	0.0037	0.0038	0.0039	0.0040	0.0041	0.0043	0.0044	0.0045	0.0047	−2.6
0.0048	0.0049	0.0051	0.0052	0.0054	0.0055	0.0057	0.0059	0.0060	0.0062	−2.5
0.0064	0.0066	0.0068	0.0069	0.0071	0.0073	0.0075	0.0078	0.0080	0.0082	−2.4
0.0084	0.0087	0.0089	0.0091	0.0094	0.0096	0.0099	0.0102	0.0104	0.0107	−2.3
0.0110	0.0113	0.0116	0.0119	0.0122	0.0125	0.0129	0.0132	0.0136	0.0139	−2.2
0.0143	0.0146	0.0150	0.0154	0.0158	0.0162	0.0166	0.0170	0.0174	0.0179	−2.1
0.0183	0.0188	0.0192	0.0197	0.0202	0.0207	0.0212	0.0217	0.0222	0.0228	−2.0
0.0233	0.0239	0.0244	0.0250	0.0256	0.0262	0.0268	0.0274	0.0281	0.0287	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	0.0329	0.0336	0.0344	0.0351	0.0359	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	0.0409	0.0418	0.0427	0.0436	0.0446	−1.7
0.0455	0.0465	0.0475	0.0485	0.0495	0.0505	0.0516	0.0526	0.0537	0.0548	−1.6
0.0559	0.0571	0.0582	0.0594	0.0606	0.0618	0.0630	0.0643	0.0655	0.0668	−1.5

Finding the exact probability - using R

```
pnorm(-1.82)
```

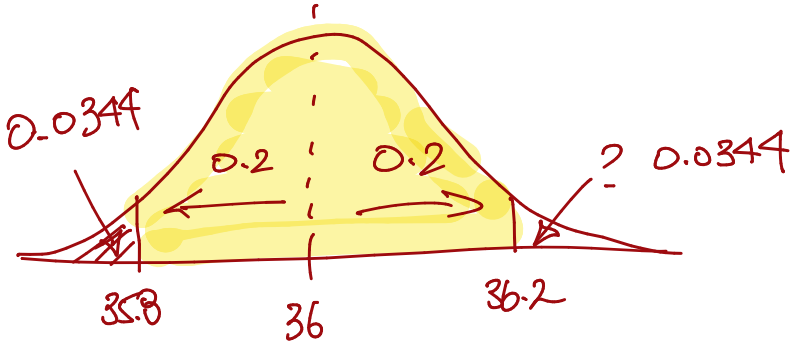
```
## [1] 0.0343795
```

Simpler!

Practice

What percentage of bottles **pass** the quality control inspection?

1. 1.82%
2. 3.44%
3. 6.88%
4. 93.12%
5. 95.56%

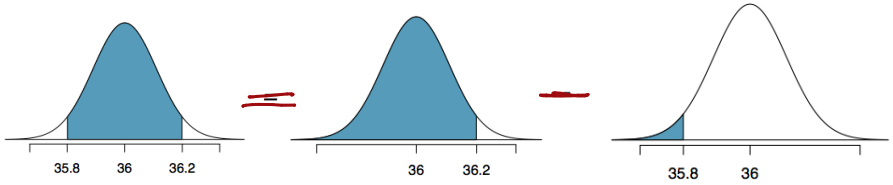


$$P[\text{pass}] = 1 - 2 \cdot P[\text{fail low}]$$
$$= 1 - 2(0.0344)$$

Practice

What percentage of bottles **pass** the quality control inspection?

1. 1.82%
2. 3.44%
3. 6.88%
4. *93.12%*
5. 95.56%



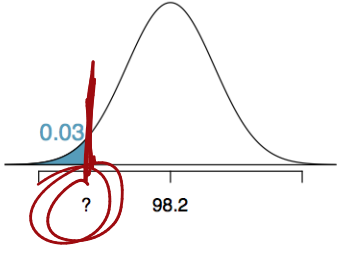
$$Z_{35.8} = \frac{35.8 - 36}{0.11} = -1.82$$

$$Z_{36.2} = \frac{36.2 - 36}{0.11} = 1.82$$

$$P(35.8 < X < 36.2) = P(-1.82 < Z < 1.82) = 0.9656 - 0.0$$

Example: Finding Cutoff Points

Body temperatures of healthy humans are distributed nearly normally with mean 98.2 ° F and standard deviation 0.73 ° F. What is the cutoff for the lowest 3% of human body temperatures?



0.09	0.08	0.07	0.06	0.05	Z
0.0233	0.0239	0.0244	0.0250	0.0256	−1.9
0.0294	0.0301	0.0307	0.0314	0.0322	−1.8
0.0367	0.0375	0.0384	0.0392	0.0401	−1.7

$P(Z < -1.88) = 0.03$

$$Z = \frac{\text{obs} - \text{mean}}{\text{SD}} \longrightarrow \frac{x - 98.2}{0.73} = -1.88$$

$$x = (-1.88 \times 0.73) + 98.2 = 96.8\ ^\circ\text{F}$$

? ⇒ pnorm(?, mean=98.2, sd=0.73)

= 0.03

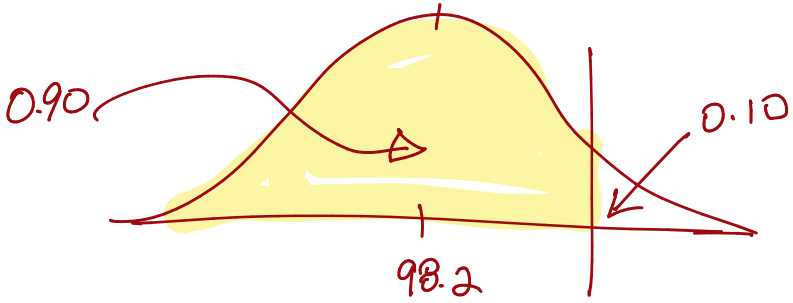
qnorm(0.03, mean = 98.2, sd = 0.73) = 96.8

qnorm(0.03) = -1.88

Practice

Body temperatures of healthy humans are distributed nearly normally with mean 98.2 ° F and standard deviation 0.73 ° F. What is the cutoff for the highest 10% of human body temperatures?

1. 97.3 ° F
2. 99.1 ° F
3. 99.4 ° F
4. 99.6 ° F



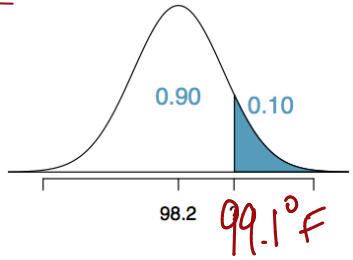
qnorm(0.90, mean = 98.2, sd = 0.73)

qnorm(0.10, mean = 98.2, sd = 0.73, lower.tail = FALSE)

Practice

Body temperatures of healthy humans are distributed nearly normally with mean 98.2 ° F and standard deviation 0.73 ° F. What is the cutoff for the highest 10% of human body temperatures?

1. 97.3 ° F
2. *99.1 ° F*
3. 99.4 ° F
4. 99.6 ° F



Z	0.05	0.06	0.07	0.08	0.09
1.0	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9115	0.9131	0.9147	0.9162	0.9177

$.10 \longrightarrow P(Z > 1.28) = 0.10$

$\frac{\text{bs} - \text{mean}}{\text{SD}} \longrightarrow \frac{x - 98.2}{0.73} = 1.28$

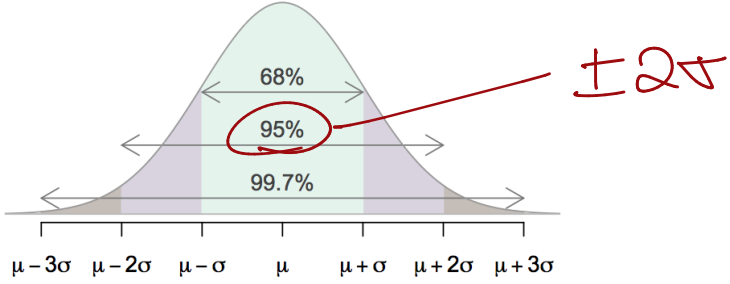
$.28 \times 0.73) + 98.2 = 99.1\ ^\circ\text{F}$

68-95-99.7 Rule

For normally distributed data,

- about 68% falls within 1 SD of the mean
- about 95% falls within 2 SD of the mean
- about 99.7% falls within 3 SD of the mean

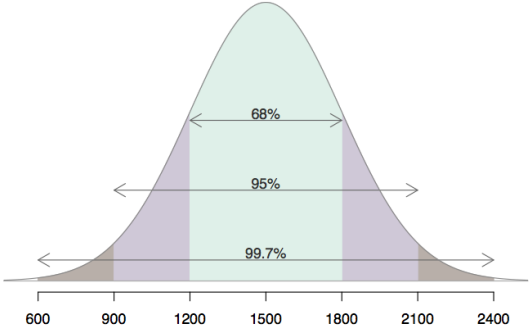
It is possible for observations to fall 4, 5 or even more standard deviations away from the mean, but these occurrences are very rare if the data are nearly normal.



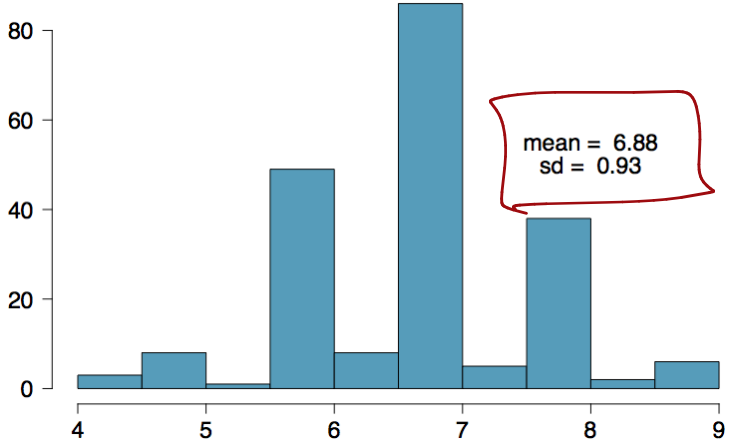
Describing variability using the 68-95-99.7 Rule

SAT scores are distributed nearly normally, with mean 1500 and standard deviation 300.

- ≈ 68% of students score between 1200 and 1800 on the SAT
- ≈ 95% of students score between 900 and 2100 on the SAT
- ≈ 99.7% of students score between 600 and 2400 on the SAT



Example: Number of nights of sleep on school nights



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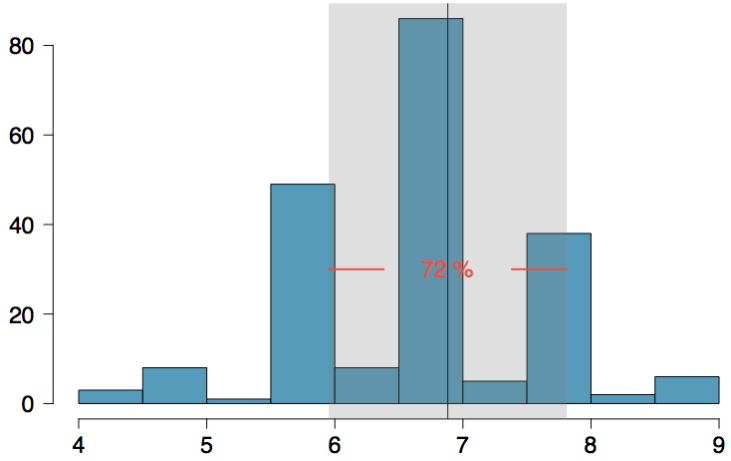
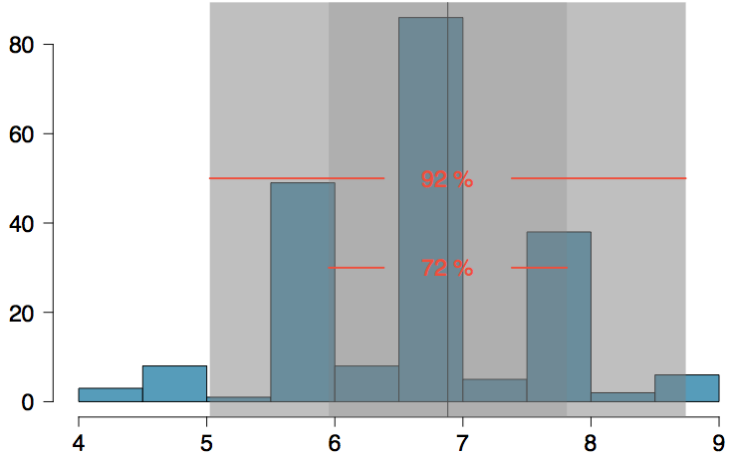


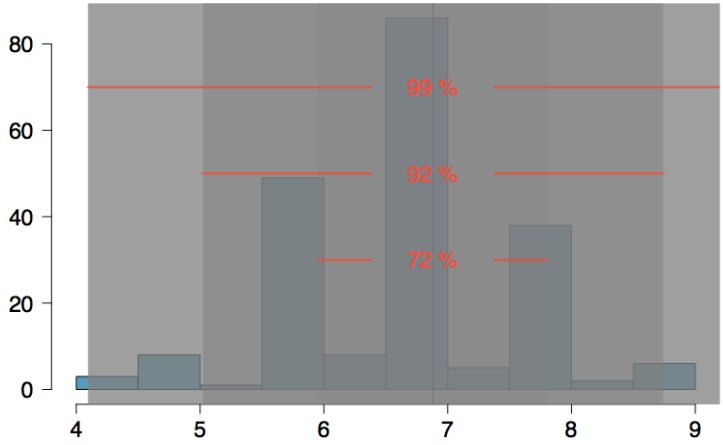
Mean = 6.88 hours, SD = 0.92 hours.
72% of the data are within 1 SD of the mean: 6.88 $\pm$ 0.93.

Example: Number of nights of sleep on school nights



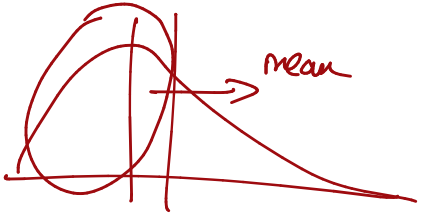
Mean = 6.88 hours, SD = 0.92 hours.
72% of the data are within 1 SD of the mean: 6.88 ± 0.93.
92% of the data are within 2 SD of the mean: $6.88 \pm 2 \times 0.93$.

Example: Number of nights of sleep on school nights



Mean = 6.88 hours, SD = 0.92 hours.
72% of the data are within 1 SD of the mean: 6.88 ± 0.93.
92% of the data are within 2 SD of the mean: $6.88 \pm 2 \times 0.93$.
99% of the data are within 3 SD of the mean: $6.88 \pm 3 \times 0.93$.

Practice



mean - mean = 0

Which of the following is **false**?

1. Majority of Z scores in a right skewed distribution are negative.
2. In a skewed distributions the Z score of the mean might be different than 0.
3. For a normal distribution, IQR is less than 2 x SD.
4. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

FALSE

IQR = 50%

$\mu \pm 2\sigma \approx 95\%$

Practice

Which of the following is **false**?

1. Majority of Z scores in a right skewed distribution are negative.
2. In a skewed distributions the Z score of the mean might be different than 0.
3. *For a normal distribution, the IQR is less than 2 x SD.*
4. Z scores are helpful for determining how unusual a data point is compared to the rest of the data in the distribution.

The Normal Approximation

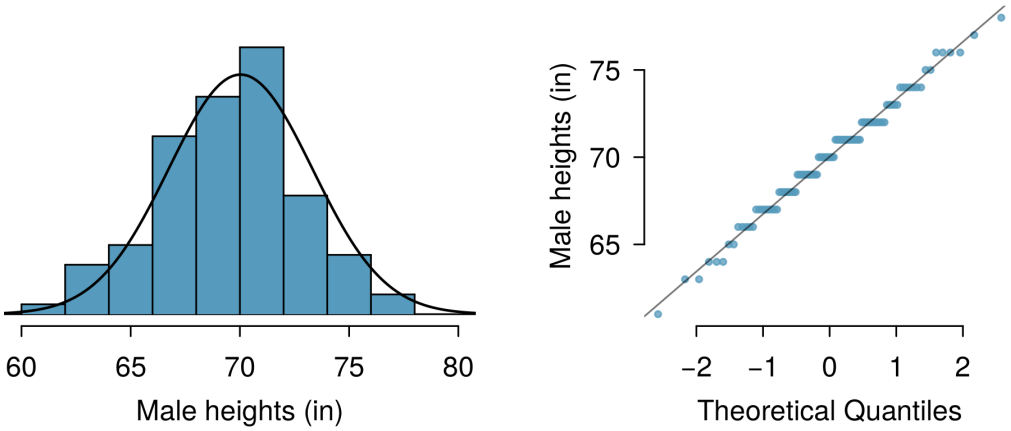
Evaluating The Normal Approximation

We often use the normal distribution as an **approximation**, taking real data and assuming it follows the normal.

How do we tell whether this is a good assumption?

Normal probability plot (QQ)

A histogram and **normal probability plot** of a sample of 100 male heights.



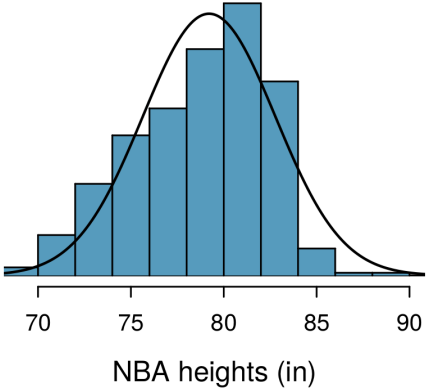
The right hand plot is also called a **quantile-quantile plot**, or **QQ plot** for short.

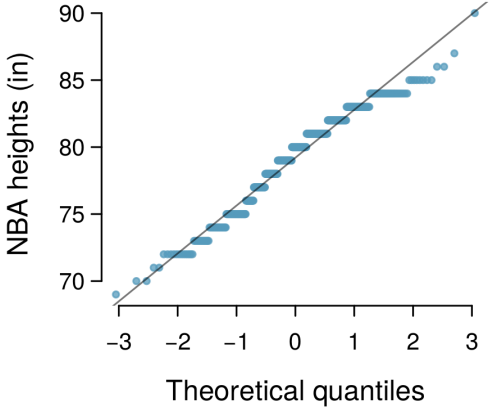
Anatomy of a normal probability plot

- Data are plotted on the y-axis of a normal probability plot, and theoretical quantiles (following a normal distribution) on the x-axis.
- If there is a linear relationship in the plot, then the data follow a nearly normal distribution.
- Constructing a normal probability plot requires calculating percentiles and corresponding z-scores for each observation, which is tedious. Therefore we generally rely on R when making these plots.

NBA Heights, 2008-09

Below is a histogram and normal probability plot for the NBA heights from the 2008-2009 season. Do these data appear to follow a normal distribution?





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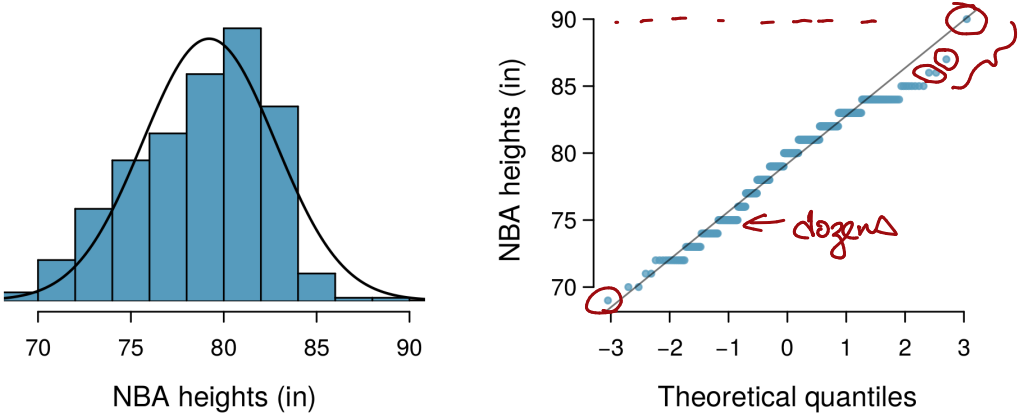
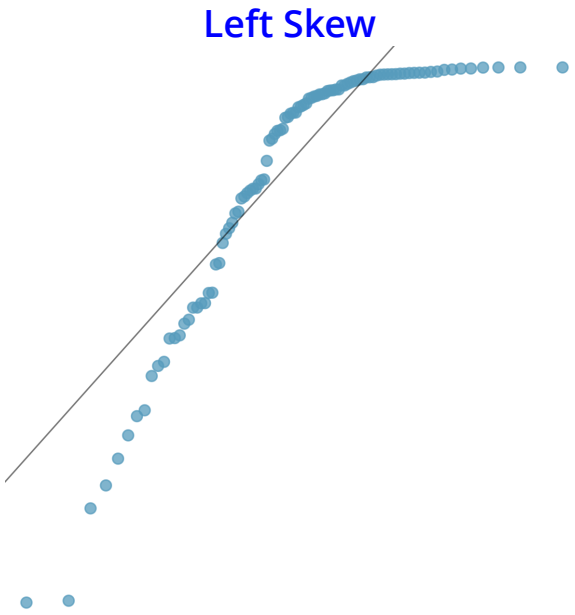


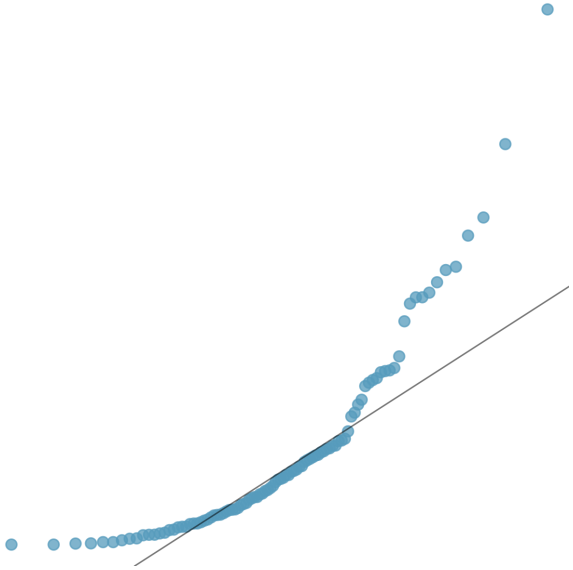
Why do the points on the normal probability have jumps?

Normal probability plot and skewness



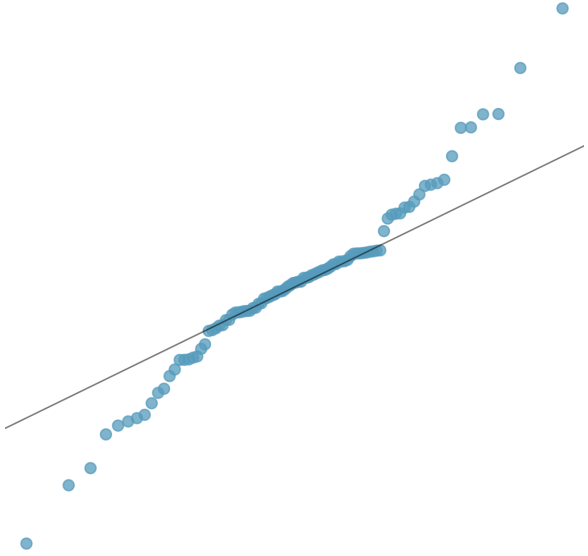
Normal probability plot and skewness

Right Skew



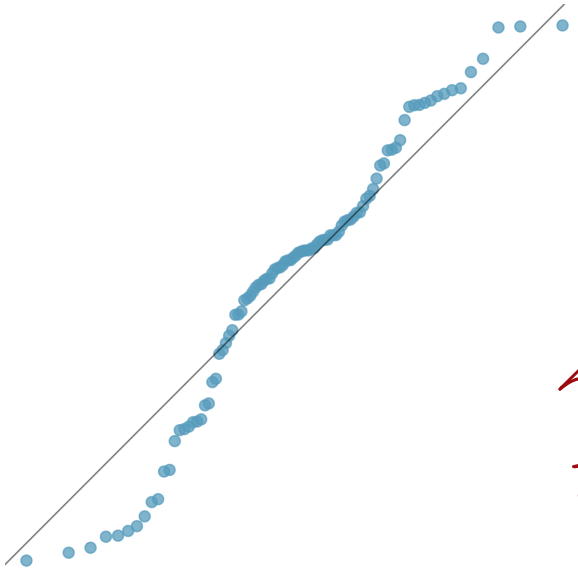
Normal probability plot and skewness

Long Tails



Normal probability plot and skewness

Short Tails



4.1 done