Math 315 - Fall 2016 - Cholesky Factorization

Instructions: You can work in groups of up to three. Please write up each solution carefully. If MATLAB is used for a problem please submit the appropriate code and output. Submit one project per group in Gradescope. One group member will log in, upload the project, and then select the group members.

In this assignment we consider solving linear systems: $A\mathbf{x} = \mathbf{b}$ where A is an $n \times n$ symmetric and positive definite matrix. First we need to define the words *symmetric* and *positive definte*. An $n \times n$ matrix A is said to be *symmetric* if

$$A = A^T,$$

and it is *positive definite* if

$$A = A^T, \ \mathbf{x}^T A\mathbf{x} > 0 \text{ for all } \mathbf{x} \neq 0.$$

The special structure of a symmetric and positive definite matrix (which we will abbreviate as SPD) allows us to compute a special LU factorization, which is called the *Cholesky factorization*. Specifically, it can be shown that an $n \times n$ matrix A is SPD if and only if it can be factored as

$$A = LL^T \text{ (called the Cholesky factorization)}$$

where L is a lower triangular matrix with positive entries on the diagonal. Pivoting is not needed to compute this factorization. The **if and only if** part of the above statement is important. This means that if we are given a matrix A, and the Cholesky factorization fails, then we know A is either not symmetric or not positive definite (or both). If it succeeds, then we know A is symmetric and positive definite.

1. For a small matrix, it is easy to compute a Cholesky factorization by hand. Consider the matrix

$$A = \begin{bmatrix} 4 & 1/2 & 1 \\ 1/2 & 17/16 & 1/4 \\ 1 & 1/4 & 33/64 \end{bmatrix}$$

To find the Cholesky factorization, do the following:

– Set $L = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}$

– Form the matrix LL^T :

$$L = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

– Now set $A = LL^T$ and solve for l_{ij} . For example, in this example we see that $l_{11}^2 = 4$, and so $l_{11} = 2$.

Continue this process, and show all steps to find (by hand) the Cholesky factorization of the matrix in 1. You can check your answer using the MATLAB function `chol`.

2. Write MATLAB code to compute the Cholesky factorization of an SPD matrix:

```
function L = MyChol(A)
    %
    %  This is a very basic implementation to compute the Cholesky
    %  factorization of an SPD matrix A.
    %
    %  Input:  A - n-by-n SPD matrix
    %
    %  Output: L - Cholesky factorization of A
    %
    %  Note: This does not properly check to see if A is SPD, so it may
    %        fail without any warnings.
    %
    n = size(A, 1);
    L = zeros(n, n);
    for k = 1:n
        L(k,k) =
        L(k+1:n, k) =
        A(k+1:n, k+1:n) =
end
```

3. Count the total number of floating point operations (FLOPS) needed by this algorithm. That is, count all floating point multiplications, divisions, additions and subtractions. You do not need to count any integer operations (e.g., k + 1), and you can ignore the computation of the square root. You must carefully explain and show your work for this problem – a single answer is not good enough.