Quiz 10

Claim: Given a graph G and a matching M with edges m, M is a maximum if and only if there does not exist an augmenting path.

Must prove two things:
1) If M is a maximum, then there is no augmenting path.
2) If there is no augmenting path, then M is a maximum.

Proof of 1)
Assume a match M is maximum, where M contains m edges. Then, M contains the greatest possible number of edges that a match can contain for graph G with edges n, such that $m < n$.
Now suppose there exists an augmenting path A whose set of edges contain all edges in M. This implies there are at least two free vertices that can be connected using an alternating path P with contains M. If an augmentation is performed, then the new match M’ will contain m+1 edges. This would imply M is not maximum, which is a contradiction. Now suppose there exists an augmenting path A from one free vertex to another, where A does not contain all the edges is M. If an augmentation is performed, the new match M’, which consists of the edges of M not in A and the new edges after the augmentation, will also contain m+1 edges. This also implies M is not maximum, which is a contradiction, since we assumed M to be maximum. Therefore it follows that if M is maximum, then there will not exist an augmenting path.

Proof of 2) (Prove the contrapositive)
Assume two things. Assume that M is not a maximum and contains m edges. Also assume that there are at least two free vertices v and w in the graph. Then, it follows that there will be an augmenting path A which connects v and w, where v → v’ is an edge adjacent to v not in M and w’ → w is also not it M. If N is a maximum which contains n edges, and an augmentation were performed, then the augmentation path N’ would contain n+1 edges, which is a contradiction, since we assume that N is a maximum. It follows that there will exist an augmenting path A from v to w such that performing an augmentation will yield a match with m+1 edges, which is valid since M is not a maximum. Therefore if M is not a maximum, then there is an augmenting path. Since this statement is true, it follows that the contrapositive is also true. Therefore, if there is no augmenting path, then M is a maximum.

Therefore, since the two required statements are proven to be true, it follows that M is a maximum if and only if there is no augmenting path.