# Leapfrogging Vortices Report

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## 1 Introduction

A vortex ring is a torus-shaped vortex in a fluid where the fluid spins around a circular, closed-loop, imaginary axis. As this happens, the entire ”ring” moves forward as well.

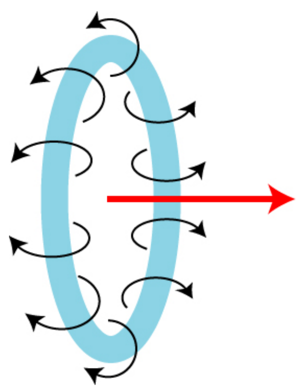

Figure 1: A visualization of a vortex ring.[1]

If timed correctly, one can have ”leapfrogging” vortices by putting one vortex ring right behind another. These vortex rings move in the same direction, with the back vortex ring periodically surpassing the front vortex ring (hence, the ”leapfrogging” effect). In this report, I will study the motion of leapfrogging vortices by creating a 2D simulation of this phenomenon.

## 2 Methods

Each of these vortex rings produces a velocity field that acts on both itself and the other vortex ring. This is what propels the vortex rings forward. The simplified 2D simulation of leapfrogging vortices is accomplished by imagining a plane cutting through the two vortex rings, as shown in fig. 2. Each vortex

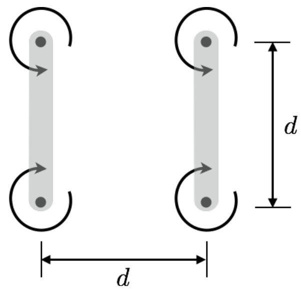


Figure 2: A pair of leapfrogging vortices.

ring intersects the plane at two points. Each of these points represents a 2D vortex with a 2D velocity field.

The tangential velocity of the flow around a vortex is given by eq. (1).

$$V_\theta = \frac{\Gamma}{2\pi r} \tag{1}$$

$\Gamma$ is the circulation, or strength of the vortex, and $r$ is the distance from the vortex center. I used the vector form of eq. (1), which is given by eq. (2).

$$\vec{V} = \frac{\vec{\Gamma} \times \vec{r}}{2\pi r^2} \tag{2}$$

As shown in fig. 2, I began with two identical vortex rings, with the distance between them equal to their diameter. To simulate the motion of these leapfrogging vortices, I discretized the motion path of the 2D vortices. I assumed a linear motion path during each discrete time interval. My method for tracing out the motion path can be summarized by the following steps:

1. **Find the position, $\vec{r}$, of each vortex with respect to every other vortex at a certain point in time.**

2. **Find the velocity, $\vec{V}$, of each vortex at that point in time.** This was accomplished by individually computing the velocity of each vortex induced by every other vortex using equation (2). The total velocity of a vortex is then just the sum of all the computed induced velocities on that vortex.

3. **Find the approximate displacement, $\Delta\vec{x}$, of each vortex over one time interval.** This is done by using the equation $\Delta\vec{x} = \vec{V}\Delta t$, where $\Delta t$ is the time interval length.

4. **Update the positions, $\vec{x}$, of each vortex.** Use $\vec{x}_{new} = \vec{x}_{old} + \Delta\vec{x}$.

5. **Repeat steps 1 through 4.** Stop when the traced-out path of motion is satisfactory.

# 3 Results

I first used a time interval length of $\Delta t = 0.01$, a starting distance between vortices of $d = 1.0$, and a vortex strength of $\Gamma = [0, 0, 1]^\top$. The simulated 2D path of the leapfrogging vortices with 4000 time steps is shown in fig. 3. The two vortex rings start at the left of the figure and progressively move to the right, leapfrogging each other along the way

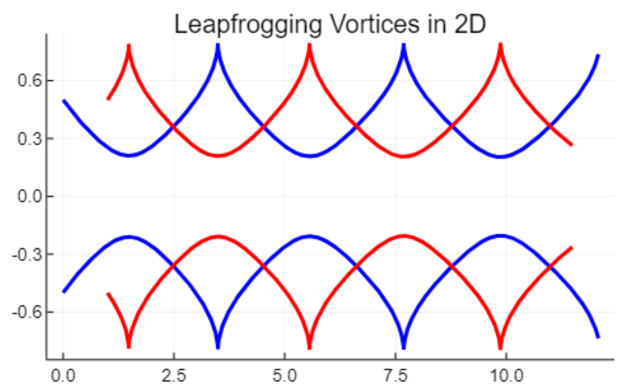


Figure 3: A 2D simulation showing the path of two leapfrogging vortices. The two colors, blue and red, represent the two different vortex rings. The lines trace out the paths of the top and bottom edges of each vortex ring (see fig. 2).

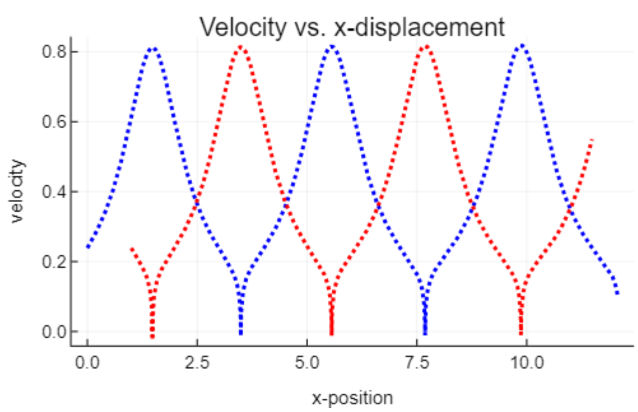


Figure 4: Velocity vs. x-displacement plot for each of the two vortex rings, blue and red.

In fig. 4, one can see how the forward velocity of the vortex rings in the

$x$-direction change as they leapfrog one another. The vortex rings are traveling the fastest when they are surrounded by the other ring, and they are traveling the slowest when they are surrounding the other ring. In other words, the maximum and minimum velocities occur in the moment when one ring passes the other (i.e. right when they "leapfrog"). This is to be expected, since that is the moment when each velocity vector from the different velocity fields is pointing exactly in the $x$- or negative $x$-direction for each vortex. This results the velocity vectors fully adding or canceling out with each other.

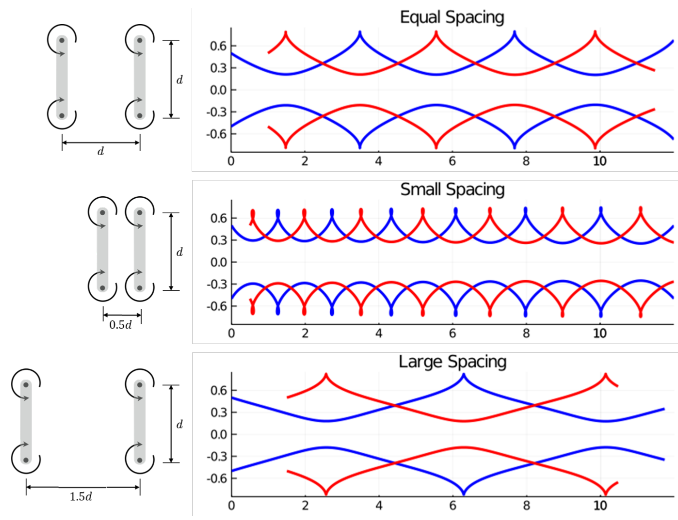


Figure 5: A visualization of how the initial spacing affects the path of the vortex rings. As in fig. 3, the two colors, blue and red, represent the two different vortex rings, and the lines trace out the paths of the top and bottom edges of each vortex ring.

Adjusting the starting positions of the vortex rings affects their paths of motion. As seen in fig. 5, the smaller the initial spacing between the two vortex rings, the more frequently they leapfrog. As the spacing increases, the frequency of leapfrogging decreases.

In addition, the three simulations in fig. 5 were set up to represent the same amount of elapsed time. The pair of vortex rings that were initially closer together seemed to travel farther, which may suggest that they travel faster on average than more spaced out vortex rings. However, a more sophisticated model and a more accurate motion-path-approximation method may be needed to confirm this.

If the two vortex rings are not aligned well, then the leapfrogging pattern breaks down very quickly, as seen in fig. 6.

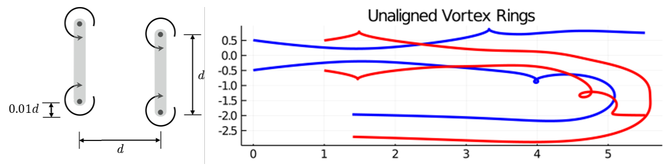


Figure 6: A visualization of the motion path for unaligned vortex rings.

# 4 Conclusion

As I created and experimented with the 2D leapfrogging-vortex-rings simulation, I learned that the behavior of vortex rings is very sensitive to initial conditions. Varying the initial distance between the vortex rings caused them to move at different speeds and leapfrog with different frequencies. Also, just a slight misalignment of the center of the vortex rings resulted in seemingly unpredictable, chaotic movement. A potential next step would be to test my results with a 3D simulation using a more accurate model for fluid flow around a vortex.

# References

[1] https://skullsinthestars.com/2012/08/28/physics-demonstrations-vortex-cannon