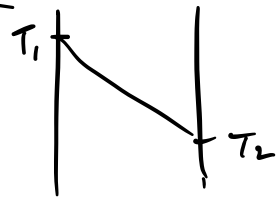
1D steady-state

$$\int \frac{d^2T}{dx^2} = \int 0$$

$$\int \frac{dT}{dx} = \int C_1$$



$$T(x) = C_1 x + C_2$$

BCs: $T(x=0) = T_1$ $T(x=L) = T_2$

$$T(x) = C_1 x + T_1$$

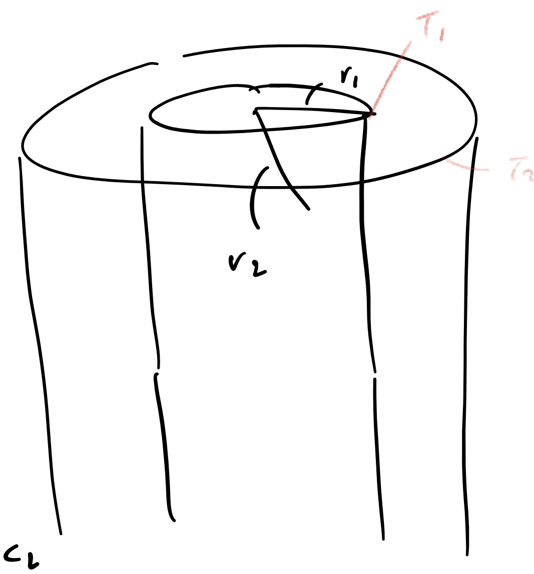
$$T_2 = C_1 L + T_1 \qquad C_1 = \frac{T_2 - T_1}{L}$$

$$\boxed{T(x) = T_1 + \left(\frac{T_2 - T_1}{L}\right)x}$$

HOLLOW CYLINDER

$$\frac{d^2T}{dr^2} + \boxed{\frac{1}{r}\frac{dT}{dr}} = 0$$

$$\underbrace{r\frac{d^2T}{dr^2} + \frac{dT}{dr}} = 0$$



$$\int \frac{d}{dr}\left(r\frac{dT}{dr}\right) = \int 0$$

$$r\frac{dT}{dr} = C_1$$

$$\int \frac{dT}{dr} = \int \frac{C_1}{r}$$

$$T(r) = C_1 \ln r + C_2$$

BC's: $T(r_1) = T_1$ $T(r_2) = T_2$

$$T_1 = C_1 \ln r_1 + C_2$$

$$T_2 = C_2 \ln r_2 + C_2$$

$$C_1 = -\frac{(T_1 - T_2)}{\ln(r_2/r_1)} \qquad C_2 = T_1 + \frac{(T_1 - T_2)}{\ln(r_2/r_1)}\ln\left(\frac{r}{r_1}\right)$$

→ 2D steady-state

$$\underbrace{\nabla^2 T}_{} = 0$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

Typically, one can "guess" the solution to be of form $e^{\lambda x}$ $\quad e^x$, $\sin x$, $\cos x$

Back to 1D: $\frac{d^2 y}{dx^2} + y = 0$

guess $y = e^{\lambda x}$

$$\lambda^2 e^{\lambda x} + e^{\lambda x} = 0$$

$$(\lambda^2 + 1) e^{\lambda x} = 0$$

$$\lambda = \sqrt{-1} = \pm i$$

$$y = e^{\pm i x}$$

$$y = c_1 e^{ix} + c_2 e^{-ix}$$

$$y = c_3 \cos x + c_4 \sin x$$

BC's: $y(x=0) = 0$, $y(\pi) = 0$

$$c_3 \cdot 1 + 0 = 0$$

$$c_3 \cdot (-1) + 0 = 0$$

$c_3 = 0 \qquad c_4$: anything

BC's: $y(0) = 0$, $y(L) = 0$

$$y(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} \leftarrow$$

$$= c_3 \cos \lambda x + c_4 \sin \lambda x$$

$$c_4 \sin(\lambda L) = 0$$

↳ satisfied if $\sin(\lambda L) = 0$

periodic : $\lambda_n = \frac{n\pi}{L}$

$n = 1, 2, 3, \ldots$

"CHARACTERISTIC VALUES"

$$Y_n = \underbrace{A_n}_{\text{arbitrary}} \cdot \underbrace{\phi_n(x)}$$

arbitrary non-zero constant

"CHARACTERISTIC FUNCTION"

$$\phi_n(x) = \sin\left(\frac{n\pi x}{L}\right)$$

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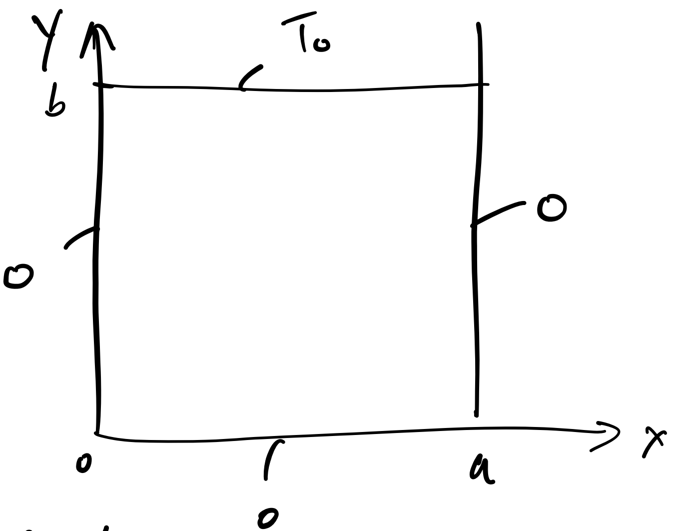
① schematic / BC's / assumptions

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$



BC's

$T(0, y) = 0$

$T(a, y) = 0$

$T(x, 0) = 0$

$T(x, b) = T_0$

② SoV

assume soln is in form $T(x,y) = X(x)\,Y(y)$

$$\frac{\partial^2 (X \cdot Y)}{\partial x^2} + \frac{\partial^2 (X \cdot Y)}{\partial y^2} = 0$$

$$Y \frac{\partial^2 X}{\partial x^2} + X \frac{\partial^2 Y}{\partial y^2} = 0$$

$$-\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = \lambda^2$$

$$\frac{\partial^2 X}{\partial x^2} = -\lambda^2 X \qquad \frac{\partial^2 Y}{\partial y^2} = \lambda^2 Y$$

③ Solve.

let $X(x) = e^{\alpha x}$

$$\alpha^2 e^{\alpha x} = -\lambda^2 e^{\alpha x}$$

$$e^{\alpha x}(\lambda^2 + \alpha^2) = 0$$

$$\alpha = \pm \lambda i$$

$$X(x) = c_1 e^{i\lambda x} + c_2 e^{-i\lambda x}$$

$$= c_3 \sin(\lambda x) + c_4 \cos(\lambda x)$$

$$Y(y) = e^{\alpha y}$$

$$\alpha^2 e^{\alpha y} - \lambda^2 Y = 0$$

$$e^{\alpha y}(\alpha^2 - \lambda^2) = 0$$

$$\alpha = \pm \lambda$$

$$Y(y) = c_5 e^{\lambda y} + c_6 e^{-\lambda y}$$

④ Apply BCs.

a) $T(0,y) = 0$

$$C_4 \cos(0) = 0$$
$$C_4 = 0$$

b) $T(x,0) = 0$

$$C_5 + C_6 = 0$$
$$C_5 = -C_6$$

c) $T(a,y) = 0$

$$C_3 \sin(\lambda a) = 0$$
$$\lambda = \frac{n\pi}{a}, \quad n = 1, 2, 3 \ldots$$

$$T(x,y) = X(x) \cdot Y(y)$$
$$= C \cdot \sin\left(\frac{n\pi x}{a}\right)\underbrace{\left(e^{-n\pi y/a} - e^{n\pi y/a}\right)}_{\sinh\left(\frac{n\pi y}{a}\right)}$$

$n = 1, 2, 3 \ldots$

$$T(x,y) = \sum_{n=1}^{\infty} C_n \sin(\lambda_n x) \sinh(\lambda_n y)$$

$$\boxed{= \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)}$$

$$a_n = C_n \cdot \sinh(\lambda_n y)$$

⑤ FOURIER TRANSFORM.

- what is a_n?

TABLE 4.1

RECALL

$$\frac{d^2X}{dx^2} + \lambda^2 X = 0 \qquad \begin{cases} X(0) = 0 \\ X(a) = 0 \end{cases}$$

has following char. values & func.

$$\left[\begin{array}{l} \phi_n(x) = \sin(\lambda_n x) \\ \lambda_n = \dfrac{n\pi}{a} \qquad n = 1, 2, 3 \ldots \end{array} \right.$$

$$\boxed{a_n = \frac{2}{a} \int_0^a f_1(x) \sin(\lambda_n x)\, dx}$$

$$c_n = \frac{2}{a} \frac{1}{\sinh(\lambda_n y)} \int_0^a f_1(x) \sin(\lambda_n x)\, dx$$

If B.C. $T(x, b) = T_0 = f_1(x)$

(constant

$$C_n = \frac{2}{\cancel{a} \sinh(\lambda_n b)} \underbrace{\int_0^a T_0 \sin\left(\frac{n\pi x}{a}\right) dx}_{\frac{T_0 \cdot \cancel{a}}{n\pi} \cdot \left[-\cos(\pi n) - (-\cos(0)) \right]}$$

$1 - \cos(n\pi)$

$n = 1, 2, 3 \ldots$

$\cos(n\pi) = (-1)^n$

$$c_n = \frac{2 \cdot T_0}{n\pi} \cdot \frac{1-(-1)^n}{\sinh(\lambda_n b)}$$

$$\boxed{T(x,y) = \left(\frac{2T_0}{\pi}\right) \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n} \cdot \frac{\sin\left(\frac{n\pi x}{a}\right) \sinh\left(\frac{n\pi y}{a}\right)}{\sinh\left(\frac{n\pi b}{a}\right)}}$$

at $x =$ Does it converge?

$y =$