MI628 - Inferência Causal

Lista 01.

01.

	$Y=1$	$Y=0$
$Z=1$	p_{11}	p_{10}
$Z=0$	p_{01}	p_{00}

$p_{zy} = P(Z=z, Y=y)$

$P(A \cap B) = P(A)P(B) \iff P(A|B) = \frac{P(A \cap B)}{P(B)} = P(A)$

a) $Z \perp\!\!\!\perp Y$, $RD=0$, $RR=1$ e $OR=1$ são afirmações equivalentes.

$Z \perp\!\!\!\perp Y \iff P(Y=y, Z=z) = P(Y=y) \cdot P(Z=z)$ e $P(Y=y|Z=z) = P(Y=y)$

$RD = P(Y=1|Z=1) - P(Y=1|Z=0)$

$Z \perp\!\!\!\perp Y \iff RD = 0$.

(1) $Z \perp\!\!\!\perp Y \Rightarrow RD = 0$

$RD = P(Y=1|Z=1) - P(Y=1|Z=0) \overset{Z \perp\!\!\!\perp Y}{=} P(Y=1) - P(Y=1) = 0$

(2) $RD = 0 \Rightarrow Z \perp\!\!\!\perp Y$

$RD = 0 \Rightarrow P(Y=1|Z=1) = P(Y=1|Z=0) \Rightarrow \frac{P(Y=1, Z=1)}{P(Z=1)} = \frac{P(Y=1, Z=0)}{P(Z=0)}$

Como $P(Z=1) = 1 - P(Z=0)$, a igualdade só se satisfaz se $P(Y=1, Z=1) = P(Y=1)P(Z=1)$ e $P(Y=1, Z=0) = P(Y=1)P(Z=0)$

$\Rightarrow Z \perp\!\!\!\perp Y$ ∎

$RD = 0 \iff RR = 1$

(1) $RD = 0 \Rightarrow RR = 1$: $RD = 0 \Rightarrow P(Y=1|Z=1) = P(Y=1|Z=0)$

$\Rightarrow RR = \frac{P(Y=1|Z=1)}{P(Y=1|Z=0)} = 1$

(2) $RR = 1 \Rightarrow RD = 0$: $RR = 1 \Rightarrow P(Y=1|Z=1) = P(Y=1|Z=0) \Rightarrow RD = 0$ ∎

$RR = 1 \iff OR = 1$

(1) $RR = 1 \Rightarrow OR = 1$

$RR = 1 \Rightarrow P(Y=1|Z=1) = P(Y=1|Z=0) \Rightarrow OR = \frac{P(Y=1|Z=1) / \overbrace{P(Y=0|Z=1)}^{1-P(Y=1|Z=1)}}{P(Y=1|Z=0) / \underbrace{P(Y=0|Z=0)}_{1-P(Y=0|Z=0)}} = 1$

(2) $OR = 1 \Rightarrow RR = 1$

$OR = 1 \Rightarrow P(Y=1|Z=1) = P(Y=1|Z=0) \Rightarrow RR = 1$ ∎

b) Se $p_{zy} > 0 \quad \forall z, y = 0,1 \Rightarrow RD > 0 \Leftrightarrow RR > 1 \Leftrightarrow OR > 1$

$$RD = P(Y=1|Z=1) - P(Y=1|Z=0) = \frac{p_{11}}{p_{11}+p_{10}} - \frac{p_{01}}{p_{01}+p_{00}} > 0$$

$$\Rightarrow \frac{p_{11}}{p_{11}+p_{10}} > \frac{p_{01}}{p_{01}+p_{00}}$$

$$RR = \frac{P(Y=1|Z=1)}{P(Y=1|Z=0)} = \frac{p_{11}}{p_{11}+p_{10}} \Big/ \frac{p_{01}}{p_{01}+p_{00}} > 1$$

$$\Rightarrow \frac{p_{11}}{p_{11}+p_{10}} > \frac{p_{01}}{p_{01}+p_{00}}$$

$$OR = \frac{P(Y=1|Z=1)/P(Y=0|Z=1)}{P(Y=1|Z=0)/P(Y=0|Z=0)} > 1 \Rightarrow \frac{P(Y=1|Z=1)}{P(Y=0|Z=1)} > \frac{P(Y=1|Z=0)}{P(Y=0|Z=0)}$$

$$\Rightarrow \frac{P(Y=1|Z=1)}{1-P(Y=1|Z=1)} > \frac{P(Y=1|Z=0)}{1-P(Y=1|Z=0)} \Rightarrow P(Y=1|Z=1) > P(Y=1|Z=0)$$

$$\Rightarrow \frac{p_{11}}{p_{11}+p_{10}} > \frac{p_{01}}{p_{01}+p_{00}}$$

c) $OR \approx RR$ se $P(Y=1|Z=1)$ e $P(Y=1|Z=0)$ são pequenos.

$P(Y=1|Z=1)$ e $P(Y=1|Z=0)$ pequenos $\Rightarrow OR \approx RR$

$$OR = \frac{P(Y=1|Z=1) / P(Y=0|Z=1)}{P(Y=1|Z=0) / P(Y=0|Z=0)}$$

$$RR = \frac{P(Y=1|Z=1)}{P(Y=1|Z=0)}$$

se $p \approx 0$

$$\underbrace{\frac{p}{1-p} = p + p^2 + \dots}_{\text{expansão Taylor}} \approx p$$

$$OR = \frac{P(Y=1|Z=1) / 1 - P(Y=1|Z=1)}{P(Y=1|Z=0) / 1 - P(Y=1|Z=0)}$$

se $P(Y=1|Z=1)$ pequeno, $1 - P(Y=1|Z=1) \to 1$

" $P(Y=1|Z=0)$ " $1 - P(Y=1|Z=0) \to 1$

$$\Rightarrow OR \approx \frac{P(Y=1|Z=1)}{P(Y=1|Z=0)} \approx RR$$

02.

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \overbrace{1}^{\Sigma_{22}} & \overbrace{\rho_{XY} \quad \rho_{XZ}}^{\Sigma_{21}} \\ \underbrace{\begin{matrix}\rho_{XY} \\ \rho_{XZ}\end{matrix}}_{\Sigma_{12}} & \underbrace{\begin{matrix} 1 & \rho_{YZ} \\ \rho_{YZ} & 1 \end{matrix}}_{\Sigma_{11}} \end{pmatrix} \right)$$

a) $$\rho_{YZ|X} = \frac{\rho_{YZ} - \rho_{XY}\rho_{ZX}}{\sqrt{1-\rho_{XY}^2}\sqrt{1-\rho_{ZX}^2}}$$

$X \sim N(\underset{\mu_2}{0}, \underset{\Sigma_{22}}{1})$ $\quad Y \sim N(0,1)$ $\quad Z \sim N(0,1)$

$$\begin{pmatrix} Y \\ Z \end{pmatrix} \sim N\left(\underbrace{\begin{pmatrix} 0 \\ 0 \end{pmatrix}}_{\mu_1}, \underbrace{\begin{pmatrix} 1 & \rho_{YZ} \\ \rho_{YZ} & 1 \end{pmatrix}}_{\Sigma_{11}} \right)$$ $\Rightarrow$ Capítulo A.1.2.

$$(Y,Z) | X = x \sim N\left(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(x - \mu_2), \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} \right)$$

$$\Sigma_1 = \begin{pmatrix} 1 & \rho_{YZ} \\ \rho_{YZ} & 1 \end{pmatrix} - \begin{pmatrix} \rho_{XY} \\ \rho_{XZ} \end{pmatrix} \cdot \begin{pmatrix} \rho_{XY} & \rho_{XZ} \end{pmatrix} = \begin{pmatrix} 1 & \rho_{YZ} \\ \rho_{YZ} & 1 \end{pmatrix} - \begin{pmatrix} \rho_{XY}^2 & \rho_{XY}\rho_{XZ} \\ \rho_{XZ}\rho_{XY} & \rho_{XZ}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1-\rho_{XY}^2 & \rho_{YZ}-\rho_{XY}\rho_{XZ} \\ \rho_{YZ}-\rho_{XY}\rho_{XZ} & 1-\rho_{XZ}^2 \end{pmatrix}$$

$$\rho_{YZ|X} = \frac{Cov(Y,Z|X=x)}{\sqrt{Var(Y|X=x)\,Var(Z|X=x)}}$$

$$= \frac{\rho_{YZ}-\rho_{XY}\rho_{XZ}}{\sqrt{(1-\rho_{XY}^2)(1-\rho_{XZ}^2)}} \blacksquare$$

b) Exemplo numérico em que $\rho_{YZ} > 0$ mas $\rho_{YZ|X} < 0$.

Seja $\rho_{XY} = 0.5$, $\rho_{XZ} = 0.80$, $\rho_{YZ} = 0.3$

$$\rho_{YZ|X} = \frac{0.3 - 0.5\,0.8}{\sqrt{(1-0.5^2)(1-0.8^2)}} = \frac{-0.1}{\sqrt{0.75\,0.36}} = \frac{-0.1}{\sqrt{0.27}} \approx -0.19.$$

04. $Y(0) \sim N(0,1)$ $\quad \tau = -0{,}5 + Y(0)$ $\quad Y(1) = Y(0) + \tau$

$Z = 1$ se $\tau \geq 0$

$Z = 0$ se $\tau < 0$

$Y = ZY(1) + (1-Z)Y(0)$ $\quad Y(1) = 2Y(0) - 0{,}5$

$Y(1) \sim N(-0{,}5\,;\,4)$

a) $E(Y|Z=1) - E(Y|Z=0)$

$Z=1$ se $\tau = -0{,}5 + Y(0) \geq 0 \Rightarrow Y(0) \geq 0{,}5 \Rightarrow Y(1) \geq 0{,}5$

$$E(Y|Z=1) = E(Y(1)\,|\,Y(1) \geq 0{,}5) = -0{,}5 - 2\,\frac{\overset{0}{\cancel{\phi(\infty)}} - \phi(1/2)}{\overset{1}{\cancel{\Phi(\infty)}} - \Phi(1/2)}$$

$$\cong -0{,}5 - 2 \cdot \frac{-0{,}35}{1-0{,}69} \cong -0{,}5 + 2{,}26 = 1{,}76$$

$Z=0$ se $\tau = -0{,}5 + Y(0) < 0 \Rightarrow Y(0) < 0{,}5$

$$E(Y|Z=0) = E(Y(0)\,|\,Y(0) < 0{,}5) = -\frac{\phi(0{,}5) - \overset{0}{\cancel{\phi(-\infty)}}}{\Phi(0{,}5) - \underset{0}{\cancel{\Phi(-\infty)}}} \cong -0{,}51$$

$$E(Y|Z=1) - E(Y|Z=0) = 1{,}76 + 0{,}51 = 2{,}27$$