Operations Research I: Models & Applications
Nonlinear Programming

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Road map

- **Introduction**.
- The EOQ model.
- Portfolio optimization.
- Linearizing maximum/minimum functions.
- Linearizing products of decision variables.

Example: pricing a single good

- A retailer buys one product at a unit cost c.
- It chooses a unit retail price p.
- The demand is a function of p: $D(p) = a - bp$.
- How to formulate the problem of finding the profit-maximizing price?
  - Parameters: $a > 0, b > 0, c > 0$.
  - Decision variable: p.
  - Constraint: $p \geq 0$.
  - Formulation:

$$\begin{aligned} \max_{p} \quad & (p-c)(a-bp) \\ \text{s.t.} \quad & p \geq 0 \end{aligned}$$

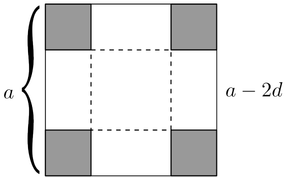
or

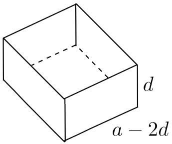
$$\max_{p \geq 0} (p-c)(a-bp).$$

Example: folding a piece of paper

- We are given a piece of square paper whose edge length is a.
- We want to cut down four small squares, each with edge length d, at the four corners.
- We then fold this paper to create a container.
- How to choose d to maximize the volume of the container?

$$\max_{d \in [0, \frac{a}{2}]} (a - 2d)^2 d.$$





Example: locating a hospital

- In a country, there are n cities, each lies at location (x_i, y_i).
- We want to locate a hospital at location (x, y) to minimize the average Euclidean distance from the cities to the hospital.

$$\min_{x,y} \sum_{i=1}^{n} \sqrt{(x - x_i)^2 + (y - y_i)^2}.$$

Nonlinear Programming

- In all the three examples, the programs are by nature **nonlinear**.
  - Because the trade off can only be modeled in a nonlinear way.
- In general, a **nonlinear program** (NLP) can be formulated as

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x) \\ \text{s.t.} \quad & g_i(x) \leq b_i \quad \forall i = 1, ..., m. \end{aligned}$$

  - $x \in \mathbb{R}^n$: there are n decision variables.
  - There are m constraints.
  - This is an LP if f and g_is are all linear in x.
  - This is an NLP if at least one of f and g_is is nonlinear in x.
- The study of formulating and optimizing NLPs is **Nonlinear Programming** (also abbreviated as NLP).
  - Formulation is easy but optimization is hard.

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Motivating example

- IM Airline uses 500 taillights per year. It purchases these taillights from a manufacturer at a unit price \$500.
- Taillights are consumed at a **constant rate** throughout a year.
- Whenever IM Airline places an order, an **ordering cost** of \$5 is incurred regardless of the order quantity.
- The **holding cost** is 2 cents per taillight per month.
- IM Airline wants to minimize the total cost, which is the sum of ordering, purchasing, and holding costs.
- How much to order? When to order?
  - What is the benefit of having a small or large order?

The EOQ model

- IM Airline's question may be answered with the economic order quantity (EOQ) model.
- We look for the order quantity that is the most economic.
  - We look for a **balance** between the ordering cost and holding cost.
- Technically, we will formulate an NLP whose optimal solution is the optimal order quantity.
- Assumptions for the (most basic) EOQ model:
  - Demand is deterministic and occurs at a constant rate.
  - Regardless the order quantity, a fixed ordering cost is incurred.
  - No shortage is allowed.
  - The ordering lead time is zero.
  - The inventory holding cost is constant.

Parameters and the decision variable

- Parameters:

$$\begin{aligned} D &= \text{annual demand (units)}, \\ K &= \text{unit ordering cost (\$)}, \\ h &= \text{unit holding cost per year (\$), and} \\ p &= \text{unit purchasing cost (\$)}. \end{aligned}$$

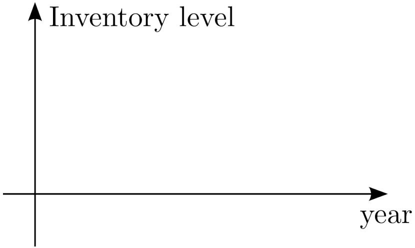
- Decision variable:

$$q = \text{order quantity per order (units)}.$$

- Objective: Minimizing annual total cost.
- For all our calculations, we will use **one year** as our time unit. Therefore, D can be treated as the demand **rate**.

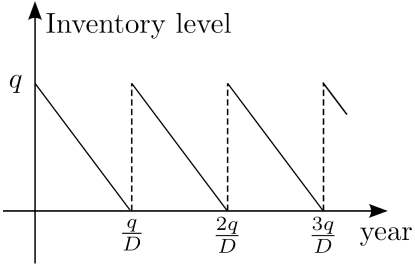
Inventory level

- To formulate the problem, we need to understand how the **inventory level** is affected by our decision.
  - The number of inventory we have on hand.
- Because there is no ordering lead time, we will always place an order when the inventory level is zero.
- As inventory is consumed at a constant rate, the inventory level will change by time like this:



Inventory level by time

- The same situation will **repeat** again and again:



- In average, how many units are stored?

Annual costs

- Annual holding cost $= h \times \frac{q}{2} = \frac{hq}{2}$.
  - For one year, the length of the time period is 1 and the inventory level is $\frac{q}{2}$ **in average**.
- Annual purchasing cost $= pD$.
  - We need to buy D units regardless the order quantity q.
- Annual ordering cost $= K \times \frac{D}{q} = \frac{KD}{q}$.
  - The number of orders in a year is $\frac{D}{q}$.
- The NLP for optimizing the ordering decision is

$$\min_{q \geq 0} \ \frac{KD}{q} + pD + \frac{hq}{2}.$$

- As pD is just a constant, a more relevant objective function is $TC(q) = \frac{KD}{q} + \frac{hq}{2}$.

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Motivating example

- We are going to invest \$100,000 in three stocks:

Stock	Current price	Expected future price
1	\$50	\$55
2	\$40	\$50
3	\$25	\$20

- How to allocate our budget?
  - What if we want to maximize our expected profit?

Maximize the expected profit

- If we want to maximize our expected profit, we may let x_i be the share of stock i purchased and formulate the following linear program

$$\begin{array}{rrcrcrcl} \max & 55x_1 & + & 50x_2 & + & 20x_3 & & \\ \text{s.t.} & 50x_1 & + & 40x_2 & + & 25x_3 & \leq & 100000 \\ & & & x_1 \geq 0, & & x_2 \geq 0, & x_3 & \geq 0. \end{array}$$

- The best strategy may be easily obtained:
  - We should purchase 2,500 shares of stock 2.
  - Our expected profit will be \$125,000.

Considering the risk

- Sometimes we consider not only expected profit but also **risk**.
- There are plenty of ways to measure risk.
- The Nobel Economics Prize Laureates in 1990, Markowitz and Sharpe, suggest:
  - The total revenue is random.
  - The larger the **variance** of the total revenue, the higher is the risk.
- We may **minimize the total variance** while **ensuring a certain expected revenue**.

Variance

- Let X be a random variable, μ be its expected value, and x_i be the ith possible realization, and $\Pr(X = x_i)$ be the probability for x_i to occur. The variance of X is

$$\text{Var}(X) = \sum_{i=1}^{n} \Pr(X = x_i)(x_i - \mu)^2.$$

- Let's assume the future price for stock 1 may be \$65 or \$45, each with the probability 50%.
  - If we buy one share, the variance is $\frac{1}{2}(65 - 55)^2 + \frac{1}{2}(45 - 55)^2 = 100$.
  - The variance of buying two shares is $\frac{1}{2}(130 - 110)^2 + \frac{1}{2}(90 - 110)^2 = 400$.
  - The variance of buying x_1 shares is $100x_1^2$.
- In general, $\text{Var}(bX) = b^2\text{Var}(X)$ for all $b > 0$.

Minimizing the risk

- For our example, let the variances of buying one share or stocks 1, 2, and 3 be 100, 1600, and 100, respectively.
- Accordingly, when we buy x_i shares of stock i, the variance of the total revenue is

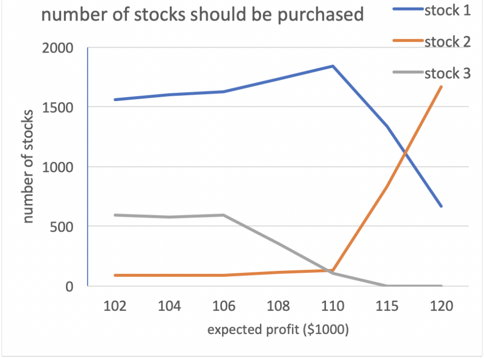
$$100x_1^2 + 1600x_2^2 + 100x_3^2.$$

- If the minimum required expected revenue is R, we may formulate the **nonlinear program**

$$\begin{array}{rrcrcrcl}
\min & 100x_1^2 & + & 1600x_2^2 & + & 100x_3^2 & & \\
\text{s.t.} & 50x_1 & + & 40x_2 & + & 25x_3 & \leq & 100000 \\
 & 55x_1 & + & 50x_2 & + & 20x_3 & \geq & R \\
 & & & x_i \geq 0 & & \forall i = 1, ..., 3. & &
\end{array}$$

Managerial implications

- Given different values of R, we may get **different optimal portfolios**.
  - Buying stock 1 is a must.
  - Sometimes even buying stock 3 is necessary.



Managerial implications

- Given different values of R, we can get different optimal portfolios.
  - The higher expected revenue we want, the higher the risk is.

Compact formulation

- We invest B in n stocks. The minimum required expected revenue is R.
- For stock i, the current price is p_i, expected future price is μ_i, and variance of buying one share is σ_i^2.
- Let x_i be the shares of stock i we buy.
- The compact formulation is

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} \sigma_i^2 x_i^2 \\ \text{s.t.} \quad & \sum_{i=1}^{n} p_i x_i \leq B \\ & \sum_{i=1}^{n} u_i x_i \geq R \\ & x_i \geq 0 \quad \forall i = 1, \cdots, n. \end{aligned}$$

Correlation among stock prices

- The price among stocks are typically correlated.
- Let σ_{ij} be the **covariance** between stocks i and j.
- The extended formulation is

$$\begin{aligned}
\min \quad & \sum_{i=1}^{n} \sigma_i^2 x_i^2 + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{n} \sigma_{ij} x_i x_j \\
\text{s.t.} \quad & \sum_{i=1}^{n} p_i x_i \leq B \\
& \sum_{i=1}^{n} u_i x_i \geq R \\
& x_i \geq 0 \quad \forall i = 1, ..., n.
\end{aligned}$$

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Fair allocation: the problem

- Suppose that we want to allocate \$1000 to two persons in a **fair** way.
- We adopt the following measurement of fairness: The smaller the difference between the two amounts, the fairer the allocation is.
- Obviously the answer is to give each person \$500.
- May we formulate a linear program to solve this problem?

Fair allocation: the first attempt

- Let x_i be the amount allocated to person i, $i = 1, 2$.
- Is the following formulation correct?

$$\begin{aligned} \min \quad & x_2 - x_1 \\ \text{s.t.} \quad & x_1 + x_2 = 1000 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

Fair allocation: the second attempt

- Let x_i be the amount allocated to person i, $i = 1, 2$.
- The following formulation is correct:

$$\begin{aligned} \min \quad & |x_2 - x_1| \\ \text{s.t.} \quad & x_1 + x_2 = 1000 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- However, the absolute function $|\cdot|$ is **nonlinear**!
- Is it possible to linearize this problem as a linear program?

Linearizing the second attempt

- First, let w be the absolute difference: $w = |x_2 - x_1|$:

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w = |x_2 - x_1| \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- We may change this equality constraint to an inequality:

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w \geq |x_2 - x_1| \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

Why?

Linearizing the second attempt

- Now, notice that $|x_2 - x_1| = \max\{x_2 - x_1, x_1 - x_2\}$ and

$$w \geq \max\{x_2 - x_1, x_1 - x_2\} \quad \Leftrightarrow \quad w \geq x_2 - x_1 \text{ and } w \geq x_1 - x_2.$$

- Therefore, the linear program we want is

$$\begin{array}{rl} \min & w \\ \text{s.t.} & x_1 + x_2 = 1000 \\ & w \geq x_2 - x_1 \\ & w \geq x_1 - x_2 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{array}$$

- May we solve this LP and get the $(500, 500)$ allocation?

Solving the linear program

- Consider the LP

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & x_1 + x_2 = 1000 \\ & w \geq x_2 - x_1 \\ & w \geq x_1 - x_2 \\ & x_i \geq 0 \quad \forall i = 1, 2. \end{aligned}$$

- The equality constraint means that $x_2 = 1000 - x_1$:

$$\begin{aligned} \min \quad & w \\ \text{s.t.} \quad & w \geq 1000 - 2x_1 \\ & w \geq 2x_1 - 1000 \\ & x_1 \geq 0. \end{aligned}$$

- Would you graphically solve the LP?

Linearizing constraints

- The technique we just applied can be generalized.
- When a **maximum** function is at the **smaller** side of an inequality:

$$y \geq \max\{x_1, x_2\} \quad \Leftrightarrow \quad y \geq x_1 \text{ and } y \geq x_2.$$

- y, x_1, and x_2 can be variables, parameters, or a function of them:

$$\begin{aligned} & y + x_1 + 3 \geq \max\{x_1 - x_3, 2x_2 + 4\} \\ \Leftrightarrow \quad & y + x_1 + 3 \geq x_1 - x_3 \text{ and } y + x_1 + 3 \geq 2x_2 + 4. \end{aligned}$$

- There may be more than two terms in the maximum function:

$$y \geq \max_{i=1,\ldots,n} \{x_i\} \quad \Leftrightarrow \quad y \geq x_i \quad \forall i = 1, ..., n.$$

Linearizing constraints

- A **minimum** function at the **larger** side can also be linearized.

$$
\begin{aligned}
& y + x_1 \leq \min\{x_1 - x_3, 2x_2 + 4, 0\} \\
\Leftrightarrow \quad & y + x_1 \leq x_1 - x_3,\ y + x_1 \leq 2x_2 + 4,\ \text{and } y + x_1 \leq 0.
\end{aligned}
$$

- This technique **does not** apply to:
  - A maximum function at the larger side: $y \leq \max\{x_1, x_2\}$ is not equivalent to $y \leq x_1$ and $y \leq x_2$.
  - A minimum function at the smaller side: $y \geq \min\{x_1, x_2\}$ is not equivalent to $y \geq x_1$ and $y \geq x_2$.
  - A maximum or minimum function in an equality.

Linearizing the objective function

- When we **minimize a maximum function**:

$$\min\ \max\{x_1, x_2\} \quad \Leftrightarrow \quad \begin{array}{rl} \min & w \\ \text{s.t.} & w \geq x_1 \\ & w \geq x_2. \end{array}$$

  - x_1 and x_2 can be variables, parameters, or a function of them.
  - There may be other constraints.
  - The objective function may contain other terms.
- Similarly, when we **maximize a minimum function**:

$$\begin{array}{rl} \max & \min\{x_1, x_2, 2x_3 + 5\} + x_4 \\ \text{s.t.} & 2x_1 + x_2 - x_4 \leq x_3. \end{array} \quad \Leftrightarrow \quad \begin{array}{rl} \max & w + x_4 \\ \text{s.t.} & w \leq x_1 \\ & w \leq x_2 \\ & w \leq 2x_3 + 5 \\ & 2x_1 + x_2 - x_4 \leq x_3. \end{array}$$

Linearizing the objective function

- This technique does not apply to:
  - Maximizing a maximum function.
  - Minimizing a minimum function.
- Finally, an **absolute function** is just a maximum function:

$$|x| = \max\{x, -x\}.$$

  - Minimizing an absolute function can be linearized.
  - An absolute function at the smaller side of an inequality can be linearized.

Example: hospital location revisited

- In a country, there are n cities, each lies at location (x_i, y_i).
- We want to locate a hospital at location (x, y) to minimize the average **Manhattan distance** from the cities to the hospital.

$$\min_{x,y} \sum_{i=1}^{n} \Big(|x - x_i| + |y - y_i| \Big).$$

- This may be linearized to

$$\begin{aligned} \min \quad & \sum_{i=1}^{n} (u_i + v_i) \\ \text{s.t.} \quad & u_i \geq x - x_i, u_i \geq x_i - x \quad \forall i = 1, ..., n \\ & v_i \geq y - y_i, v_i \geq y_i - y \quad \forall i = 1, ..., n \end{aligned}$$

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Linearizing products of decision variables

- In many cases, we see **products** of decision variables.
- This can be linearized if the two variables are:
  - A binary one and a continuous variable.
  - Two binary ones.
- This cannot be linearized if the variables are both continuous.
- Let's use examples to see how to do this.

Scenario 1A

- A company makes and sells two products with two limited resources.
- Making each product requires a setup cost.
- Making both products results in some **reduction** on the setup cost.
- The formulation:

$$
\begin{aligned}
\max \quad & 10x_1 + 12x_2 - 20z_1 - 25z_2 + 10z_1z_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 \leq 3z_1 \\
& x_2 \leq 4z_2 \\
& x_1, x_2 \geq 0 \\
& z_1, z_2 \in \{0, 1\}.
\end{aligned}
$$

- May we linearize z_1z_2?

Linearization for Scenario 1A

- Let's introduce $w = z_1 z_2$.
- As w appears in a maximization objective function, it suffices to introduce $w \leq z_1$ and $w \leq z_2$ to make $w = z_1 z_2$.
- In the formulation:

$$
\begin{array}{rl}
\max & 10x_1 + 12x_2 - 20z_1 \\
 & -25z_2 + 10z_1 z_2 \\
\text{s.t.} & ...
\end{array}
\qquad
\begin{array}{rl}
\max & 10x_1 + 12x_2 - 20z_1 \\
 & -25z_2 + 10w \\
\text{s.t.} & ... \\
 & w \leq z_1 \\
 & w \leq z_2 \\
 & w \in \{0, 1\}.
\end{array}
$$

Scenario 1B

- A company makes and sells two products with two limited resources.
- Making each product requires a setup cost.
- Making both products results in some **additional** setup cost.
- The formulation:

$$
\begin{aligned}
\max \quad & 10x_1 + 12x_2 - 20z_1 - 25z_2 - 10z_1z_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 \leq 3z_1 \\
& x_2 \leq 4z_2 \\
& x_1, x_2 \geq 0 \\
& z_1, z_2 \in \{0, 1\}.
\end{aligned}
$$

- May we linearize z_1z_2?

Linearization for Scenario 1B

- Let's still introduce $w = z_1 z_2$.
- As w now appears in a "minimization" objective function, $w \leq z_1$ and $w \leq z_2$ does not make $w = z_1 z_2$.
- Instead, let's use $w \geq z_1 + z_2 - 1$.
- In the formulation:

$$\begin{aligned} \max \quad & 10x_1 + 12x_2 - 20z_1 \\ & -25z_2 - 10z_1z_2 \\ \text{s.t.} \quad & ... \end{aligned}$$

$$\begin{aligned} \max \quad & 10x_1 + 12x_2 - 20z_1 \\ & -25z_2 - 10w \\ \text{s.t.} \quad & ... \\ & w \geq z_1 + z_2 - 1 \\ & w \in \{0, 1\}. \end{aligned}$$

Scenario 1C with linearization

- What if a product term appears in a **constraint**?
- It matters whether it appears at the "larger" or "smaller" side.
- If it is at the "**larger**" side, it may be linearized as if it appears in a maximization objective function (because the product term should be 1 only if both terms are 1).
- For example:

$$\begin{array}{rl} \max & ... \\ \text{s.t.} & x \leq 5 z_1 z_2 \\ & x \geq 0 \\ & z_1, z_2 \in \{0, 1\}. \end{array} \qquad \begin{array}{rl} \max & ... \\ \text{s.t.} & x \leq 5w \\ & x \geq 0 \\ & z_1, z_2 \in \{0, 1\} \\ & w \leq z_1, w \leq z_2 \\ & w \in \{0, 1\}. \end{array}$$

Scenario 1D with linearization

- If the product term is at the "**smaller**" side, it may be linearized as if it appears in a minimization objective function (because the product term cannot be 1 if either term is 1).
- For example:

$$\begin{array}{rl} \max & ... \\ \text{s.t.} & x \geq 5 z_1 z_2 \\ & x \geq 0 \\ & z_1, z_2 \in \{0, 1\}. \end{array} \qquad \begin{array}{rl} \max & ... \\ \text{s.t.} & x \geq 5w \\ & x \geq 0 \\ & z_1, z_2 \in \{0, 1\} \\ & w \geq z_1 + z_2 - 1 \\ & w \in \{0, 1\}. \end{array}$$

Scenario 2A

- A company makes and sells two products with two limited resources.
- **Doing the business** requires a **fixed payment** to the local government. If the payment is not made, the products cannot be sold regardless of the production quantity.
- The formulation:

$$
\begin{array}{rl}
\max & (10x_1 + 12x_2)z - 15z \\
\text{s.t.} & 2x_1 + x_2 \leq 6 \\
& x_1 + 2x_2 \leq 8 \\
& x_1, x_2 \geq 0 \\
& z \in \{0, 1\}.
\end{array}
$$

- May we linearize $x_1 z$ and $x_2 z$?

Linearization for Scenario 2A

- Hopefully we may make $w_1 = x_1 z$ and $w_2 = x_2 z$.
- For w_1, however, we cannot simply impose $w_1 \leq x_1$ and $w_1 \leq z$.
  - The latter is too tight!
  - We should "**remove**" the constraint when $z = 1$. In other words, the RHS should contain a value that is an **upper bound** of x_1.
  - In this example, 3 works (why?).
- In the formulation:

$$
\begin{array}{rl}
\max & (10x_1 + 12x_2)z - 15z \\
\text{s.t.} & ...
\end{array}
\qquad
\begin{array}{rl}
\max & 10w_1 + 12w_2 - 15z \\
\text{s.t.} & ... \\
& w_1 \leq x_1, w_1 \leq 3z \\
& w_2 \leq x_2, w_2 \leq 4z \\
& z \in \{0, 1\}.
\end{array}
$$

Scenario 2B

- A company may run two production processes to fulfill the demands for two products if it **accepts an order**.
- The formulation:

$$\begin{aligned} \max \quad & 50z - (10x_1 + 12x_2)z \\ \text{s.t.} \quad & 2x_1 + x_2 \geq 6 \\ & x_1 + 2x_2 \geq 8 \\ & x_1, x_2 \geq 0 \\ & z \in \{0, 1\}. \end{aligned}$$

- May we linearize $x_1 z$ and $x_2 z$?

Linearization for Scenario 2B

- Let's still introduce $w_1 = x_1 z$ and $w_2 = x_2 z$.
- As w now appears in a "minimization" objective function, w should be lower bounded rather than upper bounded.
- Let's use $w_1 \geq x_1 - 8(1-z)$ and $w_2 \geq x_2 - 6(1-z)$ and
- In the formulation:

$$\begin{array}{rl} \max & 50z - (10x_1 + 12x_2)z \\ \text{s.t.} & 2x_1 + x_2 \geq 6 \\ & x_1 + 2x_2 \geq 8 \\ & x_1, x_2 \geq 0 \\ & z \in \{0, 1\}. \end{array} \qquad \begin{array}{rl} \max & 50z - 10w_1 - 12w_2 \\ \text{s.t.} & \ldots \\ & w_1 \geq x_1 - 8(1-z) \\ & w_2 \geq x_2 - 6(1-z) \\ & w_1, w_2 \geq 0. \end{array}$$

Scenario 2C with linearization

- When a product term appears at the "larger" side of a **constraint**, it may be linearized as if it appears in a maximization objective function (should be upper bounded).
- For example:

$$
\begin{aligned}
\max \quad & \ldots \\
\text{s.t.} \quad & x_1 z \geq 5x_2 \\
& x_1 + x_2 \leq 10 \\
& x_1, x_2 \geq 0 \\
& z \in \{0, 1\}.
\end{aligned}
\qquad\qquad
\begin{aligned}
\max \quad & \ldots \\
\text{s.t.} \quad & w \geq 5x_2 \\
& x_1 + x_2 \leq 10 \\
& x_1, x_2 \geq 0 \\
& z \in \{0, 1\} \\
& w \leq x_1, w \leq 10z \\
& w \geq 0.
\end{aligned}
$$

Scenario 2D with linearization

- When a product term appears at the "smaller" side of a **constraint**, it may be linearized as if it appears in a minimization objective function (should be lower bounded).
- For example:

$$\begin{aligned} \max \quad & ... \\ \text{s.t.} \quad & x_1 z \leq 5x_2 \\ & x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \\ & z \in \{0, 1\}. \end{aligned}$$

$$\begin{aligned} \max \quad & ... \\ \text{s.t.} \quad & w \leq 5x_2 \\ & x_2 \leq 10 \\ & x_1, x_2 \geq 0 \\ & z \in \{0, 1\} \\ & w \geq x_1 - 10(1 - z) \\ & w \geq 0. \end{aligned}$$

Concluding remarks

- Why linearization?
- Roughly speaking, to **solve** a mathematical program:
  - Solving a linear program is easy.
  - Solving a linear integer program is doable.
  - Solving a nonlinear program can be hard.
  - Solving a nonlinear integer program is typically very hard.
- Ways to solve mathematical programs (and the difficulty for solving each type of programs) will be introduced in other courses/modules.
- In general, when facing an optimization problem, we should try to formulate a program that "can be (easily) solved."
  - That is why linearization is important.
  - To apply Operations Research in practice, being able to estimate the **solvability** of the formulation is also important.