Physics-Informed Neural Networks (PINNs)

Krishna Kumar

University of Texas at Austin

krishnak@utexas.edu

Overview

Outline

Learning Objectives

- Understand why standard neural networks fail for physics problems
- Learn how to incorporate physics into neural network training
- Master automatic differentiation for computing derivatives
- Compare data-driven vs physics-informed approaches

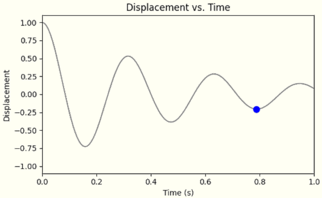
Open Notebook: PINN

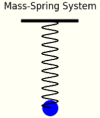
The Problem: A Damped Harmonic Oscillator

A mass m on a spring (constant k) with damping (coefficient c). The displacement $u(t)$ satisfies:

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = 0$$

with initial conditions: $u(0) = 1$, $\frac{du}{dt}(0) = 0$.





A classic physics problem to illustrate PINNs.

The Challenge: Reconstruct the full solution $u(t)$ from a few sparse, noisy data points.

Stage 1: The Data-Only Approach

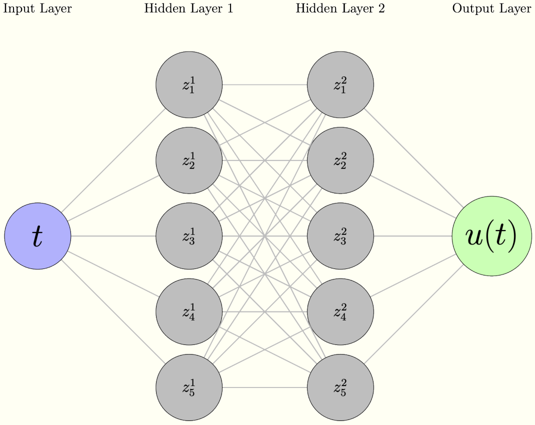
Idea: Train a standard neural network to fit the sparse data.

Loss Function: Mean Squared Error

$$\mathcal{L}_{\text{data}}(\theta) = \frac{1}{N}\sum_{i=1}^{N} |\hat{u}_\theta(t_i) - u_i|^2$$

Architecture:

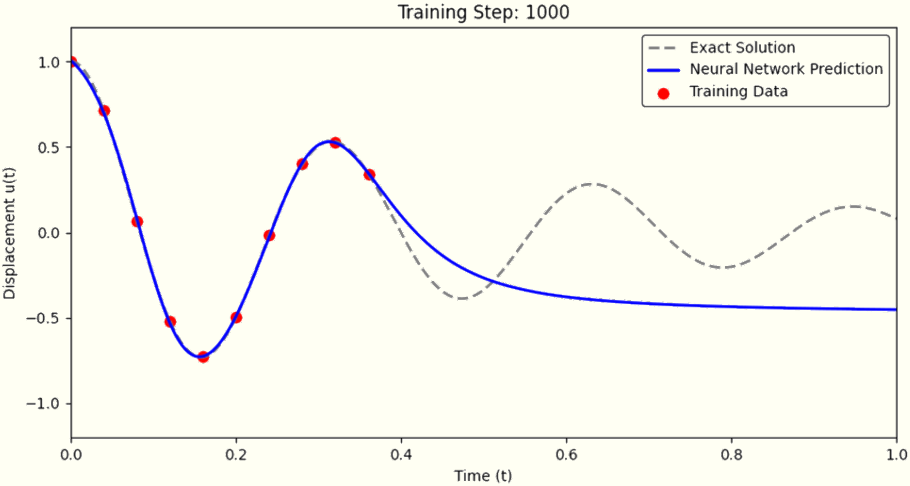
- Input: Time t
- Hidden Layers: Tanh activations
- Output: Displacement $\hat{u}_\theta(t)$



Standard NN for function fitting.

The Failure of the Data-Only Approach

Result: The network fits the training points but fails catastrophically between them.



The network overfits to the sparse data and does not respect the underlying physics.

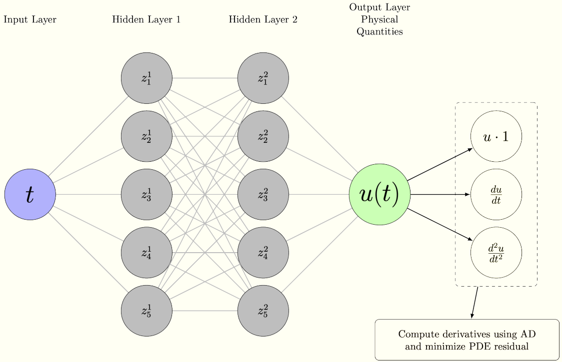
Stage 2: Enter Physics-Informed Neural Networks

The Key Insight: Don't just fit data. Enforce the differential equation itself!

Physics Residual: We define a residual based on the ODE:

$$\mathcal{R}_\theta(t) = m\frac{d^2\hat{u}_\theta}{dt^2} + c\frac{d\hat{u}_\theta}{dt} + k\hat{u}_\theta$$

If the solution is correct, $\mathcal{R}_\theta(t)$ should be zero.



PINN architecture with physics loss.

The Complete PINN Loss Function

The total loss is a combination of data fit and physics enforcement.

$$\mathcal{L}_{\text{total}}(\theta) = \mathcal{L}_{\text{data}}(\theta) + \lambda \mathcal{L}_{\text{physics}}(\theta)$$

Data Loss

Ensures the solution passes through the measurements.

$$\mathcal{L}_{\text{data}}(\theta) = \frac{1}{N_{\text{data}}} \sum_{i=1}^{N_{\text{data}}} |\hat{u}_\theta(t_i) - u_i|^2$$

Physics Loss

Ensures the solution obeys the ODE at random "collocation" points.

$$\mathcal{L}_{\text{physics}}(\theta) = \frac{1}{N_{\text{colloc}}} \sum_{j=1}^{N_{\text{colloc}}} |\mathcal{R}_\theta(t_j)|^2$$

The Secret Weapon: Automatic Differentiation (AD)

Critical Question: How do we compute $\frac{d\hat{u}_\theta}{dt}$ and $\frac{d^2\hat{u}_\theta}{dt^2}$?

Answer: Automatic Differentiation

AD provides **exact** derivatives of the neural network output with respect to its input, by applying the chain rule through the computational graph.

- No finite difference errors.
- Computationally efficient (especially reverse-mode AD).
- Built into modern frameworks (PyTorch, TensorFlow, JAX).

Example: For $u(t) = \sin(t)$, AD can compute $u'(t) = \cos(t)$ and $u''(t) = -\sin(t)$ to machine precision.

Theoretical Foundation: UAT for Sobolev Spaces

Classical UAT: NNs can approximate any *continuous function*.
Problem: For PDEs, we need to approximate functions **and their derivatives**.

Extended Universal Approximation Theorem

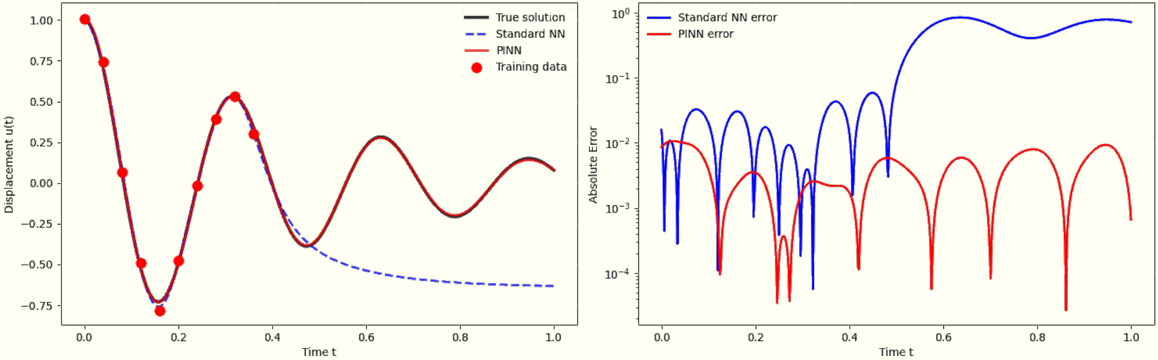
Neural networks with sufficiently smooth activation functions (e.g., tanh, not ReLU) can approximate functions in **Sobolev spaces** $H^k(\Omega)$.

$$\|u - \hat{u}_\theta\|_{H^k} < \epsilon$$

The Sobolev norm $\|u\|_{H^k}^2 = \sum_{|\alpha| \leq k} \|D^\alpha u\|_{L^2}^2$ measures the error in the function and all its derivatives up to order k.

Why this matters: For a k^{th}-order ODE/PDE, we need an activation function that is at least k times differentiable (C^k). For our 2nd-order oscillator, we need a C^2 activation like tanh.

The Moment of Truth: Standard NN vs. PINN

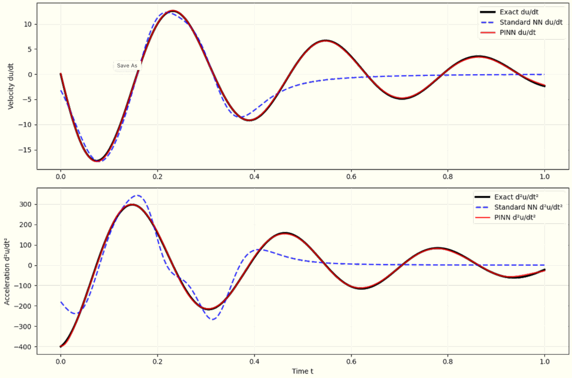


Placeholder for the comparison plot from the notebook.

Observation:

- **Standard NN:** Fits data points, but fails to generalize.
- **PINN:** Fits data points AND follows the physics, resulting in a globally accurate solution.

The ultimate test: Does the PINN learn physically consistent derivatives?

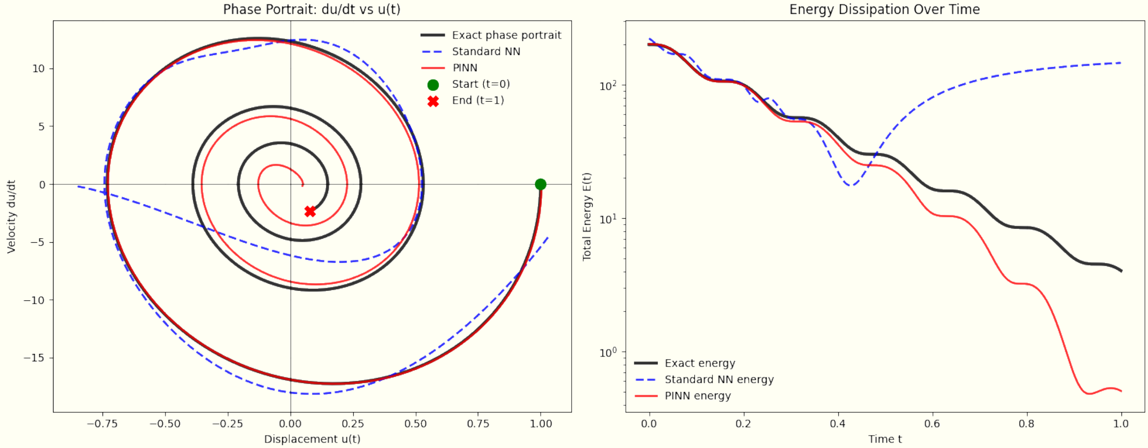


Derivative plots.

Result: The PINN learns the correct velocity (du/dt) and acceleration (d^2u/dt^2).

Deep Dive: Derivative and Phase Portrait Analysis

The ultimate test: Does the PINN learn physically consistent derivatives?



Phase portrait plots.

Outline

The 1D Poisson Problem

We now tackle a boundary value problem, the 1D Poisson equation:

$$\frac{d^2u}{dx^2} + \pi \sin(\pi x) = 0, \quad \text{for } x \in [0, 1]$$

with Dirichlet boundary conditions (BCs):

$$u(0) = 0 \quad \text{and} \quad u(1) = 0$$

Goal: Train a PINN to find the solution using only the governing equation and its BCs.

▸ Open Notebook: 1D Poisson

Method 1: Soft Constraints

Treat the boundary conditions as another component of the loss function.

Total Loss:

$$\mathcal{L}_{\text{total}} = \mathcal{L}_{\text{PDE}} + \lambda_{BC}\mathcal{L}_{BC}$$

Boundary Loss

A mean squared error term that penalizes violations of the BCs.

$$\mathcal{L}_{BC} = \frac{1}{N_{BC}} \sum_{i=1}^{N_{BC}} \left| \hat{u}_\theta(x_i) - u_{BC} \right|^2$$

Pros & Cons

+ **Flexible:** Easy to implement for any type of BC (Dirichlet, Neumann, etc.).
- **Approximate:** Satisfaction is not guaranteed, only encouraged.
- **Tuning:** Requires careful tuning of the weight λ_{BC}.

Method 2: Hard Constraints

Modify the network architecture to satisfy the BCs *by construction*.
For our problem with $u(0) = 0$ and $u(1) = 0$, we can define a trial solution $\tilde{u}(x)$:

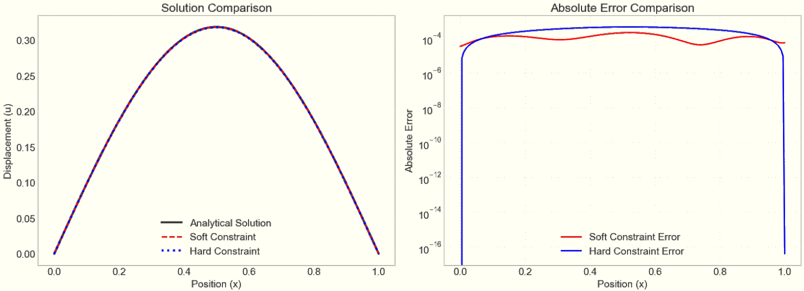
$$\tilde{u}(x) = \underbrace{x(1-x)}_{D(x)} \cdot \underbrace{\mathrm{NN}(x;\theta)}_{\text{Network Output}}$$

The distance function $D(x)$ is zero at the boundaries, forcing $\tilde{u}(x)$ to be zero there, regardless of the network's output.

Pros & Cons

+ **Exact:** BCs are satisfied perfectly.
+ **Simpler Loss:** No need for $\mathcal{L}_{BC}$ or λ_{BC}, leading to more stable training.
- **Inflexible:** Requires designing a specific trial function for the problem's geometry and BCs, which can be difficult for complex cases.

Comparison: Soft vs. Hard Constraints



Conclusion:

- Both methods achieve high accuracy.
- The hard constraint method shows slightly lower error and, by design, has zero error at the boundaries.
- For simple geometries and Dirichlet BCs, **hard constraints are often superior**.
- For complex problems, **soft constraints offer greater versatility**.

Outline

Forward vs. Inverse Problems

Forward Problem

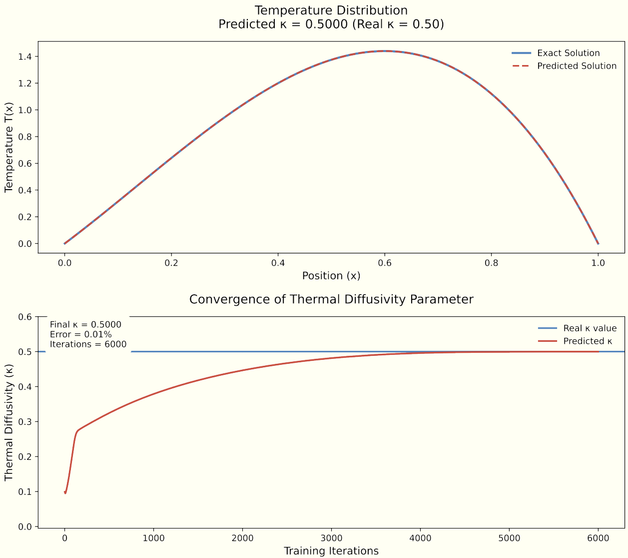
- **Given:** Full physical model (equations + parameters).
- **Find:** The solution $u(x, t)$.
- *Example: Simulate temperature given thermal conductivity.*

Inverse Problem

- **Given:** Sparse measurements of the solution $u(x, t)$.
- **Find:** Unknown physical parameters in the model.
- *Example: Infer thermal conductivity from temperature measurements.*

▸ Open Notebook: Inverse Heat

Forward vs. Inverse Problems



The PINN Approach to Inverse Problems

The Key Insight: Treat the unknown physical parameters as additional trainable variables in the network.

Problem: 1D steady-state heat conduction.

$$-k\frac{d^2 T}{dx^2} = f(x)$$

Here, the thermal diffusivity k is **unknown**.

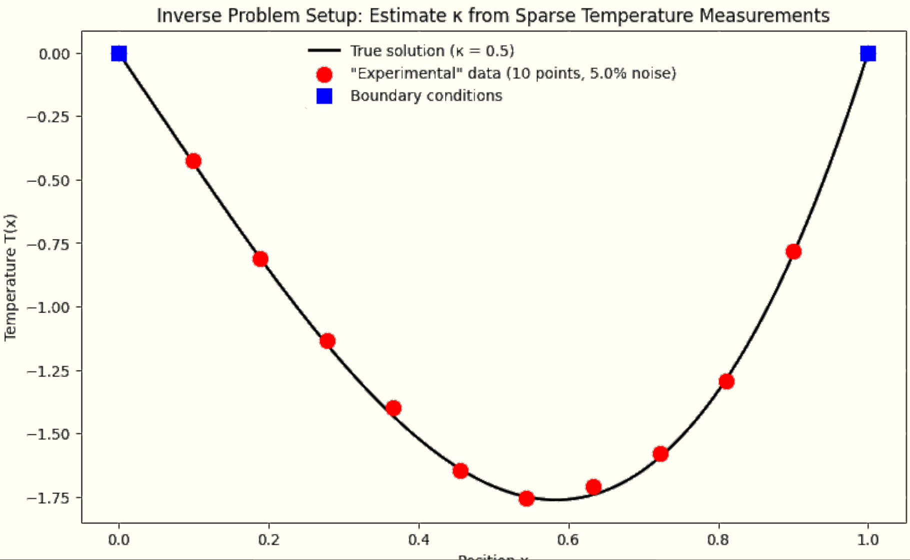
PINN Framework:

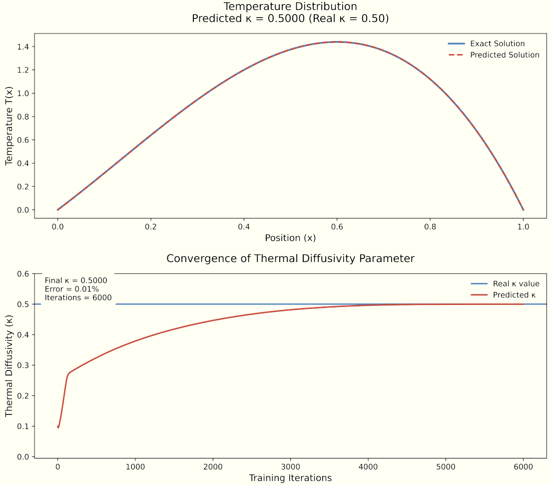
- The neural network learns the temperature field: $\hat{T}_\theta(x)$.
- A new trainable parameter is introduced: $\hat{k}$.
- The optimizer updates both the network weights θ and the parameter $\hat{k}$ simultaneously.

Loss Function: $\mathcal{L}(\theta, \hat{k}) = \mathcal{L}_{\text{data}} + \mathcal{L}_{\text{PDE}} + \mathcal{L}_{\text{BC}}$ where the physics loss now includes the trainable parameter $\hat{k}$:

$$\mathcal{L}_{\text{PDE}} = \frac{1}{N_f}\sum\left|-\hat{k}\frac{d^2\hat{T}_\theta}{dx^2}(x_j) - f(x_j)\right|^2$$

Implementation: Parameter Estimation





Parameter convergence and temperature field

- The estimated parameter $\hat{k}$ converges to the true value.
- The PINN simultaneously reconstructs the full, continuous temperature field accurately.
- This is achieved from very sparse and noisy data, showcasing the regularizing effect of the physics loss.

Summary: Why PINNs are Powerful

1. **Regularization Effect:**
   - Physics constraints prevent overfitting and guide the solution in data-sparse regions.
2. **Data Efficiency:**
   - Physics provides a strong inductive bias, allowing PINNs to learn from very few measurements.
3. **Accurate Derivatives:**
   - Automatic differentiation provides exact derivatives, which are learned correctly as a consequence of enforcing the physics.
4. **Versatility:**
   - The same framework can solve forward problems, inverse problems, and handle various boundary conditions.

PINNs = Universal Function Approx. + Physics Constraints + Auto. Diff.

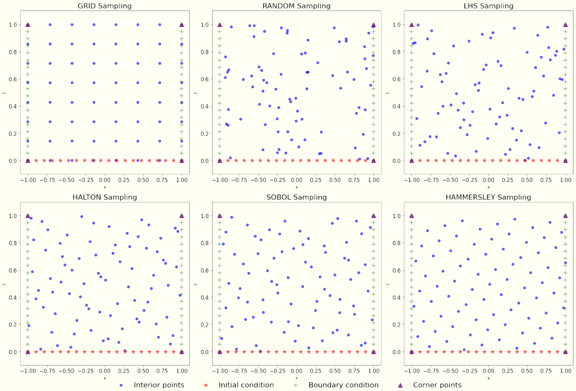
Outline

Collocation Points: Where to Enforce Physics

What are collocation points?

- Points where we evaluate PDE residual
- Do not need measurement data
- Distributed throughout domain
- More points $\rightarrow$ better physics enforcement

Key question: How should we distribute these points for optimal performance?



Different collocation sampling strategies

Collocation Sampling Strategies

Uniform Grid:

- Simple to implement
- Good coverage
- Curse of dimensionality

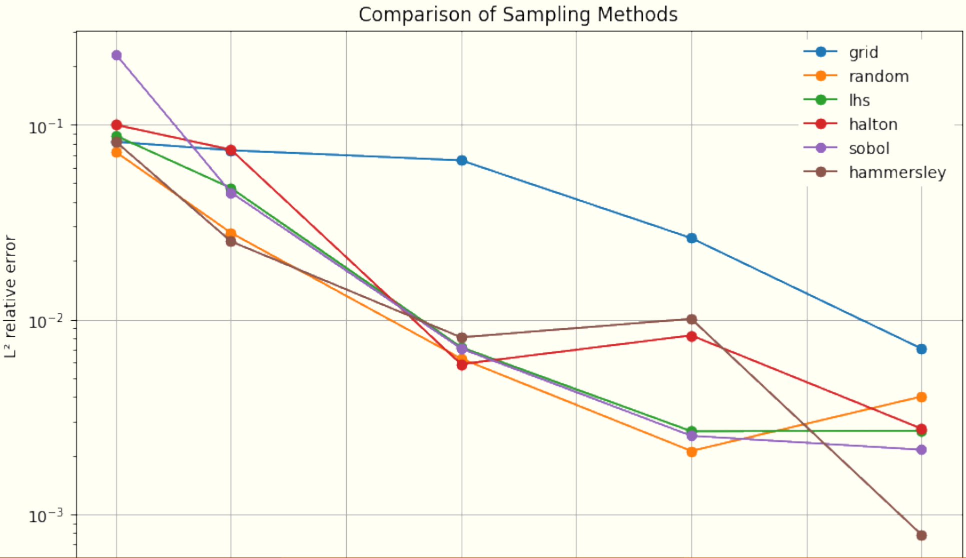
Random (Monte Carlo):

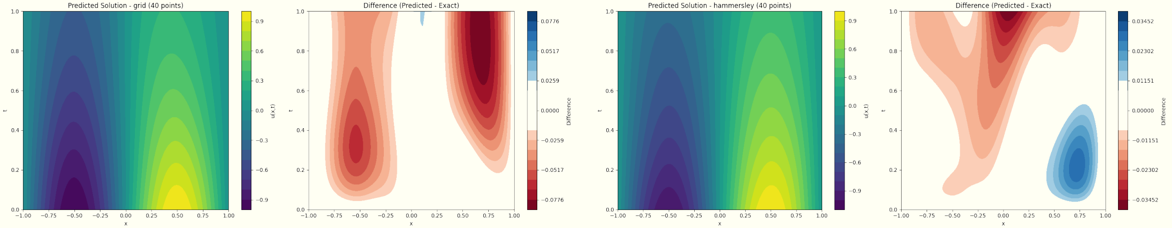
- Dimension-independent
- May cluster/leave gaps
- Easy to add points

Quasi-Random:

- Better coverage than random
- Low discrepancy sequences
- Hammersley, Sobol, Halton

Collocation Sampling Strategies





Error with uniform grid sampling

Error with Hammersley sampling

Observations:

- Quasi-random sampling often outperforms uniform/random
- Better space-filling properties lead to lower errors
- Particularly important in higher dimensions

Outline

The Loss Balancing Challenge

The Problem

Different loss terms can have vastly different magnitudes and gradients

Total PINN loss:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + \lambda_{\text{PDE}}\mathcal{L}_{\text{PDE}} + \lambda_{\text{BC}}\mathcal{L}_{\text{BC}} + \lambda_{\text{IC}}\mathcal{L}_{\text{IC}}$$

Challenges:

- Data loss: Often $O(10^{-3})$ to $O(10^{-1})$
- PDE residual: Can be $O(10^{2})$ to $O(10^{4})$ initially
- Boundary conditions: Variable scale
- Poor balance $\rightarrow$ training instability or failure

Question: How to choose λ values optimally?

Gradient-Based Adaptive Weighting

Key idea: Balance the gradients of different loss terms

Algorithm (Wang et al., 2021):

1. Compute gradient statistics:

$$\bar{g}_i = \frac{1}{|\theta|} \sum_{\theta} |\nabla_\theta \mathcal{L}_i|$$

2. Update weights to balance gradients:

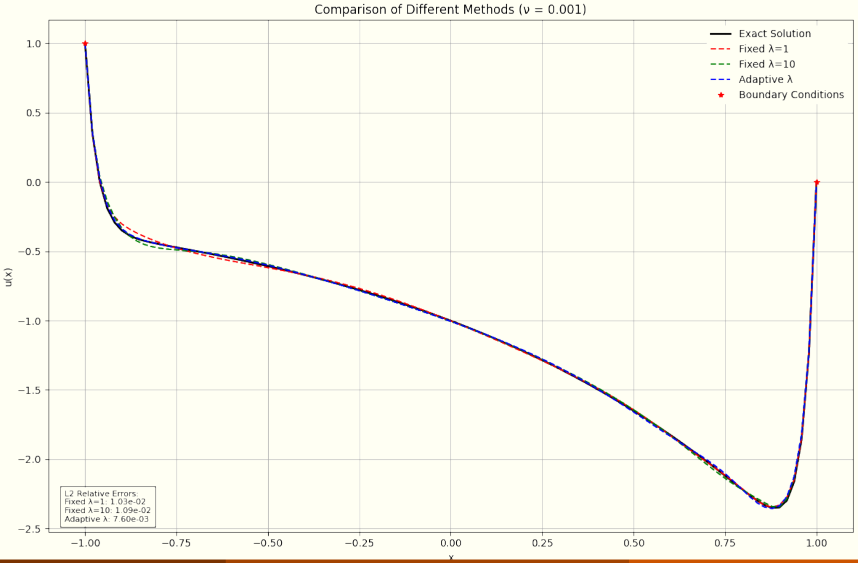
$$\lambda_i^{(k+1)} = \lambda_i^{(k)} \cdot \left(\frac{\max_j \bar{g}_j}{\bar{g}_i} \right)^{\alpha}$$

3. Apply exponential moving average for stability

Benefits:

- Prevents gradient imbalance
- Improves convergence speed
- Reduces manual tuning

Adaptive Weights in Action



Outline

The Burgers Equation: A Nonlinear PDE Challenge

Viscous Burgers equation:

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

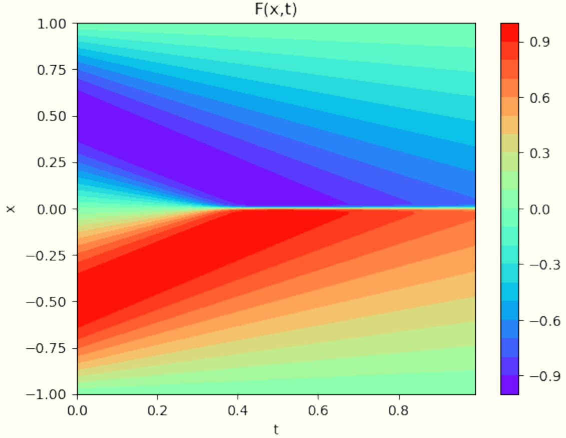
Domain: $(x, t) \in [-1, 1] \times [0, 1]$

Initial condition:

$$u(x, 0) = -\sin(\pi x)$$

Boundary conditions:

$$u(-1, t) = u(1, t) = 0$$



Burgers equation solution showing shock formation

PINN for Burgers Equation

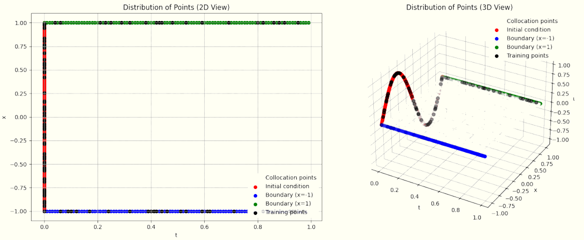
Network architecture:

- Input: (x, t)
- Output: $u(x, t)$
- Hidden layers: 8×20 neurons
- Activation: Tanh

Loss function:

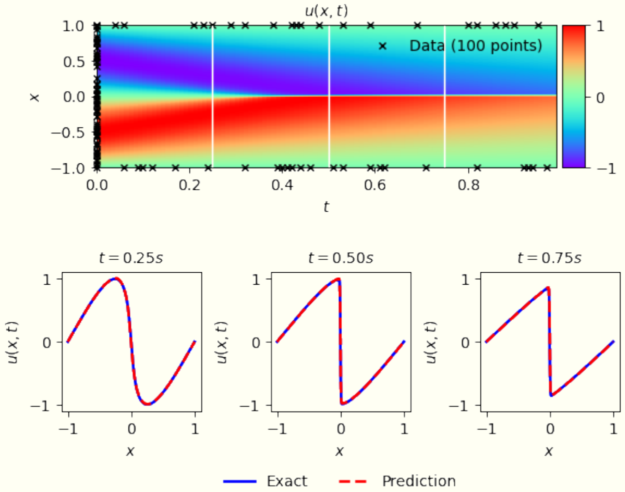
$$\mathcal{L} = \mathcal{L}_{\text{PDE}} + \mathcal{L}_{\text{IC}} + \mathcal{L}_{\text{BC}}$$

$$\mathcal{L}_{\text{PDE}} = \frac{1}{N_f} \sum \left| \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} \right|^2$$



Collocation points for Burgers equation

Burgers Equation: Forward Problem Results



Performance metrics:

- Relative L^2 error: $< 1\%$
- Training time: 5 minutes on GPU
- No mesh required
- Captures shock accurately

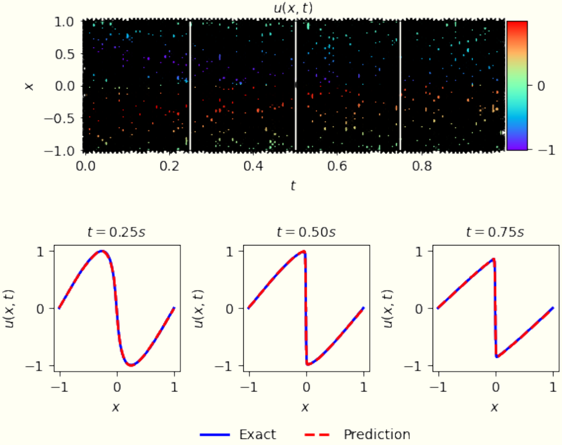
Problem setup:

- Unknown viscosity ν
- Sparse measurements of $u(x, t)$
- Goal: Recover ν and full solution

Modified loss:

$$\mathcal{L} = \mathcal{L}_{\text{data}} + \lambda \mathcal{L}_{\text{PDE}}(\nu)$$

where ν is a trainable parameter



Inverse problem: recovering solution from sparse data

Outline

Why Discrete-Time PINNs?

Limitations of Continuous-Time Approach:

- **Computational Cost:** $O(N_x \times N_t)$ collocation points
- **Memory Issues:** Storing entire space-time solution
- **Training Difficulty:** Long-time behavior hard to capture
- **Stiff Problems:** Need very dense time sampling

Example: For Burgers equation on $[-1, 1] \times [0, 10]$:

- Continuous: $100 \times 1000 = 100{,}000$ collocation points
- Discrete: 100 spatial points per time step

Solution

Use time-stepping methods within PINN framework to evolve solution sequentially

Continuous vs Discrete Time Approaches

Continuous-Time PINNs:

- Treat time as another input: $(x, t) \to u(x, t)$
- Learn entire space-time solution
- Many collocation points in time
- Can become expensive for long simulations

Discrete-Time PINNs:

- Leverage classical time-stepping schemes
- Learn mapping from $t^n \to t^{n+1}$
- Only spatial collocation needed
- More efficient for long time simulations

Key Idea: Integrate Runge-Kutta methods within the PINN framework

Runge-Kutta Time-Stepping Foundation

For time-dependent PDEs in semi-discrete form:

$$\frac{du}{dt} = \mathcal{N}[u](t)$$

A q-stage Runge-Kutta method computes:

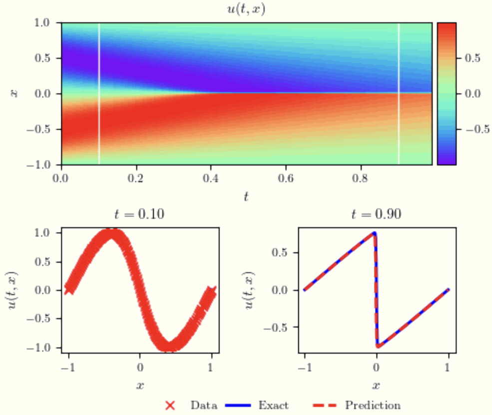
$$U^i = u^n + \Delta t \sum_{j=1}^{q} a_{ij} \mathcal{N}[U^j], \quad i = 1, \ldots, q$$

$$u^{n+1} = u^n + \Delta t \sum_{j=1}^{q} b_j \mathcal{N}[U^j]$$

Where:

- U^i: intermediate stage values
- $\{a_{ij}\}, \{b_j\}, \{c_j\}$: RK coefficients (Butcher tableau)
- Implicit if $a_{ij} \neq 0$ for $j \geq i$

Discrete-Time PINN Architecture



Network outputs q intermediate stages and final solution at t^{n+1}

Network output: $\mathcal{N}\mathcal{N}(x;\theta) = [U^1_{NN}, \ldots, U^q_{NN}, u^{n+1}_{NN}]^T$

Training Discrete-Time PINNs

Loss Function: Enforce consistency with RK scheme

The RK equations provide multiple estimates of $u^n(x)$:

$$u^n(x) \approx U^i_{NN}(x) - \Delta t \sum_{j=1}^{q} a_{ij} \mathcal{N}[U^j_{NN}](x), \quad i = 1, \ldots, q$$

$$u^n(x) \approx u^{n+1}_{NN}(x) - \Delta t \sum_{j=1}^{q} b_j \mathcal{N}[U^j_{NN}](x)$$

Total Loss:

$$\mathcal{L}(\theta) = \frac{1}{N_n(q+1)} \sum_{i=1}^{q+1} \sum_{k=1}^{N_n} |u^n_{i,NN}(x_k) - u^n(x_k)|^2$$

- All estimates should equal the known $u^n(x)$
- Minimizing discrepancy enforces the physics
- Spatial derivatives computed via AD

Algorithm: Single-Step Discrete-Time PINN

Input: Solution $u^n(x)$ at time t^n, time step Δt
Input: RK coefficients $\{a_{ij}\}, \{b_j\}$ for q stages
Initialize network $\mathcal{NN}(x;\theta) : x \to [U^1, \ldots, U^q, u^{n+1}]$
while not converged **do**
  Sample spatial points $\{x_k\}_{k=1}^{N_n}$ and- $[U^1_{NN}, \ldots, U^q_{NN}, u^{n+1}_{NN}] \leftarrow \mathcal{NN}(x_k;\theta)$
  for $j = 1$ to q **do**
    Compute $\mathcal{N}[U^j_{NN}] = -U^j_{NN}\frac{\partial U^j_{NN}}{\partial x} + \nu\frac{\partial^2 U^j_{NN}}{\partial x^2}$
  end for
  for $i = 1$ to q **do**
    $u^n_{i,est} = U^i_{NN} - \Delta t \sum_j a_{ij}\mathcal{N}[U^j_{NN}]$
  end for
  $u^n_{q+1,est} = u^{n+1}_{NN} - \Delta t \sum_j b_j \mathcal{N}[U^j_{NN}]$ and $\mathcal{L} = \frac{1}{(q+1)N_n}\sum_{i,k}|u^n_{i,est}(x_k) - u^n(x_k)|^2$
  Update θ using gradient descent on $\mathcal{L}$
end while
Output: $u^{n+1}(x) = u^{n+1}(x;\theta^*)$

Algorithm: Adaptive Time-Stepping

Algorithm Adaptive Discrete-Time PINN

```
Input: Initial u^0(x), target time T, tolerance ε
Initialize t = 0, Δt = Δt_0
while t < T do
    Train PINN for step t → t + Δt (Algorithm 1)
    Compute error estimate E using embedded RK method:
        E = ||u_high^{n+1} − u_low^{n+1}||_2
    if E < ε then
        Accept step: u^{n+1} = u_high^{n+1}; t = t + Δt
        Adjust: Δt_new = 0.9Δt(ε/E)^{1/p}
    else
        Reject step, retrain with smaller Δt and update Δt = 0.5Δt
    end if
    Δt = min(Δt_new, T − t)
end while
```

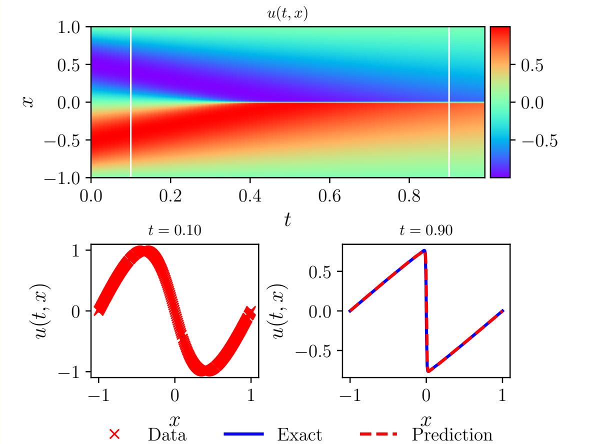
Input: Initial $u^0(x)$, target time T, tolerance ϵ

Initialize $t = 0$, $\Delta t = \Delta t_0$

while $t < T$ **do**

- Train PINN for step $t \to t + \Delta t$ (Algorithm 1)
- Compute error estimate E using embedded RK method:
  - $E = \|u^{n+1}_{high} - u^{n+1}_{low}\|_2$
- **if** $E < \epsilon$ **then**
  - Accept step: $u^{n+1} = u^{n+1}_{high}$; $t = t + \Delta t$
  - Adjust: $\Delta t_{new} = 0.9\Delta t(\epsilon/E)^{1/p}$
- **else**
  - Reject step, retrain with smaller Δt and update $\Delta t = 0.5\Delta t$
- **end if**
- $\Delta t = \min(\Delta t_{new}, T - t)$

end while



Discrete-time PINN prediction for Burgers equation using implicit Runge-Kutta

Key Results:

- High accuracy with large time steps
- Stable evolution over long times
- Captures shock formation accurately

Advantages of Discrete-Time PINNs

Computational Efficiency:

- Loss evaluated only over spatial domain
- No time-dimension collocation points
- Drastically reduced training points

Accuracy Control:

- High-order temporal accuracy via stages
- Adding stages only increases output layer
- Less costly than deeper networks

Stability:

- Implicit RK schemes (e.g., Gauss-Legendre)
- Stable for large time steps
- Essential for stiff problems

Leverages Numerical Analysis:

- Decades of research on time-stepping
- Proven stability properties
- Well-understood error bounds

Outline

From Strong Form to Energy Minimization

Many physical systems are governed by variational principles:

Principle of Minimum Potential Energy

The true solution $u(\mathbf{x})$ minimizes the total potential energy functional $\Pi(u)$

Total Potential Energy:

$$\Pi(u) = \underbrace{\int_\Omega \Psi(\epsilon(u)) d\Omega}_{\text{Internal Energy}} - \underbrace{\int_\Omega f \cdot u \, d\Omega}_{\text{Body Forces}} - \underbrace{\int_{\Gamma_N} t \cdot u \, d\Gamma}_{\text{Surface Tractions}}$$

Deep Ritz Method: Train PINN by minimizing energy instead of PDE residual

Deriving the Energy Formulation

Starting from the strong form (equilibrium equations):

$$\nabla \cdot \sigma + f = 0 \text{ in } \Omega$$

Step 1: Multiply by test function v and integrate:

$$\int_\Omega (\nabla \cdot \sigma) \cdot v \, d\Omega + \int_\Omega f \cdot v \, d\Omega = 0$$

Step 2: Apply integration by parts:

$$-\int_\Omega \sigma : \nabla v \, d\Omega + \int_\Gamma (\sigma \cdot n) \cdot v \, d\Gamma + \int_\Omega f \cdot v \, d\Omega = 0$$

Step 3: For $v = \delta u$ (virtual displacement), this becomes:

$$\delta\Pi = 0 \quad \Rightarrow \quad \text{Stationarity of } \Pi(u)$$

Implementation: Strain Energy Density

For linear elasticity with Lamé parameters λ, μ:

Strain energy density:

$$\Psi(\epsilon) = \frac{\lambda}{2}(\mathrm{tr}(\epsilon))^2 + \mu\,\mathrm{tr}(\epsilon^2)$$

Expanding:

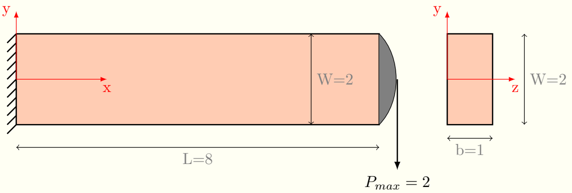
$$\Psi = \frac{\lambda}{2}(\epsilon_{xx} + \epsilon_{yy})^2 + \mu(\epsilon_{xx}^2 + \epsilon_{yy}^2 + 2\epsilon_{xy}^2)$$

```python
def strain_energy_density(epsilon_xx, epsilon_yy, epsilon_xy):
    # First term: lambda*(trace)^2
    psi_1 = 0.5 * lambda_ * (epsilon_xx + epsilon_yy)**2

    # Second term: mu*tr(epsilon^2)
    psi_2 = mu * (epsilon_xx**2 + epsilon_yy**2 + 2*epsilon_xy**2)

    return psi_1 + psi_2
```

Computing Total Potential Energy

```python
def potential_energy(x_colloc, y_colloc, x_bound, y_bound):
    # Compute strains at collocation points
    epsilon_xx, epsilon_yy, epsilon_xy = strain(x_colloc, y_colloc)

    # Strain energy density
    psi = strain_energy_density(epsilon_xx, epsilon_yy, epsilon_xy)

    # Numerical integration over domain
    dx = L / (Nx - 1)
    dy = W / (Ny - 1)
    internal_energy = (psi * dx * dy).sum()

    # External work on boundary
    u_bound = net_u(x_bound, y_bound)
    v_bound = net_v(x_bound, y_bound)
    t_x = traction_x(y_bound)
    t_y = traction_y(y_bound)

    external_work = ((t_x * u_bound + t_y * v_bound) * dy).sum()

    # Total potential energy
    return internal_energy - external_work
```

Example: 2D Cantilever Beam



Cantilever beam: fixed at $x = 0$, loaded at $x = L$

Problem Setup:

- Linear elasticity
- Fixed end: $u(0, y) = g_u(y)$, $v(0, y) = g_v(y)$
- Free end: traction $\mathbf{t}$

Approach:

- Strong BC enforcement
- Energy minimization

Strong BC Enforcement with Trial Functions

Problem: Enforce $u(x = 0, y) = g_u(y)$ and $v(x = 0, y) = g_v(y)$
Solution: Construct trial functions that satisfy BCs by design

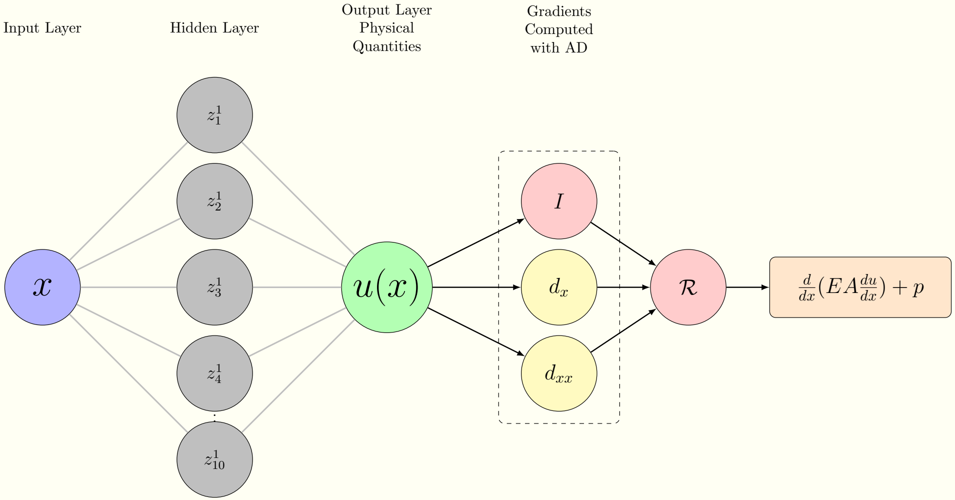
$$\tilde{u}(x, y) = g_u(y) + \mathcal{D}(x) \cdot \hat{u}_{NN}(x, y)$$
$$\tilde{v}(x, y) = g_v(y) + \mathcal{D}(x) \cdot \hat{v}_{NN}(x, y)$$

Where:

- $\mathcal{D}(x)$: Distance function, zero at boundary
- $g_u(y), g_v(y)$: Prescribed boundary values (from beam theory)
- $\hat{u}_{NN}, \hat{v}_{NN}$: Network outputs

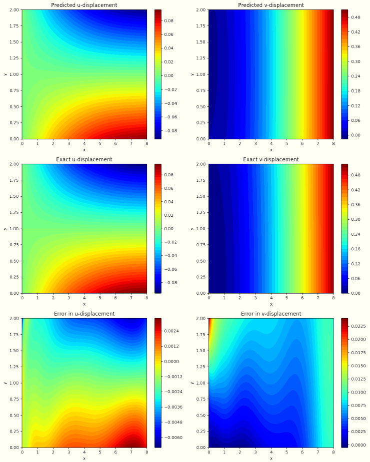
Example: For cantilever beam, $\mathcal{D}(x) = x/L$ (simple linear function)

PINN Architecture for Elasticity



Architecture for linear elasticity using energy minimization

Results: 2D Cantilever Beam

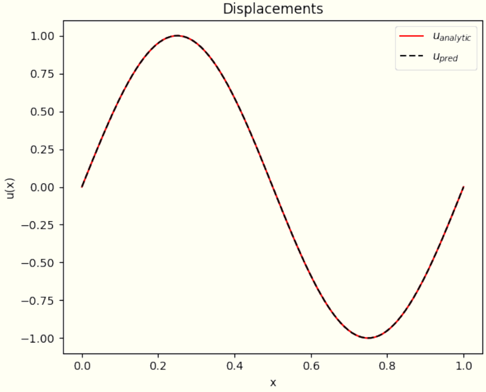


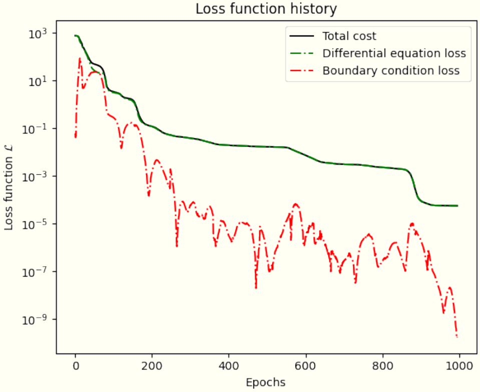
Error distribution using strong BC enforcement and energy minimization

Performance:

- Relative L^2 error: 2.4×10^{-3}
- Exact satisfaction of fixed-end BCs
- Slightly larger errors near loaded end (expected)
- No BC loss terms needed!

Static Beam: Complete Solution





Displacement and stress fields computed using energy minimization

Observations:

- Smooth displacement fields

Advantages of Energy Minimization

Physical Meaning:

- Direct minimization of physical quantity
- Guaranteed energy conservation
- Natural handling of constraints

Improved Stability:

- Convex for linear problems
- Smoother loss landscape
- Faster convergence

Computational Benefits:

- Only first derivatives needed
- Better conditioned optimization
- No need for BC loss terms

Applicability:

- Conservative systems
- Elasticity, electrostatics
- Not for dissipative systems

When applicable, energy minimization often outperforms strong form

Outline

Implementation Best Practices

Architecture design:

- Start with 4-8 layers, 20-50 neurons
- Tanh or Swish for smooth problems
- Adaptive activation functions for shocks
- Skip connections for deep networks

Training strategies:

- Adam optimizer with learning rate decay
- Start with $\lambda = 1$, then adapt
- Quasi-random collocation points
- Mini-batching for large problems

Common pitfalls:

- Imbalanced loss terms
- Too few collocation points
- Wrong activation for problem type
- Ignoring boundary conditions

Debugging tips:

- Visualize loss components separately
- Check gradient flow
- Start with manufactured solutions
- Verify BC/IC satisfaction

When to Use PINNs

PINNs excel at:

- Inverse problems
- Data assimilation
- High-dimensional PDEs
- Irregular geometries
- Parameter discovery
- Uncertainty quantification

Consider alternatives when:

- Need guaranteed accuracy
- Have simple, regular geometry
- Require real-time solutions
- Conservation is critical
- Problem is well-suited to FEM/FDM

PINNs complement, not replace, traditional methods

Questions?

Thank you!

Contact:

Krishna Kumar

krishnak@utexas.edu

University of Texas at Austin