Graphs of Polytopes

By

William Espenschied

Submitted to the Department of Mathematics and the
Faculty of the Graduate School of the University of Kansas
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

Committee members

Margaret Bayer, Chairperson

Jeremy Martin

Jack Porter

Saul Stahl

Nancy Kinnersley

Date defended:

The Dissertation Committee for William Espenschied certifies
that this is the approved version of the following dissertation :

Graphs of Polytopes

Margaret Bayer, Chairperson

Date approved: ____________________

Abstract

The graph of a polytope is the graph whose vertex set is the set of vertices of the polytope, and whose edge set is the set of edges of the polytope. Several problems concerning graphs of polytopes are discussed. The primary result being a set of bounds on the maximal size of an anticlique (sometimes called a coclique, or independent set) of a polytope based on its dimension and number of vertices.

There are two results concerning properties of polytopes which are named after David Gale. The first of which is that the Gale diagram of a join of polytopes is the direct sum of the Gale diagrams of the polytopes and dually, that the Gale diagram of a direct sum of polytopes is the join of their Gale diagrams. The second being that if two polytopes satisfy a weakened form of Gale's evenness condition, then so does their product.

It is shown, by other means, that, with only two exceptions, the complete bipartite graphs are never graphs of polytopes. The techniques developed throughout are then used to show that the complete 3-partite graph $K_{1,n,m}$ is the graph of a polytope if and only if $K_{n,m}$ is the graph of a polytope. It is then shown that $K_{2,2,3}$ and $K_{2,2,4}$ are never graphs of polytopes. A conjecture is then stated as to precisely when a complete multipartite graph is the graph of a polytope.

Contents

List of Figures

List of Tables

Chapter 1

Polytopes

This chapter will provide relevant background material in the theory of convex polytopes. Proofs of any theorems which appear can be found in McMullen & Shephard (1971) (unless otherwise noted).

The set $\{1,2,\ldots,n\}$ will be denoted $[n]$. Throughout, $\mathbb{R}^d$ will denote the real inner product space of column vectors of length d with real entries and the Euclidean inner product. The set of vectors $\{\boldsymbol{e}_i \mid i \in [d]\} \subseteq \mathbb{R}^d$ is called the *standard basis* and $\boldsymbol{e}_i$ has jth entry $(\boldsymbol{e}_i)_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$.
The vectors $\mathbf{0}_d, \mathbf{1}_d \in \mathbb{R}^d$ have jth entry $(\mathbf{0})_j = 0$, and $(\mathbf{1})_j = 1$ respectively. When no confusion may arise, these vectors shall be denoted simply $\mathbf{0}$ and $\mathbf{1}$. If $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, then $\langle \boldsymbol{x}, \boldsymbol{y} \rangle$ will denote the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$, $\|\boldsymbol{x}\| = \langle \boldsymbol{x}, \boldsymbol{x} \rangle^{1/2}$ is the Euclidean norm, and $B_\varepsilon(\boldsymbol{x}) = \{\boldsymbol{y} \mid \|\boldsymbol{x} - \boldsymbol{y}\| \leq \varepsilon\}$ is the ball of radius $\varepsilon > 0$ centered at $\boldsymbol{x}$.

1.1 Hulls; or, The Parts That Touch the Water

If $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k\} \subseteq \mathbb{R}^n$, then a vector $\boldsymbol{v}$ is an *affine combination* of $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$ if there are elements $\lambda_i \in \mathbb{R}$ such that $\boldsymbol{v} = \sum_{i \in [k]} \lambda_i \boldsymbol{x}_i$ and $\sum_{i \in [k]} \lambda_i = 1$. If $A \subseteq \mathbb{R}^n$, then the *affine hull* of A is the set

$$\operatorname{aff}(A) = \left\{ \sum_{i\in[k]} \lambda_i \boldsymbol{x}_i \;\middle|\; \boldsymbol{x}_i \in A,\, \lambda_i \in \mathbb{R} \text{ and } \sum_{i\in[k]} \lambda_i = 1 \right\}.$$

A set which is closed under taking affine combinations of its elements (that is, $\operatorname{aff}(A) = A$) is called an *affine set*. Every affine subset of $\mathbb{R}^n$ is a translation of a linear subspace of $\mathbb{R}^n$. If A is an affine set, and L is a linear subspace of $\mathbb{R}^n$ such that $A = L + \boldsymbol{b}$, then the *dimension* of A is $\dim A = \dim L$. The intersection of a family of affine sets is again an affine set. For example: $\dim\varnothing = -1$; $\dim\{\boldsymbol{x}\} = 0$; and $\dim\operatorname{aff}\{\boldsymbol{x},\boldsymbol{y}\} = 1$ for $\boldsymbol{x} \neq \boldsymbol{y}$.

A set $K \subseteq \mathbb{R}^n$ is *convex* if for each $\boldsymbol{x},\boldsymbol{y} \in K$ the set $\{\lambda\boldsymbol{x} + (1-\lambda)\boldsymbol{y} \mid 0 \leq \lambda \leq 1\} \subseteq K$. If $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k\} \subseteq \mathbb{R}^n$, then a vector $\boldsymbol{v}$ is a *convex combination* of $\boldsymbol{x}_1, \boldsymbol{x}_2, \dots, \boldsymbol{x}_k$ if there are elements $\lambda_i \in \mathbb{R}$ such that $\boldsymbol{v} = \sum_{i\in[k]} \lambda_i \boldsymbol{x}_i$, with $\lambda_i \geq 0$ and $\sum_{i\in[k]} \lambda_i = 1$. If $A \subseteq \mathbb{R}^n$, then the *convex hull* of A is the set

$$\operatorname{conv}(A) = \left\{ \sum_{i\in[k]} \lambda_i \boldsymbol{x}_i \;\middle|\; \boldsymbol{x}_i \in A,\, \lambda_i \geq 0 \text{ and } \sum_{i\in[k]} \lambda_i = 1 \right\}.$$

If K is convex, then $\operatorname{conv} K = K$. For each set A, $\operatorname{conv} A \subseteq \operatorname{aff} A$. The intersection of a family of convex sets is again a convex set. The *dimension* of a convex set K is the dimension of its affine hull, $\dim K = \dim\operatorname{aff} K$.

Both aff and conv are closure operators, that is:

- $A \subseteq \operatorname{aff} A$ and $A \subseteq \operatorname{conv} A$;
- if $A \subseteq B$, then $\operatorname{aff} A \subseteq \operatorname{aff} B$ and $\operatorname{conv} A \subseteq \operatorname{conv} B$; and
- $\operatorname{aff}(\operatorname{aff} A) = \operatorname{aff} A$ and $\operatorname{conv}(\operatorname{conv} A) = \operatorname{conv} A$.

Both $\operatorname{aff} A$ and $\operatorname{conv} A$ are minimal in the sense that $\operatorname{aff} A$ is the intersection of all affine sets which contain A, and $\operatorname{conv} A$ is the intersection of all convex sets which contain A.

If $H \subseteq \mathbb{R}^n$ is an affine set with $\dim H = n-1$, then H is called a *hyperplane*, and there is a vector $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $\langle \boldsymbol{x}_0, \boldsymbol{x}_0 \rangle = 1$ and there is some $b \in \mathbb{R}$ with $\langle \boldsymbol{x}_0, \boldsymbol{t} \rangle = b$ for each $t \in H$ ($\boldsymbol{x}_0$ is not unique unless $n = 0$ (if $n \neq 0$, then there are exactly two such vectors, and the other one is $-\boldsymbol{x}_0$)).

The pair $(H, \boldsymbol{x}_0)$ is called an *oriented hyperplane*, and the *positive* and *negative closed halfspaces* of H are defined by, respectively, $H^+ = \{y \in \mathbb{R}^n \mid \langle \boldsymbol{x}_0, \boldsymbol{y} \rangle \geq b\}$ and $H^- = \{y \in \mathbb{R}^n \mid \langle \boldsymbol{x}_0, \boldsymbol{y} \rangle \leq b\}$. Thus $H = H^+ \cap H^-$. The positive and negative *open half-spaces* $H^{(+)}$ and $H^{(-)}$ are defined by making the defining inequalities strict. A set of the form H^+ or H^- is called a *closed half-space*.

1.2 Polytopes (not 'Galeorhinus Galeus')

In this section convex polytopes[1] will be defined in two different ways. The proof of equivalence of these two definitions can be found in Brøndsted (1983), Grünbaum (2003), McMullen & Shephard (1971) or Ziegler (1995).

An $\mathcal{H}$*-polytope* is a bounded set of the form $P = \bigcap \mathscr{H}$ where

$$\mathscr{H} = \left\{H_1^+, H_2^+, \ldots, H_k^+, H_{k+1}^-, H_{k+2}^-, \ldots, H_\ell^-\right\}$$

is a finite collection of closed halfspaces. A $\mathcal{V}$*-polytope* is the convex hull of a finite set of points in some Euclidean space $\mathbb{R}^k$. A set P is an $\mathcal{H}$-polytope if and only if it is a $\mathcal{V}$-polytope. Henceforth the prefix will be omitted, and P shall be called a polytope. If P is a polytope, then the set $\operatorname{vert} P$ is the (unique) minimal (under inclusion) set such that $P = \operatorname{conv}(\operatorname{vert} P)$.

The dimension of a polytope is its dimension as a convex set. A polytope of dimension d is called a d*-polytope*. If P is a d-polytope, then a hyperplane H is called a *supporting hyperplane* of P if $P \cap H \neq \varnothing$ and either $P \subseteq H^+$ or $P \subseteq H^-$. A *face* of a polytope P is a subset F of P such that one of the following holds:

1. $F = \varnothing$; or

2. there is some supporting hyperplane H of P with $F = H \cap P$; or

3. $F = P$.

[1] Throughout, all polytopes will be convex so the adjective "convex" shall be omitted.

If F is a face of a polytope P, then F is itself a polytope since $F = \operatorname{conv}(F \cap \operatorname{vert} P)$, further $\operatorname{vert} F = F \cap \operatorname{vert} P$. If $\dim F = r$, then F is called an *r-face* of P. A face of a face of a polytope is again a face of the polytope. A face of a polytope is called *proper* if it is neither empty, nor the whole polytope. If P is a d-polytope, then a face of dimension

- 0 is called a *vertex*,
- 1 is called an *edge*,
- $d-2$ is called a *ridge*,
- $d-1$ is called a *facet*.

If P is a polytope, and $\{\boldsymbol{x}\}$ is a vertex of P, then no distinction is made between the vertex $\{\boldsymbol{x}\}$ and the point $\boldsymbol{x}$. If P is a polytope, then $\operatorname{vert} P = \{\boldsymbol{x} \in P \mid \boldsymbol{x} \text{ is a vertex of } P\}$. Each face is the intersection of all facets which contain it.

When ordered by inclusion, the faces of a d-polytope P form a lattice of rank $d+1$, called the *face lattice* of P and denoted $\mathscr{F}(P)$. The join of two faces is the minimal face containing both of them, and the meet of two faces is their intersection. Two polytopes are said to be *combinatorially equivalent* if their face lattices are isomorphic. Combinatorial equivalence is an equivalence relation on the class of all polytopes. The *ith face number* $f_i(P)$ of P is the number of faces F of P with $\dim F = i$. The *f-vector* of P is the vector $\boldsymbol{f}(P) = (f_0(P), f_1(P), \ldots, f_{d-1}(P))$. The nonproper faces correspond to the numbers $f_{-1}(P) = 1 = f_d(P)$.

If $X \subseteq \mathbb{R}^d$, then the *relative interior* of X is the set

$$\operatorname{relint} X = \{\boldsymbol{x} \in X \mid \text{there is some } \varepsilon > 0 \text{ such that } B_\varepsilon(\boldsymbol{x}) \cap \operatorname{aff}(X) \subseteq X\}\,.$$

If P is a k-polytope in $\mathbb{R}^d$ such that $\mathbf{0} \in \operatorname{relint} P$, then the *polar dual* of P, is the set

$$P^\Delta = \{\boldsymbol{x} \in \operatorname{aff} P \mid \langle \boldsymbol{x}, \boldsymbol{y} \rangle \leq 1 \text{ for all } \boldsymbol{y} \in P\}\,.$$

The polar dual of a polytope is again a polytope, and $\dim P^{\Delta} = \dim P$. Thus any polytope which is combinatorially equivalent to P^{Δ} is called a *dual* of P. Furthermore, $(P^{\Delta})^{\Delta} = P$, and $\mathscr{F}(P^{\Delta})$ is the dual (as a lattice) of $\mathscr{F}(P)$. Thus $f_i(P) = f_{d-i-1}(P^{\Delta})$.

If $P \subseteq \mathbb{R}^d$, and $\boldsymbol{x} \in \mathbb{R}^d$, then

$$P + \boldsymbol{x} = \{\boldsymbol{p} + \boldsymbol{x} \mid \boldsymbol{p} \in P\}$$

is called the *translation* of P by $\boldsymbol{x}$. Similarly, define $P - \boldsymbol{x} = P + (-\boldsymbol{x})$. Any translation of any polytope is again a polytope, and is combinatorially equivalent to P. Furthermore, the equality $\mathrm{vert}(P+\boldsymbol{x}) = \mathrm{vert}(P) + \boldsymbol{x}$ holds. If $P \in \mathbb{R}^d$ is a k-polytope and $\boldsymbol{0} \notin \mathrm{relint}\, P$ with $\boldsymbol{x} \in \mathrm{relint}\, P$, then a *dual* of P is any polytope which is combinatorially equivalent to $(P - \boldsymbol{x})^{\Delta}$. This definition is independent of the choice of $\boldsymbol{x} \in \mathrm{relint}\, P$.

1.3 Examples of Polytopes

1.3.1 Simplices

The *standard d-simplex* is

$$\Delta_d = \mathrm{conv}\,\{\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{d+1}\} = \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} \;\middle|\; \langle \boldsymbol{x}, \boldsymbol{1}\rangle = 1, \text{ and } \langle \boldsymbol{x}, \boldsymbol{e}_i\rangle \geq 0\right\}.$$

A d-polytope which is combinatorially equivalent to the standard d-simplex is called a *d-simplex*.

All d-simplices have $d+1$ vertices, and moreover any d-polytope with $d+1$ vertices is a d-simplex. The face lattice of a d-simplex is the Boolean lattice on $d+1$ elements. Equivalently, if P is a simplex, then for any $A \subseteq \mathrm{vert}\, P$, the set $\mathrm{conv}\, A$ is a face of P. A 0-dimensional simplex is a point; a 1-simplex is a line segment; a 2-simplex is a triangle; and a 3-simplex is a tetrahedron. The f-vector of a d-simplex is given by

$$f_k(\Delta_d) = \binom{d+1}{k+1}.$$

Consequently, the dual of a d-simplex is also a d-simplex (since $f_0(\Delta_d^{\Delta}) = f_{d-1}(\Delta_d) = d+1$).

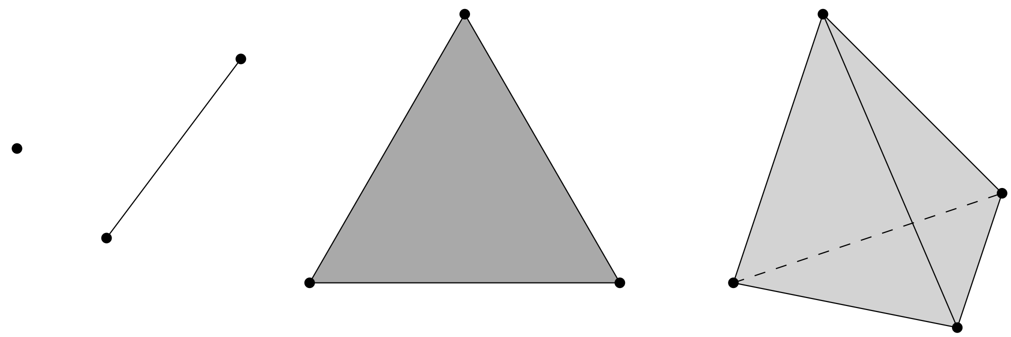


Figure 1.1: Simplices of dimensions 0, 1, 2, and 3.

1.3.2 Cubes

The *standard d-cube* is

$$Q_d = \operatorname{conv}\left\{\{-1,1\}^d\right\} = \left\{\boldsymbol{x} \in \mathbb{R}^d \;\middle|\; |\langle \boldsymbol{x}, \boldsymbol{e}_i \rangle| \leq 1\right\}.$$

A d-polytope which is combinatorially equivalent to the standard d-cube is called a *d-cube*.

Any face of a cube is also a cube. A 0-cube is a point; a 1-cube is a line segment; and a 2-cube is a convex quadrilateral. Every vertex of a d-cube lies in exactly d facets. The f-vector of a d-cube is given by

$$f_k(Q_d) = 2^{d-k}\binom{d}{k}$$

for $k \in [d] \cup \{0\}$.

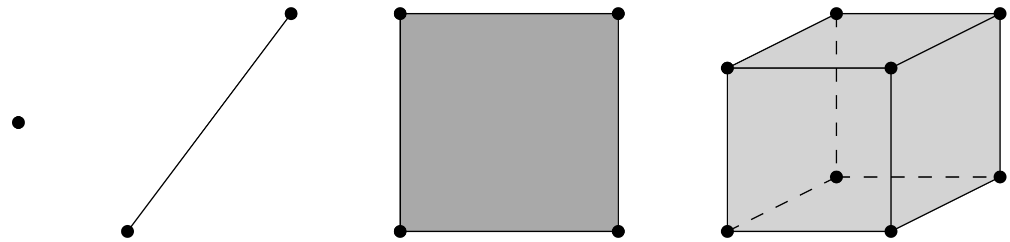


Figure 1.2: Cubes of dimensions 0, 1, 2, and 3.

1.3.3 Crosspolytopes

The *standard d-crosspolytope* is

$$X_d = \operatorname{conv}\Big(\{\boldsymbol{e}_i \mid i \in [d]\} \cup \{-\boldsymbol{e}_i \mid i \in [d]\}\Big) = \left\{ \boldsymbol{x} \in \mathbb{R}^d \;\middle|\; \sum_{i\in[d]} |\langle \boldsymbol{x}, \boldsymbol{e}_i\rangle| \leq 1 \right\} = Q_d^{\Delta}.$$

A d-polytope which is combinatorially equivalent to the standard d-crosspolytope is called a *d-crosspolytope*.

Any proper face of a crosspolytope is a simplex. A 0-crosspolytope is a point; a 1-crosspolytope is a line segment; a 2-crosspolytope is a convex quadrilateral, and a 3-crosspolytope is an octahedron. The f-vector of a d-crosspolytope is given by

$$f_k(X_d) = 2^{k+1}\binom{d}{k+1}$$

for $k \in [d-1] \cup \{-1, 0\}$.

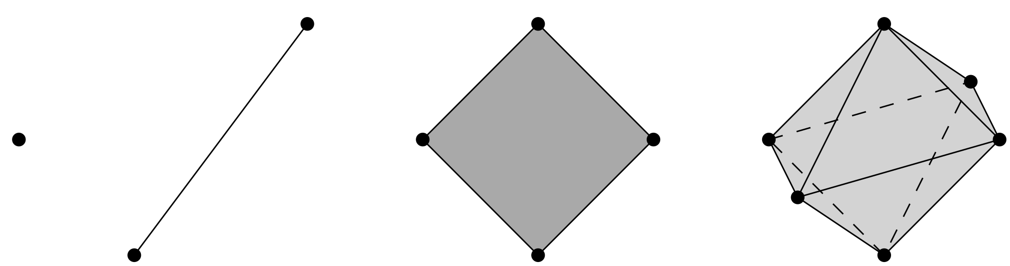


Figure 1.3: Crosspolytopes of dimensions 0, 1, 2, and 3.

1.3.4 Simplicial and Simple Polytopes

A polytope is called *simplicial* if each of its proper faces is a simplex (equivalently, each facet is a simplex). A polytope P is called *simple* if its dual P^{Δ} is simplicial.

In any simplicial polytope, a proper k-face has $k+1$ vertices, dually, in any simple polytope a proper j-face is contained in exactly $d-j$ facets. If a polytope is both simplicial and simple, then it is either a 2-polytope or a simplex. Examples of simplicial polytopes are: all 2-polytopes; the regular icosahedron; all simplices; and all crosspolytopes. Examples of simple polytopes are: all 2-polytopes; the regular dodecahedron; all simplices; and all cubes.

1.3.5 Cyclic and Gale Polytopes

Let M_d be the curve in $\mathbb{R}^d$ which is defined parametrically by the equation

$$\boldsymbol{r}(t) = \begin{bmatrix} t \\ t^2 \\ \vdots \\ t^d \end{bmatrix},$$

for $t \in \mathbb{R}$. If V is any set of $n > d$ distinct points on M_d, say $V = \{\boldsymbol{r}(t_i) \mid t_1 < t_2 < \cdots < t_n\}$, then $\operatorname{conv} V$ is a d-dimensional polytope with n vertices (that is, $V = \operatorname{vert}(\operatorname{conv} V)$). All such polytopes with n vertices in dimension d are combinatorially equivalent and any polytope which is combinatorially equivalent to such a polytope will be denoted by $C_d(n)$ and called a *cyclic polytope*.

All cyclic polytopes are simplicial, and, for $d \geq 3$, are simple if and only if $n = d+1$ (in which case they are simplices). Since the combinatorial type of $C_d(n)$ is independent of the choices for the t_i, the vertex $\boldsymbol{r}(t_i)$ will be identified with the integer i (assuming that $t_1 < t_2 < \cdots < t_n$), and the face $\operatorname{conv}\{i_1, i_2, \ldots, i_k\}$ with the set $\{i_1, i_2, \ldots, i_k\}$ (with $i_1 i_2 \ldots i_k$ if there is no chance for any confusion).

Theorem 1 (Gale's Evenness Condition)**.** *A subset $S \subseteq [n]$ with $|S| = d$ forms a facet of $C_d(n)$ if and only if*

$$|\{k \mid k \in S \text{ and } i < k < j\}| \text{ is even for all } i < j, \text{with} \{i, j\} \cap S = \varnothing.$$

Example 2. Table 1.1 shows a method of visualizing Gale's evenness condition for the polytope $C_3(6)$. The facets are given by the rows of the table, and in each row, an asterisk in column i denotes that i is in the facet, and a dash denotes an element of [6] which is not in the facet. Gale's evenness condition can then be interpreted as: between any two dashes there are an even number of asterisks.

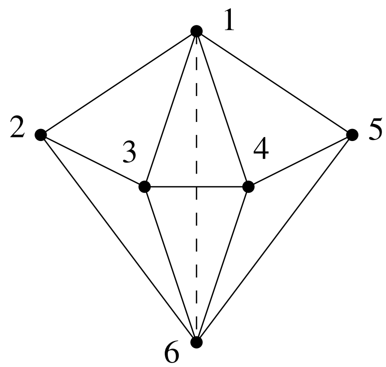


Figure 1.4: The polytope $C_3(6)$

facet	1	2	3	4	5	6
123	*	*	*	-	-	-
126	*	*	-	-	-	*
134	*	-	*	*	-	-
145	*	-	-	*	*	-
156	*	-	-	-	*	*
236	-	*	*	-	-	*
346	-	-	*	*	-	*
456	-	-	-	*	*	*

Table 1.1: A chart to determine the facets of $C_3(6)$

If P is a d-polytope and there is an ordering of $\operatorname{vert} P$ such that for each facet F of P, $\operatorname{vert} F$ satisfies Gale's evenness condition (other than the requirement $|\operatorname{vert} F| = d$), then P is said to be a *Gale polytope*. Gale polytopes will be developed more in Chapter 6.

Any subset $S \subseteq [n]$ with $|S| \leq \frac{d}{2}$ forms a face of $C_d(n)$. In particular, for $d > 3$, the convex hull of any two vertices of $C_d(n)$ forms an edge of $C_d(n)$. The f-vector of $C_d(n)$ is given by

$$f_k(C_{2d}(n)) = \sum_{j=1}^{d} \frac{n}{n-j} \binom{n-j}{j} \binom{j}{k+1-j} \qquad \text{if } k \in [2d-1] \cup \{0\}$$

$$f_k(C_{2d+1}(n)) = \sum_{j=0}^{d} \frac{k+2}{n-j} \binom{n-j}{j+1} \binom{j+1}{k+1-j} \qquad \text{if } k \in [2d] \cup \{0\}$$

The following theorem shall not be used; it is however included as an example of the usefulness of cyclic polytopes (for a proof see either McMullen & Shephard (1971), or Ziegler (1995)).

Upper Bound Theorem (McMullen 1971)**.** *If P is a d-polytope with n vertices, then*

$$f_k(P) \leq f_k(C_d(n))$$

for each $k \in [d] \cup \{-1, 0\}$.

1.4 Neoteric Polytopes From Erstwhile Polytopes

1.4.1 Pyramids

If $P \subseteq \mathbb{R}^n$ is a d-polytope (assuming $n > d$), and $\boldsymbol{v} \in \mathbb{R}^n \setminus \operatorname{aff} P$, then $\operatorname{conv}(P \cup \{\boldsymbol{v}\})$ is a $(d+1)$-polytope which is called a *pyramid* over P with *apex* $\boldsymbol{v}$. If $P \subseteq \mathbb{R}^d$, then P can be embedded in $\mathbb{R}^{d+1}$ by appending a -1 to the end of each point in P. Thus, one way of realizing $\operatorname{pyr}(P)$ is as:

$$\operatorname{conv}\left(\left\{\begin{bmatrix} \boldsymbol{x} \\ -1 \end{bmatrix} \;\middle|\; \boldsymbol{x} \in P\right\} \cup \left\{\begin{bmatrix} \mathbf{0}_d \\ 1 \end{bmatrix}\right\}\right).$$

The faces of $\operatorname{pyr}(P) = \operatorname{conv}(P \cup \{\boldsymbol{v}\})$ are the copies of the faces of P (which are not P themselves), as well as the pyramids over these faces with apex $\boldsymbol{v}$ (this includes $\{\boldsymbol{v}\}$ since $\{\boldsymbol{v}\} = \operatorname{pyr}(\varnothing)$). Thus

$$f_k(\operatorname{pyr} P) = f_k(P) + f_{k-1}(P)$$

where $f_j(P) = 0$ if either $j < -1$ or $j > d$. As an example, a d-simplex is a pyramid over a $(d-1)$-simplex.

1.4.2 Prisms

If $P \subseteq \mathbb{R}^n$ is a d-polytope, and $P' \subseteq \mathbb{R}^n$ is a translation of P such that $\operatorname{aff}(P) \cap \operatorname{aff}(P') = \varnothing$ (this requires that $n > d$), then a *prism* with base P is a polytope $\operatorname{prism}(P)$ combinatorially equivalent to the polytope $\operatorname{conv}(P \cup P')$. The proper faces of a prism are:

1. proper faces of either P or its translate; as well as

2. prisms over proper faces of P; as well as

3. the polytope P or the translate of P.

Hence

$$f_k(\operatorname{prism} P) = \begin{cases} 1 & \text{if } k \in \{-1, d+1\} \\ 2f_k(P) & \text{if } k = 0 \\ 2f_k(P) + f_{k-1}(P) & \text{if } k \in [d] \end{cases}$$

are the face numbers of a prism over a d-polytope P. As an example, a d-cube is a prism over a $(d-1)$-cube.

1.4.3 Bipyramids

If $P \subseteq \mathbb{R}^n$ is a d-polytope, and $Q \subseteq \mathbb{R}^n$ is a 1-polytope such that $\operatorname{relint}(P) \cap \operatorname{relint}(Q)$ is a single point, then any $(d+1)$-polytope which is combinatorially equivalent to the $(d+1)$-polytope $\operatorname{conv}(P \cup Q)$ is called a *bipyramid* over P. If $P \subseteq \mathbb{R}^d$ is a d-polytope with $\mathbf{0} \in \operatorname{relint} P$, then a bipyramid over P can be obtained in $\mathbb{R}^{d+1}$ as

$$\operatorname{conv}\left(\left\{\begin{bmatrix} \boldsymbol{x} \\ 0 \end{bmatrix} \middle| \, \boldsymbol{x} \in P\right\} \cup \left\{\begin{bmatrix} \mathbf{0}_d \\ 1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_d \\ -1 \end{bmatrix}\right\}\right).$$

If P is a d-polytope, and $Q = \operatorname{conv}\{\boldsymbol{a}, \boldsymbol{b}\}$ is a 1-polytope such that $B = \operatorname{conv}(P \cup Q)$ is a bipyramid over P, then the proper faces of B are:

1. the vertices $\boldsymbol{a}$ or $\boldsymbol{b}$; as well as
2. proper faces of P; as well as
3. pyramids over a proper face of P with apex either $\boldsymbol{a}$ or $\boldsymbol{b}$.

Hence

$$f_k(B) = \begin{cases} 1 & \text{if } k \in \{-1, d+1\} \\ 2f_{k-1}(P) + f_k(P) & \text{if } k \in [d-1] \cup \{0\} \\ 2f_{k-1}(P) & \text{if } k = d \end{cases}$$

are the face numbers of a bipyramid B over a d-polytope P. As an example, a d-crosspolytope is a bipyramid over a $(d-1)$-crosspolytope. An alternative definition could be that a bipyramid over a polytope P is the dual of a prism over a dual of P.

1.4.4 Product

If P is a d_1-polytope with $\operatorname{vert} P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^{d_1}$, and Q is a d_2-polytope with $\operatorname{vert} Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\} \subseteq \mathbb{R}^{d_2}$, then the *(Cartesian) product* of P and Q, denoted $P \times Q$, is a $(d_1 + d_2)$-polytope which is combinatorially equivalent to the polytope with vertex set

$$\operatorname{vert}(P \times Q) = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \boldsymbol{q}_j \end{bmatrix} \;\middle|\; i \in [n],\, j \in [m] \right\}.$$

The nonempty faces of a product $P \times Q$ are products of nonempty faces of P with nonempty faces of Q. Hence

$$f_i(P \times Q) = \begin{cases} 1 & \text{if } i \in \{-1, d_1 + d_2\} \\ \displaystyle\sum_{\substack{j+k=i \\ j \in [d_1] \cup \{0\} \\ k \in [d_2] \cup \{0\}}} f_j(P) f_k(Q) & \text{if } i \in [d_1 + d_2 - 1] \cup \{0\} \end{cases}$$

are the face numbers of a product of the d_1-polytope P and the d_2-polytope Q. In particular, if the facets of P are $F_1, F_2, \ldots, F_s$, and the facets of Q are $G_1, G_2, \ldots, G_t$, then the facets of $P \times Q$ are

$$P \times G_1, P \times G_2, \ldots, P \times G_t, F_1 \times Q, F_2 \times Q, \ldots, F_s \times Q.$$

As an example, prisms are products of a polytope with a 1-polytope.

1.4.5 Join

In this and the following section, the fundamental definitions of the operations will be in terms of point sets rather than polytopes.

If $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^{d_1}$ and $Y = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_m\} \subseteq \mathbb{R}^{d_2}$, then the *join* of X and Y is the set

$$X \vee Y = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \boldsymbol{0}_{d_2} \\ -1 \end{bmatrix} \;\middle|\; i \in [n] \right\} \cup \left\{ \begin{bmatrix} \boldsymbol{0}_{d_1} \\ \boldsymbol{q}_j \\ 1 \end{bmatrix} \;\middle|\; j \in [m] \right\} \subseteq \mathbb{R}^{d_1+d_2+1}$$

If P and Q are polytopes of dimensions d_1 and d_2 respectively such that $\operatorname{vert} P = X$ and $\operatorname{vert} Q = Y$, then define the join of P and Q to be a polytope combinatorially equivalent to $P \vee Q = \operatorname{conv}(X \vee Y)$. In this case, $\operatorname{vert}(P \vee Q) = X \vee Y$ and $\dim(P \vee Q) = d_1 + d_2 + 1$. Intuitively, the join of two polytopes is obtained by placing both polytopes in a high enough dimensional space so that they can be arranged with their affine hulls nonintersecting, and then taking a convex hull.

The i-faces of $P \vee Q$ are joins of faces of P and faces of Q such that the sums of the dimensions of these faces is $i - 1$. Hence

$$f_i(P \vee Q) = \sum_{\substack{j+k+1=i \\ j \in [d_1] \cup \{0,-1\} \\ k \in [d_2] \cup \{0,-1\}}} f_j(P) f_k(P)$$

are the face numbers of a join of the d_1-polytope P and the d_2-polytope Q. As an example, pyramids are joins of a polytope with a 0-polytope. Moreover, the join of two simplices is again a simplex.

1.4.6 Direct Sum

If $X = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\} \subseteq \mathbb{R}^{d_1}$ and $Y = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_m\} \subseteq \mathbb{R}^{d_2}$, then the *direct sum* of X and Y is the set

$$X \oplus Y = \left\{ \begin{bmatrix} \boldsymbol{p}_i \\ \mathbf{0}_{d_2} \end{bmatrix} \;\middle|\; i \in [n] \right\} \cup \left\{ \begin{bmatrix} \mathbf{0}_{d_1} \\ \boldsymbol{q}_j \end{bmatrix} \;\middle|\; j \in [m] \right\} \subseteq \mathbb{R}^{d_1+d_2}$$

If P and Q are polytopes of dimensions d_1 and d_2 respectively such that $\operatorname{vert} P = X$ and $\operatorname{vert} Q = Y$, then define the direct sum of P and Q to be a polytope combinatorially equivalent to $P \oplus Q = \operatorname{conv}(X \oplus Y)$. In this case, $\operatorname{vert}(P \oplus Q) = X \oplus Y$ and $\dim(P \oplus Q) = d_1 + d_2$. Intuitively, the direct sum of two polytopes is obtained by placing both polytopes in a high enough dimensional space so that they can be arranged with their relative interiors intersecting in a single point, and then taking a convex hull.

The i-faces of $P \oplus Q$ are joins of faces of P and faces of Q (but not P or Q themselves) such that the sums of the dimensions of these faces is $i-1$. Hence

$$f_i(P \oplus Q) = \sum_{\substack{j+k+1=i \\ j \in [d_1-1] \cup \{0,-1\} \\ k \in [d_2-1] \cup \{0,-1\}}} f_j(P) f_k(P)$$

are the face numbers of a direct sum of the d_1-polytope P and the d_2-polytope Q. As an example, bipyramids are direct sums of a polytope with a 1-polytope. Moreover, the direct sum of two crosspolytopes is again a crosspolytope.

1.4.7 Vertex Figures

Suppose P is a polytope, $\boldsymbol{v}$ is a vertex of P, and $H = \{\boldsymbol{x} \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = k\}$ is a supporting hyperplane of $\boldsymbol{v}$ with $P \subseteq H^+$. Let $m = \min\{\langle \boldsymbol{\xi}, \boldsymbol{w} \rangle \mid \boldsymbol{w} \in \operatorname{vert}(P) \setminus \{\boldsymbol{v}\}\}$. This is a positive number since H is a supporting hyperplane of the face $\boldsymbol{v}$. Now set $J = \left\{\boldsymbol{x} \mid \langle \boldsymbol{\xi}, \boldsymbol{x} \rangle = k + \frac{m}{2}\right\}$. Then $\boldsymbol{v} \in J^{(-)}$, and $\boldsymbol{w} \in J^{(+)}$ for each other vertex $\boldsymbol{w}$ of P. Furthermore, $J \cap P$ is a polytope since the intersection of a polytope and a plane is again a polytope (this can be easily proven by using the $\mathcal{H}$-polytope formulation of the definition of a polytope). A *vertex figure of P at $\boldsymbol{v}$* is any polytope combinatorially equivalent to $P_{\boldsymbol{v}} = J \cap P$.

The face lattice of a vertex figure is isomorphic to the interval $[\boldsymbol{v}, P]$ in the face lattice of P. Thus

$$f_i(P_{\boldsymbol{v}}) = |\{F \in \mathscr{F}(P) \mid \dim F = i+1 \text{ and } \boldsymbol{v} \in F\}|.$$

Some examples of vertex figures are: vertex figures of simple polytopes are simplices of one dimension lower; vertex figures of crosspolytopes are crosspolytopes of one dimension lower; a vertex figure of a regular icosahedron is a pentagon; and a vertex figure of a pyramid at its apex is the base of the pyramid.

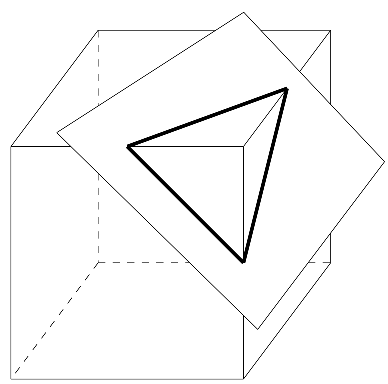


Figure 1.5: A vertex figure of a 3-cube is a triangle.

1.4.8 Kleetopes

Let P be a d-polytope in $\mathbb{R}^d$, and for each facet F of P, let $\boldsymbol{x}_F$ be the unit vector which is normal to aff(F) such that $\langle \boldsymbol{x}_F, \boldsymbol{v} \rangle \leq 0$ for each $\boldsymbol{v} \in P$. Let F_0 be a fixed facet of P. Now choose a point $\boldsymbol{x}_0$ in the region of $\mathbb{R}^d$ where $\langle \boldsymbol{x}_{F_0}, \boldsymbol{v} \rangle > 0$, and $\langle \boldsymbol{x}_F, \boldsymbol{v} \rangle < 0$ for each other F. Then $K(P;F_0)$ is any polytope which is combinatorially equivalent to conv$(P \cup \{\boldsymbol{x}_0\})$ and is independent of the choice of $\boldsymbol{x}_0$. The vertex set of $K(P;F_0)$ is vert$(P) \cup \{\boldsymbol{x}_0\}$. Geometrically, $K(P;F_0)$ is P with a shallow pyramid over F_0 "glued" to F_0. Thus

$$f_i(K(P;F_0)) = \begin{cases} f_i(P) + f_{i-1}(F_0) & \text{if } i \in [d-2] \cup \{0\} \\ f_{d-1}(P) + f_{d-2}(F_0) - 1 & \text{if } i = d-1 \\ 1 & \text{if } i \in \{-1, d\} \end{cases}$$

If $F_0, F_1, \ldots, F_n$ is a sequence of distinct facets of P, and Φ_i is the subsequence of the first $i+1$ terms, then for $i \in [n]$, define $K(P;\Phi_i) = K(K(P;\Phi_{i-1});F_i)$. The combinatorial type of the polytope $K(P;F_0,F_1,\ldots,F_n)$ is independent of the order of the sequence. Thus the sequence can be considered to just be a set of facets Φ of P. If Φ is the set of all facets of P, then $K(P;\Phi)$ is denoted P^K, and called the *Kleetope* of P. Note that the Kleetope of a simplicial polytope is a simplicial polytope, as is the Kleetope of any 3-polytope.

If P is a d-simplex, then the combinatorial type of $K(P;\Phi)$ is completely determined by $k = |\Phi|$, and thus is denoted $K(\Delta_d;k)$ for $k \leq d+1$.

Chapter 2

Graphs

This chapter will provide relevant background material in the theory of graphs.

If X is a set, and $k \in \mathbb{N}$ is a natural number (including zero), then $\binom{X}{k} = \{A \subseteq X \mid |A| = k\}$ is the set of unordered k-tuples of elements of X.

2.1 Definition and Examples

A graph is an ordered pair $G = (V, E)$ where V is a finite set, and $E \subseteq \binom{V}{2}$ is a set of unordered pairs of elements of V. The elements of $V = V(G)$ are called *vertices*, and the elements of $E = E(G)$ are called *edges*. Note that either V or E are allowed to be empty. If $V = \varnothing$, then $E = \varnothing$, and G is called the *empty graph*. If $e = \{x, y\} \in E$ and no confusion may arise, the edge e may be denoted $e = xy = yx = \{x, y\}$. Two vertices $v, w \in V$ are called *neighbors*, or *adjacent* if $vw \in E$. The *neighborhood* of a vertex v is the set $N(v) = \{w \in V \mid vw \in E\}$ of all neighbors of v. The *degree* of a vertex v is the number of edges on which v lies, that is, $\deg v = |\{e \in E \mid v \in e\}| = |N(v)|$.

Two graphs $G = (V, E)$ and $H = (W, F)$ are said to be *isomorphic* if a bijection $\phi : V \to W$ exists with the additional property: $xy \in E$ if and only if $\phi(x)\phi(y) \in F$. If G and H are isomorphic, no distinction will be made between them. It is often useful to draw a graph by depicting its vertices as points in the plane, and its edges by continuous curves between vertices.

Examples 3.

1. A *discrete* graph on n vertices is a graph isomorphic to $D_n = ([n], \varnothing)$.

2. A *cycle* of *length* $n \geq 3$ is a graph isomorphic to C_n where $V(C_n) = [n]$ and

$$E(C_n) = \{ij \mid i - j \equiv 1 \pmod{n}\}.$$

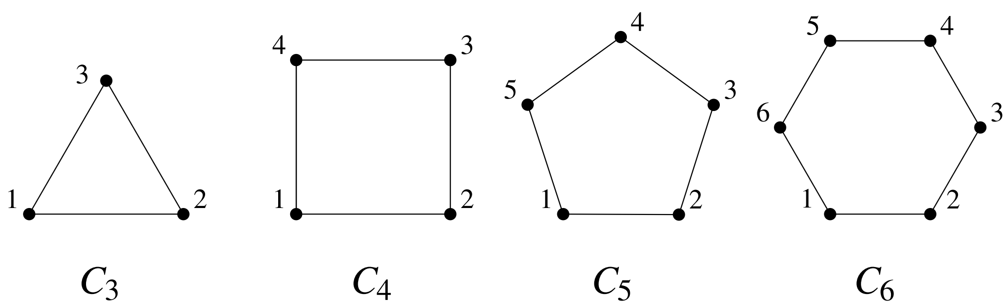


Figure 2.1: Several cycle graphs.

3. A graph is *complete* if $E = \binom{V}{2}$. If $|V| = n$, then G is denoted K_n.

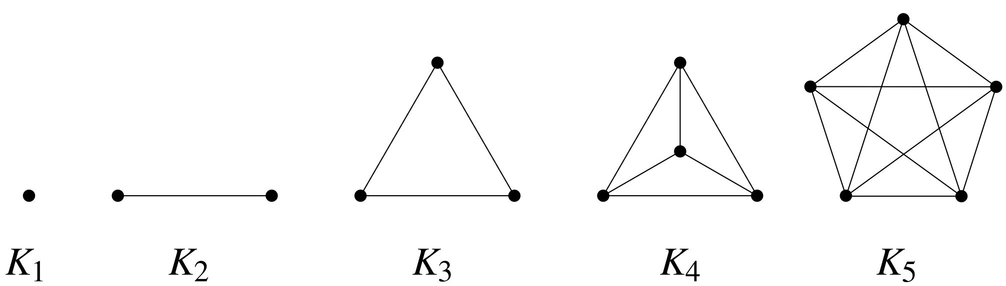
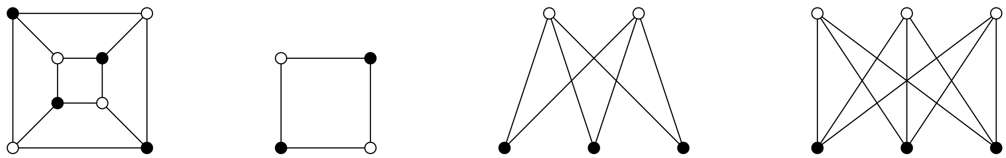


Figure 2.2: Several complete graphs.

4. A graph G is said to be *bipartite* if $V(G) = V_1 \cup V_2$ with $V_1 \neq \varnothing \neq V_2$ and $V_1 \cap V_2 = \varnothing$ such that $E(G) \cap \binom{V_1}{2} = E(G) \cap \binom{V_2}{2} = \varnothing$. If G is a bipartite graph satisfying

$$E(G) = \left\{ xy \in \binom{V}{2} \,\middle|\, x \in V_1 \text{ and } y \in V_2 \right\},$$

then G is *complete bipartite*, and is denoted $K_{n,m}$ where $|V_1| = n$ and $|V_2| = m$. Note that a complete bipartite graph must have at least one edge.



$K_{2,2}$ $K_{2,3}$ $K_{3,3}$

Figure 2.3: Several bipartite graphs. The first vertex set is colored white, and the second black.

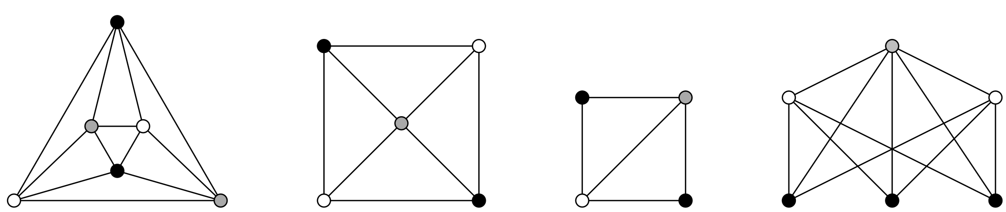
5. More generally, a graph is *n-partite* if its vertex set can be written $V(G) = \bigcup_{i \in [n]} V_i$ such that:

   (a) $V_i \neq \varnothing$ for all $i \in [n]$;

   (b) $V_i \cap V_j = \varnothing$ for $i \neq j$; and

   (c) $E(G) \cap \binom{V_i}{2} = \varnothing$ for each $i \in [n]$.

   If, additionally, $E(G) = \left\{ xy \in \binom{V}{2} \;\middle|\; \{x,y\} \not\subseteq V_i \text{ for each } i \in [n] \right\}$, then G is called *complete n-partite*, and denoted $K_{m_1,m_2,\ldots,m_n}$ where $|V_i| = m_i$ for each $i \in [n]$. Note that for any permutation σ of $[n]$ the graphs $K_{m_1,m_2,\ldots,m_n}$ and $K_{m_{\sigma 1},m_{\sigma 2},\ldots,m_{\sigma n}}$ are isomorphic. Hence complete n-partite graphs will be assumed to satisfy $m_1 \leq m_2 \leq \cdots \leq m_n$.



$K_{2,2,2}$ $K_{1,2,2}$ $K_{1,1,2}$ $K_{1,2,3}$

Figure 2.4: Several 3-partite graphs.

If G and H are graphs, then their *join* is the graph $G \vee H$ which is defined by

$$V(G \vee H) = V(G) \cup V(H)$$
$$E(G \vee H) = E(G) \cup E(H) \cup \{gh \mid g \in V(G) \text{ and } h \in V(H)\}.$$

If r, s are positive integers, then $K_{r,s} = D_r \vee D_s$, and more generally,

$$K_{r_1, r_2, \ldots, r_t} = D_{r_1} \vee D_{r_2} \vee \cdots \vee D_{r_t}.$$

2.2 Subgraphs and Minors

If $G = (V, E)$ is a graph, then a *subgraph* H of G is a graph of the form $H = (W, F)$ with $W \subseteq V$, and $F \subseteq E \cap \binom{W}{2}$. If, additionally, for all $xy \in E(G)$ the containment $\{x, y\} \subseteq V(H)$ implies the containment $xy \in E(H)$, that is, $F = E \cap \binom{W}{2}$, then H is said to be an *induced subgraph* of G. When working with graphs, subgraphs are not generally sufficient for dealing with statements of theorems; the idea of a minor of a graph is also needed:

Definitions. Let $G = (V, E)$ be a graph, and $W \subseteq V$.

1. The *restriction* of G to W, denoted by $G|_W$, is the graph defined by $V(G|_W) = W$, and $E(G|_W) = E \cap \binom{W}{2}$.

2. The *deletion* of W, denoted $G \setminus W$, is the restriction of G to $V \setminus W$, that is $G|_{V \setminus W}$.

3. If $xy = e \in E$, then the *contraction* of e denoted by G/e is the graph defined by $V(G/e) = (V \setminus \{x, y\}) \cup \{\tilde{e}\}$ and

$$E(G/e) = E(G|_{V \setminus \{x,y\}}) \cup \{\widetilde{e}z \mid xz \in E \text{ or } yz \in E \text{ for some } z \notin \{x, y\}\}$$

4. If $F \subseteq E$, then the *deletion* of F, denoted $G \setminus F$ is the graph with $V(G \setminus F) = V$ and $E(G \setminus F) = E \setminus F$.

5. A graph H is a *minor* of a graph G if H is isomorphic to a graph which is obtainable from G through a sequence of deletions and contractions.

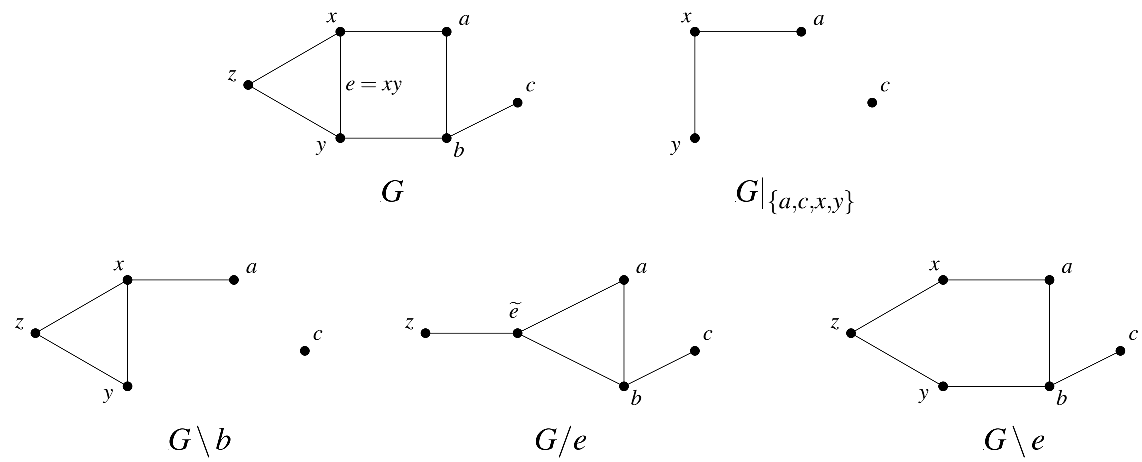


Figure 2.5: Examples of some minors of a graph G.

The word deletion is defined in two different ways above; however no confusion should arise since one definition was in regards to a set of vertices, whereas the other refers to a set of edges.

If $F = \{e_1, e_2, \ldots, e_n\} \subseteq E(G)$, then it can be shown that $(\cdots((G/e_1)/e_2)\cdots)/e_n$ is isomorphic to $(\cdots((G/e_{\sigma 1})/e_{\sigma 2})\cdots)/e_{\sigma n}$ for any permutation σ of $[n]$ (see Diestel (2010) for details). This graph is denoted G/F.

2.3 Planarity and Connectivity

When drawing a graph G in the plane, a natural question to ask is, "Is it possible to draw G such that no two edges intersect unless they share a vertex, and in that case only at the shared vertex?" If the answer to this question is yes, then the graph is called *planar*. The following gives a characterization of planarity which avoids topology. For a proof, see Wagner (1937) or Diestel (2010).

Theorem 4 (Wagner 1937)**.** *A graph G is planar if and only if G has no minor isomorphic to either K_5, or $K_{3,3}$.*

As mentioned above, the statement of Theorem 4 must be in terms of minors; the Peterson graph P (Figure 2.6) has neither a K_5, nor $K_{3,3}$ subgraph, however it does contain each as a minor. The graph K_5 can be realized by contracting each of the edges $v_i w_i$ for $i \in [5]$. While $K_{3,3}$ can be realized as $(P \setminus \{w_2 w_3, v_5 w_5\}) / \{w_4 w_5, v_2 v_5, w_1 w_2, v_3 w_3\}$.

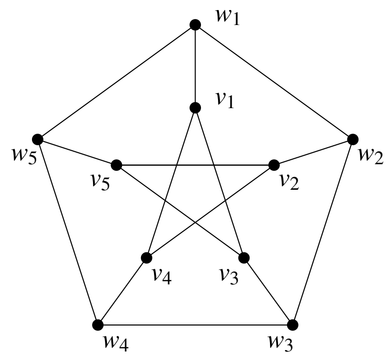


Figure 2.6: The Peterson graph.

Whenever planarity is needed, the above conditions will be checked. Thus the above definition will be treated as the definition of planarity, and the previous discussion will be reserved for geometric intuition.

A graph G is said to be *connected* if either $|V| = 1$, or for any pair $x, y \in V$ there exists a sequence of vertices $x = x_1, x_2, \ldots, x_{r+1} = y$ such that $x_i x_{i+1} \in E$ for $i \in [r]$. If, additionally, $x_i \neq x_j$ for $i \neq j$, then $x_1, x_2, \ldots, x_{r+1}$ is called a *path* from x to y of *length* r with *endpoints* x and y. If $d \in \mathbb{N}$, then a graph G is said to be *d-connected* if $|V| > d$, and for $W \subseteq V$ with $|W| < d$ the graph $G \setminus W$ is connected. Every non-empty graph is 0-connected, and the 1-connected graphs are exactly the connected graphs with at least two vertices.

Two paths $x_1, x_2, \ldots, x_i$ and $y_1, y_2, \ldots, y_j$ are said to be *disjoint* if the only intersections of the paths are at endpoints of each. The following theorem, gives a characterization of d-connectedness in terms of disjoint paths. For a proof, see either Whitney (1932) or Diestel (2010).

Theorem 5 (Whitney 1932)**.** *A graph G is d-connected if and only if for all $a, b \in V$ with $a \neq b$ there exist d pairwise disjoint paths from a to b.*

Example 6. The graph $K_{2,2,2}$ is 4-connected. Let $V(K_{2,2,2}) = \{a_{1,1}, a_{2,1}, a_{3,1}, a_{1,2}, a_{2,2}, a_{3,2}\}$, and $E(K_{2,2,2}) = \{a_{i,j} a_{k,l} \mid i \neq k\}$. Then a pair of vertices $a_{i,j} a_{k,l}$ is either adjacent, or not. In the latter case, we can assume, without loss of generality that $i = j = k = 1$, and $l = 2$. The following are four pairwise disjoint paths from $a_{1,1}$ to $a_{1,2}$:

$$a_{1,1}, a_{2,1}, a_{1,2} \qquad a_{1,1}, a_{2,2}, a_{1,2}$$

$$a_{1,1}, a_{3,1}, a_{1,2} \qquad a_{1,1}, a_{3,2}, a_{1,2}$$

If, on the other hand, $i \neq k$, then assume again that $i = j = 1$, and without loss of generality, assume $k = 3$ and $l = 2$. Then the following are four pairwise disjoint paths from $a_{1,1}$ to $a_{3,2}$:

$$a_{1,1}, a_{3,2} \qquad a_{1,1}, a_{2,1}, a_{3,2}$$

$$a_{1,1}, a_{2,2}, a_{3,2} \qquad a_{1,1}, a_{3,1}, a_{1,2}, a_{3,2}.$$

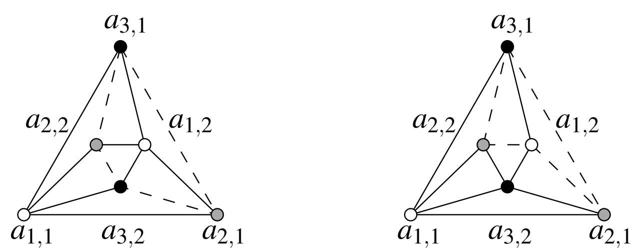


Figure 2.7: $K_{2,2,2}$ is 4-connected.

If G is a graph, then the *connectivity* of G is $\kappa(G) = \max\{k \mid G \text{ is } k\text{-connected}\}$, and the *Hadwiger number* of G is $h(G) = \max\{n \mid K_n \text{ is a minor of } G\}$. These two numbers will be important in the next section.

2.4 Graphs of Polytopes

If P is a polytope, then the *graph* of P, denoted $\mathscr{G}(P)$, is the graph with $V(\mathscr{G}(P)) = \operatorname{vert} P$, and $E(\mathscr{G}(P)) = \left\{ \boldsymbol{xy} \in \binom{\operatorname{vert} P}{2} \;\middle|\; \operatorname{conv}\{\boldsymbol{x}, \boldsymbol{y}\} \text{ is an edge of } P \right\}$. The following theorem is a special case of a more general theorem proven in Grünbaum (2003).

Theorem 7 (Grünbaum)**.** *If P is a d-polytope, then $\mathscr{G}(P)$ has a K_{d+1} minor.*

Proof. If $\dim P = -1$, then $\mathscr{G}(P)$ is empty, i.e., K_0. If $\dim P = 0$, then $\mathscr{G}(P)$ has one vertex and no edges , i.e., K_1. If $\dim P = 1$, then $\mathscr{G}(P)$ has two vertices and one edge , i.e., K_2. The proof now proceeds by induction on d.

Suppose for some $d \geq 2$ that K_d is a minor of every $(d-1)$-polytope. Let P be a d-polytope and F a facet of P. By the induction hypothesis, F has a K_d minor. Fix one such minor. Each vertex in this minor comes from a vertex of F. Let $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ be a set of vertices of F such that each $\boldsymbol{v}_i$ corresponds to a different vertex in the complete minor. Since $\dim P \geq 2$, each $\boldsymbol{v}_i$ is on an edge with another vertex $\boldsymbol{w}_i \in \operatorname{vert} P \setminus \operatorname{vert} F$ (it is possible that $\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$ has cardinality less than n). Now, in $\mathscr{G}(P)$ first perform a sequence of deletions and contractions to obtain the chosen K_d minor of $\mathscr{G}(F)$, and then contract all edges in this new graph which are not incident to any vertex in the copy of K_d. This leaves the K_d minor and one other vertex, which is adjacent to each vertex in the K_d minor. Thus a K_{d+1} minor of $\mathscr{G}(P)$ has been constructed. □

The following theorem is about how connected the graph of a d-polytope is, and was proved in Balinski (1961) using linear programming techniques.

Theorem 8 (Balinski 1961)**.** *If P is a d-polytope, then $\mathscr{G}(P)$ is d-connected.*

The converse is, however, not true, the 3-dimensional crosspolytope has graph $K_{2,2,2}$, and is 4-connected (see Example 6). It is also planar, and therefore does not contain a K_5 minor. Thus it is not the graph of any 4-dimensional polytope.

Theorem 9. *If F is a face of a polytope P, then $\mathscr{G}(F)$ is an induced subgraph of $\mathscr{G}(P)$.*

Proof. If $F = \varnothing$, then $\mathscr{G}(F)$ is the empty graph which is an induced subgraph of any graph. If $F = P$, then $\mathscr{G}(F) = \mathscr{G}(P)$ and any graph is an induced subgraph of itself.

Thus, suppose F is a proper face of P, and write

$$\varnothing \neq \operatorname{vert} F = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subsetneq \operatorname{vert} P$$

If $\boldsymbol{v}_i\boldsymbol{v}_j \in E(\mathscr{G}(P))$, then there is a hyperplane H such that $P \cap H = \operatorname{conv}\{\boldsymbol{v}_i, \boldsymbol{v}_j\}$ and $P \subseteq H^+$.

Hence,

$$\begin{aligned}
\operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\} &= \operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\} \cap H \\
&\subseteq \operatorname{conv}\left\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\right\} \cap H \\
&= F \cap H \\
&\subseteq P \cap H \\
&= \operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}
\end{aligned}$$

where the first equality follows since $\operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\} \subseteq H$. Therefore equality holds throughout and hence $F \cap H = \operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$. Whence H is a supporting hyperplane of $\operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$ for the polytope F. Thence $\boldsymbol{v}_i\boldsymbol{v}_j \in E(\mathscr{G}(F))$. □

If G is a graph, and P a d-polytope with $\mathscr{G}(P) = G$, then G is said to be *d-realizable*. A major open problem in the theory of polytopes is to give a complete characterization of the graphs which are d-realizable for a fixed d. One can similarly ask, for a fixed graph G, for which d is G d-realizable? The cyclic polytopes provide examples of graphs which are d-realizable for multiple values of d. Grünbaum asks in Grünbaum (2003) a question which, in the case of graphs becomes: if G is both d-realizable and d'-realizable, then is it d''-realizable for every d'' between d and d'.

The following theorem gives a complete characterization of which graphs are graphs of 3-polytopes. Proofs can be found in Steinitz (1922), Steinitz & Rademacher (1976), Grünbaum (2003), or Ziegler (1995).

Theorem 10 (Steinitz 1922)**.** *A graph G is the graph of a* 3*-polytope if and only if G is both planar and* 3*-connected.*

A complete characterization of the graphs of polytopes with dimension less than or equal to three can be found in Table 2.1.

Dimension	Characterization of Graph
-1	The only (-1)-dimensional polytope is the empty polytope, and it has the empty graph.
0	The only combinatorial type of 0-dimensional polytope is a single vertex, which has graph K_1.
1	The only combinatorial type of 1-dimensional polytope is a closed line segment, which has graph K_2.
2	The 2-polytopes are exactly the convex n-gons lying in a plane, and therefore have graphs which are cycle graphs. Further, every cycle graph is the graph of a 2-polytope.
3	Theorem 10 gives a complete characterization of the graphs of 3-polytopes as the planar 3-connected graphs.

Table 2.1: Characterizations of graphs of d-polytopes for $d \leq 3$.

Chapter 3

Gale Transformations and Diagrams

A Gale transformation is a method of encoding all of the combinatorial data for a d-dimensional set of points with cardinality $n \geq d+1$ into a space of dimension $n-d-1$. This is exceptionally useful if n is not much larger than d. Usually this process is only applied to the vertex set of a polytope.

3.1 Definition of Gale Transformation

Let P be a d-polytope in $\mathbb{R}^d$ with $\operatorname{vert} P = V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$ and consider the set of affine dependencies of V, that is,

$$\operatorname{dep} V = \left\{ (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n \;\middle|\; \sum_{i \in [n]} \lambda_i \boldsymbol{v}_i = 0 \text{ and } \sum_{i \in [n]} \lambda_i = 0 \right\}.$$

Note that $\operatorname{dep} V$ is an $(n-d-1)$-dimensional vector space. Let $\{\boldsymbol{a}_1, \boldsymbol{a}_2, \dots, \boldsymbol{a}_{n-d-1}\}$ be a basis of $\operatorname{dep} V$, and write $\boldsymbol{a}_i = (\alpha_{i,1}, \alpha_{i,2}, \dots, \alpha_{i,n})$ for $i \in [n-d-1]$. Now, let A be the $(n-d-1) \times n$ matrix whose ith row is $\boldsymbol{a}_i$ for $i \in [n-d-1]$, and let $\overline{\boldsymbol{v}}_j$ be the jth column of A for $j \in [n]$. Then the (multi)set $\overline{V} = \{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \dots, \overline{\boldsymbol{v}}_n\} \subseteq \mathbb{R}^{n-d-1}$ is a *Gale transformation* of V with $\overline{\boldsymbol{v}}_i$ corresponding to $\boldsymbol{v}_i$. Define as well, for a subset $X \subseteq V$ the (multi)set $\overline{X} = \left\{ \overline{\boldsymbol{v}} \in \overline{V} \;\middle|\; \boldsymbol{v} \in X \right\}$.

Note that there is not a unique Gale transformation for a given vertex set since a choice of basis was necessary. This does nothing to detract from the usefulness of the construction however. Also, note that the definition did not depend on the convexity of the polytope, only that $\dim \operatorname{aff} V = d$, thus Gale diagrams can be defined for point sets satisfying this condition

3.1.1 Computing a Gale Transformation

One method for actually computing a Gale transformation is as follows:

Let P be a d-polytope in $\mathbb{R}^d$ with $\operatorname{vert} P = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$ ordered such that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{d+1}$ are affinely independent. Then

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \left[I_{d+1} \,\middle|\, N \right]$$

where N is some $(d+1) \times (n-d-1)$ matrix. Setting

$$\left[-N^{\mathrm{T}} \,\middle|\, I_{n-d-1} \right] = \begin{bmatrix} \overline{\boldsymbol{v}}_1 & \overline{\boldsymbol{v}}_2 & \cdots & \overline{\boldsymbol{v}}_n \end{bmatrix},$$

yields a Gale transformation of $\operatorname{vert} P$, that is, the ordered (multi)set $\{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_n\}$.

3.2 A Preliminary Result

Theorem 11. *If P is a d-polytope with $V = \operatorname{vert}(P) = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^d$ and $F \subseteq V$, then* $\operatorname{conv} F$ *is a face of P if and only if* $\operatorname{conv}(V \setminus F) \cap \operatorname{aff}(F) = \varnothing$.

Proof. Suppose, without loss of generality, $F = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_k\}$ and $\operatorname{conv}(F)$ is a face of P. Further, let H be a supporting hyperplane of F, say, $H = \{\boldsymbol{w} \mid \langle \boldsymbol{h}, \boldsymbol{w} \rangle = t\}$ with $P \subseteq H^+$. Then the inclusions $\boldsymbol{v}_j \in P \cap \left(\mathbb{R}^d \setminus H\right)$ for all $j > k$ imply that $\langle \boldsymbol{h}, \boldsymbol{v}_j \rangle > t$ for $j > k$. Let $\boldsymbol{x} \in \operatorname{conv}(V \setminus F)$ with

$$\boldsymbol{x} = \sum_{j \in [n] \setminus [k]} \alpha_j \boldsymbol{v}_j, \qquad \sum_{j \in [n] \setminus [k]} \alpha_j = 1, \qquad \alpha_j \geq 0 \text{ for all } j \in [n] \setminus [k].$$

Then

$$\langle \boldsymbol{h}, \boldsymbol{x} \rangle = \sum_{j \in [n] \setminus [k]} \alpha_j \langle \boldsymbol{h}, \boldsymbol{v}_j \rangle > \sum_{j \in [n] \setminus [k]} \alpha_j t = t.$$

Thus $\boldsymbol{x} \in H^{(+)} = H^+ \setminus H$. Therefore, the inclusion $\mathrm{aff}(F) \subseteq H$ implies $\mathrm{conv}(V \setminus F) \cap \mathrm{aff}(F) = \varnothing$.

On the other hand, suppose $\mathrm{conv}(V \setminus F) \cap \mathrm{aff}(F) = \varnothing$, and let $\boldsymbol{y}_0 \in \mathrm{conv}(V \setminus F)$. Then

$$\inf_{\boldsymbol{x} \in \mathrm{aff}(F)} \|\boldsymbol{x} - \boldsymbol{y}_0\|$$

is attained at $\boldsymbol{x}_0$ when $\mathrm{aff}\{\boldsymbol{x}_0, \boldsymbol{y}_0\}$ is perpendicular to $\mathrm{aff}(F)$. Let H be the hyperplane through $\boldsymbol{x}_0$ normal to $\boldsymbol{x}_0 - \boldsymbol{y}_0$. Then

1. $\mathrm{conv}(V \setminus F) \cap H = \varnothing$ and

2. $\mathrm{aff}(H) \subseteq H$.

Thus $H \cap P = \mathrm{conv}\, F$ is a face of P. $\square$

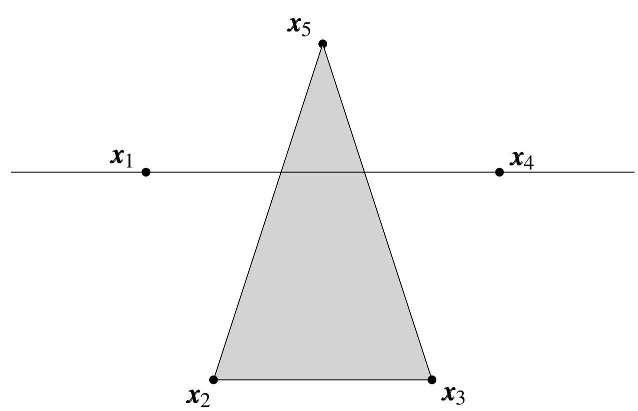


Figure 3.1: The set $\mathrm{conv}\{\boldsymbol{x}_1, \boldsymbol{x}_4\}$ is not a face of $\mathrm{conv}\{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5\}$

3.3 What is it good for?

It turns out that it is more convenient to talk about the complement of a face of a polytope than the face itself. This section follows McMullen & Shephard (1971).

Definition. Let P be a polytope with vertex set V. A subset X of the vertices is called a *coface* if $\operatorname{conv}(V \setminus X)$ is a face of P.

Definition. Let X be a set of points in $\mathbb{R}^d$, and $\boldsymbol{x}$ be any point of $\mathbb{R}^d$. Then X is said to *capture* $\boldsymbol{x}$ if $\boldsymbol{x} \in \operatorname{relint}\operatorname{conv} X$, that is, $\boldsymbol{x}$ is in the relative interior of the convex hull of X.

Theorem 12. *Let P be a polytope. Then $X \subseteq \operatorname{vert} P$ is a coface of P if and only if either $\overline{X}$ captures the origin or $X = \varnothing$.*

Proof. Let P be a d-polytope with $V = \operatorname{vert} P = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \subseteq \mathbb{R}^d$.

If $X = \varnothing$, then $\operatorname{vert}(P) \setminus X = \operatorname{vert} P$, and $P = \operatorname{conv}\operatorname{vert} P$ is a face of P. Thus, suppose $X \neq \varnothing$.

Let $I = \{i_1, i_2, \dots, i_r\} \subseteq [n]$, $J = [n] \setminus I$, and $X = \{\boldsymbol{v}_i \mid i \in I\}$. Also, let

$$\begin{bmatrix} \overline{\boldsymbol{v}}_1 \\ \overline{\boldsymbol{v}}_2 \\ \vdots \\ \overline{\boldsymbol{v}}_n \end{bmatrix} = \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_{n-d-1} \end{bmatrix}$$

where the $\overline{\boldsymbol{v}}_i$'s are regarded as row vectors, and the $\boldsymbol{w}_i$'s are regarded as column vectors.

Suppose $\mathbf{0} \notin \operatorname{relint}\operatorname{conv} \overline{X}$. Then there is some hyperplane $H = \{\boldsymbol{y} \mid \langle \boldsymbol{\mu}, \boldsymbol{y} \rangle = 0\} \subseteq \mathbb{R}^{n-d-1}$ with $\overline{\boldsymbol{v}}_i \in H^+$ for each $i \in I$ and there is some $i_0 \in I$ with $\boldsymbol{v}_{i_0} \in H^{(+)}$.

For each $i \in [n]$, let $\alpha_i = \langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_i \rangle$ and $S = \sum_{i \in I} \alpha_i$. The inequalities $\langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_i \rangle \geq 0$ for $i \in I$ and $\langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_{i_0} \rangle > 0$ imply that $S > 0$. Therefore, let $\lambda_i = \alpha_i / S$, and $\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}^{\mathrm{T}}$. Note that

$\boldsymbol{\lambda} \in \operatorname{span}\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n-d-1}\} = \operatorname{dep}(V)$; indeed:

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} = \frac{1}{S} \begin{bmatrix} \langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_1 \rangle \\ \langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_2 \rangle \\ \vdots \\ \langle \boldsymbol{\mu}, \overline{\boldsymbol{v}}_n \rangle \end{bmatrix} = \frac{1}{S} \begin{bmatrix} \overline{\boldsymbol{v}}_1 \\ \overline{\boldsymbol{v}}_2 \\ \vdots \\ \overline{\boldsymbol{v}}_n \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-d-1} \end{bmatrix} = \frac{1}{S} \begin{bmatrix} \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_{n-d-1} \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{n-d-1} \end{bmatrix}$$
$$= \frac{1}{S} \sum_{i \in [n-d-1]} \mu_i \boldsymbol{w}_i.$$

Ergo, $\sum_{i \in [n]} \lambda_i \boldsymbol{v}_i = \boldsymbol{0}$ and $\sum_{i \in [n]} \lambda_i = 0$. Set

$$\boldsymbol{z} = \sum_{i \in I} \lambda_i \boldsymbol{v}_i = \sum_{i \in J} (-\lambda_i) \boldsymbol{v}_i$$

and note that

$$\sum_{i \in I} \lambda_i = \sum_{i \in I} \frac{\alpha_i}{S} = 1$$

and $\lambda_i \geq 0$ for $i \in I$. Hence, $\boldsymbol{z} \in \operatorname{conv} X$. Further, $\sum_{i \in J} (-\lambda_i) = 1$ as well. Whence $\boldsymbol{z} \in \operatorname{aff}(V \setminus X)$. Thence, $\operatorname{aff}(V \setminus X) \cap \operatorname{conv}(X) \neq \varnothing$. Therefore, by Theorem 11 from Section 3.2, X is not a coface of P.

The proof of the converse can be found in Grünbaum (2003), McMullen & Shephard (1971), or Thomas (2006). □

If P is a d-simplex in $\mathbb{R}^d$, then $|\operatorname{vert} P| = d+1$, and thus a Gale transformation of $\operatorname{vert} P$ is a subset of $\mathbb{R}^0 = \{0\}$. Thus the only Gale transformation of a d-simplex is the multiset $\{0, 0, \ldots, 0\}$ of $d+1$ zeroes.

Corollary 1. *If $\overline{V}$ is a Gale transformation of the vertices $V = \operatorname{vert} P$ of a d-polytope $P \subseteq \mathbb{R}^d$, and H is a hyperplane in $\mathbb{R}^{n-d-1}$ which contains $\boldsymbol{0}$, then either $\left|\overline{V} \cap H^{(+)}\right| \geq 2$, or P is a simplex.*

Proof. Let P be a d-polytope in $\mathbb{R}^d$ which is not a simplex, and $V = \operatorname{vert} P = \{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2, \ldots, \overline{\boldsymbol{v}}_n\}$ be its vertex set. Then each $\boldsymbol{v}_i$ is a face of P, so that $V_i = V \setminus \{\boldsymbol{v}_i\}$ is a coface of P. Hence, $\overline{V}_i$ captures the origin. If there were some hyperplane H containing $\mathbf{0}$ such that $\overline{V} \cap H^{(+)} = \{\overline{\boldsymbol{v}}_i\}$, then this could not happen. $\square$

If P is a polytope with vertex set $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, then a pair of vertices $\boldsymbol{v}_i, \boldsymbol{v}_j$ is a *nonedge* of P if $\operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$ is not a face of P.

Corollary 2. *If $\overline{V}$ is a Gale transformation of the vertices $V = \operatorname{vert} P$ of a d-polytope $P \subseteq \mathbb{R}^d$, then a pair of vertices $\boldsymbol{v}_i, \boldsymbol{v}_j$ forms a nonedge of P if and only if there is some hyperplane H such that $\overline{V} \cap H^{(+)} = \left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$*

Proof. **PUT IN PROOF!!** $\square$

The next theorem gives two conditions which, together, guarantee that a set of n points in $\mathbb{R}^{n-d-1}$ is a Gale transformation of the vertex set of some polytope.

Theorem 13. *Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ be a set of points in $\mathbb{R}^k$ such that*

1. *$\sum_{i \in [n]} \boldsymbol{x}_i = \mathbf{0}$ and*
2. *for all hyperplanes H containing $\mathbf{0}$ each open half-space contains at least two points of X.*

Then X is a Gale transformation of the vertex set of some d-polytope.

Proof. Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ be a set of points in $\mathbb{R}^k$ satisfying the two conditions. Also, let

$$A = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \end{bmatrix}$$

be the matrix whose ith column is $\boldsymbol{x}_i$.

The second condition guarantees that X cannot be contained in any hyperplane containing the origin. Thus $\dim \operatorname{span} X = k$, and therefore by the Rank-Nullity Theorem of Linear Algebra the dimension of the kernel of A is $\dim \ker A = n - k$.

Note that by the first condition $\mathbf{1}_n \in \ker A$, and hence the set $\{\mathbf{1}_n\}$ can be extended to a basis of $\ker A$, say $\{\mathbf{1}_n, \widetilde{\boldsymbol{y}}_1, \widetilde{\boldsymbol{y}}_2, \ldots, \widetilde{\boldsymbol{y}}_{n-k-1}\}$. Let

$$B = \begin{bmatrix} \mathbf{1}_n & \widetilde{\boldsymbol{y}}_1 & \widetilde{\boldsymbol{y}}_2 & \cdots & \widetilde{\boldsymbol{y}}_{n-k-1} \end{bmatrix} = \begin{bmatrix} \widetilde{\boldsymbol{x}}_1 \\ \widetilde{\boldsymbol{x}}_2 \\ \vdots \\ \widetilde{\boldsymbol{x}}_n \end{bmatrix}.$$

Then, by definition, $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ is a Gale transformation of the set $\widetilde{X} = \{\widetilde{\boldsymbol{x}}_1, \widetilde{\boldsymbol{x}}_2, \ldots, \widetilde{\boldsymbol{x}}_n\}$. Now, by the second condition, and the previous theorem each $\widetilde{\boldsymbol{x}}_i$ is a vertex of $\operatorname{conv}\widetilde{X}$, and therefore $\widetilde{X}$ is the vertex set of some polytope. $\square$

3.4 Gale Diagrams

If $\overline{V}$ is a Gale diagram of the vertex set V of some polytope, and only the face lattice of the polytope is being considered, then the condition $\sum_{\overline{\boldsymbol{v}} \in \overline{V}} \overline{\boldsymbol{v}} = \mathbf{0}$ is superfluous. In light of this, the following definitions are made.

Definition. Suppose $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^k$ and $Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_n\} \subseteq \mathbb{R}^k$ both capture $\mathbf{0}$. Then X and Y are called *consubstantial* if for each $J \subseteq [n]$ the sets $\{\boldsymbol{x}_j \mid j \in J\}$ and $\{\boldsymbol{y}_j \mid j \in J\}$ either both capture, or both do not capture $\mathbf{0}$.

Consubstantiality is an equivalence relation, and for a fixed polytope P the Gale transformations of $\operatorname{vert} P$ all lie in the same equivalence class. However, if P is not a simplex, then there are sets consubstantial to a Gale transformation of $\operatorname{vert} P$ which are not themselves Gale transformations.

Definition. Suppose $P \subseteq \mathbb{R}^d$ is a d-polytope with n vertices, and G is a Gale transformation of $\operatorname{vert} P$. If $X \subseteq \mathbb{R}^{n-d-1}$ and G are consubstantial, then X is called a *Gale diagram* of P, and the equivalence class of all Gale diagrams of P is denoted $\operatorname{gale}(P)$.

The following theorem shows the usefulness of Gale diagrams.

Theorem 14. *Two polytopes P, Q are combinatorially equivalent if and only if* $\operatorname{gale}(P) = \operatorname{gale}(Q)$.

Proof. Suppose P and Q are combinatorially equivalent, that is, there is an isomorphism of face lattices $\varphi\colon \mathscr{F}(P) \to \mathscr{F}(Q)$; therefore $|\operatorname{vert} P| = |\operatorname{vert} Q|$. Write $\operatorname{vert} P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ and $\operatorname{vert} Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\}$ ordered such that $\varphi(\boldsymbol{p}_i) = \boldsymbol{q}_i$ for each $i \in [n]$.

The set $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \in \operatorname{gale}(P)$ is a Gale diagram of P if and only if for each $I \subseteq [n]$ such that $\{\boldsymbol{g}_i \mid i \in I\}$ captures the $\mathbf{0}$ the set $\{\boldsymbol{p}_i \mid i \in I\}$ is a coface of P. This happens if and only if $\{\boldsymbol{q}_i \mid i \in I\}$ is a coface of Q (via the isomorphism φ), which happens if and only if $G \in \operatorname{gale}(Q)$. Thus $\operatorname{gale}(Q) = \operatorname{gale}(P)$.

On the other hand, suppose that $\operatorname{gale}(P) = \operatorname{gale}(Q)$ (and therefore that $|\operatorname{vert} P| = |\operatorname{vert} Q|$). Also, let $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \in \operatorname{gale}(P) = \operatorname{gale}(Q)$, and order the sets $\operatorname{vert} P = \{\boldsymbol{p}_1, \boldsymbol{p}_2, \ldots, \boldsymbol{p}_n\}$ and $\operatorname{vert} Q = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \ldots, \boldsymbol{q}_n\}$ such that $\boldsymbol{g}_i$ corresponds to both $\boldsymbol{p}_i$ and $\boldsymbol{q}_i$ for each $i \in [n]$.

Define $\vartheta\colon 2^{\operatorname{vert} P} \to 2^{\operatorname{vert} Q}$ (where 2^X denotes the power set of the set X) by, for $i \subseteq [n]$, $\vartheta(\{\boldsymbol{p}_i \mid i \in I\}) = \{\boldsymbol{q}_i \mid i \in I\}$. Then $\{\boldsymbol{p}_i \mid i \in I\}$ is a face of P if and only if $\{\boldsymbol{g}_i \mid i \in [n] \setminus I\}$ captures $\mathbf{0}$. This happens if and only if $\vartheta(\{\boldsymbol{p}_i \mid i \in I\}) = \{\boldsymbol{q}_i \mid i \in I\}$ is a face of Q. Hence ϑ is an invertible map which sends faces of P to faces of Q. Furthermore, $\left\{\boldsymbol{p}_j \;\middle|\; j \in J\right\} \subseteq \{\boldsymbol{p}_i \mid i \in I\}$ if and only if $J \subseteq I$ if and only if

$$\vartheta(\left\{\boldsymbol{p}_j \;\middle|\; j \in J\right\}) = \left\{\boldsymbol{q}_j \;\middle|\; j \in J\right\} \subseteq \{\boldsymbol{q}_i \mid i \in I\} = \vartheta(\{\boldsymbol{p}_i \mid i \in I\}),$$

whence ϑ is order preserving. Thence, ϑ induces an isomorphism $\varphi\colon \mathscr{F}(P) \to \mathscr{F}(Q)$. $\square$

The following theorem follows immediately from Theorem 13. It is the condition that the program in Appendix A uses to check whether or not a set of points is a Gale diagram of some polytope.

Theorem 15. *If $n \geq 0$ and $d \geq -1$ are integers such that $n \geq d+1$, then a set of points $G = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\} \subseteq \mathbb{R}^{n-d-1}$ is a Gale diagram of some d-polytope P with $|\operatorname{vert} P| = n$ if and only if for each hyperplane $H \subseteq \mathbb{R}^{n-d-1}$ with $\mathbf{0} \in H$ the cardinality $\left|G \cap H^{(+)}\right| \geq 2$.*

In practice, one "only" needs to check the hyperplanes through the origin which are the span of $n-d-2$ points in G, that is, at most $\binom{n}{n-d-2} = \binom{n}{d+2}$ distinct hyperplanes. Further, since both orientations of a hyperplane need to be checked, after computing $\left|G \cap H^{(+)}\right|$, one can immediately compute $\left|G \cap H^{(-)}\right|$.

Theorem 16. *If* $\Gamma \in \operatorname{gale}(P)$*, then* P *is a pyramid with apex* $\boldsymbol{x}$ *if and only if* $\overline{\boldsymbol{x}} = \boldsymbol{0}$.

Proof. The point $\overline{\boldsymbol{x}} = \boldsymbol{0}$ if and only if $\operatorname{relint}\operatorname{conv}\{\overline{\boldsymbol{x}}\} = \{\boldsymbol{0}\}$ if and only if $\operatorname{conv}(\operatorname{vert}(P) \setminus \{\boldsymbol{x}\})$ is a face of P. Hence P is a pyramid with apex $\boldsymbol{x}$.

$\square$

Suppose $\overline{V} = \{\overline{\boldsymbol{x}}_1, \overline{\boldsymbol{x}}_2, \ldots, \overline{\boldsymbol{x}}_n\}$ is a Gale transformation of a d-polytope $P \subseteq \mathbb{R}^d$ with vertex set $\operatorname{vert} P = V = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\}$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\} \subseteq \mathbb{R}$ is a set of positive real numbers. Then $B = \{\alpha_1\overline{\boldsymbol{x}}_1, \alpha_2\overline{\boldsymbol{x}}_2, \ldots, \alpha_n\overline{\boldsymbol{x}}_n\}$ is a Gale diagram of P. Similarly, if $\Gamma = \{\boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_n\}$ is a Gale diagram of P, then there are positive real numbers $\beta_1, \beta_2, \ldots, \beta_n$ such that $\sum_{i \in [n]} \beta_i \boldsymbol{g}_i = \boldsymbol{0}$. Therefore, $H = \{\beta_1\boldsymbol{g}_1, \beta_2\boldsymbol{g}_2, \ldots, \beta_n\boldsymbol{g}_n\} \in \operatorname{gale}(P)$ is a Gale transformation of P. Hence one can easily move between Gale transformations and Gale diagrams if necessary.

3.5 Standard Gale Diagrams

Gale diagrams (like most things) are easier to work with when they lie in a nice subset of the ambient space[1]. Points corresponding to apices of pyramids cannot be moved in a Gale diagram (Theorem 16), however each other point in a Gale diagram can be moved along the ray from the origin passing through that point (and possibly through a larger set), thus a natural construction is to performing the rescaling

$$\boldsymbol{g}_i \longmapsto \begin{cases} \boldsymbol{0} & \text{if } \boldsymbol{g}_i = \boldsymbol{0} \\ \boldsymbol{g}_i / \|\boldsymbol{g}_i\| & \text{if } \boldsymbol{g}_i \neq \boldsymbol{0} \end{cases}$$

[1]Here, 'nice' has the completely circular meaning of a subset which makes a Gale diagram easy to work with.

which places all points corresponding to non-apices on the unit sphere $\mathbb{S}^{n-d-2} \subseteq \mathbb{R}^{n-d-1}$. A Gale diagram which is a subset of $\mathbb{S}^{n-d-2} \cup \{\mathbf{0}\}$ is called a *standard* Gale diagram. Standard Gale diagrams are useful aids in proving theorems like:

Theorem 17. *There are* $\lfloor d^2/4 \rfloor$ *distinct combinatorial types of* d*-polytopes with* $d+2$ *vertices,* $\lfloor d/2 \rfloor$ *of which are simplicial.*

Proof. The following is only a proof of the first part of the theorem.

Let $n(d)$ be the number of combinatorial types of d-polytopes with $d+2$ vertices.

A d-dimensional polytope with $d+2$ vertices has a 1-dimensional Gale diagram, and is therefore completely characterized by two numbers; the number of points on each side of the origin. The number of points at the origin is $d+2$ minus the sum of these two numbers. Thus $n(d)$ can be determined by counting the number of unordered pairs $\{a,b\}$ with $2 \leq a \leq b \leq \lfloor d/2 \rfloor$. This number is $\lfloor d/2 \rfloor \lceil d/2 \rceil = \lfloor d^2/4 \rfloor$. $\square$

The proof of the second part of the theorem follows from the the fact that if Γ is a Gale diagram of a d-polytope P with n vertices, then P is a simplicial poltope if and only if for every hyperplane $H \in \mathbb{R}^{n-d-1}$ containing $\mathbf{0}$ the following holds:

$$\mathbf{0} \notin \operatorname{relint} \operatorname{conv}(\Gamma \cap H).$$

The proof of which can be found in McMullen & Shephard (1971).

3.6 Examples

3.6.1 Crosspolytopes

Let $\boldsymbol{e}_i$ be the ith unit basis vector in $\mathbb{R}^d$ and consider $X_d = \operatorname{conv}\{\pm\boldsymbol{e}_1, \pm\boldsymbol{e}_2, \ldots, \pm\boldsymbol{e}_d\}$, the standard d-crosspolytope. Computing a Gale transform of X_d requires finding a basis for the null-space of the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_{d-1} & \boldsymbol{e}_d & -\boldsymbol{e}_d & -\boldsymbol{e}_{d-1} & \cdots & -\boldsymbol{e}_1 \end{bmatrix}.$$

First, note that this is a $(d+1)\times(2d)$ matrix with rank $d+1$. Thus, the null-space has dimension $d-1$. The set $\{\boldsymbol{e}_i + \boldsymbol{e}_{2d+1-i} - \boldsymbol{e}_1 - \boldsymbol{e}_{2d} \mid i \in [d] \setminus \{1\}\}$ of $d-1$ vectors in $\mathbb{R}^{2d}$ is linearly independent, and is a subset of the null-space. Thus this set forms a basis. Forming the matrix which has these vectors for its columns, and extracting the row vectors yields the Gale transformation $\{-\mathbf{1}, \boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_{d-2}, \boldsymbol{e}_{d-1}\boldsymbol{e}_{d-1}, \boldsymbol{e}_{d-2}, \ldots, \boldsymbol{e}_2, \boldsymbol{e}_1, -\mathbf{1}\} \subseteq \mathbb{R}^{d-1}$. See Figure 3.2 for an illustration of the 3-dimensional case.

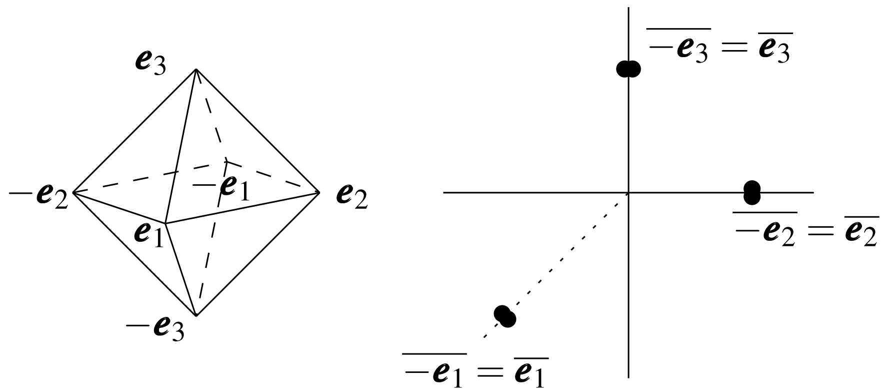


Figure 3.2: The polytope X_3 and a Gale transformation of its vertices.

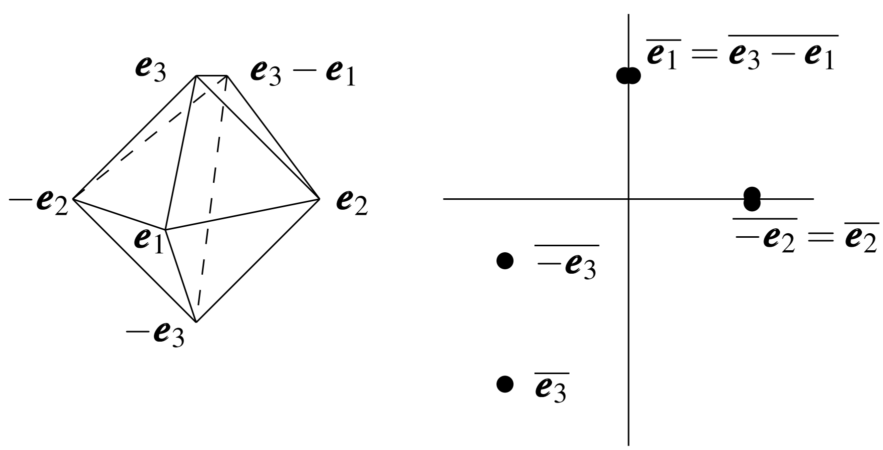


Figure 3.3: A 3-dimensional crosspolytope and a Gale transformation of its vertices.

Thus a Gale diagram of the d-dimensional crosspolytope can be given by the vertices of a $(d-1)$-simplex with the origin in its relative interior where each point occurs twice and corresponds to a pair of vertices which are not joined by an edge. These are not all of the Gale diagrams of crosspolytopes however; any doubled point can be split into two points as long as the two points are not moved "too far" apart [2] (compare Figures 3.2 and 3.3).

3.6.2 Prisms over Simplices

A prism over a d-simplex is a $(d+1)$-dimensional polytope with $2d+2$ vertices. It thus has a d-dimensional Gale diagram.

Let P be the prism over a d-simplex with vertices

$$\begin{bmatrix} \boldsymbol{e}_1 \\ 1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{e}_2 \\ 1 \end{bmatrix}, \dots \begin{bmatrix} \boldsymbol{e}_d \\ 1 \end{bmatrix}, \begin{bmatrix} -\mathbf{1} \\ 1 \end{bmatrix}, \begin{bmatrix} -\mathbf{1} \\ -1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{e}_d \\ -1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{e}_{d-1} \\ -1 \end{bmatrix}, \dots \begin{bmatrix} \boldsymbol{e}_1 \\ -1 \end{bmatrix},$$

where $\boldsymbol{e}_i$ is the ith standard basis vector in $\mathbb{R}^d$.

Then the vectors

$$\begin{bmatrix} -\boldsymbol{e}_i^{\mathrm{T}} & 1 & -1 & \boldsymbol{e}_i^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}$$

form a basis for the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \boldsymbol{e}_1 & \boldsymbol{e}_2 & \cdots & \boldsymbol{e}_d & -\mathbf{1} & -\mathbf{1} & \boldsymbol{e}_d & \boldsymbol{e}_{d-1} & \cdots & \boldsymbol{e}_1 \\ 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 \end{bmatrix}$$

so that the set

$$\{-\boldsymbol{e}_1, -\boldsymbol{e}_2, \dots, \mathbf{1}, -\mathbf{1}, \boldsymbol{e}_d, \boldsymbol{e}_{d-1}, \dots, \boldsymbol{e}_1\} \subseteq \mathbb{R}^d$$

[2] Here, "too far" has nothing to do with distance. It is entirely possible that one point can be moved freely within an unbounded region if all other points are held fixed, e.g., in Figure 3.3 the point $\overline{\boldsymbol{e}_3}$ could be moved anywhere within the open third quadrant without changing the combinatorial type of the polytope.

is a Gale transform of P.

In general a Gale diagram of P is given by the vertices of a simplex with the origin in its relative interior, along with the negatives of these points. The pairs of points which are antipodal are points of the polytope which correspond to the same point in the original simplex.

Chapter 4

Oriented Matroids

What is an oriented matroid?

In this chapter only oriented matroids which arise from point configurations in a Euclidean space will be considered.

This section will closely follow Ziegler (1995) section 6.3.

4.1 Title goes here

Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n\} \subseteq \mathbb{R}^n$, and suppose that $\operatorname{aff} X = \mathbb{R}^n$. Denote by $[X]$ the matrix whose ith column is $\boldsymbol{x}_i$. Consider the set of affine dependencies of X, that is:

$$\operatorname{dep} X = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{R}^n \;\middle|\; [X]\boldsymbol{\lambda} = \mathbf{0} \text{ and } \sum_{i \in [n]} \lambda_i = 0 \right\}.$$

Each $\boldsymbol{\lambda} \in \operatorname{dep} X \setminus \{\mathbf{0}\}$ corresponds to a point $\widetilde{\boldsymbol{\lambda}}$ which lies in $\operatorname{conv} X$ in the following way:

Let $\boldsymbol{\lambda} \in \operatorname{dep}(X) \setminus \{\mathbf{0}\}$. Define the positive and negative parts of $\boldsymbol{\lambda}$ to be the sets of coordinates for which $\boldsymbol{\lambda}$ is positive or negative respectively:

$$\mathrm{P}(\boldsymbol{\lambda}) = \{i \mid \lambda_i > 0\} \qquad\qquad \mathrm{N}(\boldsymbol{\lambda}) = \{i \mid \lambda_i < 0\}.$$

Now define $\lambda_+ = \sum_{i \in \mathrm{P}(\boldsymbol{\lambda})} \lambda_i$ and $\lambda_- = \sum_{i \in \mathrm{N}(\boldsymbol{\lambda})} \lambda_i$. The equality $\sum \lambda_i = 0$ implies $\lambda_+ = -\lambda_-$; further, $\boldsymbol{\lambda} \neq \mathbf{0}$ implies $\lambda_+ \neq 0$. Finally, the equality $[X]\boldsymbol{\lambda} = \mathbf{0}$ yields that

$$\sum_{i \in P(\boldsymbol{\lambda})} \frac{\lambda_i}{\lambda_+} \boldsymbol{x}_i = - \sum_{i \in N(\boldsymbol{\lambda})} \frac{\lambda_i}{\lambda_+} \boldsymbol{x}_i.$$

Call this point $\widetilde{\boldsymbol{\lambda}}$. The previous paragraph shows that

$$\widetilde{\boldsymbol{\lambda}} \in \operatorname{conv}\{\boldsymbol{x}_i \mid i \in P(\boldsymbol{\lambda})\} \cap \operatorname{conv}\{\boldsymbol{x}_i \mid i \in N(\boldsymbol{\lambda})\}.$$

That is, $\widetilde{\boldsymbol{\lambda}}$ is a point in a Radon partition of X. Conversely each point in each Radon partition of X corresponds to at least one affine dependence of X.

(((((Say this more eloquently))))) Stuff about only signed vectors mattering in the Radon partitions.

If $\boldsymbol{x}$ is a vector in some Euclidean space $\mathbb{R}^m$, then define the signed vector of $\boldsymbol{x}$ to be the vector $\operatorname{sign}(\boldsymbol{x}) \in \{-, 0, +\}^m$ with coordinates

$$\operatorname{sign}(\boldsymbol{x})_i = \begin{cases} - & \text{if } x_i < 0 \\ 0 & \text{if } x_i = 0 \\ + & \text{if } x_i > 0. \end{cases}$$

Denote the collection of all signed vectors of affine dependencies of X by $\mathscr{V}(X)$. That is,

$$\mathscr{V}(X) = \{\operatorname{sign}(\boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in \operatorname{dep}(X)\}.$$

If $\boldsymbol{v}$ is a sign vector, denote by $-\boldsymbol{v}$ the sign vector given by: $\operatorname{sign}(-\boldsymbol{v})_i = +$, if $\boldsymbol{v}_i < 0$; $\operatorname{sign}(-\boldsymbol{v})_i = 0$, if $\boldsymbol{v}_i = 0$; $\operatorname{sign}(-\boldsymbol{v})_i = -$, if $\boldsymbol{v}_i > 0$. The support of an n-dimensional sign vector $\boldsymbol{v}$ is the set $\operatorname{supp} \boldsymbol{v} = \{i \in [n] \mid \boldsymbol{v}_i \neq 0\}$. If V is a set of sign vectors, then $\operatorname{supp} V = \cup_{\boldsymbol{v} \in V} \operatorname{supp} \boldsymbol{v}$. A signed vector $\boldsymbol{v} \in V$ is said to be of *minimal support* if for each $\boldsymbol{w} \in V$ the inclusion $\operatorname{supp} \boldsymbol{w} \subseteq \operatorname{supp} \boldsymbol{v}$

implies $\operatorname{supp} \boldsymbol{w} = \operatorname{supp} \boldsymbol{v}$. A signed vector $\boldsymbol{v} \in \mathscr{V}(X)$ of minimal support is called a *circuit* of X. The set of all circuits of X is denoted by

$$\mathscr{C}(X) = \{\boldsymbol{v} \in \mathscr{V}(X) \mid \boldsymbol{v} \text{ is of minimal support}\}.$$

Also define the set of *signed covectors* to be

$$\mathscr{V}^*(X) = \left\{ \operatorname{sign}(\boldsymbol{c}[X] - z\mathbf{1}) \;\middle|\; \boldsymbol{c} \in (\mathbb{R}^d)^*, z \in \mathbb{R} \right\}$$

where $\mathbf{1}$ is the vector whose entries are each 1. As well as the *signed cocircuits*

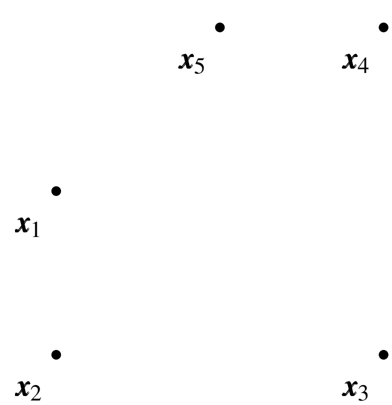
$$\mathscr{C}^*(X) = \{\boldsymbol{v} \in \mathscr{V}^*(X) \mid \boldsymbol{v} \text{ is of minimal support}\}.$$

4.2 Example

Let $X = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{x}_4, \boldsymbol{x}_5\} \subseteq \mathbb{R}^2$ where

$$\boldsymbol{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \qquad \boldsymbol{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \boldsymbol{x}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \qquad \boldsymbol{x}_5 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

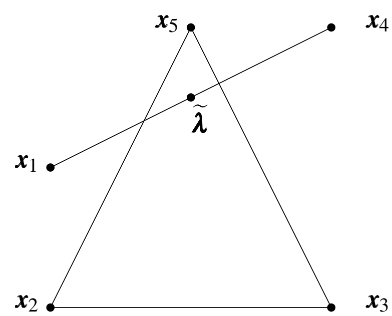
and consider the affine dependence $\boldsymbol{\lambda} = (4, -1, -1, 4, -6)^{\mathrm{T}}$. In this case, $P(\boldsymbol{\lambda}) = \{1, 4\}$, $N(\boldsymbol{\lambda}) =$



$\{2,3,5\}$ and $\lambda_+ = 8$. Hence

$$\widetilde{\boldsymbol{\lambda}} = \frac{\lambda_1 x_1 + \lambda_4 x_4}{\lambda_+} = \begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$$

which is a point along the line segment from $\boldsymbol{x}_1$ to $\boldsymbol{x}_4$ as well as a point inside the triangle with vertices $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_5$.



Further, $\operatorname{sign}(\boldsymbol{\lambda}) = \begin{bmatrix} + & - & - & + & - \end{bmatrix}^{\mathrm{T}}$ and

$$\begin{aligned}\mathscr{V}(X) = \Big\{ &\pm\begin{bmatrix} 0 & + & - & + & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & - & + & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & 0 & + & - \end{bmatrix}^{\mathrm{T}}, \\ &\pm\begin{bmatrix} + & - & + & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & + & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & + & - & + \end{bmatrix}^{\mathrm{T}}, \\ &\pm\begin{bmatrix} + & - & + & 0 & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & + & + & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & + & - \end{bmatrix}^{\mathrm{T}}, \\ &\pm\begin{bmatrix} + & + & - & + & - \end{bmatrix}^{\mathrm{T}} \Big\}\end{aligned}$$

$$\begin{aligned}\mathscr{C}(X) = \Big\{ &\pm\begin{bmatrix} 0 & + & - & + & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & 0 & + & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & + & - & 0 \end{bmatrix}^{\mathrm{T}}, \\ &\pm\begin{bmatrix} + & - & + & 0 & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & + & - \end{bmatrix}^{\mathrm{T}} \Big\}\end{aligned}$$

$$
\begin{aligned}
\mathscr{V}^*(X) = \Big\{ & \pm\begin{bmatrix} 0 & 0 & + & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & - & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & 0 & - & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} 0 & + & + & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & + & 0 & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & + & + & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} 0 & + & + & + & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & + & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & - & - & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & - & - & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & - & - & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & - & 0 & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & - & - & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & 0 & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & - & + & + & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & 0 & - & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & - & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & 0 & - & 0 & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & 0 & + & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & 0 & + & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & - & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & - & - & 0 \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & - & - & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & - & 0 & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & - & + & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & 0 & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & 0 & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & 0 & - & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & 0 & 0 & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & 0 & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & - & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & + & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & - & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & 0 & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & + & 0 & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & 0 & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & + & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & + & + & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & + & + & + \end{bmatrix}^{\mathrm{T}} \Big\}
\end{aligned}
$$

$$
\begin{aligned}
\mathscr{C}^*(X) = \Big\{ & \pm\begin{bmatrix} 0 & 0 & + & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & 0 & - & - \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} 0 & + & + & 0 & - \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} 0 & + & + & + & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & 0 & - & 0 & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & 0 & 0 & + & + \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & 0 & - & 0 \end{bmatrix}^{\mathrm{T}}, \pm\begin{bmatrix} + & + & 0 & 0 & + \end{bmatrix}^{\mathrm{T}}, \\
& \pm\begin{bmatrix} + & + & + & 0 & 0 \end{bmatrix}^{\mathrm{T}} \Big\}
\end{aligned}
$$

4.3 The Dual of an Oriented Matroid

Chapter 5

Modifying Gale Diagrams

This chapter explores certain operations on Gale diagrams and how these operations affect the polytopes.

5.1 Direct Sums and Free Joins

Theorem 18. *If P and Q are polytopes, $\Gamma \in \mathrm{gale}(P)$, and $\Lambda \in \mathrm{gale}(Q)$, then*

$$\Gamma \oplus \Lambda \in \mathrm{gale}(P \vee Q)$$

and

$$\Gamma \vee \Lambda \in \mathrm{gale}(P \oplus Q).$$

Proof. Suppose the following:

$$\begin{aligned} &\dim P = d_1; & &\dim Q = d_2; \\ &P \subseteq \mathbb{R}^{d_1}; & &Q \subseteq \mathbb{R}^{d_2}; \\ &\operatorname{vert} P = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}; & &\operatorname{vert} Q = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}; \\ &\mathbf{0}_{d_1} \in \operatorname{relint} P; & &\mathbf{0}_{d_2} \in \operatorname{relint} Q; \\ &\dim \operatorname{aff}\left\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_{d_1+1}\right\} = d_1; & &\dim \operatorname{aff}\left\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{d_2+1}\right\} = d_2; \end{aligned}$$

and

$$\operatorname{rref}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix} = \left[I_{d_1+1} \,\middle|\, A \right]; \qquad \operatorname{rref}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_m \end{bmatrix} = \left[I_{d_2+1} \,\middle|\, B \right].$$

Then order the vertices of $P \vee Q$ as follows:

$$\left\{ \begin{bmatrix} \boldsymbol{v}_1 \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} \boldsymbol{v}_{d_1+1} \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{d_1} \\ \boldsymbol{w}_1 \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{d_1} \\ \boldsymbol{w}_{d_2+1} \\ 1 \end{bmatrix}, \begin{bmatrix} \boldsymbol{v}_{d_1+2} \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \ldots, \begin{bmatrix} \boldsymbol{v}_n \\ \mathbf{0}_{d_2} \\ -1 \end{bmatrix}, \begin{bmatrix} \mathbf{0}_{d_1} \\ \boldsymbol{w}_{d_2+2} \\ 1 \end{bmatrix}, \ldots, \begin{bmatrix} \mathbf{0}_{d_1} \\ \boldsymbol{w}_m \\ 1 \end{bmatrix} \right\}.$$

Here, the first $d_1 + d_2 + 2 = \dim(P \vee Q) + 1$ vertices are affinely independent. Now, use the techniques of Section 3.1.1 to compute a Gale transform of $P \vee Q$ as follows: Form the matrix

$$\begin{bmatrix} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{d_1+1} & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} & \boldsymbol{v}_{d_1+2} & \cdots & \boldsymbol{v}_n & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} \\ \mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{d_2+1} & \mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \boldsymbol{w}_{d_2+1} & \cdots & \boldsymbol{w}_m \\ -1 & \cdots & -1 & 1 & \cdots & 1 & -1 & \cdots & -1 & 1 & \cdots & 1 \end{bmatrix}$$

First, add the first row to the last row, and then perform, in the first $d_1 + 1$ rows, the operations which transform the matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \boldsymbol{v}_2 & \cdots & \boldsymbol{v}_n \end{bmatrix}$ into $\left[I_{d_1+1} \,\middle|\, A\right]$. Note that in the $1,2$ and $1,4$ blocks each column will be the same. Call this common (d_1+1)-dimensional vector $\boldsymbol{v}$. Further, denote an $r \times s$ matrix with all entries 0 by $[0]_{\substack{r\\s}}$. Then,

$$M = \operatorname{rref}\left[\begin{array}{ccc|ccc|ccc|ccc} 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ \boldsymbol{v}_1 & \cdots & \boldsymbol{v}_{d_1+1} & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} & \boldsymbol{v}_{d_1+2} & \cdots & \boldsymbol{v}_n & \mathbf{0}_{d_1} & \cdots & \mathbf{0}_{d_1} \\ \hline \mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \boldsymbol{w}_1 & \cdots & \boldsymbol{w}_{d_2+1} & \mathbf{0}_{d_2} & \cdots & \mathbf{0}_{d_2} & \boldsymbol{w}_{d_2+2} & \cdots & \boldsymbol{w}_m \\ -1 & \cdots & -1 & 1 & \cdots & 1 & -1 & \cdots & -1 & 1 & \cdots & 1 \end{array}\right]$$

$$= \operatorname{rref}\left[\begin{array}{c|ccc|c|ccc} I_{d_1+1} & \boldsymbol{v} & \cdots & \boldsymbol{v} & A & \boldsymbol{v} & \cdots & \boldsymbol{v} \\ \hline [0]_{\substack{d_2+1\\d_1+1}} & \begin{matrix}\boldsymbol{w}_1 \\ 2\end{matrix} & \begin{matrix}\cdots \\ \cdots\end{matrix} & \begin{matrix}\boldsymbol{w}_{d_2+1} \\ 2\end{matrix} & [0]_{\substack{d_2+1\\n-d_1+1}} & \begin{matrix}\boldsymbol{w}_{d_2+2} \\ 2\end{matrix} & \begin{matrix}\cdots \\ \cdots\end{matrix} & \begin{matrix}\boldsymbol{w}_m \\ 2\end{matrix} \end{array}\right].$$

Next, use the bottom row of the matrix to turn the $1,2$ and $1,4$ blocks into all zeros. Then move the bottom row to the (d_1+2)th row, divide it by 2 and shift each of the remaining rows down, that is:

$$M = \operatorname{rref}\left[\begin{array}{c|ccc|c|ccc} I_{d_1+1} & & [0]_{\substack{d_1+1\\d_2+1}} & & A & & [0]_{\substack{d_1+1\\m-d_2-1}} & \\ \hline [0]_{\substack{d_2+1\\d_1+1}} & \begin{matrix}1 \\ \boldsymbol{w}_1\end{matrix} & \begin{matrix}\cdots \\ \cdots\end{matrix} & \begin{matrix}1 \\ \boldsymbol{w}_{d_2+1}\end{matrix} & [0]_{\substack{d_2+1\\n-d_1+1}} & \begin{matrix}1 \\ \boldsymbol{w}_{d_2+2}\end{matrix} & \begin{matrix}\cdots \\ \cdots\end{matrix} & \begin{matrix}1 \\ \boldsymbol{w}_m\end{matrix} \end{array}\right].$$

Now, in the bottom d_2+1 rows, perform the row operations which transform the matrix $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ \boldsymbol{w}_1 & \boldsymbol{w}_2 & \cdots & \boldsymbol{w}_m \end{bmatrix}$ into $\left[I_{d_2+1} \,\middle|\, B\right]$. Thus

$$M = \left[\begin{array}{c|c|c} I_{d_1+d_2+2} & A & [0] \\ \cline{2-3} & [0] & B \end{array}\right]$$

and so a Gale transform of $P \vee Q$ is given by the multiset of the columns of the matrix

$$\left[\begin{array}{c|c|c|c} -A^{\mathrm{T}} & [0] & I & [0] \\ \hline [0] & -B^{\mathrm{T}} & [0] & I \end{array}\right]$$

Which are exactly the elements of $\Gamma \oplus \Lambda$.

The second result holds by oriented matroid duality. $\square$

Example: Structure of d-polytopes with $d+2$ vertices. They are pyramids over direct sums of simplices.

5.2 Adding Points to Gale Diagrams

The first thing one should do in a section entitled “Adding Points to Gale Diagrams" is discuss under what conditions adding a point to a gale diagram yields a gale diagram. Thus, suppose $G \in \mathrm{gale}(P)$ is a Gale diagram of some d-polytope P with n vertices, and let $\boldsymbol{x} \in \mathbb{R}^{n-d-1}$ be any point. Then $G \cup \{\boldsymbol{x}\}$ is a Gale diagram of some $(d+1)$-polytope with $n+1$ vertices. This is true since adding a point cannot cause a hyperplane to not have two points in both open half-spaces. It should be clear that the location of the new point does matter in determining the new polytope.

Points can be added to Gale diagrams in many ways. Unfortunately, it is possible for two Gale diagrams $G, H \in \mathrm{gale}(P)$ to add a point to both G and H in the same way, yet not have the two new polytopes be combinatorially equivalent.

As an example; consider the crosspolytope X_3. Figure 5.1 shows two consubstantial standard Gale diagrams for X_3. If a point $\overline{1'}$ is added which is antipodal to $\overline{1}$ in each of these Gale diagrams, then Figures 5.2 and 5.3 show the graphs of the polytopes with each of these Gale diagrams. Let P be a polytope with the Gale diagram in Figure 5.2, and Q be a polytope with the Gale diagram in Figure 5.3. Note that $\mathrm{conv}\{2,5\}$ is not an edge in P, whereas in Q it is. Hence, P and Q cannot be combinatorially equivalent.

The facets of P are the 3-dimensional pyramids 23456 and 12345 each with base 2345, as well

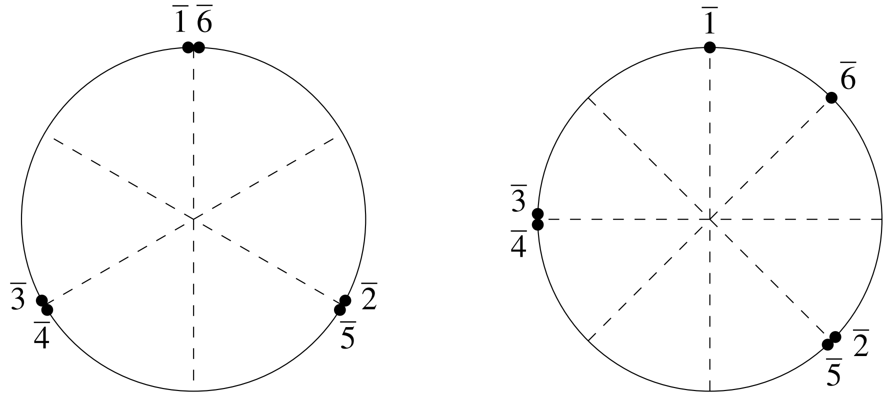


Figure 5.1: Two Gale diagrams of X_3.

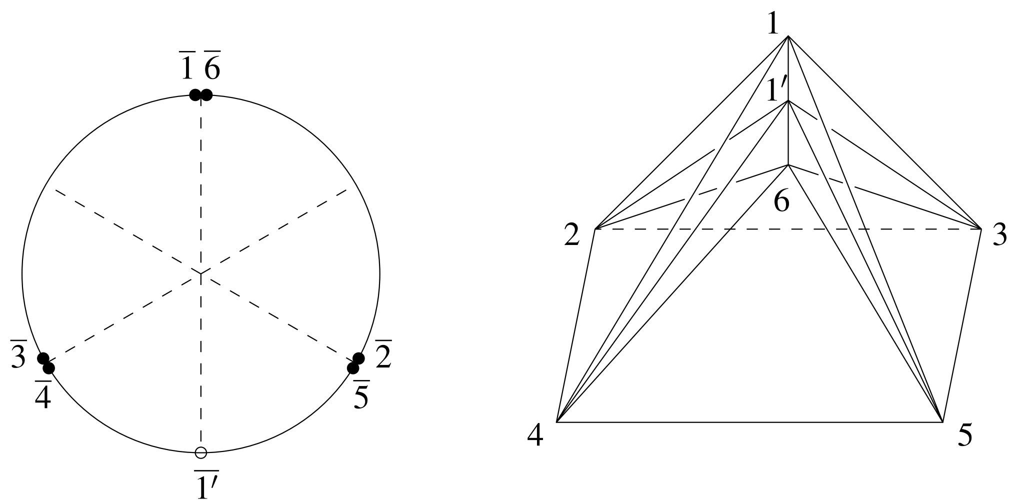


Figure 5.2: Adding an antipode $\overline{1'}$ to the point $\overline{1}$ in a Gale diagram of X_3. (cf. Figure 5.3)

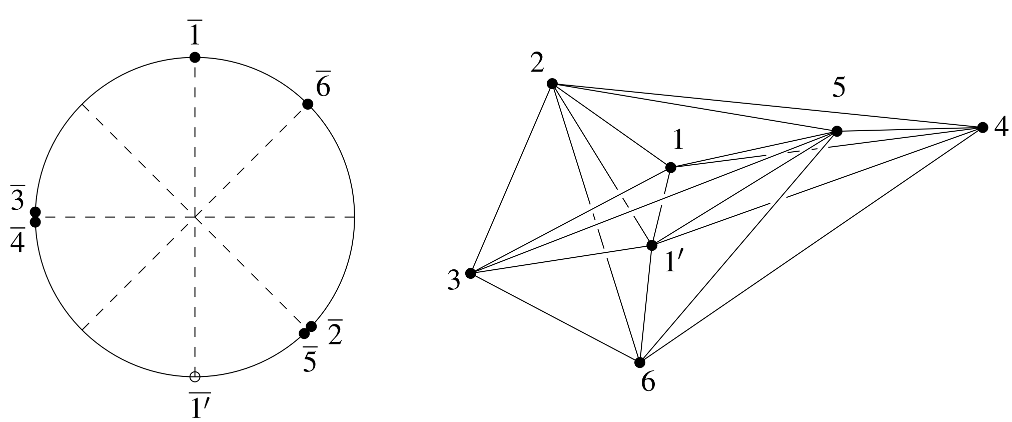


Figure 5.3: Adding an antipode $\overline{1'}$ to the point $\overline{1}$ in a Gale diagram of X_3. (cf. Figure 5.2)

as the following 3-simplices.

$11'45$	$11'35$	$11'24$	$11'23$
$1'456$	$1'356$	$1'246$	$1'236$

The facets of Q are the triangular bipyramid 23456 with base 256, as well as the following 3-simplices.

$11'45$	$11'35$	$11'24$	$11'23$
$1'456$	$1'356$	$1'246$	$1'236$
1245	1235		

On the other hand,

Chapter 6

Gale Polytopes

Recall Gale's evenness condition:

Theorem 19 (Gale's Evenness Condition)**.** *A subset $S \subseteq [n]$ with $|S| = d$ forms a facet of $C_d(n)$ if and only if*

$$|\{k \mid k \in S \textit{ and } i < k < j\}| \textit{ is even for all } i < j,\ \{i, j\} \cap S = \varnothing$$

It is reasonable to ask if a noncyclic polytope satisfies a weakened form of Gale's evenness condition, namely:

For P a d-polytope, with vertex set $\text{vert}(P) = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\}$, if $S \subseteq [n]$ and F is a facet of P with $\text{vert}(F) = \{\boldsymbol{v}_s \mid s \in S\}$, then

$$|\{k \mid k \in S \text{ and } i < k < j\}| \text{ is even for all } i < j,\ \{i, j\} \cap S = \varnothing$$

Definition. A polytope P is *Gale* if there is an ordering of $\text{vert}(P)$ such that each facet satisfies Gale's evenness condition.

An equivalent characterization is as follows:

Let P be a polytope, $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n$ be an ordering of $\text{vert}(P)$ and $S = \{\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \dots, \boldsymbol{v}_{i_k}\} \subseteq \text{vert}(P)$ be such that $i_1 < i_2 < \cdots < i_k$. Then a *contiguous block* of S is a subset G of S such that if $t = \min\left\{i_j \;\middle|\; \boldsymbol{v}_{i_j} \in G\right\}$, then there is some $r \in \mathbb{N}$ such that $G = \{\boldsymbol{v}_t, \boldsymbol{v}_{t+1}, \dots, \boldsymbol{v}_{t+r-1}\}$ and $\boldsymbol{v}_{t+r} \notin S$.

In this case, the *length* of a contiguous block is $|G| = r$.

Definition. A polytope P is *Gale* if there is an ordering of $\mathrm{vert}(P)$ such that if F is a facet of P, then for each contiguous block G of $\mathrm{vert}\,F$ one of the following holds:

1. the length of G is even; or

2. $\boldsymbol{v}_1 \in G$; or

3. $\boldsymbol{v}_n \in G$.

Such an ordering of the vertices is called a *Gale ordering*. A contiguous block which contains $\boldsymbol{v}_1$ is called an *initial block*. A contiguous block which contains $\boldsymbol{v}_n$ is called a *terminal block*. A contiguous block which is neither initial, nor terminal is called *internal*.

In terms of contiguous blocks, a polytope is Gale if and only if there is an ordering of the vertices such that in each facet the only contiguous blocks of odd length are either initial, or terminal. Note that a facet need not have either an initial block, or a terminal block.

An immediate consequence of either definition is: If P is a Gale polytope with n vertices, and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a Gale ordering of $\mathrm{vert}(P)$, then so is $\boldsymbol{v}_n, \boldsymbol{v}_{n-1}, \ldots, \boldsymbol{v}_1$. Note that all cyclic polytopes are Gale, and therefore all polytopes of dimension less than or equal to 2 are Gale as well.

Theorem 20. *If P is a Gale polytope, then so is* $\mathrm{pyr}(P)$*, a pyramid over P.*

Proof. Let $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ be a Gale ordering of $\mathrm{vert}(P)$ and $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{n+1}$ be the ordering of $\mathrm{vert}(\mathrm{pyr}(P))$ with $\boldsymbol{w}_i$ corresponding to $\boldsymbol{v}_i$ for $i \in [n]$, and with $\boldsymbol{w}_{n+1}$ as the apex of the pyramid.

Let F be a face of $\mathrm{pyr}(P)$. Then either $\mathrm{vert}(F) = \{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n\}$, or $\boldsymbol{w}_{n+1} \in \mathrm{vert}(F)$. In the first case, F contains only an initial block.

In the second case, write $\mathrm{vert}(F) = \{\boldsymbol{w}_{i_1}, \boldsymbol{w}_{i_2}, \ldots, \boldsymbol{w}_{i_k}, \boldsymbol{w}_{n+1}\}$ with $i_1 < i_2 < \cdots < i_k < n+1$. Then either $i_k = n$, or $i_k < n$. If $i_k = n$, then each contiguous block G of $\{\boldsymbol{w}_{i_1}, \boldsymbol{w}_{i_2}, \ldots, \boldsymbol{w}_{i_k}\}$ satisfies at least one of the following: G is internal of even length; G is initial; or $\boldsymbol{w}_n \in G$. Thus, after adding in $\boldsymbol{w}_{n+1}$, each contiguous block G' satisfies one of the following respectively: G' is internal of even length; G' is initial; or G' is terminal. In the case that $i_k < n$, each contiguous block of $\{\boldsymbol{w}_{i_1}, \boldsymbol{w}_{i_2}, \ldots, \boldsymbol{w}_{i_k}\}$ is either initial, or is of even length. Ergo, when $\boldsymbol{w}_{n+1}$ is added the only change is that a terminal block of length 1 is added. □

Theorem 21. *If P is a Gale polytope, then it can have at most two disjoint facets with an odd number of vertices.*

Proof. Let P be a Gale polytope, and F be a facet of P which has an odd number of vertices.

Consider the contiguous blocks of F. since F has an odd number of vertices, it must have either an initial, or terminal block. If this were not the case, then there would be an internal block with an odd number of vertices. □

Example 22. The above theorem shows that the regular dodecahedron is not a Gale polytope. Label the vertices as in figure 6.1. The following are the vertex sets of three disjoint facets $\{1,2,3,4,5\}$, $\{7,11,12,16,17\}$, and $\{9,13,14,18,19\}$.

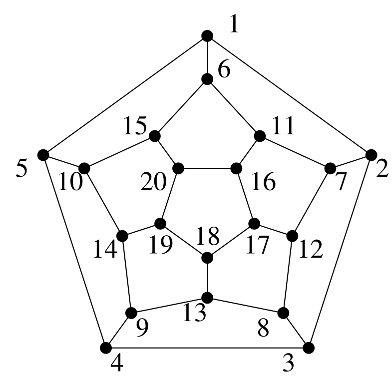


Figure 6.1: The regular dodecahedron is not a Gale polytope.

If P is a 3-polytope with five or fewer vertices then it is one of Δ_3, pyr$(C_4(2))$, or $C_5(3)$ each of which is Gale. However, there is a 3-polytope with six vertices which is not Gale. The following theorem is useful in demonstrating this.

Theorem 23. *If P is a simplicial Gale polytope, then P is a cyclic polytope.*

Proof. First, recall that if $\text{vert}(C_d(n)) = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$, then the set facets of $C_d(n)$ is

$$X = \left\{\text{conv}\left\{\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \ldots, \boldsymbol{v}_{i_{d+1}}\right\} \;\middle|\; \left\{\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \ldots, \boldsymbol{v}_{i_{d+1}}\right\} \text{ satisfies Gale's evenness condition}\right\}.$$

Let P be a simplicial Gale polytope of dimension d with $|\text{vert}\, P| = n$. Then there is some ordering $\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_n$ of $\text{vert}\, P$ such that if $\text{conv}\left\{\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_{i_{d+1}}\right\}$ is a facet of P, then the set $\text{conv}\left\{\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \ldots, \boldsymbol{v}_{i_{d+1}}\right\}$ is a facet of $C_d(n)$ since the set of facets of $C_d(n)$ is X. Thus, the map $\boldsymbol{w}_i \mapsto \boldsymbol{v}_i$ induces a map $\varphi\colon \mathscr{F}(P) \to \mathscr{F}(C_d(n))$ from the face lattice of P to that of $C_d(n)$ which is an injection.

Suppose $F = \text{conv}\left\{\boldsymbol{v}_{i_1}, \boldsymbol{v}_{i_2}, \ldots, \boldsymbol{v}_{i_{d+1}}\right\}$ is a facet of $C_d(n)$ such that $\varphi^{-1}F$ is not a facet of P. Let G be any facet of P. Then there is a sequence $F = F_1, F_2, \ldots, F_r = \varphi G$ of facets of $C_d(n)$ such that $|\text{vert}(F_i) \cap \text{vert}(F_{i+1})| = d$ for each $i \in [r-1]$. Thus, there is some t such that $\varphi^{-1}F_1, \varphi^{-1}F_2, \ldots, \varphi^{-1}F_t$ are not facets of P, but $\varphi^{-1}F_{t+1}$ is.

Let $R = F_t \cap F_{t+1}$. Then $R \in \mathscr{F}(C_d(n))$ since $\mathscr{F}(C_d(n))$ is a lattice, and $\varphi^{-1}R \in \mathscr{F}(P)$ since P is simplicial.

Since P is a polytope, there is some $K \in \mathscr{F}(P)$ such that $\varphi^{-1}R \subsetneq K \subsetneq P$ with $K \neq F_{t+1}$ (Ziegler, 1995, page 57). Similarly, if $L \in \mathscr{F}(C_d(n))$ such that $R \subsetneq L \subsetneq C_d(n)$, then either $L = F_t$, or $L = F_{t+1}$. Since φ is an injection, it must be the case that $\varphi K = F_t$. Thus φ is an isomorphism, and P is cyclic. □

Thus, for example, X_3 is not a Gale polytope since each vertex of X_3 lies on four edges, whereas $C_6(3)$ has two vertices which each lie on only three edges.

Since simplices are cyclic polytopes, and cyclic polytopes are simplicial, each facet of a cyclic polytope satisfies Gale's evenness condition. However, this need not happen for a general Gale polytope. The following is an example of a Gale polytope with a facet which is not Gale.

Example 24. Consider the 5-dimensional cyclic polytope with vertices

$$\boldsymbol{x}_1 = \begin{bmatrix} -4 & 16 & -64 & 256 & -1024 \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{x}_5 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix}^{\mathrm{T}}$$
$$\boldsymbol{x}_2 = \begin{bmatrix} -2 & 4 & -8 & 16 & -32 \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{x}_6 = \begin{bmatrix} 2 & 4 & 8 & 16 & 32 \end{bmatrix}^{\mathrm{T}}$$
$$\boldsymbol{x}_3 = \begin{bmatrix} -1 & 1 & -1 & 1 & -1 \end{bmatrix}^{\mathrm{T}} \qquad \boldsymbol{x}_7 = \begin{bmatrix} 4 & 16 & 64 & 256 & 1024 \end{bmatrix}^{\mathrm{T}}.$$
$$\boldsymbol{x}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \end{bmatrix}^{\mathrm{T}}$$

Now, intersect with the hyperplane $H = \left\{ \boldsymbol{\xi} \mid \langle \boldsymbol{\xi}, \boldsymbol{e}_2 \rangle = \frac{1}{2} \right\}$. This gives a 4-dimensional simplicial polytope P (in $\mathbb{R}^5$) with vertices

$$z_1 = \begin{bmatrix} -\frac{1}{8} & \frac{1}{2} & -2 & 8 & -32 \end{bmatrix}^{\mathrm{T}} \qquad z_4 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}^{\mathrm{T}}$$
$$z_2 = \begin{bmatrix} -\frac{1}{4} & \frac{1}{2} & -1 & 2 & -4 \end{bmatrix}^{\mathrm{T}} \qquad z_5 = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & 1 & 2 & 4 \end{bmatrix}^{\mathrm{T}}$$
$$z_3 = \begin{bmatrix} -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}^{\mathrm{T}} \qquad z_6 = \begin{bmatrix} \frac{1}{8} & \frac{1}{2} & 2 & 8 & 32 \end{bmatrix}^{\mathrm{T}}.$$

The facets of this polytope are the convex hulls of the following sets:

$$\{z_1, z_2, z_3, z_4\} \qquad \{z_1, z_2, z_3, z_6\} \qquad \{z_1, z_2, z_4, z_6\} \qquad \{z_1, z_3, z_5, z_6\}$$
$$\{z_1, z_4, z_5, z_6\} \qquad \{z_2, z_3, z_4, z_6\} \qquad \{z_3, z_4, z_5, z_6\} \qquad .$$

Let Q be a pyramid over P with apex $\boldsymbol{a}$. Then Q is a Gale polytope with Gale ordering

$$z_1, z_2, z_3, \boldsymbol{a}, z_4, z_5, z_6.$$

However, P (which is the base of the pyramid Q, and therefore also a facet of Q) is not Gale since it is a simplicial polytope with 7 vertices, but only 8 facets. The polytope P is therefore not cyclic since $C_4(7)$ has 12 facets.

Theorem 25. *If P is a Gale d-polytope, and Q is a Gale d'-polytope, then $P \times Q$ is a Gale $(d+d')$-polytope.*

Proof. Suppose $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n$ is a Gale ordering of $\operatorname{vert}(P)$, and $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_m$ is a Gale ordering of $\operatorname{vert} Q$. Then define an ordering of $\operatorname{vert}(P \times Q)$ as follows:

Let $(V_1, \boldsymbol{w}_i)$ be the sequence $(\boldsymbol{v}_1, \boldsymbol{w}_i), (\boldsymbol{v}_2, \boldsymbol{w}_i), \ldots, (\boldsymbol{v}_n, \boldsymbol{w}_i)$, and $(V_{-1}, \boldsymbol{w}_i)$ be the sequence

$$(\boldsymbol{v}_n, \boldsymbol{w}_i), (\boldsymbol{v}_{n-1}, \boldsymbol{w}_i), \ldots, (\boldsymbol{v}_1, \boldsymbol{w}_i).$$

Define similarly, $(B, \boldsymbol{w}_i)$ and $(B_{-1}, \boldsymbol{w}_i)$ for B a contiguous set of vertices of P.

Claim: $(V_1, \boldsymbol{w}_0), (V_{-1}, \boldsymbol{w}_1), \ldots, (V_{(-1)^m}, \boldsymbol{w}_m)$ is a Gale ordering of $P \times Q$.

First, consider the facets of $P \times Q$ of the form $P \times G$ where G is a facet of Q. The vertex $\boldsymbol{w}_j$ is a vertex of G if and only if $(\boldsymbol{v}_i, \boldsymbol{w}_j)$ is a vertex of the facet $P \times G$ for each $\boldsymbol{v}_i \in \operatorname{vert}(P)$. In particular, the sequence $(V_{(-1)^j}, \boldsymbol{w}_j)$ is part of a contiguous block of $P \times G$.

If n is even, then each of these parts is of even length, and hence every contiguous block of $P \times G$ is even. If n is odd, then consider the contiguous blocks of G. If G has an initial block $\boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_t$, then $P \times G$ has initial block $(V_1, \boldsymbol{w}_0), (V_{-1}, \boldsymbol{w}_1), \ldots, (V_{(-1)^t}, \boldsymbol{w}_t)$ (since $(\boldsymbol{v}_1, \boldsymbol{w}_{t+1})$ is not a vertex of $P \times G$). A similar statement applies for terminal blocks.

Suppose that G has an internal block $\boldsymbol{v}_k, \boldsymbol{v}_{k+1}, \ldots, \boldsymbol{v}_{k+s-1}$. Since this is an internal block, s must be even. Thus the sequence $(V_{(-1)^k}, \boldsymbol{w}_k), (V_{(-1)^{k+1}}, \boldsymbol{w}_{k+1}), \ldots, (V_{(-1)^{k+s-1}}, \boldsymbol{w}_{k+s-1})$ is an internal contiguous block of $P \times G$ of length $|\operatorname{vert}(P)| \cdot s$ (which is an even number). This follows since each element of the above sequence is a vertex of $P \times G$, and none of the vertices

$$(\boldsymbol{v}_1, \boldsymbol{w}_{k-1}), (\boldsymbol{v}_n, \boldsymbol{w}_{k-1}), (\boldsymbol{v}_n, \boldsymbol{w}_{k+s}), (\boldsymbol{v}_1, \boldsymbol{w}_{k+s})$$

is a vertex of $P \times G$.

Now, consider the facets of $P \times Q$ of the form $F \times Q$ with F a facet of P. In this case, similar to above, $\boldsymbol{v}_i \in F$ if and only if $(\boldsymbol{v}_i, \boldsymbol{w}_j) \in F \times Q$ for each $j \in [m] \cup \{0\}$. The sequence

$B = \boldsymbol{v}_i, \boldsymbol{v}_{i+1}, \ldots, \boldsymbol{v}_{i+k-1}$ is part of a contiguous block of F if and only if

$$\begin{cases} (B_1, \boldsymbol{w}_j) = (\boldsymbol{v}_i, \boldsymbol{w}_j), (\boldsymbol{v}_{i+1}, w_j), \ldots, (\boldsymbol{v}_{i+k-1}, \boldsymbol{w}_j) & , j \text{ even} \\ (B_{-1}, \boldsymbol{w}_j) = (\boldsymbol{v}_{i+k-1}, \boldsymbol{w}_j), (\boldsymbol{v}_{i+k-2}, w_j), \ldots, (\boldsymbol{v}_i, \boldsymbol{w}_j) & , j \text{ odd} \end{cases}$$

is part of a contiguous block of $F \times Q$. If B is an internal block of F, then the above are internal blocks of $F \times Q$.

Suppose that B is an initial block of F (in particular, $i = 1$, and B has length k). Then:

- $(B_1, \boldsymbol{w}_1)$ is an initial block of $F \times Q$;
- if m is odd, then $(B_{-1}, \boldsymbol{w}_m)$ is a terminal block of $F \times Q$;
- if m is even, then $(B_{-1}, \boldsymbol{w}_{m-1}), (B_1, \boldsymbol{w}_m)$ is an internal block of length $2k$.

Further, $(B_{-1}, \boldsymbol{w}_{2\ell-1}), (B_1, \boldsymbol{w}_{2\ell})$ is an internal block of length $2k$.

The case that B is a terminal block is handled similarly. $\square$

Chapter 7

Anticliques in Graphs of Polytopes

Let $G = (V,E)$ be a graph. An *anticlique* is a subset A of V (of cardinality at least 2) such that $E \cap \binom{A}{2} = \varnothing$. If $|A| = k$, then A is said to be a k-anticlique. A 2-anticlique is simply a nonedge.

Recall that if G is the graph of a d-polytope P with n vertices, and $\Gamma \in \text{gale}(P)$, then $\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$ is a nonedge of P if and only if there is some hyperplane $H_{i,j} \subseteq \mathbb{R}^{n-d-1}$ containing $\mathbf{0}$ such that $H_{i,j}^{(+)} \cap \Gamma = \left\{\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_j\right\}$. Such a hyperplane is called a *separating* hyperplane. The normal vector to $H_{i,j}$ which has positive inner product with $\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_j$ and norm 1 is denoted $\boldsymbol{x}_{i,j}$.

If P is a d-polytope with $d+k$ vertices, then Theorem 8 in Section 2.4 implies that each vertex of $\mathscr{G}(P)$ must have degree at least d. If a graph H with $d+k$ vertices has a $(k+1)$-anticlique, then each vertex in the anticlique has degree at most $d-1$, and hence H is not d-realizable by Balinski's theorem. However, the graph $\left([d+k], \binom{[d+k]}{2}\right) \setminus \binom{[k]}{2}$ is d-connected, and furthermore has a K_{d+1} minor (the induced subgraph on the vertex set $\{k, k+1, \ldots, k+d\}$ is complete) and thus *could* be the graph of a d-polytope. It will be shown that this cannot happen. More precisely for $k \geq 2$, let

$$f(d,k) = \max\{n \mid \text{there is some } d\text{-polytope } P \text{ with } d+k \text{ vertices and an } n\text{-anticlique}\}.$$

Then the goal is to show that $f(d,k) < k$ for $k > 2$.

7.1 An Upper Bound on f

First note that f is weakly increasing in both arguments.

Theorem 26.

1. $f(d,k) \leq f(d+1,k)$

2. $f(d,k) \leq f(d,k+1)$

Proof. Let P be a d-polytope with $d+k$ vertices, and an $f(d,k)$-anticlique.

1. The $(d+1)$-polytope $\mathrm{pyr}(P)$ has $(d+k)+1 = (d+1)+k$ vertices and an $f(d,k)$-anticlique.

2. Let F be any facet of P. The d-polytope $K(P;F)$ (defined in section 1.4.8) has $d+(k+1)$ vertices and an $f(d,k)$-anticlique.

$\square$

Recall that a standard Gale diagram is one in which each point is either on the unit sphere, or at the origin. Further if a point in the Gale diagram is at the origin, then the polytope is a pyramid with apex the corresponding point. Since the main question is concerned with sizes of anticliques, and pyramiding increases the dimension of a polytope and its number of vertices by 1 while not changing the size of a maximal anticlique, nothing is lost, or gained in the assumption that a polytope is not a pyramid.

Theorem 27. *If: P is a d-polytope with vertex set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_n\}$; $A = \{i_1, i_2, \ldots, i_k\} \subseteq [n]$; $\{\boldsymbol{v}_i \mid i \in A\}$ is an anticlique; Γ is a standard Gale diagram of P; and $\{i, j\} \subseteq A$ then*

1. $\overline{\boldsymbol{v}}_i \neq \mathbf{0}$.

2. $\overline{\boldsymbol{v}}_i \neq -\overline{\boldsymbol{v}}_j$.

if $k \geq 3$, then

3. *if $\overline{\boldsymbol{v}}_i = \overline{\boldsymbol{v}}_j$, then $i = j$.*

Proof.

1. If $\overline{\boldsymbol{v}}_i = \mathbf{0}$, then $\operatorname{conv}\{\boldsymbol{v}_i, \boldsymbol{v}_t\}$ is a face of P for each $t \in [n] \setminus \{i\}$. In particular, $\operatorname{conv}\left\{\boldsymbol{v}_i, \boldsymbol{v}_j\right\}$ is an edge of $G(P)$.

2. If $\overline{\boldsymbol{v}}_i \neq -\overline{\boldsymbol{v}}_j$,and $H_{i,j}$ is a separating hyperplane for $\overline{\boldsymbol{v}}_i, \overline{\boldsymbol{v}}_j$ with normal vector $\boldsymbol{x}_{i,j}$, then

$$0 < \left\langle \boldsymbol{x}_{i,j}, \overline{\boldsymbol{v}}_i \right\rangle = -\left\langle \boldsymbol{x}_{i,j}, \overline{\boldsymbol{v}}_j \right\rangle < 0.$$

3. Let $H_{a,b}$ be a separating hyperplane for $\overline{\boldsymbol{v}}_a, \overline{\boldsymbol{v}}_b$ with normal $\boldsymbol{x}_{a,b}$ for each pair $a, b \in A$. Suppose $\overline{\boldsymbol{v}}_i = \overline{\boldsymbol{v}}_j$ with $i \neq j$. Then for $r \in [k] \setminus \{i, j\}$,

$$0 < \left\langle \boldsymbol{x}_{j,r}, \overline{\boldsymbol{v}}_j \right\rangle = \left\langle \boldsymbol{x}_{j,r}, \overline{\boldsymbol{v}}_i \right\rangle \leq 0.$$

   Hence $i = j$.

$\square$

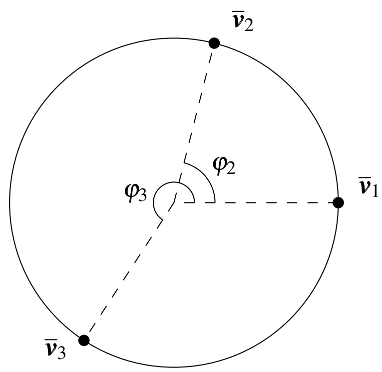
Theorem 28. $f(d,2) = 2$

Proof. If P is a d-polytope with $d+2$ vertices, then P has a 1-dimensional Gale diagram. Since a nonedge requires that there be a separating hyperplane, and in this case there is only one possible hyperplane, any vertex can have at most one nonneighbor. $\square$

Theorem 29. $f(d,3) < 3$

Proof. Let P be a d-polytope with $d+3$ vertices, and let Γ be a standard Gale diagram of P. Suppose that P has a 3-anticlique $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, and let $\boldsymbol{v}_4$ be any other vertex of P. Note that since $\Gamma \subseteq \mathbb{S}^1$ (assuming again that P is not a pyramid), by writing each $\overline{\boldsymbol{v}} \in \Gamma$ in polar coordinates only the angle $\varphi \in [0, 2\pi)$ is necessary to specify $\overline{\boldsymbol{v}}$. Thus, suppose $\overline{\boldsymbol{v}}_i$ makes angle φ_i with the positive x-axis for $i \in [4]$. By Theorem 27 part 3, none of $\varphi_1, \varphi_2, \varphi_3$ are equal, so assume $\varphi_1 < \varphi_2 < \varphi_3$. Further, if $\varphi_1 > 0$, then rotating each point in Γ clockwise by an angle of φ_1 produces a consubstantial standard Gale diagram, so assume

$$0 = \varphi_1 < \varphi_2 < \varphi_3 < 2\pi.$$



Since $\boldsymbol{v}_1\boldsymbol{v}_2$ is not an edge of P, there is some hyperplane H_3 through the origin such that $H_3^{(+)} \cap \Gamma = \{\overline{\boldsymbol{v}}_1, \overline{\boldsymbol{v}}_2\}$. Therefore, $\varphi_4 \notin [0, \varphi_2]$. Similarly, $\varphi_4 \notin [\varphi_2, \varphi_3]$, and $\varphi_4 \notin [\varphi_3, 2\pi)$. Since Γ is a standard Gale diagram, this means that $\overline{\boldsymbol{w}} = \mathbf{0}$. But this contradicts the assumption that P is not a pyramid. $\square$

Lemma 30. *If*

- *P is a d-polytope with vertex set $V \cup Y$ of cardinality $d+k$,*
- *$|V| = d$ and $|Y| = k$,*
- *Y is an anticlique of $\mathscr{G}(P)$, and*
- *$\boldsymbol{v} \in V$, then*

$\boldsymbol{vy}$ is an edge of $\mathscr{G}(P)$ for each $y \in Y$.

Proof. Let $\boldsymbol{v} \in V$, $y \in Y$. Since P is a d-polytope, it is d-connected, therefore each vertex of $G(P)$ is adjacent to at least d other vertices. But since Y is an anticlique, $\boldsymbol{y}$ is adjacent to at most d other vertices. Hence $\boldsymbol{y}$ is adjacent to exactly d other vertices, and since none of these other vertices is in Y, $\boldsymbol{vy}$ is an edge of $G(P)$. $\square$

Lemma 31. *If P is a d-polytope, $\boldsymbol{v} \in \operatorname{vert} P$ with $\deg \boldsymbol{v} = d$, and*

$$V = \{\boldsymbol{x} \in \operatorname{vert} P \mid \boldsymbol{vx} \in E(G(P))\} = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d\},$$

then for each subset X of V with cardinality $d-1$ there is a unique facet F of P with $X \subseteq \operatorname{vert} F$.

Proof. Since P is a d-polytope, each vertex lies in at least d facets. Let F be a facet containing $\boldsymbol{v}$. Since F is a $(d-1)$-polytope, each vertex of $G(F)$ has degree at least $d-1$. Thus $\boldsymbol{v}$ must have either $d-1$, or d neighbors in F. If $\boldsymbol{v}$ had each of its d neighbors in F, then $\boldsymbol{v}$ would only lie in one facet. Thus each facet in which $\boldsymbol{v}$ lies contains exactly $d-1$ neighbors of $\boldsymbol{v}$. Further, since $\boldsymbol{v}$ lies in at least d facets, each element of $\binom{V}{2}$ determines a unique facet of P. $\square$

Notice that this argument also shows that the set V above is affinely independent, and that not all vertices of V lie in any one face of P.

The

Theorem 32. *For $k > 2$, $f(d,k) < k$.*

Proof. Proceed by induction on d. For the base case, the graph of a 2-polytope with n vertices is a cycle of length n, a maximal anticlique can be obtained by taking every other vertex. Thereore $f(2,k) = 1 + \lfloor k/2 \rfloor < k$. Thus, suppose for some d that $f(d-1,k) < k$ when $k > 2$.

Let P be a d-polytope with $d+k$ vertices and, without loss of generality, suppose that P is not a pyramid. Suppose further, that P has a k-anticlique $Y = \{\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_k\}$, write $\operatorname{vert}(P) = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d\} \cup Y$, let $V = \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_d\}$, and $V_i = V \setminus \{\boldsymbol{v}_i\}$ for $i \in [d]$. Fix some $i \in [d]$, and for each $j \in [k]$, let $F_{i,j}$ be the facet of P which contains $V_i \cup \{\boldsymbol{y}_j\}$ (Lemmata 30 and 31), and G_i be the smallest face of P which contains V_i.

Notice that $G_i \subseteq F_{i,j}$ for each $j \in [k]$, and that G_i is either a facet, or a ridge since V is an affinely independent set.

1. (G_i is a facet.) In this case, for each $j \in [k]$, $G_i = F_{i,j} = \operatorname{conv}(Y \cup V_i)$. Thus P is a pyramid over G_i. This is a contradiction.

2. (G_i is a ridge.) In this case, Y can be partitioned into two nonempty sets

$$Y_1 = \left\{\boldsymbol{y}_{a_1}, \boldsymbol{y}_{a_2}, \ldots, \boldsymbol{y}_{a_r}\right\} \qquad\qquad Y_2 = \left\{\boldsymbol{y}_{b_1}, \boldsymbol{y}_{b_2}, \ldots, \boldsymbol{y}_{b_s}\right\}$$

such that

$$Y_1 \subseteq F_{i,a_1} = F_1 \qquad\qquad Y_2 \subseteq F_{i,b_1} = F_2.$$

If $|Y_1| = 2$, then F_1 is a $(d-1)$-polytope with $(d-1)+2$ vertices and G_i is a facet of F_1. Thus a Gale diagram of F_1 is one dimensional with $d+1$ vertices, and both $\overline{\boldsymbol{y}}_{a_1}$ and $\overline{\boldsymbol{y}}_{a_2}$ are on the same side of the origin. However, $\mathbf{0} \in \operatorname{relint} \operatorname{conv}\left(\overline{\operatorname{vert}(F_1) \setminus Y_1}\right)$. Similarly, $|Y_2| \neq 2$.

If $|Y_1| \geq 3$, then F_{i,a_1} is a $(d-1)$-polytope with $(d-1)+|Y_1|$ vertices, and a $|Y_1|$-anticlique. This contradicts the inductive hypothesis.

Hence $|Y_1| = |Y_2| = 1$. Whence P is a d-polytope with $d+2$ vertices and a 2-anticlique. Thence $k = 2$.

□

7.2 A Lower Bound on f

Lemma 33. *If*

- *P is a d-polytope with $d+k$ vertices,*
- *$\boldsymbol{v}$ is a vertex of P such that each facet containing $\boldsymbol{v}$ is a simplex, and $\boldsymbol{v}$ is contained in exactly d facets, and*
- *$\boldsymbol{v}$ is contained in a q-anticlique of P, then*

$f(d,k+i) \geq q+i-1$ *for* $i \in [d]$.

Note that such a vertex exists, for example, if P is a simplex (in which case each vertex has this property) or if P is of the form $P = K(P';F)$ where F is a facet of P' which is a simplex[1], so the Lemma is not vacuous.

Proof. Let $F_1, F_2, \ldots, F_d$ be the facets of P containing $\boldsymbol{v}$, and set

$$\begin{aligned} P_0(\boldsymbol{v}) &= P \\ P_i(\boldsymbol{v}) &= K(P;F_1,F_2,\ldots,F_i) \\ \{\boldsymbol{v}_i\} &= \mathrm{vert}(P_i(\boldsymbol{v})) \setminus \mathrm{vert}(P_{i-1}(\boldsymbol{v})). \end{aligned}$$

Further, let A' be a q-anticlique of P which contains $\boldsymbol{v}$, and set $A = A' \setminus \{\boldsymbol{v}\}$.

For a fixed $i \in [d]$, since the neighbors of $\boldsymbol{v}_j$ (in $\mathscr{G}(P_i(\boldsymbol{v}))$) are a subset of the neighbors of $\boldsymbol{v}$ (along with $\boldsymbol{v}$ itself), the set $A \cup \{\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_i\}$ is a $(q-i-1)$-anticlique in $\mathscr{G}(P_i(\boldsymbol{v}))$. □

Corollary 1. $k-1-\left\lfloor \dfrac{k-3}{d} \right\rfloor \leq f(d,k) \leq k-1$

Proof. Repeatedly performing the process outlined in the Lemma, starting with a simplex, yields the lower bound. □

Notice that if $\lfloor (k-3)/d \rfloor = 0$, then the upper and lower bounds on f agree.

7.3 The Value of f in Dimension 3

7.3.1 Euler's Theorem

Euler's theorem states that the f-vector of a d-polytope lies on a certain hyperplane in $\mathbb{R}^{d+1}$.

Theorem 34 (Euler).

[1] You really just need that all of the facets of F are simplices, right?

7.3.2 Induced Bipartite Graphs

In dimension 3, Corollary 1 says that $k-1-\lfloor (k-3)/3 \rfloor \leq f(3,k) \leq k-1$. In this case, the lower bound can be rewritten as $k-1-\lfloor (k-3)/d \rfloor = \lceil 2k/3 \rceil$.

If P is a 3-polytope and A is an anticlique of $G = \mathscr{G}(P)$, then A induces a bipartite graph G_A whose vertex set is that of G, and whose edge set is the set of edges in G which contain a vertex of the anticlique A. That is:

$$V(G_A) = V(G)$$

$$E(G_A) = \{ax \mid a \in A\} \cap E(G).$$

Note that G_A is planar since G is planar, and G_A is a subgraph of G.

d \ k	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
2	2	2	3	3	4	4	5	5	6	6	7	7	8	8	9
3	2	2	3	4	4	5	6	6	7	8	8	9	10	10	11
4	2	2	3	4	5	**5**	**6**	**7**	**8**	**8**	**9**	**10**	**11**	**11**	**12**
5	2	2	3	4	5	6	**6**	**7**	**8**	**9**	**10**	**10**	**11**	**12**	**13**
6	2	2	3	4	5	6	7	**7**	**8**	**9**	**10**	**11**	**12**	**12**	**13**
7	2	2	3	4	5	6	7	8	**8**	**9**	**10**	**11**	**12**	**13**	**14**
8	2	2	3	4	5	6	7	8	9	**9**	**10**	**11**	**12**	**13**	**14**
9	2	2	3	4	5	6	7	8	9	10	**10**	**11**	**12**	**13**	**14**
10	2	2	3	4	5	6	7	8	9	10	11	**11**	**12**	**13**	**14**
11	2	2	3	4	5	6	7	8	9	10	11	12	**12**	**13**	**14**

Table 7.1: Values of $f(d,k)$. Numbers in bold are lower bounds.

Chapter 8

Complete Multipartite Graphs as Graphs of Polytopes

When is a complete multipartite graph realizable?

8.1 Complete Bipartite Graphs as Graphs of Polytopes

This section will answer the question "For which values of p,q,d is the graph $K_{p,q}$ d-realizable?".

If $\mathscr{G}$ is a class of graphs with the property:

$$\text{If } G \in \mathscr{G}\text{, and } H \text{ is an induced subgraph of } G\text{, then } H \in \mathscr{G}.$$

then $\mathscr{G}$ is called a *hereditary* class of graphs.

Example 35.

1. The class of all discrete graphs is hereditary, and is denoted

$$\mathscr{D}_n = \{D_n \mid n \in \mathbb{N}\}$$

References

Balinski, M. L. (1961). On the graph structure of convex polyhedra in n-space. *Pacific J. Math.*, 11, 431–434.

Barvinok, A., Lee, S. J., & Novik, I. (2013a). Explicit constructions of centrally symmetric k-neighborly polytopes and large strictly antipodal sets. *Discrete Comput. Geom.*, 49(3), 429–443.

Barvinok, A., Lee, S. J., & Novik, I. (2013b). Neighborliness of the symmetric moment curve. *Mathematika*, 59(1), 223–249.

Björner, A., Las Vergnas, M., Sturmfels, B., White, N., & Ziegler, G. M. (1999). *Oriented matroids*, volume 46 of *Encyclopedia of Mathematics and its Applications*. Cambridge: Cambridge University Press, second edition.

Brøndsted, A. (1983). *An introduction to convex polytopes*, volume 90 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Diestel, R. (2010). *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Heidelberg: Springer, fourth edition.

Grünbaum, B. (2003). *Convex polytopes*, volume 221 of *Graduate Texts in Mathematics*. New York: Springer-Verlag, second edition. Prepared and with a preface by Volker Kaibel, Victor Klee and Günter M. Ziegler.

Halin, R. (1966). Zu einem Problem von B. Grünbaum. *Arch. Math. (Basel)*, 17, 566–568.

Henk, M., Richter-Gebert, J., & Ziegler, G. M. (1997). Basic properties of convex polytopes. In *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl. (pp. 243–270). Boca Raton, FL: CRC.

Hibi, T. (1992). *Algebraic Combinatorics on Convex Polytopes*. Glebe, N.S.W., Australia: Carslaw Publications.

Kalai, G. (1988). A simple way to tell a simple polytope from its graph. *J. Combin. Theory Ser. A*, 49(2), 381–383.

McMullen, P. & Shephard, G. C. (1971). *Convex polytopes and the upper bound conjecture*. London: Cambridge University Press. Prepared in collaboration with J. E. Reeve and A. A. Ball, London Mathematical Society Lecture Note Series, 3.

Stanley, R. P. (1999). *Enumerative combinatorics. Vol. 2*, volume 62 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press. With a foreword by Gian-Carlo Rota and appendix 1 by Sergey Fomin.

Stanley, R. P. (2012). *Enumerative combinatorics. Volume 1*, volume 49 of *Cambridge Studies in Advanced Mathematics*. Cambridge: Cambridge University Press, second edition.

Steinitz, E. (1922). Polyeder und raumeinteilungen. *Enzykl. math. Wiss.*, 3, 1–139.

Steinitz, E. & Rademacher, H. (1976). *Vorlesungen über die Theorie der Polyeder unter Einschluss der Elemente der Topologie*. Berlin: Springer-Verlag. Reprint der 1934 Auflage, Grundlehren der Mathematischen Wissenschaften, No. 41.

Thomas, R. R. (2006). *Lectures in geometric combinatorics*, volume 33 of *Student Mathematical Library*. Providence, RI: American Mathematical Society. IAS/Park City Mathematical Subseries.

Wagner, K. (1937). Über eine Eigenschaft der ebenen Komplexe. *Math. Ann.*, 114(1), 570–590.

West, D. B. (1996). *Introduction to graph theory*. Upper Saddle River, NJ: Prentice Hall Inc.

Whitney, H. (1932). Congruent Graphs and the Connectivity of Graphs. *Amer. J. Math.*, 54(1), 150–168.

Ziegler, G. M. (1995). *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.

Appendix A

Python Code to Determine Whether or not a Point Configuration is a Gale Diagram

This program was written in Python version 2.7.2.

```
#################################################################
##                                                             ##
##  This program checks whether or not a point configuration   ##
##  is the Gale diagram of some polytope.                      ##
##                                                             ##
##  That is, whether for each hyperplane passsing through the  ##
##  origin there are at least two points on either side of it. ##
##                                                             ##
#################################################################
from sympy import *
from fractions import Fraction
from itertools import *
from sys import exit
```

```
#####
#   This function checks some trivial cases
#   to see if it is possible for the point
#   configuration to be a Gale diagram.
#####
def dim_check(v, g):
    if min([v,g])<0:            # If either v, or g is negative,
        return 0                # then return a command to kill
                                # the program.
    else:
        if v in set([1,2,3]):   # All polytopes with 1, 2, or 3
            if g==0:            # vertices are simplices, and
                return 1        # therefore have a Gale diagram
            else:               # of dimension 0.
                return 0
        else:
            if v-g-1>=2:        # In a general polytope P,
                return 1        # dim(P)=v-g-1.  We have already
            else:               # checked if we have a 0- or 1-
                return 0        # polytope, so we need this >=2.

#####
#   This function gets the vectors in the
#   configuration and puts them in a
#   (g)x(v) matrix.
#####
def get_matrix(height, width):
```

```
    M=Matrix(height, width, lambda i,j: 0)
    for j in range(width):
        print "Please␣input␣the␣coordinates␣of␣point␣%s:␣" % (j
            +1)
        for i in range(height):
            M[i,j]=Fraction(raw_input("␣␣␣␣Entry␣%s:␣" % (i+1)))
    return M

#####
#   This function checks whether or not a
#   matrix M has rank rnk.
#####
def rank_ok(rnk, N):
    return len(N.rref()[1])==rnk


##########################
##########################
### Start the program. ###
##########################
##########################

# Get the dimension of the point confuguration
g = int(raw_input("Dimension␣of␣point␣configuration:␣"))
# Get the number of points in the configuration
v = int(raw_input("Number␣of␣points␣in␣the␣configuration:␣"))
```

```
if dim_check(v,g)==0:
    if v==1:
        print("There is no Gale Diagram with 1 point in a %s 
          dimensional space." % (g))
        quit
    else:
        print("There is no Gale Diagram with %s points in a %s 
          dimensional space." % (v,g))
        quit
else:
    M=get_matrix(g,v)
    print M
    #print M[1,2]
    if raw_input("Is this matrix correct?(y/n) " )!="y":
        quit
    else:
        if len(M.rref()[1])!=g:
            print "All of your points lie in a hyperplane."
        else:
            if g==1:                        # In the g=1 case, we
                pos=neg=0                   # just need 2 points
                for i in range(v):          # on either side of
                    if M[i]>0:              # the origin.
                        pos+=1
                    elif M[i]<0:
                        neg+=1
                if pos>=2 and neg>=2:
```

```
        print('This is the Gale diagram of a polytope of '
              'dimension %s.' % (v-g-1))
        quit
    else:
        print('This is not the Gale diagram of a polytope.')
        quit
else:
    N=Matrix(g, g-1, lambda i,j:0)
    for comb in combinations(range(v),g-1):
        for j in comb:
            for i in range(g):
                N[i,comb.index(j)]=M[i,j]
        if len(N.rref()[1])!=g-1:
            continue
        else:
            nml=Matrix(g,1, lambda i,j:0)
            for d in range(g):
                P=N.T
                P.col_del(d)
                nml[d]=(-1)**d*P.det()
            pos=neg=0
            for k in range(v):
                ip=0
                for b in range(g):
                    ip+=nml[b]*M[b,k]
```

```
            if ip>0:
                pos+=1
            elif ip<0:
                neg+=1
        if pos<2:
            print "The hyperplane perpendicular to"
            print nml,
            print "has an open half space with", pos,
            print "point(s) contained in it."
            exit
        elif neg<2:
            print "The hyperplane perpendicular to"
            print nml,
            print "has an open half space with", neg,
            print "point(s) contained in it."
            exit
print "This is the Gale diagram of a polytope of dimension",
print v-g-1
```