Systems of Linear Equations Exercises (Pre-Reading for Data Science MSc)

1. Define a system of linear equations and give the general matrix form.

$$Ax = b$$

where A = matrix of coefficients $A \in \mathbb{R}^{m \times n}$
x = vector of unknowns $x \in \mathbb{R}^n$
b = constant vector $b \in \mathbb{R}^m$

2. Write the following system in matrix form $Ax = b$:

$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$

$$\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

3. Solve the system

$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$

using substitution.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right] \quad \begin{array}{l} y = 1 \\ x = 2 \end{array}$$

4. Solve the same system using elimination.

$$2x + y = 5$$

$$x - y = 1 \Rightarrow y = x - 1$$

$$2x + x - 1 = 5 \qquad 2 - y = 1$$

$$3x = 6 \qquad y = 1$$

$$x = 2$$

5. Solve the same system using the matrix inverse method.

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \qquad A = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$$

$$= -\frac{1}{3} \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix}$$

$$x = A^{-1} b = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -2/3 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$R_1C_1 \to 2 \qquad \therefore x = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \equiv \begin{matrix} x = 2 \\ y = 1 \end{matrix}$$

$$R_2C_1 \to 1$$

6. Perform Gaussian elimination on

$$\left[\begin{array}{cc|c} 2 & 1 & 5 \\ 1 & -1 & 1 \end{array}\right]$$

to obtain row-echelon form.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right] \qquad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right]$$

7. Identify the pivots of

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Pivots = 2 $(R_1 C_1)$ and 1 $(R_2 C_2)$

Made with Goodnotes

8. Determine the rank of the system

$$\begin{cases} 2x + 2y = 2 \\ x + y = 1 \end{cases}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 2 & 2 & 2 \end{array}\right] \quad R_2 \to R_2 - 2R_1$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 0 \end{array}\right] \quad \rho = 1$$

9. State the condition on rank and number of unknowns for: (a) unique solution, (b) infinitely many solutions, (c) no solution.

a) $\rho = n$

b) $\rho < n$

c) Inconsistent

10. For the system

$$\begin{cases} x + y + z = 2 \\ 2x + 3y + z = 5 \end{cases}$$

determine the nullity and describe the dimension of the solution space.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 3 & 1 & 5 \end{array}\right] \qquad \nu = n - \rho$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & -1 & 3 \end{array}\right] \quad \rho = 2 \quad n = 3 \quad \nu = 1$$

Dimension = 1D