

Eigenvalues and Eigenvectors Exercises (Pre-Reading for Data Science MSc)

1. Define the eigenvalue equation for a square matrix M and nonzero vector v .

$$Mv = \lambda v \quad v \neq 0$$

2. For the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix},$$

write down the characteristic equation.

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = 0$$

3. Compute the eigenvalues of

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 = 0$$
$$= \lambda^2 - 7\lambda + 10 = 0$$

$$\lambda = 2, 5$$

4. For $\lambda = 5$, solve $(A - \lambda I)v = 0$ to find an eigenvector.

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$-x + 2y = 0 \Rightarrow x = 2y$$

$$x - 2y = 0$$

$$\therefore \vec{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

5. For $\lambda = 2$, solve $(A - \lambda I)v = 0$ to find an eigenvector.

$$\begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$$\begin{aligned} 2x + 2y &= 0 \\ x + y &= 0 \end{aligned} \Rightarrow x = -y$$

$$\vec{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

6. State the number of eigenvalues an $n \times n$ matrix can have. What happens if all eigenvalues are distinct?

An $n \times n$ matrix has $\leq n$ eigenvalues.
If all are distinct, there are exactly n eigenvectors

7. Diagonalise

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

$$A = PDP^{-1}$$

$$P = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^{-1} = \frac{1}{\det(P)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

Using DP we can confirm that
 $A = PDP^{-1}$

8. Compute A^3 using diagonalisation, given the eigen-decomposition $A = PDP^{-1}$.

$$A^3 = PD^3P^{-1} \quad D^3 = \begin{bmatrix} 125 & 0 \\ 0 & 8 \end{bmatrix}$$

$$PD^3 = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 125 & 0 \\ 0 & 8 \end{bmatrix} = \begin{bmatrix} 250 & 8 \\ 125 & -8 \end{bmatrix}$$

$$\therefore PD^3P^{-1} = \begin{bmatrix} 250 & 8 \\ 125 & -8 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} 86 & 78 \\ 39 & 47 \end{bmatrix}$$

A^3

9. State two properties of symmetric matrices regarding eigenvalues and eigenvectors.

1. All eigenvalues are real
2. Eigenvectors are orthogonal

10. Explain the connection between eigenvalues/eigenvectors and PCA (principal component analysis).

In PCA, eigenvectors give principal components (directions of maximum variance). Eigenvalues give the size of variance along each component.