Matrix Exercises (Pre-Reading for Data Science MSc)

1. Write down the definition of a symmetric matrix. Verify whether the matrix

$$\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix}$$

is symmetric.

Symmetric matrix = a matrix A that satisfies

$$A = A^T$$

$$A = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \quad A^T = \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \quad \therefore \begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \text{ is symmetric}$$

2. Compute the magnitude of the vector

$$\vec{v} = \begin{bmatrix} 3 \\ 4 \\ 12 \end{bmatrix}.$$

$$||v|| = \sqrt{3^2 + 4^2 + 12^2} = 13$$

3. For the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

compute AB and BA.

$$AB = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

$$BA = \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix}$$

AB:

$R_1C_1 \rightarrow 19$

$R_1C_2 \rightarrow 22$

$R_2C_1 \rightarrow 43$

$R_2C_2 \rightarrow 50$

BA

$R_1C_1 \rightarrow 23$

$R_1C_2 \rightarrow 34$

$R_2C_1 \rightarrow 31$

$R_2C_2 \rightarrow 46$

4. Show that multiplying any matrix by the identity matrix leaves it unchanged. Verify this with

$$C = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}.$$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (2×2)

CI:

$$\left.\begin{matrix} R_1C_1 \Rightarrow 2 \\ R_1C_2 \Rightarrow -1 \\ R_2C_1 \Rightarrow 0 \\ R_2C_2 \Rightarrow 3 \end{matrix}\right\} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

IC:

$$\left.\begin{matrix} R_1C_1 \Rightarrow 2 \\ R_1C_2 \Rightarrow -1 \\ R_2C_1 \Rightarrow 0 \\ R_2C_2 \Rightarrow 3 \end{matrix}\right\} \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix}$$

5. Compute the determinant of

$$D = \begin{bmatrix} 6 & 2 \\ 9 & 3 \end{bmatrix}.$$

Explain what the result means in geometric terms.

$$\det(D) = ad - bc$$
$$= 0$$

$\therefore$ D is singular $\Rightarrow$ it collapses all space to a lower dimension, thus it is not invertible

6. For the matrix

$$E = \begin{bmatrix} 4 & 7 \\ 2 & 6 \end{bmatrix}$$

compute its inverse if it exists.

$$E^{-1} = \frac{1}{\det(E)} \times \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 6 & -7 \\ -2 & 4 \end{bmatrix}$$

7. A dataset has feature vectors represented as rows: $(2,3), (4,5), (6,7)$. Write this dataset as a matrix X. What is the dimension of X?

$$X = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 6 & 7 \end{bmatrix} \qquad X \in \mathbb{R}^{2 \times 3}$$

8. Compute the dot product of

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}.$$

How would this operation be useful in regression?

Dot Product $= 4 + 10 + 18 = 32$

Predictions in linear regression are defined as $\hat{y} = \vec{w}\vec{x}$ where $\hat{y}$ is predicted outcome, $\vec{w}$ is the weights vector, and x is the features vector.

The DP combines features and weights into a single scalar prediction.

Made with Goodnotes

9. Compute the transpose of (a)

$$F = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}.$$

(b) Why is the transpose important in the normal equation for linear regression?

(a) $F^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$

(b) In the normal equation $\hat{\beta} = (X^T X)^{-1} X^T y$:

- X is $n \times p$ (data points $\times$ features) $n \overset{p}{\Gamma}$
- Without the transpose, XXB is not valid
- $X^T X$ is $p \times p$ which fits features vector β

10. Suppose

$$X = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}, \quad \beta = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Compute $y = X\beta$ as used in linear regression.

XB:

$$\left.\begin{array}{l} R_1 C_1 \to 4 \\ R_2 C_1 \to 10 \\ R_3 C_1 \to 16 \end{array}\right\} \begin{bmatrix} 4 \\ 10 \\ 16 \end{bmatrix}$$