Linear Algebra Unit Test (Pre-Reading for Data Science MSc) #1

1. Define a symmetric matrix and give an example. State one key property of its eigenvectors.

Symmetric matrix $\rightarrow A^{-1} = A \rightarrow \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$

↪ all real eigenvalues

2. For the matrix

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix},$$

find the determinant and state whether it is invertible.

$\det(A) = 3 \rightarrow$ it is invertible as $\det(A) \neq 0$

3. Solve the system

$$\begin{cases} 2x + y = 5 \\ x - y = 1 \end{cases}$$

using Gaussian elimination.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 3 \end{array}\right] \quad \begin{array}{l} y = 1 \\ x = 2 \end{array}$$

4. State the dimension of the vector space $\mathbb{R}^4$ and give a basis.

4D

$(1,0,0,0), (0,1,0,0), (0,0,1,0)\ (0,0,0,1)$

5. Compute

$$3 \times \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

$$\begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 9 \end{bmatrix}$$

6. For $v = (3, 4)$, compute its magnitude and the unit vector in the same direction.

$\|v\| = 5$ Unit vector $= \left(\frac{3}{5}, \frac{4}{5}\right)$

7. Compute the dot product of

$$a = (1, 2, 3), \quad b = (4, -5, 6).$$

$ab = 12$

8. Find the eigenvalues of

$$B = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}.$$

$$\det \begin{bmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{bmatrix} = (4-\lambda)(3-\lambda) - 2 = 0$$

$$\lambda^2 - 7\lambda + 10 = 0$$

$$\lambda = 5, 2$$

9. For the largest eigenvalue of B, compute one corresponding eigenvector.

$\lambda = 5$

$$\begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow x = 2y$$

$$\therefore \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

10. Diagonalise

$$C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

$C = PDP^{-1}$ $\quad \lambda = 2, 3$

$\lambda = 2$ $\quad \lambda = 3$

$$\begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0 \Rightarrow \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \therefore v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\therefore P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

11. Write the covariance matrix for two variables X_1, X_2 and explain what the diagonal and off-diagonal entries represent.

Diagonals ⇒ variances
Off-diagonals ⇒ covariances

$$\mathrm{Cov}(X_1, X_2) = \begin{bmatrix} \mathrm{Var}(X_1) & \mathrm{Cov}(X_1, X_2) \\ \mathrm{Cov}(X_1, X_2) & \mathrm{Var}(X_2) \end{bmatrix}$$

12. Compute the variance of $x = (2, 4, 6, 8)$.

$$\mathrm{Var}(x) = \frac{1}{3} \sum_{i=1}^{n} (x_i - 5)^2$$

$$= \frac{1}{3}(9 + 1 + 1 + 9) = 20/3$$

13. Compute the covariance of $x = (1, 2, 3)$ and $y = (2, 4, 5)$.

$$\mathrm{Cov}(x, y) = \frac{1}{2} \sum_{i=1}^{n} (x_i - 3)(y_i - 3)$$

$$= \frac{1}{2}(2 - 1 + 0) = \frac{1}{2}$$

$$\mathrm{Cov}(x, y) = \frac{1}{2} \sum_{i=1}^{n} (x_i - 2)\left(y_i - \frac{11}{2}\right)$$

$$= 1.5$$

14. Let

$$\Sigma = \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix}.$$

Verify that it is positive semidefinite by evaluating $z^\top \Sigma z$ for $z = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$$\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 4 \end{bmatrix}$$

$R_1 C_1 \Rightarrow 4 - 4 = 0$

$R_1 C_2 \Rightarrow -2 + 6 = 4$

$$\begin{bmatrix} 0 \\ 4 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 8 \qquad 8 \geq 0 \text{ so } \Sigma \text{ is PSD}$$

15. State the rank-nullity theorem and give an example with a 2×3 matrix.

$\nu = n - p$

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right]$$

$p + \nu = n$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Rank = 2

$\nu = 1$

$\therefore n = 3$

Made with Goodnotes