Logic & Proof Exercises (Pre-Reading for Data Science MSc)

1. Define what a proposition is and give two examples, one true and one false.

Proposition $\rightarrow$ a statement that is either true (T) or false ($\perp$), never both

T = 3 is prime $\quad\quad$ $\perp$ = 9 is prime

2. Construct the truth table for $(P \vee Q) \rightarrow (\neg P)$.

P	Q	$(P \vee Q) \rightarrow (\neg P)$
T	T	$\perp$
T	$\perp$	$\perp$
$\perp$	T	T
$\perp$	$\perp$	T

3. Show, using truth tables, that $\neg(P \vee Q)$ is logically equivalent to $(\neg P \wedge \neg Q)$.

P	Q	$\neg(P \vee Q)$
$\perp$	T	$\perp$
$\perp$	$\perp$	T
T	T	$\perp$
T	$\perp$	$\perp$

P	Q	$(\neg P \wedge \neg Q)$
$\perp$	T	$\perp$
$\perp$	$\perp$	T
T	T	$\perp$
T	$\perp$	$\perp$

4. Express the statement "Every data point has a label" in symbolic form using quantifiers.

$\forall x\ (\text{DataPoint}(x) \rightarrow \text{Labelled}(x))$

5. Write the negation of: $\forall x \in \mathbb{N},\ x^2 \geq x$.

$\exists x \in \mathbb{N},\ x^2 < x$

6. Prove directly: if n is even, then n^2 is even.

$n = 2k$

$n^2 = (2k)^2$

$= 4k^2 = 2(2k^2) \rightarrow$ even

7. Prove by contrapositive: if n^2 is odd, then n is odd.

Contrapositive: If n^2 is even, $n^2 = 2(2k^2)$ which is even.

8. Prove by contradiction: $\sqrt{2}$ is irrational.

$\sqrt{2} = \frac{a}{b}$, $\quad 2 = \frac{a^2}{b^2}$

$a^2 = 2b^2$ set b^2 as k,

$a = 2k \rightarrow b$ is even $\Big\} \therefore \sqrt{2}$ is irrational.

9. Prove by induction that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

- Base case $n=1$ LHS $=1$, RHS $=1$
- Assume statement holds for k
- Prove it holds for $k+1$

$$\sum_{i=1}^{n} i = \frac{k(k+1)}{2} + k+1 = \frac{k(k+1)}{2} + \frac{2(k+1)}{2}$$

$$= \frac{(k+1)(k+2)}{2}$$

10. Prove by cases that $n^2 \equiv 0$ or $1 \pmod 4$ for all integers n.

Case 1: even

$n = 2k \Rightarrow n^2 = 4k^2 \equiv 0 \pmod 4$

Case 2: odd

$n = 2k+1 \Rightarrow n^2 = 4k^2+1 \equiv 1 \pmod 4$

11. Define the logical connectives $\wedge, \vee, \neg, \rightarrow, \leftrightarrow$.

$\wedge$ = AND

$\vee$ = OR

$\neg$ = NOT

$\rightarrow$ = Implies

$\Leftrightarrow$ = If and only if

Made with Goodnotes

12. Construct the truth table for $(P \wedge Q) \vee (\neg Q)$.

P	Q	$(P \wedge Q) \vee (\neg Q)$
T	T	T
T	⊥	T
⊥	T	⊥
⊥	⊥	T

13. Verify, using truth tables, that $(P \to Q) \equiv (\neg P \vee Q)$.

P	Q	$P \Rightarrow Q$	$(\neg P \vee Q)$
T	T	T	T
T	⊥	⊥	⊥
⊥	T	T	T
⊥	⊥	T	T

14. Translate into symbolic logic: "There exists a student who has passed every exam."

$\exists x,\ \text{Student}(x) \wedge \text{PassedAll}(x)$

15. Write the negation of: $\exists x \in \mathbb{Z},\ x^2 = 2$.

$\forall x \in \mathbb{Z},\ x^2 \neq 2$

16. Prove by contradiction: there is no integer solution to $2n = 3m + 1$.

$2n = 3m + 1 \Rightarrow 2n \equiv 1 \pmod 3$

But $2n \equiv 0$ or $2 \pmod 3$

Contradiction $\Rightarrow$ no solutions

17. Give a constructive proof that there exists an even prime number.

2 is prime and even

18. Provide a non-constructive proof that there exist irrational a, b such that a^b is rational.

Consider $a = \sqrt{2}$ $b = \sqrt{2}$

$a^b = \sqrt{2}^{\sqrt{2}} \rightarrow$ irrational

Set $a \rightarrow \sqrt{2}^{\sqrt{2}}$

$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2$

19. Prove by induction that $2^n \geq n + 1$ for all $n \geq 0$.

- Base case $n = 0$, $2^0 = 1 \geqslant 0 + 1$
- Assume $2^k \geqslant k + 1$
- Therefore $2^{k+1} = 2 \times 2^k \geqslant 2(k+1) \geqslant (k+1) + 1$ holds