Reasoning About the World

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First InveniaCon
27 September 2017

Reasoning

If the butler killed the man, then there must be a pistol.

There is no pistol.

Therefore, the butler did not kill the man.

If the butler killed the man, then there must be a pistol.

There is no pistol.

Therefore, the butler did not kill the man.

If the cook killed the man, then there must be a knife.

There is a knife.

Therefore, the cook killed the man.

If $\overbrace{\text{the butler killed the man}}^{B}$, then $\overbrace{\text{there must be a pistol}}^{P}$.

There is no pistol.

Therefore, the butler did not kill the man.

If $\overbrace{\text{the cook killed the man}}^{C}$, then $\overbrace{\text{there must be a knife}}^{K}$.

There is a knife.

Therefore, the cook killed the man.

If B, then P.

$\overline{P}$.

Therefore, $\overline{P}$.

If C, then K.

K.

Therefore, C.

$$B \implies P$$

$$\overline{P}$$

$$\therefore \overline{B}$$

$$C \implies K$$

$$K$$

$$\therefore C$$

valid:	$B \implies P$
(modus tollens)	$\overline{P}$
	$\therefore \overline{B}$
invalid:	$C \implies K$
(logical fallacy)	K
	$\therefore C$

valid:	$B \implies P$
(modus tollens)	$\overline{P}$
	$\therefore \overline{B}$
?	$C \implies K$
	K
	$\therefore C$ becomes more plausible

valid:	$B \implies P$
(modus tollens)	$\overline{P}$
	$\therefore \overline{B}$
?	$C \implies K$ becomes more plausible
	K
	$\therefore C$ becomes more plausible

? $B \implies P$ becomes more plausible

$\overline{P}$

$\therefore \overline{B}$ becomes more plausible

? $C \implies K$ becomes more plausible

K

$\therefore C$ becomes more plausible

- Propositions have a **degree of plausibility**.

Plausible Reasoning

- Propositions have a **degree of plausibility**.
- Reasoning depends on **background information**.

Notation (Plausibility)

$(A \mid X)$: plausibility of A given background information X.

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Goal: figure out what exactly plausibility is.

Representation of Plausibility

Assumption (Representation)

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- Plausibility is ordered.

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- Plausibility is ordered.
- Between any two plausibilities, we can find another plausibility.

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- Between any two plausibilities, we can find another plausibility.

Lemma (Representation)

Plausibility can be represented by **real numbers**.

Truth

Truth

Assumption (Truth)

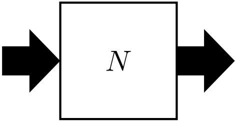
- There exists a plausibility T such that $(A \mid X) \le \mathsf{T}$ for all A.
- $(\text{tautology} \mid X) = \mathsf{T}$.

Negation

Negation

$$(\overline{A} \mid X)$$

Negation

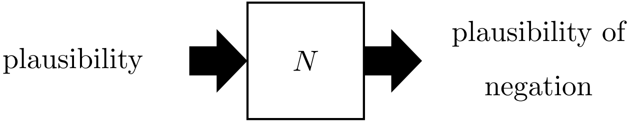


$(\overline{A} \mid X)$

Negation

$$(A \mid X) \Rightarrow \boxed{N} \Rightarrow (\overline{A} \mid X)$$

Negation



Assumption (Negation)

There exists a decreasing function N such that

$$(\overline{A} \,|\, X) = N(A \,|\, X)$$

for all A.

Negation

- Define $\mathsf{F} = N(\mathsf{T})$.

Negation

- Define $\mathsf{F} = N(\mathsf{T})$.
- $\mathsf{F} \leq (A \,|\, X) \leq \mathsf{T}$

Negation

- Define $\mathsf{F} = N(\mathsf{T})$.
- $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$:
  - $(\overline{A} \mid X) \leq \mathsf{T}$. (Definition of T)

Negation

- Define $\mathsf{F} = N(\mathsf{T})$.
- $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$:
  - $(\overline{A} \mid X) \leq \mathsf{T}$. (Definition of T)

  $\Rightarrow$ $N(\overline{A} \mid X) \geq N(\mathsf{T})$. ($N$ is decreasing)

Negation

- Define $\mathsf{F} = N(\mathsf{T})$.
- $\mathsf{F} \leq (A \,|\, X) \leq \mathsf{T}$:
    - $(\overline{A} \,|\, X) \leq \mathsf{T}$. (Definition of T)

    $\Rightarrow$ $N(\overline{A} \,|\, X) \geq N(\mathsf{T})$. ($N$ is decreasing)

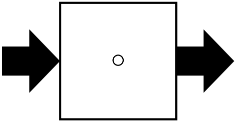
    $\Rightarrow$ $(A \,|\, X) \geq \mathsf{F}$. (Definition of N and F)

    QED.

Conjunction

$$(AB \mid X)$$

Conjunction



$(AB \mid X)$

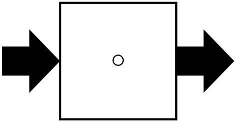
Conjunction

$(A \mid X)$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX)$



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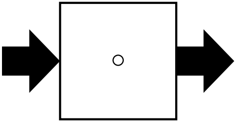


$(AB \mid X)$

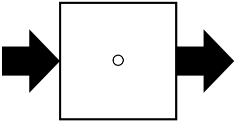
Conjunction

$(A \,|\, X)$

$(A \,|\, BX)$

$(B \,|\, X)$

$(B \,|\, AX)$



$(AB \,|\, X)$

$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{brown hair,} \\
AB &= \text{a blue eye and brown hair.}
\end{aligned}
$$

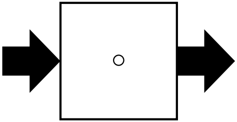
Conjunction

$(A \mid X) = \text{high}$

$(A \mid BX)$

$(B \mid X) = \text{high}$

$(B \mid AX)$



$(AB \mid X) = \text{high}$

$$A = \text{a blue eye,}$$

$$B = \text{brown hair,}$$

$$AB = \text{a blue eye and brown hair.}$$

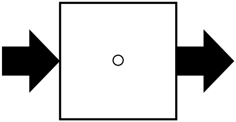
Conjunction

$(A \mid X)$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX)$



$(AB \mid X)$

$$A = \text{a blue eye,}$$

$$B = \text{a green eye,}$$

$$AB = \text{a blue eye and a green eye.}$$

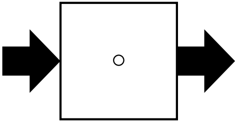
Conjunction

$(A \mid X) = \text{high}$

$(A \mid BX)$

$(B \mid X) = \text{high}$

$(B \mid AX)$



$(AB \mid X) = \text{low}$

$$A = \text{a blue eye},$$

$$B = \text{a green eye},$$

$$AB = \text{a blue eye and a green eye}.$$

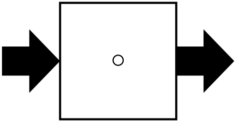
Conjunction

$(A \mid X)$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX)$



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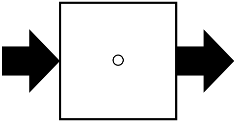


$(AB \mid X)$

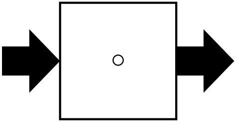
Conjunction

$(A \mid X)$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX)$



$(AB \mid X)$

$$A = \text{a blue eye,}$$

$$B = \text{a green eye,}$$

$$AB = \text{a blue eye and a green eye.}$$

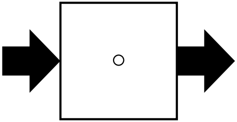
Conjunction

$(A \mid X) = \text{high}$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX) = \text{low}$



$(AB \mid X) = \text{low}$

$$A = \text{a blue eye},$$

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Conjunction

$(A \mid X)$

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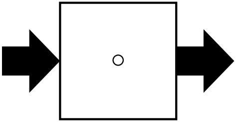
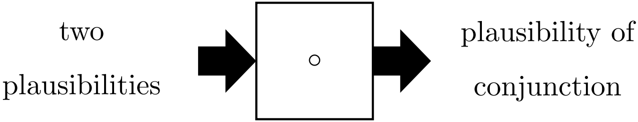
$(B \mid AX)$



$(AB \mid X)$

Conjunction



Conjunction

Assumption (Conjunction)

There exists a function $\circ$ such that

$$(AB \mid X) = (A \mid X) \circ (B \mid AX)$$

for all A and B.

Conjunction

- $x \circ \mathsf{T} =$

Conjunction

- $x \circ \mathsf{T} = x$

Conjunction

- $x \circ \mathsf{T} = x$:
    - $(A \mid X) = (A(B + \overline{B}) \mid X)$.

Conjunction

- $x \circ \mathsf{T} = x$:
    - $(A \mid X) = (A(B + \overline{B}) \mid X)$.
    - $(A(B + \overline{B}) \mid X) = (A \mid X) \circ (B + \overline{B} \mid AX)$. (Definition of $\circ$)

Conjunction

- $x \circ \mathsf{T} = x$:
  - $(A \mid X) = (A(B + \overline{B}) \mid X)$.
  - $(A(B + \overline{B}) \mid X) = (A \mid X) \circ (B + \overline{B} \mid AX)$. (Definition of $\circ$)
  - $(B + \overline{B} \mid AX) = \mathsf{T}$. (Definition of T)

Conjunction

- $x \circ \mathsf{T} = x$:
  - $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.
  - $(A(B + \overline{B}) \,|\, X) = (A \,|\, X) \circ (B + \overline{B} \,|\, AX)$. (Definition of $\circ$)
  - $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

  $\Rightarrow$ $(A \,|\, X) = (A \,|\, X) \circ \mathsf{T}$.

  QED.

Plausible Reasoning

Conjunction

- $x \circ \mathsf{F} =$

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$:

$$\Rightarrow \; (\overline{A\overline{A}} \,|\, X) = \mathsf{T}. \qquad \text{(Definition of } \mathsf{T}\text{)}$$

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$:

$$\Rightarrow \; (\overline{A\overline{A}} \,|\, X) = \mathsf{T}. \qquad \text{(Definition of } \mathsf{T}\text{)}$$

$$\Rightarrow \; N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T}).$$

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$:

$\Rightarrow\ (\overline{A\overline{A}} \,|\, X) = \mathsf{T}.$ (Definition of T)

$\Rightarrow\ N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T}).$

$\Rightarrow\ (A\overline{A} \,|\, X) = \mathsf{F}.$ (Definitions of N and F)

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$:

$$\Rightarrow (\overline{A\overline{A}} \,|\, X) = \mathsf{T}. \qquad \text{(Definition of } \mathsf{T}\text{)}$$

$$\Rightarrow N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T}).$$

$$\Rightarrow (A\overline{A} \,|\, X) = \mathsf{F}. \qquad \text{(Definitions of } N \text{ and } \mathsf{F}\text{)}$$

  - $\underbrace{(A\overline{A} \,|\, X)}_{\mathsf{F}} = (A \,|\, X) \circ (\overline{A} \,|\, AX).$ (Definition of $\circ$)

Conjunction

- $x \circ \mathsf{F} = \mathsf{F}$:

  $\Rightarrow (\overline{A\overline{A}} \,|\, X) = \mathsf{T}.$ (Definition of T)

  $\Rightarrow N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T}).$

  $\Rightarrow (A\overline{A} \,|\, X) = \mathsf{F}.$ (Definitions of N and F)

  - $\underbrace{(A\overline{A} \,|\, X)}_{\mathsf{F}} = (A \,|\, X) \circ (\overline{A} \,|\, AX).$ (Definition of $\circ$)
  - $(\overline{A} \,|\, AX) = \mathsf{F}.$

- $x \circ \mathsf{F} = \mathsf{F}$:

  $\Rightarrow$ $(\overline{A\overline{A}} \,|\, X) = \mathsf{T}$. (Definition of T)

  $\Rightarrow$ $N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T})$.

  $\Rightarrow$ $(A\overline{A} \,|\, X) = \mathsf{F}$. (Definitions of N and F)

  - $\underbrace{(A\overline{A} \,|\, X)}_{\mathsf{F}} = (A \,|\, X) \circ (\overline{A} \,|\, AX)$. (Definition of $\circ$)
  - $(\overline{A} \,|\, AX) = \mathsf{F}$.

  $\Rightarrow$ $\mathsf{F} = (A \,|\, X) \circ \mathsf{F}$.

  QED.

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$(ABC \mid X)$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$(ABC \mid X) = (A(BC) \mid X)$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$\begin{aligned}(ABC \mid X) &= (A(BC) \mid X)\\ &= (A \mid X) \circ (BC \mid AX)\end{aligned}$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big),
\end{aligned}
$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big),
\end{aligned}
$$

$$
(ABC \mid X) = ((AB)C \mid X)
$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big),
\end{aligned}
$$

$$
\begin{aligned}
(ABC \mid X) &= ((AB)C \mid X) \\
&= (AB \mid X) \circ (C \mid ABX)
\end{aligned}
$$

Conjunction

- $x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big), \\
(ABC \mid X) &= ((AB)C \mid X) \\
&= (AB \mid X) \circ (C \mid ABX) \\
&= \Big((A \mid X) \circ (B \mid AX) \Big) \circ (C \mid ABX). \text{ QED.}
\end{aligned}
$$

Conjunction

$$x \circ \mathsf{T} = \mathsf{T} \circ x = x$$

$$x \circ \mathsf{F} = \mathsf{F} \circ x = \mathsf{F}$$

$$x \circ (y \circ z) = (x \circ y) \circ z$$

Conjunction

$$x \circ \mathsf{T} = \mathsf{T} \circ x = x \qquad x \cdot 1 = 1 \cdot x = x$$

$$x \circ \mathsf{F} = \mathsf{F} \circ x = \mathsf{F} \qquad x \cdot 0 = 0 \cdot x = 0$$

$$x \circ (y \circ z) = (x \circ y) \circ z \qquad x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

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  $\Rightarrow \circ \cong \times.$

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

- $p(AB \mid X) = p((A \mid X) \circ (B \mid AX)) = p(A \mid X)p(B \mid AX)$

  $\Rightarrow \circ \cong \times.$
- $p(B \mid AX) = \dfrac{p(AB \mid X)}{p(A \mid X)}.$

- $p(\mathsf{T}) = 1$

- $p(\mathsf{T}) = 1$:
    - $(A \mid X) = (A(B + \overline{B}) \mid X)$.

- $p(\mathsf{T}) = 1$:
  - $(A \mid X) = (A(B + \overline{B}) \mid X)$.

  $\Rightarrow\ p(A \mid X) = p(A(B + \overline{B}) \mid AX)$.

- $p(\mathsf{T}) = 1$:
  - $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

  $\Rightarrow\ p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

  $\Rightarrow\ p(A \,|\, X) = p(A \,|\, X)p(B + \overline{B} \,|\, AX)$. (Product Rule)

- $p(\mathsf{T}) = 1$:
  - $(A \mid X) = (A(B + \overline{B}) \mid X)$.

$\Rightarrow$ $p(A \mid X) = p(A(B + \overline{B}) \mid AX)$.

$\Rightarrow$ $p(A \mid X) = p(A \mid X)p(B + \overline{B} \mid AX)$. (Product Rule)

  - $(B + \overline{B} \mid AX) = \mathsf{T}$. (Definition of T)

- $p(\mathsf{T}) = 1$:
    - $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

    $\Rightarrow\ p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

    $\Rightarrow\ p(A \,|\, X) = p(A \,|\, X)p(B + \overline{B} \,|\, AX)$. (Product Rule)

    - $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

    $\Rightarrow\ p(A \,|\, X) = p(A \,|\, X)p(\mathsf{T})$.

- $p(\mathsf{T}) = 1$:
  - $(A \mid X) = (A(B + \overline{B}) \mid X)$.

  $\Rightarrow\ p(A \mid X) = p(A(B + \overline{B}) \mid AX)$.

  $\Rightarrow\ p(A \mid X) = p(A \mid X)p(B + \overline{B} \mid AX)$. (Product Rule)
  - $(B + \overline{B} \mid AX) = \mathsf{T}$. (Definition of T)

  $\Rightarrow\ p(A \mid X) = p(A \mid X)p(\mathsf{T})$.

  $\Rightarrow\ 1 = p(\mathsf{T})$.

  QED.

- $p(\mathsf{T}) = 1$:
  - $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

  $\Rightarrow \; p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

  $\Rightarrow \; p(A \,|\, X) = p(A \,|\, X)p(B + \overline{B} \,|\, AX)$. (Product Rule)
  - $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

  $\Rightarrow \; p(A \,|\, X) = p(A \,|\, X)p(\mathsf{T})$.

  $\Rightarrow \; 1 = p(\mathsf{T})$.

  QED.
- $p(\mathsf{F}) = 0$.

- $0 \leq p(A \mid X) \leq 1$

- $0 \leq p(A \mid X) \leq 1$:
    - $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$.

- $0 \leq p(A \mid X) \leq 1$:
    - $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$.

  $\Rightarrow\ p(\mathsf{F}) \leq p(A \mid X) \leq p(\mathsf{T})$. ($p$ is strictly increasing)

- $0 \leq p(A \mid X) \leq 1$:
    - $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$.

$\Rightarrow$ $p(\mathsf{F}) \leq p(A \mid X) \leq p(\mathsf{T})$. ($p$ is strictly increasing)

$\Rightarrow$ $0 \leq p(A \mid X) \leq 1$.

QED.

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \,|\, X) = 1 - p(A \,|\, X)$$

for all A.

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \mid X) = 1 - p(A \mid X)$$

for all A.

- $p(\overline{A} \mid X) = p(N(A \mid X)) = 1 - p(A \mid X)$.

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \mid X) = 1 - p(A \mid X)$$

for all A.

- $p(\overline{A} \mid X) = p(N(A \mid X)) = 1 - p(A \mid X)$.

  $\Rightarrow\ N \cong 1 - \,\cdot\,.$

Cox's Theorem

Theorem (Cox)

Plausibility is probability.

Revisited

valid:	$B \implies P$	$B \implies P$
(modus tollens)	$\overline{P}$	$\overline{P}$
	$\therefore \overline{B}$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$\overbrace{B \implies P}^{X}$	$\overbrace{B \implies P}^{X}$
(modus tollens)	$\overline{P}$	$\overline{P}$
	$\therefore \overline{B}$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$(P \mid B X)$	$\overbrace{B \implies P}^{X}$
(modus tollens)	$\overline{P}$	$\overline{P}$
	$\therefore \overline{B}$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$(P \mid B X) = \mathsf{T}$	$\overbrace{B \implies P}^{X}$
(modus tollens)	$\overline{P}$	$\overline{P}$
	$\therefore \overline{B}$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$p(P \mid BX) = 1$	$\overbrace{B \implies P}^{X}$
(modus tollens)	$\overline{P}$	$\overline{P}$
	$\therefore \overline{B}$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$p(P \mid BX) = 1$	$\overbrace{B \implies P}^{X}$
(modus tollens)		$\overline{P}$
	$p(B \mid \overline{P}X) = \ldots$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

$$p(B \mid \overline{P} X)$$

Revisited

$$p(B \mid \overline{P}X) = \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

Revisited

$$p(B \mid \overline{P}X) = \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

$$= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

Revisited

$$
\begin{aligned}
p(B \mid \overline{P}X) &= \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{(1 - p(P \mid BX))p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Sum Rule)}
\end{aligned}
$$

Revisited

$$
\begin{aligned}
p(B \mid \overline{P}X) &= \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{(1 - p(P \mid BX))p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Sum Rule)} \\
&= \frac{(1 - 1)p(B \mid X)}{p(\overline{P} \mid X)} && (X = (B \implies P))
\end{aligned}
$$

Revisited

$$
\begin{aligned}
p(B \mid \overline{P} X) &= \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{p(\overline{P} \mid BX) p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{(1 - p(P \mid BX)) p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Sum Rule)} \\
&= \frac{(1 - 1) p(B \mid X)}{p(\overline{P} \mid X)} && (X = (B \implies P)) \\
&= 0.
\end{aligned}
$$

Revisited

valid:	$p(P \mid BX) = 1$	$\overbrace{B \implies P}^{X}$
(modus tollens)		$\overline{P}$
	$p(B \mid \overline{P}X) = 0$	$\therefore \overline{B}$
invalid:	$C \implies K$	$C \implies K$
(logical fallacy)	K	K
	$\therefore C$	$\therefore C$

Revisited

valid:	$p(P \mid BX) = 1$	$\overbrace{B \implies P}^{X}$
(modus tollens)		$\overline{P}$
	$p(B \mid \overline{P}X) = 0$	$\therefore \overline{B}$
invalid:	$p(K \mid CY) = 1$	$\overbrace{C \implies K}^{Y}$
(logical fallacy)		K
	$p(C \mid KY) = \ldots$	$\therefore C$

Revisited

$$p(C \mid K Y)$$

Revisited

$$p(C \mid KY) = \frac{p(CK \mid Y)}{p(K \mid Y)} \qquad \text{(Product Rule)}$$

Revisited

$$p(C \mid KY) = \frac{p(CK \mid Y)}{p(K \mid Y)} \qquad \text{(Product Rule)}$$

$$= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} \qquad \text{(Product Rule)}$$

Revisited

$$
\begin{aligned}
p(C \mid KY) &= \frac{p(CK \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{1 \cdot p(C \mid Y)}{p(K \mid Y)} && (Y = (C \implies K))
\end{aligned}
$$

Revisited

$$
\begin{aligned}
p(C \mid KY) &= \frac{p(CK \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{1 \cdot p(C \mid Y)}{p(K \mid Y)} && (Y = (C \implies K)) \\
&= \frac{p(C \mid Y)}{p(K \mid Y)}.
\end{aligned}
$$

Revisited

valid: (modus tollens)

$$p(P \mid BX) = 1$$

$$p(B \mid \overline{P}X) = 0$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid: (logical fallacy)

$$p(K \mid CY) = 1$$

$$p(C \mid KY) = \frac{p(C \mid Y)}{p(K \mid Y)}$$

$$\overbrace{C \implies K}^{Y}$$
$$K$$
$$\therefore C$$

Revisited

valid: $p(P \mid BX) = 1$ $\quad \overbrace{B \implies P}^{X}$

(modus tollens) $\quad \overline{P}$

$p(B \mid \overline{P}X) = 0$ $\quad \therefore \overline{B}$

invalid: $p(K \mid CY) = 1$ $\quad \overbrace{C \implies K}^{Y}$

(logical fallacy) $\quad K$

$p(C \mid KY) \geq p(C \mid Y)$ $\quad \therefore C$

Revisited

valid: $$p(P \mid BX) = 1 \qquad \overbrace{B \implies P}^{X}$$

(modus tollens) $$\overline{P}$$

$$p(B \mid \overline{P}X) = 0 \qquad \therefore \overline{B}$$

$$p(C \mid KY) = \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)}$$

Revisited

$$p(B \mid \overline{P}X) = \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)}$$

$$p(C \mid KY) = \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)}$$

Plausibility **Probability**

$$(A \mid X) \xrightarrow{\quad p \quad} p(A \mid X)$$

Plausibility **Probability**

$$(A \mid X) \xleftarrow{\quad p^{-1} \quad} p(A \mid X)$$

"It is clear that, not only is the quantitative use of the rules of probability theory as extended logic the only sound way to conduct inference; it is the *failure* to follow those rules strictly that has for many years been leading to unnecessary errors, paradoxes, and controversies." (Jaynes, 2003, p. 143)