Agreeing to Disagree

Wessel Bruinsma

9 June 2019

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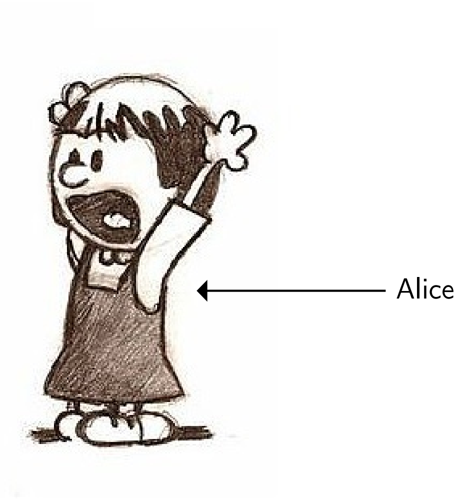

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Alice disagrees

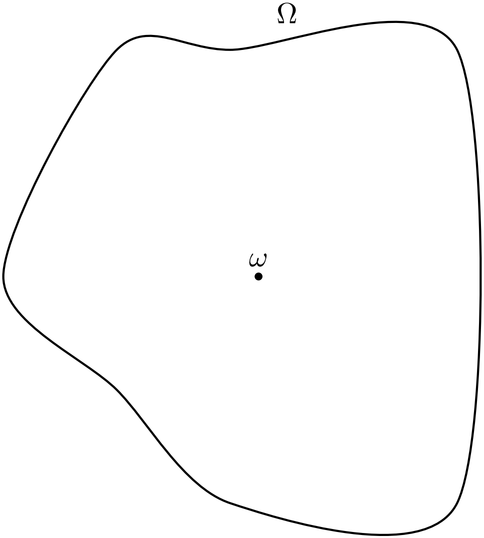
Image from relativelyinteresting.com/win-argument-according-science/.

Contents

A Model of Knowledge

Alice's World

Ω:
states of Alice's world.



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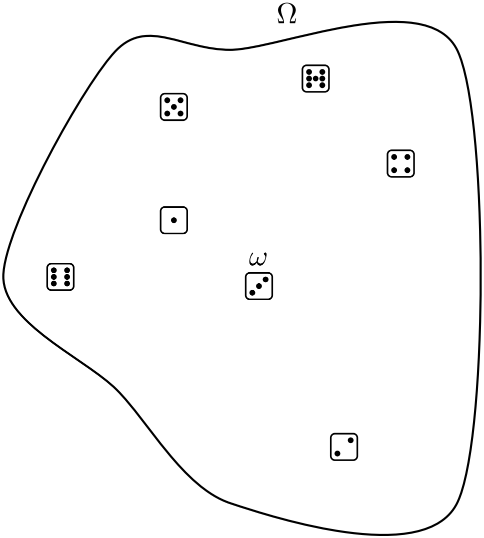

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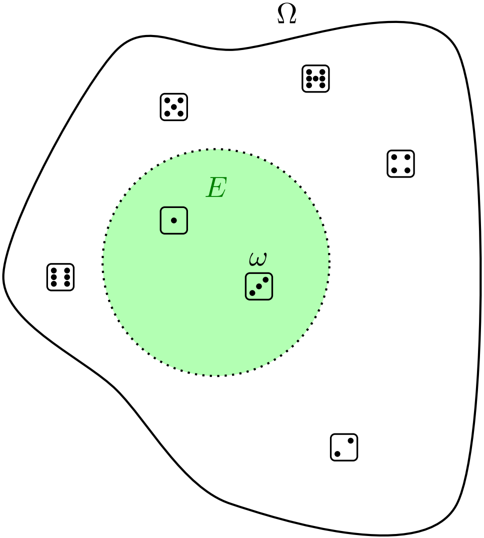

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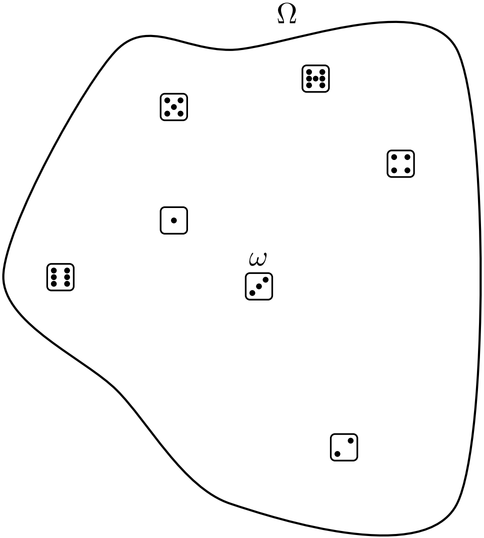

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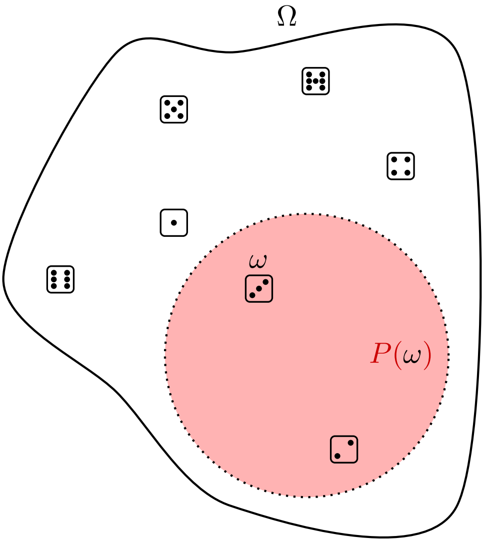

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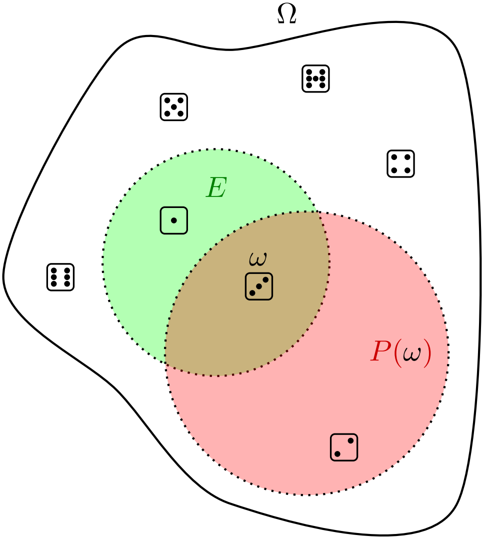

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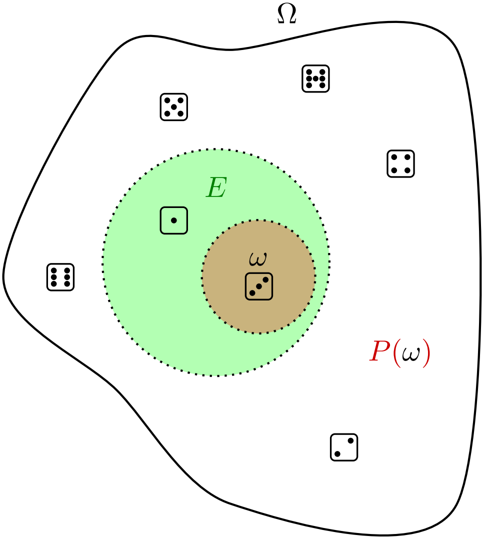

Alice's World (2)

$P(\omega) \subseteq E$:

at ω, Alice *knows* E.

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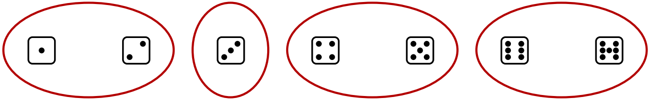
Alice's knowledge function:
$K(E) = \{\omega : \text{Alice knows } E\}$.

The Rare Die

$P_{\mathsf{Alice}}(\,\cdot\,)$:



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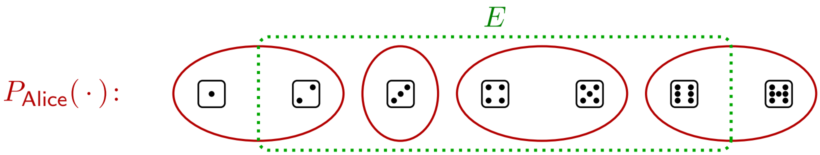

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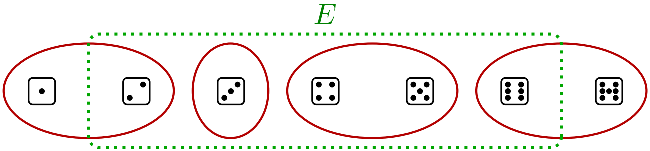

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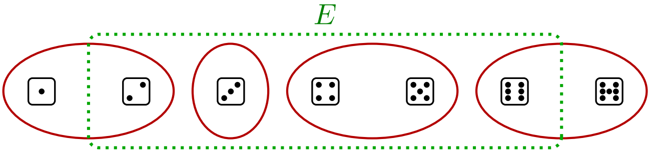

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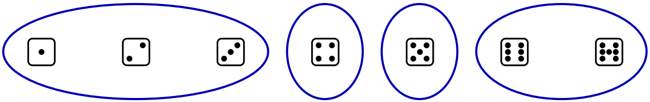

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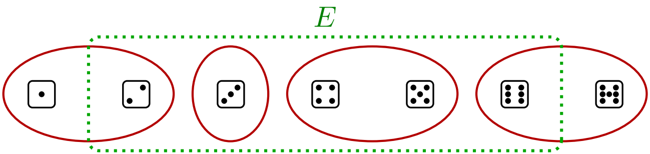

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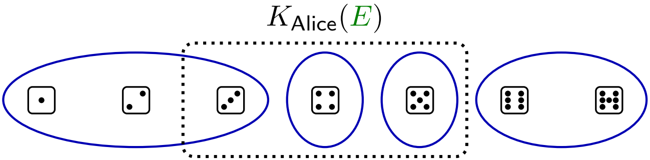

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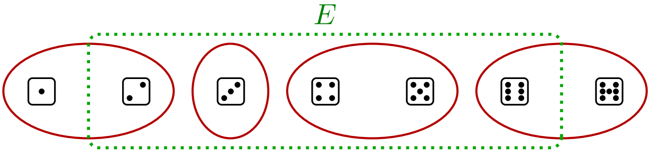

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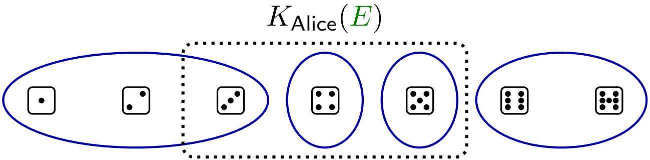

$$K_{\mathsf{Bob}}(K_{\mathsf{Alice}}(E)) = \{⚃, ⚄\}.$$

Common Knowledge

$\omega \in K_{\mathsf{Alice}}(E)$:

Alice knows E.

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$$\vdots$$

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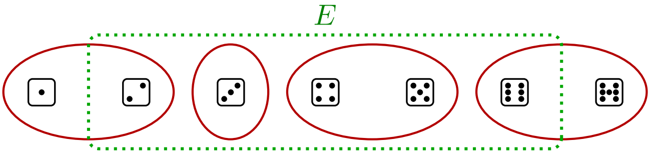

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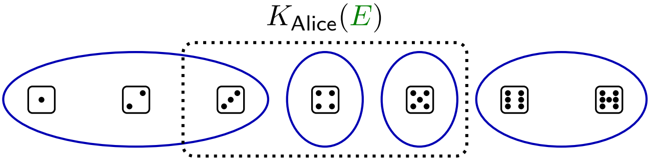

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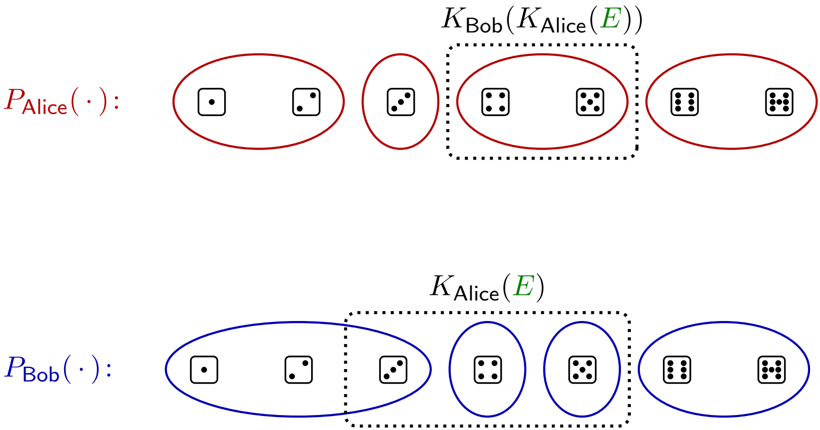

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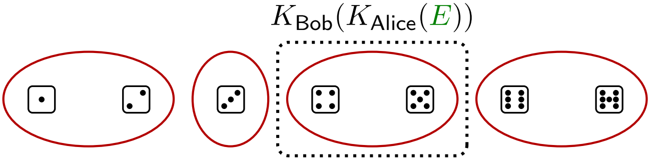

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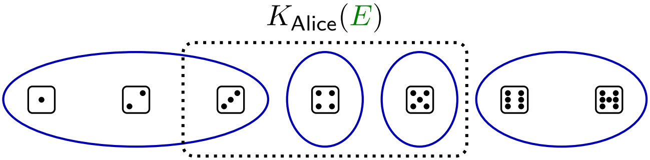

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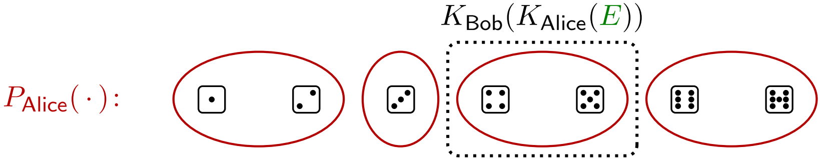

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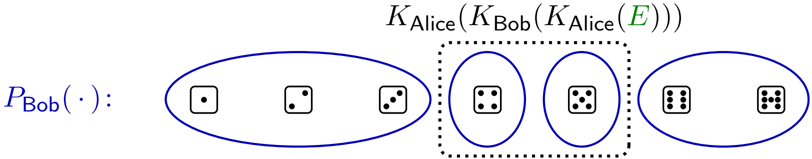

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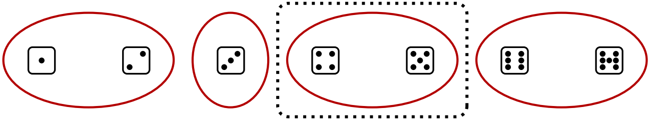

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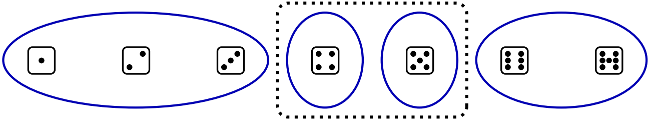

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The Exciting Bit

Aumann's Agreement Theorem

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- Alice and Bob cannot agree to disagree.

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- $\mu(E \mid F) = q_{\text{Bob}}$.
- $q_{\text{Alice}} = q_{\text{Bob}}$. $\blacksquare$

Disagreement is Unpredictable

- Alice's estimate of some random variable X:

$$\mathbb{E}_{\textsf{Alice}}(X) = \mathbb{E}(X \mid P_{\textsf{Alice}}(\omega)).$$

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Theorem ( )

$$\mathbb{E}_{\mathsf{Alice}}(\mathbb{E}_{\mathsf{Bob}'}(X)) < \mathbb{E}_{\mathsf{Alice}}(X)$$

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Theorem (Hanson (2002))

It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

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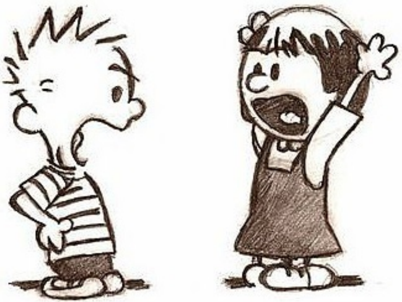
It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

- Alice cannot anticipate the direction of Bob's disagreement.



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Questioning our Assumptions

The Common Prior Assumption

Do we really have a common prior?

Principle of Monotonicity

$$F \subseteq E \implies K(F) \subseteq K(E).$$

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$$K(\text{know axioms}) \subseteq K(\text{know theorems}).$$

Principle of Substitution

$$F = E \implies K(F) = K(E).$$

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$$K(\text{triangle is equilateral}) = K(\text{triangle is equiangular}).$$

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The state-space model of knowledge respects extensional equality, but disregards the intentional dimension.

"We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious..."

—Aumann, in his original paper (Aumann, 1976)

Conclusion

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Alice disagrees

Image from `relativelyinteresting.com/win-argument-according-science/`.

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- Common prior

Alice disagrees

Image from relativelyinteresting.com/win-argument-according-science/.

Conclusion

- Common prior
- Accept model of knowledge

Alice disagrees

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Appendix

References

Aumann, R. J. (1976). Agreeing to disagree. *Annals of Statistics*, *4*(6), 1236–1239.

Hanson, R. (2002). Disagreement is unpredictable. *Economics Letters*, *77*(3), 365–369.