Spectral Methods in Gaussian Modelling

Topic 3: Variational Inference

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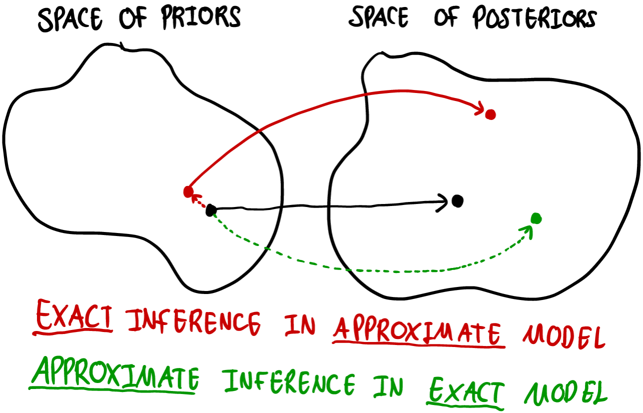
- Should perform **approximate** inference in **exact** model.
  - No overfitting!

Approximate Inference (2)



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- How to construct $q(f)$?

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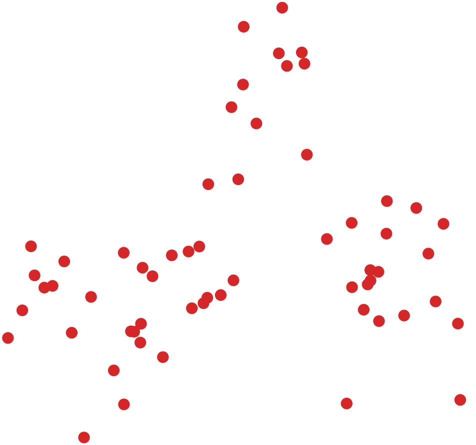

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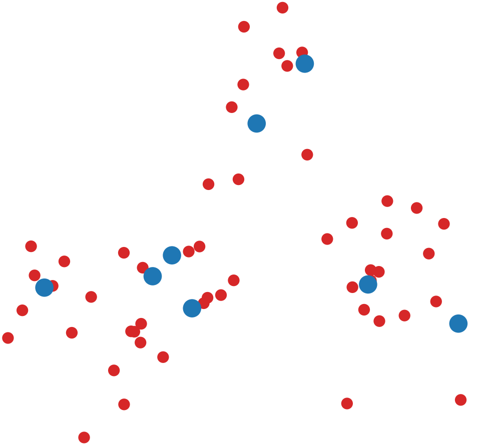

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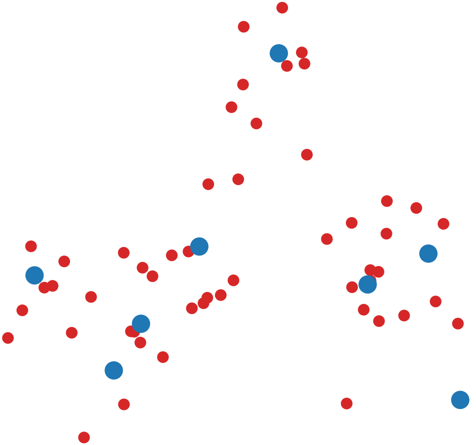

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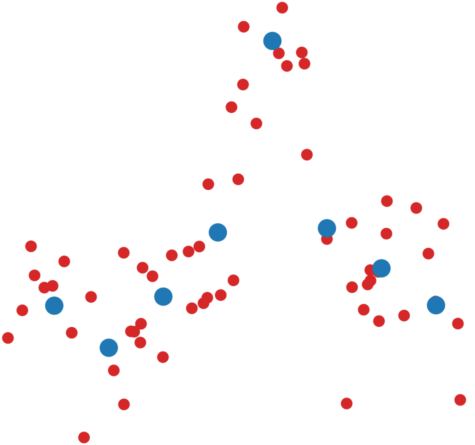

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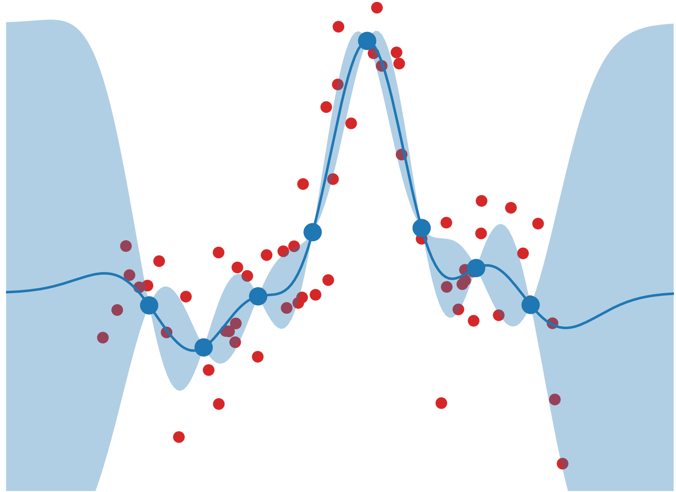

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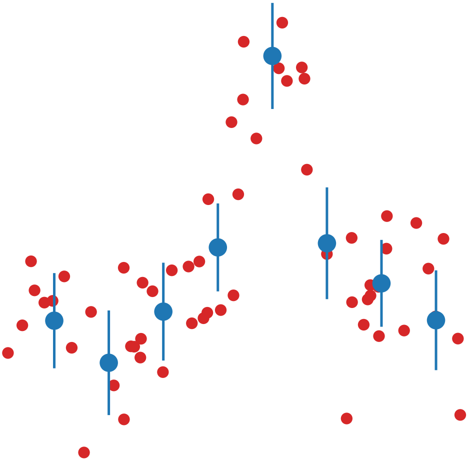

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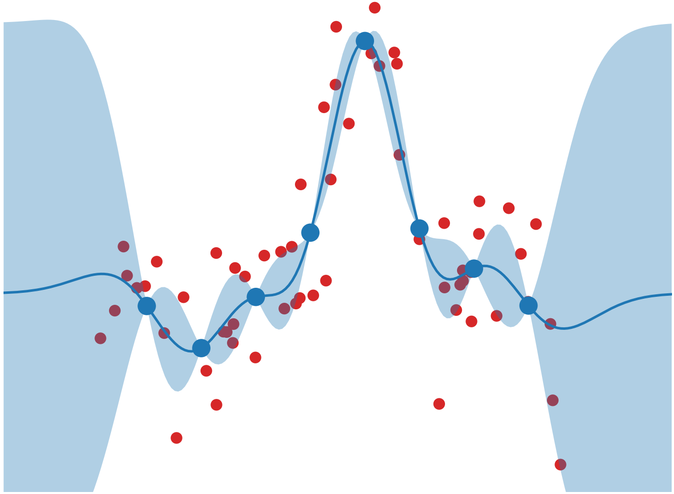

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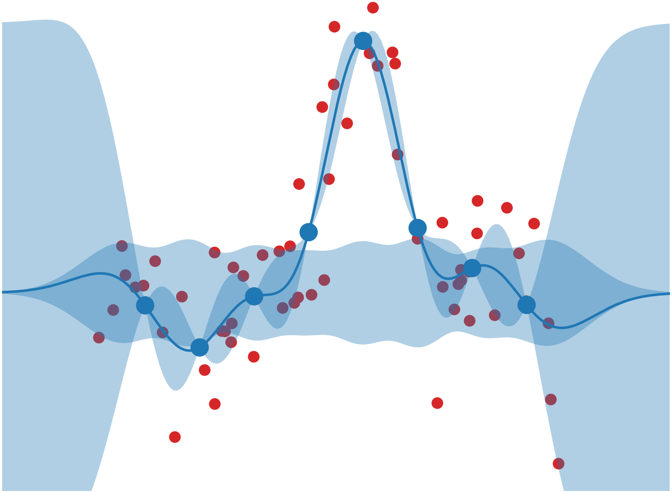

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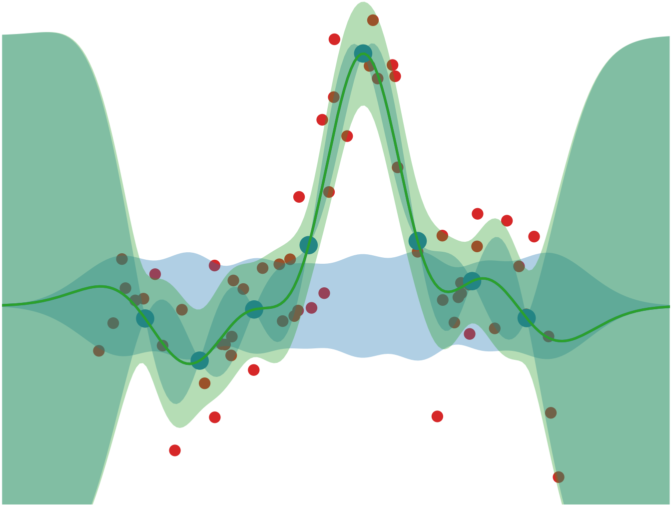

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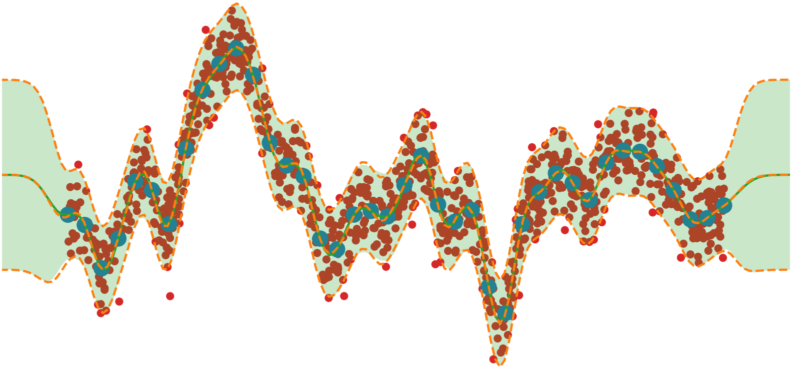
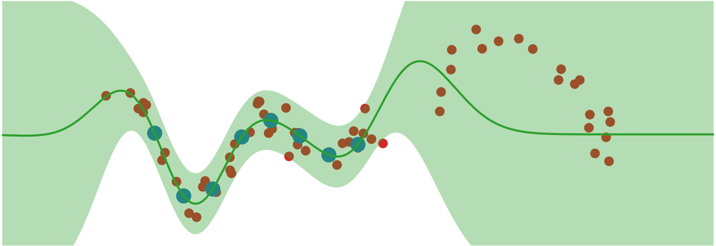
Example

- $N = 1000$ and $M = 40$; $25\times$ compression!



Inducing Points Versus SSA

- Inducing points: **local in time/space**



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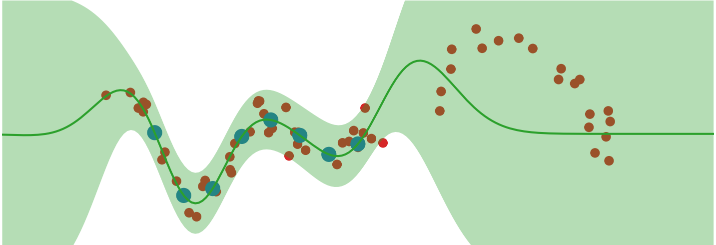

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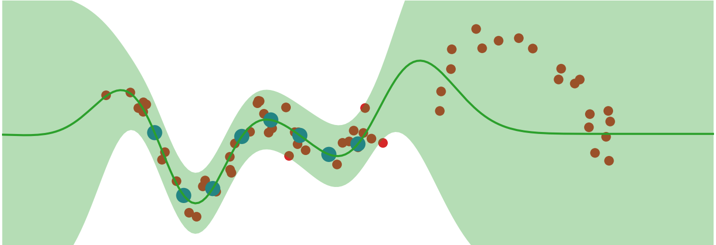

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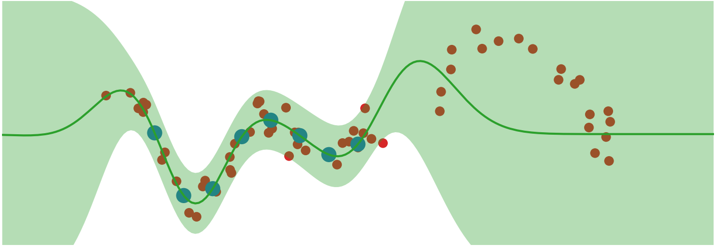

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- SSA: **local in spectrum**

Inducing Points Versus SSA (2)

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+ Appealing approximative construction

– Hidden $O(N^3)$ scaling

SSA:

+ Representative power

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Yes: Variational Fourier Features (VFFs)!

(Hensman et al., 2016)

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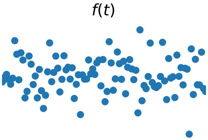

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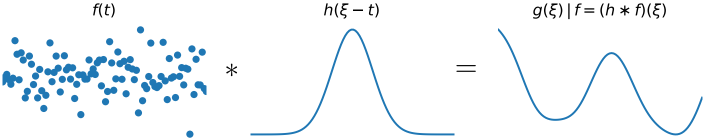

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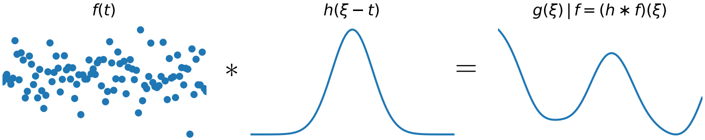

- In other cases, **improve** $q(f)$.

Variational Fourier Features

- Predictive mean in SSA:

$$\hat{f}^{(\mathrm{SSA})}(t) = \sum_{i=1}^{M} \alpha_i \cos(2\pi\xi_i t) + \sum_{i=1}^{M} \beta_i \sin(2\pi\xi_i t).$$

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- VFFs: engineer h such that $\hat{f}^{(\mathrm{VFF})}$ is also a Fourier expansion.

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Attempt 2: $g(\xi) \mid f = \int_{a}^{b} e^{-2\pi\iota\xi(t-a)} f(t)\,\mathrm{d}t.$ ($L^2[a,b]$-VFFs)

- Works, but edge effects near boundaries.

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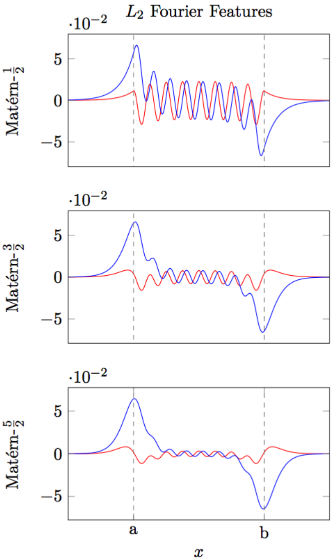

(Figure taken from Hensman et al. (2016).)

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where $(\phi_i)_{i=1}^{\infty}$ and $(\lambda_i)_{i=1}^{\infty}$ e.f.'s and e.v.'s of

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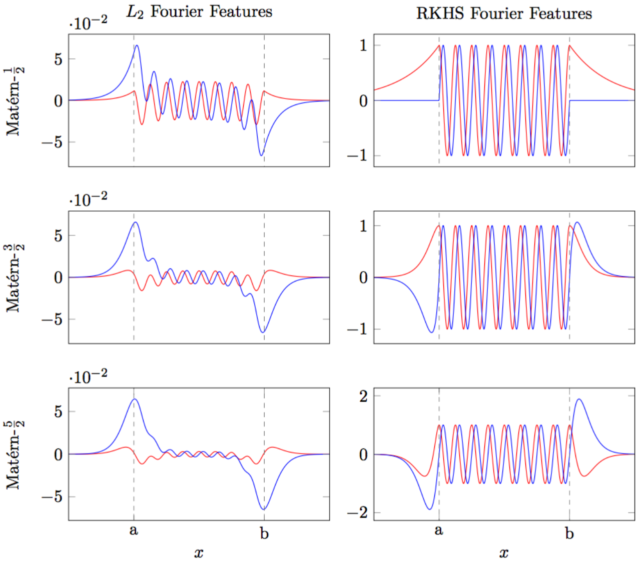
Variational Fourier Features (6)

- Works for Matérn kernels of half-integer order!

$\Rightarrow$ VFFs (RKHS-VFFs)

Variational Fourier Features (7)



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- Recent result by Burt et al. (2018):

Theorem

Fix $\varepsilon > 0$ and $\delta > 0$. Let $(t_i)_{i=1}^{\infty}$ be sampled i.i.d. from $\mathcal{N}(0, \alpha)$, let k be an exponentiated-quadratic kernel, and let μ have density $\mathcal{N}(0, \beta)$ with $\beta > 2\alpha$. Then there are $\tilde{N}$ and $\tilde{C}$ such that, for all $N > \tilde{N}$, the inter-domain point method with $M = \tilde{C} \log N$ eigenfunction inducing features achieves $\mathrm{D_{KL}}(q(f) \,\|\, p(f \mid \mathcal{D})) \leq \varepsilon$ with probability at least $1 - \delta$.

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- Nice behaviour:

$$\mathbb{E}[u_i u_j] = 1 \text{ if } i = j \text{ else } 0, \qquad \mathbb{E}[f(t_i)u_j] = \sqrt{\lambda_j}\phi_j(t_i).$$

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$$u_i \mid f = \frac{1}{\sqrt{\lambda_i}} \int f(t)\phi_i(t)\,\mathrm{d}\mu(t).$$

- Nice behaviour:

$$\mathbb{E}[u_i u_j] = 1 \text{ if } i = j \text{ else } 0, \qquad \mathbb{E}[f(t_i)u_j] = \sqrt{\lambda_j}\phi_j(t_i).$$

- Key quantity:

$$c = \operatorname{tr}\,(K_{ff} - K_{fu}K_{uu}^{-1}K_{fu}),$$

$$(K_{ff})_{ij} = \mathbb{E}[f(t_i)f(t_j)], \qquad (K_{fu})_{ij} = \mathbb{E}[f(t_i)u_j],$$
$$(K_{uu})_{ij} = \mathbb{E}[u_i u_j].$$

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Rates of Convergence (3)

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- Bound follows from application of Chebyshev's to

$$\frac{1}{N}c = \sum_{m=M+1}^{\infty} \lambda_m \left[\frac{1}{N}\sum_{i=1}^{N} \phi_m^2(t_i)\right]$$

combined with $\displaystyle\sum_{m=M+1}^{\infty} \lambda_m = O(A^M)$ for some $A \in (0, 1)$.

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- Can design inducing point methods amenable to convergence analysis.