Tea: Points and Circles

Wessel Bruinsma

University of Cambridge, CBL

10 February 2020

Japanese puzzle designer Naoki Inaba (2008):

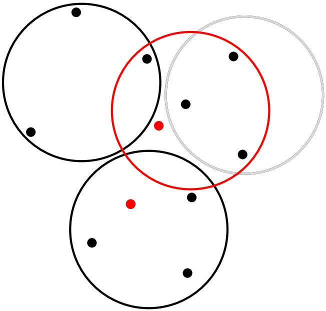
http://inabapuzzle.com/hirameki/suuri_4.html

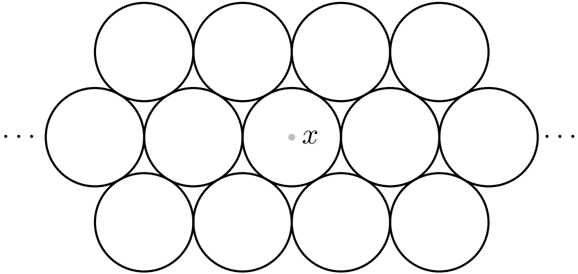
Can

10 points randomly positioned in the plane

always

be covered by non-intersecting unit circles?





- Fix 10 arbitrary points. Configuration of circles?
- Reasonable candidate: hexagonal circle packing. Offset x?
- Use probability to show that appropriate x always exists!
    - Technique called probabilistic method, pioneered by Paul Erdős.
    1. Consider random offset $x \sim \mathrm{Unif}$.
    2. Show that $\mathbb{P}(\text{all points covered}) > 0$.
- Key observation:

$$\mathbb{P}(\text{a single point covered}) = \frac{\pi}{\sqrt{12}} \approx 0.9069.$$

Proof of $\mathbb{P}(\text{all points covered}) > 0$

$$N = \text{number of points covered} = \sum_{i=1}^{10} \mathbb{1}(\text{point } i \text{ covered}).$$

$$\mathbb{E}[N] = \sum_{i=1}^{10} \mathbb{E}[\mathbb{1}(\text{point } i \text{ covered})] = \sum_{i=1}^{10} \mathbb{P}(\text{point } i \text{ covered}) \approx 9.069.$$

$$\mathbb{E}[N] = \sum_{i=1}^{10} \mathbb{P}(N = i)\, i. \qquad \text{(weighted average of } 1, \ldots, 10\text{)}$$

$$\implies \mathbb{P}(N = 10) > 0.$$

Therefore, with positive probability, $x \sim \mathrm{Unif}$ covers all 10 points.
In particular, an x that covers all 10 points exists!

Wrapping Up

- Yes, we can always cover 10 points with disjoint unit circles!
- Shown with the probabilistic method (Paul Erdős).

- Aloupis et al. (2012) refine the argument to 12 points.

Slides: `https://wessel.page.link/points-and-circles`.

Appendix

References

Aloupis, G., Hearn, R. A., Iwasawa, H., & Uehara, R. (2012). Covering points with disjoint unit disks. In *34th Canadian conference on computational geometry*.