Agreeing to Disagree

Wessel Bruinsma

1 November 2018

Image from `relativelyinteresting.com/win-argument-according-science/`.

Alice

Image from relativelyinteresting.com/win-argument-according-science/.

Alice disagrees

Image from relativelyinteresting.com/win-argument-according-science/.

I. A Model of Knowledge

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II. The Exciting Bit

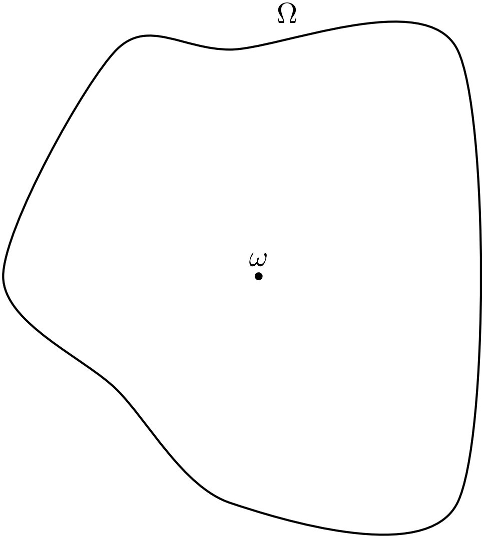
III. Questioning our Assumptions

A Model of Knowledge

Alice's World

Ω:

states of Alice's world.



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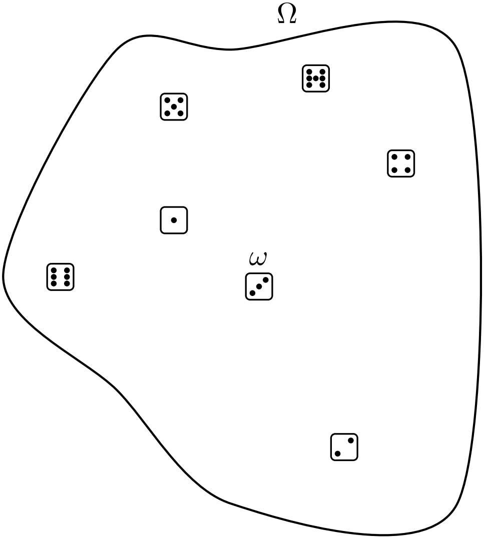

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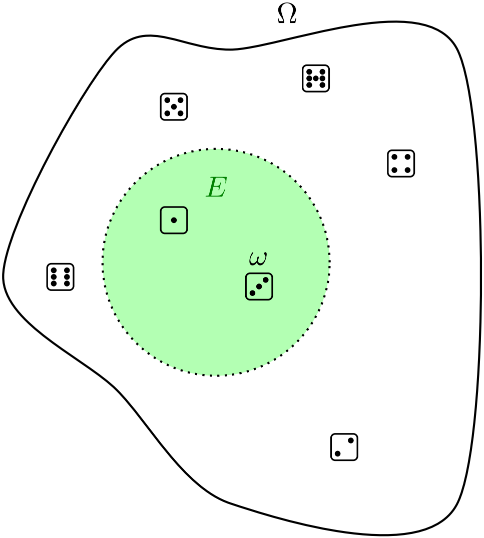

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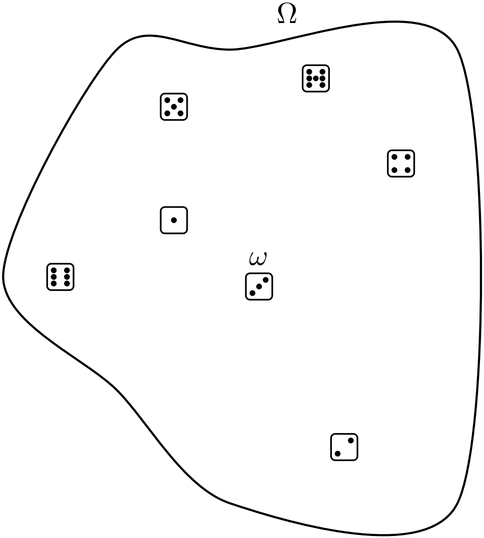

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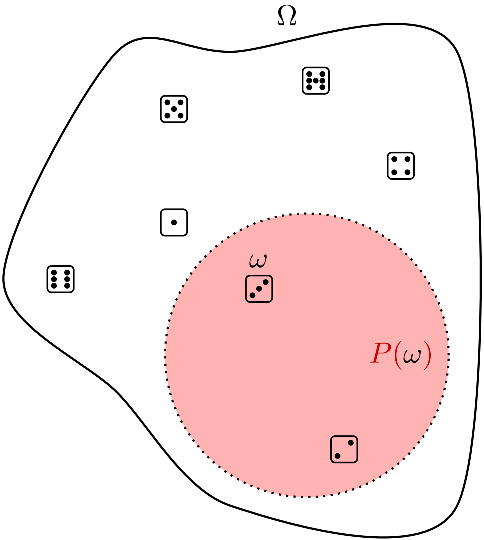

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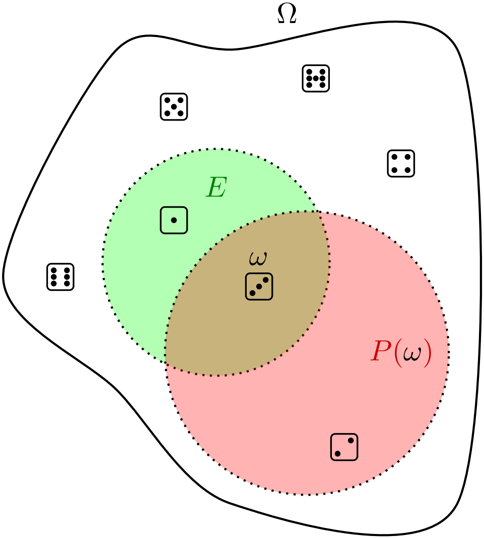

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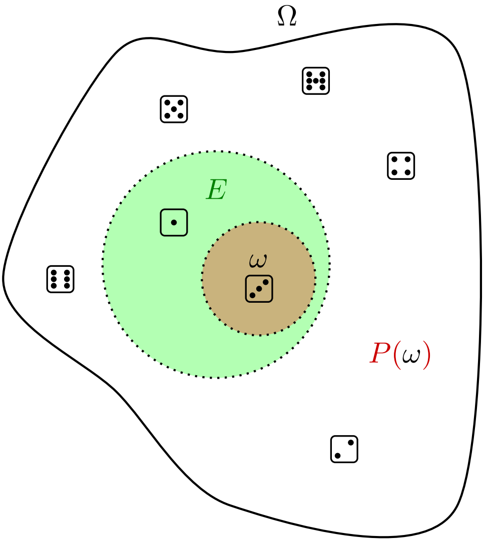

Alice's World (2)

$P(\omega) \subseteq E$:

at ω, Alice *knows* E.

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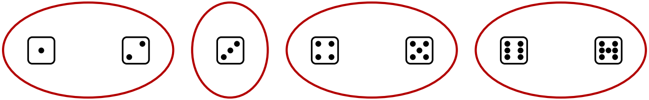
Alice's knowledge function:

$K(E) = \{\omega : \text{Alice knows } E\}$.

$P_{\text{Alice}}(\cdot):$



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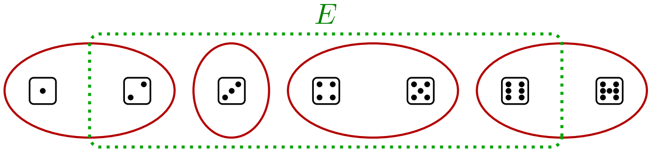
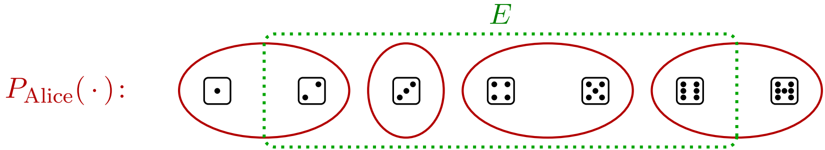
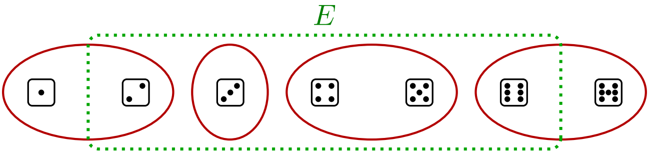


The Rare Die



$$K_{\text{Alice}}(E) = \{⚂, ⚃, ⚄\}.$$

$P_{\mathrm{Alice}}(\cdot):$



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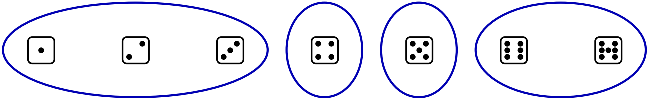
$P_{\mathrm{Bob}}(\cdot):$



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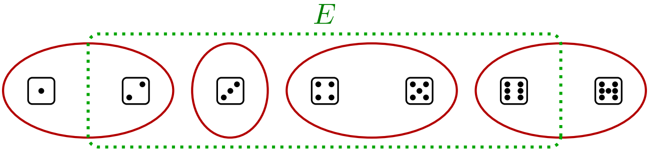

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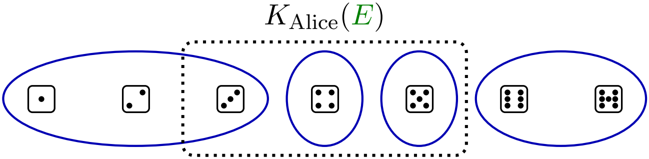

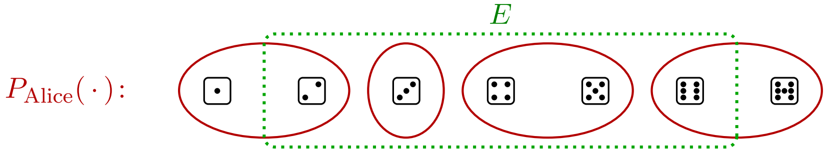

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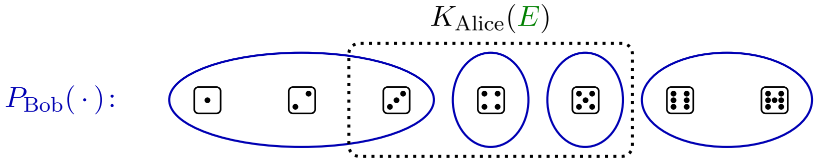

$$K_{\text{Bob}}(K_{\text{Alice}}(E)) = \{⚃, ⚄\}.$$

$\omega \in K_{\text{Alice}}(E)$:

Alice knows E.

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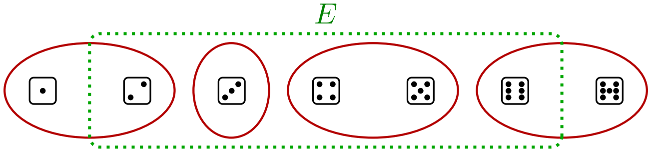
$$\vdots$$

At ω, E is **common knowledge** between Alice and Bob.

$P_{\text{Alice}}(\cdot)$:



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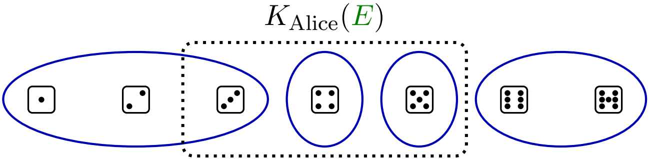

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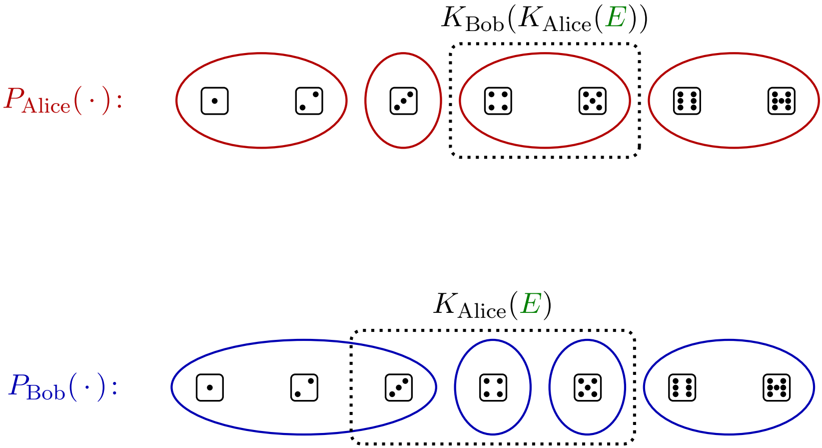

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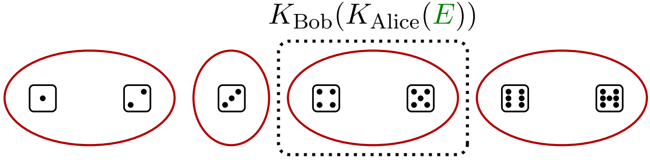

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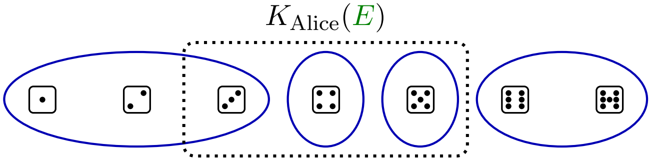

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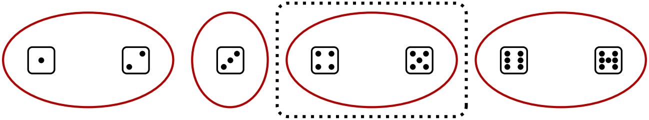

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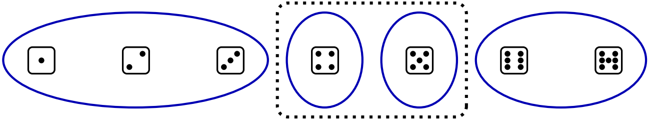

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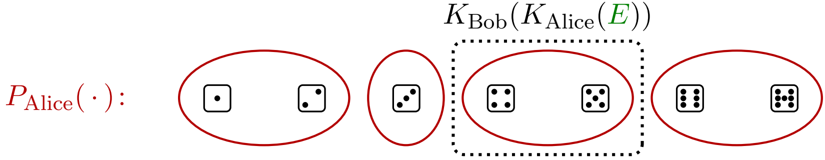

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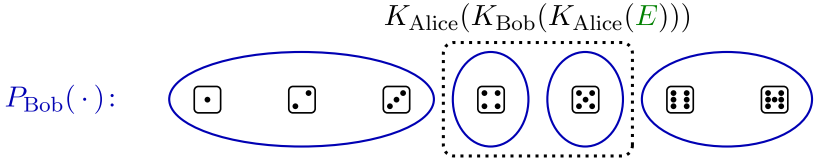

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The Exciting Bit

- Alice and Bob have a common prior μ.

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If $\mu(E \mid P_{\text{Alice}}(\omega))$ and $\mu(E \mid P_{\text{Bob}}(\omega))$ are common knowledge between Alice and Bob, then these beliefs must be equal.

- Alice and Bob cannot agree to disagree.

Sketch of Proof

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- $q_{\text{Alice}} = q_{\text{Bob}}$. $\blacksquare$

Disagreement is Unpredictable

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$$\mathbb{E}_{\text{Alice}}(X) = \mathbb{E}(X \mid P_{\text{Alice}}(\omega)).$$

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- Alice's estimate of future Bob's estimate: $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X))$.

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Theorem ( )

$$\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$$

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Theorem (Hanson [Han02])

It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

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It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

- Alice cannot anticipate the direction of Bob's disagreement.

Image from relativelyinteresting.com/win-argument-according-science/.

Questioning our Assumptions

Do we really have a common prior?

Principle of Monotonicity

$$F \subseteq E \implies K(F) \subseteq K(E).$$

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$$K(\text{know axioms}) \subseteq K(\text{know theorems}).$$

Principle of Substitution

$$F = E \implies K(F) = K(E).$$

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$$K(\text{triangle is equilateral}) = K(\text{triangle is equiangular}).$$

$$\boxed{F = E} \implies K(F) = K(E).$$

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$$F = E?$$

Extension Versus Intension

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The state-space model of knowledge respects extensional equality, but disregards the intentional dimension.

"We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious..."

—Aumann, in his original paper [Aum76]

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- Accept model of knowledge

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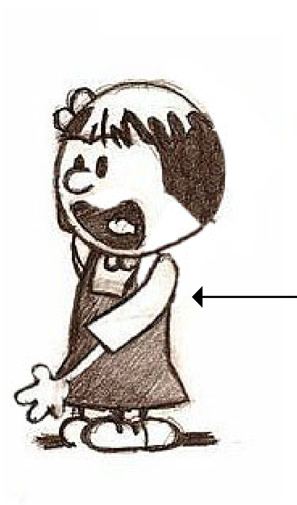

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Appendix

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