Spectral Methods in Gaussian Modelling

Topic 4: Spectrum Estimation

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Spectrum Estimation

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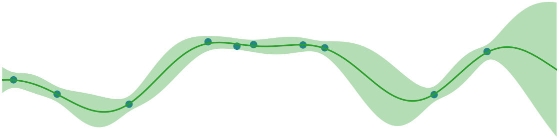
- Estimation difficult: samples limited, noisy, and nonuniform.



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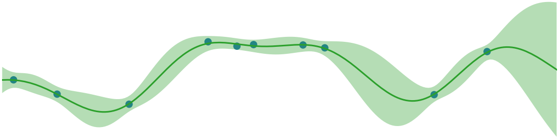

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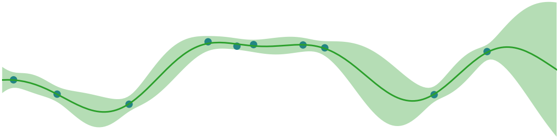

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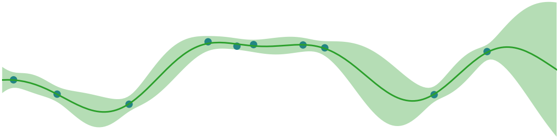

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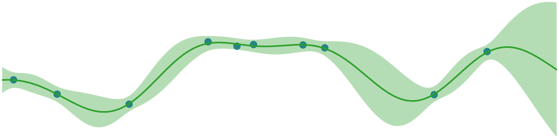

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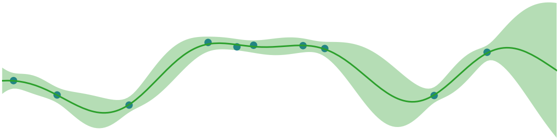

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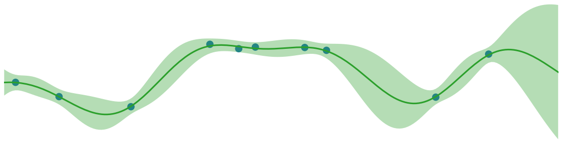

- Estimators:
  - parametric methods (SSA, SMK),
  - nonparametric methods (GPCM).
- Novel model by Tobar (2018).

Bayesian Nonparametric Spectral Estimation

- Generative model:

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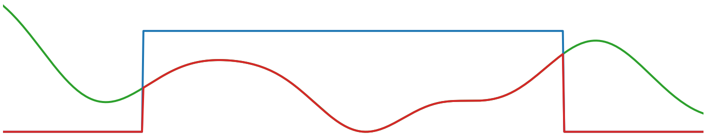

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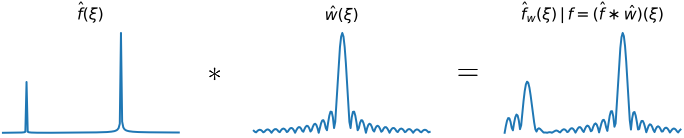

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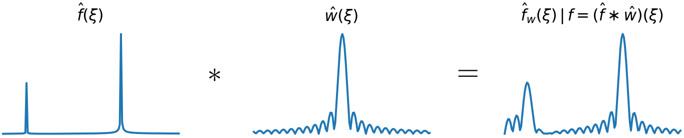

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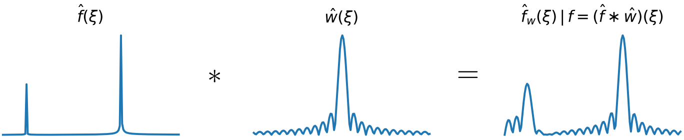

- Window functions:
  - algorithmic purposes (here: ensure integrability),
  - acquisition devices (e.g., sampling: $w = \mathrm{III}$).

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- Generative model:

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- Choices:
  - $w(t) = \exp(-\alpha\pi^2 t^2)$, (tractability)
  - $k =$ SMK or EQ in simple cases.

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\begin{aligned}
k_{\mathrm{Re}\,\hat{f}_w}(\xi,\xi') \\
k_{y(\mathrm{Re}\,\hat{f}_w)}(t,\xi) \\
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k_{\mathrm{Re}\,\hat{f}_w}(\xi,\xi') &= \frac{1}{2}(k_{\hat{f}_w}(\xi,\xi') + k_{\hat{f}_w}(\xi,-\xi')),\\
k_{y(\mathrm{Re}\,\hat{f}_w)}(t,\xi) &= \mathrm{Re}\,k_{y\hat{f}_w}(t,\xi),\\
k_{\mathrm{Im}\,\hat{f}_w}(\xi,\xi') &= \frac{1}{2}(k_{\hat{f}_w}(\xi,\xi') - k_{\hat{f}_w}(\xi,-\xi')),\\
k_{y(\mathrm{Im}\,\hat{f}_w)}(t,\xi) &= \mathrm{Im}\,k_{y\hat{f}_w}(t,\xi).
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$$\hat{r}_w(\xi+u,\xi'+u) = \mathcal{N}\big(\xi-\xi';0,\alpha\big)\mathcal{N}\big(u;\tfrac{1}{2}(\xi+\xi'),\tfrac{1}{4}\alpha\big).$$

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$$k_{\hat{f}_w}(\xi, \xi') = \mathcal{N}\big(\xi - \xi'; 0, \alpha\big)\Big(\hat{k}(u) * \mathcal{N}\big(u; 0, \tfrac{1}{4}\alpha\big)\Big)\big(\tfrac{1}{2}(\xi + \xi')\big)$$

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Interpretation of Posterior Mean

$$\mathbb{E}[\hat{f}_w(\xi) \mid e] = \int \hat{k}(u) \left(\sum_{i=1}^{N} e^{-2\pi \imath u t_i} (K_e^{-1} e)_i \right) \mathcal{N}\big(u; \xi, \tfrac{1}{2}\alpha\big) \, \mathrm{d}u.$$

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- Interpretation: DFT of whitened observations, weighted by prior, then smoothed due to window.

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- Interpretation: DFT of whitened observations, weighted by prior, then smoothed due to window.
- If prior uninformative, $K_e \approx I$, then weighted DFT in the limit:

$$\lim_{\alpha \to 0} \mathbb{E}[\hat{f}_w(\xi) \mid e] \approx \hat{k}(\xi) \sum_{i=1}^{N} e^{-2\pi\iota \xi t_i} e_i.$$

Comparison with Lomb–Scargle

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– Optimisation per ξ

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- Gaussian prior on A and B: LS recovers BNSE in the limit.

Experiment: Two Heart Rate Signals

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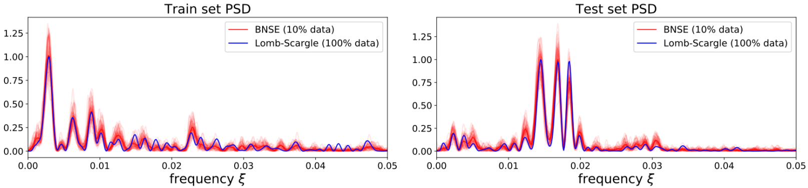

(Figure taken from Tobar (2018).)

Conclusion

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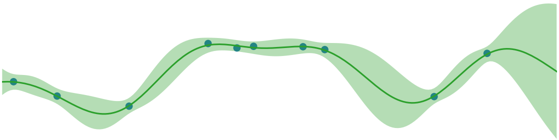

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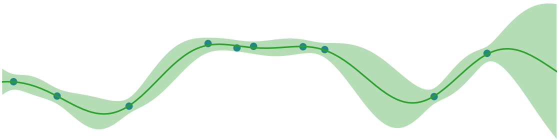

- Closed-form estimate of PSD.

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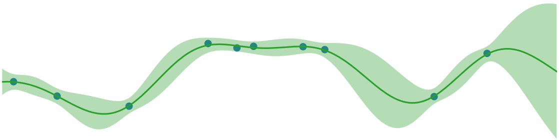

- Closed-form estimate of PSD.

$\Rightarrow$ Can optimise to find periodicities.

Appendix

References

Tobar, F. (2018). Bayesian nonparametric spectral estimation. In S. Bengio, H. Wallach, H. Larochelle, K. Grauman, N. Cesa-Bianchi, & R. Garnett (Eds.), *Advances in neural information processing systems 31*, Curran Associates, Inc. eprint: `https://arxiv.org/abs/1809.02196`