Agreeing to Disagree

Wessel Bruinsma

19 June 2018

Image from `relativelyinteresting.com/win-argument-according-science/`.

Alice

Image from `relativelyinteresting.com/win-argument-according-science/`.

Alice disagrees

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I. A Model of Knowledge

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II. The Exciting Bit

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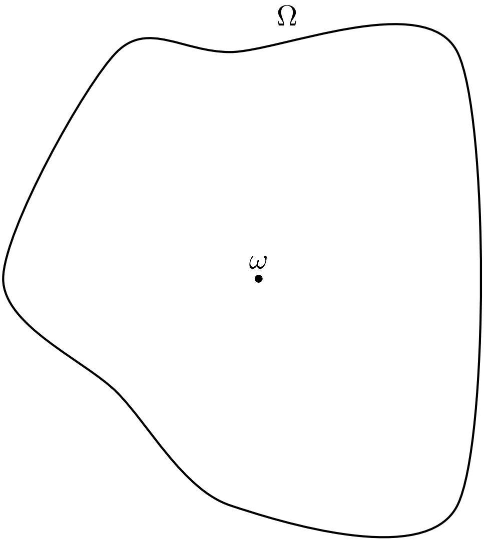
II. The Exciting Bit
III. Questioning our Assumptions

A Model of Knowledge

Ω:

states of Alice's world.



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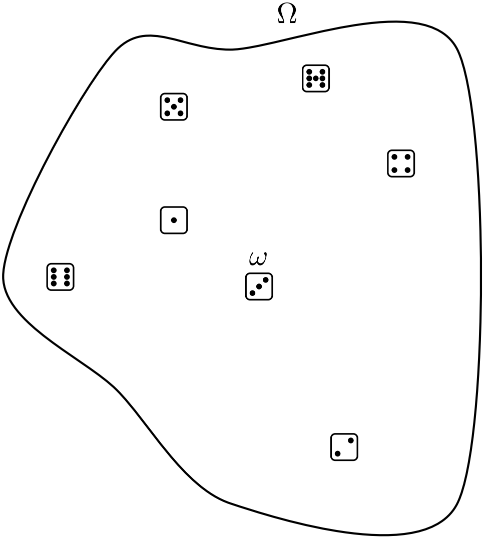

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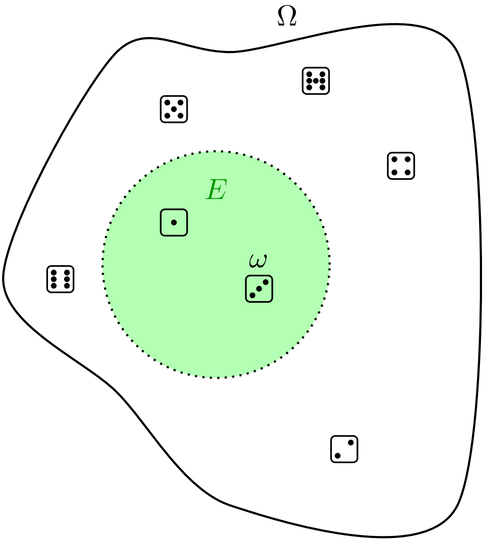
$E \subseteq \Omega$:

event.



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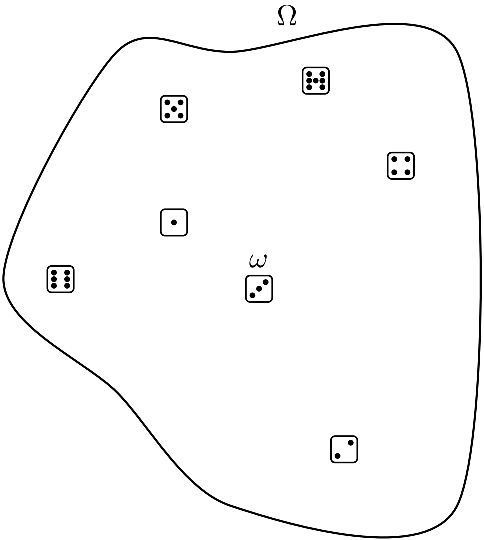

Alice's World

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states of Alice's world.

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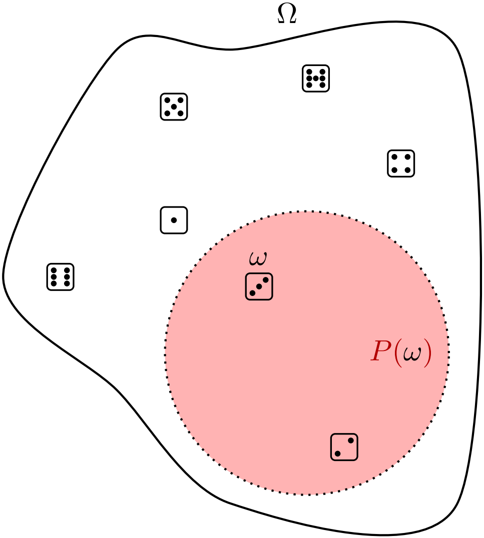
$P(\omega) \subseteq \Omega$:

Alice's knowledge.



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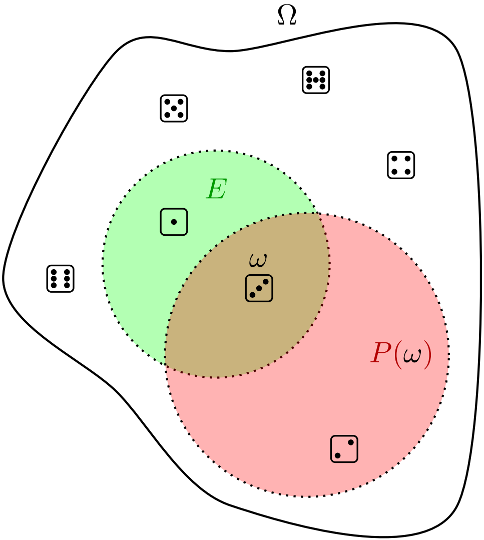

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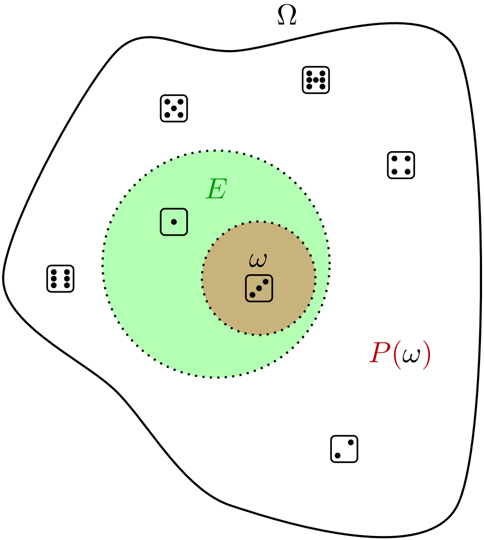

$P(\omega) \subseteq E$:

at ω, Alice *knows* E.

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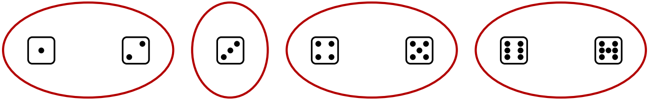
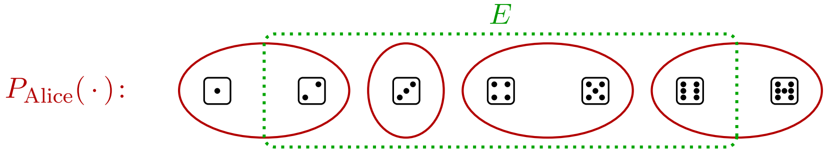
Alice's knowledge function:

$K(E) = \{\, \omega : \text{Alice knows } E \,\}$.

The Rare Die

$P_{\text{Alice}}(\cdot)$:





E
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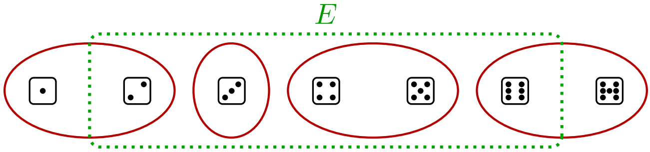

$K_{\text{Alice}}(E) = \{$ ⚂, ⚃, ⚄ $\}$.

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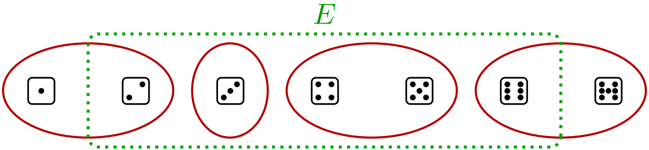

$K_{\text{Alice}}(E) = \{$ ⚂, ⚃, ⚄ $\}$.

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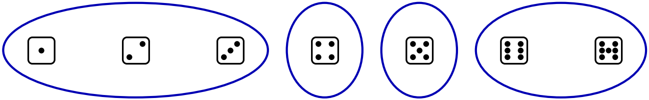

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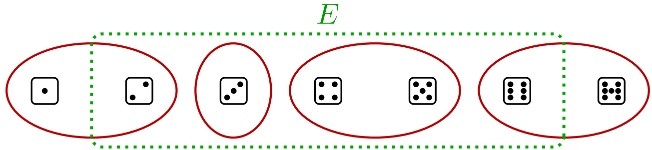

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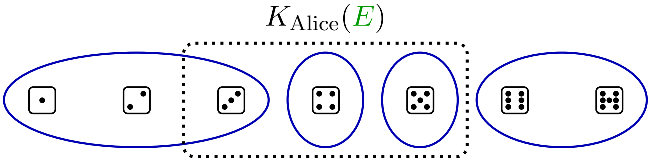

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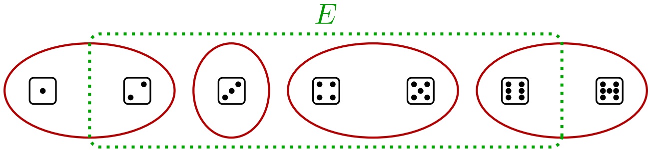

$$K_{\text{Alice}}(E) = \{\,⚂, ⚃, ⚄\,\}.$$

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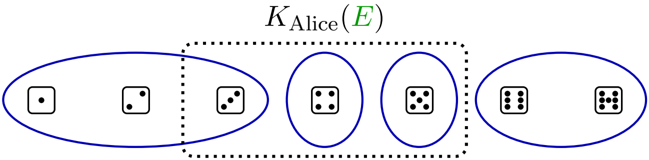

$$K_{\text{Bob}}(K_{\text{Alice}}(E)) = \{\,⚃, ⚄\,\}.$$

$\omega \in K_{\text{Alice}}(E)$:

Alice knows E.

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$\omega \in K_{\text{Bob}}(K_{\text{Alice}}(E))$:

Bob knows that Alice knows E.

Common Knowledge

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$$\vdots$$

At ω, E is **common knowledge** between Alice and Bob.

The Rare Die (2)

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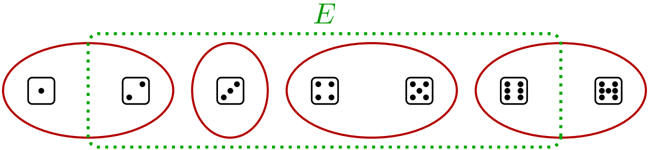

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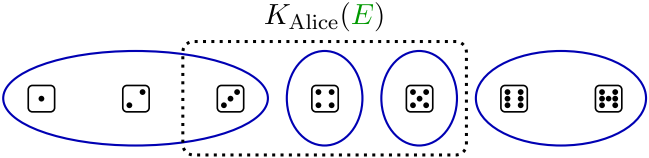

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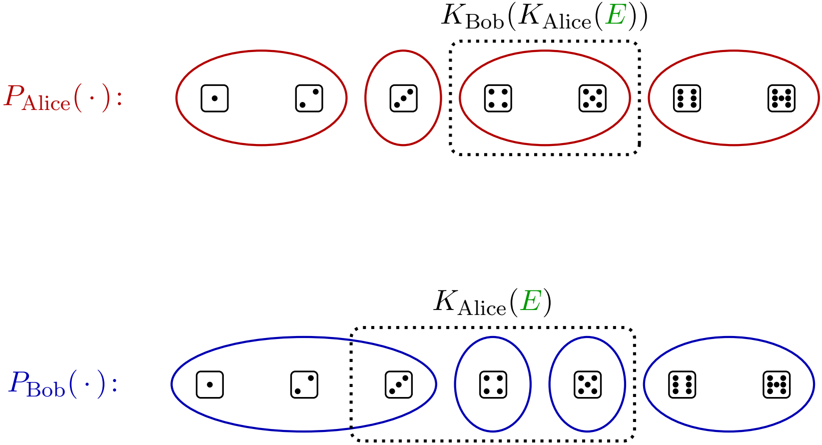

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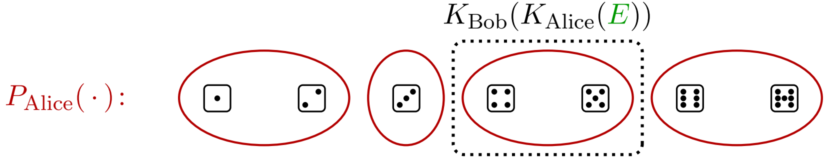

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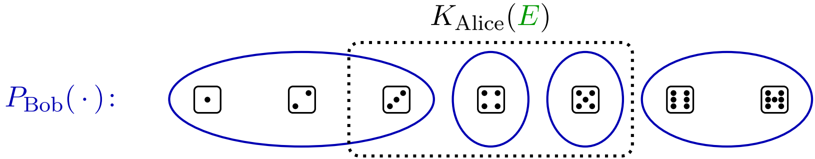

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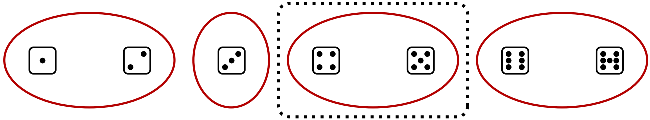

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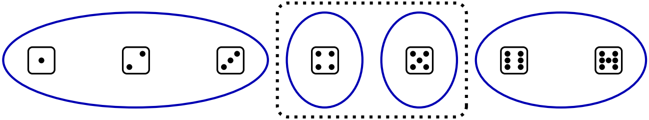

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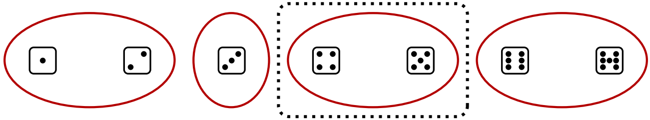

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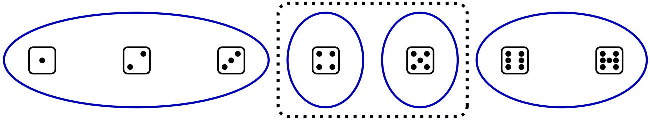

$$K_{\text{Bob}}(K_{\text{Alice}}(K_{\text{Bob}}(K_{\text{Alice}}(E)))) = \{\text{⚃}, \text{⚄}\}.$$

The Exciting Bit

- Alice and Bob have a common prior μ.

Aumann's Agreement Theorem

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- Alice and Bob cannot agree to disagree.

Sketch of Proof

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- $\mu(E \mid F) = q_{\text{Bob}}$.
- $q_{\text{Alice}} = q_{\text{Bob}}$. $\blacksquare$

Disagreement is Unpredictable

- Alice's estimate of some random variable X:

$$\mathbb{E}_{\text{Alice}}(X) = \mathbb{E}(X \mid P_{\text{Alice}}(\omega)).$$

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Theorem ( )

$$\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$$

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Theorem (Hanson [Han02])

It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

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It cannot be that $\mathbb{E}_{\text{Alice}}(\mathbb{E}_{\text{Bob}'}(X)) < \mathbb{E}_{\text{Alice}}(X)$ (or "$>$") is common knowledge between Alice and Bob.

- Alice cannot anticipate the direction of Bob's disagreement.

Image from `relativelyinteresting.com/win-argument-according-science/`.

Questioning our Assumptions

Do we really have a common prior?

Principle of Monotonicity

$$F \subseteq E \implies K(F) \subseteq K(E).$$

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$$K(\text{know axioms}) \subseteq K(\text{know theorems}).$$

Principle of Substitution

$$F = E \implies K(F) = K(E).$$

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$$K(\text{triangle is equilateral}) = K(\text{triangle is equiangular}).$$

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Extension Versus Intension

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The state-space model of knowledge respects extensional equality, but disregards the intentional dimension.

"We publish this observation with some diffidence, since once one has the appropriate framework, it is mathematically trivial. Intuitively, though, it is not quite obvious..."

—Aumann, in his original paper [Aum76]

Alice disagrees

Image from relativelyinteresting.com/win-argument-according-science/.

- Common prior

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Image from `relativelyinteresting.com/win-argument-according-science/`.

- Common prior
- Accept model of knowledge

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Appendix

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