Reasoning About the World

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Invenia Labs, First InveniaCon

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Reasoning

If the butler killed the man, then there must be a pistol.
There is no pistol.
Therefore, the butler did not kill the man.

Reasoning

If the butler killed the man, then there must be a pistol.
There is no pistol.
Therefore, the butler did not kill the man.

If the cook killed the man, then there must be a knife.
There is a knife.
Therefore, the cook killed the man.

Reasoning

If $\overbrace{\text{the butler killed the man}}^{B}$, then $\overbrace{\text{there must be a pistol}}^{P}$.
There is no pistol.
Therefore, the butler did not kill the man.

If $\overbrace{\text{the cook killed the man}}^{C}$, then $\overbrace{\text{there must be a knife}}^{K}$.
There is a knife.
Therefore, the cook killed the man.

If B, then P.
$\overline{P}$.
Therefore, $\overline{P}$.

If C, then K.
K.
Therefore, C.

Reasoning

$$B \implies P$$
$$\overline{P}$$
$$\therefore \overline{B}$$

$$C \implies K$$
$$K$$
$$\therefore C$$

Reasoning

valid:
(modus tollens)

$$B \implies P$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid:
(logical fallacy)

$$C \implies K$$
$$K$$
$$\therefore C$$

valid:
(modus tollens)

$$B \implies P$$
$$\overline{P}$$
$$\therefore \overline{B}$$

?

$$C \implies K$$
$$K$$
$\therefore C$ becomes more plausible

Reasoning

valid:
(modus tollens)

$$B \implies P$$
$$\overline{P}$$
$$\therefore \overline{B}$$

?

$C \implies K$ becomes more plausible
K
$\therefore C$ becomes more plausible

?

$$\begin{array}{l} B \implies P \text{ becomes more plausible} \\ \overline{P} \\ \therefore \overline{B} \text{ becomes more plausible} \end{array}$$

?

$$\begin{array}{l} C \implies K \text{ becomes more plausible} \\ K \\ \therefore C \text{ becomes more plausible} \end{array}$$

Plausible Reasoning

- Propositions have a degree of plausibility.

Plausible Reasoning

- Propositions have a degree of plausibility.
- Reasoning depends on background information.

Notation (Plausibility)

$(A \mid X)$: plausibility of A given background information X.

Plausible Reasoning

Notation (Plausibility)

$(A \mid X)$: plausibility of A given background information X.

Goal: figure out what exactly plausibility is.

Plausible Reasoning: Representation of Plausibility

Assumption (Representation)

Plausible Reasoning: Representation of Plausibility

Assumption (Representation)

- Plausibility is ordered.

Plausible Reasoning: Representation of Plausibility

Assumption (Representation)

- Plausibility is ordered.
- Between any two plausibilities, we can find another plausibility.

Plausible Reasoning: Representation of Plausibility

Assumption (Representation)

- Plausibility is ordered.
- Between any two plausibilities, we can find another plausibility.

Lemma (Representation)

Plausibility can be represented by real numbers.

Plausible Reasoning: Truth

Plausible Reasoning: Truth

Assumption (Truth)

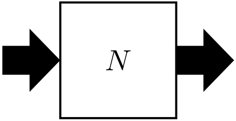
- There exists a plausibility T such that $(A \mid X) \leq \mathsf{T}$ for all A.
- $(\text{tautology} \mid X) = \mathsf{T}$.

Plausible Reasoning: Negation

Plausible Reasoning: Negation

$$(\overline{A} \mid X)$$

Plausible Reasoning: Negation

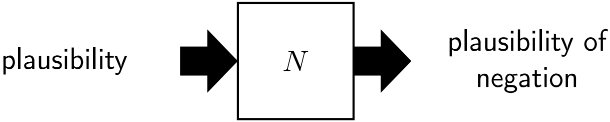


$(\overline{A} \mid X)$

Plausible Reasoning: Negation

$$(A \mid X) \quad \Rightarrow \boxed{N} \Rightarrow \quad (\overline{A} \mid X)$$

Plausible Reasoning: Negation



Plausible Reasoning: Negation

Assumption (Negation)

There exists a decreasing function N such that

$$(\overline{A} \mid X) = N(A \mid X)$$

for all A.

Plausible Reasoning: Negation

Define $\mathsf{F} = N(\mathsf{T})$.

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Then $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$

Plausible Reasoning: Negation

Define $\mathsf{F} = N(\mathsf{T})$.

Then $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$:

- $(\overline{A} \mid X) \leq \mathsf{T}$. (Definition of T)

Define $\mathsf{F} = N(\mathsf{T})$.

Then $\mathsf{F} \leq (A \,|\, X) \leq \mathsf{T}$:

- $(\overline{A} \,|\, X) \leq \mathsf{T}$. (Definition of T)

$\Rightarrow$ $N(\overline{A} \,|\, X) \geq N(\mathsf{T})$. ($N$ is decreasing)

Plausible Reasoning: Negation

Define $\mathsf{F} = N(\mathsf{T})$.

Then $\mathsf{F} \leq (A \,|\, X) \leq \mathsf{T}$:

- $(\overline{A} \,|\, X) \leq \mathsf{T}$. (Definition of T)

$\Rightarrow$ $N(\overline{A} \,|\, X) \geq N(\mathsf{T})$. ($N$ is decreasing)

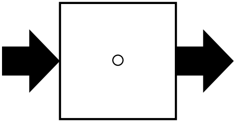
$\Rightarrow$ $(A \,|\, X) \geq \mathsf{F}$. (Definition of N and F)

QED.

Plausible Reasoning: Conjunction

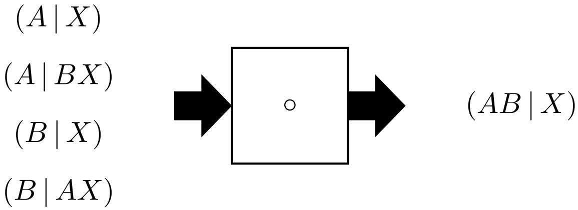
$$(AB \,|\, X)$$

Plausible Reasoning: Conjunction

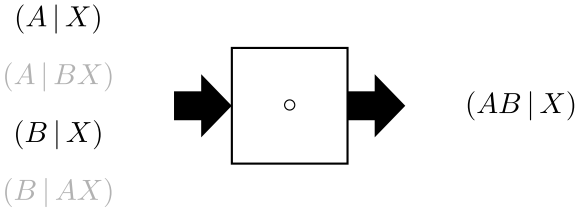


$(AB \,|\, X)$

Plausible Reasoning: Conjunction



Plausible Reasoning: Conjunction



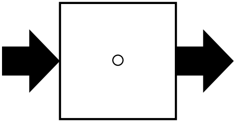
Plausible Reasoning: Conjunction

$(A \,|\, X)$

$(A \,|\, BX)$

$(B \,|\, X)$

$(B \,|\, AX)$



$(AB \,|\, X)$

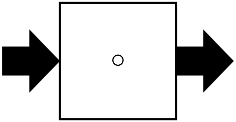
$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{brown hair,} \\
AB &= \text{a blue eye and brown hair.}
\end{aligned}
$$

$(A \mid X) = \text{high}$

$(A \mid BX)$

$(B \mid X) = \text{high}$

$(B \mid AX)$



$(AB \mid X) = \text{high}$

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\begin{aligned}
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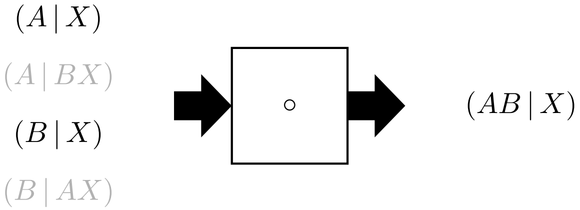
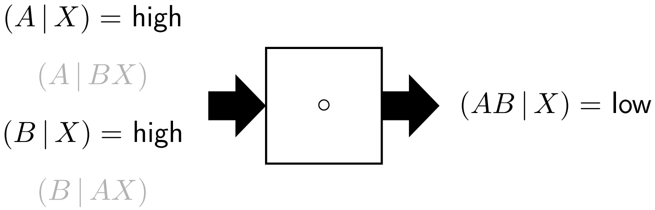
Plausible Reasoning: Conjunction



$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{a green eye,} \\
AB &= \text{a blue eye and a green eye.}
\end{aligned}
$$

Plausible Reasoning: Conjunction



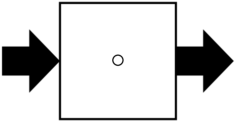
$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{a green eye,} \\
AB &= \text{a blue eye and a green eye.}
\end{aligned}
$$

$(A \mid X)$

$(A \mid BX)$

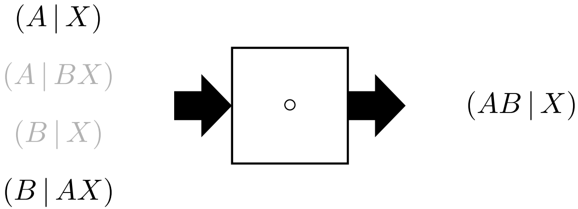
$(B \mid X)$

$(B \mid AX)$



$(AB \mid X)$

Plausible Reasoning: Conjunction



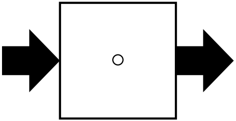
Plausible Reasoning: Conjunction

$(A \,|\, X)$

$(A \,|\, BX)$

$(B \,|\, X)$

$(B \,|\, AX)$



$(AB \,|\, X)$

$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{a green eye,} \\
AB &= \text{a blue eye and a green eye.}
\end{aligned}
$$

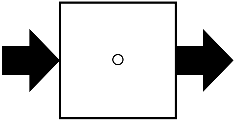
Plausible Reasoning: Conjunction

$(A \mid X) = \text{high}$

$(A \mid BX)$

$(B \mid X)$

$(B \mid AX) = \text{low}$



$(AB \mid X) = \text{low}$

$$
\begin{aligned}
A &= \text{a blue eye,} \\
B &= \text{a green eye,} \\
AB &= \text{a blue eye and a green eye.}
\end{aligned}
$$

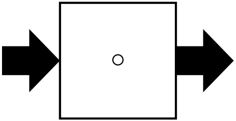
Plausible Reasoning: Conjunction

$(A \,|\, X)$

$(A \,|\, BX)$

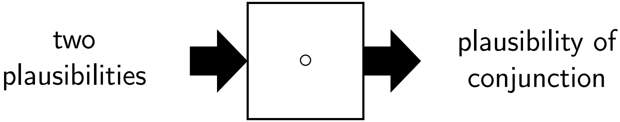
$(B \,|\, X)$

$(B \,|\, AX)$



$(AB \,|\, X)$

Plausible Reasoning: Conjunction



Plausible Reasoning: Conjunction

Assumption (Conjunction)

There exists a function $\circ$ such that

$$(AB \mid X) = (A \mid X) \circ (B \mid AX)$$

for all A and B.

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} =$

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} = x$

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} = x$:

- $(A \mid X) = (A(B + \overline{B}) \mid X)$.

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} = x$:

- $(A \mid X) = (A(B + \overline{B}) \mid X)$.
- $(A(B + \overline{B}) \mid X) = (A \mid X) \circ (B + \overline{B} \mid AX)$. (Definition of $\circ$)

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} = x$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.
- $(A(B + \overline{B}) \,|\, X) = (A \,|\, X) \circ (B + \overline{B} \,|\, AX)$. (Definition of $\circ$)
- $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

Plausible Reasoning: Conjunction

$x \circ \mathsf{T} = x$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.
- $(A(B + \overline{B}) \,|\, X) = (A \,|\, X) \circ (B + \overline{B} \,|\, AX)$. (Definition of $\circ$)
- $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $(A \,|\, X) = (A \,|\, X) \circ \mathsf{T}$.

QED.

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} =$

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} = \mathsf{F}$

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} = \mathsf{F}$:

$\Rightarrow \ (\overline{A\overline{A}} \mid X) = \mathsf{T}.$ (Definition of T)

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} = \mathsf{F}$:

$\Rightarrow$ $(\overline{A\overline{A}} \,|\, X) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T})$.

$x \circ \mathsf{F} = \mathsf{F}$:

$\Rightarrow$ $(\overline{A\overline{A}} \mid X) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $N(\overline{A\overline{A}} \mid X) = N(\mathsf{T})$.

$\Rightarrow$ $(A\overline{A} \mid X) = \mathsf{F}$. (Definitions of N and F)

$x \circ \mathsf{F} = \mathsf{F}$:

- $\Rightarrow$ $(\overline{A\overline{A}} \,|\, X) = \mathsf{T}$. (Definition of T)
- $\Rightarrow$ $N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T})$.
- $\Rightarrow$ $(A\overline{A} \,|\, X) = \mathsf{F}$. (Definitions of N and F)
- $\underbrace{(A\overline{A} \,|\, X)}_{\mathsf{F}} = (A \,|\, X) \circ (\overline{A} \,|\, AX)$. (Definition of $\circ$)

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} = \mathsf{F}$:

- $\Rightarrow \; (\overline{A\overline{A}} \mid X) = \mathsf{T}.$ (Definition of T)
- $\Rightarrow \; N(\overline{A\overline{A}} \mid X) = N(\mathsf{T}).$
- $\Rightarrow \; (A\overline{A} \mid X) = \mathsf{F}.$ (Definitions of N and F)
  - $\underbrace{(A\overline{A} \mid X)}_{\mathsf{F}} = (A \mid X) \circ (\overline{A} \mid AX).$ (Definition of $\circ$)
  - $(\overline{A} \mid AX) = \mathsf{F}.$

Plausible Reasoning: Conjunction

$x \circ \mathsf{F} = \mathsf{F}$:

$\Rightarrow$ $(\overline{A\overline{A}} \,|\, X) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $N(\overline{A\overline{A}} \,|\, X) = N(\mathsf{T})$.

$\Rightarrow$ $(A\overline{A} \,|\, X) = \mathsf{F}$. (Definitions of N and F)

- $\underbrace{(A\overline{A} \,|\, X)}_{\mathsf{F}} = (A \,|\, X) \circ (\overline{A} \,|\, AX)$. (Definition of $\circ$)
- $(\overline{A} \,|\, AX) = \mathsf{F}$.

$\Rightarrow$ $\mathsf{F} = (A \,|\, X) \circ \mathsf{F}$.

QED.

Plausible Reasoning: Conjunction

$$x \circ (y \circ z) = (x \circ y) \circ z$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$(ABC \,|\, X)$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$(ABC \,|\, X) = (A(BC) \,|\, X)$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$\begin{aligned} (ABC \mid X) &= (A(BC) \mid X) \\ &= (A \mid X) \circ (BC \mid AX) \end{aligned}$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big),
\end{aligned}
$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \,|\, X) &= (A(BC) \,|\, X) \\
&= (A \,|\, X) \circ (BC \,|\, AX) \\
&= (A \,|\, X) \circ \Big((B \,|\, AX) \circ (C \,|\, ABX) \Big), \\
(ABC \,|\, X) &= ((AB)C \,|\, X)
\end{aligned}
$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \mid X) &= (A(BC) \mid X) \\
&= (A \mid X) \circ (BC \mid AX) \\
&= (A \mid X) \circ \Big((B \mid AX) \circ (C \mid ABX) \Big), \\
(ABC \mid X) &= ((AB)C \mid X) \\
&= (AB \mid X) \circ (C \mid ABX)
\end{aligned}
$$

Plausible Reasoning: Conjunction

$x \circ (y \circ z) = (x \circ y) \circ z$:

$$
\begin{aligned}
(ABC \,|\, X) &= (A(BC) \,|\, X) \\
&= (A \,|\, X) \circ (BC \,|\, AX) \\
&= (A \,|\, X) \circ \Big((B \,|\, AX) \circ (C \,|\, ABX) \Big), \\
(ABC \,|\, X) &= ((AB)C \,|\, X) \\
&= (AB \,|\, X) \circ (C \,|\, ABX) \\
&= \Big((A \,|\, X) \circ (B \,|\, AX) \Big) \circ (C \,|\, ABX). \text{ QED.}
\end{aligned}
$$

Plausible Reasoning: Conjunction

$$x \circ \mathsf{T} = \mathsf{T} \circ x = x$$
$$x \circ \mathsf{F} = \mathsf{F} \circ x = \mathsf{F}$$
$$x \circ (y \circ z) = (x \circ y) \circ z$$

Plausible Reasoning: Conjunction

$$x \circ \mathsf{T} = \mathsf{T} \circ x = x$$
$$x \circ \mathsf{F} = \mathsf{F} \circ x = \mathsf{F}$$
$$x \circ (y \circ z) = (x \circ y) \circ z$$

$$x \cdot 1 = 1 \cdot x = x$$
$$x \cdot 0 = 0 \cdot x = 0$$
$$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

Cox's Theorem

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

Cox's Theorem

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

- $p(AB \mid X) = p((A \mid X) \circ (B \mid AX)) = p(A \mid X)p(B \mid AX)$

Cox's Theorem

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

- $p(AB \mid X) = p((A \mid X) \circ (B \mid AX)) = p(A \mid X)p(B \mid AX)$
  $\Rightarrow$ $\circ \cong \times$.

Lemma (Product Rule)

There exists a nonnegative, strictly increasing function p such that

$$p(AB \mid X) = p(A \mid X)p(B \mid AX)$$

for all A and B.

- $p(AB \mid X) = p((A \mid X) \circ (B \mid AX)) = p(A \mid X)p(B \mid AX)$
  $\Rightarrow \ \circ \cong \times$.
- $p(B \mid AX) = \dfrac{p(AB \mid X)}{p(A \mid X)}$.

Cox's Theorem

$p(\mathsf{T}) = 1$

$p(\mathsf{T}) = 1$:

- $(A \mid X) = (A(B + \overline{B}) \mid X)$.

Cox's Theorem

$p(\mathsf{T}) = 1$:

- $(A \mid X) = (A(B + \overline{B}) \mid X)$.

$\Rightarrow$ $p(A \mid X) = p(A(B + \overline{B}) \mid AX)$.

Cox's Theorem

$p(\mathsf{T}) = 1$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

$\Rightarrow$ $p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X) p(B + \overline{B} \,|\, AX)$. (Product Rule)

$p(\mathsf{T}) = 1$:

- $(A \mid X) = (A(B + \overline{B}) \mid X)$.

$\Rightarrow$ $p(A \mid X) = p(A(B + \overline{B}) \mid AX)$.

$\Rightarrow$ $p(A \mid X) = p(A \mid X) p(B + \overline{B} \mid AX)$. (Product Rule)

- $(B + \overline{B} \mid AX) = \mathsf{T}$. (Definition of T)

Cox's Theorem

$p(\mathsf{T}) = 1$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

$\Rightarrow$ $p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X)p(B + \overline{B} \,|\, AX)$. (Product Rule)

- $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X)p(\mathsf{T})$.

Cox's Theorem

$p(\mathsf{T}) = 1$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

$\Rightarrow$ $p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X) p(B + \overline{B} \,|\, AX)$. (Product Rule)

- $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X) p(\mathsf{T})$.

$\Rightarrow$ $1 = p(\mathsf{T})$.

QED.

Cox's Theorem

$p(\mathsf{T}) = 1$:

- $(A \,|\, X) = (A(B + \overline{B}) \,|\, X)$.

$\Rightarrow$ $p(A \,|\, X) = p(A(B + \overline{B}) \,|\, AX)$.

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X)p(B + \overline{B} \,|\, AX)$. (Product Rule)

- $(B + \overline{B} \,|\, AX) = \mathsf{T}$. (Definition of T)

$\Rightarrow$ $p(A \,|\, X) = p(A \,|\, X)p(\mathsf{T})$.

$\Rightarrow$ $1 = p(\mathsf{T})$.

QED.

$p(\mathsf{F}) = 0$.

Cox's Theorem

$$0 \leq p(A \mid X) \leq 1$$

Cox's Theorem

$0 \leq p(A \mid X) \leq 1$:

- F $\leq (A \mid X) \leq$ T.

$0 \leq p(A \mid X) \leq 1$:

- $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$.

$\Rightarrow$ $p(\mathsf{F}) \leq p(A \mid X) \leq p(\mathsf{T})$. ($p$ is strictly increasing)

Cox's Theorem

$0 \leq p(A \mid X) \leq 1$:

- $\mathsf{F} \leq (A \mid X) \leq \mathsf{T}$.

$\Rightarrow$ $p(\mathsf{F}) \leq p(A \mid X) \leq p(\mathsf{T})$. ($p$ is strictly increasing)

$\Rightarrow$ $0 \leq p(A \mid X) \leq 1$.

QED.

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \mid X) = 1 - p(A \mid X)$$

for all A.

Cox's Theorem

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \,|\, X) = 1 - p(A \,|\, X)$$

for all A.

- $p(\overline{A} \,|\, X) = p(N(A \,|\, X)) = 1 - p(A \,|\, X)$.

Cox's Theorem

Lemma (Sum Rule)

It holds that

$$p(\overline{A} \mid X) = 1 - p(A \mid X)$$

for all A.

- $p(\overline{A} \mid X) = p(N(A \mid X)) = 1 - p(A \mid X)$.
  $\Rightarrow$ $N \cong 1 - \bullet$.

Cox's Theorem

Theorem (Cox)

Plausibility is probability.

Reasoning: Revisited

valid: (modus tollens)	$B \implies P$ $\overline{P}$ $\therefore \overline{B}$	$B \implies P$ $\overline{P}$ $\therefore \overline{B}$
invalid: (logical fallacy)	$C \implies K$ K $\therefore C$	$C \implies K$ K $\therefore C$

Reasoning: Revisited

valid:
(modus tollens)

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid:
(logical fallacy)

$$C \implies K$$
$$K$$
$$\therefore C$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

$$C \implies K$$
$$K$$
$$\therefore C$$

Reasoning: Revisited

valid: (modus tollens)	$(P \mid BX)$ $\overline{P}$ $\therefore \overline{B}$	$\overbrace{B \implies P}^{X}$ $\overline{P}$ $\therefore \overline{B}$
invalid: (logical fallacy)	$C \implies K$ K $\therefore C$	$C \implies K$ K $\therefore C$

Reasoning: Revisited

valid: (modus tollens)

$$(P \mid BX) = \mathsf{T}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid: (logical fallacy)

$$C \implies K$$
$$K$$
$$\therefore C$$

$$C \implies K$$
$$K$$
$$\therefore C$$

valid: (modus tollens)

$$p(P \mid B X) = 1$$
$$\overline{P}$$
$$\therefore \overline{B}$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid: (logical fallacy)

$$C \implies K$$
$$K$$
$$\therefore C$$

$$C \implies K$$
$$K$$
$$\therefore C$$

Reasoning: Revisited

valid: (modus tollens)	$p(P \mid BX) = 1$ $p(B \mid \overline{P}X) = \ldots$	$\overbrace{B \implies P}^{X}$ $\overline{P}$ $\therefore \overline{B}$
invalid: (logical fallacy)	$C \implies K$ K $\therefore C$	$C \implies K$ K $\therefore C$

Reasoning: Revisited

$$p(B \mid \overline{P} X)$$

Reasoning: Revisited

$$p(B \mid \overline{P}X) = \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

Reasoning: Revisited

$$p(B \mid \overline{P}X) = \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

$$= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

Reasoning: Revisited

$$p(B \mid \overline{P}X) = \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

$$= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Product Rule)}$$

$$= \frac{(1 - p(P \mid BX))p(B \mid X)}{p(\overline{P} \mid X)} \qquad \text{(Sum Rule)}$$

Reasoning: Revisited

$$
\begin{aligned}
p(B \mid \overline{P} X) &= \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{p(\overline{P} \mid BX) p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{(1 - p(P \mid BX)) p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Sum Rule)} \\
&= \frac{(1 - 1) p(B \mid X)}{p(\overline{P} \mid X)} && (X = (B \implies P))
\end{aligned}
$$

Reasoning: Revisited

$$
\begin{aligned}
p(B \mid \overline{P}X) &= \frac{p(B\overline{P} \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{p(\overline{P} \mid BX)p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Product Rule)} \\
&= \frac{(1 - p(P \mid BX))p(B \mid X)}{p(\overline{P} \mid X)} && \text{(Sum Rule)} \\
&= \frac{(1 - 1)p(B \mid X)}{p(\overline{P} \mid X)} && (X = (B \implies P)) \\
&= 0.
\end{aligned}
$$

Reasoning: Revisited

valid:
(modus tollens)

$$p(P \mid BX) = 1 \qquad \overbrace{B \implies P}^{X}$$

$$\overline{P}$$

$$p(B \mid \overline{P}X) = 0 \qquad \therefore \overline{B}$$

invalid:
(logical fallacy)

$$C \implies K \qquad C \implies K$$

$$K \qquad K$$

$$\therefore C \qquad \therefore C$$

Reasoning: Revisited

valid: (modus tollens)	$p(P \mid BX) = 1$ $p(B \mid \overline{P}X) = 0$	$\overbrace{B \implies P}^{X}$ $\overline{P}$ $\therefore \overline{B}$
invalid: (logical fallacy)	$p(K \mid CY) = 1$ $p(C \mid KY) = \ldots$	$\overbrace{C \implies K}^{Y}$ K $\therefore C$

Reasoning: Revisited

$$p(C \mid KY)$$

Reasoning: Revisited

$$p(C \mid KY) = \frac{p(CK \mid Y)}{p(K \mid Y)} \qquad \text{(Product Rule)}$$

Reasoning: Revisited

$$
\begin{aligned}
p(C \mid KY) &= \frac{p(CK \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} && \text{(Product Rule)}
\end{aligned}
$$

Reasoning: Revisited

$$
\begin{aligned}
p(C \mid KY) &= \frac{p(CK \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{1 \cdot p(C \mid Y)}{p(K \mid Y)} && (Y = (C \implies K))
\end{aligned}
$$

Reasoning: Revisited

$$
\begin{aligned}
p(C \mid KY) &= \frac{p(CK \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)} && \text{(Product Rule)} \\
&= \frac{1 \cdot p(C \mid Y)}{p(K \mid Y)} && (Y = (C \implies K)) \\
&= \frac{p(C \mid Y)}{p(K \mid Y)}.
\end{aligned}
$$

Reasoning: Revisited

valid:
(modus tollens)

$$p(P \mid BX) = 1$$

$$p(B \mid \overline{P}X) = 0$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid:
(logical fallacy)

$$p(K \mid CY) = 1$$

$$p(C \mid KY) = \frac{p(C \mid Y)}{p(K \mid Y)}$$

$$\overbrace{C \implies K}^{Y}$$
$$K$$
$$\therefore C$$

Reasoning: Revisited

valid:
(modus tollens)

$$p(P \mid BX) = 1$$

$$p(B \mid \overline{P}X) = 0$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

invalid:
(logical fallacy)

$$p(K \mid CY) = 1$$

$$p(C \mid KY) \geq p(C \mid Y)$$

$$\overbrace{C \implies K}^{Y}$$
$$K$$
$$\therefore C$$

Reasoning: Revisited

valid:
(modus tollens)

$$p(P \mid BX) = 1$$

$$p(B \mid \overline{P}X) = 0$$

$$\overbrace{B \implies P}^{X}$$
$$\overline{P}$$
$$\therefore \overline{B}$$

$$p(C \mid KY) = \frac{p(K \mid CY)p(C \mid Y)}{p(K \mid Y)}$$

Reasoning: Revisited

$$p(B \mid \overline{P} X) = \frac{p(\overline{P} \mid B X) p(B \mid X)}{p(\overline{P} \mid X)}$$

$$p(C \mid K Y) = \frac{p(K \mid C Y) p(C \mid Y)}{p(K \mid Y)}$$

Interpretation of Probability

Plausibility

Probability

$$(A \mid X) \xrightarrow{\quad p \quad} p(A \mid X)$$

Plausibility Probability

$$(A \mid X) \xleftarrow{\quad p^{-1} \quad} p(A \mid X)$$

It is clear that, not only is the quantitative use of the rules of probability theory as extended logic the only sound way to conduct inference; it is the failure to follow those rules strictly that has for many years been leading to unnecessary errors, paradoxes, and controversies.

(Jaynes, 2003, p. 143)