SDS 386D HW1

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#1. Write the joint density:

$f(x,u) \propto M^{-1} 1(u < l(x)/M)\, h(x)\, 1(u \leq 1)$

where $l(x) = (-\log x)^2 \cdot x^2(1-x)/6$

$h(x) = 6x(1-x)$

Find a bound M for $l(x)$ on $(0,1)$

Although the analytical supremum of $l(x)$ is not easy to find, we can plot $l(x)$ numerically and pick $M = \frac{1}{60}$, which is not far from the sharp bound.

My algorithm:

1. Obtain X from p.d.f $h(x)$:

   the c.d.f of $h(x)$ is $3x^2 - 2x^3 =: H(x)$

   $H^{-1}(x)$ on $(0,1)$ is implemented by ~~using~~ a binary search between $(0,1)$ so that it can be easily adapted to higher order polynomials

   and obtain U from a uniform distribution on $(0,1)$

2. check whether $u < \frac{l(x)}{M}$

   if this holds, accept X as a sample from $f(x)$

   else: reject X and sample again.

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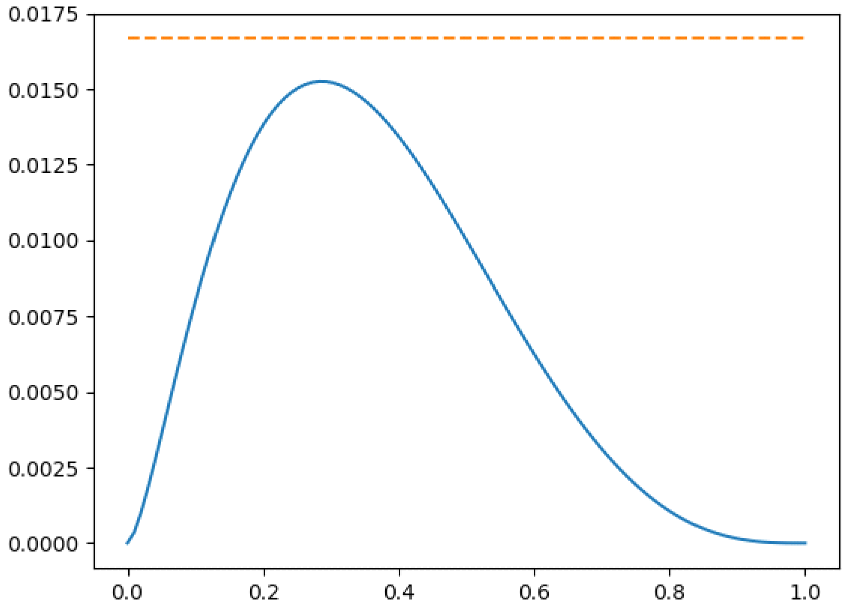

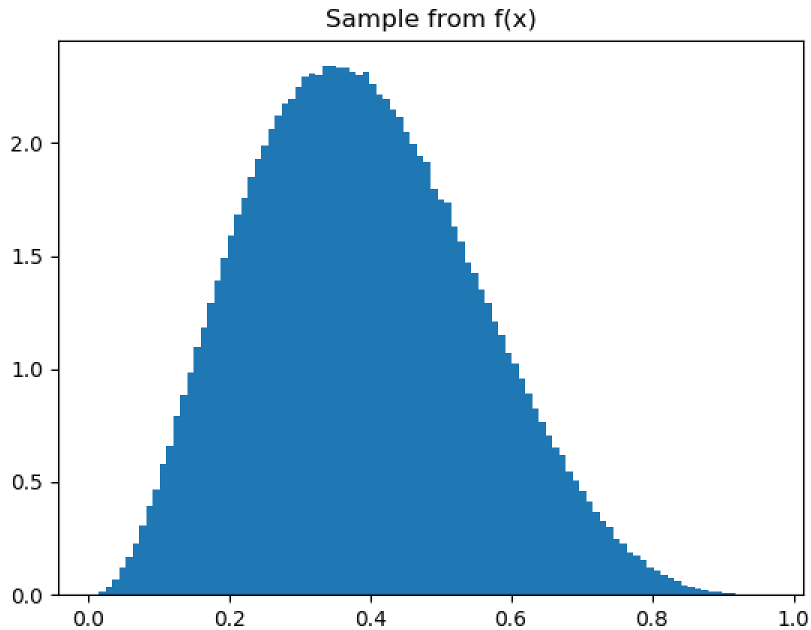

Run the rejection sampling algorithm until a million samples are generated. The samples drawn are shown in the above histogram. We can see that it fits well with curve of f(x).

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Acceptance probability:

The numerical test shows that the acceptance probability is 51.03%.

To get the actual probability, we need to know the normalizing constant in this problem

$$c = \int_0^1 (-\log x)^2 x^3 (1-x)^2$$

This can be obtained by { numerical integration, Monte Carlo integration, Analytically

$c = \frac{919}{108000}$ from the third method and a MC integration using samples from $f(x)$ confirms this result.

$\Rightarrow \Pr\left(u < \frac{l(x)}{M}\right) = \frac{c}{M} \approx 0.5105$ which is consistent with the test result.

An implementation of adaptive rejection sampling is also tested: this implementation follows the method described in the lecture. The proposed piecewise probability enveloping the original probability is restricted to have have 50 pieces at maximum.

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Generate a million accepted samples as in rejection sampling. Only 15089 samples are rejected throughout the whole process and the histogram is similar.

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#2. Use the sampling method from #1, a MC integration can be carried out on this problem:

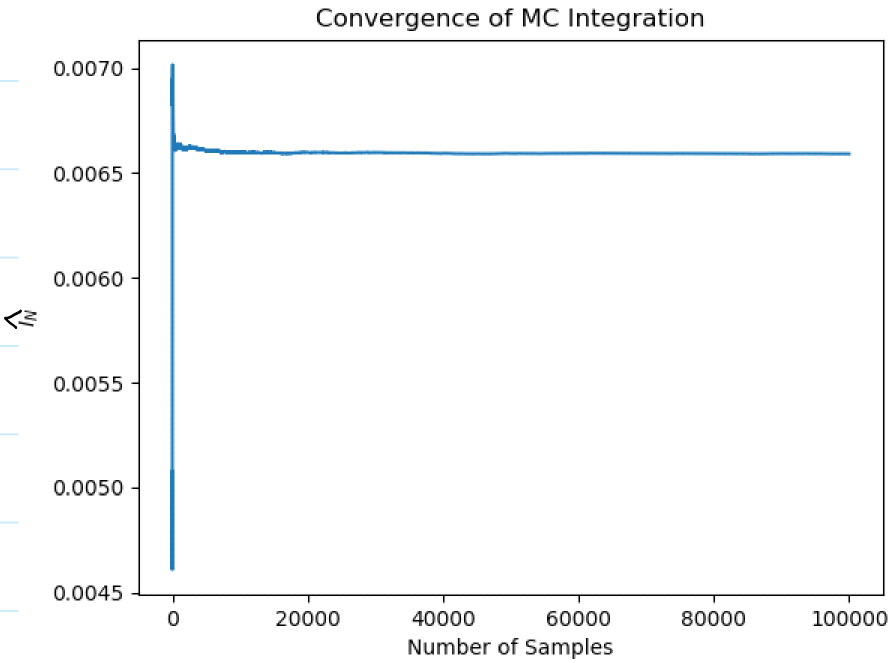
$$I = \int_0^1 c\, f(x)\, (1-x)^{1/2}\, dx.$$

where $c = 919/108000$

Take 10^5 samples from $f(x)$ and use

$$\hat{I}_N = \frac{1}{N}\sum_{i=1}^{N} c\,(1-X_i)^{1/2}$$

The numerical ~~cost~~ gives 0.00659 which is close to the exact solution.



As shown in the convergence plot, the MC approx converges to the exact integration as $N \uparrow$

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Variance:

From lecture, the variance of $\hat{I}_N$ is

$$Var(\hat{I}_N) = \frac{1}{N} Var(g(X_1))$$

$$= \frac{1}{N} \cdot \left[E(g(X)^2) - E(g(X))^2 \right]$$

$$= \frac{1}{N} \left[\int g^2(X) f(X) \, dX - \left(\int g(X) f(x) dx \right)^2 \right]$$

$$\approx \frac{1}{N} \left[\sum_{i=1}^{N} g^2(X_i) / N - \hat{I}_N^2 \right]$$ where X_i is from p.d.f $f(x)$.

Numerical test shows the $Var(\hat{I}_N)$ at $N=1000$ is around 8×10^{-7} (Using the same sample $\{X_i\}$ and $\{g(X_i)\}$)

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#3. $l(x) = f(x)/h(x) = \pi(1+x^2) \cdot \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$

$$0 = l'(x) = \pi(1+x^2)\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}(-x) + \pi(2x)\frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2}$$

$$\Rightarrow \quad (1+x^2)x = 2x$$

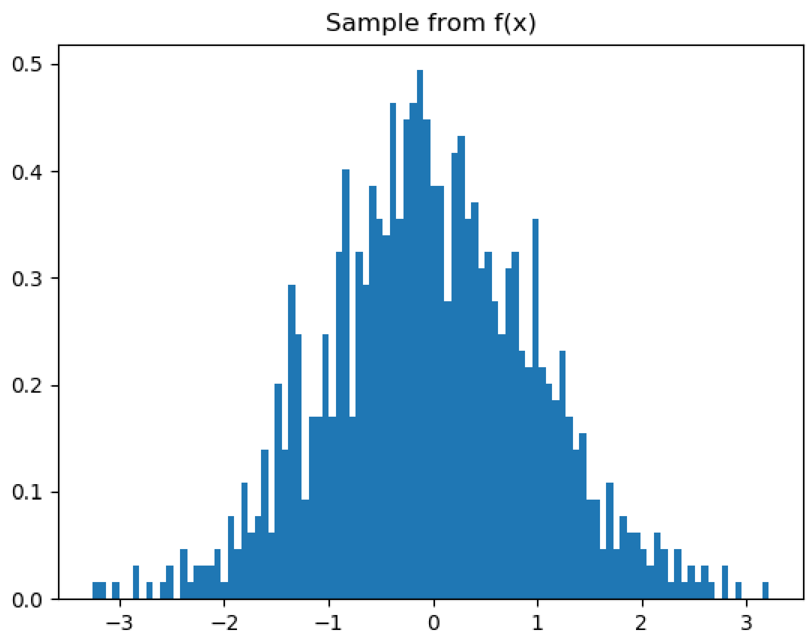
$x=0$ is a saddle point

$x \neq 0$: $\quad 1+x^2 = 2 \quad x = \pm 1$

$$l(\pm 1) = \sqrt{\frac{2\pi}{e}} = M$$

$$P_r = \frac{1}{M} = \sqrt{\frac{e}{2\pi}} \approx 65.8\%$$

Get a 1000 sample from simulation : the acceptance probability is about 64% ~ 68%, and the histogram is shown below. As the number of samples is increased, the acceptance probability will converge to the exact value.



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