Skolem's Paradox

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July 23, 2025

Abstract

In this paper, we will look at Skolem's paradox, including necessary concepts and definitions, why it seems like an issue for mathematics, its mathematical resolution, and philosophical interpretations.

1 Introduction

In the nineteenth century, an important realization was made: all of mathematics can be encoded encapsulated by sets, with the only relation being $\in$, denoting set membership. With this realization, we came up with a relatively unassuming list of axioms of set theory called Zermelo-Frankel set theory (ZFC). With ZFC being the basis for all of mathematics, sets became a crucial and, technically, the only object of study. One important relation for sets is being the same size. Intuitively, two sets are the same size if and only if you can pair off each of the elements of the sets so that no member of either set is left out after the pairing. Surprisingly, Cantor showed in the late nineteenth century that there are different sizes of infinity. Slightly more precisely, for any set A, the set of all subsets of A, written 2^A[1], (because for any finite set A with n elements, the

[1] In general, for sets A, B, B^A denotes the set of all functions from A to B. There is a natural bijection between 2^A interpreted as the set of all functions from A to $2 = \{0, 1\}$ as constructed in set theory and the set 2^A interpreted as the power set of A. The correspondence is $\Psi(f) = \{x \in A \mid f(x) = 1\}$, whose

power set of A has exactly 2^n elements) is larger than A. However, the the Löwenheim-Skolem theorems tells us that there is a model of ZFC that is countable, meaning has the same size as the set of all natural numbers $\mathbb{N}$. On the face of it, we seem to have arrived at a contradiction–on the one hand, we know that $2^{\mathbb{N}}$ must be larger than $\mathbb{N}$ by Cantor's theorem. On the other hand, in our model of ZFC, there are only countably many sets, so surely $2^{\mathbb{N}}$, being a subset of a countable model[2] must be countable as well. In this paper, we will make essential notions precise, explore how this paradox is resolved mathematically, and finally explore some philosophical issues posed by the Löwenheim-Skolem theorems.

2 Mathematical Perspective

As mentioned in the previous section, as sets are a fundamental object of study, we wanted a precise notion of what it means for two sets to have the same size. Cantor made this relation precise by saying sets A and B have the same size if there exists a bijection $f : A \to B$. What is a bijection, then? A bijection $f : A \to B$ is a function[3] from one set to the other that has an inverse $g : B \to A$, meaning that $g \circ f = \mathrm{id}_A$ and $f \circ g = \mathrm{id}_B$, where $\circ$ denotes the composition of functions and id stands for the identity function, i.e. $\mathrm{id}_A(x) = x$ for every x in A, and likewise for id_B. Equivalently, f is a bijection if f is both injective and surjective, where the former condition means that for any x and y in A, $f(x) = f(y)$ only if $x = y$, and the latter condition means that for any z in B, there exists some x in A such that $f(x) = z$. This definition is also equivalent to the existence of a set of pairs of elements whose first element is a member of A, whose second element is a member of B, and is such that every element of A and B is in exactly

inverse is given by $\Psi^{-1}(A')(x) = \begin{cases} 0, & \text{if } x \notin A' \\ 1, & \text{if } x \in A' \end{cases}$

[2]More precisely, a subset of the domain of the model, where informally a model can be thought of as a collection of things called the domain, relations, functions, and constants that obey a set of rules

[3]As all of mathematics can be encoded in set theory, it is indeed the case that a function is a specific type of set, but the precise definition will be omitted as it is unimportant for our purposes and unenlightening. All that matters to us is that a function $f : A \to B$ is such that for any set $x \in A$, $f(x)$ is a unique set in B.

one pair. With this definition of what it means for two sets to have the same size, Cantor showed that for any set A, 2^A does not have the same size as A. The proof is interesting and quick, so we will see the proof, which is by contradiction.

Proof. Suppose for a contradiction that for some set A, we have a bijection $f : A \to 2^A$. Then, in particular, f is surjective. We will now show that there cannot exist a surjective function $f : A \to 2^A$. If such a function f did exist, we let $D = \{x \in A \mid x \notin f(x)\}$ be the set of all elements of A that are not members of the subset they are assigned to by f. We will now show that there does not exist any element $x \in A$ such that $f(x) = D$. Suppose for a contradiction that there were some $x \in A$ with $f(x) = D$. Now we consider whether or not x is a member of D. If $x \in D$, then $x \notin f(x)$ by definition of D. But as $f(x) = D$, we have $x \notin D$, which is a contradiction. On the contrary, if $x \notin D$, as $D = f(x)$, we have $x \notin f(x)$, and thus $x \in D$ by definition of D, also reaching a contradiction. This shows our supposition that there is no $x \in A$ with $f(x) = D$, so f is not surjective. $\square$

Further, we can find a subset of 2^A that is the same size as A, namely by constructing the function $f : A \to 2^A$ so that $f(x) = \{x\}$ for every $x \in A$. Thus, A is smaller than 2^A. Using this definition of size, we say an infinite set is countable if it has the same size as $\mathbb{N}$ and is said to be uncountable otherwise.

Next, we move on to models. Given a language L, and L-structure A is a choice of a set, called the domain of A, and for each symbol in L, an interpretation on A, where the symbols are allowed to be relations, functions, and constants. Then, for a theory T in the language L (a set of L-sentences), a model of T is an L-structure M such that every sentence in T is true in M. That M is a model of T is denoted by $M \models T$. We say a structure is countable if its domain is. The Löwenheim-Skolem theorems essentially tell us that there is a countable model of ZFC, which seems to contradict Cantor's theorem that ZFC models "there is an uncountable set", expressed as ZFC $\models \exists x U(x)$ where $U(x)$ is shorthand for the precise mathematical description of what it means to be uncountable.[4] This is because if we let M be a countable model of ZFC and x be some uncountable set, then $x \subset M$, but M is countable, so x must also be countable!

The paradox may be strengthened by taking strengthened theorems, which ensure that the countable model's interpretation of membership agrees with the actual notion of set membership, i.e., $M \models x \in y$ if and only if $x \in y$, and also that M's interpretation of $f : \mathbb{N} \to x$ is a bijection agrees with the actual meaning of $f : \mathbb{N} \to x$ being a bijection. Thus, the paradox does not stem from the model's interpretation of the membership relation or the model's interpretation of what bijection means; instead, it must stem from the model's interpretation of its quantifiers. There are, in fact, two interpretations of $U(x)$, one being that there do not exist any bijections between x and $\mathbb{N}$ in the actual universe of sets, which we will call the English interpretation, and the other being that in the domain of M, there do not exist any bijections between x and $\mathbb{N}$, which we will call the model interpretation. In other words, on the English interpretation, the existential quantifier in $U(x) \equiv \nexists f : \mathbb{N} \to x$ "that is a bijection", scopes over the entire actual universe of sets to search for such an f, whereas in the model interpretation, the existential quantifier just scopes over the domain of M. Thus, it is possible for $M \models U(x)$ to be true, while in actuality, $U(x)$ is false because it is possible that all bijections between $\mathbb{N}$ and x lie outside of M.

Similarly, if M is a countable model of ZFC, then a set $\mathbb{R}$ of the real numbers inside the model M exists. However, there are uncountably many real numbers, so what M interprets as the real numbers is a subset of the actual set of real numbers that lie inside M, hence the real numbers in M are missing quite a few real numbers that lie in the actual set of real numbers. Therefore, from the mathematical perspective, there is nothing paradoxical about the assertion that there is a countable model of ZFC that has uncountable sets because the first instance of countable is relative to the actual universe of sets, whereas the instance of uncountable is a notion relative to the model we are referring to, which departs from notions of countability from the perspective of the ac-

[4] One potential description is as follows: $(\exists g \in A^B)(\forall a \in A)(\forall b \in B)(g(f(a)) = a \wedge f(g(b)) = b)$

tual universe. It is worth noting, however, that Skolem's results rely on the assumption that we are working with first-order logic. In 1930, Zermelo showed that second-order models of ZFC correctly compute power sets and sizes of sets. Further, in 1987, Tennant and McCarty showed that the Löwenheim-Skolem theorem fails in constructivist set theory. Thus, the Löwenheim-Skolem theorem is a feature of first-order models of set theory.[Bay25]

3 Philosophical Perspective

Set theory has long been a central aspect of mathematics, leading to various philosophical interpretations of what sets truly are. One view that Skolem favored is known as the algebraic conception of sets. According to this perspective, sets do not exist as independent objects we can discover; instead, they are merely the objects within the universes or models of set theory, defined by the axioms we choose. It is important to understand that there is no single, distinguished universe of sets because numerous models of set theory exist. What matters most are the axioms, interpreted as relations we impose.

In contrast, another popular philosophical view in mathematics is realism, which posits that mathematical objects exist objectively outside human activity. Platonism is a common subview of realism, which asserts that mathematical objects exist independently in an abstract realm. According to these perspectives, sets are real entities that mathematicians discover rather than invent, and there is a single, objective universe of sets.

To clarify the differences, the realist view argues that an intended model of set theory exists in an abstract realm, and our axioms of set theory attempt to describe this universe of sets. On the other hand, from the algebraic perspective, the intended models of set theory are simply those models that satisfy our chosen axioms. This latter view was also supported by Hilbert, who claimed that we can replace abstract concepts like lines, points, and planes with concrete objects like beer mugs, tables, and chairs as long as these ordinary objects maintain the correct relationships with one another.

For the remainder of the paper, we will discuss the traditional interpretation of Skolem's attack on set theory. First, we need to discuss what it means for a property to be relative and absolute, being opposite concepts. A property is relative if it cannot be precisely captured by set theory. For instance, the Löwenheim-Skolem theorems show that countability is a relative notion since $\{x \mid M \models y \in x\}$ is countable, and yet $M \models U(x)$ for a model M of ZFC. However, one can show that for any model M of ZFC, a set $x \in M$ satisfies $(\forall y \in M)(y \notin x)$ if and only if $\{y \mid M \models y \in x\}$ is empty. Thus, being empty is an absolute concept–while the empty set in one model of ZFC may be nonempty in another, we can still pin down the set-theoretic property of being empty in a precise way that is independent of the model.

In the same way, we can pin down what it means for a set to be of size n for any $n \in \mathbb{N}$.[Bay25] However, we cannot pin down the concept of countability or uncountability in the same way that is independent of a given model, at least within first-order logic due to the Löwenheim-Skolem theorems. Thus, there are set theoretical concepts that we cannot pinpoint precisely based on algebraic conception.

Skolem saw this as a problem because he thought that classical paradoxes of set theory, such as Russel's paradox, showed that the algebraic interpretation of automatization was the only reasonable interpretation, but under the algebraic conception, we cannot use first-order logic to pin down what is meant by countable or uncountable. Skolem thought that the classical paradoxes highlighted that our intuitions about sets are not good enough, so the only reliable way of understanding our axioms is algebraic, which is bad because this means that some set-theoretic notions are inherently relative.

To me, one of the two weakest parts of Skolem's argument is that the algebraic conception of automatization is the only acceptable view. Although the classical paradoxes do show that our naive intuitions about sets may be incorrect, I do not think this shows that there is no intended universe of sets we are trying to describe with our axioms; rather, just that we may have to do some work to understand the objects we are trying to capture. Moreover, there are a couple of pleasing consequences if we accept Platonism. Firstly, we can avoid relativism because, according to Platonism, a single, unique universe of sets $V exists$. Because of this, even though there exist models of ZFC that internally think $2^{\mathbb{N}}$ is uncountable, we can, from the perspective of V, see that $2^{\mathbb{N}}$[5] is countable because we have an absolute framework. To highlight the difference in philosophy, the algebraic approach posits that all models of our axioms are on equal footing, so there is no such absolute perspective from which we can view uncountability. I also find Platonism appealing because of the incredible success of mathematics–if mathematical objects exist as abstract entities, then it seems reasonable that by studying these objects, we can come to understand reality better. On the other hand, if the best we can do is give a list of axioms and then study the things that obey these axioms, it does seem rather remarkable that we can understand reality by just studying algebraic structures that we define arbitrarily.

Another weak point of Skolem's argument is that the Löwenheim-Skolem theorems rely heavily on first-order models of ZFC. Indeed, if we strengthen our language by allowing for second-order logic or weaken our language by requiring constructive mathematics, the Löwenheim-Skolem theorems are avoided. Thus, it is implicit in Skolem's argument that only first-order models of set theory are acceptable. While second-order theories fail to have some nice model-theoretic properties that first-order theories have, such as compactness, they have other nice properties, like more expressibility, because second-order languages can quantify overall properties. Because of this heightened expressive power of language, second-order theories can precisely pin down notions of the size of sets. Ultimately, I think a much more robust argument needs to be made in favor of first-order logic instead of second-order or even constructive mathematics for Skolem's argument to go through.

Ultimately, while the Löwenheim-Skolem theorems seem to pose an issue for ZFC, a mathematical resolution resolves the paradox. While Skolem and contemporaries

[5]Here $2^{\mathbb{N}}$ is the power set in the countable model, not from the perspective of V

have argued that the relativity of uncountability serves as a problem for the theory, I think there are too many other important philosophical assumptions that need more of a case to be accepted, such as whether the algebraic conception of axiomatization is the correct interpretation or whether first-order logic is really needed as opposed to second-order logic or constructivism.

References

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